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FROM AZTEC DIAMONDS TO PYRAMIDS: STEEP TILINGS

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ABSTRACT. We introduce a family of domino tilings that includes tilings of the Aztec diamond and pyramid partitions as special cases. These tilings live in a strip of $\mathbb{Z}^2$ of the form $1 \leq x - y \leq 2\ell$ for some integer $\ell \geq 1$, and are parametrized by a binary word $w \in \{+,-\}^{2\ell}$ that encodes some periodicity conditions at infinity. Aztec diamond and pyramid partitions correspond respectively to $w = (+-)^\ell$ and to the limit case $w = +^\infty -^\infty$. For each word w and for different types of boundary conditions, we obtain a nice product formula for the generating function of the associated tilings with respect to the number of flips, that admits a natural multivariate generalization. The main tools are a bijective correspondence with sequences of interlaced partitions and the vertex operator formalism (which we slightly extend in order to handle Littlewood-type identities). In probabilistic terms our tilings map to Schur processes of different types (standard, Pfaffian and periodic). We also introduce a more general model that interpolates between domino tilings and plane partitions.

1. Introduction

The Aztec diamond of order ℓ consists of all unit squares of the square lattice that lie completely within the region $|x| + |y| \leq \ell + 1$, where ℓ is a fixed integer. The Aztec diamond theorem states that the number of domino tilings of the Aztec diamond of order ℓ is $2^{\ell(\ell+1)/2}$ [EKLP92a] (Figure 1 displays two among the 1024 domino tilings of the Aztec diamond of order 4). A more precise result [EKLP92b] states that the generating polynomial of these tilings with respect to the minimal number of flips needed to obtain a tiling from the one with all horizontal tiles is $\prod_{i=1}^{\ell}(1 + q^{2i-1})^{\ell+1-i}$. See below for the definition of a flip.

A pyramid partition is an infinite heap of bricks of size $2 \times 2 \times 1$ in $\mathbb{R}^3$, as shown on Figure 2. A pyramid partition has a finite number of maximal bricks and each brick rests upon two side-by-side bricks, and is rotated 90° from the bricks immediately below it. The *fundamental pyramid partition* is the pyramid partition with a unique maximal brick. We denote by a_n the number of pyramid partitions obtained from the fundamental pyramid partition after the removal of n bricks.

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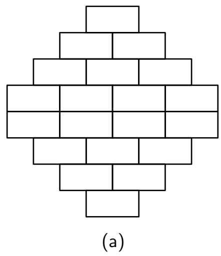

FIGURE 1. Two domino tilings of the Aztec diamond of size 4: the minimal tiling (a), and another tiling (b).

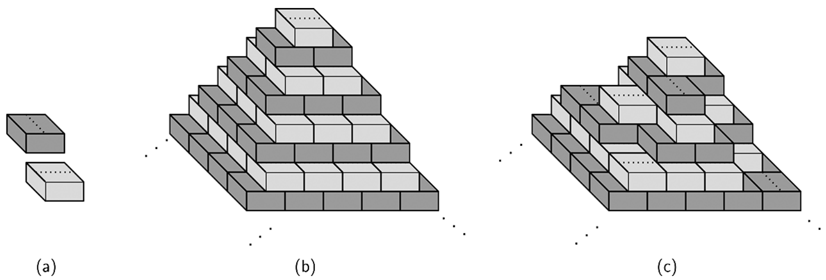

FIGURE 2. (a) The two types of bricks used in the construction of pyramid partitions; (b) the fundamental pyramid partition, from which all others are obtained by removing some bricks; (c) a pyramid partition.

Kenyon [Ken05] and Szendrői [Sze08] conjectured the beautiful formula

$$\sum_n a_n q^n = \prod_{k\geq 1} \frac{(1+q^{2k-1})^{2k-1}}{(1-q^{2k})^{2k}} \tag{1}$$

which was proved by Young in two ways: first by using the domino shuffling algorithm [You09], then by using vertex operators [You10].

Aztec diamond and pyramid partitions are closely related. Indeed a pyramid partition can be seen as a domino tiling of the whole plane; see Figure 3. In this setting the removal of a brick corresponds to the flip of two dominos. The goal of this paper is to show that these are indeed part of the same family of tilings that we call *steep tilings*. For good measure we also encompass the so-called plane overpartitions [CSV11].

Informally speaking, steep tilings are domino tilings of an oblique strip, i.e., a strip tilted by 45° in the square lattice, that satisfy a "steepness" condition which amounts to the presence of "frozen regions" (more precisely, periodic repetitions of the same pattern) sufficiently far away along the strip in both infinite directions. We may furthermore consider different types of boundary conditions along the two rims of the strip, namely pure, free, mixed and periodic boundary conditions (Aztec diamonds and pyramid partitions corresponding to the pure case). For a given asymptotic pattern and for each type of boundary conditions, we are able to derive an elegant product formula for the generating function of the associated

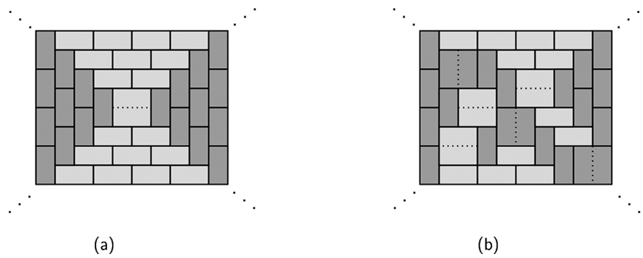

FIGURE 3. (a) The fundamental pyramid partition viewed from the top. It induces a domino tiling of the plane. (b) The pyramid partition of Figure 2(c) viewed from the top.

tilings (in the case of pure boundary conditions it can be interpreted as a hook-length type formula). Our derivation is based on a bijection between steep tilings and sequences of interlaced partitions, and the vertex operator formalism which allows for efficient computations. Let us mention that, in probabilistic terms, the sequences of interlaced partitions that we encounter form Schur processes, either in their original form [OR03] for pure boundary conditions, in their Pfaffian variant [BR05] for mixed boundary conditions, in their periodic variant [Bor07] for periodic boundary conditions, or finally in a seemingly new "reflected" variant for (doubly) free boundary conditions. The present article however focuses on the combinatorial results, and probabilistic implications will be explored in subsequent papers.

We now present the organization of the paper. Section 2 is devoted to the basic definitions of steep tilings (Section 2.1), boundary conditions and flips (Section 2.2), which we need to state our main results (Section 2.3). In Section 3, we discuss the bijections between steep tilings and other combinatorial objects: particle configurations (Section 3.1), sequences of integer partitions (Section 3.2) and height functions (Section 3.3). In Section 4, we highlight some special cases: domino tilings of Aztec diamond tilings (Section 4.1), pyramid partitions (Section 4.2) and plane overpartitions (Section 4.3). In Section 5 we compute generating functions of steep tilings via the vertex operator formalism: we first treat the case of so-called pure boundary conditions (Section 5.1) before discussing arbitrary prescribed boundary conditions (Section 5.2) and finally free boundary conditions (Section 5.1), where we provide a new vertex operator derivation of the Littlewood identity. In Section 6 we consider the case of periodic boundary conditions (i.e. cylindric steep tilings). In Section 7, we define a more general model that interpolates between steep tilings and (reverse) plane partitions: after the definition of the model in terms of matchings (Section 7.1), we turn to the discussion of pure boundary conditions and flips (Section 7.2) before the enumerative results (Section 7.3); we then reinterpret the extended model in terms of tilings (Section 7.4) and explain how it specializes to steep tilings (Section 7.5) and plane partitions (Section 7.6). Concluding remarks are gathered in Section 8.

2. Steep tilings of an oblique strip, and the main results

In this section we define steep tilings and present our main results. Several notions given here will become more transparent in Section 3 where we will study

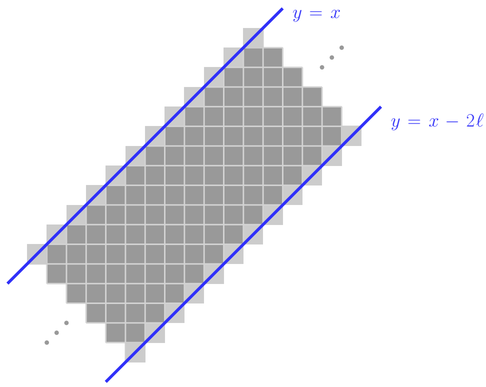

FIGURE 4. A portion of the oblique strip of width $2\ell = 10$. The region R covered by dominos necessarily includes the whole darker region, and may or may not include the lighter boundary squares.

in greater depth the structure of these objects, and in particular their connection with height functions.

2.1. **Steep tilings of an oblique strip.** Let us start by describing the general family of domino tilings we are interested in. Recall that a *domino* is a 2×1 (horizontal domino) or 1×2 (vertical domino) rectangle whose corners have integer coordinates. Fix a positive integer ℓ, and consider the *oblique strip* of width 2ℓ which is the region of the Cartesian plane comprised between the lines $y = x$ and $y = x - 2\ell$. A *tiling* of the oblique strip is a set of dominos whose interiors are disjoint, and whose union R, which we call the *tiled region*, is "almost" the oblique strip in the sense that

$$\{(x,y) \in \mathbb{R}^2, |x-y-\ell| \leq \ell - 1\} \subset R \subset \{(x,y) \in \mathbb{R}^2, |x-y-\ell| \leq \ell + 1\}; \tag{2}$$

see Figure 4. We are forced to use this slightly unusual definition for a tiling since the oblique strip itself clearly cannot be obtained as a union of dominos, as it is not a union of unit squares with integer corners. Observe that R necessarily contains every point of the oblique strip with integer coordinates.

Following a classical terminology [CEP96], we say that a horizontal (resp. vertical) domino is *north-going* (resp. *east-going*) if the sum of the coordinates of its top left corner is odd, and *south-going* (resp. *west-going*) otherwise. We are interested in tilings of the oblique strip which are *steep* in the following sense: going towards infinity in the north-east (resp. south-west) direction, we eventually encounter only north- or east-going (resp. south- or west-going) dominos. Figure 5 displays an example of such a tiling. The reason for which we use the term "steep" is that the associated height functions (to be defined in Section 3.3) grow eventually at the maximal possible slope.

Note that any domino covering a square crossed by the boundary $y = x$ (resp. $y = x - 2\ell$) is either north- or east-going (resp. south- or west-going) and thus, sufficiently far away in the south-west direction (resp. north-east direction), all such squares are uncovered. A further property of steep tilings is that they are eventually periodic in both directions, as expressed by the following proposition.

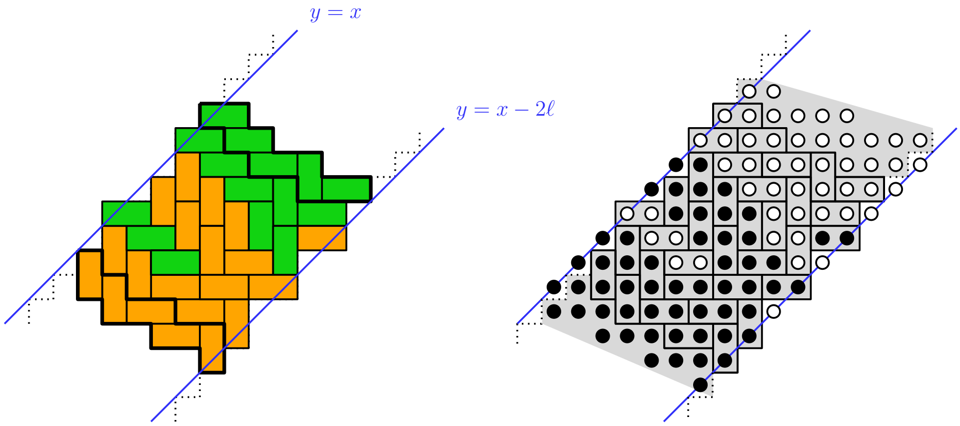

FIGURE 5. Left: a steep tiling of the oblique strip of width $2\ell = 10$. North- and east-going (resp. south- and west-going) dominos are represented in green (resp. orange). Outside of the displayed region, the tiling is obtained by repeating the "fundamental patterns" surrounded by thick lines. Right: the associated particle configuration, as defined in Section 3.

Proposition 1. *Given a steep tiling of the oblique strip of width 2ℓ, there exists a unique word $w = (w_1, \dots, w_{2\ell})$ on the alphabet $\{+, -\}$ and an integer A such that, for all $k \in \{1, \dots, \ell\}$, the following hold:*

- *for all $x > A$, $(x, x - 2k)$ is the bottom right corner of a domino which is north-going if $w_{2k-1} = +$ and east-going if $w_{2k-1} = -$,*
- *for all $x < -A$, $(x, x - 2k + 2)$ is the top left corner of a domino which is west-going if $w_{2k} = +$ and south-going if $w_{2k} = -$.*

Proof. Pick A' large enough so that the region $x > A'$ only contains north- or east-going dominos and the region $x < -A'$ only contains south- or west-going dominos. For $k \in \{1, \dots, \ell\}$ and $x > A'$, consider the unit square with bottom right corner $(x, x - 2k)$. It is necessarily included in the tiled region by (2), and thus covered either by a north-going or an east-going domino. We then set $w_{2k-1}^{(x)} = +$ in the former case and $w_{2k-1}^{(x)} = -$ in the latter. Observe that we cannot have $w_{2k-1}^{(x)} = -$ and $w_{2k-1}^{(x+1)} = +$ as otherwise the corresponding dominos would overlap. This ensures that the sequence $(w_{2k-1}^{(x)})_{x>A'}$ is eventually constant, with value $w_{2k-1} \in \{+, -\}$. Similarly, by considering the unit square with top left corner $(x, x-2k+2)$, $x < -A'$, we define a sequence $(w_{2k}^{(x)})_{x<-A'}$ which is eventually constant with value w_{2k}. The proposition follows by taking A large enough. $\square$

Example 1. The steep tiling of Figure 5 corresponds to the word $w = (+ + + + + - - - + +)$.

The word w of Proposition 1 is called the *asymptotic data* of the steep tiling. We denote by $\mathcal{T}_w$ the set of steep tilings of asymptotic data w, considered up to translation along the direction $(1, 1)$.

2.2. **Boundary conditions and flips.** Let us now introduce a few further definitions needed to state our main results. First, we discuss the different types of "boundary conditions" that we may impose on steep tilings. What we call boundary conditions correspond actually to the shape of the tiled region, since by (2) only the unit squares centered on the lines $y = x$ and $y = x - 2\ell$ are in an unspecified (covered/uncovered) state; see again Figure 4. Recall that the steepness condition imposes that all unit squares centered on the line $y = x$ (resp. $y = x - 2\ell$) are eventually uncovered when going towards infinity in the south-west (resp. north-east) direction, and conversely are eventually covered in the opposite direction (as a consequence of Proposition 1). When we impose no further restriction on the shape, we say that we have *free boundary conditions*. A steep tiling is called *pure* if there is no "gap" between uncovered squares on each of the two boundaries, i.e., if there exists two half-integers a, b such that the following two conditions hold:

(a) the unit square centered at (x, x) is covered if $x \geq a$ and uncovered if $x < a$,
(b) the unit square centered at $(x, x - 2\ell)$ is covered if $x \leq b$ and uncovered if $x > b$.

In this case we say of course that we have *pure boundary conditions*. We will see (Remark 1 page 5933) that the quantity $b - a$ is determined by the word w. We denote by $\mathcal{T}_w^0$ the set of pure steep tilings of asymptotic data w, considered up to translation along the direction $(1, 1)$. We may also have *mixed boundary conditions* if (a) holds but not (b), or vice versa. Finally, *periodic boundary conditions* correspond to the case where the shape of the tiled region is such that the two boundaries "fit" into one another (precisely, there exists an integer c such that, for each half-integer x, the unit square centered at (x, x) is covered if and only if the unit square centered at $(x + c, x + c - 2\ell)$ is uncovered). Upon identifying the two boundaries, we obtain a *cylindric steep tiling*.

