STABILITY RESULTS FOR SECTIONS OF CONVEX BODIES

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Abstract. It is shown by Makai, Martini, and ´Odor that a convex body $K$, all of whose maximal sections pass through the origin, must be origin-symmetric. We prove a stability version of this result. We also discuss a theorem of Koldobsky and Shane about determination of convex bodies by fractional derivatives of the parallel section function and establish the corresponding stability result.

1. Introduction

Let $K$ be a convex body in $\mathbb{R}^n$, i.e. a compact convex set with non-empty interior. Throughout the paper, we assume all convex bodies include the origin as an interior point. Now, we say $K$ is origin-symmetric if $K = -K$. The parallel section function of $K$ in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \text{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}), \quad t \in \mathbb{R}. $$

Here, $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0 \}$ is the hyperplane passing through the origin and orthogonal to the vector $\xi$.

For the study of central sections it is often more natural to consider a larger class of bodies than the class of convex bodies. For $x \in \mathbb{R}^n$, let $[0, x]$ denote the closed line segment connecting $x$ to the origin. A star body $K$ in $\mathbb{R}^n$ is a compact set such that $[0, x] \subset K$ for every $x \in K$, and whose radial function defined by

$$\rho_K(\xi) = \max\{a \geq 0 : a\xi \in K\}, \quad \xi \in S^{n-1}, $$

is positive and continuous. Geometrically, $\rho_K(\xi)$ is the distance from the origin to the point on the boundary in the direction of $\xi$. Every convex body (with the origin in its interior) is a star body. The intersection body of a star body $K$ is the star body $IK$ with radial function

$$\rho_{IK}(\xi) = \text{vol}_{n-1}(K \cap \xi^\perp), \quad \xi \in S^{n-1}. $$

Intersection bodies were introduced by Lutwak in [10] and have been actively studied since then. For example, they played a crucial role in the solution of the Busemann-Petty problem (see [8] for details).

The cross-section body of a convex body $K$ is the star body $CK$ with radial function

$$\rho_{CK}(\xi) = \max_{t \in \mathbb{R}} A_{K,\xi}(t), \quad \xi \in S^{n-1}. $$

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Cross-section bodies were introduced by Martini \cite{12}. For properties of these bodies and related questions see \cite{2}, \cite{4}, \cite{5}, \cite{11}, \cite{13}, \cite{14}.

Brunn’s theorem asserts that the origin-symmetry of a convex body $K$ implies

$$A_{K,\xi}(0) = \max_{t \in \mathbb{R}} A_{K,\xi}(t)$$

for all $\xi \in S^{n-1}$. In other words, $CK = IK$. The converse statement was proved by Makai, Martini and Ondor \cite{11}.

**Theorem 1** (Makai, Martini and Ondor). Let $K$ be a convex body in $\mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$, $K \cap \xi^\perp$ has maximal $(n-1)$-dimensional volume amongst all the hyperplane sections of $K$ perpendicular to $\xi$. Then $K$ is origin-symmetric.

Theorem 1 ensures that when a convex body $K$ is such that $CK = IK$, it is origin-symmetric. The goal of the present paper is to provide a stability version of this theorem. For star bodies $K$ and $L$ in $\mathbb{R}^n$, the radial metric is defined as

$$\rho(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$ 

We prove the following result. The notation $B^n_2(r)$ is used for the Euclidean ball in $\mathbb{R}^n$ with radius $r > 0$ centred at the origin.

**Theorem 2.** Let $K$ be a convex body in $\mathbb{R}^n$ such that

$$B^n_2(r) \subset K \subset B^n_2(R)$$

for some $r, R > 0$. If there exists $0 < \varepsilon < \min \left\{ \left( \frac{\sqrt{3}r}{6\sqrt{3}\pi r + 32\pi} \right)^2, \frac{r^2}{16} \right\}$ so that

$$\rho(CK, IK) \leq \varepsilon,$$

then

$$\rho(K, -K) \leq C(n, r, R) \varepsilon^q \quad \text{where} \quad q = \begin{cases} \frac{1}{2} & \text{if } n = 2, \\ \frac{1}{2(n+1)} & \text{if } n = 3, 4, \\ \frac{1}{(n-2)(n+1)} & \text{if } n \geq 5. \end{cases}$$

Here, $C(n, r, R) > 0$ are constants depending on the dimension, $r$, and $R$.

**Remark.** In the proof of Theorem 2 we give the explicit dependency of $C(n, r, R)$ on $r$ and $R$.

The following corollary is a straightforward consequence of the Lipschitz property of the parallel section function (Lemma 9) and Theorem 2. Roughly speaking, if for every direction $\xi \in S^{n-1}$, the convex body $K$ has a maximal section perpendicular to $\xi$ that is close to the origin, then $K$ is close to being origin-symmetric.

**Corollary 3.** Let $K$ be a convex body in $\mathbb{R}^n$ such that

$$B^n_2(r) \subset K \subset B^n_2(R)$$

for some $r, R > 0$. Let $L = L(n)$ be the constant given in Lemma 9. If there exists

$$0 < \varepsilon < \min \left\{ \frac{r}{2}, \frac{3r^3}{LR^{n-1}(6\sqrt{3}\pi r + 32\pi)^2}, \frac{r^3}{16LR^{n-1}} \right\}$$

then

$$\rho(K, -K) \leq C(n, r, R) \varepsilon^q \quad \text{where} \quad q = \begin{cases} \frac{1}{2} & \text{if } n = 2, \\ \frac{1}{2(n+1)} & \text{if } n = 3, 4, \\ \frac{1}{(n-2)(n+1)} & \text{if } n \geq 5. \end{cases}$$

Here, $C(n, r, R) > 0$ are constants depending on the dimension, $r$, and $R$.\[\]
so that, for each direction \( \xi \in S^{n-1} \), \( A_{K,\xi} \) attains its maximum at some \( t = t(\xi) \) with \( |t(\xi)| \leq \varepsilon \), then

\[
\rho(K, -K) \leq \tilde{C}(n,r,R) \varepsilon^q.
\]

Here, \( \tilde{C}(n,r,R) > 0 \) are constants depending on the dimension, \( r \), and \( R \), and \( q = q(n) \) is the same as in Theorem 2.

The proof of Theorem 2 is given in Section 4 and consists of a sequence of lemmas from Section 3. The main idea is the following. If \( K \) is of class \( C^\infty \), then we use Brunn’s theorem and an integral formula from [3] to show that \( \rho(CK, IK) \) being small implies that \( \int_{S^{n-1}} |A'_{K,\xi}(0)|^2 d\xi \) is also small. (Recall that \( K \) is called \( m \)-smooth or \( C^m \) if \( \rho_K \in C^m(S^{n-1}) \).) If \( K \) is not smooth, we approximate it by smooth bodies, for which the above integral is small. Then we use the Fourier transform techniques from [15] and the tools of spherical harmonics similar to those from [6] to finish the proof.

As we will see below, the same methods can be used to obtain a stability version of a result of Koldobsky and Shane [9]. It is well known that the knowledge of \( A_{K,\xi}(0) \) for all \( \xi \in S^{n-1} \) is not sufficient for determining the body \( K \) uniquely, unless \( K \) is origin-symmetric. However, Koldobsky and Shane have shown that if \( A_{K,\xi}(0) \) is replaced by a fractional derivative of non-integer order of the function \( A_{K,\xi}(t) \) at \( t = 0 \), then this information does determine the body uniquely.

**Theorem 4** (Koldobsky and Shane). Let \( K \) and \( L \) be convex bodies in \( \mathbb{R}^n \). Let \(-1 < p < n - 1\) be a non-integer, and let \( m \) be an integer greater than \( p \). If \( K \) and \( L \) are \( m \)-smooth and

\[
A_{K,\xi}^{(p)}(0) = A_{L,\xi}^{(p)}(0)
\]

for all \( \xi \in S^{n-1} \), then

\[
K = L.
\]

The following is our stability result.

**Theorem 5.** Let \( K \) and \( L \) be convex bodies in \( \mathbb{R}^n \) such that

\[
B^n_2(r) \subset K \subset B^n_2(R) \quad \text{and} \quad B^n_2(r) \subset L \subset B^n_2(R)
\]

for some \( r, R > 0 \). Let \(-1 < p < n - 1\) be a non-integer, and let \( m \) be an integer greater than \( p \). If \( K \) and \( L \) are \( m \)-smooth and

\[
\sup_{\xi \in S^{n-1}} \left| A_{K,\xi}^{(p)}(0) - A_{L,\xi}^{(p)}(0) \right| \leq \varepsilon
\]

for some \( 0 < \varepsilon < 1 \), then

\[
\rho(K, L) \leq C(n,p,r,R) \varepsilon^q \quad \text{where} \quad q = \begin{cases} 
\frac{2}{n+1} & \text{if } n \leq 2p + 2, \\
\frac{4}{(n-2p)(n+1)} & \text{if } n > 2p + 2.
\end{cases}
\]

Here, \( C(n,p,r,R) > 0 \) are constants depending on the dimension, \( p \), \( r \), and \( R \).

**Remark.** In the proof of Theorem 5 we give the explicit dependency of \( C(n,p,r,R) \) on \( r \) and \( R \). Furthermore, our second result remains true when \( p \) is a non-integer greater than \( n - 1 \). However, considering such values for \( p \) would make our arguments less clear.
2. Preliminaries

Throughout our paper, the constants
\[ \kappa_n := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} \quad \text{and} \quad \omega_n := n \cdot \kappa_n \]
give the volume and surface area of the unit Euclidean ball in \( \mathbb{R}^n \), where \( \Gamma \) denotes the Gamma function. Whenever we integrate over Borel subsets of the sphere \( S^{n-1} \), we are using non-normalized spherical measure; that is, the \((n-1)\)-dimensional Hausdorff measure on \( \mathbb{R}^n \), scaled so that the measure of \( S^{n-1} \) is \( \omega_n \).

