

ON SOME $2D$ ORTHOGONAL q -POLYNOMIALS

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ABSTRACT. We introduce two q -analogues of the $2D$ -Hermite polynomials which are functions of two complex variables. We derive explicit formulas, orthogonality relations, raising and lowering operator relations, generating functions, and Rodrigues formulas for both families. We also introduce a q - $2D$ analogue of the disk polynomials (Zernike polynomials) and derive similar formulas for them as well, including evaluating certain connection coefficients. Some of the generating functions may be related to Rogers–Ramanujan type identities.

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1. INTRODUCTION

The $2D$ -Hermite (or complex Hermite) polynomials $\{H_{m,n}(z_1, z_2)\}_{m,n=0}^{\infty}$,

$$(1.1) \quad H_{m,n}(z_1, z_2) = \sum_{k=0}^{m \wedge n} (-1)^k k! \binom{m}{k} \binom{n}{k} z_1^{m-k} z_2^{n-k},$$

were introduced in [24]. Recently several mathematical physicists studied these polynomials from mathematical and physical points of view, [1], [7], [15], [16], [17]. Their combinatorics were studied in [21], [22], and in [20]. Ismail [20] proved a Kibble–Slepian type multilinear generating function for these polynomials, while the present authors gave a new proof together with a proof of the original Kibble–Slepian formula for Hermite polynomials in the forthcoming work [23]. Relevant references on the $2D$ -Hermite polynomials are [27], [29], [30], [31], and [32].

This work introduces two q -analogues of the $2D$ -Hermite polynomials denoted by $\{H_{m,n}(z_1, z_2|q)\}$ and $\{h_{m,n}(z_1, z_2|q)\}$ and a q - $2D$ sequence of ultraspherical polynomials. The polynomials $\{H_{m,n}(z_1, z_2|q)\}$ and $\{h_{m,n}(z_1, z_2|q)\}$ transform to each other as $q \rightarrow 1/q$. We produce one orthogonality measure for the first family while we give infinitely many orthogonality measures for the second family. An orthogonality measure is given for the q - $2D$ ultraspherical polynomials. We also find the raising and lowering operators for both families of q - $2D$ Hermite polynomials together with Sturm–Liouville equations which they satisfy.

In Section 2 we collect all the preliminary results used throughout the paper. Section 3 treats the polynomials $\{H_{m,n}(z_1, z_2|q)\}$, while Section 4 treats the second family $\{h_{m,n}(z_1, z_2|q)\}$. In Section 5 we first give the definition of a set of orthogonal polynomials in two variables, the disk polynomials [8, §2.3]. They are also known as Zernike polynomials [34]. The rest of Section 5 contains the definition and properties of the q - $2D$ ultraspherical polynomials denoted by $\{p_{m,n}(z\bar{z}; b|q)\}$. These are q -analogues of the disk polynomials. They constitute a q -analogue which is different from the one introduced by Floris in [10]–[11]; see also [12]. Section 6 has several applications of the results obtained in the earlier sections including multilinear generating functions. In Section 7 we establish moment type representations for $\{H_{m,n}(z_1, z_2|q)\}$ and $\{h_{m,n}(z_1, z_2|q)\}$ and give closed form expressions for the connection coefficients in the expansion of $\{H_{m,n}(z_1, z_2|q)\}$ (respectively $\{h_{m,n}(z_1, z_2|q)\}$) in $\{h_{m,n}(z_1, z_2|q)\}$ (respectively $\{H_{m,n}(z_1, z_2|q)\}$). In addition we give a two dimensional q -analogue of the generating function [19, (4.6.29)]

$$(1.2) \quad \sum_{n=0}^{\infty} H_{m+n}(x) \frac{t^n}{n!} = \exp(2xt - t^2) H_m(x - t).$$

A formula that may have ramifications on the theory of partitions is formula (7.10). In Section 8 we show that the zero sets of $\{H_{m,n}(z, \bar{z}|q)\}$, $\{h_{m,n}(z, \bar{z}|q)\}$ and $\{p_{m,n}(z, \bar{z}; b|q)\}$ are concentric circles in \mathbb{C} centered at $z = 0$. We also show that the limiting distribution of the zeros of $\{H_{m,n}(z, \bar{z}|q)\}$ and $\{p_{m,n}(z, \bar{z}; b|q)\}$ coincide with the support of their measures of orthogonality. The polynomials $\{h_{m,n}(z, \bar{z}|q)\}$ are orthogonal on bounded sets with respect to different measures. We describe their zero sets as $m, n \rightarrow \infty$. The asymptotics involves the zeros of the Ramanujan function to be defined in (2.14). This is similar to the one variable q -polynomials in [18]. In Section 9 we show that certain matrices whose entries are formed by $2D$ -polynomials are positive definite.

This is the first part in a series of papers on the subject of 2D orthogonal polynomials where we study several new families of orthogonal polynomials.

2. PRELIMINARIES

In this section we collect all the formulas used in the later sections and mention some of the notation. We shall follow the notation and terminology for special functions and q -series in [5], [14], [19], and [25]. We assume the reader is familiar with the notation of q -shifted factorials as well as the unilateral and bilateral basic hypergeometric functions ${}_r\phi_s$ and ${}_r\psi_r$. Moreover we use the notation

$$(2.1) \quad \{x\} = \text{the fractional part of } x, \quad [x] = x - \{x\},$$

$$(2.2) \quad m \wedge n = \min \{m, n\}.$$

The q -difference and dilation operators are

$$(2.3) \quad (D_q f)(z) = \frac{f(z) - f(qz)}{z - qz}, \quad z \neq 0, \quad \text{and} \quad (\eta_q f)(x) = f(qx),$$

respectively. If the dependence on z is important we shall use $D_{q,z}$ and $\eta_{q,z}$ instead of D_q and η_q , respectively. The Leibniz rule for D_q is

$$(2.4) \quad D_q^n (fg)(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k f(x) \eta_q^k D_q^{n-k} g(x).$$

The q -binomial theorem is [14, (II.3)]

$$(2.5) \quad \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}, \quad |z| < 1.$$

The terminating case is

$$(2.6) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-z)^k = (z; q)_n.$$

Two important special and limiting cases are the Euler identities [14, (II.1)–(II.2)]

$$(2.7) \quad \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}},$$

$$(2.8) \quad \sum_{n=0}^{\infty} \frac{(-z)^n}{(q; q)_n} q^{\binom{n}{2}} = (z; q)_{\infty}.$$

The q -integral is [14, §1.11]

$$(2.9) \quad \int_0^{\infty} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

The q -Laguerre polynomials are [25, 3.21.1)]

$$(2.10) \quad \begin{aligned} L_n^{(\alpha)}(x; q) &= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \middle| q, -q^{n+\alpha+1}x \right) \\ &= \frac{1}{(q; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, -x \\ q^{\alpha+1} \end{matrix} \middle| q, q^{n+\alpha+1} \right). \end{aligned}$$

Their moment problem is indeterminate; that is, there are infinitely many orthogonality measures with respect to which the q -Laguerre polynomials are orthogonal.

For a treatment of the q -Laguerre polynomials and the corresponding moment problem we refer the interested reader to [19, §21.8]. The little q -Laguerre, also known as Wall polynomials, are defined by [25, (3.20.1)]

$$(2.11) \quad p_n(x; a|q) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q, qx \right) = \frac{1}{(q^{-n}/a; q)_n} {}_2\phi_0 \left(\begin{matrix} q^{-n}, 1/x \\ - \end{matrix} \middle| q, \frac{x}{a} \right).$$

See also Section 11 of Chapter VI in [6].

The q -Bessel functions $J_\nu^{(2)}$ and $I_\nu^{(2)}$ are defined by

$$(2.12) \quad J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n q^{n(n+\nu)}}{(q, q^{\nu+1}; q)_n} \left(\frac{z}{2}\right)^{\nu+2n},$$

$$(2.13) \quad I_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^\infty \frac{q^{n(n+\nu)}}{(q, q^{\nu+1}; q)_n} \left(\frac{z}{2}\right)^{\nu+2n},$$

respectively, [19], [14]. The Ramanujan function is [19]

$$(2.14) \quad A_q(z) = \sum_{n=0}^\infty \frac{q^{n^2}}{(q; q)_n} (-z)^n.$$

It has only positive zeros. It appeared in Ramanujan’s Lost Notebook [26] with some statements about the asymptotics of its zeros.

Garrett, Ismail, and Stanton [13] generalized the Rogers–Ramanujan identities to

$$(2.15) \quad \sum_{n=0}^\infty \frac{q^{n^2+mn}}{(q; q)_n} = \frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q, q^4, q^5)_\infty} + \frac{(-1)^{m+1} q^{-\binom{m}{2}} b_m(q)}{(q^2, q^3, q^5)_\infty}$$

where

$$(2.16) \quad a_m(q) = \sum_j q^{j^2+j} \begin{bmatrix} m-j-2 \\ j \end{bmatrix}_q, \quad b_m(q) = \sum_j q^{j^2} \begin{bmatrix} m-j-1 \\ j \end{bmatrix}_q.$$

The polynomials $a_m(q)$ and $b_m(q)$ were considered by Schur in conjunction with his proof of the Rogers–Ramanujan identities; see [2] and [13] for details. We shall refer to $a_m(q)$ and $b_m(q)$ as the Schur polynomials.

Let $q = e^{-2k^2}$ and $|q| < 1$, the Ramanujan’s identities are

$$(2.17) \quad \int_{-\infty}^\infty e^{-x^2+2mx} (-aqe^{2kx}, -bqe^{-2kx}; q)_\infty dx = \frac{\sqrt{\pi} (abq; q)_\infty e^{m^2}}{(ae^{2mk} \sqrt{q}, be^{-2mk} \sqrt{q}; q)_\infty}$$

[14, Ex. 6.15)(i)] and

$$(2.18) \quad \int_{-\infty}^\infty \frac{e^{-x^2+2mx} dx}{(ae^{2ikx} \sqrt{q}, be^{-2ikx} \sqrt{q}; q)_\infty} = \frac{\sqrt{\pi} e^{m^2} (-aqe^{2imk}, -bqe^{-2imk}; q)_\infty}{(abq; q)_\infty}$$

[14, Ex. 6.15)(ii)]. For $0 < q < 1$, $\Re(a+c) > 0$ and $\Re(b-c) > 0$, Ramanujan extended the beta integral on $(0, \infty)$ to the following integrals:

$$(2.19) \quad \int_0^\infty \frac{(-tq^b, -q^{a+1}/t; q)_\infty t^{c-1} d_q t}{(-t, -q/t; q)_\infty (1-q)} = \frac{(q, -q^c, -q^{1-c}, q^{a+b}; q)_\infty}{(-1, -q, q^{a+c}, q^{b-c}; q)_\infty}$$

[14, E. 6.17(i)] and

$$(2.20) \quad \int_0^\infty \frac{(-tq^b, -q^{a+1}/t; q)_\infty t^{c-1} dt}{(-t, -q/t; q)_\infty} = \frac{\Gamma(c) \Gamma(1-c) (q^c, q^{1-c}, q^{a+b}; q)_\infty}{(q, q^{a+c}, q^{b-c}; q)_\infty}$$

[14, Ex. 6.17(ii)]. Then,

$$(2.21) \quad \int_0^\infty \frac{t^{c-1} d_q t}{(-t, -q/t; q)_\infty (1-q)} = \frac{(q, -q^c, -q^{1-c}; q)_\infty}{(-1, -q; q)_\infty}$$

and

$$(2.22) \quad \int_0^\infty \frac{t^{c-1} dt}{(-t, -q/t; q)_\infty} = \frac{\Gamma(c) \Gamma(1-c) (q^c, q^{1-c}; q)_\infty}{(q; q)_\infty}.$$

The Askey–Roy integral is [14, (4.11.1)]

$$(2.23) \quad \int_{-\pi}^\pi \frac{(ce^{i\theta}/\beta, qe^{i\theta}/\alpha, \alpha e^{-i\theta}, q\beta e^{-i\theta}/c; q)_\infty d\theta}{(ae^{i\theta}, be^{i\theta}, \alpha e^{-i\theta}, \beta e^{-i\theta}; q)_\infty} \frac{1}{2\pi} \\ = \frac{(ab\alpha\beta, c, q/c, \alpha/\beta, q\beta/c\alpha; q)_\infty}{(\alpha\alpha, a\beta, b\alpha, b\beta, q; q)_\infty}.$$

3. FIRST q -ANALOGUE

The first q -analogue of $\{H_{m,n}(z_1, z_2)\}$ is defined by

$$(3.1) \quad H_{m,n}(z_1, z_2|q) = \sum_{k=0}^{m \wedge n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} (q; q)_k z_1^{m-k} z_2^{n-k}.$$

Clearly,

$$(3.2) \quad H_{m,n}(z_2, z_1|q) = H_{n,m}(z_1, z_2|q).$$

Theorem 1. *The polynomials $\{H_{m,n}(z_1, z_2|q)\}$ satisfy the relations*

$$(3.3) \quad \sum_{m,n=0}^\infty H_{m,n}(z_1, z_2|q) \frac{u^m v^n}{(q; q)_m (q; q)_n} = \frac{(uv; q)_\infty}{(uz_1, vz_2; q)_\infty},$$

$$(3.4) \quad H_{m,n}(qz_1, z_2|q) = H_{m,n}(z_1, z_2|q) - z_1(1 - q^m) H_{m-1,n}(z_1, z_2|q),$$

$$(3.5) \quad H_{m,n}(z_1, qz_2|q) = H_{m,n}(z_1, z_2|q) - z_2(1 - q^n) H_{m,n-1}(z_1, z_2|q),$$

$$(3.6)$$

$$H_{m,n}(qz_1, z_2|q)q^{-m} = H_{m,n}(z_1, z_2|q) - q^{-1}(1 - q^m)(1 - q^n) H_{m-1,n-1}(z_1, z_2|q),$$

$$(3.7)$$

$$H_{m,n}(z_1, qz_2|q)q^{-n} = H_{m,n}(z_1, z_2|q) - q^{-1}(1 - q^m)(1 - q^n) H_{m-1,n-1}(z_1, z_2|q),$$

$$(3.8) \quad z_1 H_{m,n}(z_1, z_2|q) = q^m(1 - q^n) H_{m,n-1}(z_1, z_2|q) + H_{m+1,n}(z_1, z_2|q),$$

$$z_2 H_{m,n}(z_1, z_2|q) = q^n(1 - q^m) H_{m-1,n}(z_1, z_2|q) + H_{m,n+1}(z_1, z_2|q).$$

Moreover they have the operational representation

$$(3.9) \quad H_{m,n}(z_1, z_2|q) = ((1 - q)^2 D_{q,z_1} D_{q,z_2}; q)_\infty z_1^m z_2^n.$$

Before proving Theorem 1 we consider some of its implications. We note that (3.4) and (3.5) are the lowering operator relations

$$(3.10) \quad D_{q,z_1} H_{m,n}(z_1, z_2|q) = \frac{1 - q^m}{1 - q} H_{m-1,n}(z_1, z_2|q),$$

$$(3.11) \quad D_{q,z_2} H_{m,n}(z_1, z_2|q) = \frac{1 - q^n}{1 - q} H_{m,n-1}(z_1, z_2|q),$$

respectively. Moreover we observe that (3.6)–(3.7) imply the symmetry relation

$$(3.12) \quad H_{m,n}(qz_1, z_2|q)q^{-m} = H_{m,n}(z_1, qz_2|q)q^{-n}.$$

Indeed (3.12) can be proved directly from the generating function (3.3). Finally we record a possible connection between the generating function (3.3) and partitions. Let $M(m, n)$ denote the number of partitions of a positive integer n with crank $= m$. Andrews and Garvan [4] established the generating function

$$(3.13) \quad \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n)z^m q^n = \frac{(q; q)_{\infty}}{(qz, q/z; q)_{\infty}}.$$

It is clear that (3.13) is a special case of our generating function (3.3). This suggests that there may be a more refined statistic defined on partitions which will have the generating function (3.3).

Proof of Theorem 1. The generating function follows from (3.1) and the Euler sums (2.7)–(2.8). (3.4) and (3.5) follow from

$$\frac{(uv; q)_{\infty}}{(uz_1q, vz_2; q)_{\infty}} = (1 - uz_1) \frac{(uv; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}}$$

and

$$\frac{(uv; q)_{\infty}}{(uz_1, vz_2q; q)_{\infty}} = (1 - vz_2) \frac{(uv; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}};$$

(3.6) and (3.7) follow from

$$\frac{(uq^{-1}v; q)_{\infty}}{(uq^{-1}z_1q, vz_2; q)_{\infty}} = (1 - q^{-1}uv) \frac{(uv; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}}$$

and

$$\frac{(uvq^{-1}; q)_{\infty}}{(uz_1, vq^{-1}z_2q; q)_{\infty}} = (1 - q^{-1}uv) \frac{(uv; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}}.$$

The first 3-term recurrence follows from (3.4) and (3.6); similarly, the second one can be obtained from (3.5) and (3.7). It can be proved directly. It is clear that $z_1 H_{m,n}(z_1, z_2|q) - H_{m+1,n}(z_1, z_2|q)$ is

$$\begin{aligned} & \sum_{k=0}^{(m+1) \wedge n} \left\{ \begin{bmatrix} m \\ k \end{bmatrix}_q - \begin{bmatrix} m+1 \\ k \end{bmatrix}_q \right\} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} (q; q)_k z_1^{m+1-k} z_2^{n-k} \\ &= - \sum_{k=1}^{m \wedge n} q^{m+1-k} \begin{bmatrix} m \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} (q; q)_k z_1^{m+1-k} z_2^{n-k}, \end{aligned}$$

which gives the first recurrence relation after replacing k by $k + 1$. The proof of the second recurrence relation is similar. The representation of (3.9) follows by expanding $((1 - q)^2 D_{q,z_1} D_{q,z_2}; q)_{\infty}$ using (2.8). □

Theorem 2. *The polynomials satisfy the Rodrigues type formula*

$$(3.14) \quad H_{m,n}(z_1, z_2|q) = \frac{(1 - 1/q)^{m+n} q^{mn}}{(qz_1z_2; q)_\infty} D_{q^{-1}, z_2}^m D_{q^{-1}, z_1}^n ((qz_1z_2; q)_\infty)$$

and the raising relations

$$(3.15) \quad H_{m+1,n}(z_1, z_2|q) = q^n \frac{1 - 1/q}{(qz_1z_2; q)_\infty} D_{q^{-1}, z_2} ((qz_1z_2; q)_\infty H_{m,n}(z_1, z_2|q)),$$

$$(3.16) \quad H_{m,n+1}(z_1, z_2|q) = q^m \frac{1 - 1/q}{(qz_1z_2; q)_\infty} D_{q^{-1}, z_1} ((qz_1z_2; q)_\infty H_{m,n}(z_1, z_2|q)).$$

Moreover the polynomials $\{H_{m+1,n}(z_1, z_2|q)\}$ have the multiplication formula

$$(3.17) \quad H_{m,n}(az_1, bz_2|q) = \sum_{j=0}^{m \wedge n} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{H_{m-j, n-j}(z_1, z_2|q)}{a^{j-m} b^{j-n}} (q, 1/ab; q)_j (q; q)_j.$$

Proof. It is clear that $(1 - 1/q)D_{q^{-1}, z_1}(qz_1z_2; q)_\infty = z_2(qz_1z_2; q)_\infty$. Therefore the right-hand side of (3.14) is

$$\begin{aligned} & q^{nm} (1 - 1/q)^m D_{q^{-1}, z_2}^m [z_2^n (qz_1z_2; q)_\infty] \\ &= q^{nm} (1 - 1/q)^m \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_{q^{-1}} \frac{z_1^k}{(1 - 1/q)^k} (q^{-k} z_2)^{n-m+k} \frac{(q^{-n}; q)_{m-k}}{(1 - 1/q)^{m-k}} \\ &= q^{nm} \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_{q^{-1}} z_1^{m-k} z_2^{n-k} q^{-(m-k)(n-k)} (q^{-n}; q)_k = H_{m,n}(z_1, z_2|q), \end{aligned}$$

and we have proved (3.14). Formulas (3.15) and (3.16) follow directly from (3.14). The generating function (3.3) implies

$$(3.18) \quad \sum_{n=0}^{\infty} H_{m,n}(z_1, z_2|q) \frac{u^m v^n}{(q; q)_m (q; q)_n} = \frac{(uv; q)_\infty}{(abuv; q)_\infty} \frac{(abuv; q)_\infty}{(uz_1, vz_2; q)_\infty}$$

and (3.17) follow from the q -binomial theorem (2.5). □

In the next section we shall introduce the polynomials $\{h_{m,n}(z_1, z_2|q)\}$; see (4.1) and (4.5). We also note (4.23), which indicates their relation to the q -Laguerre polynomials [25]. We now show a connection between the polynomials $\{H_{m,n}(z_1, z_2|q)\}$ and the little q -Jacobi polynomials [25]:

$$\begin{aligned} H_{m,n}(z_1, z_2|q) &= q^{mn} i^{m+n} h_{m,n}(z_1/i, z_2/i|q^{-1}) \\ &= q^{mn} (q^{-1}; q^{-1})_n z_1^{m-n} L_n^{(m-n)}(-z_1z_2; q^{-1}) \\ &= q^{mn} (q^{n-m-1}; q^{-1})_n z_1^{m-n} p_n \left(z_1z_2, q^{m-n} \middle| q \right) \\ &= (-1)^n \frac{(q; q)_m q^{\binom{n}{2}}}{(q; q)_{m-n}} z_1^{m-n} p_n \left(z_1z_2, q^{m-n} \middle| q \right) \end{aligned}$$

or

$$(3.19) \quad H_{m,n}(z_1, z_2|q) = (-1)^n \frac{(q; q)_m q^{\binom{n}{2}}}{(q; q)_{m-n}} z_1^{m-n} p_n \left(z_1z_2, q^{m-n} \middle| q \right),$$

where $p_n(x; q^\alpha|q)$ is the little q -Laguerre or Wall's polynomials [25]

$$p_n(x; a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q; qx\right).$$

They satisfy the discrete orthogonality relation

$$\sum_{k=0}^\infty a^k q^k (q^{k+1}; q)_\infty p_m(q^k; a|q) p_n(q^k; a|q) = \frac{(q; q)_\infty (aq)^n (q; q)_n}{(aq; q)_\infty (aq; q)_\infty} \delta_{m,n},$$

where $m, n \in \mathbb{N}_0$ and $0 < a < q^{-1}$.