We now introduce the notion of flip. A *flip* is the operation which consists in replacing a pair of horizontal dominos forming a 2×2 block by a pair of vertical dominos, or vice versa. A flip can be horizontal-to-vertical or vertical-to-horizontal with obvious definitions. We say that the flip is centered on the k-th diagonal if the center of the 2×2 block lies on the diagonal $y = x - k$, for $0 < k < 2\ell$. In the case of free boundary conditions, we also consider *boundary flips* centered on the 0-th or on the 2ℓ-th diagonals, where only one domino covering a boundary square is rotated (see Figure 10 below) and where the shape of the tiled region is modified. In the case of periodic boundary conditions, boundary flips must be performed simultaneously on both sides in order to preserve periodicity (see the discussion in Section 6). In all cases, a vertical-to-horizontal flip centered on the k-th diagonal with k even and a horizontal-to-vertical flip centered on the k-th diagonal with k odd are called *ascendent*, other flips being called *descendent*. We will see in the next section that for each word $w \in \{+, -\}^{2\ell}$, there exists a unique element of $\mathcal{T}_w^0$, called the *minimal tiling*, such that every element of $\mathcal{T}_w$ (resp. $\mathcal{T}_w^0$) can be obtained from it using a sequence of ascendent flips (resp. ascendent non-boundary flips). Such sequences turn out to have the smallest possible length among all possible sequences of flips between the minimal tiling and the tiling at hand, and furthermore for each $0 \leq k \leq 2\ell$ the number of flips centered on the k-th diagonal is independent of the chosen sequence.

2.3. **Main results.** We are now ready to state our main theorems. We first treat the simplest case of pure tilings.

Theorem 2. *Let $w \in \{+,-\}^{2\ell}$ be a word. Let $T_w(q)$ be the generating function of pure steep tilings of asymptotic data w, where the exponent of q records the minimal number of flips needed to obtain a tiling from the minimal one. Then one has*

$$T_w(q) = \prod_{\substack{i<j \\ w_i=+,\ w_j=- \\ i-j \text{ odd}}} \left(1+q^{j-i}\right) \prod_{\substack{i<j \\ w_i=+,\ w_j=- \\ i-j \text{ even}}} \frac{1}{1-q^{j-i}}. \tag{3}$$

Theorem 3. *Let $w \in \{+,-\}^{2\ell}$ be a word. Let $T_w \equiv T_w(x_1,\dots,x_{2\ell-1})$ be the generating function of pure steep tilings of asymptotic data w, where the exponent of the variable x_i records the number of flips centered on the i-th diagonal in a shortest sequence of flips from the minimal tiling. Then one has:*

$$T_w = \prod_{\substack{i<j \\ w_i=+,\ w_j=- \\ i-j \text{ odd}}} (1+x_i x_{i+1}\dots x_{j-1}) \prod_{\substack{i<j \\ w_i=+,\ w_j=- \\ i-j \text{ even}}} \frac{1}{1-x_i x_{i+1}\dots x_{j-1}}. \tag{4}$$

Of course Theorem 2 is a direct consequence of Theorem 3 (which will be proved in Section 5.1), by letting $x_i = q$ for each i.

Note that these theorems are hook formulas. Indeed, given a word w, one can form the Young diagram $\lambda(w)$ delimited by the path whose i-th step is south if $w_i = +$ and west otherwise. Then Theorem 2 exactly states that

$$T_w(q) = \prod_{c\in\lambda(w)} (1+\epsilon(c)q^{h(c)})^{\epsilon(c)}, \tag{5}$$

where the product is over all cells of $\lambda(w)$, $h(c)$ denotes the hook-length of the cell c (i.e. the number of cells to the right or under c plus one) and $\epsilon(c) = (-1)^{h(c)+1}$. In particular this shows that one can remove the $-$'s (resp. the $+$'s) placed at the beginning (resp. the end) of the word w without changing the value of the generating function, since this does not change the shape of the Young diagram. This fact is easily interpreted geometrically, as one can check that such letters induce regions of the oblique strip where the tiling is entirely fixed in all configurations. Note also that the odd hooks give a term to the numerator of T_w and the even hooks a term to the denominator of T_w. Now, it is easily seen that for any ℓ the only Young diagram with ℓ parts and only odd hooks is the staircase shape $(\ell, \ell-1,\dots,2,1)$. Therefore for a given ℓ, the only family of pure steep tilings whose generating function is a polynomial is the one with asymptotic data $w = (+-)^\ell$ (upon removing the possible trivial leading $-$'s and and trailing $+$'s). As we will see in Section 4.1, this corresponds to tilings of the Aztec diamond of size ℓ (and as a consequence of Theorem 3 we recover a formula due to Stanley; see Remark 2 page 5935).

We then deal with the free and periodic boundary conditions. Here we only state the univariate analogues of Theorem 2, but multivariate formulas analogous to that of Theorem 3 are given in the respective Sections 5.3 and 6. Let us introduce the shorthand notation

$$\varphi_{i,j}(x) = \begin{cases} 1+x & \text{if } j-i \text{ is odd,} \\ 1/(1-x) & \text{if } j-i \text{ is even,} \end{cases} \tag{6}$$

so that (3) may be rewritten in the simpler form

$$T_w(q) = \prod_{\substack{i<j \\ w_i=+,\ w_j=-}} \varphi_{i,j}(q^{j-i}). \tag{7}$$

Theorem 4. *Let $w \in \{+,-\}^{2\ell}$ be a word. Let $F_w(q)$ and $M_w(q)$ be the generating functions of steep tilings of asymptotic data w and with respectively free and mixed (pure-free) boundary conditions, where the exponent of q records the minimal number of flips needed to obtain a tiling from the minimal one. Then one has*

$$M_w(q) = T_w(q) \prod_{i:\, w_i=+} \frac{1}{1-q^{m_i}} \prod_{\substack{i<j \\ w_i=w_j=+}} \varphi_{i,j}(q^{m_i+m_j}) \tag{8}$$

and

$$F_w(q) = T_w(q) \prod_{k=0}^{\infty} \left(\frac{1}{1-q^{(k+1)L}} \prod_{i} \frac{1}{1-q^{kL+m_i}} \prod_{i<j} \varphi_{i,j}(q^{2kL+m_i+m_j}) \right) \tag{9}$$

where $T_w(q)$ and $\varphi_{i,j}(\cdot)$ are as above, and where we use further shorthand notation $L = 2\ell+1$ and

$$m_i = \begin{cases} 2\ell+1-i & \text{if } w_i = + \\ i & \text{if } w_i = - \end{cases} \qquad (i = 1, \ldots, 2\ell). \tag{10}$$

Theorem 5. *Let $w \in \{+,-\}^{2\ell}$ be a word containing at least one $+$ and one $-$. Let $C_w(q)$ be the generating function of cylindric steep tilings of asymptotic data w, where the exponent of q records the minimal number of flips needed to obtain a tiling from the minimal one. Then one has*

$$C_w(q) = T_w(q) \prod_{k=1}^{\infty} \left(\frac{1}{1-q^{2k\ell}} \prod_{\substack{i,j \\ w_i=+,\ w_j=-}} \varphi_{i,j}(q^{2k\ell+j-i}) \right). \tag{11}$$

Those two theorems will be proved in the respective Sections 5.3 and 6.

3. The fundamental bijection

The purpose of this section is to establish a general bijection between steep tilings and sequences of interlaced partitions. The connection is best visualized by introducing particle configurations as an intermediate step. We will also discuss other avatars of the same objects, namely height functions, which are convenient to understand the flips.

3.1. Particle configurations.

A *site* is a point $(x, y) \in (\mathbb{Z} + \frac{1}{2})^2$ (i.e. the center of a unit square with integer corners) such that $0 \le x - y \le 2\ell$. Each site may be occupied by zero or one particle (graphically, we represent empty and occupied sites by the respective symbols $\circ$ and $\bullet$).

Given a steep tiling of the oblique strip, we define a *particle configuration* as follows. If a site is covered by a north- or east-going domino, or if it is uncovered and belongs to the line $y = x - 2\ell$, then we declare it empty. Conversely, if a site is covered by a south- or west-going domino, or if it is uncovered and belongs to the line $y = x$, then we declare it occupied. Condition (2) ensures that we have defined the state of all sites. The convention for uncovered sites is consistent if we think of them as being covered by "external" dominos. See Figure 5 for an example.

The steepness condition implies that, sufficiently far away in the north-east direction, all sites are empty and that conversely, sufficiently far away in the south-west direction, all sites are occupied. In particular, if we fix an integer $m \in \{0, \dots, 2\ell\}$, we may canonically label the occupied sites along the diagonal $y = x - m$ by *positive* integers, starting from the "highest" one. Their abcissae form a strictly decreasing sequence $(x_{m;n})_{n\geq 1}$ of half-integers such that $x_{m;n}+n$ is eventually constant. Conversely, labelling the empty sites along the same diagonal starting from the "lowest" one, their abcissae form a strictly increasing sequence $(x'_{m;n})_{n\geq 1}$ such that $x'_{m;n} - n$ is eventually constant. Actually, since the two sequences span two disjoint sets whose union is $\mathbb{Z} + \frac{1}{2}$, there exists an integer c_m (the "charge") such that

$$c_m = \lim_{n\to\infty} \left(x_{m;n} + n - \frac{1}{2} \right) = \lim_{n\to\infty} \left(x'_{m;n} - n + \frac{1}{2} \right). \tag{12}$$

By examining the rules for constructing the particle configuration, we readily see that, for all $k \in \{1, \dots, \ell\}$ and $n \geq 1$,

$$x_{2k;n} - x_{2k-1;n} \in \{0, 1\} \tag{13}$$

(0 corresponds to a west-going domino, 1 to a south-going one) and

$$x'_{2k-1;n} - x'_{2k-2;n} \in \{0, 1\} \tag{14}$$

(0 corresponds to an east-going domino, 1 to a north-going one). Proposition 1 implies that these quantities are eventually constant as $n \to \infty$, and by (12) we deduce that

$$c_{2k} - c_{2k-1} = \begin{cases} 0 & \text{if } w_{2k} = +, \\ 1 & \text{if } w_{2k} = -, \end{cases} \tag{15}$$

and

$$c_{2k-1} - c_{2k-2} = \begin{cases} 1 & \text{if } w_{2k-1} = +, \\ 0 & \text{if } w_{2k-1} = -. \end{cases} \tag{16}$$

Note finally that the shape of the tiled region is entirely coded by the states (empty or occupied) of the sites along the diagonals $y = x$ and $y = x - 2\ell$. In particular, the tiling is pure (as defined in Section 2.2) if and only if $x_{0;n} = x_{0;n+1} + 1$ and $x_{2\ell;n} = x_{2\ell;n+1} + 1$ for all $n \geq 1$.

3.2. **Sequences of interlaced partitions.** The particles along a diagonal $y = x - m$ form a so-called "Maya diagram" [MJD00], which classically codes an integer partition $\lambda^{(m)}$ via

$$\lambda^{(m)}_n = x_{m;n} + n - \frac{1}{2} - c_m \tag{17}$$

(indeed the sequence $(\lambda^{(m)}_n)_{n\geq 1}$ thus defined is clearly a non-increasing sequence of integers which vanishes eventually, i.e., an integer partition). Empty sites code for the conjugate partition via

$$(\lambda^{(m)})'_n = -x'_{m;n} + n - \frac{1}{2} + c_m. \tag{18}$$

As a straightforward consequence of the relations (13), (14), (15) and (16), we find that, for all $k \in \{1, \dots, \ell\}$ and $n \geq 1$,

$$\lambda_n^{(2k)} - \lambda_n^{(2k-1)} \in \begin{cases} \{0,1\} & \text{if } w_{2k} = +, \\ \{-1,0\} & \text{if } w_{2k} = -, \end{cases} \tag{19}$$

and

$$(\lambda^{(2k-1)})'_n - (\lambda^{(2k-2)})'_n \in \begin{cases} \{0,1\} & \text{if } w_{2k-1} = +, \\ \{-1,0\} & \text{if } w_{2k-1} = -. \end{cases} \tag{20}$$

Let us now recall some classical terminology about integer partitions [Sta99]. Two partitions λ and μ form a *skew shape* λ/μ if the Young diagram of λ contains that of μ, the remaining cells forming the *skew diagram* of λ/μ. This skew diagram is called a *horizontal (resp. vertical) strip* if no two of its cells are in the same column (resp. row), in that case we write $\lambda \prec \mu$ or $\mu \succ \lambda$ (resp. $\lambda \prec' \mu$ or $\mu \succ' \lambda$), and we say colloquially that λ and μ are *interlaced* (be it horizontally or vertically).

We then immediately deduce from (19) and (20) that the $\lambda^{(m)}$'s form a sequence of interlaced partitions. More precisely, the Young diagrams of $\lambda^{(2k)}$ and $\lambda^{(2k-1)}$ differ by a vertical strip (which is either added or removed depending on the sign w_{2k}), while those of $\lambda^{(2k-1)}$ and $\lambda^{(2k-2)}$ differ by a horizontal strip (added or removed depending on w_{2k-1}). At this stage, we should mention that the coding is not bijective since two tilings differing by a translation along the direction $(1,1)$ yield the same sequence of partitions. This is because the right hand sides of (17) and (18) are invariant if we shift all particle positions by a constant (recall the definition (12) of c_m). We say that a steep tiling is *centered* if it has $c_0 = 0$. Any steep tiling differs from a centered one by a (unique) translation. We arrive at the following:

Proposition 6 (Fundamental bijection). *Given a word $w \in \{+,-\}^{2\ell}$, the above construction defines a bijection between the set of centered steep tilings with asymptotic data w and the set of sequences of partitions $(\lambda^{(0)}, \dots, \lambda^{(2\ell)})$ such that, for all $k \in \{1, \dots, \ell\}$,*

- $\lambda^{(2k-2)} \prec \lambda^{(2k-1)}$ *if $w_{2k-1} = +$, and $\lambda^{(2k-2)} \succ \lambda^{(2k-1)}$ if $w_{2k-1} = -$,*
- $\lambda^{(2k-1)} \prec' \lambda^{(2k)}$ *if $w_{2k} = +$, and $\lambda^{(2k-1)} \succ' \lambda^{(2k)}$ if $w_{2k} = -$.*

Furthermore, the bijection has the following properties:

A. *The precise shape of the tiled region is determined by the initial and final partitions $\lambda^{(0)}$ and $\lambda^{(2\ell)}$. In particular, the tiling is pure if and only if $\lambda^{(0)} = \lambda^{(2\ell)} = \emptyset$.*
B. *For $m = 1, \dots, 2\ell$, the absolute value of $|\lambda^{(m)}| - |\lambda^{(m-1)}|$ counts the number of dominos whose centers are on the line $y = x - m + 1/2$ and whose orientations are opposite to the asymptotic one; see Table* 1.
C. *For $m = 0, \dots, 2\ell$, $|\lambda^{(m)}|$ counts the number of flips centered on the m-th diagonal in any minimal sequence of flips between the tiling at hand and the minimal tiling $T^w_{\min}$ corresponding to the sequence $(\emptyset, \emptyset, \dots, \emptyset)$.*

Proof. To prove the bijectivity of the construction, we simply exhibit the inverse mapping, and leave it to the reader to check the details. From (15) and (16), we may recover the value of c_m for all $m \in \{1, \dots, 2\ell\}$ starting with the data of $c_0 = 0$ and w. By (17) and (18), the sequence of partitions then determines all

TABLE 1. The absolute value of the difference between the sizes of $\lambda^{(m-1)}$ and of $\lambda^{(m)}$ is equal to the number of dominos whose centers are on the $y = x - m + 1/2$ and have the above orientations and types.