Let \( K \) be a convex body in \( \mathbb{R}^n \) containing the origin in its interior. The maximal section function of \( K \) is defined by
\[ m_K(\xi) = \max_{t \in \mathbb{R}} \text{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}) = \max_{t \in \mathbb{R}} A_{K,\xi}(t), \quad \xi \in S^{n-1}. \]

Note that \( m_K \) is simply the radial function for the cross-section body \( CK \). For each \( \xi \in S^{n-1} \), let \( t_K(\xi) \in \mathbb{R} \) be the closest to zero number such that \( A_{K,\xi}(t_K(\xi)) = m_K(\xi) \).

Towards the proof of our first stability result, we use the formula
\[ f_K(t) := \frac{1}{\omega_n} \int_{S^{n-1}} A_{K,\xi}(t) \, d\xi \]
\[ = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)} \int_{K \cap \{|x| \geq |t|\}} \frac{1}{|x|} \left(1 - \frac{t^2}{|x|^2}\right)^{\frac{n-3}{2}} \, dx; \]
(1)

refer to Lemma 1.2 in [3] or Lemma 1 in [1] for the proof.

The Minkowski functional of \( K \) is defined by
\[ \|x\|_K = \min\{a \geq 0 : x \in aK\}, \quad x \in \mathbb{R}^n. \]

It easy to see that \( \rho_K(\xi) = \|\xi\|_K^{-1} \) for \( \xi \in S^{n-1} \). The latter also allows us to consider \( \rho_K \) as a homogeneous degree \(-1\) function on \( \mathbb{R}^n \setminus \{0\} \). The support function of \( K \) is defined by
\[ h_K(x) = \sup_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^n. \]

The function \( h_K \) is the Minkowski functional for the polar body \( K^o \) associated with \( K \). Given another convex body \( L \) in \( \mathbb{R}^n \), define
\[ \delta_2(K, L) = \left(\int_{S^{n-1}} |h_K(\xi) - h_L(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \]
and
\[ \delta_\infty(K, L) = \sup_{\xi \in S^{n-1}} |h_K(\xi) - h_L(\xi)|. \]

These functions are, respectively, the \( L^2 \) and Hausdorff metrics for convex bodies in \( \mathbb{R}^n \). The following theorem, due to Vitale [17], relates these metrics; refer to Proposition 2.3.1 in [7] for the proof.

**Theorem 6.** Let \( K \) and \( L \) be convex bodies in \( \mathbb{R}^n \), and let \( D \) denote the diameter of \( K \cup L \). Then
\[ \frac{2\kappa_{n-1} D^{1-n}}{n(n + 1)} \delta_\infty(K, L)^{n+1} \leq \delta_2(K, L)^2 \leq \omega_n \delta_\infty(K, L)^2. \]
Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be any $n$-tuple of non-negative integers. We will use the notation
\[ [\alpha] := \sum_{j=1}^{n} \alpha_j \]
to define the differential operator
\[ \frac{\partial^{[\alpha]}}{\partial x^{\alpha}} := \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \]

We let $S(\mathbb{R}^n)$ denote the space of Schwartz test functions; that is, functions in $C^\infty(\mathbb{R}^n)$ for which all derivatives decay faster than any rational function. The Fourier transform of $\phi \in S(\mathbb{R}^n)$ is a test function $\mathcal{F}\phi$ defined by
\[ \mathcal{F}\phi(x) = \hat{\phi}(x) = \int_{\mathbb{R}^n} \phi(y) e^{-i(x,y)} \, dy, \quad x \in \mathbb{R}^n. \]
The continuous dual of $S(\mathbb{R}^n)$ is denoted as $S'(\mathbb{R}^n)$, and elements of $S'(\mathbb{R}^n)$ are referred to as distributions. The action of $f \in S'(\mathbb{R}^n)$ on a test function $\phi$ is denoted as $\langle f, \phi \rangle$. The Fourier transform of $f$ is a distribution $\hat{f}$ defined by
\[ \langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle, \quad \phi \in S(\mathbb{R}^n); \]
$\hat{f}$ is well-defined as a distribution because $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is a continuous and linear bijection.

For any $f \in C(S^{n-1})$ and $p \in \mathbb{C}$, the $-n+p$ homogeneous extension of $f$ is given by
\[ f_p(x) = |x|^{-n+p} f \left( \frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}. \]
When $Rp > 0$, $f_p$ is locally integrable on $\mathbb{R}^n$ with at most polynomial growth at infinity. In this case, $f_p$ is a distribution on $S(\mathbb{R}^n)$ acting by integration, and we may consider its Fourier transform. Goodey, Yaskin, and Yaskina show in [6] that, for $f \in C^\infty(S^{n-1})$, the additional restriction $Rp < n$ ensures that the action of $\hat{f}_p$ is also by integration, with $\hat{f}_p \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

We make extensive use of the mapping $I_p : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ defined in [6], which sends a function $f$ to the restriction of $\hat{f}_p$ to $S^{n-1}$. For $0 < Rp < n$ and $m \in \mathbb{Z}^{\geq 0}$, Goodey, Yaskin and Yaskina show $I_p$ has an eigenvalue $\lambda_m(n, p)$ whose eigenspace includes all spherical harmonics of degree $m$ and dimension $n$. These eigenvalues are given explicitly in the following lemma; refer to [6] for the proof.

**Lemma 7.** If $0 < Rp < n$, then the eigenvalues $\lambda_m(n, p)$ are given by
\[ \lambda_m(n, p) = \frac{2^p \pi^{\frac{n}{2}} (-1)^{\frac{n-1}{2}} \Gamma \left( \frac{m+p}{2} \right)}{\Gamma \left( \frac{m+n-2}{2} \right)} \] if $m$ is even,
and
\[ \lambda_m(n, p) = i \frac{2^p \pi^{\frac{n}{2}} (-1)^{\frac{n-1}{2}} \Gamma \left( \frac{m+p}{2} \right)}{\Gamma \left( \frac{m+n-2}{2} \right)} \] if $m$ is odd.

The spherical gradient of $f \in C(S^{n-1})$ is the restriction of $\nabla f \left( \frac{x}{|x|} \right)$ to $S^{n-1}$. It is denoted by $\nabla_0 f$. 

An extensive discussion on spherical harmonics is given in \[7\]. A spherical harmonic \( Q \) of dimension \( n \) is a harmonic and homogeneous polynomial in \( n \) variables whose domain is restricted to \( S^{n-1} \). We say \( Q \) is of degree \( m \) if the corresponding polynomial has degree \( m \). The collection \( \mathcal{H}_m^n \) of all spherical harmonics with dimension \( n \) and degree \( m \) is a finite dimensional Hilbert space with respect to the inner product for \( L^2(S^{n-1}) \). If, for each \( m \in \mathbb{Z}\geq 0 \), \( B_m \) is an orthonormal basis for \( \mathcal{H}_m^n \), then the union of all \( B_m \) is an orthonormal basis for \( L^2(S^{n-1}) \). Given \( f \in L^2(S^{n-1}) \), and defining

\[
\sum_{Q \in B_m} \langle f, Q \rangle Q =: Q_m \in \mathcal{H}_m^n,
\]

we call \( \sum_{m=0}^{\infty} Q_m \) the condensed harmonic expansion for \( f \). The condensed harmonic expansion does not depend on the particular orthonormal bases chosen for each \( \mathcal{H}_m^n \).

Let \( m \in \mathbb{N} \cup \{0\} \), and let \( h : \mathbb{R} \rightarrow \mathbb{C} \) be an integrable function which is \( m \)-smooth in a neighbourhood of the origin. For \( p \in \mathbb{C} \setminus \mathbb{Z} \) such that \(-1 < \Re p < m\), we define the fractional derivative of the order \( p \) of \( h \) at zero as

\[
h^{(p)}(0) = \frac{1}{\Gamma(-p)} \int_0^1 t^{-1-p} \left( h(-t) - \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k!} t^k \right) dt + \frac{1}{\Gamma(-p)} \int_1^\infty t^{-1-p} h(-t) dt + \frac{1}{\Gamma(-p)} \sum_{k=0}^{m-1} \frac{(-1)^k h^{(k)}(0)}{k! (k-p)}.
\]

Given the simple poles of the Gamma function, the fractional derivatives of \( h \) at zero may be analytically extended to the integer values \( 0, \ldots, m-1 \), and they will agree with the classical derivatives.

Let \( K \) be an infinitely smooth convex body. By Lemma 2.4 in \[8\], \( A_{K,\xi} \) is infinitely smooth in a neighbourhood of \( t = 0 \) which is uniform with respect to \( \xi \in S^{n-1} \). With the exception of a sign difference, the equality

\[
A_{K,\xi}^{(p)}(0) = \frac{\cos \left( \frac{\pi p}{2} \right)}{2\pi(n-1-p)} \left( \|x\|_K^{-n+1+p} + \|x\|_K^{-n+1+p} \right)^\wedge (\xi)
+ \frac{\sin \left( \frac{\pi p}{2} \right)}{2\pi(n-1-p)} \left( \|x\|_K^{-n+1+p} - \|x\|_K^{-n+1+p} \right)^\wedge (\xi)
\]

was proven by Ryabogin and Yaskin in \[15\] for all \( \xi \in S^{n-1} \) and \( p \in \mathbb{C} \) such that \(-1 < \Re(p) < n-1 \). The sign difference results from their use of \( h(x) \) rather than \( h(-x) \) in the definition of fractional derivatives.