Theorem 3. *The polynomials $\{H_{m,n}(z, \bar{z}|q)\}$ satisfy the orthogonality*

$$(3.20) \quad \int_{\mathbb{C}} H_{m,n}(z, \bar{z}|q) \overline{H_{s,t}(z, \bar{z}|q)} d\mu(z, \bar{z}) = \frac{q^{mn} (q; q)_m (q; q)_n}{(q; q)_\infty} \delta_{m,s} \delta_{n,t},$$

where

$$d\mu(z, \bar{z}) = \frac{d\theta}{2\pi} \otimes \sum_{k=0}^\infty \frac{q^k}{(q; q)_k} \delta(r - q^{k/2}),$$

and $z = re^{i\theta}$, $r \in \mathbb{R}^+$, $\theta \in [0, 2\pi]$, $m, n, s, t \in \mathbb{N}_0$.

Proof. We may assume that $m \geq n$ because of the symmetry (3.2). Then apply (3.19) and change into polar coordinates to get

$$\begin{aligned} & \int_0^{2\pi} H_{m,n}(z, \bar{z}|q) \overline{H_{s,t}(z, \bar{z}|q)} d\mu(z, \bar{z}) \\ &= (-1)^{n+t} \frac{(q; q)_m q^{\binom{n}{2}} (q; q)_s q^{\binom{t}{2}}}{(q; q)_{m-n} (q; q)_{s-t}} \int_0^{2\pi} e^{i\theta(m-n+t-s)} \frac{d\theta}{2\pi} \\ & \times \sum_{k=0}^\infty \frac{q^k}{(q; q)_k} p_n\left(r^2, q^{m-n} \middle| q\right) p_t\left(r^2, q^{s-t} \middle| q\right) r^{m-n+s-t} \delta(r - q^{k/2}) \\ &= (-1)^{n+t} \frac{(q; q)_m q^{\binom{n}{2}} (q; q)_s q^{\binom{t}{2}}}{(q; q)_{m-n} (q; q)_{s-t}} \delta_{m-n+t-s, 0} \\ & \times \sum_{k=0}^\infty \frac{q^{k(1+m-n)}}{(q; q)_k} p_n\left(q^k, q^{m-n} \middle| q\right) p_t\left(q^k, q^{m-n} \middle| q\right) \\ &= \frac{q^{mn} (q; q)_m (q; q)_n}{(q; q)_\infty} \delta_{m,s} \delta_{n,t}. \end{aligned}$$

This completes the proof of the orthogonality relation. □

It is clear that the orthogonality relation (3.20) and the generating function (3.3) imply the q -beta integral

$$(3.21) \quad \int_{\mathbb{C}} \frac{d\mu(z, \bar{z})}{(u_1 z, v_1 \bar{z}, v_2 z, u_2 \bar{z}; q)_\infty} = \frac{(u_1 u_2 v_1 v_2; q)_\infty}{(q, u_1 u_2, v_1 v_2, u_1 v_1, u_2 v_2; q)_\infty}.$$

The large degree asymptotics of $H_{m,n}(z, \bar{z}|q)$ are straightforward. Indeed (3.1) and Tannery's theorem show that

$$(3.22) \quad \lim_{m \rightarrow \infty} z_1^{-m} H_{m,n}(z_1, z_2|q) = z_2^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-z_1 z_2)^{-k} = z_2^n (1/z_1 z_2; q)_n,$$

where we used the q binomial theorem (2.6) in the last step. Similarly

$$(3.23) \quad \lim_{n \rightarrow \infty} z_2^{-n} H_{m,n}(z_1, z_2|q) = z_1^m(1/z_1 z_2; q)_m.$$

One can similarly show that

$$(3.24) \quad \lim_{m,n \rightarrow \infty} z_1^{-m} z_2^{-n} H_{m,n}(z_1, z_2|q) = (1/z_1 z_2; q)_\infty.$$

It must be noted that the convergence in (3.22)–(3.24) is uniform on compact subsets of the z_1 and z_2 planes.

Theorem 4. *The polynomials $\{H_{m,n}(z_1, z_2|q)\}$ have the generating function*

$$(3.25) \quad \begin{aligned} & \sum_{m,n=0}^\infty H_{m,n}(z_1, z_2|q) \frac{u^m(a/u; q)_n v^n(b/v; q)_n}{(q; q)_m (q; q)_n} \\ &= \frac{(az_1 \cdot bz_2; q)_\infty}{(uz_1 v z_2; q)_\infty} {}_2\phi_2 \left(\begin{matrix} a/u, b/v \\ az_1, bz_2 \end{matrix} \middle| q; uv \right). \end{aligned}$$

Proof. From the explicit representation (3.1) it follows that the right-hand side of (3.25) is equal to

$$\begin{aligned} & \sum_{m \geq k \geq 0, n \geq k \geq 0}^\infty (-1)^k q^{\binom{k}{2}} \frac{u^m(a/u; q)_m v^n(b/v; q)_n}{(q; q)_k (q; q)_{m-k} (q; q)_{n-k}} z_1^{m-k} z_2^{n-k} \\ &= \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} u^k(a/u; q)_k v^k(b/v; q)_k \\ & \quad \times \sum_{m=0}^\infty \frac{(uz_1)^m (aq^k/u; q)_m}{(q; q)_m} \sum_{n=0}^\infty \frac{(vz_2)^n (bq^k/v; q)_n}{(q; q)_n} \\ &= \sum_{k=0}^\infty \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} u^k(a/u; q)_k v^k(b/v; q)_k \frac{(az_1 q^k, bz_2 q^k; q)_\infty}{(uz_1, v z_2; q)_\infty}, \end{aligned}$$

and the theorem follows. □

The polynomials $\{H_{m,n}(z_1, z_2|q)\}$ have an additional orthogonality relation, which we now record.

Theorem 5. *We have the orthogonality relation*

$$(3.26) \quad \begin{aligned} & \sum_{j=0}^p \sum_{k=0}^s \int_0^\pi (q, e^{2i\theta}, e^{-2i\theta}; q)_\infty \frac{H_{j,k}(re^{i\theta}, re^{-i\theta}|q) H_{s-k,p-j}(re^{i\theta}, re^{-i\theta}|q)}{(q; q)_j (q; q)_k (q; q)_{s-k} (q; q)_{p-j}} \frac{d\theta}{\pi} \\ &= \frac{r^{2p}(1/r^2; q)_p}{(q; q)_p} {}_1\phi_1 \left(\begin{matrix} q^{-s} \\ q^{1-p} r^2 \end{matrix} \middle| q, q \right) \delta_{s,p}. \end{aligned}$$

Proof. A special case of the Askey–Wilson integral is [19], [14]

$$(3.27) \quad \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty} d\theta = \frac{\pi}{(q, ab; q)_\infty}.$$

Therefore

$$\begin{aligned} & \frac{(q; q)_\infty}{\pi} \int_0^\pi (e^{2i\theta}, e^{-2i\theta}; q)_\infty \\ & \quad \times \left[\sum_{j,k,m,n=0}^\infty \frac{H_{j,k}(re^{i\theta}, re^{-i\theta}|q)H_{m,n}(re^{i\theta}, re^{-i\theta}|q)}{(q; q)_j(q; q)_k(q; q)_m(q; q)_n} u^j v^k v^m u^n \right] d\theta \\ & = \frac{(q; q)_\infty}{\pi} \int_0^\pi \frac{(uv, uv; q)_\infty (e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ure^{i\theta}, vre^{-i\theta}, vre^{i\theta}, ure^{-i\theta}; q)_\infty} d\theta = \frac{(uv, uv; q)_\infty}{(uvr^2; q)_\infty} \\ & = \sum_{s,t=0}^\infty \frac{(-1)^s q^{\binom{s}{2}} (1/r^2; q)_t}{(q; q)_s (q; q)_t} r^{2t} (uv)^{s+t}. \end{aligned}$$

Therefore $j + n$ must equal $k + m = p$, say. The coefficient of $(uv)^p$ in the above expression is

$$r^{2p} \sum_{s=0}^p \frac{(-1)^s q^{\binom{s}{2}} (1/r^2; q)_{p-s}}{(q; q)_s (q; q)_{p-s}} r^{-2s} = \frac{r^{2p} (1/r^2; q)_p}{(q; q)_p} {}_1\phi_1 \left(\begin{matrix} q^{-s} \\ q^{1-p} r^2 \end{matrix} \middle| q, q \right),$$

and the theorem follows. □

The next theorem gives sharp bounds on the zeros of $H_{m,n}(z_1, z_2|q)$.

Theorem 6. *Let $a, b, c, d \geq 0$, $\tau = \tau(m, n; a, b, c, d) = \lfloor (a + c)m + (b + d)n \rfloor$ and $\chi = \chi(m, n; a, b, c, d) = \{(a + c)m + (b + d)n\}$, for $0 < \tau(m, n; a, b, c, d) < m \wedge n$. Then*

$$\lim_{m,n \rightarrow \infty} \frac{(q; q)_\infty H_{m,n} \left(z_1 q^{am+bn-\frac{1}{4}}, z_2 q^{cm+dn-\frac{1}{4}} | q \right) (-z_1 z_2)^\tau}{z_1^m z_2^n q^{am^2+(b+c)mn+dn^2-(m+n)/4-\tau^2/2-\tau\chi}} = \theta_4 \left(z_1 z_2 q^\chi; q^{1/2} \right)$$

holds uniformly on compact subsets of the z_1 and z_2 planes.

Proof. The proof follows from

$$\begin{aligned} & \frac{H_{m,n}(z_1 q^{-1/4}, z_2 q^{-1/4}|q)}{z_1^m z_2^n} (q; q)_\infty q^{(m+n)/4} \\ & = \frac{1}{(q^{m+1}, q^{n+1}; q)_\infty} \sum_{k=0}^{m \wedge n} q^{k^2/2} (-z_1 z_2)^{-k} (q^{k+1}, q^{m-k+1}, q^{n-k+1}; q)_\infty. \end{aligned}$$

□

The application of this theorem to derive sharp bounds for the zeros of $p_{m,n}$ will be done in Section 8.

4. A SECOND q -ANALOGUE

Our second q -analogue is defined by the explicit representation

$$\begin{aligned} & h_{m,n}(z_1, z_2|q) \\ (4.1) \quad & := (-1)^n (q^{m-n+1}; q)_n z_1^{m-n} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ q^{m-n+1} \end{matrix} \middle| q; -q^{m+1} z_1 z_2 \right) \\ & = q^{mn} z_1^m z_2^n \sum_{j=0}^\infty \frac{(q^{-m}, q^{-n}; q)_j}{(q; q)_j} \left(\frac{-q}{z_1 z_2} \right)^j. \end{aligned}$$

Equivalently

$$(4.2) \quad h_{m,n}(z_1, z_2|q) = \sum_{j=0}^{m \wedge n} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q q^{(m-j)(n-j)} (-1)^j (q; q)_j z_1^{m-j} z_2^{n-j}.$$

Note the $(m, n) - (z_1, z_2)$ symmetry

$$(4.3) \quad h_{m,n}(z_1, z_2|q) = h_{n,m}(z_2, z_1|q).$$

It is easy to see that

$$(4.4) \quad \begin{bmatrix} m \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-m)} \begin{bmatrix} m \\ n \end{bmatrix}_q.$$

Therefore

$$(4.5) \quad h_{m,n}(z_1, z_2|1/q) = q^{-mn} i^{-m-n} H_{m,n}(iz_1, iz_2|q).$$

Theorem 7. *The polynomials $\{h_{m,n}(z_1, z_2|q)\}$ have the following properties:*

$$(4.6) \quad \sum_{m,n=0}^{\infty} \frac{h_{m,n}(z_1, z_2|q)}{(q; q)_m (q; q)_n} q^{(m-n)^2/2} u^m v^n = \frac{(-q^{1/2}uz_1, -q^{1/2}vz_2; q)_{\infty}}{(-uv; q)_{\infty}},$$

$$(4.7) \quad h_{m,n}(z_1q^{-1}, z_2|q) = h_{m,n}(z_1, z_2|q) + z_1(1 - q^m)q^{-m}h_{m-1,n}(z_1, z_2|q),$$

$$(4.8) \quad h_{m,n}(z_1, z_2q^{-1}|q) = h_{m,n}(z_1, z_2|q) + z_2(1 - q^n)q^{-n}h_{m,n-1}(z_1, z_2|q),$$

$$(4.9) \quad q^m h_{m,n}(z_1/q, z_2|q) = h_{m,n}(z_1, z_2|q) + (1 - q^m)(1 - q^n)q^{1-m-n}h_{m-1,n-1}(z_1, z_2|q),$$

$$(4.10) \quad q^n h_{m,n}(z_1, z_2/q|q) = h_{m,n}(z_1, z_2|q) + (1 - q^m)(1 - q^n)q^{1-m-n}h_{m-1,n-1}(z_1, z_2|q),$$

$$(4.11) \quad q^n z_1 h_{m,n}(z_1, z_2|q) = h_{m+1,n}(z_1, z_2|q) + (1 - q^n)h_{m,n-1}(z_1, z_2|q),$$

and

$$(4.12) \quad q^m z_2 h_{m,n}(z_1, z_2|q) = h_{m,n+1}(z_1, z_2|q) + (1 - q^m)h_{m-1,n}(z_1, z_2|q).$$

Moreover they have the Rodrigues type formula

$$(4.13) \quad h_{m,n}(z_1, z_2|q) = (q - 1)^{m+n} (-z_1 z_2; q)_{\infty} D_{q, z_2}^m D_{q, z_1}^n \frac{1}{(-z_1 z_2; q)_{\infty}}.$$

Furthermore we also have the operational formula

$$(4.14) \quad h_{m,n}(z_1, z_2|q) = \frac{q^{mn}}{(-q^{-1}(1 - q)^2 D_{q^{-1}, z_1} D_{q^{-1}, z_2}; q)_{\infty}} z_1^m z_2^n.$$

Before proving Theorem 7 we explore some of its consequences. First note that (4.7)–(4.8) describe lowering operators. Indeed they can be written as

$$(4.15) \quad D_{q^{-1}, z_1} h_{m,n}(z_1, z_2|q) = \frac{q^{-m} - 1}{q^{-1} - 1} h_{m-1,n}(z_1, z_2|q)$$

and

$$(4.16) \quad D_{q^{-1}, z_2} h_{m,n}(z_1, z_2|q) = \frac{q^{-n} - 1}{q^{-1} - 1} h_{m,n-1}(z_1, z_2|q),$$

respectively. The raising operators come from the Rodrigues type formula (4.13). We have

$$(4.17) \quad h_{m+1,n}(z_1, z_2|q) = (q - 1)(-z_1z_2; q)_\infty D_{q,z_1} \left(\frac{h_{m,n}(z_1, z_2|q)}{(-z_1z_2; q)_\infty} \right),$$

$$(4.18) \quad h_{m,n+1}(z_1, z_2|q) = (q - 1)(-z_1z_2; q)_\infty D_{q,z_2} \left(\frac{h_{m,n}(z_1, z_2|q)}{(-z_1z_2; q)_\infty} \right).$$

Proof of Theorem 7. The generating function (4.6) follows from

$$\begin{aligned} & \sum_{m,n=0}^\infty \frac{h_{m,n}(z_1, z_2|q)}{(q; q)_m (q; q)_n} q^{(m-n)^2/2} u^m v^n \\ &= \sum_{j=0}^\infty \frac{(-uv)^j}{(q; q)_j} \sum_{m=j}^\infty \frac{(uz_1)^{m-j} q^{(m-j)^2/2}}{(q; q)_{m-j}} \sum_{n=j}^\infty \frac{(vz_2)^{n-j} q^{(n-j)^2/2}}{(q; q)_{n-j}} \\ &= \sum_{j=0}^\infty \frac{(-uv)^j}{(q; q)_j} \left(-q^{1/2}uz_1, -q^{1/2}vz_2; q \right)_\infty \\ &= \frac{\left(-q^{1/2}uz_1, -q^{1/2}vz_2; q \right)_\infty}{(-uv; q)_\infty}. \end{aligned}$$

Let $z_1 \rightarrow z_1/q$ in (4.6). Then

$$\frac{\left(-q^{1/2}uz_1, -q^{1/2}vz_2; q \right)_\infty}{(-quv; q)_\infty} = \left(1 + uz_1q^{-1/2} \right) \frac{\left(-q^{1/2}uz_1, -q^{1/2}vz_2; q \right)_\infty}{(-quv; q)_\infty}$$

implies (4.7). Letting $z_2 \rightarrow z_2/q$ in (4.6) we get (4.8).