Parity of m	w_m	Orientation (type)
odd	$+$	vertical (east-going)
odd	$-$	horizontal (north-going)
even	$+$	horizontal (south-going)
even	$-$	vertical (west-going)

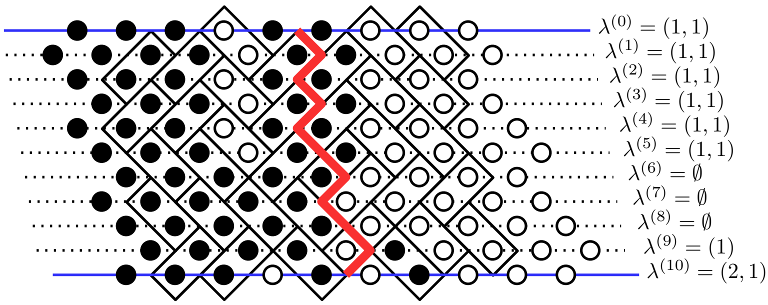

FIGURE 6. The particle configuration of Figure 5 after a 45° rotation. On each line one reads the Maya diagram of an integer partition. The thick (red) path indicates the frontier between empty and occupied sites for the minimal configuration of the same asymptotic data, corresponding to the case when all partitions are empty (see also Figure 9).

positions of occupied and empty sites in the particle configuration. Finally, the domino positions may be read off (13) and (14).

The properties A and B are easy to check. The property C results from the discussion of height function in the forthcoming subsection. ☐

Example 2. The steep tiling of Figure 5 corresponds to the sequence

$$\lambda^{(0)} = \lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = \lambda^{(4)} = \lambda^{(5)} = (1,1),$$

$$\lambda^{(6)} = \lambda^{(7)} = \lambda^{(8)} = \emptyset, \quad \lambda^{(9)} = (1), \quad \lambda^{(10)} = (2,1).$$

For the convenience of the reader, we provide in Figure 6 a rotated version of the particle configuration of Figure 5, on which the Maya diagrams of the partitions and their interlacing are easier to visualize.

3.3. **Height functions.** To complete the picture, let us discuss the height function associated to a steep tiling of the oblique strip. Let V be the set of integer points in the oblique strip, namely

$$V = \{(x, y) \in \mathbb{Z}^2, 0 \leq x - y \leq 2\ell\}. \tag{21}$$

Consider the oriented graph with vertex set V where each vertex $(x, y) \in V$ such that $x - y$ is odd has exactly two outgoing edges pointing to $(x \pm 1, y)$, and exactly

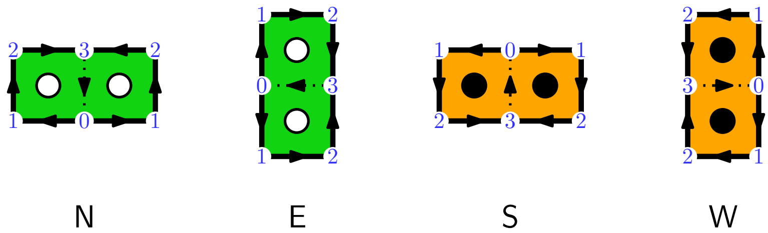

FIGURE 7. Variation of the height function around each type of domino (numbers represent the height difference with respect to the "lowest" vertex around the domino). The local particle configuration is displayed.

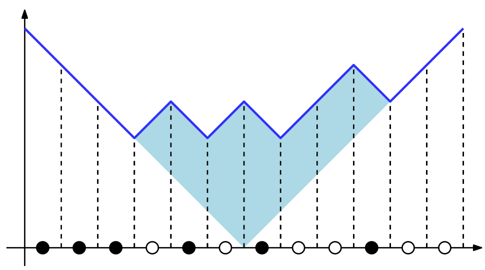

FIGURE 8. Russian representation of the partition $(4, 2, 1)$, and the associated Maya diagram.

two incoming edges originating from $(x, y \pm 1)$. Following Thurston [Thu90], we define a *height function* as a function $H : V \to \mathbb{Z}$ such that, for any oriented edge $(x, y) \to (x', y')$, we have

$$H(x', y') - H(x, y) \in \{1, -3\}. \tag{22}$$

It is easily seen that height functions (considered up to an additive constant) are in bijection with domino tilings: each edge with a height difference of -3 corresponds to a domino; see Figure 7. Note that boundary dominos (i.e. dominos overlapping either the line $y = x$ or the line $y = x - 2\ell$) have one of their corners not in V, but no information is lost in restricting the height function to V.

Interestingly, the height function is closely related to the particle configuration: as is apparent on Figure 7, the height difference between the vertex at the top right and that at the bottom left of an empty (resp. occupied) site is $+2$ (resp. -2). Hence, for any $m \in \{0, \dots, 2\ell\}$, the graph of the function $x \mapsto H(x, x-m)$ coincides (up to scaling and translation) with the "Russian" representation of the partition $\lambda^{(m)}$; see Figure 8. By translating Proposition 1 in the language of height functions we find that, for $|x|$ large enough, we have

$$H(x, x - m) = 2|x - c_m| + h_m \tag{23}$$

where c_m is defined as in (12) and $h_m - h_{m+1} = 1$ if $w_m = +$, -1 if $w_m = -$ (we may fix $h_0 = 0$ since the height function is defined modulo an additive constant). Observe that the height function grows eventually at the maximal possible slope, which is why we call the corresponding tiling "steep". For fixed asymptotic data, the lowest possible height function $H^w_{\min}$ is achieved when (23) holds for all x, which

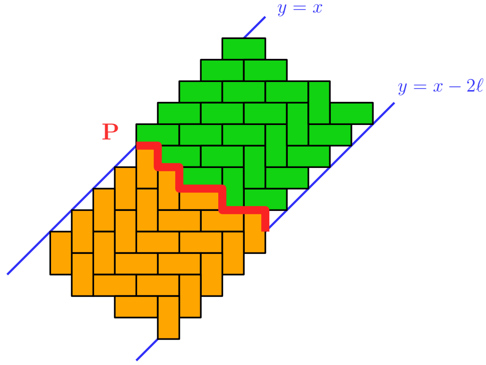

FIGURE 9. The minimal tiling $T^w_{\min}$ corresponding to the asymptotic data $w = (+ + + + + - - - ++)$. The minimal tiling is separated into two regions, one filled with only south- or west- going (orange) dominos, and the other one filled with only north- or east- going (green) dominos, corresponding respectively to occupied and unoccupied sites in the particle configuration.

corresponds to having $\lambda^{(m)} = \emptyset$ for all m. The corresponding centered tiling $T^w_{\min}$ is the *minimal* tiling of asymptotic data w; see Figure 9.

Remark 1. In the minimal tiling $T^w_{\min}$, both the region covered by north- or east-going dominos and that covered by east- or south-going ones are connected, and the frontier between them is made of a finite path P; see Figure 9. By (23), P intersects the line $x = y - m$ at the point

$$P_m = (c_m, c_m - m) = \left(\frac{m - \sum_{j=1}^m w_j (-1)^j}{2}, \frac{-m - \sum_{j=1}^m w_j (-1)^j}{2} \right). \tag{24}$$

In particular, P_0 and $P_{2\ell}$ mark the limit between uncovered and covered boundary squares in $T^w_{\min}$, thus in any centered pure steep tiling.

As pointed out by Elkies et al. [EKLP92a] in the context of the Aztec diamond, and understood by Propp [Pro93] in a much broader context, height functions play a crucial role in the study of flips. Recall that here a flip consists in replacing a pair of horizontal dominos forming a 2×2 block by a pair of vertical dominos, or vice versa. In the context of tilings of the oblique strip, we also consider “boundary flips” where we rotate a boundary domino adjacent to an uncovered square (thus changing the shape of the tiled region). As is apparent on Figure 10, a flip modifies the height function precisely at one vertex, where the height is increased or decreased by 4 (this is the smallest possible change, since the difference between two height functions is constant modulo 4). Given a tiling T with height function H, it may be shown along the same lines as in [EKLP92a] that the minimal number of flips needed to pass from $T^w_{\min}$ to T is equal to

$$r_w(T) = \sum_{(x,y) \in V} |H(x,y) - H^w_{\min}(x,y)|/4. \tag{25}$$

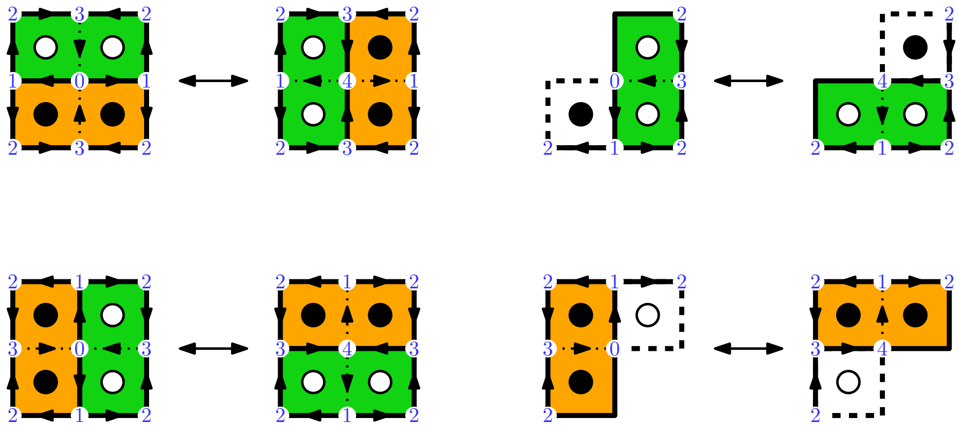

FIGURE 10. The effect of flips on the height function and on the particle configuration (left: bulk case, right: boundary case). The flips are ascendent from left to right and descendent from right to left.

By (23), this number is finite if and only if T is centered and has asymptotic data w, and we then have

$$r_w(T) = \sum_{m=0}^{2\ell} |\lambda^{(m)}| \tag{26}$$

where $(\lambda^{(0)}, \dots, \lambda^{(2\ell)})$ is the sequence of interlaced partitions associated to T, and where $|\lambda^{(m)}|$ denotes the size of the partition $\lambda^{(m)}$. By an easy refinement of (25), we find that $|\lambda^{(m)}|$ precisely counts the number of flips made on the diagonal $x = y - m$ (which are boundary flips for $m = 0$ or 2ℓ) in any shortest sequence of flips from $T^w_{\min}$ to T. This establishes the property C in Proposition 6.

4. Some particular cases

We now discuss a few particularly interesting cases of the bijection.

4.1. Aztec diamond.

Domino tilings of the Aztec diamond of size ℓ are obtained by considering the word $w = (+-)^\ell = + - + - \cdots + -$ (ℓ times):

Proposition 7. *There is a one-to-one correspondence between domino tilings of the Aztec diamond of size ℓ and sequences of partitions $(\lambda^{(0)}, \dots, \lambda^{(2\ell)})$ such that*

$$\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \succ' \lambda^{(2)} \prec \lambda^{(3)} \succ' \cdots \prec \lambda^{(2\ell-1)} \succ' \lambda^{(2\ell)} = \emptyset. \tag{27}$$

This fact was also observed by Dan Betea (personal communication).

Proof. By translating a tiling of the Aztec diamond, and completing it in the way displayed on Figure 11, we obtain a centered pure steep tiling with asymptotic data $w = (+-)^\ell$. By Proposition 6, it corresponds to a sequence of partitions satisfying (27) with $\lambda^{(0)} = \lambda^{(2\ell)} = \emptyset$.

Conversely, we need to show that a steep tiling obtained from a sequence of partitions satisfying (27) is "frozen" outside the Aztec diamond. While this may be checked directly by reasoning on dominos, let us here exploit the correspondence

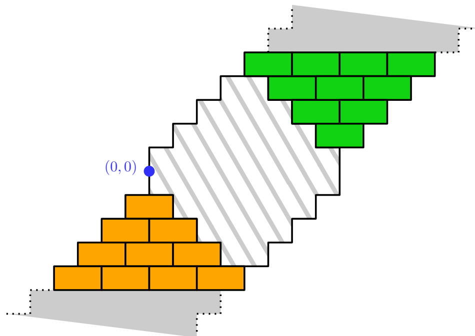

FIGURE 11. A tiling of the Aztec diamond of size ℓ (here $\ell = 4$) may be completed into a steep tiling with asymptotic data $(+-)^\ell$.

with partitions: observe that (the Young diagram of) $\lambda^{(1)}$ has at most one row (since $\lambda^{(1)} \succ \emptyset$), and so does $\lambda^{(2)}$ (since $\lambda^{(1)} \succ' \lambda^{(2)}$). Hence, $\lambda^{(3)}$ and $\lambda^{(4)}$ have at most two rows (since $\lambda^{(2)} \prec \lambda^{(3)} \succ' \lambda^{(4)}$), and so we find by induction that, for all $k \in \{1, \dots, \ell\}$, $\lambda^{(2k-1)}$ and $\lambda^{(2k)}$ have at most k rows. Similarly, by reasoning on (27) backwards, we find that $\lambda^{(2k-1)}$ and $\lambda^{(2k-2)}$ have at most $\ell + 1 - k$ columns. Translating these constraints in the particle language, we find that all sites strictly above the line $x + y = 2\ell$ are empty, and all sites strictly below the line $x + y = 0$ are occupied. Thus, the height function coincides with $H^w_{\min}$ for $x + y \geq 2\ell$ or $x + y \leq 0$, so the tiling coincides in these regions with the minimal tiling $T^w_{\min}$, which here consists only of horizontal dominos, as wanted. $\square$

Remark 2. Using the property B in Proposition 6, it is not difficult to see that the multivariate weighting scheme of Theorem 3 is equivalent to the so-called Stanley's weighing scheme for the Aztec diamond [Pro97, Yan91]. Indeed, attaching a weight x_i ($i = 1, \dots, 2\ell - 1$) to each flip centered on the i-th diagonal is tantamount to attaching a weight z_j ($j = 1, \dots, 2\ell$) to each vertical domino whose center is on the line $y = x - j + 1/2$, upon imposing the relation

$$x_{2k-1} = z_{2k-1} z_{2k}, \qquad x_{2k} = \frac{1}{z_{2k} z_{2k+1}}. \tag{28}$$

Thus, Theorem 3 implies that

$$T_{(+-)^\ell} = \prod_{\substack{1 \leq i < j \leq 2\ell \\ i \text{ odd}, j \text{ even}}} (1 + z_i z_j) \tag{29}$$

which is equivalent to Stanley's formula (where weights for horizontal dominos can be set to 1 without loss of generality).

4.2. **Pyramid partitions.** They can be recovered by considering centered pure steep tilings with asymptotic data $w = +^\ell -^\ell$ (that is, $+$ repeated ℓ times then $-$ repeated ℓ times) and then letting $\ell \to +\infty$. Let ℓ be a fixed odd integer and denote by P_ℓ the set of pyramid partitions that we can obtain from the fundamental partition given on Figure 3 where the center of the brick on the top is $(0, 0)$ and where one can only take off bricks that lie inside the strip $-\ell \leq x - y \leq \ell$ (note that, by our conventions, the removal of a brick actually corresponds to an ascendent domino flip). It is straightforward to see the following proposition.

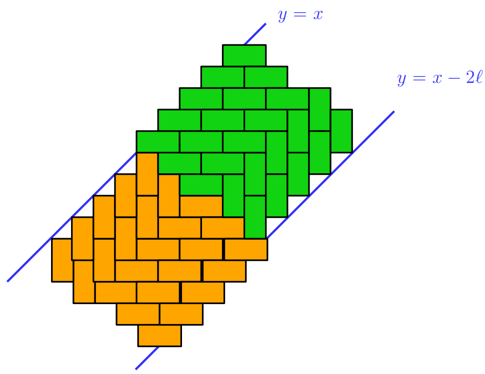

FIGURE 12. The minimal tiling $T^w_{\min}$ corresponding to the asymptotic data $w = (++++++------)$.