3. Auxiliary results

We first prove some auxiliary lemmas.

**Lemma 8.** Let \( m \) be a non-negative integer. Let \( K \) be an \( m \)-smooth convex body in \( \mathbb{R}^n \) such that

\[
B_2^n(r) \subset K \subset B_2^n(R)
\]

for some \( r, R > 0 \). There exists a family \( \{K_\delta\}_{0 < \delta < 1} \) of infinitely smooth convex bodies in \( \mathbb{R}^n \) which approximate \( K \) in the radial metric as \( \delta \) approaches zero, with

\[
B_2^n((1+\delta)^{-1}r) \subset K_\delta \subset B_2^n((1-\delta)^{-1}R).
\]
Furthermore,

\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \sup_{|t| \leq \frac{1}{4}} |A_{K,\xi}(t) - A_{K,\xi}(t)| = 0,
\]

and

\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| A_{K,\xi}^{(p)}(0) - A_{K,\xi}^{(p)}(0) \right| = 0
\]

for every \( p \in \mathbb{R}, -1 < p \leq m \).

**Proof.** For each \( 0 < \delta < 1 \), let \( \phi_{\delta} : [0, \infty) \to [0, \infty) \) be a \( C^\infty \) function with support contained in \([\delta/2, \delta]\), and

\[
\int_{\mathbb{R}^n} \phi_{\delta}(|z|) \, dz = 1.
\]

It follows from Theorem 3.3.1 in [10] that there is a family \( \{K_{\delta}\}_{0 < \delta < 1} \) of \( C^\infty \) convex bodies in \( \mathbb{R}^n \) such that

\[
\|x\|_{K_{\delta}} = \int_{\mathbb{R}^n} \|x + |x|z\|_{K} \phi_{\delta}(|z|) \, dz
\]

and

\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \|\xi\|_{K_{\delta}} - \|\xi\|_{K} \right| = 0.
\]

For each \( \xi \in S^{n-1} \) and \( z \in \mathbb{R}^n \) with \( |z| \leq \delta \), we have

\[
\|\xi + |\xi|z\|_{K} = \|\xi + z\|_{K_\lambda} = \lambda \eta \|\xi\|_{K}
\]

for some \( \eta \in S^{n-1} \) and \( 0 < 1 - \delta \leq \lambda \leq 1 + \delta \). It then follows from the support of \( \phi_{\delta} \) and the inequality \( R^{-1} \leq \|\eta\|_K \leq r^{-1} \) that

\[
\|\xi\|_{K_{\delta}} = \int_{\mathbb{R}^n} \|\xi + z\|_{K} \phi_{\delta}(|z|) \, dz \leq (1 + \delta)r^{-1}
\]

and

\[
\|\xi\|_{K_{\delta}} = \int_{\mathbb{R}^n} \|\xi + z\|_{K} \phi_{\delta}(|z|) \, dz \geq (1 - \delta)R^{-1},
\]

which gives

\[
B_2^2((1 + \delta)^{-1}r) \subset K_{\delta} \subset B_2^2((1 - \delta)^{-1}R).
\]

This containment, with the limit of the difference of Minkowski functionals above, implies

(3) \[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \rho_{K_{\delta}}(\xi) - \rho_{K}(\xi) \right| = 0.
\]

Therefore, \( \{K_{\delta}\}_{0 < \delta < 1} \) approximate \( K \) with respect to the radial metric.

Furthermore, the radial functions \( \{\rho_{K_{\delta}}\}_{0 < \delta < 1} \) approximate \( \rho_{K} \) in \( C^m(S^{n-1}) \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be any \( n \)-tuple of non-negative integers such that \( 1 \leq ||\alpha|| \leq m \), and consider the function

\[
f(y, z) := \frac{\partial^{[\alpha]} \|x + |x|z\|_{K}}{\partial x^{\alpha}} \bigg|_{x=y}.
\]

Observe that \( f \) is uniformly continuous on

\[
\{y \in \mathbb{R}^n, 2^{-1} \leq |y| \leq 2\} \times \{z \in \mathbb{R}^n, |z| \leq 2^{-1}\}
\]
since $K$ is $m$-smooth. Therefore, we have
\[
\frac{\partial^{[\alpha]}_{\partial x\alpha}}{\partial x^\alpha}(\|x\|_K - \|x\|_K) \bigg|_{x = \xi} = \int_{\mathbb{R}^n} \frac{\partial^{[\alpha]}_{\partial x\alpha}}{\partial x^\alpha}(\|x + z\|_K - \|x\|_K) \bigg|_{x = \xi} \phi_\delta(|z|) \, dz
\]
for all $\xi \in S^{n-1}$ and $\delta < 1/2$, which implies
\[
\sup_{\xi \in S^{n-1}} \left| \frac{\partial^{[\alpha]}_{\partial x\alpha}}{\partial x^\alpha}(\|x\|_K - \|x\|_K) \bigg|_{x = \xi} \right| \leq \sup_{\xi \in S^{n-1}} \sup_{|z| < \delta} \left| f(\xi, z) - f(\xi, 0) \right|.
\]
Noting that $|(\xi, z) - (\xi, 0)| = |z| < \delta$, the uniform continuity of $f$ then implies
\[
(4) \quad \lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \frac{\partial^{[\alpha]}_{\partial x\alpha}}{\partial x^\alpha}(\|x\|_K - \|x\|_K) \bigg|_{x = \xi} \right| = 0.
\]
It follows from the relation $\rho_K(x) = \|x\|_K^{-1}$ that $\frac{\partial^{[\alpha]}_{\partial x\alpha}}{\partial x^\alpha}\rho_K \bigg|_{x = \xi}$ may be expressed as a finite linear combination of terms of the form
\[
\rho^{d+1}_K(\xi) \prod_{j=0}^d \frac{\partial^{[\beta_j]}_{\partial x^{[\beta_j]}}}{{\partial x^{[\beta_j]}}} \|x\|_K \bigg|_{x = \xi},
\]
where $d \in \mathbb{Z}^{\geq 0}$, and each $\beta_j$ is an $n$-tuple of non-negative integers such that $[\beta_j] \geq 1$ and $[\alpha] = \sum_{j=0}^d [\beta_j]$. Of course, $\frac{\partial^{[\alpha]}_{\partial x\alpha}}{\partial x^\alpha}\rho_K \bigg|_{x = \xi}$ may be expressed similarly. Equations (3) and (4) then imply
\[
(5) \quad \lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \left| \frac{\partial^{[\alpha]}_{\partial x\alpha}}{\partial x^\alpha}(\rho_K - \rho_K) \bigg|_{x = \xi} \right| = 0,
\]
once we note that $\rho_K$ and the partial derivatives of $\|x\|_K$, up to order $m$, are bounded on $S^{n-1}$.

Our next step is to uniformly approximate the parallel section function $A_{K,\xi}$. Fix $\xi \in S^{n-1}$, and define the hyperplane
\[
H_t = \xi^\perp + t\xi
\]
for any $t \in \mathbb{R}$ such that $|t| < r$. Let $S^{n-2}$ denote the Euclidean sphere in $H_t$ centred at $t\xi$, and let $\rho_{K \cap H_t}$ denote the radial function for $K \cap H_t$ with respect to $t\xi$ on $S^{n-2}$. Then, for $|t| < r$,
\[
(6) \quad A_{K,\xi}(t) = \frac{1}{n-1} \int_{S^{n-2}} \rho_{K \cap H_t}^{n-1}(\theta) \, d\theta.
\]
For $|t| < r/2$ and $0 < \delta < 1$, $A_{K,\xi}(t)$ may be expressed similarly. Fixing $\theta \in S^{n-2}$, and with angles $\alpha$ and $\beta$ as in Figure [1] we have
\[
|\rho_{K \cap H_t}(\theta) - \rho_{K \cap H_t}(\theta)| \leq \frac{\sin \beta}{\sin \alpha} |\rho_K(\eta_1) - \rho_K(\eta_1)|.
\]
By restricting to $|t| \leq r/4$, $\alpha$ may be bounded away from zero and $\pi$. Indeed, if $\alpha < \pi/2$, then
\[
\tan \alpha \geq \frac{r/2 - |t|}{R} \geq \frac{r}{4R},
\]
and if $\alpha > \pi/2$, then
\[
\tan(\pi - \alpha) \geq \frac{r/2 + |t|}{R} \geq \frac{r}{2R}.
\]
Figure 1. The diagrams represent two extremes: when the angle \( \alpha \) is small (\( \alpha < \pi / 2 \)), and when it is large (\( \alpha > \pi / 2 \)). The point \( O \) represents the origin in \( \mathbb{R}^n \), and \( |OT| = t \) where \( 0 \leq t \leq r/4 \). The points \( A \) and \( C \) are the boundary points for \( K \) and \( K_\delta \) in the direction \( \theta \), with two obvious possibilities: either \( |TA| = \rho_{K \cap H_t}(\theta) \) and \( |TC| = \rho_{K_\delta \cap H_t}(\theta) \), or the opposite. The point \( B \) is a boundary point for the same convex body as \( A \), but in the direction \( \eta_1 \). The point \( D \) lies outside the convex body for which \( A \) and \( B \) are boundary points.