Let $u \rightarrow uq, z_1 \rightarrow z_1/q$ in (4.6); from

$$\frac{\left(-q^{1/2}uz_1, -q^{1/2}vz_2; q \right)_\infty}{(-q^2uv; q)_\infty} = (1 + quv) \frac{\left(-q^{1/2}uz_1, -q^{1/2}vz_2; q \right)_\infty}{(-quv; q)_\infty}$$

we get (4.9). Similarly let $v \rightarrow vq, z_2 \rightarrow z_2/q$ in (4.6) to get (4.10). To prove (4.13) we first note that

$$D_{q,z_1} \frac{1}{(-z_1z_2; q)_\infty} = -\frac{z_2}{1 - q} \frac{1}{(-z_1z_2; q)_\infty}.$$

Therefore the right-hand side of (4.13) is

$$\begin{aligned} & (q - 1)^m (-z_1z_2; q)_\infty D_{q,z_2}^m \left(\frac{1}{(-z_1z_2; q)_\infty} z_2^n \right) \\ &= (-1)^m \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q (-z_1)^k \eta_{q,z_2}^k \frac{(q; q)_n}{(q; q)_{n-m+k}} z_2^{n-m+k} \\ &= \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_q z_1^k (-1)^{m-k} \frac{(q; q)_n}{(q; q)_{n-m+k}} z_2^{n-m+k} q^{k(n-m+k)}. \end{aligned}$$

Replacing k by $m - k$ we get the left-hand side of (4.13). We now come to (4.14). It is clear that

$$D_{q^{-1}}^k z^r = \frac{q^{-r} - 1}{q^{-1} - 1} \dots \frac{q^{k-r-1} - 1}{q^{-1} - 1} z^{r-k} = \frac{(q; q)_r}{(q; q)_{r-k}} \frac{q^{\binom{k+1}{2} - kr}}{(1 - q)^k} z^{r-k}.$$

The right-hand side of (4.14) is

$$q^{mn} \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k (q; q)_k q^{k^2 - km - kn} z_1^{m-k} z_2^{n-k},$$

and the proof is complete. □

We give multiplication formulas for the polynomials $\{h_{m,n}(z_1, z_2; q)\}$ in the following theorem.

Theorem 8. *We have*

$$(4.19) \quad h_{m,n}(az_1, bz_2; q) = \sum_{j=0}^{m \wedge n} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q (q, ab; q)_j a^{m-j} b^{n-j} h_{m-j, n-j}(z_1, z_2; q).$$

Proof. The desired result follows from the generating function (4.6) and the q -binomial theorem (2.5). □

We now identify the q -Sturm–Liouville problems whose eigenvalues are $\{h_{m,n}(z_1, z_2|q)\}$. Combine (4.15) and (4.17) to obtain

$$(4.20) \quad -\frac{1}{w(z_1 z_2)} D_{q, z_1} (w(z_1 z_2) D_{q^{-1}, z_1} h_{m,n}(z_1, z_2|q)) = \frac{1 - q^m}{(1 - q)^2} q^{1-m} h_{m,n}(z_1, z_2|q),$$

where

$$(4.21) \quad w(z) := 1/(-z; q)_{\infty}.$$

Similarly

$$(4.22) \quad -\frac{1}{w(z_1 z_2)} D_{q, z_2} (w(z_1 z_2) D_{q^{-1}, z_2} h_{m,n}(z_1, z_2|q)) = \frac{1 - q^n}{(1 - q)^2} q^{1-n} h_{m,n}(z_1, z_2|q).$$

We note the relation

$$(4.23) \quad h_{m,n}(z_1, z_2|q) = (-1)^n (q; q)_n z_1^{m-n} L_n^{(m-n)}(z_1 z_2; q).$$

Theorem 9. *The polynomials $\{h_{m,n}(z, \bar{z}|q)\}$ satisfy the orthogonality*

$$(4.24) \quad \int_{\mathbb{R}^2} h_{m,n}(z, \bar{z}|q) \overline{h_{s,t}(z, \bar{z}|q)} \frac{dx dy}{(-z\bar{z}; q)_{\infty}} = \frac{\pi \log q^{-1} (q; q)_m (q; q)_n}{q^{(m-n)^2/2 + (m+n)/2}} \delta_{m,s} \delta_{n,t},$$

where $m, n, s, t \in \mathbb{N}_0$.

Proof. There is no loss of generality in assuming that $m \geq n$ because of the symmetry property (4.3). Apply (4.23) and change into polar coordinates to get

$$\begin{aligned} & \int_{\mathbb{R}^2} h_{m,n}(z, \bar{z}|q) \overline{h_{s,t}(z, \bar{z}|q)} \frac{dx dy}{(-z\bar{z}; q)_\infty} \\ &= (-1)^{n+t} (q; q)_n (q; q)_t \int_0^{2\pi} e^{i\theta(m-n+t-s)} d\theta \\ &\times \int_0^\infty L_n^{(m-n)}(r^2; q) L_t^{(s-t)}(r^2; q) \frac{r^{m-n+s-t+1} dr}{(-r^2; q)_\infty} \\ &= (-1)^{n+t} \pi (q; q)_n (q; q)_t \delta_{m-n+t-s,0} \\ &\times \int_0^\infty L_n^{(m-n)}(x; q) L_t^{(m-n)}(x; q) \frac{x^{(m-n)} dx}{(-x; q)_\infty} \\ &= (-1)^{n+t} \pi (q; q)_n (q; q)_t \delta_{m-n+t-s,0} \\ &\times (\log q^{-1}) q^{-(m-n)^2/2-(m+n)/2} (q^{n+1}; q)_{m-n} \delta_{n,t} \\ &= \frac{\pi (\log q^{-1}) (q; q)_m (q; q)_n}{q^{(m-n)^2/2+(m+n)/2}} \delta_{m,s} \delta_{n,t}, \end{aligned}$$

and the proof is complete. □

Note that the orthogonality relation (4.24) and the generating function (4.6) give rise to the integral

$$\begin{aligned} (4.25) \quad & \int_{\mathbb{R}^2} \frac{(-q^{1/2}u_1z, -q^{1/2}v_1\bar{z}, -q^{1/2}v_2z, -q^{1/2}u_2\bar{z}; q)_\infty}{(-z\bar{z}; q)_\infty} dx dy \\ &= \pi \ln q^{-1} \frac{(-u_1v_1, -u_2v_2, -u_1u_2, -v_1v_2; q)_\infty}{(u_1u_2v_1v_2; q)_\infty}. \end{aligned}$$

It is clear that (4.25) is a q -beta integral.

Since the associated moment problem for $L_n^{(\alpha)}(x; q)$ is indeterminate, there exist infinitely many orthogonal measures. Let $x^\alpha d\mu(x)$ be such a measure, for example,

$$\begin{aligned} d\mu(x) &= x^{-\alpha} w_{QL}(x; \alpha, c, \lambda) dx, \quad \alpha, \lambda, c > 0, \\ d\mu(x) &= \frac{x}{(-x; q)_\infty} \delta(x - cq^y), \quad y \in \mathbb{Z}, c, \alpha + 1 > 0, \end{aligned}$$

etc. It is clear that our proof shows that

$$d\sigma(re^{i\theta}, re^{-i\theta}) = \frac{1}{2} d\theta d\mu(r^2), \quad r \in \mathbb{R}^+, \theta \in [0, 2\pi],$$

is also an orthogonal measure for $h_{m,n}(re^{i\theta}, re^{-i\theta}|q)$ where $r \in \mathbb{R}^+, \theta \in [0, 2\pi]$.

Theorem 10. Assume that $q = e^{-2k^2}$ and $|q| < 1$. Then we have

$$\begin{aligned} (4.26) \quad & (-aqe^{2imk}, -bqe^{-2imk}; q)_\infty = \sum_{s,t=0}^\infty \frac{H_{s,t}(a, b|q)}{(q; q)_s (q; q)_t} q^{\frac{(s-t)^2}{2}} \left(q^{\frac{1}{2}}e^{2imk}\right)^s \left(q^{\frac{1}{2}}e^{-2imk}\right)^t, \\ (4.27) \quad & \frac{(ab; q)_\infty}{(-ab; ae^{2mk}, be^{-2mk}; q)_\infty} = \sum_{s,t=0}^\infty \frac{h_{s,t}(a, b|q)}{(q; q)_s (q; q)_t} \left(q^{\frac{1}{2}}e^{mk}\right)^s \left(q^{\frac{1}{2}}e^{-mk}\right)^t, \end{aligned}$$

and for any $0 < q < 1$

$$(4.28) \quad \frac{(q^{a+b}; q)_\infty}{(-q^{a+b}, q^{a+c}, q^{b-c}; q)_\infty} = \sum_{m,n=0}^\infty \frac{h_{m,n}(q^a, q^b|q)}{(q; q)_m (q; q)_n} q^{c(m-n)},$$

where $\Re(a + c) > 0$ and $\Re(b - c) > 0$.

Proof. Equation (4.28) could be proved either from equations (2.19), (2.21) or from equations (2.20), (2.22):

$$\begin{aligned} & \frac{(q, -q^c, -q^{1-c}, q^{a+b}; q)_\infty}{(-1, -q, -q^{a+b}, q^{a+c}, q^{b-c}; q)_\infty} = \int_0^\infty \frac{(-q^{a+1}/t, -tq^b; q)_\infty t^{c-1} d_q t}{(-q^{a+b} - t, -q/t; q)_\infty (1 - q)} \\ &= \sum_{m,n=0}^\infty \frac{h_{m,n}(q^{a+\frac{1}{2}}, q^{b-\frac{1}{2}}|q)}{(q; q)_m (q; q)_n} q^{\frac{(m-n)^2}{2}} \int_0^\infty \frac{t^{c+n-m-1} d_q t}{(-t, -q/t; q)_\infty (1 - q)} \\ &= \sum_{m,n=0}^\infty \frac{h_{m,n}(q^{a+\frac{1}{2}}, q^{b-\frac{1}{2}}|q)}{(q; q)_m (q; q)_n} q^{\frac{(m-n)^2}{2}} \frac{(q, -q^{c+n-m}, -q^{m-n-c+1}; q)_\infty}{(-1, -q; q)_\infty}. \end{aligned}$$

Without loss of generality we may assume that $m \geq n$:

$$\begin{aligned} & \frac{(q^{a+b}, q)_\infty}{(-q^{a+b}, q^{a+c}, q^{b-c}; q)_\infty} \\ &= \sum_{m,n=0}^\infty \frac{h_{m,n}(q^a, q^b|q)}{(q; q)_m (q; q)_n} q^{\frac{(m-n)^2}{2}} \frac{(-q^{c+n-m+\frac{1}{2}}, -q^{m-n-c+\frac{1}{2}}; q)_\infty}{(-q^{c+\frac{1}{2}}, -q^{\frac{1}{2}-c}; q)_\infty} \\ &= \sum_{m,n=0}^\infty \frac{h_{m,n}(q^a, q^b|q)}{(q; q)_m (q; q)_n} q^{c(m-n)}. \end{aligned}$$

For $q = e^{-2k^2}$ and $|q| < 1$, observe that

$$\begin{aligned} & \sqrt{\pi} e^{m^2} (-aqe^{2imk}, -bqe^{-2imk}; q)_\infty = \int_{-\infty}^\infty \frac{(abq; q)_\infty e^{-x^2+2mx} dx}{(ae^{2ikx} \sqrt{q}, be^{-2ikx} \sqrt{q}; q)_\infty} \\ &= \sum_{s,t=0}^\infty \frac{H_{s,t}(a, b|q)}{(q; q)_s (q; q)_t} q^{(s+t)/2} \int_{-\infty}^\infty e^{-x^2+2mx+2iksx-2ikt x} dx \\ &= \sqrt{\pi} e^{m^2} \sum_{s,t=0}^\infty \frac{H_{s,t}(a, b|q)}{(q; q)_s (q; q)_t} q^{(s-t)^2/2} \left(q^{1/2} e^{2imk}\right)^s \left(q^{1/2} e^{-2imk}\right)^t. \end{aligned}$$

Equation (4.27) is proved in a similar fashion. □

From (4.23) and (3.19) we get

Corollary 11. *If $q = e^{-2k^2}$ and $|q| < 1$, then we have*

$$(4.29) \quad \begin{aligned} &(-aqe^{2imk}, -bqe^{-2imk}; q)_\infty \\ &= \sum_{s,t=0}^\infty \frac{p_t \left(ab, q^{s-t} \middle| q \right)}{(q; q)_s (q; q)_t} q^{\frac{(s-t)^2}{2}} \left(aq^{\frac{1}{2}} e^{2imk} \right)^s \left(a^{-1} q^{\frac{1}{2}} e^{-2imk} \right)^t, \end{aligned}$$

$$(4.30) \quad \begin{aligned} &\frac{(ab; q)_\infty}{(-ab; ae^{2mk}, be^{-2mk}; q)_\infty} \\ &= \sum_{s,t=0}^\infty \frac{L_t^{(s-t)}(ab; q)}{(q; q)_s} \left(aq^{\frac{1}{2}} e^{mk} \right)^s \left(-a^{-1} q^{\frac{1}{2}} e^{-mk} \right)^t, \end{aligned}$$

and for any $0 < q < 1$

$$(4.31) \quad \frac{(q^{a+b}; q)_\infty}{(-q^{a+b}, q^{a+c}, q^{b-c}; q)_\infty} = \sum_{m,n=0}^\infty \frac{L_n^{(m-n)}(q^{a+b}; q)}{(q; q)_m} (-1)^n q^{(a+c)(m-n)},$$

where $\Re(a + c) > 0$ and $\Re(b - c) > 0$, where $L_n^{(\alpha)}(x; q)$ and $p_n(x, a|q)$ are the q -Laguerre and little q -Laguerre polynomials respectively.

The next theorem gives the Plancherel–Rotach asymptotics of the polynomials $\{h_{m,n}(z_1, z_2)\}$.

Theorem 12. *For $a, b \in \mathbb{C}$ and $0 < \epsilon < 1$ the following asymptotic result holds uniformly when w_1, w_2 are in compact subsets of the complex plane:*

$$(4.32) \quad \lim_{m,n \rightarrow \infty} \frac{h_{m,n}(w_1 q^{-am-bn}, w_2 q^{-(1-a)m-(1-b)n} | q) (q; q)_\infty^2}{w_1^m w_2^n q^{(a-b)mn-am^2+(b-1)n^2}} = A_q \left(\frac{1}{w_1 w_2} \right).$$

Proof. From definition (4.2) it follows that

$$\begin{aligned} &\frac{h_{m,n}(w_1 q^{-am-bn}, w_2 q^{-(1-a)m-(1-b)n} | q) (q; q)_\infty^2}{w_1^m w_2^n q^{(a-b)mn-am^2+(b-1)n^2}} \\ &= \sum_{j=0}^{m \wedge n} \frac{(-w_1 w_2)^{-j} q^{j^2}}{(q; q)_j} (q^{m-j+1}, q^{n-j+1}; q)_\infty \rightarrow A_q \left(\frac{1}{w_1 w_2} \right), \end{aligned}$$

as $m, n \rightarrow \infty$. □

5. 2D q -ULTRASPHERICAL POLYNOMIALS

The 2D-ultraspherical polynomials are

$$(5.1) \quad C_{m,n}^\nu(z_1, z_2) = \sum_{k=0}^{m \wedge n} \binom{m}{k} \binom{n}{k} k! (-1)^k (\nu)_{m+n-k} z_1^{m-k} z_2^{n-k}, \quad \nu > -1.$$

They are also known as the disk polynomials or the Zernike polynomials [8].

It is clear that

$$(5.2) \quad C_{m,n}^\nu(z_1, z_2) = (\nu)_{m+n} z_1^m z_2^n {}_2F_1 \left(\begin{matrix} -m, -n \\ -m - n - \nu + 1 \end{matrix} \middle| \frac{1}{z_1 z_2} \right),$$

which is a constant multiple of the disk polynomials of Section 2.3 in [8].

They have the generating function

$$(5.3) \quad \sum_{m,n=0}^{\infty} C_{m,n}^{\nu}(z_1, z_2) \frac{u^m v^n}{(m! n!)} = (1 - uz_1 - vz_2 + uv)^{-\nu},$$

whose proof is an exercise in the application of the binomial theorem. Next we solve the connection relation between $C_{m,n}^{\nu}(z_1, z_2)$ and $H_{m,n}(z_1, z_2)$. We claim that

$$(5.4) \quad \begin{aligned} C_{m,n}^{\nu}(z_1, z_2) &= \sum_{p=0}^{m \wedge n} \frac{(\nu)_{m+n} m! n!}{p! (m-p)! (n-p)!} H_{m-p,n-p}(z_1 z_2) {}_1F_1(-p; -\nu - m - n + 1; -1). \end{aligned}$$

Write the right-hand side of (5.3) as

$$\begin{aligned} \int_0^{\infty} \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-t+tz_1+tvz_2-tuv} dt &= \int_0^{\infty} \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-t+t(t-1)uv} \sum_{r,s=0}^{\infty} H_{r,s}(z_1 z_2) \frac{u^r v^s}{r! s!} t^{r+s} dt \\ &= \sum_{r,s=0}^{\infty} H_{r,s}(z_1 z_2) \frac{u^r v^s}{r! s!} \sum_{j,k=0}^{\infty} \frac{(-1)^j (uv)^{j+k} \Gamma(\nu+r+s+j+2k)}{j! k! \Gamma(\nu)}. \end{aligned}$$

Equating coefficients of $u^m v^n$ we see that $m = r + j + k, n = s + j + k$. Let $p = j + k$. Now (5.4) follows after some manipulations.

It is clear from the generating function (5.3) that the disk polynomials have the convolution property

$$(5.5) \quad C_{m,n}^{\nu+\mu}(z_1, z_2) = \sum_{j=0}^m \sum_{k=0}^n C_{j,k}^{\mu}(z_1, z_2) C_{m-j,n-k}^{\nu}(z_1, z_2).$$

Floris [11] (see also [10] and [12]) introduced the following q -analogue of the disk polynomials. For $\alpha, \beta > -1$ and $l, m \in \mathbb{Z}_+$, the Floris q -disk polynomials $R_{l,m}^{(\alpha)}(z, z^*; q^2)$ are defined by

$$R_{l,m}^{(\alpha)}(z, z^*; q^2) = \begin{cases} z^{l-m} P_m^{(\alpha, l-m)}(1 - zz^*; q^2), & l \geq m, \\ P_m^{(\alpha, m-l)}(1 - zz^*; q^2) (z^*)^{m-l}, & l \leq m, \end{cases}$$

where

$$z^* z = q^2 z z^* + 1 - q^2$$

and

$$P_m^{(\alpha, \beta)}(x; q) = p_m(x; q^{\alpha}, q^{\beta}; q)$$

is the little q -Jacobi polynomials. The q -disk polynomials $R_{l,m}^{(\alpha)}(z, z^*; q^2)$ satisfy

$$R_{l,m}^{(\alpha)}(z, z^*; q^2)^* = R_{m,l}^{(\alpha)}(z, z^*; q^2)$$

and the orthogonality relation

$$\begin{aligned} \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} R_{l,m}^{(\alpha)}(e^{i\theta} z, e^{-i\theta} z^*; q^2)^* R_{l',m'}^{(\alpha)}(e^{i\theta} z, e^{-i\theta} z^*; q^2) d\theta (1 - zz^*)^{\alpha} d_{q^2}(1 - zz^*) \\ = \frac{(1 - q^2) (q^2; q^2)_l (q^2; q^2)_m q^{2m(\alpha+1)} \delta_{ll'} \delta_{mm'}}{(1 - q^{2(\alpha+l+m+1)}) (q^{2(\alpha+1)}; q^2)_l (q^{2(\alpha+1)}; q^2)_m}, \end{aligned}$$

for $\alpha > -1, l, l', m, m' \in \mathbb{Z}_+$.