Proposition 8 (See also [You10, Lemma 5.9])**.** *There is a one-to-one correspondence between pyramid partitions in P_ℓ and pure (centered steep) tilings with asymptotic data $w = (+^\ell -^\ell)$. Equivalently, there is a bijection between pyramid partitions in P_ℓ and sequences of partitions $(\lambda^{(0)}, \dots, \lambda^{(2\ell)})$ such that*

(30)
$$\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \lambda^{(3)} \cdots \prec \lambda^{(\ell)} \succ' \lambda^{(\ell+1)} \succ \cdots \succ \lambda^{(2\ell-1)} \succ' \lambda^{(2\ell)} = \emptyset.$$

Indeed, one can easily see that the empty pyramid partition restricted to the oblique strip of width 2ℓ is nothing but $T^{+^\ell -^\ell}_{\min}$ translated by $(-\ell, \ell)$; see Figure 12. Theorem 2 gives

$$T_{(+^\ell -^\ell)}(q) = \prod_{i=1}^{\ell} (1 + q^{2i-1})^{2\min(i,\ell+1-i)-1} \prod_{i=1}^{\ell-1} \frac{1}{(1-q^{2i})^{2\min(i,\ell-i)}}. \tag{31}$$

It is immediate to see that when $\ell \to +\infty$, we recover the generating function of pyramid partitions (1). A similar construction holds when ℓ is an even integer (the "central block" is rotated by 90°).

4.3. **Plane overpartitions.** We now provide an example of non-pure steep tilings. A *plane overpartition* [CSV11] is a plane partition where in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of an integer can be overlined or not and all the others are overlined. An example of a plane overpartition of shape $(4, 4, 2, 1)$ is

$$\begin{matrix} 2 & 2 & \bar{2} & \bar{1} \\ \bar{2} & 1 & 1 & \bar{1} \\ \bar{2} & \bar{1} & & \\ 1 & & & \end{matrix} \tag{32}$$

It is easily seen that a plane overpartition of shape λ containing integers at most ℓ is in bijection with a sequence of interlaced partitions such that

$$\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \prec' \lambda^{(2)} \prec \lambda^{(3)} \prec' \lambda^{(4)} \prec \cdots \prec \lambda^{(2\ell-1)} \prec' \lambda^{(2\ell)} = \lambda. \tag{33}$$

Indeed, for $i = 1, \dots, \ell$, the horizontal strip $\lambda^{(2i-1)}/\lambda^{(2i-2)}$ (resp. the vertical strip $\lambda^{(2i)}/\lambda^{(2i-1)}$) is formed by the non-overlined entries (resp. the overlined entries) equal to $\ell + 1 - i$.

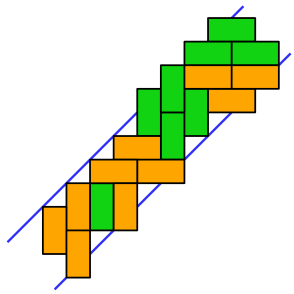

FIGURE 13. The steep tiling corresponding to the plane overpartition of equation (32).

By Proposition 6, plane overpartitions are then in bijection with some (non-pure) steep tilings with asymptotic data $+^{2\ell}$; see Figure 13 for an example. These tilings are essentially the same as those discussed in [CSV11, Section 4].

Note that, upon reversing and conjugating the sequence (33), a plane overpartition may as well be coded by a sequence

$$\lambda' = \mu^{(0)} \succ \mu^{(1)} \succ' \mu^{(2)} \succ \mu^{(3)} \succ' \mu^{(4)} \succ \cdots \succ \mu^{(2\ell-1)} \succ' \mu^{(2\ell)} = \emptyset \tag{34}$$

corresponding to a steep tiling with asymptotic data $-^{2\ell}$ (which is actually the previous tiling rotated by 180°). We then observe that pairs of plane overpartitions whose shape are conjugate to one another are in bijection with pyramid partitions: this is seen by concatenating their associated sequences of interlaced partitions (upon using the convention (33) for the first plane overpartition and the convention (34) for the other one), or alternatively by "assembling" their associated domino tilings into one another.

Remark 3. As pointed out by Sunil Chhita, the case where λ is a rectangular shape corresponds to domino tilings of the so-called double Aztec diamond [AJvM14, ACJvM15]. Compare for instance [AJvM14, Figure 1] with [CSV11, Figures 8 and 11] (where one shall think of all outgoing red lines at the bottom being moved as much as possible to the right in the case where λ is a rectangular shape).

5. Enumeration via the vertex operator formalism

5.1. **Pure steep tilings.** The purpose of this section is to establish Theorem 3, which implies Theorem 2 upon taking $x_i = q$ for all i. By our general bijection (Proposition 6), the enumeration of elements of $\mathcal{T}_w$ is equivalent to the one of interlaced sequences of partitions with empty boundary conditions. This amounts to computing the partition function of a *Schur process* [OR03, Bor11], which is easily done using the *vertex operator formalism*; see e.g. [Kac90, Oko01]. Let us now briefly recall this formalism.

We work over the vector space of formal sums of partitions, with basis $\{|\lambda\rangle, \lambda \in \Lambda\}$ and dual basis $\{\langle\mu|, \mu \in \Lambda\}$, where Λ is the set of all integer partitions (here we use the convenient bra-ket notation). We consider the *vertex operators* $\Gamma_+(t)$ and

$\Gamma_-(t)$ defined by

$$\begin{aligned} \Gamma_+(t)|\lambda\rangle &= \sum_{\mu:\ \mu\prec\lambda} t^{|\lambda|-|\mu|}|\mu\rangle, \\ \Gamma_-(t)|\lambda\rangle &= \sum_{\mu:\ \mu\succ\lambda} t^{|\mu|-|\lambda|}|\mu\rangle, \end{aligned} \tag{35}$$

where t must be seen as a formal variable. Following [You10], we also introduce the conjugate vertex operators $\Gamma'_+(t)$ and $\Gamma'_-(t)$ defined by

$$\begin{aligned} \Gamma'_+(t)|\lambda\rangle &= \sum_{\mu:\ \mu\prec'\lambda} t^{|\lambda|-|\mu|}|\mu\rangle, \\ \Gamma'_-(t)|\lambda\rangle &= \sum_{\mu:\ \mu\succ'\lambda} t^{|\mu|-|\lambda|}|\mu\rangle. \end{aligned} \tag{36}$$

Note that $\Gamma'_\pm(t) = \omega\Gamma_\pm(t)\omega$ where $\omega : |\lambda\rangle \longmapsto |\lambda'\rangle$ is the conjugation of partitions.

Lemma 9. *Fix a word $w \in \{+,-\}^{2\ell}$, and α, β two partitions. Let $T_{w,\alpha,\beta} \equiv T_{w,\alpha,\beta}(x_1, x_2, \ldots, x_{2\ell})$ be the generating function of sequences $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(2\ell)})$ of partitions that are interlaced as described in Proposition 6, with $\lambda^{(0)} = \alpha$ and $\lambda^{(2\ell)} = \beta$, and where the exponent of the variable x_i $(1 \le i \le 2\ell)$ records the size of the partition $\lambda^{(i)}$ minus that of α. Then $T_{w,\alpha,\beta}$ is given by*

$$T_{w,\alpha,\beta} = \langle\alpha| \prod_{i=1}^{\ell} \Gamma_{w_{2i-1}}(y_{2i-1}^{w_{2i-1}})\Gamma'_{w_{2i}}(y_{2i}^{w_{2i}})|\beta\rangle \tag{37}$$

with $y_i = x_i x_{i+1} \ldots x_{2\ell}$. Here and in the sequel, $y_i^{w_i}$ shall be understood as y_i if $w_i = +$, and $1/y_i$ if $w_i = -$.

Proof. By the definition of the operators $\Gamma_\pm$ and $\Gamma'_\pm$, we have

$$\langle\alpha| \prod_{i=1}^{\ell} \Gamma_{w_{2i-1}}(q_{2i-1})\Gamma'_{w_{2i}}(q_{2i})|\beta\rangle = \sum \prod_{i=1}^{2\ell} q_i^{w_i(|\lambda^{(i)}|-|\lambda^{(i-1)}|)} \tag{38}$$

where the sum runs over all sequences of partitions $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(2\ell)})$ satisfying the interlacing conditions of Proposition 6. Taking $q_i = y_i^{w_i}$, we obtain the correct weight, since $q_i^{w_i} = y_i$, and since in the product $\prod_{i=1}^{2\ell}(x_i x_{i+1} \ldots x_{2\ell})^{|\lambda^{(i)}|-|\lambda^{(i-1)}|}$ the exponent of x_i is $\sum_{j=1}^{i} |\lambda^{(j)}| - |\lambda^{(j-1)}| = |\lambda^{(i)}| - |\alpha|$. ☐

Vertex operators are known (see e.g. [You10, Lemma 3.3]) to satisfy the following non-trivial commutation relations:

$$\begin{aligned} \Gamma_+(t)\Gamma_-(u) &= \frac{1}{1-tu}\Gamma_-(u)\Gamma_+(t), \\ \Gamma'_+(t)\Gamma'_-(u) &= \frac{1}{1-tu}\Gamma'_-(u)\Gamma'_+(t), \\ \Gamma_+(t)\Gamma'_-(u) &= (1+tu)\Gamma'_-(u)\Gamma_+(t), \\ \Gamma'_+(t)\Gamma_-(u) &= (1+tu)\Gamma_-(u)\Gamma'_+(t), \end{aligned} \tag{39}$$

while other commutation relations are trivial (namely, two vertex operators with the same sign in index commute together). Note the following more compact way of rewriting (39): fix two symbols $\Gamma^{\triangleleft}, \Gamma^{\diamond} \in \{\Gamma, \Gamma'\}$; then

$$\Gamma_+^{\triangleleft}(t)\Gamma_-^{\diamond}(u) = (1+\epsilon t u)^{\epsilon}\, \Gamma_-^{\diamond}(u)\Gamma_+^{\triangleleft}(t), \tag{40}$$

where $\epsilon = -1$ if $\Gamma^{\triangleleft} = \Gamma^{\diamond}$ and $\epsilon = +1$ otherwise.

Proof of Theorem 3. By Proposition 6, the desired generating function T_w of pure steep tilings is nothing but $T_{w,\emptyset,\emptyset}$, as expressed by Lemma 9 in terms of vertex operators (note that T_w does not depend on $x_{2\ell}$). We may now evaluate the right hand side of (37): note first that for any weights $z_1, \ldots, z_{2\ell}$, symbols $(\Gamma^{\diamond_i})_{1\le i\le 2\ell} \in \{\Gamma, \Gamma'\}^{2\ell}$ and m between 1 and 2ℓ, we have

$$\langle\emptyset| \prod_{i=1}^{m} \Gamma_-^{\diamond_i}(z_i) \prod_{i=m+1}^{2\ell} \Gamma_+^{\diamond_i}(z_i) |\emptyset\rangle = 1, \tag{41}$$

since by definition $\Gamma_+^{\diamond}|\emptyset\rangle = |\emptyset\rangle$ and $\langle\emptyset|\Gamma_-^{\diamond} = \langle\emptyset|$ for any $\Gamma^{\diamond} \in \{\Gamma, \Gamma'\}$. Therefore we can evaluate (37) by "moving" all the Γ_- and Γ'_- operators to the left using the commutation relations (39). We will end up with a multiplicative prefactor coming from the commutation relations, and a remaining scalar product of the form (41) which evaluates to 1.

In the process of moving these operators to the left, for each $i < j$ such that $w_i = +$ and $w_j = -$ we have to exchange the operators $\Gamma_+^{\triangleleft}(y_i)$ and $\Gamma_-^{\diamond}(y_j^{-1})$ where $\Gamma^{\triangleleft}$ (resp. $\Gamma^{\diamond}$) is equal to Γ if i (resp. j) is odd, and to Γ' if i (resp. j) is even. By (40), the multiplicative contribution of this exchange is equal to

$$\left(1 + \epsilon_{i,j}\frac{y_i}{y_j}\right)^{\epsilon_{i,j}} = (1 + \epsilon_{i,j} x_i x_{i+1} \ldots x_{j-1})^{\epsilon_{i,j}}, \tag{42}$$

where $\epsilon_{i,j} = -1$ if i and j have the same parity, and $\epsilon_{i,j} = 1$ otherwise. The desired expression (4) follows. ☐

Remark 4. As pointed out by Paul Zinn-Justin [ZJ12], it is also possible to relate the vertex operator formalism to domino tilings via the six-vertex model on the free-fermion line, the product $\Gamma_+(t)\Gamma'_-(u)$ corresponding essentially to the transfer matrix of this model.

Remark 5. In the case of domino tilings of the Aztec diamond, the vertex operator computation can be related to domino shuffling [EKLP92b]. Indeed, the commutation relation between $\Gamma_+(t)$ and $\Gamma'_-(u)$ can be derived bijectively via domino shuffling (we leave this as a pleasant exercise to the reader) and from it we deduce the relation

$$\begin{aligned}(43)\quad &\langle\emptyset|\Gamma_+(z_1)\Gamma'_-(z_2)\Gamma_+(z_3)\Gamma'_-(z_4)\Gamma_+(z_5)\Gamma'_-(z_6)\cdots\Gamma_+(z_{2\ell-1})\Gamma'_-(z_{2\ell})|\emptyset\rangle \\ &= \prod_{i=1}^{\ell}(1 + z_{2i-1}z_{2i}) \times \langle\emptyset|\Gamma_+(z_1)\Gamma'_-(z_4)\Gamma_+(z_3)\Gamma'_-(z_6)\Gamma_+(z_5)\cdots\Gamma'_-(z_{2\ell})|\emptyset\rangle\end{aligned}$$

which can be interpreted combinatorially as a 2^ℓ-to-1 correspondence between tilings of the Aztec diamonds of orders ℓ and $\ell - 1$. We readily recover Stanley's formula (29) by induction.

5.2. **General steep tilings with prescribed boundary conditions.** We now consider the enumeration of not necessarily pure steep tilings with prescribed boundary conditions (i.e. the tiled region is fixed): by Proposition 6 and Lemma 9 this amounts to evaluating the right hand side of (37) when $(\alpha, \beta) \neq (\emptyset, \emptyset)$.

Let us first consider the case of plane overpartitions discussed in Section 4.3, where $w = +^{2\ell}$, $\alpha = \emptyset$ and $\beta = \lambda$. The corresponding generating function is

$$
\begin{aligned}
T_{+^{2\ell},\emptyset,\lambda} &= \langle \emptyset | \Gamma_+(y_1)\Gamma'_+(y_2)\Gamma_+(y_3)\Gamma'_+(y_4)\cdots\Gamma_+(y_{2\ell-1})\Gamma'_+(y_{2\ell})|\lambda\rangle \\
&= s_\lambda(y_1, y_3, \ldots, y_{2\ell-1}/y_2, y_4, \ldots, y_{2\ell})
\end{aligned}
\tag{44}
$$

where $s_\lambda(\cdot/\cdot)$ is a *super Schur function*, also known as a hook Schur function; see [Kra96] and the references therein. Indeed, the second equality of (44) may be obtained by moving all Γ'_+ to the right (since all operators commute with each other) and observing that the resulting expression counts some super semistandard tableaux, also called (ℓ, ℓ)-semistandard tableaux [Rem84]. Alternatively, a direct bijection between plane overpartitions and super semistandard tableaux was given in [CSV11, Remark 1]. If we specialize $x_i = q$ for all i, so that $y_i = q^{2\ell+1-i}$, the generating function specializes to $s_\lambda(q^2, q^4, \ldots, q^{2\ell}/q, q^3, \ldots, q^{2\ell-1})$: to the best of our knowledge, there is no known hook-content-type formula for this specialization, except in the $\ell \to \infty$ limit where we have [Kra96, CSV11]

$$
s_\lambda(aq^2, aq^4, \ldots/bq, bq^3, \ldots) = q^{2|\lambda|} \prod_{\rho \in \lambda} \frac{a + bq^{2c(\rho)-1}}{1 - q^{2h(\rho)}}.
\tag{45}
$$

Here the product is over all cells ρ of the Young diagram of λ, $h(\rho)$ and $c(\rho)$ being respectively the hook-length and the content of ρ, and a and b are extra parameters that in our context count the respective numbers of east- and south-going dominos. (The $\ell \to \infty$ limit is well defined if the steep tilings of asymptotic data $+^{2\ell}$ are first translated by $(-2\ell, 2\ell)$ so that they eventually fill the whole $y \geq x$ half-plane as $\ell \to \infty$. Those tilings are nothing but "half-pyramid partitions".)