Therefore

\[
0 < \arctan \left( \frac{r}{4R} \right) \leq \alpha \leq \pi - \arctan \left( \frac{r}{4R} \right) < \pi.
\]

We now have

\[
(7) \quad |\rho_{K \cap H_t}(\theta) - \rho_{K_\delta \cap H_t}(\theta)| \leq \frac{1}{\sin \left( \arctan \left( \frac{r}{4R} \right) \right)} \sup_{\eta \in S^{n-1}} |\rho_K(\eta) - \rho_{K_\delta}(\eta)|,
\]

where the upper bound is independent of \( \xi \in S^{n-1} \), \( t \) with \( |t| \leq r/4 \), and \( \theta \in S^{n-2} \). This inequality, the integral expression (6), and equation (3) imply

\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}, |t| \leq \frac{r}{4}} \sup_{\text{all } \theta} |A_{K,\xi}(t) - A_{K_\delta,\xi}(t)| = 0.
\]

Lemma 2.4 in [8] establishes the existence of a small neighbourhood of \( t = 0 \), independent of \( \xi \in S^{n-1} \), on which \( A_{K,\xi} \) is \( m \)-smooth. The following is an elaboration of Koldobsky’s proof, so that we may uniformly approximate the derivatives of \( A_{K,\xi} \). Again fix \( \xi \in S^{n-1} \), and fix \( \theta \in S^{n-2} \subset H_t \). Let \( \rho_{K,\theta} \) denote the \( m \)-smooth restriction of \( \rho_K \) to the two dimensional plane spanned by \( \xi \) and \( \theta \), and consider \( \rho_{K,\theta} \) as a function on \( [0, 2\pi] \), where the angle is measured from the positive \( \theta \)-axis.
A right triangle then gives the equation
\[ \rho_{K \cap H_t}(\theta) + t^2 = \rho_{K,\theta}^2 \left( \arctan \left( \frac{t}{\rho_{K \cap H_t}(\theta)} \right) \right), \]
which we can use to implicitly differentiate \( y(t) := \rho_{K \cap H_t}(\theta) \) as a function of \( t \).
Indeed,
\[
F(t, y) := y^2 + t^2 - \rho_{K,\theta}^2 \left( \arctan \left( \frac{t}{y} \right) \right)
\]
is differentiable away from \( y = 0 \), with
\[
F_y(t, y) = 2y + \frac{2t}{y^2 + t^2} \rho_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) \rho'_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right).
\]
The containment \( B_{\mathbb{R}}^n(r) \subset K \subset B_{\mathbb{R}}^n(R) \) implies \( \rho_{K,\theta} \) is bounded above on \( S^{n-1} \) by \( R \), and
\[
\rho_{K \cap H_t}(\theta) \geq \frac{\sqrt{15}r}{4}
\]
for \(|t| \leq r/4\). If
\[
M = 1 + \sup_{\xi \in S^{n-1}} \left| \nabla_o \rho_K(\xi) \right| < \infty
\]
and \( \lambda \in \mathbb{R} \) is a constant such that
\[
0 < \lambda < \min \left\{ \frac{15\sqrt{15}r^3}{128RM}, \frac{r}{4} \right\},
\]
then
\[
\left| F_y(t, \rho_{K \cap H_t}(\theta)) \right| > \frac{\sqrt{15}r}{4}
\]
for \(|t| \leq \lambda \). Therefore, by the Implicit Function Theorem, \( y(t) = \rho_{K \cap H_t}(\theta) \) is differentiable on \((-\lambda, \lambda)\), with
\[
y'(t) = \frac{\rho_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) \rho'_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) (y^2 + t^2)^{-1} y - t}{y + t\rho_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) \rho'_{K,\theta} \left( \arctan \left( \frac{t}{y} \right) \right) (y^2 + t^2)^{-1}}.
\]
Recursion shows that \( \rho_{K \cap H_t}(\theta) \) is \( m \)-smooth on \((-\lambda, \lambda)\), independent of \( \xi \in S^{n-1} \) and \( \theta \in S^{n-2} \). It follows from the integral expression \([5]\) that \( A_{K,\xi} \) is \( m \)-smooth on \((-\lambda, \lambda)\) for every \( \xi \in S^{n-1} \). This argument also shows that \( A_{K,\delta,\xi} \) is \( m \)-smooth on the same interval, for \( \delta > 0 \) small enough. Using the resulting expressions for the derivatives of \( A_{K,\xi} \) and \( A_{K,\delta,\xi} \), and applying equations \([3]\), \([5]\), and the inequality \([7]\), we have
\[
\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \sup_{|t| \leq \lambda} \left| A_{K,\xi}^{(k)}(t) - A_{K,\delta,\xi}^{(k)}(t) \right| = 0
\]
for \( k = 1, \ldots, m \).
Finally, for any $p \in \mathbb{R}$ such that $-1 < p < m$ and $p \neq 0, 1, \ldots, m - 1$, we will uniformly approximate $A_{K, \xi}^{(p)}(0)$. With $\lambda > 0$ as chosen above, we have

$$A_{K, \xi}^{(p)}(0) = \frac{1}{\Gamma(-p)} \int_0^\lambda t^{-1-p} \left( A_{K, \xi}(t) - \sum_{k=0}^{m-1} \frac{(-1)^k A_{K, \xi}^{(k)}(0)}{k!} t^k \right) dt + \frac{1}{\Gamma(-p)} \int_\lambda^\infty t^{-1-p} A_{K, \xi}(t) dt + \frac{1}{\Gamma(-p)} \sum_{k=0}^{m-1} \frac{(-1)^k \lambda^{k-p} A_{K, \xi}^{(k)}(0)}{k! (k - p)}.$$  

The first integral in this equation can be rewritten as

$$\int_0^\lambda t^{-1-p} \int_0^t A_{K, \xi}^{(m)}(z) \frac{1}{(m-1)!} (t-z)^{m-1} \, dz \, dt,$$

using the integral form of the remainder in Taylor’s Theorem. We also have

$$\int_\lambda^\infty t^{-1-p} A_{K, \xi}(t) dt = \int_{K \cap \{x, -\xi\geq \lambda\}} \langle x, -\xi \rangle^{-1-p} \, dx$$

$$= \int_{B_K(\xi)} \langle \eta, -\xi \rangle^{-1-p} \int_{\lambda(\eta, -\xi)^{-1}} \rho_K^{-2-p} \, dr \, d\eta$$

$$= \frac{1}{n-1-p} \int_{B_K(\xi)} \left( \langle \eta, -\xi \rangle^{-1-p} \rho_K^{n-1-p}(\eta) - \lambda^{n-1-p}(\eta, -\xi)^{-n} \right) \, d\eta,$$

where

$$B_K(\xi) = \left\{ \eta \in S^{n-1} \mid \langle \eta, \xi \rangle < 0 \text{ and } \rho_K(\eta) \geq \lambda \langle \eta, -\xi \rangle^{-1} \right\}.$$

Therefore, with the set $B_{K, \delta}(\xi)$ defined similarly, we have

$$\left| A_{K, \xi}^{(p)}(0) - A_{K, \delta, \xi}^{(p)}(0) \right| \cdot |\Gamma(-p)| \leq \frac{1}{(m-1)!} \left( \sup_{|z| \leq \lambda} \left| A_{K, \xi}^{(m)}(z) - A_{K, \delta, \xi}^{(m)}(z) \right| \right) \int_0^\lambda \int_0^t t^{-1-p} (t-z)^{m-1} \, dz \, dt$$

$$+ \left( \sup_{\eta \in S^{n-1}} \left| \rho_K^{n-1-p}(\eta) - \rho_{K, \delta}^{n-1-p}(\eta) \right| \right) \int_{B_K(\xi) \cap B_{K, \delta}(\xi)} \frac{\langle \eta, -\xi \rangle^{-1-p}}{|\eta - \xi|^{n-1-p}} \, d\eta$$

$$+ \left( \frac{\langle \eta, -\xi \rangle^{-1-p} \rho_{K, \delta}^{n-1-p}(\eta) - \lambda^{n-1-p}(\eta, -\xi)^{-n}}{n-1-p} \right) \int_{B_{K, \delta}(\xi) \setminus B_K(\xi)} \, d\eta$$

$$+ \sum_{k=0}^{m-1} \frac{\lambda^{k-p}}{k! (k-p)} \left| A_{K, \xi}^{(k)}(0) - A_{K, \delta, \xi}^{(k)}(0) \right|,$$

for $\delta > 0$ small enough. The integrals in expressions (8) and (9) are finite, with

$$\int_0^\lambda \int_0^t t^{-1-p} (t-z)^{m-1} \, dz \, dt = \frac{\lambda^{m-p}}{m (m-p)},$$
Lemma 9. Let $K$ be a convex body in $\mathbb{R}^n$ such that

$$B_2^n(r) \subset K \subset B_2^n(R)$$

for some $r, R > 0$. If

$$L(n) = 8(n-1)^{n-1} \pi^{\frac{n-1}{2}} \left[ \Gamma \left( \frac{n+1}{2} \right) \right]^{-1},$$

since $p$ is a non-integer less than $m$, and

$$\int_{B_K(\xi) \cap B_{K_\delta}(\xi)} \langle \eta, -\xi \rangle^{-1-p} d\eta \leq \left( \frac{R}{\lambda} \right)^{1+p} \omega_n.$$  

Furthermore, the integrands in expression (10) and (11) are bounded above by

$$\left( \frac{2R}{\lambda} \right)^{1+p} (2R)^{n-1-p} + \lambda^{n-1-p} \left( \frac{2R}{\lambda} \right)^n \quad \text{if } p < n - 1,$$

and

$$\left( \frac{2R}{\lambda} \right)^{1+p} \left( \frac{r}{2} \right)^{n-1-p} + \lambda^{n-1-p} \left( \frac{2R}{\lambda} \right)^n \quad \text{if } p > n - 1,$$

noting that $B_2^n(r/2) \subset K_\delta \subset B_2^n(2R)$ for $\delta < 1/2$.