We now introduce our q -analogue of these polynomials. For $0 < q < 1$ and $b < q^{-1}$, let us define

$$(5.6) \quad p_{m,n}(z_1, z_2; b|q) = \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k}{2}} (q; q)_k (bq; q)_{m+n-k}}{(-1)^k z_1^{k-m} z_2^{k-n}}.$$

It is clear that

$$(5.7) \quad p_{m,n}(z_2, z_1; b|q) = p_{n,m}(z_1, z_2; b|q).$$

Then for $m \geq n$ we have

$$(5.8) \quad p_{m,n}(z_1, z_2; b|q) = (-1)^n q^{\binom{n}{2}} (bq; q)_m (q^{m-n+1}; q)_n z_1^{m-n} p_n(z_1 z_2; q^{m-n}, b|q),$$

where $p_n(x; a, b|q)$ are the little Jacobi polynomials.

Theorem 13. For $0 < q < 1$ and $b < q^{-1}$, the polynomials $\{p_{m,n}(z, \bar{z}; b|q)\}$ satisfy the orthogonality

$$(5.9) \quad \int_{\mathbb{C}} p_{m,n}(z, \bar{z}; b|q) \overline{p_{s,t}(z, \bar{z}; b|q)} d\mu(z, \bar{z}) = \frac{(bq; q)_{\infty} q^{mn} (q, bq; q)_m (q, bq; q)_n}{(q; q)_{\infty} 1 - bq^{m+n+1}} \delta_{m,s} \delta_{n,t},$$

where

$$(5.10) \quad d\mu(z, \bar{z}) = \frac{d\theta}{2\pi} \otimes \sum_{k=0}^{\infty} \frac{(bq; q)_k q^k}{(q; q)_k} \delta(r - q^{k/2}),$$

$z = re^{i\theta}, r \in \mathbb{R}^+, \theta \in [0, 2\pi]$ and $m, n, s, t \in \mathbb{N}_0$.

Proof. Because of the symmetry (5.7), we may assume that $m \geq n$. We first use polar coordinates, then apply (5.8) and the orthogonality relation of the little Jacobi polynomials to get

$$\begin{aligned} & \int_{\mathbb{C}} p_{m,n}(z, \bar{z}; b|q) \overline{p_{s,t}(z, \bar{z}; b|q)} d\mu(z, \bar{z}) \\ &= (-1)^{n+t} q^{\binom{n}{2} + \binom{t}{2}} (bq; q)_m (bq; q)_s (q^{m-n+1}; q)_n (q^{s-t+1}; q)_t \\ & \times \int_0^{2\pi} e^{i\theta(m-n+t-s)} \frac{d\theta}{2\pi} \sum_{k=0}^{\infty} \frac{(bq; q)_k q^k r^{m-n+s-t}}{(q; q)_k} \\ & \times \delta(r - q^{k/2}) p_n(r^2; q^{m-n}, b|q) p_t(r^2; q^{s-t}, b|q) \\ &= (-1)^{n+t} q^{\binom{n}{2} + \binom{t}{2}} (bq; q)_m (bq; q)_s (q^{m-n+1}; q)_n (q^{m-n+1}; q)_t \\ & \times \sum_{k=0}^{\infty} \frac{(bq; q)_k q^{k(m-n+1)}}{(q; q)_k} p_n(q^k; q^{m-n}, b|q) p_t(q^k; q^{m-n}, b|q) \delta_{m-n+t-s,0} \\ &= \frac{(bq; q)_{\infty} q^{mn} (q, bq; q)_m (q, bq; q)_n}{(q; q)_{\infty} 1 - bq^{m+n+1}} \delta_{m,s} \delta_{n,t}. \end{aligned}$$

This completes the proof. □

Theorem 14. *The polynomials $\{p_{m,n}(z_1, z_2; b|q)\}$ have the generating function*

$$(5.11) \quad \sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n = \frac{(bq, uv; q)_{\infty}}{(uz_1, z_2v; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2 \\ uv \end{matrix} \middle| q; bq \right).$$

For $bq, cq < 1$ and $b \neq 0$, the connection relation between the q -2D ultraspherical polynomials is given by

$$(5.12) \quad \frac{p_{m,n}(z_1, z_2; b|q)}{(bq; q)_{\infty}} = \sum_{j=0}^{\infty} \frac{\left(\frac{c}{b}; q\right)_j}{(q; q)_j} \left(bq^{\frac{m+n}{2}+1}\right)^j \frac{p_{m,n}\left(z_1q^{\frac{j}{2}}, z_2q^{\frac{j}{2}}; c|q\right)}{(cq; q)_{\infty}}.$$

The connection relation between the q -2D ultraspherical and q -2D Hermite is given by

$$(5.13) \quad \frac{p_{m,n}(z_1, z_2; b|q)}{(bq; q)_{\infty}} = \sum_{j=0}^{\infty} \frac{(bq^{(m+n)/2+1})^j}{(q; q)_j} H_{m,n}\left(z_1q^{j/2}, z_2q^{j/2}|q\right).$$

Moreover we have the inverse relation

$$(5.14) \quad H_{m,n}(z_1, z_2|q) = \sum_{k=0}^{\infty} \frac{(-bq^{(m+n)/2+1})^k}{(bq; q)_{\infty} (q; q)_k} q^{\binom{k}{2}} p_{m,n}\left(z_1q^{k/2}, z_2q^{k/2}; b|q\right).$$

Proof. Using the explicit definition (5.8) we see that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\ &= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-uv)^k}{(q; q)_k} \sum_{m,n \geq k} \frac{(bq; q)_{m+n-k} (z_1u)^{m-k} (z_2v)^{n-k}}{(q; q)_{m-k} (q; q)_{n-k}} \\ &= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-uv)^k}{(q; q)_k} \sum_{m,n=0}^{\infty} \frac{(bq; q)_{m+n+k} (z_1u)^m (z_2v)^n}{(q; q)_m (q; q)_n} \\ &= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-uv)^k}{(q; q)_k} \sum_{m,n=0}^{\infty} \frac{(z_1u)^m (z_2v)^n}{(q; q)_m (q; q)_n} \frac{(bq; q)_{\infty}}{(bq^{m+n+k+1}; q)_{\infty}}. \end{aligned}$$

We then expand $1/(bq^{m+n+k+1}; q)_{\infty}$ using (2.8) and write the above expression as

$$\begin{aligned} &= (bq; q)_{\infty} \sum_{j=0}^{\infty} \frac{(bq)^j}{(q; q)_j} \sum_{m,n=0}^{\infty} \frac{(z_1uq^j)^m (z_2vq^j)^n}{(q; q)_m (q; q)_n} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (-uvq^j)^k}{(q; q)_k} \\ &= (bq; q)_{\infty} \sum_{j=0}^{\infty} \frac{(bq)^j}{(q; q)_j} \sum_{m,n=0}^{\infty} \frac{(z_1uq^j)^m (z_2vq^j)^n}{(q; q)_m (q; q)_n} (uvq^j; q)_{\infty} \\ &= (bq, uv; q)_{\infty} \sum_{j=0}^{\infty} \frac{(bq)^j}{(q, uv; q)_j} \frac{1}{(uz_1q^j, z_2vq^j; q)_{\infty}} \\ &= \frac{(bq, uv; q)_{\infty}}{(uz_1, z_2v; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(uz_1, z_2v; q)_j (bq)^j}{(q, uv; q)_j} = \frac{(bq, uv; q)_{\infty}}{(uz_1, z_2v; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2 \\ uv \end{matrix} \middle| q; bq \right). \end{aligned}$$

From the above calculations we see that the generating function could also be written as

$$\sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n = (bq; q)_{\infty} \sum_{j=0}^{\infty} \frac{(bq)^j}{(q; q)_j} \frac{(uvq^j; q)_{\infty}}{(uz_1q^j, z_2vq^j; q)_{\infty}},$$

and we find that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\ &= (bq; q)_{\infty} \sum_{m,n=0}^{\infty} \frac{u^m v^n}{(q; q)_m (q; q)_n} \sum_{j=0}^{\infty} \frac{(bq^{(m+n)/2+1})^j}{(q; q)_j} H_{m,n}(z_1q^{j/2}, z_2q^{j/2}|q), \end{aligned}$$

which gives (5.13). We now prove (5.14). From $\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} = \delta_{n,0}$ we get

$$\begin{aligned} & \frac{1}{(bq; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-bq^{(m+n)/2+1})^k}{(q; q)_k} q^{\binom{k}{2}} p_{m,n}(z_1q^{k/2}, z_2q^{k/2}; b|q) \\ &= \sum_{j,k=0}^{\infty} \frac{(-bq^{(m+n)/2+1})^{k+j}}{(q; q)_k (q; q)_j} q^{\binom{k}{2}} H_{m,n}(z_1q^{(k+j)/2}, z_2q^{(k+j)/2}|q) \\ &= \sum_{\ell=0}^{\infty} \frac{(-bq^{(m+n)/2+1})^{\ell}}{(q; q)_{\ell}} H_{m,n}(z_1q^{\ell/2}, z_2q^{\ell/2}|q) \sum_{k=0}^{\ell} \begin{bmatrix} \ell \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} \\ &= H_{m,n}(z_1, z_2|q). \end{aligned}$$

The connection formula between $p_{m,n}(z_1, z_2; b|q)$ polynomials can be proved directly by observing that

$$\begin{aligned} p_{m,n}(z_1, z_2; b|q) &= \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (q; q)_k (-1)^k z_1^{m-k} z_2^{n-k} \\ &\quad \times (cq; q)_{m+n-k} \frac{(bq; q)_{\infty}}{(cq; q)_{\infty}} \frac{(cq^{m+n-k+1}; q)_{\infty}}{(bq^{m+n-k+1}; q)_{\infty}} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (q; q)_k (-1)^k z_1^{m-k} z_2^{n-k} \\ &\quad \times (cq; q)_{m+n-k} \frac{(bq; q)_{\infty}}{(cq; q)_{\infty}} \sum_{j=0}^{\infty} \frac{\binom{c}{b}; q}_j}{(q; q)_j} (bq^{m+n-k+1})^j \\ &= \frac{(bq; q)_{\infty}}{(cq; q)_{\infty}} \sum_{j=0}^{\infty} \frac{\binom{c}{b}; q}_j}{(q; q)_j} (bq^{\frac{m+n}{2}+1})^j \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \\ &\quad \times q^{\binom{k}{2}} (q; q)_k (-1)^k (cq; q)_{m+n-k} (z_1q^{\frac{j}{2}})^{m-k} (z_2q^{\frac{j}{2}})^{n-k} \\ &= \frac{(bq; q)_{\infty}}{(cq; q)_{\infty}} \sum_{j=0}^{\infty} \frac{\binom{c}{b}; q}_j}{(q; q)_j} (bq^{\frac{m+n}{2}+1})^j p_{m,n}(z_1q^{\frac{j}{2}}, z_2q^{\frac{j}{2}}; c|q). \end{aligned}$$

This completes the proof of our theorem. □

Let us rewrite (5.11) in the form

$$(5.15) \quad {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2 \\ uv \end{matrix} \middle| q; bq \right) = \frac{(uz_1, z_2v; q)_\infty}{(bq, uv; q)_\infty} \sum_{m,n=0}^\infty \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n.$$

Theorem 15. *The polynomials $\{p_{m,n}(z_1, z_2; b|q)\}$ satisfy the following properties:*

$$(5.16) \quad D_{q,z_1} p_{m,n}(z_1, z_2; b|q) = \frac{(1-bq)}{1-q} (1-q^m) p_{m-1,n}(z_1, z_2; bq|q),$$

$$(5.17) \quad D_{q,z_2} p_{m,n}(z_1, z_2; b|q) = \frac{(1-bq)}{1-q} (1-q^n) p_{m,n-1}(z_1, z_2; bq|q),$$

$$(5.18) \quad D_{q^{-1},z_1} \left(\frac{(qz_1z_2; q)_\infty}{(bqz_1z_2; q)_\infty} p_{m,n}(z_1, z_2; b|q) \right) = \frac{(qz_1z_2; q)_\infty p_{m,n+1}(z_1, z_2; bq^{-1}|q)}{q^{m-1}(q-1)(bz_1z_2; q)_\infty},$$

$$(5.19) \quad D_{q^{-1},z_2} \left(\frac{(qz_1z_2; q)_\infty}{(bqz_1z_2; q)_\infty} p_{m,n}(z_1, z_2; b|q) \right) = \frac{(qz_1z_2; q)_\infty p_{m+1,n}(z_1, z_2; bq^{-1}|q)}{q^{n-1}(q-1)(bz_1z_2; q)_\infty},$$

$$(5.20) \quad \begin{aligned} p_{m,n}(z_1q, z_2; b|q) - bq^{n+m-1} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1q, z_2; b|q) \\ = p_{m,n}(z_1, z_2; b|q) q^m - q^{m-1} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1, z_2; b|q), \end{aligned}$$

$$(5.21) \quad \begin{aligned} p_{m,n}(z_1q, z_2; b|q) - bq^{2m-1} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1, z_2q; b|q) \\ = p_{m,n}(z_1, z_2; b|q) - q^{m-1} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1, z_2; b|q), \end{aligned}$$

$$(5.22) \quad \begin{aligned} p_{m,n}(z_1, z_2q; b|q) - bq^{2n-1} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1q, z_2; b|q) \\ = q^n p_{m,n}(z_1, z_2; b|q) - q^{n-1} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1, z_2; b|q), \end{aligned}$$

$$(5.23) \quad \begin{aligned} p_{m,n}(z_1, z_2q; b|q) - bq^{m+n} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1, z_2q; b|q) \\ = q^n p_{m,n}(z_1, z_2; b|q) - q^{n-1} (1-q^m) (1-q^n) p_{m-1,n-1}(z_1, z_2; b|q), \end{aligned}$$

$$(5.24) \quad \begin{aligned} p_{m,n}(z_1q, z_2; b|q) - bq^{n+1} z_1 (1-q^m) p_{m-1,n}(z_1q, z_2; b|q) \\ = p_{m,n}(z_1, z_2; b|q) - z_1 (1-q^m) p_{m-1,n}(z_1, z_2; b|q), \end{aligned}$$

$$(5.25) \quad \begin{aligned} p_{m,n}(z_1q, z_2; b|q) - bq^m z_1 (1-q^m) p_{m-1,n}(z_1, z_2q; b|q) \\ = p_{m,n}(z_1, z_2; b|q) - z_1 (1-q^m) p_{m-1,n}(z_1, z_2; b|q), \end{aligned}$$

$$(5.26) \quad \begin{aligned} (1 + bq^{m+n} z_1) p_{m,n}(z_1, z_2; b|q) \\ = z_1 p_{m-1,n}(z_1, z_2; b|q) - q^{m-1} (1-q^n) (1-bq^n) p_{m-1,n-1}(z_1, z_2; b|q), \end{aligned}$$

$$(5.27) \quad \begin{aligned} (1 - q^{m-n}) p_{m+1,n+1}(z_1, z_2; b|q) \\ = z_1 (1 - bq^{m+1}) (1 - q^{m+1}) p_{m,n+1}(z_1, z_2; b|q) \\ - z_2 q^{m-n} (1 - bq^{n+1}) (1 - q^{n+1}) p_{m+1,n}(z_1, z_2; b|q), \end{aligned}$$

$$(5.28) \quad \begin{aligned} p_{m+1,n+1}(z_1q, z_2; b|q) - p_{m+1,n+1}(z_1, z_2q; b|q) \\ = z_2 (1 - bq^{m+2}) (1 - q^{n+1}) p_{m+1,n}(z_1q, z_2; b|q) \\ - z_1 (1 - bq^{m+1}) (1 - q^{m+1}) p_{m,n+1}(z_1, z_2q; b|q), \end{aligned}$$

$$(5.29) \quad \begin{aligned} & z_2 (1 - q^n) p_{m,n-1} (z_1 q, z_2; b|q) - z_1 (1 - q^m) p_{m-1,n} (z_1, z_2 q; b|q) \\ & = z_2 (1 - q^n) p_{m,n-1} (z_1, z_2; b|q) - z_1 (1 - q^m) p_{m-1,n} (z_1, z_2; b|q), \end{aligned}$$

$$(5.30) \quad \begin{aligned} & qz_1 z_2 (p_{m,n} (z_1 q, z_2; b|q) - p_{m,n} (z_1, z_2 q; b|q)) \\ & + z_1 z_2 (1 - q^m) (1 - q^n) (p_{m-1,n-1} (z_1 q, z_2; b|q) - p_{m-1,n-1} (z_1, z_2 q; b|q)) \\ & = qz_1 z_2^2 (1 - q^n) (p_{m,n-1} (z_1 q, z_2; b|q) - bp_{m,n-1} (z_1 q, z_2 q; b|q)) \\ & - qz_1^2 z_2 (1 - q^m) (p_{m-1,n} (z_1, z_2 q; b|q) - bp_{m-1,n} (z_1 q, z_2 q; b|q)) \\ & + z_1 (1 - q^m) (p_{m-1,n} (z_1 q, z_2; b|q) - p_{m-1,n} (z_1 q, z_2 q; b|q)) \\ & - z_2 (1 - q^n) (p_{m,n-1} (z_1, z_2 q; b|q) - p_{m,n-1} (z_1 q, z_2 q; b|q)). \end{aligned}$$

Proof. Applying the difference operator D_{q,z_1} to the form (5.15) of the generating function we find that

$$\begin{aligned} \sum_{m,n=0}^{\infty} \frac{D_{q,z_1} p_{m,n} (z_1, z_2; b|q)}{(q; q)_m (q; q)_n (bq; q)_{\infty}} u^m v^n &= \sum_{j=0}^{\infty} \frac{(bq)^j}{(q; q)_j} \frac{(uvq^j; q)_{\infty}}{(z_2 vq^j; q)_{\infty}} D_{q,z_1} \frac{1}{(z_1 uq^j; q)_{\infty}} \\ &= \frac{u}{1-q} \sum_{j=0}^{\infty} \frac{(bq^2)^j}{(q; q)_j} \frac{(uvq^j; q)_{\infty}}{(uz_1 q^j, z_2 vq^j; q)_{\infty}} \\ &= \frac{u}{1-q} \sum_{m,n=0}^{\infty} \frac{p_{m,n} (z_1, z_2; bq|q)}{(bq^2; q)_{\infty} (q; q)_m (q; q)_n} u^m v^n \end{aligned}$$

to get (5.16); (5.17) is obtained similarly. On the other hand, from (5.8) and the backward shift operator we get (5.18), and by the symmetry (5.7) we have (5.19). From Heine’s contiguous relation (17.6.17) in [3] we get

$$(5.31) \quad \begin{aligned} & {}_2\phi_1 \left(\begin{matrix} uq^{-1}z_1q, vz_2 \\ uq^{-1}v \end{matrix} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2 \\ uv \end{matrix} \middle| q; bq \right) \\ & = uvb \frac{(1-uz_1)(1-vz_2)}{(1-uvq^{-1})(1-uv)} {}_2\phi_1 \left(\begin{matrix} uz_1q, vqz_2 \\ uvq \end{matrix} \middle| q; bq \right), \end{aligned}$$

$$(5.32) \quad \begin{aligned} & {}_2\phi_1 \left(\begin{matrix} uq^{-1}z_1q, vz_2 \\ uq^{-1}v \end{matrix} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2 \\ uv \end{matrix} \middle| q; bq \right) \\ & = uvb \frac{(1-uz_1)(1-vz_2)}{(1-uvq^{-1})(1-uv)} {}_2\phi_1 \left(\begin{matrix} uqz_1, vz_2q \\ uqv \end{matrix} \middle| q; bq \right), \end{aligned}$$

$$(5.33) \quad \begin{aligned} & {}_2\phi_1 \left(\begin{matrix} uz_1, vq^{-1}z_2q \\ uvq^{-1} \end{matrix} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2 \\ uv \end{matrix} \middle| q; bq \right) \\ & = uvb \frac{(1-uz_1)(1-vz_2)}{(1-uvq^{-1})(1-uv)} {}_2\phi_1 \left(\begin{matrix} uz_1q, vqz_2 \\ uvq \end{matrix} \middle| q; bq \right) \end{aligned}$$

and

$$(5.34) \quad \begin{aligned} & {}_2\phi_1 \left(\begin{matrix} uz_1, vq^{-1}z_2q \\ uvq^{-1} \end{matrix} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2 \\ uv \end{matrix} \middle| q; bq \right) \\ & = uvb \frac{(1-uz_1)(1-vz_2)}{(1-uvq^{-1})(1-uv)} {}_2\phi_1 \left(\begin{matrix} uqz_1, vqz_2 \\ uqv \end{matrix} \middle| q; bq \right). \end{aligned}$$

Applying (5.15) we get (5.20) from (5.31), (5.21) from (5.32), (5.22) from (5.33) and (5.23) from (5.34).