Remark 6. If we set $a_i = c_i = q^{2i-1}$ and $b_i = d_{i+1} = q^{2i}$ in the Cauchy identity for super Schur functions [Rem84]

$$
\sum_\lambda s_\lambda(a/b) s_\lambda(c/d) = \prod_{i,j} \frac{(1 + a_i d_j)(1 + b_i c_j)}{(1 - a_i c_j)(1 - b_i d_j)},
\tag{46}
$$

then we get back the generating function of pyramid partitions in $P_{2\ell}$. This can be related to the fact, already noted in Section 4.3, that pairs of plane overpartitions with compatible shapes are in correspondence with pyramid partitions.

Let us now discuss the more general situations. For $w = +^{2\ell}$ and arbitrary boundary conditions α and β, we have

$$
\begin{aligned}
T_{+^{2\ell},\alpha,\beta} &= \langle \alpha | \Gamma_+(y_1)\Gamma'_+(y_2)\Gamma_+(y_3)\Gamma'_+(y_4)\cdots\Gamma_+(y_{2\ell-1})\Gamma'_+(y_{2\ell})|\beta\rangle \\
&= s_{\beta/\alpha}(y_1, y_3, \ldots, y_{2\ell-1}/y_2, y_4, \ldots, y_{2\ell})
\end{aligned}
\tag{47}
$$

where $s_{\beta/\alpha}(\cdot, \cdot)$ is a skew super Schur function (or (ℓ, ℓ)-hook skew Schur function). Let us instead consider a general word $w = \{+, -\}^\ell$ and boundary conditions of the form $\alpha = \emptyset$, $\beta = \lambda$. Let $i_1 < i_2 < \cdots < i_n$ and $i'_1 < i'_2 < \cdots < i'_m$ denote the respectively odd and even positions of the $+$'s in w. Upon moving all Γ_- and Γ'_- to the left in (37) where they are "absorbed" by $\langle \emptyset |$, we find that

$$
T_{w,\emptyset,\lambda} = T_w s_\lambda(y_{i_1}, y_{i_2}, \ldots, y_{i_n}/y_{i'_1}, y_{i'_2}, \ldots, y_{i'_m}).
\tag{48}
$$

Finally, for general w, α and β, we may recast $T_{w,\alpha,\beta}/T_w$ as a (finite) sum of the form $\sum_\nu s_{\alpha/\nu}(\cdot/\cdot)s_{\beta/\nu}(\cdot/\cdot)$. Writing down an explicit formula is left to the interested reader. We are not aware of any specialization of the super Schur functions besides (45) that would provide a "nice" formula for $T_{w,\alpha,\beta}$ for generic w or α.

5.3. **General steep tilings with free boundary conditions.** We now wish to study steep tilings with "free" boundary conditions, i.e., the tiled region is not prescribed, or in other words we do not specify the first and last element of their corresponding sequences of interlaced partitions. We thus consider generating functions of the form

$$F_w(u,v) = \sum_{\alpha,\beta} u^{|\alpha|} v^{|\beta|} T_{w,\alpha,\beta} \tag{49}$$

which count *all* steep tilings with asymptotic data w, the exponents of u and v recording the number of boundary flips needed on both sides to obtain a given tiling from the pure minimal tiling T^w_{min}. Interestingly, it is possible to obtain a nice expression for $F_w(s,t)$ by the vertex operator formalism, as we will now explain.

Introducing the *free boundary states*

$$\langle \underline{u}| = \sum_\lambda u^{|\lambda|}\langle\lambda| \qquad \text{and} \qquad |\underline{v}\rangle = \sum_\lambda v^{|\lambda|}|\lambda\rangle, \tag{50}$$

where the sums range over all partitions, it immediately follows from (37) that

$$F_w(u,v) = \langle \underline{u}| \prod_{i=1}^{\ell} \Gamma_{w_{2i-1}}(y_{2i-1}^{w_{2i-1}})\Gamma'_{w_{2i}}(y_{2i}^{w_{2i}})\, |\underline{v}\rangle, \tag{51}$$

where we recall that $y_i = x_i x_{i+1}\cdots x_{2\ell}$. To evaluate this expression, it is necessary to understand how the Γ operators act on the free boundary states.

Proposition 10 (Reflection relations)**.** *We have*

$$\begin{aligned}
\Gamma_+(t)\,|\underline{v}\rangle &= \frac{1}{1-tv}\Gamma_-(tv^2)\,|\underline{v}\rangle,\\
\Gamma'_+(t)\,|\underline{v}\rangle &= \frac{1}{1-tv}\Gamma'_-(tv^2)\,|\underline{v}\rangle,\\
\langle\underline{u}|\,\Gamma_-(t) &= \frac{1}{1-tu}\langle\underline{u}|\,\Gamma_+(tu^2),\\
\langle\underline{u}|\,\Gamma'_-(t) &= \frac{1}{1-tu}\langle\underline{u}|\,\Gamma'_+(tu^2).
\end{aligned} \tag{52}$$

Proof. These amount to [Mac95, I.5, Ex. 27(a), (3)] but let us provide here a combinatorial derivation. It is sufficient to establish the first relation, which implies the others by conjugation and duality. This amounts to proving that, for any partition μ, we have

$$\sum_{\lambda:\lambda\succ\mu} t^{|\lambda/\mu|} v^{|\lambda|} = \frac{1}{1-tv}\sum_{\nu:\nu\prec\mu} (tv^2)^{|\mu/\nu|} v^{|\nu|}. \tag{53}$$

Given λ such that $\lambda\succ\mu$, set $\nu_i = \mu_i + \mu_{i+1} - \lambda_{i+1}$ and $k = \lambda_1 - \mu_1$: it is readily checked that ν is a partition such that $\nu\prec\mu$, satisfying $|\lambda/\mu| = |\mu/\nu| + k$, and that the mapping $\lambda\mapsto(\nu,k)$ is bijective (k being an arbitrary non-negative integer). The wanted identity follows. ∎

We now apply the reflection relations to the evaluation of (51). As a warmup, let us consider the case of plane overpartitions where $u = 0$ (so that the left boundary state is $\langle\emptyset|$) and $w = +^{2\ell}$: it is straightforward to check that

$$F_{(+^{2\ell})}(0,v) = \prod_{i=1}^{2\ell} \frac{1}{1-vy_i} \prod_{\substack{1\le i<j\le 2\ell \\ j-i \text{ odd}}} (1+v^2y_iy_j) \prod_{\substack{1\le i<j\le 2\ell \\ j-i \text{ even}}} \frac{1}{1-v^2y_iy_j} \tag{54}$$

("bounce" the Γ_+/Γ'_+ on $|\underline{v}\rangle$ then move the resulting Γ_-/Γ'_- to the left where they are "absorbed" by $\langle\emptyset|$, and collect all factors obtained on the way).

Remark 7. Recalling that $F_{(+^{2\ell})}(0,v) = \sum_\lambda v^{|\lambda|} s_\lambda(y_1,y_3,\ldots/y_2,y_4,\ldots)$, the expression (54) is actually equivalent to the so-called Littlewood identity [Mac95, I.5, Ex. 4] (recovered by taking $v = 1$ and, say, $y_{2i-1} = z_i$ and $y_{2i} = 0$ for $i = 1,\ldots,\ell$).

Remark 8. We recover the generating function of plane overpartitions of arbitrary shape computed in [CSV11] by taking the appropriate weight specialization, namely $v = 1$ and, for $i = 1,\ldots,\ell$, $y_{2i-1} = q^{\ell+1-i}$ and $y_{2i} = aq^{\ell+1-i}$. (Note that we do not quite recover [CSV11, Theorem 6] which contains a typo, but the correct formula which is on the second line of the first equation before [CSV11, Theorem 13].)

Before writing down formulas in more general situations, let us recall the shorthand notation

$$\varphi_{i,j}(x) = \begin{cases} 1+x & \text{if } j-i \text{ is odd,} \\ 1/(1-x) & \text{if } j-i \text{ is even.} \end{cases} \tag{55}$$

Keeping $u = 0$ (mixed boundary conditions) but taking w general, we easily obtain

$$F_w(0,v) = \prod_{i:\ w_i=+} \frac{1}{1-vy_i} \prod_{\substack{i<j \\ w_i=+,\ w_j=-}} \varphi_{i,j}(y_i/y_j) \prod_{\substack{i<j \\ w_i=w_j=+}} \varphi_{i,j}(v^2y_iy_j). \tag{56}$$

By taking $v = 1$ and $x_i = q$ for all i, i.e., $y_i = q^{2\ell+1-i}$, we obtain the expression (8) announced in Theorem 4.

Slightly more involved expressions, involving infinite products, arise when considering the case where both u and v are non-zero. Again we begin with the case $w = +^{2\ell}$. The strategy to evaluate (51) in this case is to pick each Γ_+/Γ'_+ (say, successively from left to right), move it to the right and bounce it on $|\underline{v}\rangle$, move the resulting Γ_-/Γ'_- to the left and bounce it on $\langle\underline{u}|$, then finally put back the resulting Γ_+/Γ'_+ into place; see Figure 14 for an illustration. In this process, we collect some factors arising from the bounces and the crossings between operators with different indices, and we end up with the same expression as at the beginning except that every Γ has its parameter multiplied by u^2v^2. In more explicit terms, we have

$$\begin{aligned} &\langle\underline{u}|\,\Gamma_+(y_1)\Gamma'_+(y_2)\cdots\Gamma_+(y_{2\ell-1})\Gamma'_+(y_{2\ell})\,|\underline{v}\rangle \\ &\quad= \prod_{i=1}^{2\ell} \frac{1}{(1-vy_i)(1-uv^2y_i)} \prod_{1\le i<j\le 2\ell} \varphi_{i,j}(v^2y_iy_j)\varphi_{i,j}(u^2v^4y_iy_j) \\ &\qquad\quad \times \langle\underline{u}|\,\Gamma_+(u^2v^2y_1)\Gamma'_+(u^2v^2y_2)\cdots\Gamma_+(u^2v^2y_{2\ell-1})\Gamma'_+(u^2v^2y_{2\ell})\,|\underline{v}\rangle. \end{aligned} \tag{57}$$

Upon iterating this relation k times, we pull out more factors, with a remaining product of Γ_+/Γ'_+ operators with parameters of the form $(uv)^{2k}y_i$. But we have

$$\lim_{k\to\infty} \Gamma_+((uv)^{2k}y_i) = \lim_{k\to\infty} \Gamma'_+((uv)^{2k}y_i) = 1 \tag{58}$$

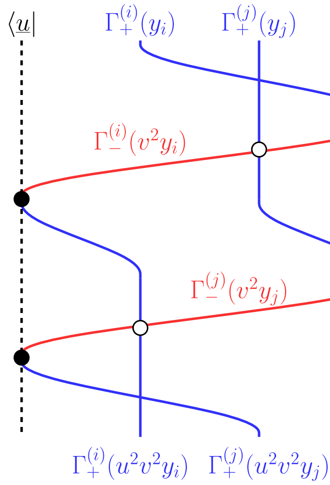

FIGURE 14. Schematic picture of the computation of (51) in the case $w = +^{2\ell}$. We get a factor contributing to (57) when an operator bounces on the boundary (black circles) or when a Γ_+/Γ'_+ crosses a Γ_-/Γ'_- (white circles). Here we only represent the trajectories of two operators (with $\Gamma^{(k)} = \Gamma$ if k is odd and Γ' otherwise).

(where the Γ shall be viewed as infinite matrices whose coefficients are formal power series), hence the remaining product tends to $\langle \underline{u}|\underline{v}\rangle = \prod_{k\geq 1} 1/(1-u^k v^k)$. Rearranging the factors, we end up with the expression
(59)
$$F_{(+^{2\ell})}(u,v) = \prod_{k=1}^{\infty} \left(\frac{1}{1-u^k v^k} \prod_{i=1}^{2\ell} \frac{1}{1-u^{k-1}v^k y_i} \prod_{1\leq i<j\leq 2\ell} \varphi_{i,j}(u^{2k-2}v^{2k}y_i y_j) \right).$$
Finally, for general w, a straightforward adaptation of our strategy yields

$$\begin{aligned}
(60) \quad F_w(u,v) = \prod_{k=1}^{\infty} \Bigg(& \frac{1}{1-u^k v^k} \prod_{i:\, w_i=+} \frac{1}{1-u^{k-1}v^k y_i} \prod_{i:\, w_i=-} \frac{1}{1-u^k v^{k-1}/y_i} \\
& \times \prod_{\substack{1\leq i<j\leq 2\ell \\ w_i=+,\ w_j=-}} \varphi_{i,j}(u^{2k-2}v^{2k-2}y_i/y_j) \prod_{\substack{1\leq i<j\leq 2\ell \\ w_i=w_j=+}} \varphi_{i,j}(u^{2k-2}v^{2k}y_i y_j) \\
& \times \prod_{\substack{1\leq i<j\leq 2\ell \\ w_i=-,\ w_j=+}} \varphi_{i,j}(u^{2k}v^{2k}y_j/y_i) \prod_{\substack{1\leq i<j\leq 2\ell \\ w_i=w_j=-}} \varphi_{i,j}(u^{2k}v^{2k-2}/(y_i y_j)) \Bigg).
\end{aligned}$$

The reader might be wary of the divisions by y_i or y_j, but recall from Lemma 9 that $y_i = x_i x_{i+1}\cdots x_{2\ell}$ where x_i records the size difference between $\lambda^{(i)}$ and $\alpha = \lambda^{(0)}$. To obtain a bona fide power series we need to do the change of variables $u \to u y_0$, and we may then even set $u = v = 1$ without trouble to obtain the "true" generating function of all steep tilings counted with a weight x_i per flip on the i-th diagonal. In particular, if we take $x_i = q$ for all i, hence $y_i = q^{2\ell+1-i}$, $u = q^{2\ell+1}$ and $v = 1$ in

(60), then by rearranging the products we obtain the expression (9) announced in Theorem 4, whose proof is now complete.

Remark 9. The other equations in [Mac95, I.5, Ex. 27] suggest other types of "free" boundary conditions. For instance, if we consider the state

$$|\underline{\tilde{v}}\rangle = \sum_{\lambda \text{ even}} v^{|\lambda|} |\lambda\rangle = \Gamma_-(-v)|\underline{v}\rangle \tag{61}$$

(where a partition is said to be even if all its parts are even), then we have the modified reflection relations

$$\Gamma_+(t)\,|\underline{\tilde{v}}\rangle = \frac{1}{1-t^2v^2}\Gamma_-(tv^2)\,|\underline{\tilde{v}}\rangle, \qquad \Gamma'_+(t)\,|\underline{\tilde{v}}\rangle = \Gamma'_-(tv^2)\,|\underline{\tilde{v}}\rangle \tag{62}$$

and we may then obtain different product formulas for steep tilings with such boundary condition. Considering instead a sum over partitions of the form $(\alpha_1 - 1, \ldots, \alpha_p - 1|\alpha_1, \ldots, \alpha_p)$ in Frobenius notation, we obtain a boundary state that mutates a Γ_+ into a Γ'_- and a Γ'_+ into a Γ_+, up to factors (it would be interesting to have a combinatorial proof of this fact).

