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ISOMETRIC DILATIONS AND H^∞ CALCULUS FOR BOUNDED ANALYTIC SEMIGROUPS AND RITT OPERATORS

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Abstract. We show that any bounded analytic semigroup on L^p (with $1 < p < \infty$) whose negative generator admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$ can be dilated into a bounded analytic semigroup $(R_t)_{t\geqslant 0}$ on a bigger L^p-space in such a way that R_t is a positive contraction for any $t \geqslant 0$. We also establish a discrete analogue for Ritt operators and consider the case when L^p-spaces are replaced by more general Banach spaces. In connection with these functional calculus issues, we study isometric dilations of bounded continuous representations of amenable groups on Banach spaces and establish various generalizations of Dixmier's unitarization theorem.

1. Introduction

In [Wei01, Remark 4.c], Weis showed that if $(T_t)_{t\geqslant 0}$ is a bounded analytic semigroup on a space $L^p(\Omega)$ (with $1 < p < \infty$) such that each operator $T_t \colon L^p(\Omega) \to L^p(\Omega)$ is a positive contraction, then its negative generator A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some angle $0 < \theta < \frac{\pi}{2}$. Here and later on in this paper,

$$\Sigma_\theta = \{z \in \mathbb{C}^* \ : \ |\mathrm{Arg}(z)| < \theta\} \tag{1.1}$$

denotes the open sector of angle 2θ around the positive real axis $\mathbb{R}_+^*$. The first main result of this paper is the following converse, which says that this class of semigroups generates *by dilation* the class of all bounded analytic semigroups on an L^p-space whose negative generator admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$. More precisely, we will prove the following.

Theorem 1.1. *Let Ω be a measure space and let $1 < p < \infty$. Let $(T_t)_{t\geqslant 0}$ be a bounded analytic semigroup on $L^p(\Omega)$ and assume that its negative generator admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $0 < \theta < \frac{\pi}{2}$. Then there exist a measure space Ω', a bounded analytic semigroup $(R_t)_{t\geqslant 0}$ on the space $L^p(\Omega')$ such that each $R_t \colon L^p(\Omega') \to L^p(\Omega')$ is a positive contraction, and two bounded operators $J \colon L^p(\Omega) \to L^p(\Omega')$ and $Q \colon L^p(\Omega') \to L^p(\Omega)$ such that*

$$T_t = QR_tJ, \qquad \text{for all } t \geqslant 0.$$

Note that in the above situation, J is an isomorphic embedding whereas JQ is a bounded projection. Hence the new space $L^p(\Omega')$ can be seen as a bigger space than the initial $L^p(\Omega)$, containing the latter as a complemented subspace.

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Theorem 1.1 improves a recent result by the second named author on the structure of L^p-semigroups with a bounded H^∞ functional calculus [Fac14]. Together with Weis's theorem, this provides a complete characterization of bounded H^∞ functional calculus on L^p-spaces. This characterization should be regarded as an L^p-analogue of the theorem from [LM98b] which says that if $(T_t)_{t\geqslant 0}$ is a bounded analytic semigroup on some Hilbert space, with negative generator A, then A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some angle $0<\theta<\frac{\pi}{2}$ if and only if $(T_t)_{t\geqslant 0}$ is similar to a contractive semigroup.

H^∞ functional calculus is a very useful and important tool in various areas: harmonic analysis of semigroups, multiplier theory, Kato's square root problem, maximal regularity in parabolic equations, control theory, etc. It grew up from the two fundamental papers [McI86, CDMY96]. For detailed information we refer the reader to [Haa06], [KW04], to the survey papers [Are04], [LM98a], [LM07] and [Wei06], and to the references therein.

We will also establish an analogue of Theorem 1.1 for Ritt operators. This class of operators can be regarded as the discrete analogue of the class of bounded analytic semigroups. Ritt operators have a natural and fruitful notion of H^∞ functional calculus with respect to Stolz domains B_γ; see Section 2 below and [LM14]. We will show that a Ritt operator $T\colon L^p(\Omega)\to L^p(\Omega)$ has a bounded $H^\infty(B_\gamma)$ functional calculus for some $0<\gamma<\frac{\pi}{2}$ (if and) only if there exist a measure space Ω', a contractive and positive Ritt operator $R\colon L^p(\Omega')\to L^p(\Omega')$, and two bounded operators $J\colon L^p(\Omega)\to L^p(\Omega')$ and $Q\colon L^p(\Omega')\to L^p(\Omega)$ such that $T^n=QR^nJ$ for all integers $n\geqslant 0$.