From the contiguous relation (17.6.18) in [3] we get

$$(5.35) \quad \begin{aligned} {}_2\phi_1 \left(\begin{array}{c} uz_1q, vz_2 \\ uv \end{array} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{array}{c} uz_1, vz_2 \\ uv \end{array} \middle| q; bq \right) \\ = uz_1bq \frac{1-vz_2}{1-uv} {}_2\phi_1 \left(\begin{array}{c} uz_1q, vqz_2 \\ uvq \end{array} \middle| q; bq \right) \end{aligned}$$

and

$$(5.36) \quad \begin{aligned} {}_2\phi_1 \left(\begin{array}{c} uz_1q, vz_2 \\ uv \end{array} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{array}{c} uz_1, vz_2 \\ uv \end{array} \middle| q; bq \right) \\ = uz_1bq \frac{1-vz_2}{1-uv} {}_2\phi_1 \left(\begin{array}{c} uqz_1, vz_2q \\ uqv \end{array} \middle| q; bq \right). \end{aligned}$$

Applying (5.15) to (5.27) we get (5.24); applying (5.15) to (5.36) we get (5.25).

From the fourth contiguous relation [3, (17.6.19)] we get

$$(5.37) \quad \begin{aligned} {}_2\phi_1 \left(\begin{array}{c} uqz_1, vz_2 \\ uqv \end{array} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{array}{c} uz_1, vz_2 \\ uv \end{array} \middle| q; bq \right) \\ = bq \frac{(1-vz_2)(uz_1-uv)}{(1-uv)(1-uvq)} {}_2\phi_1 \left(\begin{array}{c} uqz_1, vqz_2 \\ uqvq \end{array} \middle| q; bq \right), \end{aligned}$$

and we apply (5.15) to (5.37) to get (5.26).

From the fourth contiguous relation [3, (17.6.20)] we get

$$(5.38) \quad \begin{aligned} {}_2\phi_1 \left(\begin{array}{c} uqz_1, vq^{-1}z_2 \\ uv \end{array} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{array}{c} uz_1, vz_2 \\ uv \end{array} \middle| q; bq \right) \\ = b \frac{(uz_1q - vz_2)}{(1-uv)} {}_2\phi_1 \left(\begin{array}{c} uqz_1, vz_2 \\ uqv \end{array} \middle| q; bq \right) \end{aligned}$$

and

$$(5.39) \quad \begin{aligned} {}_2\phi_1 \left(\begin{array}{c} uz_1q, vz_2q^{-1} \\ uv \end{array} \middle| q; bq \right) - {}_2\phi_1 \left(\begin{array}{c} uz_1, vz_2 \\ uv \end{array} \middle| q; bq \right) \\ = b \frac{(uz_1q - vz_2)}{(1-uv)} {}_2\phi_1 \left(\begin{array}{c} uqz_1, vz_2 \\ uqv \end{array} \middle| q; bq \right). \end{aligned}$$

From the contiguous relation (17.6.21) we get

$$(5.40) \quad \begin{aligned} & vz_2(1-uz_1) {}_2\phi_1 \left(\begin{array}{c} uz_1q, vz_2 \\ uv \end{array} \middle| q; bq \right) \\ & - uz_1(1-vz_2) {}_2\phi_1 \left(\begin{array}{c} uz_1, vz_2q \\ uv \end{array} \middle| q; bq \right) \\ & = (vz_2 - uz_1) {}_2\phi_1 \left(\begin{array}{c} uz_1, vz_2 \\ uv \end{array} \middle| q; bq \right). \end{aligned}$$

Then apply (5.15) to get (5.29):

$$\begin{aligned} & vz_2 \sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1q, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\ & - uz_1 \sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2q; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\ & = (vz_2 - uz_1) \sum_{m,n=0}^{\infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n. \end{aligned}$$

From (5.38) and (5.39) we get (5.27) and (5.28) respectively. From the contiguous relation (17.6.22) in [3] we obtain

$$\begin{aligned} & (uz_1 - z_1z_2q) {}_2\phi_1 \left(\begin{matrix} uz_1q, vz_2 \\ uv \end{matrix} \middle| q; bq \right) \\ (5.41) \quad & - (vz_2 - z_1z_2q) {}_2\phi_1 \left(\begin{matrix} uz_1, vz_2q \\ uv \end{matrix} \middle| q; bq \right) \\ & = (uz_1 - vz_2) (1 - bz_1z_2q) {}_2\phi_1 \left(\begin{matrix} uz_1q, vz_2q \\ uv \end{matrix} \middle| q; bq \right), \end{aligned}$$

which gives (5.30). □

The inversion transformation of quanta $q \rightarrow q^{-1}$ in (4.5) relates the properties of one family of polynomials for $q > 1$ to the properties of another family of polynomials with $0 < q < 1$. The polynomials $p_{m,n}(z_1, z_2; b|q)$ are essentially invariant under the quanta inversion transformation,

$$\begin{aligned} & p_{m,n}(z_1, z_2; b|q^{-1}) \\ & = (-b)^{m+n} \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-b)^{-k} (q; q)_k \left(\frac{q}{b}; q\right)_{m+n-k} \\ & \quad \times q^{k(k-m)+k(k-n)-\binom{k}{2}-\binom{m+n-k+1}{2}} z_1^{m-k} z_2^{n-k} \\ & = (-b)^{m+n} q^{-(\frac{m+n+1}{2})} \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(-\frac{q}{b}\right)^k (q; q)_k \left(\frac{q}{b}; q\right)_{m+n-k} z_1^{m-k} z_2^{n-k} \\ & = (-1)^{m+n} \left(\frac{b}{q}\right)^{(1-\alpha)m+\alpha n} q^{-(\frac{m+n}{2})} p_{m,n} \left(\left(\frac{b}{q}\right)^{\alpha} z_1, \left(\frac{b}{q}\right)^{1-\alpha} z_2; \frac{1}{b}|q \right). \end{aligned}$$

Therefore, we have established the symmetry

$$(5.42) \quad p_{m,n}(z_1, z_2; b|q^{-1}) = \frac{(bq^{-1})^{(1-\alpha)m+\alpha n}}{(-1)^{m+n} q^{\binom{m+n}{2}}} p_{m,n} \left((b/q)^{\alpha} z_1, (b/q)^{1-\alpha} z_2; 1/b|q \right),$$

for $\alpha \in \mathbb{C}$.

We now come to the asymptotics of $p_{m,n}(z_1, z_2; b|q)$.

Theorem 16. *Let $z_1, z_2 \in \mathbb{C}$, $bq < 1$ and $z_1z_2 \neq 0$. Then we have*

$$(5.43) \quad \lim_{m,n \rightarrow \infty} \frac{p_{m,n}(z_1, z_2; b|q)}{(bq; q)_{\infty} z_1^m z_2^n} = \left(\frac{1}{z_1z_2}; q \right)_{\infty}$$

uniformly on compact subsets of the z_1 and z_2 planes.

The theorem follows from definition (5.6) and Tannery’s theorem.

6. APPLICATIONS

Theorem 17. *Let $|t_i x_i| < \sqrt{q}$ for $i = 1, 2, 3, 4$. Then we have*

$$\begin{aligned}
 & \frac{(t_1 x_1 \sqrt{q}, t_2 x_2 \sqrt{q}, t_3 x_3 \sqrt{q}, t_4 x_4 \sqrt{q}, x_1 x_2, x_3 x_4, t_1 t_2 t_3 t_4 x_1 x_2 x_3 x_4 q^2; q)_\infty}{(t_1 t_2 x_1 x_2, t_1 t_4 x_1 x_4, t_2 t_3 x_2 x_3, t_3 t_4 x_3 x_4, -x_1 x_2 x_3 x_4; q)_\infty} \\
 (6.1) \quad &= \sum \frac{h_{m_1, n_1}(t_1 t_3, t_2 t_4 | q)}{(q; q)_{m_1} (q; q)_{n_1}} q^{((m_1 - n_1)^2 + m_1 + n_1 + (m_1 + m_2 + m_3 - n_1 - n_2 - n_3)^2) / 2} \\
 & \times \frac{H_{m_2, n_2}(t_1, t_2 | q) H_{m_3, n_3}(t_3, t_4 | q) x_1^{m_1 + m_2} x_2^{n_1 + n_2} x_3^{m_1 + m_3} x_4^{n_1 + n_3}}{(-1)^{m_2 + m_3 - n_2 - n_3} \prod_{j=2}^3 (q; q)_{m_j} (q; q)_{n_j}},
 \end{aligned}$$

where the summation is over all the nonnegative integers $m_i, n_i, i = 1, 2, 3$, such that $m_1 + m_2 + m_3 = n_1 + n_2 + n_3$.

Proof. Observe that

$$\begin{aligned}
 & \frac{(x_1 x_2 q^{-1}; q)_\infty}{(t_1 x_1 q^{-1/2} z, t_2 x_2 q^{-1/2} / z; q)_\infty} \frac{(x_3 x_4 q^{-1}; q)_\infty}{(t_3 x_3 q^{-1/2} z, t_4 x_4 q^{-1/2} / z; q)_\infty} \\
 & \times \frac{(q^{1/2} t_1 t_3 x_1 x_3 z q^{-1}, q^{1/2} t_2 t_4 x_2 x_4 q^{-1} / z; q)_\infty}{(-x_1 x_2 x_3 x_4 q^{-2}; q)_\infty} \\
 &= \sum \frac{h_{m_1, n_1}(t_1 t_3, t_2 t_4 | q) H_{m_2, n_2}(t_1, t_2 | q) H_{m_3, n_3}(t_3, t_4 | q)}{\prod_{j=1}^3 (q; q)_{m_j} (q; q)_{n_j}} \\
 & \times q^{((m_1 - n_1)^2 - 2m_1 - 2n_1 - m_2 - n_2 - m_3 - n_3) / 2} \\
 & \times (-1)^{(m_1 - n_1)} x_1^{m_1 + m_2} x_2^{n_1 + n_2} x_3^{m_1 + m_3} x_4^{n_1 + n_3} \\
 & \times z^{(m_1 + m_2 + m_3) - (n_1 + n_2 + n_3)}.
 \end{aligned}$$

By the q -beta integral we have

$$\begin{aligned}
 & \frac{(x_1 x_2 q^{-1}, x_3 x_4 q^{-1}, t_1 t_2 t_3 t_4 x_1 x_2 x_3 x_4; q)_\infty}{(q, -x_1 x_2 x_3 x_4 q^{-2}; q)_\infty} \\
 & \times \frac{(t_1 x_1, t_2 x_2, t_3 x_3, t_4 x_4; q)_\infty}{(t_1 t_2 x_1 x_2 q^{-1}, t_1 t_4 x_1 x_4 q^{-1}, t_2 t_3 x_2 x_3 q^{-1}, t_3 t_4 x_3 x_4 q^{-1}; q)_\infty} \\
 &= \sum \frac{h_{m_1, n_1}(t_1 t_3, t_2 t_4 | q) H_{m_2, n_2}(t_1, t_2 | q) H_{m_3, n_3}(t_3, t_4 | q)}{(q; q)_\infty \prod_{j=1}^3 (q; q)_{m_j} (q; q)_{n_j}} \\
 & \times q^{((m_1 - n_1)^2 - 2m_1 - 2n_1 - m_2 - n_2 - m_3 - n_3) / 2} / (2\pi i) \\
 & \times (-1)^{(m_1 - n_1)} x_1^{m_1 + m_2} x_2^{n_1 + n_2} x_3^{m_1 + m_3} x_4^{n_1 + n_3} \\
 & \times \int_{|z|=1} z^{(m_1 + m_2 + m_3) - (n_1 + n_2 + n_3)} \left(q, q^{1/2} z, q^{1/2} / z; q \right)_\infty \frac{dz}{z} \\
 &= \sum \frac{h_{m_1, n_1}(t_1 t_3, t_2 t_4 | q) H_{m_2, n_2}(t_1, t_2 | q) H_{m_3, n_3}(t_3, t_4 | q)}{(q; q)_\infty \prod_{j=1}^3 (q; q)_{m_j} (q; q)_{n_j}} \\
 & \times q^{((m_1 - n_1)^2 - 2m_1 - 2n_1 - m_2 - n_2 - m_3 - n_3 + (m_1 + m_2 + m_3 - n_1 - n_2 - n_3)^2) / 2} \\
 & \times (-1)^{m_2 + m_3 - n_2 - n_3} x_1^{m_1 + m_2} x_2^{n_1 + n_2} x_3^{m_1 + m_3} x_4^{n_1 + n_3},
 \end{aligned}$$

where the summation is over all the nonnegative integers $m_i, n_i, i = 1, 2, 3$, such that $m_1 + m_2 + m_3 = n_1 + n_2 + n_3$, which is (6.1). \square

From (4.23) and (3.19) we obtain the following equivalent representation:

Corollary 18. *Let $|t_i x_i| < q$ for $j = 1, 2, 3, 4$. Then we have*

$$\begin{aligned}
 (6.2) \quad & \frac{(t_1 x_1, t_2 x_2, t_3 x_3, t_4 x_4, x_1 x_2, x_3 x_4, t_1 t_2 t_3 t_4 x_1 x_2 x_3 x_4; q)_\infty}{(t_1 t_2 x_1 x_2 q^{-1}, t_1 t_4 x_1 x_4 q^{-1}, t_2 t_3 x_2 x_3 q^{-1}, t_3 t_4 x_3 x_4 q^{-1}, -x_1 x_2 x_3 x_4; q)_\infty} \\
 &= \sum \frac{L_{n_1}^{(m_1 - n_1)}(t_1 t_2 t_3 t_4 q^{-2}; q)}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3}} p_n \left(t_1 t_2 q^{-1}, q^{m_2 - n_2} \middle| q \right) p_n \left(t_3 t_4 q^{-1}, q^{m_3 - n_3} \middle| q \right) \\
 &\times \begin{bmatrix} m_2 \\ n_2 \end{bmatrix}_q \begin{bmatrix} m_3 \\ n_3 \end{bmatrix}_q q^{((m_1 - n_1)^2 + n_2^2 + n_3^2 + (m_1 + m_2 + m_3 - n_1 - n_2 - n_3)^2 - (m_1 + m_2 + m_3 - 3n_1))/2} \\
 &\times (-1)^{m_2 + m_3 + n_1} t_1^{m_1 + m_2 - n_1 - n_2} t_3^{m_1 + m_3 - n_1 - n_3} x_1^{m_1 + m_2} x_2^{n_1 + n_2} x_3^{m_1 + m_3} x_4^{n_1 + n_3},
 \end{aligned}$$

where the summation is over all the nonnegative integers $m_i, n_i, i = 1, 2, 3$, such that $m_1 + m_2 + m_3 = n_1 + n_2 + n_3$.

From (3.19) we find that

$$\begin{aligned}
 \sum_{n=-\infty}^{\infty} t^n J_n^{(2)}(x; q) &= \frac{\left(\frac{x^2}{4}; q\right)_\infty \left(-\frac{x^2}{4}; q\right)_\infty}{\left(\frac{xt}{2}, \frac{x}{2t}; q\right)_\infty \left(\frac{x^2}{4}; q\right)_\infty} \\
 &= \frac{\left(-\frac{x^2}{4}; q\right)_\infty}{\left(\frac{x^2}{4}; q\right)_\infty} \sum_{j,k=0}^{\infty} \frac{H_{j,k}\left(\frac{x}{2}, \frac{x}{2} \middle| q\right)}{(q; q)_j (q; q)_k} t^{j-k} \\
 &= \frac{\left(-\frac{x^2}{4}; q\right)_\infty}{\left(\frac{x^2}{4}; q\right)_\infty} \sum_{j,k=0}^{\infty} \frac{p_k\left(\frac{x^2}{4}, q^{j-k} \middle| q\right)}{(q; q)_j (q; q)_k} \left(\frac{x}{2}\right)^{j-k} t^{j-k} \\
 &= \frac{\left(-\frac{x^2}{4}; q\right)_\infty}{\left(\frac{x^2}{4}; q\right)_\infty} \sum_{n=-\infty}^{\infty} t^n \frac{\left(\frac{x}{2}\right)^n}{(q; q)_n} \sum_{k=0}^{\infty} \frac{p_k\left(\frac{x^2}{4}, q^n \middle| q\right)}{(q, q^{n+1}; q)_k}.
 \end{aligned}$$

Then,

$$J_n^{(2)}(x; q) = \left(\frac{x}{2}\right)^n \frac{(q^{n+1}, -\frac{x^2}{4}; q)_\infty}{(q, \frac{x^2}{4}; q)_\infty} \sum_{k=0}^{\infty} \frac{p_k\left(\frac{x^2}{4}, q^n \middle| q\right)}{(q, q^{n+1}; q)_k}$$

for all $n \in \mathbb{Z}$, and

$$(6.3) \quad \frac{J_\alpha^{(2)}(2x; q)}{x^\alpha} = \frac{(q^{\alpha+1}, -x^2; q)_\infty}{(q, x^2; q)_\infty} \sum_{k=0}^{\infty} \frac{p_k\left(x^2, q^\alpha \middle| q\right)}{(q, q^{\alpha+1}; q)_k}$$

for $\alpha > 0$ by analytic continuation.

We now use the Askey–Roy integral (2.23) to derive

$$\begin{aligned} & \frac{(ab\alpha\beta, c, q/c, c\alpha/\beta, q\beta/c\alpha; q)_\infty}{(a\beta, b\alpha, q, -c^2\alpha/q\beta, -q\beta/c^2\alpha; q)_\infty} \\ &= \int_{-\pi}^{\pi} \frac{(ce^{i\theta}/\beta, c\alpha e^{-i\theta}, qe^{i\theta}/c\alpha, q\beta e^{-i\theta}/c; q)_\infty}{(-c^2\alpha/q\beta, -q\beta/c^2\alpha; q)_\infty} \\ & \times \frac{(a\alpha, b\beta; q)_\infty}{(ae^{i\theta}, \alpha e^{-i\theta}, be^{i\theta}, \beta e^{-i\theta}; q)_\infty} \frac{d\theta}{2\pi} \\ &= \sum \frac{h_{m_1, n_1}(c/\beta, c\alpha|q)}{(q; q)_{m_1} (q; q)_{n_1}} \frac{h_{m_2, n_2}(1/c\alpha, \beta/c|q)}{(q; q)_{m_2} (q; q)_{n_2}} \\ & \times \frac{H_{m_3, n_3}(a, \alpha|q)}{(q; q)_{m_3} (q; q)_{n_3}} \frac{H_{m_4, n_4}(b, \beta|q)}{(q; q)_{m_4} (q; q)_{n_4}} (-1)^{m_1+m_2+n_1+n_2} \\ & \times q^{-(m_1+m_2+n_1+n_2)/2+(m_1-n_1)^2/2+(m_2-n_2)^2/2} \\ & \times \int_{-\pi}^{\pi} e^{i\theta(m_1+m_2+m_3+m_4-n_1-n_2-n_3-n_4)} \frac{d\theta}{2\pi}, \end{aligned}$$

that is,

$$\begin{aligned} & \frac{(ab\alpha\beta, c, q/c, c\alpha/\beta, q\beta/c\alpha; q)_\infty}{(a\beta, b\alpha, q, -c^2\alpha/q\beta, -q\beta/c^2\alpha; q)_\infty} \\ (6.4) \quad &= \sum \frac{h_{m_1, n_1}(c/\beta, c\alpha|q)}{(q; q)_{m_1} (q; q)_{n_1}} \frac{h_{m_2, n_2}(1/c\alpha, \beta/c|q)}{(q; q)_{m_2} (q; q)_{n_2}} \\ & \times \frac{H_{m_3, n_3}(a, \alpha|q)}{(q; q)_{m_3} (q; q)_{n_3}} \frac{H_{m_4, n_4}(b, \beta|q)}{(q; q)_{m_4} (q; q)_{n_4}} (-1)^{m_1+m_2+n_1+n_2} \\ & \times q^{-(m_1+m_2+n_1+n_2)/2+(m_1-n_1)^2/2+(m_2-n_2)^2/2}, \end{aligned}$$

where $|q|, |\alpha|, |\beta|, |a|, |b| < 1, c\alpha\beta \neq 0$ and the summation is over all the nonnegative integers such that $m_1 + m_2 + m_3 + m_4 - n_1 - n_2 - n_3 - n_4 = 0$.