Remark 10. For a random steep tiling with mixed boundary conditions, the associated sequence of interlaced partitions forms a so-called *Pfaffian Schur process* [BR05]. The case with two free boundaries has, to the best of our knowledge, not been considered before.

6. Cylindric steep tilings

By a variant of our approach we may consider *cylindric steep tilings* of width 2ℓ: these may be viewed as domino tilings of the plane which are periodic in one direction, namely they are invariant under a translation of vector $(c, c - 2\ell)$ for some c, and which are steep in the same sense as before, that is, we only find north- or east-going (resp. south- or west-going) dominos sufficiently far away in the north-east (resp. south-west) direction. We may restrict a cylindric steep tiling to a fundamental domain by cutting it "along" the $y = x$ and $y = x - 2\ell$ lines (more precisely we cut along the lattice paths that remain closest to these lines and follow domino boundaries): we then obtain a steep tiling of the oblique strip as before, with the only additional constraint that the two boundaries must "fit" into each other. We define the asymptotic data $w \in \{+, -\}^{2\ell}$ as before.

Then, we may proceed as in Section 3 and construct the particle configuration and sequence of integer partitions associated to the tiling. Clearly, the additional constraint is that the particle configuration on the lines $y = x$ and $y = x - 2\ell$ must be the same up to translation, which implies that the associated partitions $\lambda^{(0)}$ and $\lambda^{(2\ell)}$ are equal, and that the parameter c above must be equal to $c_{2\ell} - c_0$, as defined by (12) (the cylindric steep tiling is said to be centered if $c_0 = 0$). We readily arrive at an analogue of Proposition 6, with however a caveat regarding flips. Indeed, when viewing a cylindric steep tiling as a periodic tiling, a flip consists in rotating a 2×2 block of dominos and all its translates (as we wish to preserve periodicity). When considering the tiling restricted to a fundamental domain, regular (bulk) flips are defined as before, but a boundary flip is only allowed if a corresponding flip can be and is performed on the other boundary in order to preserve the shape compatibility (this pair of moves is a flip centered on the 2ℓ-th diagonal).

Proposition 11. *Given a word $w \in \{+,-\}^{2\ell}$, there is a bijection between the set of centered cylindric steep tilings with asymptotic data w and the set of sequences of partitions $(\lambda^{(0)}, \dots, \lambda^{(2\ell)})$ with $\lambda^{(0)} = \lambda^{(2\ell)}$ and such that, for all $k \in \{1, \dots, \ell\}$,*

- $\lambda^{(2k-2)} \prec \lambda^{(2k-1)}$ *if* $w_{2k-1} = +$, *and* $\lambda^{(2k-2)} \succ \lambda^{(2k-1)}$ *if* $w_{2k-1} = -$,
- $\lambda^{(2k-1)} \prec' \lambda^{(2k)}$ *if* $w_{2k} = +$, *and* $\lambda^{(2k-1)} \succ' \lambda^{(2k)}$ *if* $w_{2k} = -$.

Furthermore, the bijection has the following properties:

B'. *For $m = 1, \dots, 2\ell$, the absolute value of $|\lambda^{(m)}| - |\lambda^{(m-1)}|$ counts the number of dominos whose centers are on the line $y = x - m + 1/2$ and whose orientations are opposite to the asymptotic one, as detailed in Table 1.*

C'. *If w contains at least one $+$ and one $-$, then for any $m = 1, \dots, 2\ell$, $|\lambda^{(m)}|$ counts the number of flips centered on the m-th diagonal in any minimal sequence of flips between the tiling at hand and the minimal tiling corresponding to the sequence $(\emptyset, \emptyset, \dots, \emptyset)$.*

Proof. The bijectivity and the property B' are obtained along the same lines as for Proposition 6. We only detail the proof of the property C' since it involves a slight subtlety. We need again to consider height functions, which typically become "multivalued" functions in the cylindric setting. Indeed, using (23) and the characterizations of c_m and h_m in terms of the asymptotic data w, we find that, for a periodic steep tiling with period $(c, c-2\ell)$ and any (x, y) we have

$$H(x,y) - H(x+c, y+c-2\ell) = h, \tag{63}$$

where $h = h_0 - h_{2\ell}$ is equal to the number of $+$ in w minus that of $-$. For $h \neq 0$, H is quasiperiodic but not periodic in the plane hence multivalued on the cylinder. Since the minimal height function $H^w_{\min}$ has the same quasiperiodicity property, the "reduced" height function $\widetilde{H} = (H - H^w_{\min})/4$ is periodic. To a cylindring steep tiling T with height function H, we associate its rank given by

$$r_w(T) = \sum_{(x,y)\in\widetilde{V}} \widetilde{H}(x,y) = \sum_{m=0}^{2\ell-1} |\lambda^{(m)}| \tag{64}$$

where $\widetilde{V} = \{(x,y) \in \mathbb{Z}^2, 0 \leq x - y < 2\ell\}$. This definition is the cylindric analogue of (25) and (26). The property C' results from the following claim: *if $r_w(T) > 0$ and w is neither $+^{2\ell}$ nor $-^{2\ell}$, then T admits a descendent flip* (reducing the rank by 1).

Such a flip can be obtained by the criterion given in the erratum of [EKLP92a]: we first consider the vertices where $\widetilde{H}$ is maximal, and we look for one among them where H is (locally) maximal. If such a vertex exists, it is not difficult to see that we may perform a descendent flip on it. But, as H is unbounded in the plane for $h \neq 0$, the existence of such a vertex is a priori not obvious. Observe first that, since $\widetilde{H}$ is periodic and $0 < r_w(T) < \infty$, $\widetilde{H}$ attains its maximal value on a non-empty subset S of $\mathbb{Z}^2$ which intersects the fundamental domain $\widetilde{V}$ at finitely many points. By contraposition, proving our claim boils down to showing that, if H admits no local maximum within S, then w is necessarily either $+^{2\ell}$ or $-^{2\ell}$.

Assuming that H admits no local maximum within S, we may construct an infinite walk $v_0, v_1, \dots$ in $\mathbb{Z}^2$ as follows: v_0 is an arbitrary element of S and, assuming that v_{i-1} has been constructed, we pick v_i as one of its nearest neighbors in $\mathbb{Z}^2$ such that $H(v_i) > H(v_{i-1})$. It is easily seen that $v_i \in S$ for all i, and since $S \cap \widetilde{V}$

is finite, the projection of the walk on the cylinder eventually intersects itself, in other words there exists $j < j'$ such that $v_{j'} - v_j = k(c, c-2\ell)$ for some $k \in \mathbb{Z}$. Note that $0 \neq H(v_{j'}) - H(v_j) \geq j' - j \geq 2\ell|k|$ by the construction of our lattice walk. But we have $H(v_{j'}) - H(v_j) = -kh$ from (63), so we conclude that $|h| \geq 2\ell$ hence $w = +^{2\ell}$ or $-^{2\ell}$ as wanted. □

Remark 11. In the case $w = +^{2\ell}$ or $-^{2\ell}$, then clearly any sequence of interlaced partitions satisfying the conditions of Proposition 11 is constant. The rank $r_w(T)$ of the corresponding tiling T is a multiple of 2ℓ, hence there is a clear obstruction to the property C' and to the presence of flips in T. This is actually not due to the cylindrical topology per se, but to the presence of "forced cycles" in the associated perfect matching; see [Pro93, Example 2.3] (in fact Jockusch's graph is isomorphic to a truncated cylindric tilted square grid of circumference 4, thus corresponds essentially to the same situation as ours).

The enumerative consequence of Proposition 11 is the following statement, which readily implies Theorem 5 by taking $x_i = q$ for all i.

Theorem 12. *Let $w \in \{+,-\}^{2\ell}$ be a word. Let $C_w \equiv C_w(x_1, \ldots, x_{2\ell})$ be the generating function of cylindric steep tilings of asymptotic data w, where the exponent of the variable x_i records the number of flips centered on the i-th diagonal in a shortest sequence of flips from the minimal tiling. Then one has*

$$C_w = \prod_{k\geq 1}\left(\frac{1}{1-y^k}\prod_{\substack{1\leq i<j\leq 2\ell\\ w_i=+,\ w_j=-}} \varphi_{i,j}(y^{k-1}x_i x_{i+1}\cdots x_{j-1}) \right. \\ \left. \times \prod_{\substack{1\leq i<j\leq 2\ell\\ w_i=-,\ w_j=+}} \varphi_{i,j}(y^{k-1}x_1x_2\cdots x_{i-1}x_jx_{j+1}\cdots x_{2\ell})\right) \tag{65}$$

where $y = x_1 \cdots x_{2\ell}$ and $\varphi_{i,j}(\cdot)$ is defined as in (6).

Remark 12. Proposition 11 actually shows that cylindric steep tilings (weighted by flips) form a *periodic Schur process*, as defined by Borodin [Bor07], and (65) is nothing but the partition function of this process.

Proof. We could obtain (65) as a suitable specialization of [Bor07, Proposition 1.1], itself a variation on [Mac95, I.5, Ex. 28(a)], but let us here provide a proof using vertex operators. By Proposition 11, we have

$$C_w = \sum_\lambda y^{|\lambda|} T_{w,\lambda,\lambda} \tag{66}$$

where $T_{w,\lambda,\lambda}$ is defined as in Lemma 9. We evaluate this quantity following a strategy similar to that used for the derivation of (59) and (60) in the case of steep tilings with free boundary conditions, but here we would like the cyclic symmetry to be manifest, which leads us to rewrite C_w in a slightly different form. Let us introduce the *energy operator* H such that $H|\lambda\rangle = |\lambda| \times |\lambda\rangle$, so that $x^H|\lambda\rangle = x^{|\lambda|}|\lambda\rangle$ with x a formal variable. Then, we have

$$C_w = \mathrm{Tr}\left[\Gamma_{w_1}(1)(x_1)^H\Gamma'_{w_2}(1)(x_2)^H \cdots \Gamma_{w_{2\ell-1}}(1)(x_{2\ell-1})^H\Gamma'_{w_{2\ell}}(1)(x_{2\ell})^H\right] \tag{67}$$

where $\mathrm{Tr}[\cdot] = \sum_\lambda \langle\lambda| \cdot |\lambda\rangle$ is the trace. The strategy is then to move, one by one, each Γ_+ or Γ'_+ to the right and "wrap" it around the cylinder (using the cyclicity of the trace) until it is back into place. To that end, we need the following easy commutation relations:

$$\Gamma_+(t)\,x^H = x^H\Gamma_+(tx), \qquad \Gamma'_+(t)\,x^H = x^H\Gamma'_+(tx), \tag{68}$$

and the usual commutation relations (39), which show that a factor $\varphi_{i,j}$ arises each time a Γ_+/Γ'_+ crosses a Γ_-/Γ'_-, with an argument equal to the product of the x_k's between them, and furthermore that the argument of each Γ_+/Γ'_+ is multiplied by y after one turn. In other words we have

$$\begin{aligned} C_w = &\prod_{\substack{1\le i<j\le 2\ell \\ w_i=+,\ w_j=-}} \varphi_{i,j}(x_i x_{i+1}\cdots x_{j-1}) \\ &\qquad\times \prod_{\substack{1\le i<j\le 2\ell \\ w_i=-,\ w_j=+}} \varphi_{i,j}(x_1x_2\cdots x_{i-1}x_j x_{j+1}\cdots x_{2\ell}) \times \widetilde{C}_w(y) \end{aligned} \tag{69}$$

where $\widetilde{C}_w(y)$ is obtained from the right hand side of (67) by replacing the argument of each Γ_+/Γ'_+ by y. We may then repeat the same strategy k times, pulling more factors times $\widetilde{C}_w(y^k)$. But then it is readily seen that

$$\lim_{k\to\infty} \widetilde{C}_w(y^k) = \mathrm{Tr}\left[(x_1\cdots x_{2\ell})^H\right] = \prod_{k\ge 1}\frac{1}{1-y^k} \tag{70}$$

from the (already noted) fact that $\lim \Gamma_+(y^k) = \lim \Gamma'_+(y^k) = 1$ as $k\to\infty$ and that the Γ_-/Γ'_- are "upper unitriangular". The wanted expression (65) follows. □

7. An extended model: Interpolation between plane partitions and domino tilings

In this section we define an extended model that is more general than steep tilings. The model gives an interpretation of any sequence of partitions interlaced with relations in $\{\prec,\succ,\prec',\succ'\}$ in terms of perfect matchings of some infinite planar graph (which also gives an interpretation in terms of tilings; see Section 7.4). The model contains both steep tilings and plane partitions as special cases. For simplicity we only deal with the case of pure boundary conditions, but there is no doubt that we may also treat more general (mixed, free, periodic) ones.

Let $k\ge 1$, and let $\diamond\in\{\succ,\prec,\succ',\prec'\}^k$ be a word. We consider the set $\mathcal{S}_\diamond$ made by sequences of partitions interlaced according to $\diamond$:

$$\mathcal{S}_\diamond := \left\{(\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(k)}),\ \text{for all } 1\le i\le k, \lambda^{(i-1)}\diamond_i\lambda^{(i)}\right\}. \tag{71}$$

Proposition 6 shows that if k is even, and if $\diamond_i\in\{\prec,\succ\}$ (resp. $\diamond_i\in\{\prec',\succ'\}$) when i is odd (resp. even), elements of $\mathcal{S}_\diamond$ are in bijection with steep tilings of a given asymptotic data. We will generalize the construction to words $\diamond$ that do not satisfy this condition.

In order to present the construction, it is convenient to take a step from the world of domino tilings to the world of matchings. Recall that a *matching* of a graph is a subset of disjoint edges. A matching is *perfect* if it covers all the vertices. It is well known (and clear) that domino tilings of a region of $\mathbb{Z}^2$ made by a union of unit squares are in bijection with perfect matchings of its dual graph.

The flip operation has a natural description in terms of matchings, that can be generalized as follows. Let G be a bipartite plane graph, let $\mathfrak{t}$ be a matching of G, and let f be a bounded face of G bordered by $2p$ edges, for some $p\ge 1$. If p of

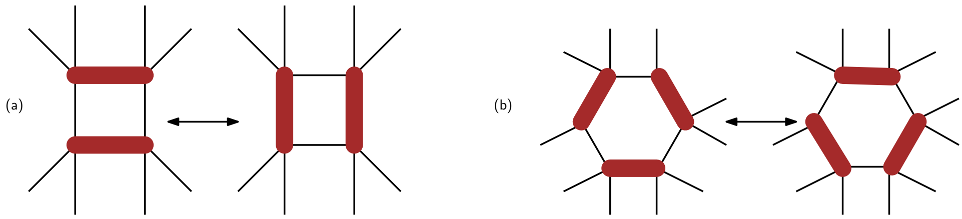

FIGURE 15. The flip operation around a face; (a) face of degree 4; (b) face of degree 6. Edges belonging to the matching are represented as bold.

these edges belong to the matching $\mathfrak{t}$, we can remove these edges from $\mathfrak{t}$ and replace them by the other p edges bordering f, thus creating a new matching $\mathfrak{t}'$ of G. This operation is called *a flip*; see Figure 15.

We now state the main result of this section. An element of $\mathcal{S}_\diamond$ is *pure* if it is such that $\lambda^{(0)} = \lambda^{(k)} = \emptyset$.

Theorem 13. *Let $k \geq 1$ and $\diamond \in \{\prec, \succ, \prec', \succ'\}^k$. There exists an infinite plane graph $G_\diamond$, and a matching $\mathfrak{m}_\diamond$ of $G_\diamond$, called* minimal*, such that pure elements of $\mathcal{S}_\diamond$ are in bijection with matchings of $G_\diamond$ that can be obtained from $\mathfrak{m}_\diamond$ by a finite sequence of flips.*

The bijection maps a sequence $\vec{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$ to a matching $\phi(\vec{\lambda})$ such that the the minimum number of flips needed to obtain $\phi(\vec{\lambda})$ from $\mathfrak{m}_\diamond$ is equal to $\sum_{i=0}^{k} |\lambda^{(i)}|$.