Theorem 1.1 and its discrete version above are proved in Section 5. We also give analogous results when L^p-spaces are replaced by other classes of Banach spaces, such as UMD spaces, noncommutative L^p-spaces, or quotients of closed subspaces of L^p-spaces.

Section 4 is devoted to closely related but different dilation results. Let $(T_t)_{t\geqslant 0}$ be a bounded analytic semigroup on a Banach space X, with negative generator A, and let $1<p<\infty$. Under mild conditions on X we show that if A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some angle $0<\theta<\frac{\pi}{2}$, then there exist a measure space Ω', a C_0-group $(U_t)_{t\in\mathbb{R}}$ of isometries on $L^p(\Omega';X)$ and two bounded operators $J\colon X\to L^p(\Omega';X)$ and $Q\colon L^p(\Omega';X)\to X$ such that $T_t=QU_tJ$ for all $t\geqslant 0$. This result improves a well-known dilation result of Fröhlich-Weis for operators with a bounded H^∞ functional calculus [FW06]. In a slightly different framework, the Fröhlich-Weis theorem yields a dilation $T_t=QU_tJ$ where $(U_t)_{t\in\mathbb{R}}$ is a *bounded* C_0-group. Passing from a bounded group to an isometric one makes a crucial difference for the applications that we develop in Section 5.

This led us to the question whether a bounded C_0-group can be dilated into an isometric one. Section 6 is devoted to this issue in the more general framework of amenable group representations. In the case of L^p-spaces, we show the following result.

Theorem 1.2. *Let G be an amenable locally compact group, let Ω be a measure space, let $1<p<\infty$ and let $\pi\colon G\to\mathcal{B}(L^p(\Omega))$ be a bounded strongly continuous representation. Then there exist a measure space Ω', a strongly continuous isometric representation $\pi'\colon G\to\mathcal{B}(L^p(\Omega'))$, and two bounded operators $J\colon L^p(\Omega)\to L^p(\Omega')$ and $Q\colon L^p(\Omega')\to L^p(\Omega)$ such that*

$$\pi(t)=Q\pi'(t)J,\qquad \textit{for all } t\in G.$$

We also show versions of this theorem when $L^p(\Omega)$ is replaced by a more general Banach space, and similarity results are established. Indeed Theorem 1.2 can be regarded as an L^p-analogue of Dixmier's unitarization theorem which says that any bounded strongly continuous representation of an amenable group on some Hilbert space is similar to a unitary one.

Section 2 is a preliminary one. It contains an introduction to sectorial operators, Ritt operators and their associated holomorphic functional calculus, and provides some background on compressions and on ordered spaces. In Section 3, we investigate 'fractional powers' of power bounded operators, following [Dun11]. These operators are crucial for the Ritt versions of our results.

We conclude this introduction with a few conventions to be used in this paper. Unless stated otherwise, the Banach spaces we consider are complex. Given any Banach spaces X, Y, we let $\mathcal{B}(X,Y)$ denote the space of all bounded operators from X into Y. We denote this space by $\mathcal{B}(X)$ when $Y = X$. We write Id_X for the identity operator on X, or simply Id if there is no ambiguity on X.

Let $1 < p < \infty$. A Banach space X is called an SQ_p-space (for subspace of quotient of L^p) provided that there exist a measure space Ω and two closed subspaces $F \subset E \subset L^p(\Omega)$ such that X is isometrically isomorphic to the quotient space E/F.

For any nonempty open set $\Sigma \subset \mathbb{C}$, we let $H^\infty(\Sigma)$ denote the algebra of all bounded analytic functions $\varphi\colon \Sigma \to \mathbb{C}$, equipped with the supremum norm

$$\|\varphi\|_{H^\infty(\Sigma)} = \sup\big\{|\varphi(z)| \,:\, z \in \Sigma\big\}.$$

For any $a \in \mathbb{C}$ and any $r > 0$, we let $D(a,r) \subset \mathbb{C}$ denote the open disc with center a and radius r. We simply denote by $\mathbb{D} = D(0,1)$ the open unit disc centered at 0.