From (4.23) and (3.19) we obtain the following equivalent representation:

$$\begin{aligned} & \frac{(ab\alpha\beta, c, q/c, c\alpha/\beta, q\beta/c\alpha; q)_\infty}{(a\beta, b\alpha, q, -c^2\alpha/q\beta, -q\beta/c^2\alpha; q)_\infty} \\ (6.5) \quad &= \sum \frac{L_{n_1}^{(m_1-n_1)}(c^2\alpha/\beta; q) q^{(m_1-n_1)^2/2}}{(-1)^{m_1} (q; q)_{m_1} (\beta/c)^{m_1-n_1} q^{(m_1+n_1)/2}} \\ & \times \frac{L_{n_2}^{(m_2-n_2)}(\beta/c^2\alpha; q) q^{(m_2-n_2)^2/2}}{(-1)^{m_2} (q; q)_{m_2} (c\alpha)^{m_2-n_2} q^{(m_2+n_2)/2}} \\ & \times \frac{p_{n_3}\left(a\alpha, q^{m_3-n_3} \middle| q\right) p_{n_4}\left(b\beta, q^{m_4-n_4} \middle| q\right)}{a^{n_3-n_3} b^{n_4-n_4} (q; q)_{m_3} (q; q)_{n_3} (q; q)_{m_4} (q; q)_{n_4}}, \end{aligned}$$

where $|q|, |\alpha|, |\beta|, |a|, |b| < 1, c\alpha\beta \neq 0$, the summation is over all the nonnegative integers such that $m_1 + m_2 + m_3 + m_4 - n_1 - n_2 - n_3 - n_4 = 0$ and $L_n^{(\alpha)}(x; q)$ and $p_n(x, a|q)$ are the q -Laguerre and little q -Laguerre polynomials respectively.

Let $a = ue^{i\phi}, b = ue^{-i\phi}, \alpha = ve^{i\psi}, \beta = ve^{-i\psi}, c = q^{1/2}$ in the Askey–Roy integral to get

$$\int_{-\pi}^{\pi} \frac{(q^{1/2}e^{i\theta}e^{i\psi}/v, q^{1/2}e^{i\theta}e^{-i\psi}/v, q^{1/2}e^{-i\theta}ve^{i\psi}, q^{1/2}e^{-i\theta}ve^{-i\psi}; q)_{\infty}}{(e^{i\phi}ue^{i\theta}, e^{-i\phi}ue^{i\theta}, ve^{-i\theta}e^{i\psi}, ve^{-i\theta}e^{-i\psi}; q)_{\infty}} \frac{d\theta}{2\pi} = \frac{(u^2v^2, q^{1/2}, q^{1/2}, q^{1/2}e^{2i\psi}, q^{1/2}e^{-2i\psi}; q)_{\infty}}{(uve^{i(\phi+\psi)}, uve^{i(\phi-\psi)}, uve^{-i(\phi-\psi)}, uve^{-i(\phi+\psi)}, q; q)_{\infty}}$$

and

$$\begin{aligned} & \frac{(u^2v^2, q^{1/2}, q^{1/2}, q^{1/2}e^{2i\psi}, q^{1/2}e^{-2i\psi}; q)_{\infty}}{(uve^{i(\phi+\psi)}, uve^{i(\phi-\psi)}, uve^{-i(\phi-\psi)}, uve^{-i(\phi+\psi)}, q; q)_{\infty}} \\ &= \sum \frac{h_{m_1}(\sinh(i\psi + \frac{\pi}{2}i)|q) h_{m_1}(\sinh(i\psi - \frac{\pi}{2}i)|q)}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4} (vi)^{m_1+m_2}} \\ &\times H_{m_3}(\cos \phi|q) H_{m_4}(\cos \psi|q) u^{m_3} v^{m_4} \\ &\times q^{(m_1^2+m_2^2)/2} \int_{-\pi}^{\pi} e^{i\theta(m_1+m_3-m_2-m_4)} \frac{d\theta}{2\pi} \\ &= \sum \frac{h_{m_1}(\sinh(i\psi + \frac{\pi}{2}i)|q) h_{m_2}(i \sinh(\psi - \frac{\pi}{2}i)|q)}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4} (vi)^{m_1+m_2}} \\ &\times q^{(m_1^2+m_2^2)/2} H_{m_3}(\cos \phi|q) H_{m_4}(\cos \psi|q) u^{m_3} v^{m_4} \end{aligned}$$

or

$$\begin{aligned} & \frac{(u^2v^2, q^{1/2}, q^{1/2}, q^{1/2}e^{\pi+2i\psi}, q^{1/2}e^{-\pi-2i\psi}; q)_{\infty}}{(uve^{i(\phi+\psi)+\pi/2}, uve^{i(\phi-\psi)-\pi/2}, uve^{-i(\phi-\psi)+\pi/2}, uve^{-i(\phi+\psi)-\pi/2}, q; q)_{\infty}} \\ &= \sum \frac{h_{m_1}(i \sin \psi|q) h_{m_2}(i \sin \psi|q) (-1)^{m_2}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4} (vi)^{m_1+m_2}} \\ &\times q^{(m_1^2+m_2^2)/2} H_{m_3}(\cos \phi|q) H_{m_4}(\cos \psi|q) u^{m_3} v^{m_4} \\ &= \sum \frac{(-1)^{m_1} q^{(m_1^2+m_2^2)/2} u^{m_3} v^{m_4-m_1-m_2}}{(q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4}} \\ &\times H_{m_1}(\sin \psi|q^{-1}) H_{m_2}(\sin \psi|q^{-1}) H_{m_3}(\cos \phi|q) H_{m_4}(\cos \psi|q) \end{aligned}$$

or

$$(6.6) \quad \frac{(u^2v^2, q^{1/2}, q^{1/2}, q^{1/2}e^{\pi+2i\psi}, q^{1/2}e^{-\pi-2i\psi}; q)_{\infty}}{(uve^{i(\phi+\psi)+\pi/2}, uve^{i(\phi-\psi)-\pi/2}, uve^{-i(\phi-\psi)+\pi/2}, uve^{-i(\phi+\psi)-\pi/2}, q; q)_{\infty}} = \sum \frac{H_{m_1}(\sin \psi|q^{-1}) H_{m_2}(\sin \psi|q^{-1}) H_{m_3}(\cos \phi|q) H_{m_4}(\cos \psi|q)}{(-1)^{m_1} q^{-(m_1^2+m_2^2)/2} (q; q)_{m_1} (q; q)_{m_2} (q; q)_{m_3} (q; q)_{m_4} v^{m_1+m_2-m_4}},$$

where the summation is over all the nonnegative integers $m_i, i = 1, 2, 3, 4$, such that $m_1 + m_3 = m_2 + m_4$.

7. ADDITIONAL RESULTS

In this section we first derive moment integral representations for $\{H_{m,n}(\zeta, \bar{\zeta}|q)\}$ and $\{h_{m,n}(\zeta, \bar{\zeta}|q)\}$. We then derive additional generating functions and expansions.

We shall use the terminating q -binomial theorem (2.6) in the form

$$(7.1) \quad \prod_{j=0}^{n-1} (a + bq^j) = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{j}{2}} a^{n-j} b^j.$$

Theorem 19. *Let $\mu(\zeta, \bar{\zeta})$, be a normalized orthogonal measure for $H_{m,n}(z, \bar{z}|q)$ and $\nu(\zeta, \bar{\zeta})$ be a normalized measure for $h_{m,n}(z, \bar{z}|)$ respectively. Then we have the integral representations*

$$(7.2) \quad H_{m,n}(z_1, z_2|q) = \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left(z_1 + i\zeta q^{\frac{1}{2}+j} \right) \prod_{k=0}^{n-1} \left(z_2 + i\bar{\zeta} q^{\frac{1}{2}+k} \right) d\nu(\zeta, \bar{\zeta})$$

and

$$(7.3) \quad q^{\frac{(m-n)^2}{2}} i^{m+n} h_{m,n}(z_1, z_2|q) = \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left(\zeta + iz_1 q^{\frac{1}{2}+j} \right) \prod_{k=0}^{n-1} \left(\bar{\zeta} + iz_2 q^{\frac{1}{2}+k} \right) d\mu(\zeta, \bar{\zeta}),$$

where $z_1, z_2 \in \mathbb{C}$ and $m, n \in \mathbb{N}_0$.

Proof. Let

$$a_{m,n}(z_1, z_2|q) = \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left(\zeta + iz_1 q^{\frac{1}{2}+j} \right) \prod_{k=0}^{n-1} \left(\bar{\zeta} + iz_2 q^{\frac{1}{2}+k} \right) d\mu(\zeta, \bar{\zeta}).$$

The form (7.1) of the q -binomial theorem implies

$$\begin{aligned} & \prod_{j=0}^{m-1} \left(\zeta + iz_1 q^{\frac{1}{2}+j} \right) \prod_{k=0}^{n-1} \left(\bar{\zeta} + iz_2 q^{\frac{1}{2}+k} \right) \\ &= \sum_{j=0}^m \sum_{k=0}^n \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{j^2+k^2}{2}} i^{j+k} z_1^j z_2^k \zeta^{m-j} \bar{\zeta}^{n-k}, \end{aligned}$$

and we find that

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{a_{m,n}(z_1, z_2|q) u^m v^n}{(q; q)_m (q; q)_n} \\ &= \int_{\mathbb{R}^2} \sum_{j,k=0}^{\infty} \frac{q^{\binom{j}{2} + \binom{k}{2}} \left(iq^{\frac{1}{2}} uz_1 \right)^j \left(iq^{\frac{1}{2}} vz_2 \right)^k}{(q; q)_j (q; q)_k} \sum_{m \geq j, n \geq k} \frac{(u\zeta)^{m-j} (v\bar{\zeta})^{n-j}}{(q; q)_{m-j} (q; q)_{n-j}} d\mu(\zeta, \bar{\zeta}) \\ &= \frac{\left(-iq^{\frac{1}{2}} uz_1, -iq^{\frac{1}{2}} vz_2; q \right)_{\infty}}{(uv; q)_{\infty}} \int_{\mathbb{R}^2} \frac{(uv; q)_{\infty} d\mu(\zeta, \bar{\zeta})}{(u\zeta, v\bar{\zeta}; q)_{\infty}} \\ &= \frac{\left(-iq^{\frac{1}{2}} uz_1, -iq^{\frac{1}{2}} vz_2; q \right)_{\infty}}{(uv; q)_{\infty}} = \sum_{m,n=0}^{\infty} \frac{h_{m,n}(z_1, z_2|q)}{(q; q)_m (q; q)_n} q^{\frac{(m-n)^2}{2}} i^{m+n} u^m v^n. \end{aligned}$$

Similarly, let

$$\begin{aligned}
 b_{m,n}(z_1, z_2|q) &= \int_{\mathbb{R}^2} \prod_{j=0}^{m-1} \left(z_1 + i\zeta q^{\frac{1}{2}+j} \right) \prod_{k=0}^{n-1} \left(z_2 + i\bar{\zeta} q^{\frac{1}{2}+k} \right) d\nu(\zeta, \bar{\zeta}) \\
 &= \int_{\mathbb{R}^2} \sum_{j=0}^m \sum_{k=0}^n \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{j^2+k^2}{2}} i^{j+k} \zeta^j \bar{\zeta}^k z_1^{m-j} z_2^{n-k} d\nu(\zeta, \bar{\zeta}).
 \end{aligned}$$

Then,

$$\begin{aligned}
 &\sum_{m,n=0}^{\infty} \frac{b_{m,n}(z_1, z_2|q) u^m v^n}{(q; q)_m (q; q)_n} \\
 &= \int_{\mathbb{R}^2} \sum_{j,k=0}^{\infty} \frac{q^{\binom{j}{2}+\binom{k}{2}} \left(i q^{\frac{1}{2}} \zeta u \right)^j \left(i q^{\frac{1}{2}} \bar{\zeta} v \right)^k}{(q; q)_m (q; q)_n} \sum_{j \geq m, k \geq n} \frac{(uz_1)^{m-j} (vz_2)^{n-k} d\nu(\zeta, \bar{\zeta})}{(q; q)_{m-j} (q; q)_{n-k}} \\
 &= \frac{(uv; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}} \int_{\mathbb{R}^2} \frac{\left(-i q^{\frac{1}{2}} \zeta u, -i q^{\frac{1}{2}} \bar{\zeta} v; q \right)_{\infty} d\nu(\zeta, \bar{\zeta})}{(uv; q)_{\infty}} \\
 &= \frac{(uv; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}}.
 \end{aligned}$$

This completes the proof. □

Remark 20. Observe that for any fixed $z_1, z_2 \neq 0$, (7.2) and (7.3) can be recast into

$$(7.4) \quad H_{m,n}(z_1, z_2|q) = z_1^m z_2^n \int_{\mathbb{R}^2} \left(-\frac{i\zeta q^{\frac{1}{2}}}{z_1}; q \right)_m \left(-\frac{i\bar{\zeta} q^{\frac{1}{2}}}{z_2}; q \right)_n d\nu(\zeta, \bar{\zeta})$$

and
(7.5)

$$q^{\frac{(m-n)^2}{2}} i^{m+n} h_{m,n}(z_1, z_2|q) = \int_{\mathbb{R}^2} \zeta^m \bar{\zeta}^n \left(-\frac{i z_1 q^{\frac{1}{2}}}{\zeta}; q \right)_m \left(-\frac{i z_2 q^{\frac{1}{2}}}{\bar{\zeta}}; q \right)_n d\mu(\zeta, \bar{\zeta}).$$

Using the relation $(a; q)_{-n} = (-qa^{-1})^n q^{\binom{n}{2}} / (qa^{-1}; q)_n$ for $n = 0, 1, \dots$ and the above equations, we can extend the definitions of $H_{m,n}(z_1, z_2|q)$ and $h_{m,n}(z_1, z_2|q)$ to all $m, n \in \mathbb{Z}$. Of course, we can use these equations and the equation $(a; q)_n = (a; q)_{\infty} / (aq^n; q)_{\infty}$ to extend the definitions of $H_{m,n}(z_1, z_2|q)$ and $h_{m,n}(z_1, z_2|q)$ to all $m, n \in \mathbb{C}$ where the integrals are convergent.

Corollary 21. *Let $a, b, z_1, z_2 \neq 0$ such that $\left| \frac{cdq}{abz_1z_2} \right| < 1$ and $cq^m, dq^m \neq 1, m \in \mathbb{N}$. Then*

$$\begin{aligned}
 (7.6) \quad &\sum_{m,n=0}^{\infty} \frac{(a; q)_m (b; q)_n q^{\frac{(m-n)^2}{2}} h_{m,n}(z_1, z_2|q)}{(q, c; q)_m (q, d; q)_n} \left(\frac{-c}{\sqrt{q}az_1} \right)^m \left(\frac{-d}{\sqrt{q}bz_2} \right)^n \\
 &= \frac{(c/a, d/b; q)_{\infty}}{(c, d; q)_{\infty}} {}_2\phi_1 \left(\begin{matrix} a, b \\ 0 \end{matrix} \middle| q, -\frac{cd}{qabz_1z_2} \right) \\
 &= \frac{(c/a, d/b, cdq^{-1}/bz_1z_2; q)_{\infty}}{(c, d, -cdq^{-1}/abz_1z_2; q)_{\infty}} {}_1\phi_1 \left(\begin{matrix} a \\ cdq^{-1}/bz_1z_2 \end{matrix} \middle| q, \frac{cd}{qaz_1z_2} \right).
 \end{aligned}$$

The equality between the left-hand side and the extreme right-hand side holds when $z_1 z_2 \neq 0$ without the assumption $\left| \frac{cdq}{abz_1 z_2} \right| < 1$. On the other hand, if $z_1, z_2 \neq 0$ and $cq^m, dq^m \neq 1, m \in \mathbb{N}$, then

$$(7.7) \quad \sum_{m,n=0}^{\infty} \frac{q^{m^2-mn+n^2} h_{m,n}(z_1, z_2|q)}{(q, cq; q)_m (q, dq; q)_n} \left(\frac{c}{z_1}\right)^m \left(\frac{d}{z_2}\right)^n = \frac{A_q\left(\frac{cd}{z_1 z_2}\right)}{(cq, dq; q)_{\infty}},$$

where A_q is the Ramanujan function defined in (2.14). Alternately the generating function (7.7) may be written as

$$(7.8) \quad \sum_{m,n=0}^{\infty} \frac{q^{m^2-mn+n^2} h_{m,n}(z_1, z_2|q)}{(q, cz_1q; q)_m (q, dz_2q; q)_n} c^m d^n = \frac{A_q(cd)}{(cz_1q, dz_2q; q)_{\infty}},$$

which holds for any $c, d, z_1, z_2 \in \mathbb{C}$ such that $cz_1q^m, dz_2q^m \neq 1, m \in \mathbb{N}$.