It is now sufficient to prove

$$\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} \chi_{B(\xi, \delta)} d\eta = 0,$$

where

$$B(\xi, \delta) = B_K(\xi) \Delta B_{K_\delta}(\xi)$$

$$= \left\{ \eta \in S^{n-1} \mid \rho_K(\eta) \geq \lambda \langle \eta, -\xi \rangle > \rho_{K_\delta}(\eta) \text{ or } \rho_{K_\delta}(\eta) \geq \lambda \langle \eta, -\xi \rangle > \rho_K(\eta) \right\}.$$  

We will prove the equivalent statement

$$\lim_{\delta \to 0} \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} \chi_{B(-\xi, \delta)} d\eta = 0,$$

where the sign of $\xi$ has changed, so that we may use Figure 1.

Towards this end, fix any $\theta \in S^{n-2}$, and consider Figure 1 specifically when $t = \lambda$. In this case,

$$\|OA\| = \rho_K(\eta_2) = \lambda \langle \eta_2, \xi \rangle^{-1} \text{ and } \|OC\| = \rho_{K_\delta}(\eta_1) = \lambda \langle \eta_1, \xi \rangle^{-1}$$

or

$$\|OC\| = \rho_K(\eta_2) = \lambda \langle \eta_2, \xi \rangle^{-1} \text{ and } \|OA\| = \rho_{K_\delta}(\eta_1) = \lambda \langle \eta_1, \xi \rangle^{-1}.$$  

Any $\eta \in B(-\xi, \delta)$ lying in the right half-plane spanned by $\xi$ and $\theta$ will lie between $\eta_1$ and $\eta_2$. Furthermore, the angle $\omega$ converges to zero as $\delta$ approaches zero, uniformly with respect to $\xi \in S^{n-1}$ and $\theta \in S^{n-2}$. Indeed, we have

$$0 \leq \sin \omega \leq \frac{2 \sin \beta \sin \gamma}{r \sin \alpha} \left| \rho_K(\eta_1) - \rho_{K_\delta}(\eta_1) \right|,$$

using the fact that both $K$ and $K_\delta$ contain a ball of radius $r/2$, and with $\sin \alpha$ uniformly bounded away from zero as before. It follows that the spherical measure of $B(-\xi, \delta)$ converges to zero as $\delta$ approaches zero, uniformly with respect to $\xi \in S^{n-1}$.  

\[\square\]

Lemma 9. Let $K$ be a convex body in $\mathbb{R}^n$ such that

$$B_2^n(r) \subset K \subset B_2^n(R)$$

for some $r, R > 0$. If

$$L(n) = 8(n-1)^{n-1} \pi^{\frac{n-1}{2}} \left[ \Gamma \left( \frac{n+1}{2} \right) \right]^{-1},$$
then
\[ |A_{K,\xi}(t) - A_{K,\xi}(s)| \leq L(n) R^{n-1} r^{-1} |t - s| \]
for all \( s, t \in [-r/2, r/2] \) and \( \xi \in S^{n-1} \).

Proof. For \( \xi \in S^{n-1} \), Brunn’s Theorem implies \( f := A_{K,\xi}^{-1} \) is concave on its support, which includes the interval \([-r, r]\). Let
\[ L_0 = \max \left\{ \left| f \left( -\frac{3r}{4} \right) - f(-r) \right|, \left| f(r) - f \left( \frac{3r}{4} \right) \right| \right\}, \]
and suppose \( s, t \in [-r/2, r/2] \) are such that \( s < t \). If
\[ \frac{f(t) - f(s)}{t - s} > 0, \]
then
\[ f \left( -\frac{3r}{4} \right) - f(-r) \geq f(s) - f \left( -\frac{3r}{4} \right) \geq f(t) - f(s) > 0; \]
otherwise, we will obtain a contradiction of the concavity of \( f \). Similarly, if
\[ \frac{f(t) - f(s)}{t - s} < 0, \]
then
\[ f(r) - f \left( \frac{3r}{4} \right) \leq f \left( \frac{3r}{4} \right) - f(t) \leq f(t) - f(s) < 0. \]

Therefore,
\[ |A_{K,\xi}^{-1}(t) - A_{K,\xi}^{-1}(s)| \leq L_0 |t - s| \]
for all \( s, t \in [-r/2, r/2] \). Now, we have
\[ |A_{K,\xi}(t) - A_{K,\xi}(s)| \leq (n - 1) \left( \max_{t_0 \in \mathbb{R}} A_{K,\xi}(t_0) \right)^{\frac{n-1}{2}} \left| A_{K,\xi}^{-1}(t) - A_{K,\xi}^{-1}(s) \right|, \]
by the Mean Value Theorem, and
\[ L_0 \leq \frac{4}{r} \cdot 2 \left( \max_{t_0 \in \mathbb{R}} A_{K,\xi}(t_0) \right)^{\frac{1}{n-1}} = \frac{8}{r} A_{K,\xi}^{-1} \left( t_K(\xi) \right). \]
Finally, since \( K \) is contained in a ball of radius \( R \), we have
\[ A_{K,\xi}(t_K(\xi)) \leq \frac{\pi^{\frac{n+1}{2}}}{\Gamma \left( \frac{n+1}{2} \right)} R^{n-1}. \]
Combining these inequalities gives
\[ |A_{K,\xi}(t) - A_{K,\xi}(s)| \leq L(n) R^{n-1} r^{-1} |t - s| \]
for all \( s, t \in [-r/2, r/2] \) and \( \xi \in S^{n-1} \).

We now prove two lemmas that will be the core of the proof of Theorem 2.
Lemma 10. Let $K$ be a convex body in $\mathbb{R}^n$ such that
\[ B^n_2(r) \subset K \subset B^n_2(R) \]
for some $r, R > 0$. Let $\{K_\delta\}_{0 < \delta < 1}$ be as in Lemma 8. If there exists $0 < \varepsilon < \frac{r^2}{16}$ so that
\[ \rho(CK, IK) \leq \varepsilon, \]
then, for $\delta > 0$ small enough,
\[ \int_{S^1} |A'_{K_\delta, \xi}(0)| \, d\xi \leq \left( 6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon} \quad \text{when } n = 2, \]
\[ \int_{S^{n-1}} |A'_{K_\delta, \xi}(0)|^2 \, d\xi \leq C(n) \left( \varepsilon + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) \sqrt{\varepsilon} \quad \text{when } n \geq 3. \]
Here, $C(n) > 0$ are constants depending only on the dimension.

Proof. By Lemma 8, we may choose $0 < \alpha < 1/2$ small enough so that for every $0 < \delta < \alpha$,
\[ \sup_{\xi \in S^{n-1}} \sup_{|t| \leq r/4} |A_{K_\delta, \xi}(t) - A_{K_\delta, \xi}(t)| \leq \varepsilon. \]
We first show that for each $0 < \delta < \alpha$ and $\xi \in S^{n-1}$, there exists a number $c_\delta(\xi)$ of which \[ |A'_{K_\delta, \xi}(c_\delta(\xi))| \leq 3\sqrt{\varepsilon}. \]
Indeed, if $\xi \in S^{n-1}$ is such that $|t_{K_\delta}(\xi)| \leq \sqrt{\varepsilon}$, then
\[ A'_{K_\delta, \xi}(t_{K_\delta}(\xi)) = 0, \]
and we may take $c_\delta(\xi) = t_{K_\delta}(\xi)$.
Assume $\xi \in S^{n-1}$ is such that $|t_{K_\delta}(\xi)| > \sqrt{\varepsilon}$. Letting $s$ denote the sign of $t_{K_\delta}(\xi)$, we have
\[
\begin{align*}
|A_{K_\delta, \xi}(s\sqrt{\varepsilon}) - A_{K_\delta, \xi}(0)| &= A_{K_\delta, \xi}(s\sqrt{\varepsilon}) - A_{K_\delta, \xi}(0) \\
&= \left( A_{K_\delta, \xi}(s\sqrt{\varepsilon}) - A_{K_\delta, \xi}(0) \right) + \left( A_{K_\delta, \xi}(s\sqrt{\varepsilon}) - A_{K_\delta, \xi}(s\sqrt{\varepsilon}) \right) \\
&\quad + \left( A_{K_\delta, \xi}(0) - A_{K_\delta, \xi}(0) \right) \\
&\leq \sup_{\xi \in S^{n-1}} \max_{t \in \mathbb{R}} |A_{K_\delta, \xi}(t) - A_{K_\delta, \xi}(0)| + 2 \sup_{\xi \in S^{n-1}} \sup_{|t| \leq r/4} |A_{K_\delta, \xi}(t) - A_{K_\delta, \xi}(t)| \\
&\leq 3\varepsilon.
\end{align*}
\]
It then follows from the Mean Value Theorem that there is a number $c_\delta(\xi)$ with $|c_\delta(\xi)| \leq \sqrt{\varepsilon}$ for which
\[ |A'_{K_\delta, \xi}(c_\delta(\xi))| = \left| \frac{A_{K_\delta, \xi}(s\sqrt{\varepsilon}) - A_{K_\delta, \xi}(0)}{s\sqrt{\varepsilon} - 0} \right| \leq 3\sqrt{\varepsilon}. \]
With the numbers $c_\delta(\xi)$ as above, for the case $n = 2$ we have
\[
\int_{S^{n-1}} |A'_{K_\delta,\xi}(0)|^2 \, d\xi \\
\leq \int_{S^{n-1}} \left( |A'_{K_\delta,\xi}(c_\delta(\xi))| + \left| \int_{c_\delta(\xi)}^0 A''_{K_\delta,\xi}(t) \, dt \right| \right) \, d\xi \\
\leq 6\pi\sqrt{\varepsilon} + \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} |A''_{K_\delta,\xi}(t)| \, dt \, d\xi.
\]
(12)