Finally given any set Ω and a subset $\Lambda \subset \Omega$, we let $\chi_\Lambda\colon \Omega \to \{0,1\}$ denote the characteristic function of Λ.

2. Preliminaries

We start this section with some classical definitions and results on sectorial operators and their associated functional calculus. The construction and basic properties below go back to [McI86] and [CDMY96]; see also [KW01], [KW04] and [Haa06] for complements. We refer to [Gol85] or [Paz83] for some background on C_0-semigroups and the subclass of bounded analytic semigroups.

Let X be a Banach space. Let $A\colon D(A) \to X$ be a closed linear operator with dense domain $D(A) \subset X$ and let $\sigma(A)$ denote its spectrum. Recall the definition (1.1). We say that A is a sectorial operator of type $\mu \in (0,\pi)$ if $\sigma(A) \subset \overline{\Sigma_\mu}$ and for any $\nu \in (\mu,\pi)$, the set

$$\Big\{zR(z,A) \,:\, z \in \mathbb{C} \setminus \overline{\Sigma_\nu}\Big\} \tag{2.1}$$

is bounded in $\mathcal{B}(X)$, with $R(z,A) = (z\mathrm{Id} - A)^{-1}$ denoting the resolvent operator. It is well-known that an operator A is a sectorial operator of type $< \frac{\pi}{2}$ if and only if $-A$ generates a bounded analytic semigroup. In this case, this semigroup is denoted by $(e^{-tA})_{t\geqslant 0}$.

For any $\theta \in (0,\pi)$, let $H_0^\infty(\Sigma_\theta)$ denote the algebra of all bounded holomorphic functions $\varphi\colon \Sigma_\theta \to \mathbb{C}$ for which there exist two positive real numbers $s, K > 0$ such

that

$$|\varphi(z)| \leqslant K \frac{|z|^s}{1+|z|^{2s}}, \qquad \text{for all } z \in \Sigma_\theta.$$

Let $0 < \mu < \theta < \pi$ and let $\varphi \in H_0^\infty(\Sigma_\theta)$. Then for any $\nu \in (\mu, \theta)$, we set

$$\varphi(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \varphi(z) R(z, A)\, dz, \tag{2.2}$$

where the boundary $\partial\Sigma_\nu$ is oriented counterclockwise. The sectoriality condition ensures that this integral is absolutely convergent and defines a bounded operator on X. Moreover by Cauchy's theorem, this definition does not depend on the choice of ν. Further, the resulting mapping

$$\begin{array}{ccc} H_0^\infty(\Sigma_\theta) & \longrightarrow & \mathcal{B}(X) \\ \varphi & \longmapsto & \varphi(A) \end{array}$$

is an algebra homomorphism which is consistent with the usual functional calculus for rational functions.

We say that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if the latter homomorphism is bounded; that is, there exists a constant $K \geqslant 0$ such that

$$\big\|\varphi(A)\big\|_{X\to X} \leqslant K\|\varphi\|_{H^\infty(\Sigma_\theta)}, \qquad \text{for all } \varphi \in H_0^\infty(\Sigma_\theta).$$

If A has dense range and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, then the above homomorphism naturally extends to a bounded homomorphism $\varphi \mapsto \varphi(A)$ from the whole space $H^\infty(\Sigma_\theta)$ into $\mathcal{B}(X)$.

For a sectorial operator A, the fractional powers A^α can be defined for any $\alpha > 0$. We refer to [Haa06, Chapter 3], [KW04] and [MCSA01] for various definitions of these operators and their basic properties. The spectral mapping theorem for fractional powers states that

$$\sigma(A^\alpha) = \big\{z^\alpha \colon z \in \sigma(A)\big\}, \quad \alpha > 0. \tag{2.3}$$

If, in addition, the operator A is bounded, then for any $\alpha > 0$ the operator A^α is bounded as well. We will frequently use the following effect of taking fractional powers on sectoriality and functional calculus.