Proof. The integral representation (7.5) and Fubini's theorem imply

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(a; q)_m (b; q)_n q^{\frac{(m-n)^2}{2}} i^{m+n} h_{m,n}(z_1, z_2|q)}{(q, c; q)_m (q, d; q)_n} \left(\frac{ic}{az_1\sqrt{q}}\right)^m \left(\frac{id}{bz_2\sqrt{q}}\right)^n \\ &= \int_{\mathbb{R}^2} \sum_{m=0}^{\infty} \frac{(a, -iz_1q^{\frac{1}{2}}/\zeta; q)_m}{(q, c; q)_m} \left(\frac{ic\zeta}{az_1\sqrt{q}}\right)^m \\ & \quad \times \sum_{n=0}^{\infty} \frac{(b, -iz_2q^{1/2}/\bar{\zeta}; q)_n}{(q, d; q)_n} \left(\frac{id\bar{\zeta}}{bz_2\sqrt{q}}\right)^n d\mu(\zeta, \bar{\zeta}), \end{aligned}$$

where we used the q -Gauss sum in the last line [14, (II.8)]. Thus the above quantity equals

$$\begin{aligned} & \frac{(c/a, d/b; q)_{\infty}}{(c, d; q)_{\infty}} \int_{\mathbb{R}^2} \frac{\left(\frac{ic\zeta}{z_1\sqrt{q}}, \frac{id\bar{\zeta}}{z_2\sqrt{q}}; q\right)_{\infty} d\mu(\zeta, \bar{\zeta})}{\left(\frac{ic\zeta}{az_1\sqrt{q}}, \frac{id\bar{\zeta}}{bz_2\sqrt{q}}; q\right)_{\infty}} \\ &= \frac{(c/a, d/b; q)_{\infty}}{(c, d; q)_{\infty}} \sum_{m,n=0}^{\infty} \frac{(a; q)_m (b; q)_n}{(q; q)_m (q; q)_n} \left(\frac{ic}{az_1\sqrt{q}}\right)^m \left(\frac{id}{bz_2\sqrt{q}}\right)^n \int_{\mathbb{R}^2} \zeta^m \bar{\zeta}^n d\mu(\zeta, \bar{\zeta}). \end{aligned}$$

It is straightforward to find that

$$\int_{\mathbb{R}^2} \zeta^m \bar{\zeta}^n d\mu(\zeta, \bar{\zeta}) = (q; q)_n \delta_{m,n},$$

whence,

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{(a; q)_m (b; q)_n q^{\frac{(m-n)^2}{2}} i^{m+n} h_{m,n}(z_1, z_2|q)}{(q, c; q)_m (q, d; q)_n} \left(\frac{ic}{az_1\sqrt{q}}\right)^m \left(\frac{id}{bz_2\sqrt{q}}\right)^n \\ &= \frac{(c/a, d/b; q)_{\infty}}{(c, d; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q; q)_n} \left(-\frac{cd}{abz_1 z_2 q}\right)^n, \end{aligned}$$

which is the first equality in (7.6). The equality between the ${}_2\phi_1$ and the ${}_1\phi_1$ is a special case of the ${}_2\phi_1 - {}_2\phi_2$ transformation [14, (III.4)].

Similarly the integral representation (7.5) and Fubini’s theorem give

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{q^{m^2-mn+n^2} h_{m,n}(z_1, z_2|q)}{(q, c; q)_m (q, d; q)_n} \left(\frac{c}{z_1 q}\right)^m \left(\frac{d}{z_2 q}\right)^n \\ &= \frac{1}{(c, d; q)_{\infty}} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} \left(\frac{-cd}{z_1 z_2 q^2}\right)^k. \end{aligned}$$

This establishes (7.7) and the proof is complete. □

Note that the ${}_1\phi_1$ representation of the generating function in Corollary 21 is an analytic continuation of the ${}_2\phi_1$ function.

Corollary 22. *We have the generating function*

$$\begin{aligned} (7.9) \quad & \sum_{m,n=0}^{\infty} \frac{q^{\binom{m}{2}} (b; q)_n q^{\frac{(m-n)^2}{2}} h_{m,n}(z_1, z_2|q)}{(q, cz_1; q)_m (q, dz_2; q)_n} \left(\frac{c}{\sqrt{q}}\right)^m \left(\frac{-d}{\sqrt{qb}}\right)^n \\ &= \frac{(dz_2/b, q; q)_{\infty}}{(cz_1, dz_2; q)_{\infty}} b^{-\nu/2} I_{\nu}^{(2)}(2\sqrt{b}; q), \end{aligned}$$

where $I_{\nu}^{(2)}$ is the modified q -Bessel function defined in (2.13) and $q^{\nu} := cdq^{-2}/b$.

Proof. Replace c and d by cz_1 and dz_2 , respectively, in (7.6); then let $a \rightarrow \infty$. □

A very interesting special case of Corollary 21 is the following theorem.

Theorem 23. *Let $cd = -q^s$, $s = 0, 1, \dots$. Then the generating function*

$$\begin{aligned} (7.10) \quad & \sum_{m,n=0}^{\infty} \frac{q^{m^2-mn+n^2} h_{m,n}(z_1, z_2|q)}{(q, cz_1 q; q)_m (q, dz_2 q; q)_n} c^m d^n \\ &= \frac{1}{(cz_1 q, dz_2 q; q)_{\infty}} \left[\frac{(-1)^s q^{-\binom{s}{2}} a_s(q)}{(q, q^4; q^5)_{\infty}} + \frac{(-1)^{s+1} q^{-\binom{s}{2}} b_s(q)}{(q^2, q^3; q^5)_{\infty}} \right] \end{aligned}$$

holds when neither cz_1 nor $dz_2 = q^{-r}$ for $r = 0, 1, \dots$.

Proof. Apply (7.8) and the Garret–Ismail–Stanton result (2.15). □

Theorem 24. *For any $m, n \in \mathbb{N}_0$ and $z_1, z_2 \in \mathbb{C}$ we have*

$$(7.11) \quad z_1^m z_2^n = \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (q; q)_k H_{m-k, n-k}(z_1, z_2|q),$$

$$(7.12) \quad z_1^m z_2^n = q^{-mn} \sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (q; q)_k q^{\binom{k}{2}} h_{m-k, n-k}(z_1, z_2|q).$$

Proof. From the generating function (3.3) we see that

$$\begin{aligned} \frac{1}{(uz_1, vz_2; q)_{\infty}} &= \frac{1}{(uv; q)_{\infty}} \sum_{m,n=0}^{\infty} \frac{H_{m,n}(z_1, z_2|q) u^m v^n}{(q; q)_m (q; q)_n} \\ &= \sum_{k=0}^{\infty} \sum_{m,n=0}^{\infty} \frac{H_{m,n}(z_1, z_2|q) u^{m+k} v^{n+k}}{(q; q)_m (q; q)_n (q; q)_k}. \end{aligned}$$

The expansion (7.11) follows from equating like powers of u and v . Similarly we apply the generating function (4.6) and find that

$$\begin{aligned} (-q^{1/2}uz_1, -q^{1/2}z_2v; q)_\infty &= (-uv; q)_\infty \sum_{m,n=0}^\infty \frac{h_{m,n}(z_1, z_2|q) q^{(m-n)^2/2} u^m v^n}{(q; q)_m (q; q)_n} \\ &= \sum_{k=0}^\infty \frac{q^{\binom{k}{2}}}{(q; q)_k} \sum_{m,n=k}^\infty \frac{h_{m-k, n-k}(z_1, z_2|q) q^{(m-n)^2/2} u^m v^n}{(q; q)_{m-k} (q; q)_{n-k}}, \end{aligned}$$

and (7.12) follows. □

Theorem 25. *The connection relations between the polynomials $\{H_{m,n}(z_1, z_2|q)\}$ and $\{h_{m,n}(z_1, z_2|q)\}$ are*

$$(7.13) \quad \begin{aligned} H_{m,n}(z_1, z_2|q) &= q^{-mn} \sum_{s=0}^{m \wedge n} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q (q; q)_s q^{\binom{s}{2}} h_{m-s, n-s}(z_1, z_2|q) \\ &\quad \times \sum_{k=0}^s \begin{bmatrix} s \\ k \end{bmatrix}_q (-1)^k q^{k(m+n-k)}, \end{aligned}$$

$$(7.14) \quad \begin{aligned} h_{m,n}(z_1, z_2|q) &= \sum_{s=0}^{m \wedge n} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q (q; q)_s H_{m-s, n-s}(z_1, z_2|q) \\ &\quad \times \sum_{k=0}^s \begin{bmatrix} s \\ k \end{bmatrix}_q (-1)^k q^{(m-k)(n-k)}. \end{aligned}$$

Proof. The theorem follows from the explicit formulas (3.1), (4.1) and the connection relations (7.11)–(7.12). □

Theorem 26. *We have the generating functions*

$$(7.15) \quad \begin{aligned} &\sum_{m,n=0}^\infty H_{m+j, n+k}(z_1, z_2|q) \frac{u^m v^n}{(q; q)_m (q; q)_n} \\ &= \frac{(uvq^{j+k}; q)_\infty}{(uz_1, vz_2; q)_\infty} \sum_{\ell=0}^{j \wedge k} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k \\ \ell \end{bmatrix}_q q^{\binom{\ell}{2}} (-1)^\ell (q; q)_\ell \\ &\quad \times z_1^{j-\ell} z_2^{k-\ell} \left(\frac{vq^\ell}{z_1}; q \right)_{j-\ell} \left(\frac{uq^j}{z_2}; q \right)_{k-\ell} (z_2v; q)_\ell \\ &= \frac{(uvq^{j+k}; q)_\infty}{(uz_1, vz_2; q)_\infty} \sum_{\ell=0}^{j \wedge k} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k \\ \ell \end{bmatrix}_q q^{\binom{\ell}{2}} (-1)^\ell (q; q)_\ell \\ &\quad \times z_1^{j-\ell} z_2^{k-\ell} \left(\frac{uq^\ell}{z_2}; q \right)_{k-\ell} \left(\frac{vq^k}{z_1}; q \right)_{j-\ell} (z_1u; q)_\ell \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} h_{m+j,n+k}(z_1, z_2|q) q^{(m-n)^2/2} \frac{u^m}{(q; q)_m} \frac{v^n}{(q; q)_n} \\
 (7.16) \quad &= \frac{(-1)^{j+k} q^{\frac{(j-k)^2}{2}} \left(-uz_1q^{k+\frac{1}{2}} - vz_2q^{j+\frac{1}{2}}, q\right)_{\infty}}{(-uv; q)_{\infty} \left(\frac{z_1q^{\frac{1}{2}}}{v}; q\right)_{k-j} \left(\frac{z_2q^{\frac{1}{2}}}{u}; q\right)_{j-k}} \sum_{\ell=0}^{j \wedge k} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k \\ \ell \end{bmatrix}_q \\
 & \times (-1)^{\ell} (q; q)_{\ell} u^{k-\ell} v^{j-\ell} \left(\frac{z_1q^{\frac{1}{2}}}{v}; q\right)_{k-\ell} \left(\frac{z_2q^{\frac{1}{2}}}{u}; q\right)_{j-\ell} (-uv; q)_{\ell}.
 \end{aligned}$$

More generally our q -disk polynomials have the generating function

$$\begin{aligned}
 & \sum_{m,n=0}^{\infty} \frac{p_{m+j,n+k}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\
 (7.17) \quad &= \frac{(bq, uvq^{j+k}; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(bq^{1+j+k})^{\ell} (uz_1, vz_2; q)_{\ell}}{(q, uvq^{j+k}; q)_{\ell}} \sum_{i=0}^{j \wedge k} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \\
 & \times (-q^{-\ell})^i z_1^{j-i} z_2^{k-i} q^{\binom{i}{2}} (q; q)_i \left(\frac{v}{z_1} q^i; q\right)_{j-i} \left(\frac{u}{z_2} q^j; q\right)_{k-i} (z_2 v q^{\ell}; q)_i \\
 &= \frac{(bq, uvq^{j+k}; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(uz_1, vz_2; q)_{\ell}}{(q, uvq^{j+k}; q)_{\ell}} (bq^{1+j+k})^{\ell} \sum_{i=0}^{j \wedge k} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \\
 & \times (-q^{-\ell})^i z_1^{j-i} z_2^{k-i} q^{\binom{i}{2}} (q; q)_i \left(\frac{uq^i}{z_2}; q\right)_{k-i} \left(\frac{vq^k}{z_1}; q\right)_{j-i} (uz_1 q^{\ell}; q)_i.
 \end{aligned}$$

Proof. Observe that for $k \in \mathbb{N}$ we have

$$D_{q,u}^k \left(\frac{u^m}{(q; q)_m} \right) = \begin{cases} 0, & m < k, \\ \frac{1}{(1-q)^k} \frac{u^{m-k}}{(q; q)_{m-k}}, & m \geq k, \end{cases}$$

$$D_{q,u}^k \left(u^m \left(\frac{a}{u}; q \right)_m \right) = \begin{cases} 0, & m < k, \\ \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{(q; q)_k}{(1-q)^k} u^{m-k} \left(\frac{a}{u}; q \right)_{m-k}, & m \geq k, \end{cases}$$

$$D_{q,u}^k ((au; q)_m) = \begin{cases} 0, & m < k, \\ \begin{bmatrix} m \\ k \end{bmatrix}_q \left(\frac{a}{q-1} \right)^k q^{\binom{k}{2}} (q; q)_k (auq^k; q)_{m-k}, & m \geq k, \end{cases}$$

and

$$D_{q,u}^k \left(\frac{(au; q)_{\infty}}{(bu; q)_{\infty}} \right) = \left(\frac{b}{1-q} \right)^k \left(\frac{a}{b}; q \right)_k \frac{(auq^k; q)_{\infty}}{(bu; q)_{\infty}}.$$

Then we apply the operator $(1 - q)^{j+k} D_{q,u}^j D_{q,v}^k$ to the generating function for $h_{m,n}(z_1, z_2|q)$ to get

$$\begin{aligned} & \sum_{m \geq j, n \geq k} h_{m,n}(z_1, z_2|q) q^{(m-n)^2/2} \frac{u^{m-j}}{(q; q)_{m-j}} \frac{v^{n-k}}{(q; q)_{n-k}} \\ &= (1 - q)^{j+k} D_{q,v}^k D_{q,u}^j \left(\frac{(-uz_1q^{1/2}, -vz_2q^{1/2}; q)_\infty}{(-uv; q)_\infty} \right) \\ &= (1 - q)^k (-1)^j (-uz_1q^{j+1/2}; q)_\infty D_{q,v}^k \left(v^j \left(\frac{z_1}{v} q^{1/2}; q \right)_j \frac{(-vz_2q^{1/2}; q)_\infty}{(-uv; q)_\infty} \right) \\ &= \frac{(-uz_1q^{j+\frac{1}{2}} - vz_2q^{k+\frac{1}{2}}; q)_\infty}{(-uv; q)_\infty (-1)^{j+k}} \sum_{\ell=0}^k \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k \\ \ell \end{bmatrix}_q (-1)^\ell (q; q)_\ell \\ &\times v^{j-\ell} \left(\frac{z_1 q^{\frac{1}{2}}}{v}; q \right)_{j-\ell} u^{k-\ell} \left(\frac{z_2 q^{\frac{1}{2}}}{u}; q \right)_{k-\ell} (-uv; q)_\ell, \end{aligned}$$

and (7.16) follows. Similarly we find

$$\begin{aligned} & \sum_{m \geq j, n \geq k}^\infty H_{m,n}(z_1, z_2|q) \frac{u^{m-j} v^{n-k}}{(q; q)_{m-j} (q; q)_{n-k}} \\ &= (1 - q)^{j+k} D_{q,v}^k D_{q,u}^j \left(\frac{(uv; q)_\infty}{(uz_1, vz_2; q)_\infty} \right) \\ &= \frac{(uvq^{j+k}; q)_\infty}{(uz_1, vz_2; q)_\infty} \sum_{\ell=0}^{j \wedge k} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k \\ \ell \end{bmatrix}_q q^{\binom{\ell}{2}} (-1)^\ell (q; q)_\ell \\ &\quad \times z_1^{j-\ell} z_2^{k-\ell} \left(\frac{vq^\ell}{z_1}; q \right)_{j-\ell} \left(\frac{uq^j}{z_2}; q \right)_{k-\ell} (z_2v; q)_\ell \end{aligned}$$

and

$$\begin{aligned} & \sum_{m \geq j, n \geq k}^\infty H_{m,n}(z_1, z_2|q) \frac{u^{m-j} v^{n-k}}{(q; q)_{m-j} (q; q)_{n-k}} \\ &= (1 - q)^{j+k} D_{q,u}^j D_{q,v}^k \left(\frac{(uv; q)_\infty}{(uz_1, vz_2; q)_\infty} \right) \\ &= \frac{(uvq^{j+k}; q)_\infty}{(uz_1, vz_2; q)_\infty} \sum_{\ell=0}^{j \wedge k} \begin{bmatrix} j \\ \ell \end{bmatrix}_q \begin{bmatrix} k \\ \ell \end{bmatrix}_q q^{\binom{\ell}{2}} (-1)^\ell (q; q)_\ell \\ &\quad \times z_1^{j-\ell} z_2^{k-\ell} \left(\frac{uq^\ell}{z_2}; q \right)_{k-\ell} \left(\frac{vq^k}{z_1}; q \right)_{j-\ell} (z_1u; q)_\ell, \end{aligned}$$

which gives (7.15).

More generally, we have

$$\begin{aligned}
 & \sum_{m \geq j, n \geq k} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_{m-j} (q; q)_{n-k}} u^{m-j} v^{n-k} \\
 = & (1-q)^{j+k} (bq; q)_\infty \sum_{\ell=0}^\infty \frac{(bq)^\ell}{(q; q)_\ell} D_{q,v}^k D_{q,u}^j \frac{(uvq^\ell; q)_\infty}{(uz_1q^\ell, z_2vq^\ell; q)_\infty} \\
 = & (1-q)^k (bq; q)_\infty z_1^j \sum_{\ell=0}^\infty \frac{(bq^{1+j})^\ell}{(q; q)_\ell (uz_1q^\ell; q)_\infty} D_{q,v}^k \left(\left(\frac{v}{z_1}; q \right)_j \frac{(uvq^{j+\ell}; q)_\infty}{(z_2vq^\ell; q)_\infty} \right) \\
 = & (bq; q)_\infty \sum_{\ell=0}^\infty \frac{(bq^{1+j+k})^\ell}{(q; q)_\ell (uz_1q^\ell; q)_\infty} \sum_{i=0}^\infty \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \\
 \times & (-1)^i z_1^{j-i} z_2^{k-i} q^{\binom{i}{2}-i\ell} (q; q)_i \left(\frac{v}{z_1} q^i; q \right)_{j-i} \left(\frac{u}{z_2} q^j; q \right)_{k-i} \frac{(uvq^{j+k+\ell}; q)_\infty}{(z_2vq^{\ell+i}; q)_\infty} \\
 = & \frac{(bq, uvq^{j+k}; q)_\infty}{(uz_1, vz_2; q)_\infty} \sum_{\ell=0}^\infty \frac{(uz_1vz_2; q)_\ell}{(q, uvq^{j+k}; q)_\ell} (bq^{1+j+k})^\ell \\
 \times & \sum_{i=0}^\infty \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q (-q^{-\ell})^i z_1^{j-i} z_2^{k-i} q^{\binom{i}{2}} (q; q)_i \left(\frac{v}{z_1} q^i; q \right)_{j-i} \left(\frac{u}{z_2} q^j; q \right)_{k-i} (z_2vq^\ell; q)_i,
 \end{aligned}$$

which gives

$$\begin{aligned}
 & \sum_{m,n=0}^\infty \frac{p_{m+j,n+k}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\
 = & \frac{(bq, uvq^{j+k}; q)_\infty}{(uz_1, vz_2; q)_\infty} \sum_{\ell=0}^\infty \frac{(bq^{1+j+k})^\ell (uz_1, vz_2; q)_\ell}{(q, uvq^{j+k}; q)_\ell} \sum_{i=0}^\infty \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \\
 \times & (-q^{-\ell})^i z_1^{j-i} z_2^{k-i} q^{\binom{i}{2}} (q; q)_i \left(\frac{v}{z_1} q^i; q \right)_{j-i} \left(\frac{u}{z_2} q^j; q \right)_{k-i} (z_2vq^\ell; q)_i.
 \end{aligned}$$

Similarly we have

$$\begin{aligned}
 & \sum_{m \geq j, n \geq k} \frac{p_{m,n}(z_1, z_2; b|q)}{(q; q)_{m-j} (q; q)_{n-k}} u^{m-j} v^{n-k} \\
 = & (1-q)^{j+k} (bq; q)_\infty \sum_{\ell=0}^\infty \frac{(bq)^\ell}{(q; q)_\ell} D_{q,u}^j \frac{1}{(uz_1q^\ell; q)_\infty} D_{q,v}^k \frac{(uvq^\ell; q)_\infty}{(z_2vq^\ell; q)_\infty} \\
 = & (1-q)^j z_2^k (bq; q)_\infty \sum_{\ell=0}^\infty \frac{(bq^{k+1})^\ell}{(q; q)_\ell (z_2vq^\ell; q)_\infty} D_{q,u}^j \left(\left(\frac{u}{z_2}; q \right)_k \frac{(uvq^{k+\ell}; q)_\infty}{(uz_1q^\ell; q)_\infty} \right) \\
 = & \frac{(bq, uvq^{j+k}; q)_\infty}{(uz_1, vz_2; q)_\infty} \sum_{\ell=0}^\infty \frac{(uz_1, vz_2; q)_\ell}{(q, uvq^{j+k}; q)_\ell} (bq^{j+k+1})^\ell \\
 \times & \sum_{i=0}^{j \wedge k} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\binom{i}{2}} (-q^{-\ell})^i (q; q)_i \\
 \times & z_1^{j-i} z_2^{k-i} \left(\frac{uq^i}{z_2}; q \right)_{k-i} \left(\frac{vq^k}{z_1}; q \right)_{j-i} (uz_1q^\ell; q)_i,
 \end{aligned}$$

which gives

$$\begin{aligned} & \sum_{m,n=0}^{\infty} \frac{p_{m+j,n+k}(z_1, z_2; b|q)}{(q; q)_m (q; q)_n} u^m v^n \\ &= \frac{(bq, uvq^{j+k}; q)_{\infty}}{(uz_1, vz_2; q)_{\infty}} \sum_{\ell=0}^{\infty} \frac{(uz_1, vz_2; q)_{\ell}}{(q, uvq^{j+k}; q)_{\ell}} (bq^{1+j+k})^{\ell} \\ & \times \sum_{i=0}^{\infty} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q (-q^{-\ell})^i z_1^{j-i} z_2^{k-i} q^{\binom{i}{2}} (q; q)_i \left(\frac{uq^i}{z_2}; q \right)_{k-i} \left(\frac{vq^k}{z_1}; q \right)_{j-i} (uz_1 q^{\ell}; q)_i, \end{aligned}$$

which establishes (7.17). □

8. ZEROS

In this section we study the zeros of the two 2D- q -Hermite polynomials and the q -analogue of the Zernike polynomials introduced in this paper. Because all polynomials factor as a function of θ times a radial function it is clear that with $z_1 = z, z_2 = \bar{z}$ the zeros of the polynomials investigated here as functions of z lie on circles.