When $0 < \delta < 1/2$, $K_\delta$ is contained in a ball of radius $2R$ and contains a ball of radius $r/2$. Lemma 9 then implies
\[
\sup_{\xi \in S^{n-1}} \sup_{t \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})} |A'_{K_\delta,\xi}(t)| \leq \frac{2L(n) (2R)^{n-1}}{r}.
\]

So, when $n \geq 3$,
\[
\int_{S^{n-1}} |A'_{K_\delta,\xi}(0)|^2 \, d\xi \\
\leq \int_{S^{n-1}} \left( |A'_{K_\delta,\xi}(c_\delta(\xi))|^2 + \left| \int_{c_\delta(\xi)}^0 2A''_{K_\delta,\xi}(t)A'_{K_\delta,\xi}(t) \, dt \right| \right) \, d\xi \\
\leq 9 \omega_n \varepsilon + \frac{4L(n) (2R)^{n-1}}{r} \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} |A''_{K_\delta,\xi}(t)| \, dt \, d\xi.
\]
(13)

Considering inequalities (12) and (13), we still need to bound
\[
\int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} |A''_{K_\delta,\xi}(t)| \, dt \, d\xi
\]
for arbitrary $n$. Rearranging the equation
\[
\frac{d^2}{dt^2} A_{K_\delta,\xi}^{n-1}(t) = \frac{d}{dt} \left( \frac{1}{n-1} A_{K_\delta,\xi}^{2-n}(t) A'_{K_\delta,\xi}(t) \right) \\
= \frac{2-n}{(n-1)^2} A_{K_\delta,\xi}^{3-2n}(t) (A'_{K_\delta,\xi}(t))^2 + \frac{1}{n-1} A_{K_\delta,\xi}^{2-n}(t) A''_{K_\delta,\xi}(t)
\]
gives
\[
A'_{K_\delta,\xi}(t) = (n-1) A_{K_\delta,\xi}^{n-2}(t) \frac{d^2}{dt^2} A_{K_\delta,\xi}^{-1}(t) + \frac{n-2}{n-1} \left( A'_{K_\delta,\xi}(t) \right)^2
\]

Brunn’s Theorem implies that the second derivative of $A_{K_\delta,\xi}^{n-1}$ is non-positive for $|t| < r$, so
\[
|A''_{K_\delta,\xi}(t)| \leq (1-n) A_{K_\delta,\xi}^{n-2}(t) \frac{d^2}{dt^2} A_{K_\delta,\xi}^{-1}(t) + \frac{n-2}{n-1} \left( A'_{K_\delta,\xi}(t) \right)^2 \\
= -A'_{K_\delta,\xi}(t) + 2 \left( \frac{n-2}{n-1} \right) \left( A'_{K_\delta,\xi}(t) \right)^2.
\]
Because $K_\delta$ contains a ball of radius $r/2$ centred at the origin, we have

$$A_{K_\delta,\xi}(t) \geq \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \left(\frac{3\pi r^2}{16}\right)^{\frac{n-1}{2}}$$

for $|t| \leq r/4$, and so

$$\frac{n-2}{n-1} \left(\frac{A'_{K_\delta,\xi}(t)}{A_{K_\delta,\xi}(t)}\right)^2 \leq \frac{n-2}{n-1} \Gamma\left(\frac{n+1}{2}\right) \left(\frac{2L(n)(2R)^{n-1}}{r}\right)^2 \left(\frac{16}{3\pi r^2}\right)^{\frac{n-1}{2}}$$

for all $|t| \leq \sqrt{\varepsilon}$, where $\bar{L}(n)$ is a constant depending only on $n$. Therefore,

$$\int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} |A''_{K_\delta,\xi}(t)| \, dt \, d\xi$$

(14)

$$\leq \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left( -A''_{K_\delta,\xi}(t) \right) dt \, d\xi + \frac{4 \omega_n \bar{L}(n) R^{2n-2}}{r^{n+1}} \sqrt{\varepsilon}.$$

We will bound the first term on the final line above using formula (14). Letting

$$\bar{C}(n) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)},$$

formula (14) becomes

$$f_{K_\delta}(t) = \bar{C}(n) \int_{S^{n-1}} \int_{|t|}^{\rho_{K_\delta}(\xi)} \frac{1}{r} \left(1 - \frac{t_2^2}{r^2}\right)^{\frac{n-3}{2}} r^{n-1} \, dr \, d\xi$$

$$= \bar{C}(n) \int_{S^{n-1}} \int_{|t|}^{\rho_{K_\delta}(\xi)} r \left(r^2 - t_2^2\right)^{\frac{n-3}{2}} \, dr \, d\xi$$

$$= \bar{C}(n) \left(\frac{n}{n-1}\right) \int_{S^{n-1}} \left(\rho_{K_\delta}^2(\xi) - t_2^2\right)^{\frac{n-1}{2}} \, d\xi.$$

The derivatives of $A_{K_\delta,\xi}$ and $(\rho_{K_\delta}^2(\xi) - t_2^2)^{\frac{n-1}{2}}$ are bounded on $(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$ uniformly with respect to $\xi \in S^{n-1}$, so

$$f'_{K_\delta}(t) = \frac{1}{\omega_n} \int_{S^{n-1}} A'_{K_\delta,\xi}(t) \, d\xi = -\bar{C}(n) t \int_{S^{n-1}} (\rho_{K_\delta}^2(\xi) - t_2^2)^{\frac{n-3}{2}} \, d\xi.$$

Observing $\bar{C}(2) = \pi^{-1}$ and using that $0 < \varepsilon < r^2/16$ and $r/2 \leq \rho_{K_\delta} \leq 2R$ for $\delta < 1/2$, we have

$$\int_{S^{n-1}} A'_{K_\delta,\xi}(\pm \sqrt{\varepsilon}) \, d\xi$$

$$= \omega_n |f'_{K_\delta}(\pm \sqrt{\varepsilon})| = \bar{C}(n) \omega_n \sqrt{\varepsilon} \int_{S^{n-1}} (\rho_{K_\delta}^2(\xi) - \varepsilon)^{\frac{n-3}{2}} \, d\xi$$

$$\leq \begin{cases} 
16 \pi \left(\sqrt{3} r\right)^{-1} \sqrt{\varepsilon} & \text{if } n = 2, \\
\bar{C}(n) \omega_n^3 (2R)^{n-3} \sqrt{\varepsilon} & \text{if } n \geq 3.
\end{cases}$$
This implies
\[
\left| \int_{S^{n-1}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} A'_{K,\xi}(t) \, dt \, d\xi \right| = \left| \int_{S^{n-1}} (A'_{K,\xi}(-\sqrt{\varepsilon}) - A'_{K,\xi}(\sqrt{\varepsilon})) \, d\xi \right|
\]
(15)
\[
\leq \begin{cases} 
32\pi (\sqrt{3}r)^{-1}\sqrt{\varepsilon} & \text{if } n = 2, \\
2\tilde{C}(n) \omega_n^2 (2R)^{n-3}\sqrt{\varepsilon} & \text{if } n \geq 3.
\end{cases}
\]

Noting that \( \tilde{L}(2) = 0 \), inequalities (12), (14), and (15) give
\[
\int_{S^1} |A'_{K,\xi}(0)| \, d\xi \leq \left(6\pi + \frac{32\pi}{\sqrt{3}r}\right)\sqrt{\varepsilon}
\]
when \( n = 2 \). For \( n \geq 3 \), inequalities (13), (14), and (15) give
\[
\int_{S^{n-1}} |A'_{K,\xi}(0)|^2 \, d\xi \leq C(n) \left(\sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}}\right)\sqrt{\varepsilon},
\]
where \( C(n) \) is a constant depending on \( n \). \( \square \)

**Lemma 11.** Let \( K \) and \( L \) be infinitely smooth convex bodies in \( \mathbb{R}^n \) such that
\[
B_2^n(r) \subset K \subset B_2^n(R) \quad \text{and} \quad B_2^n(r) \subset L \subset B_2^n(R)
\]
for some \( r, R > 0 \). Let \( p \in (0, n) \). If \( \varepsilon > 0 \) is such that
\[
\left\| I_p \left( \|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p} \right) \right\|_2 \leq \varepsilon,
\]
then when \( n \leq 2p \),
\[
\rho(K, L) \leq C(n, p) R^2 r^{\frac{3n-1+2p}{n+1}} \varepsilon^{\frac{2}{n+1}},
\]
and when \( n > 2p \),
\[
\rho(K, L) \leq C(n, p) R^2 r^{\frac{3n-1+2p}{n+1}} \left(\varepsilon^2 + \frac{R^{2(n+1-p)}}{r^{2p(n+1)}}\right)^{\frac{n-2p}{(n+2-2p)(n+1)}} \varepsilon^{\frac{2}{n+1}}.
\]

Here, \( \|\cdot\|_2 \) denotes the norm on \( L^2(S^{n-1}) \), and \( C(n, p) > 0 \) are constants depending on the dimension and \( p \).