Lemma 2.1. *Let A be sectorial of type $0 < \mu < \pi$ and let $\alpha \in (0, \frac{\pi}{\mu})$. Then A^α is sectorial of type $\alpha\mu$. If further A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (\mu, \pi)$ and $\alpha < \frac{\pi}{\theta}$, then A^α admits a bounded $H^\infty(\Sigma_{\alpha\theta})$ functional calculus.*

We now turn to Ritt operators, the second key class of operators considered in this paper. We describe their holomorphic functional calculus and present some of their main features. There is now a vast literature on this topic in which details and complements can be found; see in particular [ALM14], [Arh13], [Blu01b], [Blu01a], [LM14], [Lyu99], [NZ99], [Nev93], [Vit05] and the references therein.

An operator $T \in \mathcal{B}(X)$ is called a Ritt operator if the two sets

$$\big\{T^n \,:\, n \geqslant 0\big\} \qquad \text{and} \qquad \big\{n(T^n - T^{n-1}) \,:\, n \geqslant 1\big\} \tag{2.4}$$

are bounded. One can show that this is equivalent to the spectral inclusion

$$\sigma(T) \subset \overline{\mathbb{D}} \tag{2.5}$$

and the norm-boundedness of the set

$$\big\{(z-1)R(z,T) \,:\, |z| > 1\big\}. \tag{2.6}$$

The boundedness of (2.6) implies the existence of a constant $K \geqslant 0$ such that

$$|z-1|\big\|R(z,T)\big\|_{X\to X} \leqslant K$$

whenever $\mathrm{Re}(z) > 1$. This implies that $\mathrm{Id} - T$ is a sectorial operator of type $< \frac{\pi}{2}$. In fact, it is known that a bounded operator $T\colon X \to X$ is a Ritt operator if and only if

$$\sigma(T) \subset \mathbb{D} \cup \{1\} \qquad \text{and} \qquad \mathrm{Id} - T \text{ is a sectorial operator of type } < \tfrac{\pi}{2}; \tag{2.7}$$

see e.g. [Blu01a, Proposition 2.2].

We introduce the Stolz domains B_γ as sketched in Figure 1. Namely, for any angle $0 < \gamma < \frac{\pi}{2}$, we let B_γ be the interior of the convex hull of 1 and the disc $D(0, \sin\gamma)$. We will use the fact that for any $0 < \gamma < \frac{\pi}{2}$ there exists a positive constant C_γ such that

$$\frac{|1-z|}{1-|z|} \leqslant C_\gamma, \qquad \text{for all } z \in B_\gamma. \tag{2.8}$$

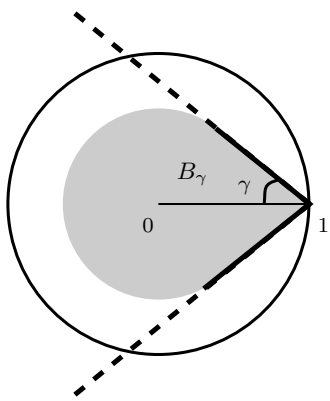

FIGURE 1. Stolz domain

It is well-known that the spectrum of any Ritt operator T is included in the closure of one of those Stolz domains. More precisely (see [LM14, Lemma 2.1]) there exists $0 < \beta < \frac{\pi}{2}$ such that

$$\sigma(T) \subset \overline{B_\beta} \tag{2.9}$$

and for any $\nu \in (\beta, \frac{\pi}{2})$, the set

$$\Big\{(z-1)R(z,T) \,:\, z \in \mathbb{C} \setminus \overline{B_\nu}\Big\} \qquad \text{is bounded.} \tag{2.10}$$

The following is an analogue of the construction (2.2). For any $\gamma \in (0, \frac{\pi}{2})$, let $H_0^\infty(B_\gamma)$ denote the space of all bounded holomorphic functions $\phi\colon B_\gamma \to \mathbb{C}$ for which there exist constants $s, K > 0$ such that $|\phi(z)| \leqslant K|1-z|^s$ for all $z \in B_\gamma$. Assume that T satisfies (2.9) and (2.10) for some $\beta < \gamma$. For $\phi \in H_0^\infty(B_\gamma)$ we set

$$\phi(T) \,=\, \frac{1}{2\pi i}\int_{\partial B_\nu} \phi(z)R(z,T)\,dz\,, \tag{2.11}$$

where $0 < \beta < \nu < \gamma$ and the boundary ∂B_ν is oriented counterclockwise. This definition does not depend on ν and is consistent with the usual functional calculus for polynomials.