Remark 13. We will actually see a more precise result. Namely, the graph $G_\diamond$ and its embedding in the plane will be such that each flip has a well-defined *abscissa*, which will be of the form $\frac{3}{2}j$ for some $j \in \{0, 1, \dots, k\}$. Then the following will be true: the number of flips at abscissa $\frac{3}{2}j$ in any shortest sequence of flips from $\mathfrak{m}_\diamond$ to $\phi(\vec{\lambda})$ is independent of the sequence, and is equal to $|\lambda^{(j)}|$. This will lead us to analogues of Theorem 3 in the general setting (see Theorem 17 below).

7.1. **The graph $G_\diamond$, and admissible matchings.** We now start the proof of Theorem 13. We first construct a graph $G_\diamond$, which is a bipartite graph embedded in the plane (Figure 16). The vertex set V of $G_\diamond$ is defined by:

$$V = \bigcup_{i=0}^{k} V_j \cup \bigcup_{i=1}^{k} W_j, \tag{72}$$

where for $0 \leq j \leq k$, $V_j = \{(\frac{3}{2}j, y), y \in \mathbb{Z} + \frac{j+1}{2}\}$, and where for $1 \leq j \leq k$,

$$W_j = \begin{cases} \{(\frac{3}{2}j - 1, y), y \in \mathbb{Z} + \frac{j+1}{2}\}, & \text{if } \diamond_j \in \{\prec, \succ\}, \\ \{(\frac{3}{2}j - \frac{1}{2}, y), y \in \mathbb{Z} + \frac{j}{2}\}, & \text{if } \diamond_j \in \{\prec', \succ'\}. \end{cases} \tag{73}$$

We then add an edge of $G_\diamond$ between any two vertices of V which differ by a vector $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$, or $(\frac{1}{2}, -\frac{1}{2})$. See Figure 16. Note that the graph $G_\diamond$ does not characterize the word $\diamond$: indeed the symbols $\prec, \succ$ ($\prec', \succ'$, respectively) play the same role in the construction.

In view of defining the matchings we are interested in, we first need to define a function Y that will play the role of a zero ordinate, local to each "column"

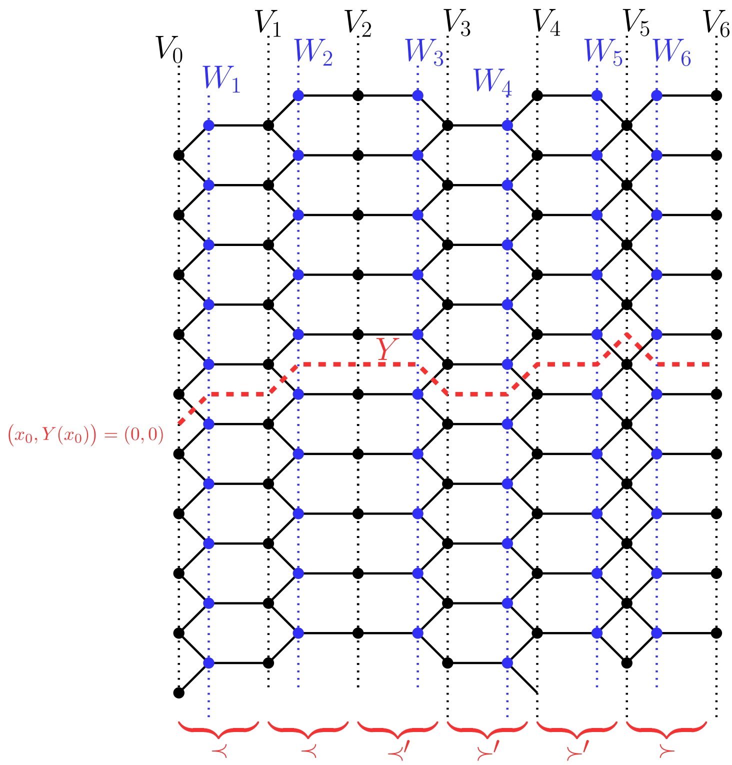

FIGURE 16. The graph $G_\diamond$ for $\diamond = \prec\prec\prec'\succ'\succ'\succ$ (and $k = 6$). Only a bounded portion is shown, the actual graph being infinite towards top and bottom. Vertical dotted lines are there to help visualize the vertex sets $V_0, W_1, \ldots, V_k, W_k$, and they are not part of the graph. The red dotted path is not part of the graph either, and represents the path $(x_j, Y(x_j))_{0\le j\le 2k}$. In particular its leftmost point is the origin $(0,0)$ of the coordinate system.

of $G_\diamond$. To this end, define $x_0 = 0$, and for $1 \le j \le k$ let x_{2k-1} (resp. x_{2k}) be the common abscissa of all vertices in W_k (resp. V_k). Then $x_0 < x_1 < \cdots < x_{2k}$ are all the abscissas of vertices appearing in $G_\diamond$. We define the function $Y : \{x_0, x_1, \ldots, x_{2k}\} \to \frac{1}{2}\mathbb{Z}$ by the fact that $Y(x_0) = 0$ and for $1 \le i \le k$:

$$Y(x_{2i}) = \begin{cases} Y(x_{2i-2}) + \frac{1}{2} & \text{if } \diamond_j \in \{\prec, \succ'\}, \\ Y(x_{2i-2}) - \frac{1}{2} & \text{if } \diamond_j \in \{\prec', \succ\}, \end{cases} \tag{74}$$

$$Y(x_{2i-1}) = \begin{cases} Y(x_{2i}) & \text{if } \diamond_j \in \{\prec, \succ\}, \\ Y(x_{2i-2}) & \text{if } \diamond_j \in \{\prec', \succ'\}. \end{cases} \tag{75}$$

Strictly speaking, only the values $Y(x_{2i})$ are needed in our construction, but defining Y on all x_i's enables one to represent it easily on pictures; see Figure 16.

All the matchings of $G_\diamond$ that we will consider are such that all the vertices of $V \setminus (V_0 \cup V_k)$ are covered by the matching. If v is a vertex of V, we will then say that v is matched *to the left* (resp. *to the right*), if either v is covered and the matching

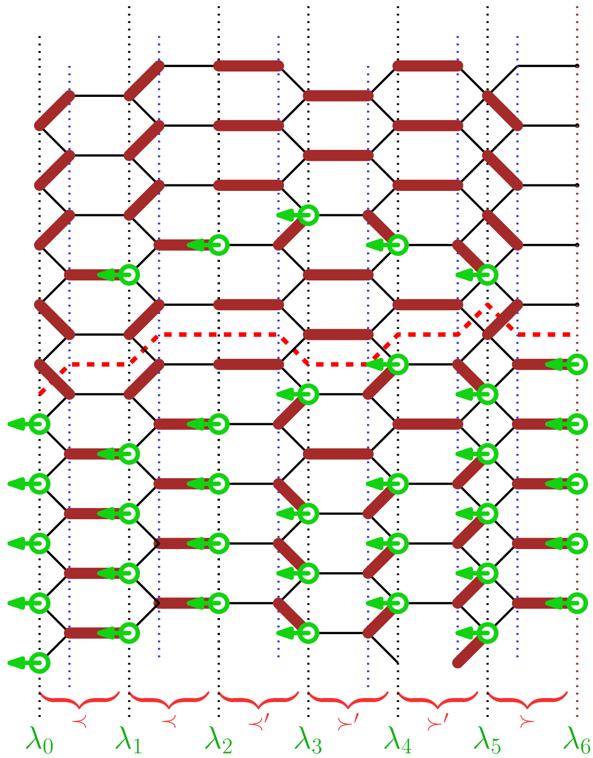

FIGURE 17. An admissible matching of the graph of Figure 16. Edges in the matching are bold (and brown). To help visualize, vertices in $V_0 \cup V_1 \cup \cdots \cup V_k$ that are matched to the left are indicated by circled (green) left-pointing arrows. Outside the displayed region, the matching continues periodically along the y direction towards top and bottom. For each $j \in \{0, 1, \ldots, k\}$, the bijection Ψ consists in interpreting the circled vertices at abscissa $\frac{3}{2}j$ as the Maya diagram of a partition. This example corresponds to the partitions $\lambda^{(0)} = \emptyset$, $\lambda^{(1)} = \lambda^{(2)} = (2)$, $\lambda^{(3)} = (3, 1)$, $\lambda^{(4)} = (2, 1)$, $\lambda^{(5)} = (1)$, $\lambda^{(6)} = \emptyset$.

connects it to a vertex to its left (resp. to its right), or if v is uncovered and belongs to V_0 (resp. to V_k). Note that this definition is consistent if one imagines that uncovered vertices are matched towards the exterior of the graph. A matching of $G_\diamond$ is *admissible* if it is such that for all $0 \leq j \leq k$, the number of vertices in V_j above the ordinate $Y(x_{2j})$ that are matched to the left is finite, and if this number equals the number of vertices in V_j below the ordinate $Y(x_{2j})$ that are matched to the right. See Figure 17.

Let $\mathfrak{m}$ be an admissible matching of $G_\diamond$, and let $j \in \{0, 1, \ldots, k\}$. Consider all the vertices of V_j that are matched to the left, from top to bottom, and let $y_{1,j} > y_{2,j} > \ldots$ be their ordinates. Then the admissibility condition ensures that the non-increasing integer sequence

$$\lambda_i^{(j)} := y_{i,j} - Y(x_{2j}) + i - \frac{1}{2} \tag{76}$$

vanishes for i large enough, i.e., that $\lambda^{(j)}$ is an integer partition. One can interpret this construction by noting that if one considers vertices in V_j that are matched to the left (resp. to the right) as occupied sites (resp. empty sites), then V_j, when read from bottom to top, is the Maya diagram of the partition $\lambda^{(j)}$; see Figure 17. We let $\Psi(\mathfrak{m}) := (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$ be the tuple of partitions thus defined.

Proposition 14. *The mapping Ψ is a bijection between admissible matchings of $G_\diamond$ and elements of $\mathcal{S}_\diamond$.*

Proof. Let us first check that $\Psi(\mathfrak{m})$ belongs to $\mathcal{S}_\diamond$. Let $j \in \{1, 2, \dots, k\}$, and suppose first that $\diamond_j \in \{\prec, \succ\}$. We claim that for $i \geq 1$, there is exactly one vertex of V_j whose ordinate lies between $y_{i,j-1}$ and $y_{i+1,j-1}$. Indeed, by construction, there are $(y_{i,j-1} - y_{i+1,j-1})$ vertices in W_j whose ordinates are in this interval, and exactly $(y_{i,j-1} - y_{i+1,j-1} - 1)$ matching edges are coming to these vertices from V_{j-1}. Thus exactly one of these vertices is matched to the right. Since a vertex in W_j is matched to the right if and only if its unique neighbor in V_j is matched to the left, this proves the claim. Equivalently, there is a unique $i' \geq 1$ such that $y_{i,j-1} > y_{i',j} > y_{i+1,j-1}$, which shows that the partitions $\lambda^{(j-1)}$ and $\lambda^{(j)}$ are interlaced, i.e., either $\lambda^{(j-1)} \prec \lambda^{(j)}$ or $\lambda^{(j-1)} \succ \lambda^{(j)}$. To determine which case we are in, take i large enough such that $\lambda_i^{(j-1)} = \lambda_i^{(j)} = 0$. Then (76) and the definition of Y show that $y_{i,j-1} - y_{i,j}$ is positive (resp. negative) if $\diamond_j = \succ$ (resp. $\diamond_j = \prec$). This proves that $\lambda^{(j-1)} \diamond_j \lambda^{(j)}$ in both cases. The case where $\diamond_j \in \{\prec', \succ'\}$ is handled exactly in the same way, and we leave it to the reader to check that $\lambda^{(j-1)} \diamond_j \lambda^{(j)}$ in this case as well, thus proving that $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)})$ belongs to $\mathcal{S}_\diamond$.

Now let $\vec{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)}) \in \mathcal{S}_\diamond$. We will prove that there exists a unique admissible matching of $G_\diamond$ such that for each j, the elements of V_j that are matched to the left are exactly the ones of ordinate $\lambda_i^{(j)} + Y(x_{2j}) - i + \frac{1}{2}$ for some $i \geq 1$. First note that imposing which elements of V_j are matched to the left for all $j \in \{0, 1, \dots, k\}$ determines uniquely the matching on horizontal edges (indeed any horizontal edge is incident to such a vertex). One thus has to show that, once these horizontal edges have been placed, there is a unique way of adding diagonal edges to the matching in order to make it admissible and respect the condition on left-matching vertices. This can be easily checked, case by case, distinguishing as before according to the value of $\diamond_j \in \{\prec, \succ, \prec', \succ'\}$. We leave it to the reader. $\square$

7.2. **The pure case, and flips.** In the general case where $\lambda^{(0)}$ and $\lambda^{(k)}$ are arbitrary, we will not go further than Proposition 14. We now focus on the pure case, i.e., we assume $\lambda^{(0)} = \lambda^{(k)} = \emptyset$. First, we define *the minimal matching* $\mathfrak{m}_\diamond$ as the image of $(\emptyset, \emptyset, \dots, \emptyset)$ by the bijection Ψ^{-1}. See Figure 18.

In order to prove Theorem 13, we first note that if F is a bounded face of $G_\diamond$, there is exactly one $j \in [\![1, k]\!]$ such that two vertices of V_j belong to F. Moreover, if the flip of the face F is possible, these two vertices are matched in different directions (one to the left, the other one to the right), and the flip exchanges these directions between the two vertices. It follows that the effect of a flip on the Maya

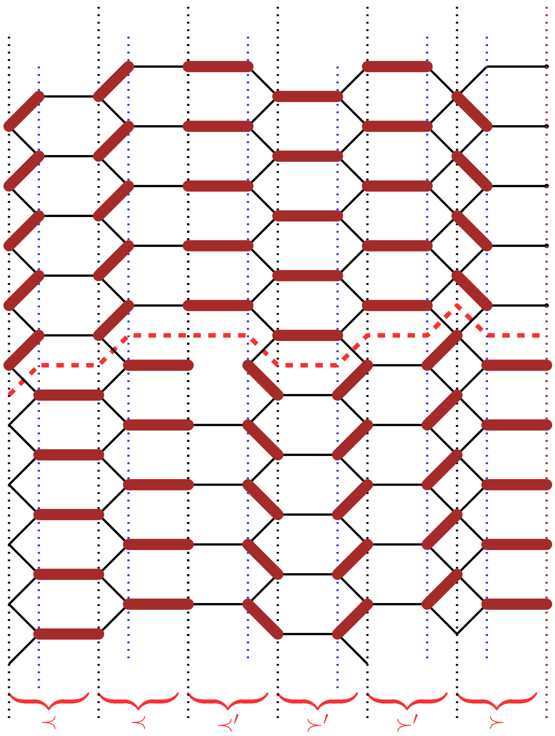

FIGURE 18. The minimal matching of the graph of Figure 16. Below the red dotted line representing Y, all the vertices in $V_0 \cup V_1 \cup \cdots \cup V_k$ are matched to the left.

diagram is to make exactly one particle jump by one position. We define a flip to be *ascendent* or *descendent* according to whether the particle jumps to the top or to the bottom, respectively. Moreover, we define of the *abscissa* of the flip to be the abscissa of this particle, i.e., $\frac{3}{2}j$. Since the jump of one particle to the top increases the quantity $\sum_{i=0}^{k} |\lambda^{(i)}|$ by exactly one, Theorem 13 is thus a direct consequence of the following lemma:

Lemma 15. *Let $\mathfrak{m}$ be a pure admissible matching which is different from the minimal one. Then it is possible to perform a descendent flip on $\mathfrak{m}$.*

Proof. This is a consequence of the general theory developed by Propp [Pro93], but let us here provide a self-contained argument. First to each admissible matching we associate a height function that associates to each bounded face F of the graph $G_\diamond$ a value $h(F) \in \mathbb{Z}$ as follows. Let F be a bounded face of $G_\diamond$, and consider the unique $j = j(F)$ such that V_j has two vertices incident to F. Let v_0 be the midpoint between these two vertices. We define $h(F) := 2h_\bullet(F) + h_\circ(F)$ where $h_\bullet(F)$ is the number of vertices of V_j that are above v_0 and that are matched to the left, and $h_\circ(F)$ is the number of vertices of V_j that are below v_0 and that are matched to the right.