**Proof.** Define the function
\[
f(\xi) := \|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p}
\]
on \( S^{n-1} \). Towards bounding the radial distance between \( K \) and \( L \) by \( \|f\|_2 \), the \( L^2(S^{n-1}) \) norm of \( f \), note that the identity
\[
\rho_K(\xi) - \rho_L(\xi) = \rho_K(\xi)\rho_L(\xi) \left(\|\xi\|_L - \|\xi\|_K\right)
\]
implies
\[
|\rho_K(\xi) - \rho_L(\xi)| \leq R^2 \|\xi\|_K - \|\xi\|_L|.
\]
By Theorem 6 we have
\[
\delta_\infty(K^\circ, L^\circ) \leq C(n) D^{\frac{n-1}{n+1}} \left(\delta_2(K^\circ, L^\circ)\right)^{\frac{2}{n+1}},
\]
Combining the above inequalities, we get

\[
\sup_{\xi \in S^{n-1}} |\|\xi\|_K - \|\xi\|_L| \leq C(n) \int_{S^{n-1}} (|\|\xi\|_K - \|\xi\|_L|^2) \, d\xi
\]

for some new constant \(C(n)\). There exists a function \(g : S^{n-1} \to \mathbb{R}\) such that

\[
(|\|\xi\|_K - \|\xi\|_L|)g(\xi) = |\|\xi\|_K^{-n+p} - \|\xi\|_L^{-n+p}|
\]

If \(\xi \in S^{n-1}\) is such that \(\|\xi\|_K \neq \|\xi\|_L\), then an application of the Mean Value Theorem to the function \(t^{-n+p}\) on the interval bounded by \(\|\xi\|_K\) and \(\|\xi\|_L\) gives

\[
|g(\xi)| \geq (n-p) \left( \max \{\|\xi\|_K, \|\xi\|_L\} \right)^{-n-1+p} \geq (n-p)r^{n+1-p}.
\]

Therefore,

\[
|\|\xi\|_K - \|\xi\|_L| \leq (n-p)^{-1}r^{-n-1+p}|f(\xi)|.
\]

Combining the above inequalities, we get

\[
(16) \quad \sup_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)| \leq C(n,p)R^2r^{-\frac{3n-4p}{n+2-p}} \|f\|_2^n
\]

for some constant \(C(n,p)\).

We now compare the \(L^2\) norm of \(f\) to that of \(I_p(f)\) by considering two separate cases based on the dimension \(n\), as in the proof of Theorem 3.6 in [6]. In both cases, we let \(\sum_{m=0}^\infty Q_m\) be the condensed harmonic expansion for \(f\), and let \(\lambda_m(n,p)\) be the eigenvalues from Lemma 7. As in [6], the condensed harmonic expansion for \(I_p f\) is then given by \(\sum_{m=0}^\infty \lambda_m(n,p)Q_m\).

Assume \(n \leq 2p\). An application of Stirling’s formula to the equations given in Lemma [7] shows that \(\lambda_m(n,p)\) diverges to infinity as \(m\) approaches infinity. The eigenvalues are also non-zero, so there is a constant \(C(n,p)\) such that \(C(n,p)|\lambda_m(n,p)|^2\) is greater than one for all \(m\). Therefore,

\[
\|f\|_2^2 = \sum_{m=0}^\infty \|Q_m\|_2^2
\]

\[
\leq C(n,p) \sum_{m=0}^\infty |\lambda_m(n,p)|^2 \|Q_m\|_2^2 = C(n,p)\|I_p f\|_2^2 \leq C(n,p)\varepsilon^2.
\]

Combining this inequality with (16) gives the first estimate in the theorem.

Assume \(n > 2p\). Hölder’s inequality gives

\[
\|f\|_2^2 = \sum_{m=0}^\infty \|Q_m\|_2^2
\]

\[
= \sum_{m=0}^\infty \left( |\lambda_m(n,p)|^{\frac{4}{n+2-2p}} \|Q_m\|_2^{\frac{4}{n+2-2p}} \right) \cdot \left( |\lambda_m(n,p)|^{-\frac{4}{n+2-2p}} \|Q_m\|_2^{-\frac{2n-4p}{n+2-2p}} \right)
\]

\[
\leq \left( \sum_{m=0}^\infty |\lambda_m(n,p)|^2 \|Q_m\|_2^2 \right)^{\frac{2}{n+2-2p}} \left( \sum_{m=0}^\infty |\lambda_m(n,p)|^{-\frac{4}{n+2-2p}} \|Q_m\|_2^{-\frac{2n-4p}{n+2-2p}} \right)^{\frac{2n-4p}{n+2-2p}},
\]

where \(C(n) > 0\) is a constant depending on \(n\), and \(D\) is the diameter of \(K^o \cup L^o\). Both \(K^o\) and \(L^o\) are contained in a ball of radius \(r^{-1}\) centred at the origin. We then have \(D \leq 2r^{-1}\), and

\[
\sup_{\xi \in S^{n-1}} |\|\xi\|_K - \|\xi\|_L| \leq C(n) \int_{S^{n-1}} (|\|\xi\|_K - \|\xi\|_L|^2) \, d\xi
\]

Therefore,

\[
|\|\xi\|_K - \|\xi\|_L| \leq (n-p)^{-1}r^{-n-1+p}|f(\xi)|.
\]

Combining the above inequalities, we get

\[
(16) \quad \sup_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)| \leq C(n,p)R^2r^{-\frac{3n-4p}{n+2-p}} \|f\|_2^n.
\]

for some constant \(C(n,p)\).
where we again note that the eigenvalues are all non-zero. It follows from Lemma 7 and Stirling’s formula that there is a constant $C(n, p)$ such that
\[
\left| \lambda_m(n, p) \right|^{\frac{-4}{n+2p}} \leq C(n, p)m^2
\]
for all $m \geq 1$, and
\[
\left| \lambda_0(n, p) \right|^{\frac{-4}{n+2p}} \leq C(n, p).
\]
Using the identity
\[
\| \nabla o f \|^2 = \sum_{m=1}^\infty m(m+n-2)\|Q_m\|^2
\]
given by Corollary 3.2.12 in [7], we then have
\[
\| f \|^2 \leq C(n, p) \left( \|I_p(f)\|^2 \right)^{\frac{-2}{n+2-2p}} \left( \|Q_0\|^2 + \|\nabla o f \|^2 \right)^{\frac{n-2p}{n+2-2p}}.
\]
The Minkowski functional of a convex body is the support function of the corresponding polar body, so
\[
\nabla o \|\xi\|^{-n+p}_K = (-n+p)\|\xi\|^{-n-1+p}_K \nabla o h_{K^o}(\xi).
\]
Because $K^o$ is contained in a ball of radius $r^{-1}$, it follows from Lemma 2.2.1 in [7] that
\[
\|\nabla o h_{K^o}(\xi)\| \leq 2r^{-1}
\]
for all $\xi \in S^{n-1}$. We now have
\[
\|\nabla o \|\xi\|^{-n+p}_K\|^2 \leq 4(n-p)^2 R^{2(n+1-p)} r^{-2}\omega_n.
\]
This constant bounds the squared $L^2$ norm of $\nabla o \|\xi\|^{-n+p}_L$ as well, so
\[
\|\nabla o f\|^2 \leq 16(n-p)^2 R^{2(n+1-p)} r^{-2}\omega_n.
\]
Therefore,
\[
\| f \|^2 \leq C(n, p) \varepsilon^{\frac{4}{n+2-2p}} \left( \varepsilon^2 + R^{2(n+1-p)} r^{-2}\right)^{\frac{n-2p}{n+2-2p}},
\]
where the constant $C(n, p) > 0$ is different from before. This inequality with (16) gives the second estimate in the theorem. \qed

## 4. Proofs of stability results

We are now ready to prove our stability results.

**Proof of Theorem 2.** Let $\{K_\delta\}_{0<\delta<1}$ be the family of smooth convex bodies from Lemma 8. We will show that $\rho(K_\delta, -K_\delta)$ is small for $0 < \delta < \alpha$, where $\alpha$ is the constant from the proof of Lemma 10. The bounds in the theorem will then follow from
\[
\rho(K, -K) \leq \lim_{\delta \to 0} \left( 2\rho(K, K_\delta) + \rho(K_\delta, -K_\delta) \right) = \lim_{\delta \to 0} \rho(K_\delta, -K_\delta).
\]
We begin by separately considering the case $n = 2$. Let the radial function $\rho_{K_\delta}$ be a function of the angle measured counter-clockwise from the positive horizontal axis. For any $\xi \in S^1$, let the angles $\phi_1$ and $\phi_2$ be functions of $t \in (-r, r)$ as indicated
Figure 2. $K_\delta$ is a convex body in $\mathbb{R}^2$, and $\xi \in S^1$.