In accordance with [LM14], we say that T has a bounded $H^\infty(B_\gamma)$ functional calculus if there exists a positive constant K such that

$$\big\|\phi(T)\big\|_{X\to X} \leqslant K\|\phi\|_{H^\infty(B_\gamma)}, \qquad \text{for all } \phi \in H_0^\infty(B_\gamma).$$

Let $\mathcal{P}$ be the algebra of all complex polynomials. We will use the following result (see [LM14, Proposition 2.5]).

Lemma 2.2. *A Ritt operator $T\colon X \to X$ admits a bounded $H^\infty(B_\gamma)$ functional calculus if (and only if) there exists a positive constant K such that*

$$\big\|\phi(T)\big\|_{X\to X} \leqslant K\|\phi\|_{H^\infty(B_\gamma)}, \qquad \textit{for all } \phi \in \mathcal{P}.$$

The following result proved in [LM14, Proposition 4.1] allows one to transfer known results from the theory of functional calculus for sectorial operators to the context of Ritt operators. Recall property (2.7).

Proposition 2.3. *Let $T\colon X \to X$ be a Ritt operator on a Banach space X. Then the following are equivalent.*

(i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*
(ii) *$\mathrm{Id} - T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$.*

We now turn to background on compressions. Let X be a Banach space and let $T\colon X \to X$ be a bounded operator. Let $F \subset E \subset X$ be closed subspaces. Assume that E and F are T-invariant; i.e. we have $T(E) \subset E$ and $T(F) \subset F$. Then T determines a bounded operator $\widetilde{T}\colon E/F \to E/F$, called a compression of T. It is characterized by the equality

$$\widetilde{T}q = qT|_E,$$

where $q\colon E \to E/F$ denotes the canonical quotient map. This clearly implies that for any complex polynomial ϕ, we have $\phi(\widetilde{T})q = q\phi(T)|_E$. Note that a compression of a compression of T is again a compression of T.

In the following, an algebraic semigroup $\mathscr{S}$ is a set supplied with an associative binary composition with identity e. A representation $\pi\colon \mathscr{S} \to \mathcal{B}(X)$ is a map satisfying $\pi(st) = \pi(s)\pi(t)$ for any $s, t \in \mathscr{S}$ and $\pi(e) = \mathrm{Id}_X$. We say that a closed subspace $E \subset X$ is π-invariant if it is $\pi(t)$-invariant for any $t \in \mathscr{S}$. If $F \subset E \subset X$ are two π-invariant closed subspaces, we let $\widetilde{\pi}\colon \mathscr{S} \to \mathcal{B}(E/F)$ be defined by $\widetilde{\pi}(t) = \widetilde{\pi(t)}$. It is plain that $\widetilde{\pi}$ is a representation of $\mathscr{S}$. This will be called a compressed representation in the sequel.

We will need the following proposition, which is a variant of [Pis01, Proposition 4.2]. See [Fac15, Proposition 5.5.6] for a proof.

Proposition 2.4. *Let $\mathscr{S}$ be an algebraic semigroup and let $\pi\colon \mathscr{S} \to \mathcal{B}(X)$ and $\rho\colon \mathscr{S} \to \mathcal{B}(Z)$ be representations of $\mathscr{S}$ on two Banach spaces X and Z. Assume that $J\colon X \to Z$ and $Q\colon Z \to X$ are two bounded operators such that*

$$\pi(t) = Q\rho(t)J, \qquad \textit{for all } t \in \mathscr{S}.$$

Then π is similar to a compression of ρ; that is, there exist ρ-invariant closed subspaces $F \subset E \subset Z$ and an isomorphism $S\colon X \to E/F$ such that $\|S\|\|S^{-1}\| \leqslant \|Q\|\|J\|$ and the compressed representation $\widetilde{\rho}\colon \mathscr{S} \to \mathcal{B}(E/F)$ satisfies

$$\pi(t) = S^{-1}\widetilde{\rho}(t)S, \qquad \textit{for all } t \in \mathscr{S}.$$

We end this section with some background on complexifications and orders. A complex Banach space X is called the complexification of a real Banach space $X_\mathbb{R}$ if $X_\mathbb{R}$ is a closed real subspace of X, we have a real direct sum decomposition $X = X_\mathbb{R} \oplus iX_\mathbb{R}$ and for any x_1, x_2 in $X_\mathbb{R}$,

$$\max\{\|x_1\|, \|x_2\|\} \leqslant \|x_1 + ix_2\|. \tag{2.12}$$

In this situation, we say that an operator $T \in \mathcal{B}(X)$ is real if T maps $X_\mathbb{R}$ into itself.