We now let $h_\emptyset$ be the height function corresponding to the minimal matching. If $\mathfrak{m}$ is a pure admissible matching different from the minimal one, then its *reduced* height function $h_{\mathrm{red}} := h - h_\emptyset$ has a positive maximum. We now choose a face F of $G_\diamond$ as follows:

1. $h_{\mathrm{red}}(F)$ is a maximum of the function h_{red}.
2. Among faces satisfying 1., $h(F)$ is a maximum of the function h.
3. Among faces satisfying 1. and 2., F is one that maximizes the quantity $d(F) - u(F)$, where $u(F)$ is the number of $i \le j(F)$ such that $\diamond_i \in \{\prec, \prec'\}$, where $d(F) = j(F) - u(F)$, and where $j(F)$ is as above the unique index such that the face F is incident to two vertices of $V_{j(F)}$.

We now claim that the face F is flippable, and that the corresponding flip is decreasing. This statement follows from a case by case analysis, distinguishing according to the nature of the face F, i.e., according to the possible choices of two symbols in $\{\prec, \succ, \prec', \succ'\}$ that give rise to the face F. We leave this verification to the reader, and observe that it suffices to prove the lemma. ☐

7.3. **Enumerative results.** The enumeration of elements of $\mathcal{S}_\diamond$ can be performed easily via the vertex operator formalism, exactly as we have proceeded for the special case treated in Section 5. Theorem 13 thus implies:

Theorem 16. *Let $\diamond$ be a word on the alphabet $\{\prec, \succ, \prec', \succ'\}$, and let $T_\diamond(q)$ be the generating function of admissible matchings of the graph $G_\diamond$ that can be obtained from the minimal matching by a finite sequence of flips, where the exponent of the variable q marks the length of a minimal such sequence. Then one has*

$$T_\diamond(t) = \prod_{\substack{i<j \\ \diamond_i \in \{\prec, \prec'\} \\ \diamond_j \in \{\succ, \succ'\}}} (1 + \epsilon_{i,j} q^{j-i})^{\epsilon_{i,j}}, \tag{77}$$

where $\epsilon_{i,j} = \begin{cases} 1 & \text{if } (\diamond_i, \diamond_j) \in \{(\prec, \succ'), (\prec', \succ)\}, \\ -1 & \text{if } (\diamond_i, \diamond_j) \in \{(\prec, \succ), (\prec', \succ')\}. \end{cases}$

Theorem 17. *Let $\diamond \in \{\prec, \succ, \prec', \succ'\}^k$, and let $S_\diamond(x_1, x_2, \dots, x_{k-1})$ be the generating function of admissible matchings of the graph $G_\diamond$ that can be obtained from the minimal matching by a finite sequence of flips, where the exponent of the variable x_i marks the number of flips at abscissa $\frac{3}{2}i$ in a sequence of minimal length. Then one has*

$$S_\diamond(x_1, x_2, \dots, x_{k-1}) = \prod_{\substack{i<j \\ \diamond_i \in \{\prec, \prec'\} \\ \diamond_j \in \{\succ, \succ'\}}} (1 + \epsilon_{i,j} x_i x_{i+1} \dots x_{j-1})^{\epsilon_{i,j}}, \tag{78}$$

where $\epsilon_{i,j}$ is as in Theorem 16.

7.4. **Interpretation as tilings.** Admissible matchings of the graph $G_\diamond$ can also be interpreted as tilings. Let $G'_\diamond$ be the dual graph of $G_\diamond$, i.e., the graph with one vertex inside each bounded face of $G_\diamond$, and edges representing face adjacencies in $G_\diamond$. Then each edge e in a matching of $G_\diamond$ can be interpreted as a tile in a tiling of $G'_\diamond$, made by the union of the two faces of $G'_\diamond$ corresponding to the two endpoints of e. In order to display this interpretation, one must first choose a way of drawing the graph $G'_\diamond$. There is not necessarily a good canonical way to do this, since for example the graph $G'_\diamond$ can have multiple edges (this is the case for the graph $G_\diamond$ of Figure 16). In any case, it is always possible to fix some drawing of the dual graph

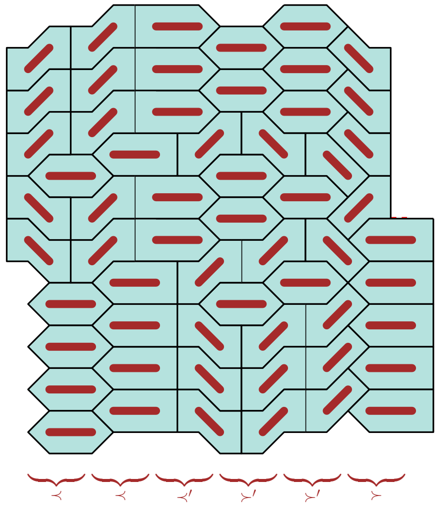

FIGURE 19. The admissible matching of Figure 17 represented as a tiling. To make the comparison easier, we have drawn inside each (blue) tile the corresponding (brown) edge of $G_\diamond$.

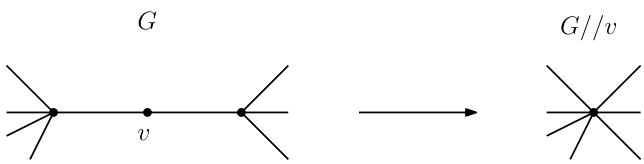

FIGURE 20. Contracting a vertex of degree 2.

$G'_\diamond$, possibly with broken lines instead of straight lines in order to take multiple edges into account. Figure 19 displays as a tiling of the matching of Figure 17. (Other displays would have been possible, would the underlying drawing of the dual graph have been different.)

7.5. **Vertex contraction, and the special case of steep tilings.** If G is a graph and v is a vertex of degree 2 of G the *contracted graph* $G//v$ is obtained from G by contracting the two edges incident to v (thus identifying v with its two neighbors). See Figure 20. It is well known, and easy to see, that perfect matchings of G are in bijection with perfect matchings of $G//v$. Now, because of the construction of $G_\diamond$, if there is a $j \in \{1, 2, \ldots, k-1\}$ such that $\diamond_j \in \{\prec, \succ\}$ and $\diamond_{j+1} \in \{\prec', \succ'\}$, the vertices in V_j all have degree 2. We can thus contract all of them according to the procedure above. If we apply this construction for all such values of j, we are left with a graph that we denote by $\widetilde{G}_\diamond$. Note that the vertex contraction removes all the bounded faces of degree 8, so that $\widetilde{G}_\diamond$ only has bounded faces of degree 4 or 6.

Now, as noted at the beginning of this section, steep tilings correspond to the case where k is even (say $k = 2\ell$), and $\diamond_j \in \{\prec, \succ\}$ for j odd and $\diamond_j \in \{\prec', \succ'\}$ for j even. In this case, all the vertices in V_j for j odd can be contracted. One easily

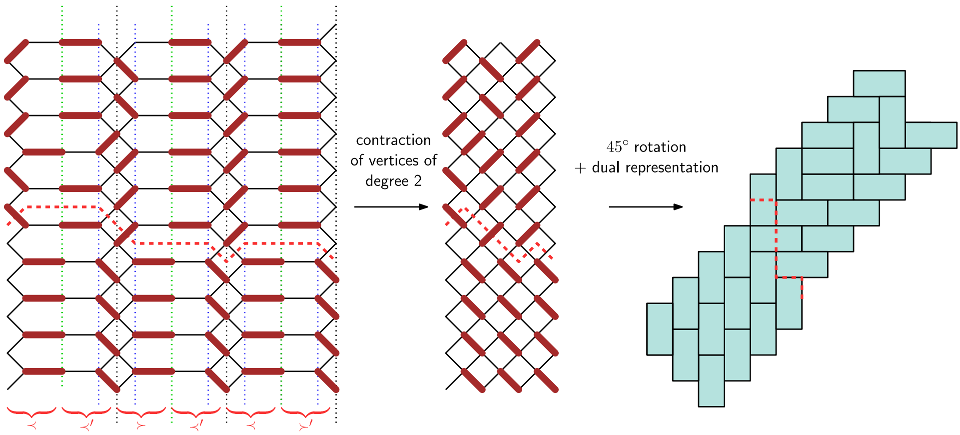

FIGURE 21. (a) A configuration from the extended model, corresponding to the word $\diamond = + +' - +' + -'$. All the vertices on the green dotted lines have degree 2. (b) After contraction of these vertices of degree 2, we obtain a portion of a (rotated) square lattice, equipped with a matching. (c) By rotating the lattice by $45°$ and looking at the dual representation in terms of dominos, we obtain a steep tiling of the oblique strip, as defined in Section 2.

sees that in this case the graph $G_\diamond$ has only bounded faces of degree 4 and 8, so that the graph $\widetilde{G}_\diamond$ has only faces of degree 4; see Figure 21. More precisely, $\widetilde{G}_\diamond$ is the portion of a square lattice that intersects an oblique strip of width 2ℓ, up to a rotation of $45°$. Since the square lattice is self-dual, the corresponding tiles will be 2×1 or 1×2 dominos (after the rotation of $45°$). We thus recover steep tilings as they were defined in the first part of this paper.

7.6. **The special case of plane partitions.** In the case of plane partitions, the correspondence presented in this section is well known; see e.g. [OR03]. A plane partition (of width 2ℓ) can be defined as a sequence of partitions $(\lambda^{(i)})_{-\ell\leq i\leq \ell}$ such that $\lambda^{(-\ell)} = \lambda^{(\ell)} = \emptyset$, and $\lambda^{(i)} \prec \lambda^{(i+1)}$ if $i < 0$ and $\lambda^{(i)} \succ \lambda^{(i+1)}$ if $i > 0$. In our setting it is a pure element of $\mathcal{S}_\diamond$ for $\diamond = \prec^\ell \succ^\ell$. By applying the construction of this section, one recovers a standard bijection between plane partitions of width 2ℓ a family of perfect matchings of the hexagonal lattice; see Figure 22(b). The dual interpretation, in terms of tilings by rhombi, is also standard; see Figure 22(c).

8. Conclusion and discussion

In this paper, we have introduced the so-called steep tilings. We have studied their combinatorial structure and their various avatars, and obtained explicit expressions for their generating function with arbitrary asymptotic data and different types of boundary conditions (pure, mixed, free and periodic). We have also introduced an extended model interpolating between domino and rhombus tilings, where similar expressions can be found. We now list a few concluding remarks, some of which indicate directions for further research.

First, our derivation of the generating functions was done using the vertex operator formalism, which yields compact proofs. It is however not very difficult to

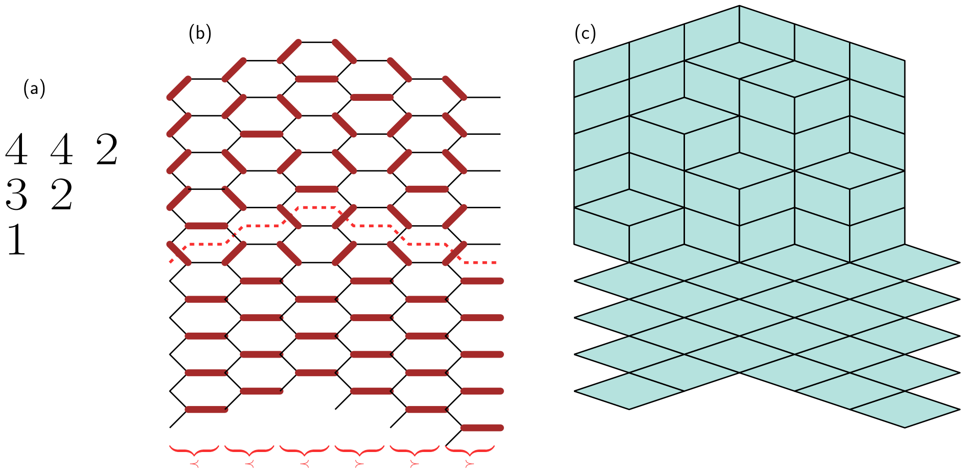

FIGURE 22. (a) A plane partition corresponding to $\lambda^{(-3)} = \lambda^{(3)} = \emptyset$, $\lambda^{(-2)} = (1)$, $\lambda^{(-1)} = (3)$, $\lambda^{(0)} = (4, 2)$, $\lambda^{(1)} = (4)$, $\lambda^{(2)} = (2)$, displayed as an array of numbers. (b) The corresponding matching on the hexagonal lattice. (c) The dual picture, as a rhombi tiling, that can also be viewed as a projected three-dimensional display of the array of (a).

convert these into *bijective* proofs, for instance by looking for bijective proofs of the commutation relations (39) which form the heart of our derivation. This is actually related to the celebrated Robinson-Schensted-Knuth correspondence and, more precisely, its reformulation in terms of growth diagrams introduced by Fomin. There is an important amount of literature devoted to that subject; see e.g. [Kra06] and the references therein. In particular, the sequences of interlaced partitions that we consider in this paper are sometimes called "oscillating supertableaux" [PP96]. Interestingly, the bijective approach yields efficient random generation algorithms for the perfect sampling of steep tilings [BBB14].

Second, beyond the computation of the partition function done in this paper, we have access to detailed statistics of random steep tilings, namely the probabilities of finding dominos of a given type at given positions. In the case of pure boundary conditions, the correlations of the associated particle system are known explicitly from the general results of [OR03] and, remarkably, it is possible to deduce from them an explicit formula for the inverse Kasteleyn matrix of steep tilings with arbitrary asymptotic data [BBC17]. This actually works in the context of the extended model of Section 7, and enables us to recover in a unified and combinatorial way results from [BF14] for rhombus tilings and from [CY14] for domino tilings of the Aztec diamond. In the case of the mixed or periodic boundary conditions, it should be possible to perform the same study as the particle correlations are known [BR05, Bor07]. In the case of free boundary conditions however, even the particle correlations are (to the best of our knowledge) unknown. Our vertex operator derivation of the partition function suggests the possible definition of a "reflected Schur process" that we would like to investigate.

Third, as in the case of plane partitions and domino tilings of the Aztec diamond, steep tilings display a "limit shape phenomenon", that may be observed experimentally via the above-mentioned random generation algorithms, and studied analytically by considering the asymptotic behaviour of domino/particle correlations. Generally speaking, all the questions that have been asked or answered in the literature about rhombus or domino tilings (such as limit shapes, with different kinds of scalings, or fluctuations, in the bulk or near the boundary, etc.) can be asked for steep tilings, which opens a wide area to be investigated.

Finally, a tantalizing question is whether it is possible to go beyond the determinantal/Schur/free-fermionic setting to study some refined statistics on steep tilings. For instance, it is well known that domino tilings of the Aztec diamond correspond to the 2-enumeration of alternating sign matrices (ASMs), and one may ask what an ASM with different "asymptotic data" looks like. Regarding sequences of interlaced partitions, non-determinantal generalizations of Schur processes are the so-called Macdonald processes [BC14]. The search for possible connections with tilings was initiated by Vuletić in her thesis [Vul09a, Vul09b], and we wonder whether this could yield new interesting statistics in the case of domino tilings. Last but not least, some refined weighting schemes for domino tilings of the Aztec diamonds (leading to a host of new fascinating limit shapes) were recently considered [CY14, DFSG14], which raises the question of a possible connection with our approach.

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