In Figure 2, if $\xi$ corresponds to the angle $\theta$, then the parallel section function for $K_\delta$ may be written as

$$A_{K_\delta, \theta}(t) = \rho_{K_\delta}(\theta + \phi_1) \sin \phi_1 + \rho_{K_\delta}(\theta - \phi_2) \sin \phi_2.$$  

Implicit differentiation of

$$\cos \phi_j = t \rho_{K_\delta}(\theta - (-1)^j \phi_j) \quad (j = 1, 2)$$

gives

$$\frac{d\phi_j}{dt} \bigg|_{t=0} = \frac{(-1)}{\rho_{K_\delta}(\theta - (-1)^j \frac{\pi}{2})}.$$  

so

$$A'_{K_\delta, \theta}(0) = -\frac{\rho'_{K_\delta}(\theta + \frac{\pi}{2})}{\rho_{K_\delta}(\theta + \frac{\pi}{2})} + \frac{\rho'_{K_\delta}(\theta - \frac{\pi}{2})}{\rho_{K_\delta}(\theta - \frac{\pi}{2})}.$$  

Since $f(\phi) := \rho_{K_\delta}(\phi + \pi/2) - \rho_{K_\delta}(\phi - \pi/2)$ is a continuous function on $[0, \pi]$ with

$$f(0) = \rho_{K_\delta}(\pi/2) - \rho_{K_\delta}(-\pi/2) = -\left(\rho_{K_\delta}(-\pi/2) - \rho_{K_\delta}(\pi/2)\right) = -f(\pi),$$

there exists an angle $\theta_0 \in [0, \pi]$ such that $\rho_{K_\delta}(\theta_0 + \pi/2) = \rho_{K_\delta}(\theta_0 - \pi/2)$. With this $\theta_0$, we get the inequality

$$\left|\int_{\theta_0}^{\theta} \left( -\frac{\rho'_{K_\delta}(\phi + \frac{\pi}{2})}{\rho_{K_\delta}(\phi + \frac{\pi}{2})} + \frac{\rho'_{K_\delta}(\phi - \frac{\pi}{2})}{\rho_{K_\delta}(\phi - \frac{\pi}{2})} \right) d\phi \right| \leq \int_{0}^{2\pi} \left| A'_{K_\delta, \phi}(0) \right| d\phi.$$  

Integrating the left side of this inequality and applying Lemma 10 to the right side, gives

$$\left| \log \left( \frac{\rho_{K_\delta}(\theta - \frac{\pi}{2})}{\rho_{K_\delta}(\theta + \frac{\pi}{2})} \right) \right| \leq \left( 6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon}.$$
This implies
\[
1 - \exp \left[ \left(6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon} \right] \leq \exp \left[ - \left(6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon} \right] - 1
\]
\[
\leq \frac{\rho_{K_\delta}(\theta - \frac{\pi}{2})^2}{\rho_{K_\delta}(\theta + \frac{\pi}{2})^2} - 1
\]
\[
\leq \exp \left[ \left(6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon} \right] - 1.
\]
It follows that
\[
2 \left( \exp \left[ \left(6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon} \right] - 1 \right) R \leq \rho_{K_\delta}(\theta - \frac{\pi}{2}) - \rho_{K_\delta}(\theta + \frac{\pi}{2})
\]
\[
\leq 2 \left( \exp \left[ \left(6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon} \right] - 1 \right) R,
\]
since \(K_\delta\) is contained in a ball of radius \(2R\). Viewing \(\rho_{K_\delta}\) again as a function of vectors, we have
\[
\sup_{\xi \in S^1} |\rho_{K_\delta}(\xi) - \rho_{K_\delta}(-\xi)| \leq 2 \left( \exp \left[ \left(6\pi + \frac{32\pi}{\sqrt{3}r} \right) \sqrt{\varepsilon} \right] - 1 \right) R.
\]
The inequality \(e^t - 1 \leq 2t\) is valid when \(0 < t < 1\); therefore, if
\[
\varepsilon < \left( \frac{\sqrt{3}r}{6\sqrt{3}\pi r + 32\pi} \right)^2,
\]
then
\[
\sup_{\xi \in S^1} |\rho_{K_\delta}(\xi) - \rho_{K_\delta}(-\xi)| \leq \left( 24\pi + \frac{128\pi}{\sqrt{3}r} \right) R \sqrt{\varepsilon}.
\]
Consider the case when \(n > 2\). For \(K_\delta\) with \(p = 1\), equation (2) becomes
\[
I_2 (||x||_{K_\delta}^{-n+2} - ||x||_{K_\delta}^{-n+2}) (\xi) = -2\pi i (n - 2) A'_{K_\delta, \xi}(0),
\]
so
\[
\|I_2 (||x||_{K_\delta}^{-n+2} - ||x||_{K_\delta}^{-n+2})\|_2 = 2\pi(n - 2) \left( \int_{S^{n-1}} |A'_{K_\delta, \xi}(0)|^2 d\xi \right) ^{\frac{1}{2}}
\]
\[
\leq \tilde{C}(n) \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) ^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}
\]
\[
\rho(K_\delta, -K_\delta) \leq C(n) \frac{R^{2n-4}}{r^{n+2}} \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) ^{\frac{1}{n+1}} \varepsilon^{\frac{1}{2(n+1)}}
\]
when \(n = 3\) or \(4\), and
\[
\rho(K_\delta, -K_\delta) \leq C(n) \left[ \left( \sqrt{\varepsilon} + \frac{R^{2n-4}}{r} + \frac{R^{3n-3}}{r^{n+2}} \right) ^{\frac{1}{2}} \right] ^{\frac{n-4}{(n-2)(n+1)}} \frac{R^{2n-4}}{r^{n+2}} \varepsilon^{\frac{1}{2(n-2)}}
\]
when \(n \geq 5\), where \(C(n) > 0\) are constants depending on the dimension. \[\square\]
We now present the proof of our second stability result.

Proof of Theorem [5]. Apply Lemma [8] to $K$ and $L$; let $\{K_\delta\}_{0<\delta<1}$ and $\{L_\delta\}_{0<\delta<1}$ be the resulting families of smooth convex bodies. For each $\delta$, define the constant

$$
\varepsilon_\delta := \sup_{\xi \in S^{n-1}} \left| A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{K,\xi}(0) \right| + \sup_{\xi \in S^{n-1}} \left| A^{(p)}_{L_\delta,\xi}(0) - A^{(p)}_{L,\xi}(0) \right| + \varepsilon.
$$

Defining the auxiliary function

$$
f_\delta(\xi) := \|\xi\|_{K_\delta}^{n+1+p} - \|\xi\|_{L_\delta}^{n+1+p},
$$

we have

$$
\cos \left( \frac{p\pi}{2} / 2 \right) I_{1+p} (f_\delta(x) + f_\delta(-x))(\xi) + i \sin \left( \frac{p\pi}{2} / 2 \right) I_{1+p} (f_\delta(x) - f_\delta(-x))(\xi)
$$

$$
= 2\pi(n - 1 - p) \left( A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{L_\delta,\xi}(0) \right) + \left( A^{(p)}_{K_\delta,-\xi}(0) - A^{(p)}_{L_\delta,-\xi}(0) \right)
$$

from equation (2). The function of $\xi$ on the left side of this equality is split into its even and odd parts, because $I_{1+p}$ preserves even and odd symmetry. Therefore,

$$
\frac{\cos \left( \frac{p\pi}{2} / 2 \right) I_{1+p} (f_\delta(x) + f_\delta(-x))(\xi)}{\pi(n - 1 - p)} = \left( A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{L_\delta,\xi}(0) \right) + \left( A^{(p)}_{K_\delta,-\xi}(0) - A^{(p)}_{L_\delta,-\xi}(0) \right)
$$

and

$$
\frac{i \sin \left( \frac{p\pi}{2} / 2 \right) I_{1+p} (f_\delta(x) - f_\delta(-x))(\xi)}{\pi(n - 1 - p)} = \left( A^{(p)}_{K_\delta,\xi}(0) - A^{(p)}_{L_\delta,\xi}(0) \right) - \left( A^{(p)}_{K_\delta,-\xi}(0) - A^{(p)}_{L_\delta,-\xi}(0) \right).
$$

By the definition of $\varepsilon_\delta$,

$$
\left| I_{1+p} (2f_\delta)(\xi) \right| \leq \left| I_{1+p} (f_\delta(x) + f_\delta(-x))(\xi) \right| + \left| I_{1+p} (f_\delta(x) - f_\delta(-x))(\xi) \right|
$$

$$
\leq \frac{2\pi(n - 1 - p)}{\cos \left( \frac{p\pi}{2} / 2 \right)} \varepsilon_\delta + \frac{2\pi(n - 1 - p)}{\sin \left( \frac{p\pi}{2} / 2 \right)} \varepsilon_\delta,
$$

which implies

$$
\| I_{1+p} (f_\delta) \|_2 \leq \frac{2\pi(n - 1 - p)}{\sqrt{\omega_n}} \left( \left| \cos \left( \frac{p\pi}{2} / 2 \right) \right| + \left| \sin \left( \frac{p\pi}{2} / 2 \right) \right| \right) \varepsilon_\delta.
$$

Both $K_\delta$ and $L_\delta$ are contained in a ball of radius $2R$ when $0 < \delta < 1/2$ and contain a ball of radius $r/2$. It now follows from Lemma [11] that

$$
\rho(K_\delta, L_\delta) \leq C(n, p) R^2 r^{\frac{3n+1+2p}{n+1}} \varepsilon_\delta^{\frac{2}{n+1}}
$$

when $n \leq 2p + 2$, and

$$
\rho(K_\delta, L_\delta) \leq C(n, p) R^2 r^{\frac{3n+1+2p}{n+1}} \left( \varepsilon_\delta^2 + \frac{R^2(n-p)}{r^2} \right) \varepsilon_\delta^{\frac{4}{n-2p(n+1)}}
$$

when $n > 2p + 2$, where $C(n, p) > 0$ are constants depending on the dimension and $p$. Finally, the bounds in the theorem statement follow from the observations

$$
\rho(K, L) \leq \lim_{\delta \to 0} \left( \rho(K_\delta, K_\delta) + \rho(L_\delta, L_\delta) + \rho(K_\delta, L_\delta) \right) = \lim_{\delta \to 0} \rho(K_\delta, L_\delta),
$$

and $\lim_{\delta \to 0} \varepsilon_\delta = \varepsilon$. \qed
REFERENCES


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