We say that X is an ordered Banach space when it is the complexification of a real Banach space $X_\mathbb{R}$ equipped with a partial order $\geqslant$ compatible with the vector space structure and such that the positive cone $\mathcal{C} = \{x \in X_\mathbb{R} : x \geqslant 0\}$ is closed and proper, that is, $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. An operator $T \in \mathcal{B}(X)$ on an ordered Banach space is called positive if $T(x) \geqslant 0$ for any $x \geqslant 0$.

An ordered Banach space X is called absolutely monotone if for any $x, y \in X_\mathbb{R}$, we have

$$-y \leqslant x \leqslant y \implies \|x\| \leqslant \|y\|. \tag{2.13}$$

Next X is called a Riesz-normed space if moreover for any $x \in X_\mathbb{R}$ and any $\epsilon > 0$, there exists $y \in X_\mathbb{R}$ such that $-y \leqslant x \leqslant y$ and $\|y\| \leqslant (1+\epsilon)\|x\|$. In this case any $x \in X_\mathbb{R}$ is the difference of two positive elements. Indeed if $-y \leqslant x \leqslant y$, then $y \geqslant 0$, $y - x \geqslant 0$ and $x = y - (y - x)$. Consequently, any positive $T \in \mathcal{B}(X)$ is real. We refer e.g. to [BR84] for more on these notions.

The class of Riesz-normed spaces includes Banach lattices, for which we refer to [LT79], and noncommutative L^p-spaces, for which we refer to [PX03].

3. Fractional powers for power bounded operators

Let X be a Banach space. A bounded operator $T\colon X \to X$ is power bounded if the set $\{T^n : n \geqslant 0\}$ is bounded in $\mathcal{B}(X)$. The spectrum of such an operator is contained in $\overline{\mathbb{D}}$, and $\mathrm{Id} - T$ is sectorial of type $\frac{\pi}{2}$; see e.g. [HT10, Lemma 3.1]. Thus we may define

$$T_\alpha \coloneqq \mathrm{Id} - (\mathrm{Id} - T)^\alpha \tag{3.1}$$

for any $\alpha > 0$. By abuse of language, these operators T_α will be called the *fractional powers of* T.

A power bounded operator is not necessarily a Ritt operator. However Dungey proved in [Dun11, Theorem 4.3] that T_α is a Ritt operator whenever $\alpha < 1$. We give an elementary proof and complements below.

Theorem 3.1. *Let X be a Banach space, let $T\colon X \to X$ be a power bounded operator and let $\alpha \in (0,1)$. Then we have the following:*

(a) *T_α is a Ritt operator.*
(b) *If T is contractive, then T_α is contractive as well.*
(c) *If X is an ordered Banach space and T is positive, then T_α is positive as well.*

Proof. We let $\alpha \in (0,1)$ and we set $g_\alpha(z) = 1 - (1-z)^\alpha$ for any $z \in \overline{\mathbb{D}}$ (with the convention that $0^\alpha = 0$). We have

$$g_\alpha(z) = \sum_{k=1}^{\infty} a_{\alpha,k} z^k \qquad \text{for all } z \in \overline{\mathbb{D}},$$

where

$$a_{\alpha,k} = (-1)^{k-1}\binom{\alpha}{k} > 0 \qquad \text{and} \qquad \sum_{k=1}^{\infty} a_{\alpha,k} = 1. \tag{3.2}$$

Using the compatibility of the holomorphic functional calculus with fractional powers (see [HT10, Proposition 3.2]), this implies that

$$T_\alpha = \mathrm{Id} - (\mathrm{Id} - T)^\alpha = \sum_{k=1}^{\infty} a_{\alpha,k} T^k. \tag{3.3}$$

Next observe that for any $z \in \overline{\mathbb{D}}$, we have

$$|g_\alpha(z)| = \left|\sum_{k=1}^{\infty} a_{\alpha,k} z^k\right| \leqslant \sum_{k=1}^{\infty} a_{\alpha,k}|z|^k \leqslant \sum_{k=1}^{\infty} a_{\alpha,k} = 1.$$