Let

$$(8.1) \quad 0 < i_1(q) < i_2(q) < \dots$$

be the zeros of $A_q(z)$.

Theorem 27. *Assume that the zeros of $H_{m,n}(z, \bar{z}|q)$ and of $h_{m,n}(z, \bar{z}|q)$ lie on the circles with radii*

$$(8.2) \quad r_1(H, m, n) > r_2(H, m, n) > \dots \quad \text{and} \quad r_1(h, m, n) > r_2(h, m, n) > \dots,$$

respectively. Moreover let the zeros of $p_{m,n}(z, \bar{z}; b|q)$ lie on the circle $|z| = r_j(p, m, n)$, $j = 1, 2, \dots$, ordered as

$$(8.3) \quad r_1(p, m, n) > r_2(p, m, n) > \dots$$

Then

$$(8.4) \quad \lim_{m,n \rightarrow \infty} r_j(H, m, n) = q^{j/2}, \quad j = 1, 2, \dots,$$

$$(8.5) \quad \lim_{m,n \rightarrow \infty} q^{(m+n)/2} r_j(h, m, n) = 1/\sqrt{i_j(q)}, \quad j = 1, 2, \dots,$$

$$(8.6) \quad \lim_{m,n \rightarrow \infty} r_j(p, m, n) = q^{j/2}, \quad j = 1, 2, \dots$$

Proof. The first part, (8.4), follows from (3.24) since its left-hand side converges to its right-hand side on compact subsets of \mathbb{C} . Similarly (8.6) follows from (5.43). Formula (8.5) follows from Theorem 12 since the limit in Theorem 12 is uniform on compact subsets of \mathbb{C} . □

It is important to note that the support of the orthogonality measure of $\{H_{m,n}(z, \bar{z}|q)\}$ and $\{p_{m,n}(z, \bar{z}; b|q)\}$ coincides with the closure of the union of the limiting circles on which the zeros lie. This is similar to the single variable case. It is not surprising that the zeros of the Ramanujan function appear in the leading terms of the asymptotics of zeros of the polynomials $\{h_{m,n}(z, \bar{z}|q)\}$. This is again similar to the single variable case.

9. POSITIVITY RESULTS

Lemma 28. For $N \in \mathbb{N}_0$, $q \in (0, 1)$ and $z \in \mathbb{C} \setminus \{0\}$, the following matrices are positive definite:

$$(9.1) \quad \left(\frac{H_{m,n}(iz, i\bar{z})}{i^{m+n}} \right)_{m,n=0}^N, \quad (q^{mn}h_{m,n}(z, \bar{z}|q^{-1}))_{m,n=0}^N,$$

$$(9.2) \quad \left(\frac{q^{\frac{(m-n)^2}{2}}h_{m,n}(iz, i\bar{z}|q)}{i^{m+n}} \right)_{m,n=0}^N, \quad (H_{m,n}(z, \bar{z}|q^{-1}))_{m,n=0}^N.$$

Proof. Observe that

$$\begin{aligned} \frac{H_{m,n}(iz, i\bar{z}|q)}{i^{m+n}} &= q^{mn}h_{m,n}(z, \bar{z}|q^{-1}) \\ &= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(q; q)_k}{|z|^{2k}} \left\{ \begin{bmatrix} m \\ k \end{bmatrix}_q z^m \right\} \cdot \overline{\left\{ \begin{bmatrix} n \\ k \end{bmatrix}_q z^n \right\}} \end{aligned}$$

and

$$\begin{aligned} \frac{q^{\frac{(m-n)^2}{2}}h_{m,n}(iz, i\bar{z}|q)}{i^{m+n}} &= q^{\frac{m^2}{2}}q^{\frac{n^2}{2}}H_{m,n}(z, \bar{z}|q^{-1}) \\ &= \sum_{j=0}^{\infty} \frac{q^{j^2}(q; q)_j}{|z|^{2j}} \left\{ \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\frac{m^2}{2}-mj}z^m \right\} \overline{\left\{ \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{n^2}{2}-nj}z^n \right\}}. \end{aligned}$$

□

Theorem 29. For $z \neq 0$ and $0 < q < 1$, there exist sequences $e_n = \{e_m^{(n)}\}_{m=0}^{\infty}$ and $f_n = \{f_m^{(n)}\}_{m=0}^{\infty}$ with $e_m^{(n)} = f_m^{(n)} = 0$ for $m > n$ such that

$$(9.3) \quad \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^j e_m^{(j)} z^m \left(-\frac{\zeta q^{\frac{1}{2}}}{z}; q \right)_m \right\} \overline{\left\{ \sum_{n=0}^k e_n^{(k)} z^n \left(-\frac{\zeta q^{\frac{1}{2}}}{z}; q \right)_n \right\}} d\nu(\zeta, \bar{\zeta}) \\ = \sum_{\ell=0}^{\infty} \frac{q^{\binom{\ell}{2}}(q; q)_{\ell}}{|z|^{2\ell}} \sum_{m=\ell}^j \begin{bmatrix} m \\ \ell \end{bmatrix}_q e_m^{(j)} z^m \sum_{n=\ell}^k \begin{bmatrix} n \\ \ell \end{bmatrix}_q e_n^{(k)} z^n = \delta_{j,k}$$

and

$$(9.4) \quad \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^j f_m^{(j)} (-\zeta)^m \left(\frac{zq^{\frac{1}{2}}}{\zeta}; q \right)_m \right\} \cdot \overline{\left\{ \sum_{n=0}^k f_n^{(k)} (-\zeta)^n \left(\frac{zq^{\frac{1}{2}}}{\zeta}; q \right)_n \right\}} d\mu(\zeta, \bar{\zeta}) \\ = \sum_{\ell=0}^{\infty} \frac{q^{\ell^2}(q; q)_{\ell}}{|z|^{2\ell}} \left\{ \sum_{m=\ell}^j \begin{bmatrix} m \\ \ell \end{bmatrix}_q q^{\frac{m^2}{2}-m\ell} z^m f_m^{(j)} \right\} \overline{\left\{ \sum_{n=\ell}^k \begin{bmatrix} n \\ \ell \end{bmatrix}_q q^{\frac{n^2}{2}-n\ell} z^n f_n^{(k)} \right\}} = \delta_{j,k}$$

for $j, k = 0, 1, 2, \dots$

Proof. Let us define the inner vector spaces

$$H(\mathbb{N}_0; z, q) = \left\{ \{c_n\}_{n=0}^{\infty} \mid c_n \in \mathbb{C}, n \in \mathbb{N}_0, \sum_{m,n=0}^{\infty} \frac{H_{m,n}(iz, i\bar{z})}{i^{m+n}} c_m \bar{c}_n < \infty \right\}$$

with

$$(\{c_n\}_{n=0}^\infty, \{d_n\}_{n=0}^\infty)_H = \sum_{m,n=0}^\infty \frac{H_{m,n}(iz, i\bar{z})}{i^{m+n}} c_m \bar{d}_n,$$

where $\{c_n\}_{n=0}^\infty, \{d_n\}_{n=0}^\infty \in H(\mathbb{N}_0; z, q)$ and

$$h(\mathbb{N}_0; z, q) = \left\{ \{c_n\}_{n=0}^\infty \mid c_n \in \mathbb{C}, n \in \mathbb{N}_0, \sum_{m,n=0}^\infty \frac{q^{\frac{(m-n)^2}{2}} h_{m,n}(iz, i\bar{z}|q)}{i^{m+n}} c_m \bar{c}_n < \infty \right\}$$

with

$$(\{c_n\}_{n=0}^\infty, \{d_n\}_{n=0}^\infty)_h = \sum_{m,n=0}^\infty \frac{q^{\frac{(m-n)^2}{2}} h_{m,n}(iz, i\bar{z}|q)}{i^{m+n}} c_m \bar{d}_n,$$

where $\{c_n\}_{n=0}^\infty, \{d_n\}_{n=0}^\infty \in h(\mathbb{N}_0; z, q)$. Then, the vectors $\{\delta_{m,n}\}_{m=0}^\infty, n = 0, 1, \dots$, are linearly independent in these spaces. For $n \in \mathbb{N}_0$, let $e_n = \{e_m^{(n)}\}_{m=0}^\infty$ and $f_n = \{f_m^{(n)}\}_{m=0}^\infty$ be the obtained orthonormal bases from the orthogonalization process in $H(\mathbb{N}_0; z, q)$ and $h(\mathbb{N}_0; z, q)$ respectively. Then it is clear that $e_m^{(n)} = f_m^{(n)} = 0$ for $m > n$. Observe that

$$\frac{H_{m,n}(iz, i\bar{z})}{i^{m+n}} = z^m \bar{z}^n \int_{\mathbb{R}^2} \left(-\frac{\zeta q^{\frac{1}{2}}}{z}; q \right)_m \left(-\frac{\bar{\zeta} q^{\frac{1}{2}}}{\bar{z}}; q \right)_n d\nu(\zeta, \bar{\zeta}).$$

Then,

$$\begin{aligned} & \left(\{e_m^{(j)}\}_{m=0}^\infty, \{e_m^{(k)}\}_{m=0}^\infty \right)_H = \sum_{m,n=0}^\infty \frac{H_{m,n}(iz, i\bar{z})}{i^{m+n}} e_m^{(j)} \overline{e_n^{(k)}} = \delta_{j,k} \\ &= \sum_{\ell=0}^\infty \frac{q^{\binom{\ell}{2}} (q; q)_\ell}{|z|^{2\ell}} \sum_{m=\ell}^j \begin{bmatrix} m \\ \ell \end{bmatrix}_q e_m^{(j)} z^m \overline{\sum_{n=\ell}^k \begin{bmatrix} n \\ \ell \end{bmatrix}_q e_n^{(k)} z^n} \\ &= \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^j e_m^{(j)} z^m \left(-\frac{\zeta q^{\frac{1}{2}}}{z}; q \right)_m \right\} \overline{\left\{ \sum_{n=0}^k e_n^{(k)} z^n \left(-\frac{\bar{\zeta} q^{\frac{1}{2}}}{\bar{z}}; q \right)_n \right\}} d\nu(\zeta, \bar{\zeta}). \end{aligned}$$

Similarly, from

$$\frac{q^{\frac{(m-n)^2}{2}} h_{m,n}(iz, i\bar{z}|q)}{i^{m+n}} = \int_{\mathbb{R}^2} (-\zeta)^m (-\bar{\zeta})^n \left(\frac{zq^{\frac{1}{2}}}{\zeta}; q \right)_m \left(\frac{\bar{z}q^{\frac{1}{2}}}{\bar{\zeta}}; q \right)_n d\mu(\zeta, \bar{\zeta})$$

we get

$$\begin{aligned} & \left(\{f_m^{(j)}\}_{m=0}^\infty, \{f_m^{(k)}\}_{m=0}^\infty \right)_h = \sum_{m,n=0}^\infty \frac{q^{\frac{(m-n)^2}{2}} h_{m,n}(iz, i\bar{z}|q)}{i^{m+n}} f_m^{(j)} \overline{f_n^{(k)}} = \delta_{j,k} \\ &= \sum_{\ell=0}^\infty \frac{q^{\ell^2} (q; q)_\ell}{|z|^{2\ell}} \left\{ \sum_{m=\ell}^j \begin{bmatrix} m \\ \ell \end{bmatrix}_q q^{\frac{m^2}{2} - m\ell} z^m f_m^{(j)} \right\} \overline{\left\{ \sum_{n=\ell}^k \begin{bmatrix} n \\ \ell \end{bmatrix}_q q^{\frac{n^2}{2} - n\ell} z^n f_n^{(k)} \right\}} \\ &= \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^j f_m^{(j)} (-\zeta)^m \left(\frac{zq^{\frac{1}{2}}}{\zeta}; q \right)_m \right\} \cdot \overline{\left\{ \sum_{n=0}^k f_n^{(k)} (-\bar{\zeta})^n \left(\frac{\bar{z}q^{\frac{1}{2}}}{\bar{\zeta}}; q \right)_n \right\}} d\mu(\zeta, \bar{\zeta}). \end{aligned}$$

□

Remark 30. If we could invert the matrices (9.1) and (9.2), then we could determine $e_n, f_n, n = 0, 1, \dots$, explicitly.

Lemma 31. For $z \cdot \zeta \neq 0$ and $m = 0, 1, \dots$ we have

$$(9.5) \quad \zeta^m = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\binom{m-j}{2}} (-z)^{m-j} \left(-\zeta q^{1/2}/z; q\right)_j z^j$$

and

$$(9.6) \quad \zeta^m = \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix}_q \left(zq^{1/2}\right)^{m-j} \left(\frac{zq^{1/2}}{\zeta}; q\right)_j \zeta^j.$$

Proof. From the q -binomial theorem we have

$$\sum_{m=0}^{\infty} \frac{\left(-\zeta q^{1/2}/z; q\right)_m (zt)^m}{(q; q)_m} = \frac{\left(-\zeta tq^{1/2}; q\right)_{\infty}}{(zt; q)_{\infty}},$$

to get

$$\begin{aligned} \left(-\zeta tq^{1/2}; q\right)_{\infty} &= \sum_{m=0}^{\infty} \frac{t^m}{(q; q)_m} q^{m^2/2} \zeta^m \\ &= \sum_{m=0}^{\infty} \frac{t^m}{(q; q)_m} \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix}_q q^{\binom{m-j}{2}} (-z)^{m-j} \left(-\zeta q^{1/2}/z; q\right)_j z^j, \end{aligned}$$

and (9.5) is obtained by matching the coefficients of t^m .

Similarly, from

$$\frac{1}{(\zeta t; q)_{\infty}} = \frac{1}{(ztq^{1/2}; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(\zeta t)^k \left(\frac{zq^{1/2}}{\zeta}; q\right)_k}{(q; q)_k}$$

we get (9.6). □

Corollary 32. Let $z \neq 0$ and $0 < q < 1$. Then for $j, k = 0, 1, \dots$ we have

$$(9.7) \quad \sum_{\ell=0}^{\infty} \frac{q^{\binom{\ell}{2}} (q; q)_{\ell}}{|z|^{2\ell}} e_j(\ell|q) e_k(\ell|q) = \frac{(q; q)_j \log q^{-1}}{q^{\binom{j+1}{2}} |z|^{2j}} \delta_{j,k}$$

and

$$(9.8) \quad \sum_{\ell=0}^{\infty} \frac{q^{\ell^2} (q; q)_{\ell}}{|z|^{2\ell}} f_j(\ell|q) f_k(\ell|q) = \frac{\delta_{j,k}}{(q^{j+1}; q)_{\infty} |z|^{2j}},$$

where

$$(9.9) \quad e_j(\ell|q) = \sum_{m=\ell}^j \begin{bmatrix} m \\ \ell \end{bmatrix}_q \begin{bmatrix} j \\ m \end{bmatrix}_q q^{\binom{j-m}{2}} (-1)^{j-m}$$

and

$$(9.10) \quad f_j(\ell|q) = \sum_{m=\ell}^j \begin{bmatrix} m \\ \ell \end{bmatrix}_q \begin{bmatrix} j \\ m \end{bmatrix}_q q^{\frac{m^2}{2} - m\ell} \left(-q^{1/2}\right)^{j-m}.$$

Proof. Observe that

$$\begin{aligned} \delta_{j,k} \frac{(q; q)_j \log q^{-1}}{q^{\binom{j+1}{2}}} &= \int_{\mathbb{R}^2} \zeta^j \bar{\zeta}^k d\nu(\zeta, \bar{\zeta}) \\ &= \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^j \begin{bmatrix} j \\ m \end{bmatrix}_q q^{\binom{j-m}{2}} (-z)^{j-m} \left(-\zeta q^{1/2}/z; q \right)_m z^m \right\} \\ &\times \overline{\left\{ \sum_{n=0}^k \begin{bmatrix} k \\ n \end{bmatrix}_q q^{\binom{k-n}{2}} (-z)^{k-n} \left(-\zeta q^{1/2}/z; q \right)_n z^n \right\}} d\nu(\zeta, \bar{\zeta}). \end{aligned}$$

Let us take

$$e_m^{(j)} = \frac{q^{\frac{j(j+1)}{4}}}{\sqrt{(q; q)_j \log q^{-1}}} \begin{bmatrix} j \\ m \end{bmatrix}_q q^{\binom{j-m}{2}} (-z)^{j-m}$$

to get (9.7) and (9.9). Similarly, from

$$\begin{aligned} \frac{\delta_{j,k}}{(q^{j+1}; q)_\infty} &= \int_{\mathbb{R}^2} \zeta^j \bar{\zeta}^k d\mu(\zeta, \bar{\zeta}) \\ &= \int_{\mathbb{R}^2} \left\{ \sum_{m=0}^j \begin{bmatrix} j \\ m \end{bmatrix}_q \left(zq^{1/2} \right)^{j-m} \left(\frac{zq^{1/2}}{\zeta}; q \right)_m \zeta^m \right\} \\ &\times \overline{\left\{ \sum_{n=0}^k \begin{bmatrix} k \\ n \end{bmatrix}_q \left(zq^{1/2} \right)^{k-n} \left(\frac{zq^{1/2}}{\zeta}; q \right)_n \zeta^n \right\}} d\mu(\zeta, \bar{\zeta}), \end{aligned}$$

we take

$$f_m^{(j)} = \sqrt{(q^{j+1}; q)_\infty} \begin{bmatrix} j \\ m \end{bmatrix}_q \left(-zq^{1/2} \right)^{j-m}$$

to obtain (9.8) and (9.10). □

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