This shows that $g_\alpha(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$. Moreover, it is easy to see that the above inequality is an equality (if and) only if $z = 1$. Hence we deduce that

$$g_\alpha(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{1\}. \tag{3.4}$$

By (2.3), we have $\sigma(T_\alpha) = g_\alpha(\sigma(T))$. Moreover since T is power bounded, we have $\sigma(T) \subset \overline{\mathbb{D}}$. We deduce that $\sigma(T_\alpha) \subset g_\alpha(\overline{\mathbb{D}})$ and hence

$$\sigma(T_\alpha) \subset \mathbb{D} \cup \{1\}.$$

Moreover the fractional power $(\mathrm{Id} - T)^\alpha$ is sectorial of type $\frac{\alpha\pi}{2} < \frac{\pi}{2}$, by Lemma 2.1. Applying (2.7), we conclude that T_α is a Ritt operator.

This completes (a). The proofs of (b) and (c) immediately follow from (3.2) and (3.3). □

We now consider the behaviour of T_α when T is a Ritt operator and $\alpha > 1$.

Proposition 3.2. *Let $T\colon X \to X$ be a Ritt operator on a Banach space X.*

(a) *For sufficiently small $\alpha > 1$, T_α is a Ritt operator.*
(b) *Assume that T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. Then for sufficiently small $\alpha > 1$, T_α admits a bounded $H^\infty(B_{\gamma'})$ functional calculus for some $\gamma' \in (0, \frac{\pi}{2})$.*

Proof. Part (a) is already contained in [Dun11, Theorem 1.3 (IV)]. For the sake of completeness, we recall the short argument. The operator $\mathrm{Id} - T$ is sectorial of type $< \frac{\pi}{2}$. Hence for $\alpha > 1$ sufficiently close to 1, the operator $\mathrm{Id} - T_\alpha = (\mathrm{Id} - T)^\alpha$ is also sectorial of type $< \frac{\pi}{2}$, by Lemma 2.1. Moreover, by (2.3) and (2.9), it is easy to see that

$$\sigma(T_\alpha) = \big\{1 - (1-z)^\alpha \ : \ z \in \sigma(T)\big\} \subset \mathbb{D} \cup \{1\}$$

for any $\alpha > 1$ sufficiently close to 1. By (2.7), we conclude that T_α is a Ritt operator.

To prove part (b), assume that T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. It follows from Proposition 2.3 that the operator $\mathrm{Id} - T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$. For sufficiently small $\alpha > 1$, it follows from Lemma 2.1 that $(\mathrm{Id} - T)^\alpha$ has a bounded $H^\infty(\Sigma_{\theta'})$ functional calculus for some $\theta' \in (0, \frac{\pi}{2})$. The converse implication of Proposition 2.3 now implies the assertion. □

We now prove an H^∞ functional calculus property on UMD Banach spaces. As is well-known, this class provides a natural setting for functional calculus and vector-valued harmonic analysis. We refer to [Bur01] and the references therein for information. We simply note for further use that for any $p \in (1,\infty)$ and any measure space Ω, the Bochner space $L^p(\Omega; X)$ is UMD if X is UMD. Next, the UMD property is stable under passing to subspaces and quotients. In particular, SQ_p-spaces are UMD for any $1 < p < \infty$. Furthermore, UMD spaces are always reflexive and the dual of any UMD space is UMD as well.

We say that an isomorphism $U\colon X \to X$ on a Banach space X is a power bounded isomorphism if

$$\{U^n \,:\, n \in \mathbb{Z}\}$$

is bounded in $\mathcal{B}(X)$. In this case, U_α is a well-defined Ritt operator for any $\alpha \in (0,1)$, by Theorem 3.1.

Theorem 3.3. *Let U be a power bounded isomorphism on a UMD Banach space X. Then for every $\alpha \in (0,1)$, the operator U_α admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*

Proof. The proof heavily relies on some results and techniques from [ALM14] concerning the shift operator. Following this paper, we consider

$$\mathbb{D}_\theta = D\big(-\mathrm{i}\cot(\theta), \tfrac{1}{\sin(\theta)}\big) \cup D\big(\mathrm{i}\cot(\theta), \tfrac{1}{\sin(\theta)}\big)$$

for any $\frac{\pi}{2} < \theta < \pi$ (see Figure 2).

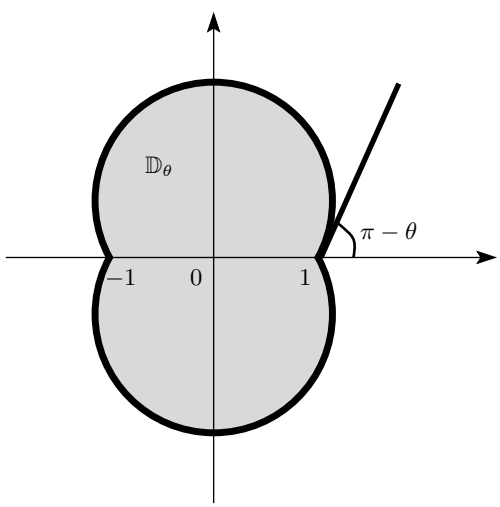

FIGURE 2. Domain $\mathbb{D}_\theta$

Let $\alpha \in (0,1)$ and let $\theta > \frac{\pi}{2}$ such that $\alpha\theta < \frac{\pi}{2}$. Let $S_X\colon \ell^p_{\mathbb{Z}}(X) \to \ell^p_{\mathbb{Z}}(X)$ denote the vector-valued shift operator given by

$$S_X\big((x_k)_{k\in\mathbb{Z}}\big) = (x_{k-1})_{k\in\mathbb{Z}}.$$

Let $K = \sup_{n\in\mathbb{Z}} \|U^n\|$. By the transference principle [BGM89, Theorem 2.8], one has

$$\big\|\varphi(U)\big\|_{X\to X} \leqslant K^2 \big\|\varphi(S_X)\big\|_{\ell^p_{\mathbb{Z}}(X)\to\ell^p_{\mathbb{Z}}(X)}$$

for any rational function φ with poles outside $\overline{\mathbb{D}_\theta}$. Further, it follows from the proof of [ALM14, Proposition 6.2] that there exists a constant $C > 0$ such that

$$\big\|\varphi(S_X)\big\|_{\ell^p_{\mathbb{Z}}(X)\to\ell^p_{\mathbb{Z}}(X)} \leqslant C\,\sup\big\{|\varphi(z)| \,:\, z \in \mathbb{D}_\theta\big\}$$

for such functions. The existence of this constant C depends on the UMD property of X. Combining these two estimates yields

$$\big\|\varphi(U)\big\|_{X\to X} \leqslant CK^2\,\sup\big\{|\varphi(z)| \,:\, z \in \mathbb{D}_\theta\big\}$$

for any rational function φ with poles outside $\overline{\mathbb{D}_\theta}$. Then the inclusion $1 - \mathbb{D}_\theta \subset \Sigma_\theta$ and the method of the proof of [ALM14, Propositions 4.7 and 6.2] imply that the sectorial operator $\mathrm{Id} - U$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Applying Lemma 2.1 we deduce that the operator $(\mathrm{Id} - U)^\alpha$ has a bounded $H^\infty(\Sigma_{\alpha\theta})$ functional calculus. Finally by Proposition 2.3, we obtain that the Ritt operator $U_\alpha = \mathrm{Id} - (\mathrm{Id} - U)^\alpha$ has a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. □

4. Positive isometric dilations of Ritt operators and analytic semigroups

The aim of this section is to improve dilation results from [FW06] and [ALM14] concerning bounded analytic semigroups or Ritt operators with a bounded H^∞ functional calculus. The main point is that we are able to construct isometric dilations, whereas the above cited papers only established isomorphic dilations. Also the class of Banach spaces on which we consider semigroups or Ritt operators is larger than the ones in [FW06, ALM14].

Our constructions will rely on abstract square functions. We start with some preliminaries on such objects in the discrete case. We let $\Omega_0 = \{-1, 1\}^{\mathbb{Z}}$ equipped with its normalized Haar measure. For any integer $k \in \mathbb{Z}$, we define ε_k by $\varepsilon_k(\omega) = \omega_k$ if $\omega = (\omega_i)_{i\in\mathbb{Z}} \in \Omega_0$. The coordinate functionals ε_k are independent Rademacher variables on the probability space Ω_0.

Let X be a Banach space and let $1 < p < \infty$. We let $\mathrm{Rad}_p(X) \subset L^p(\Omega_0; X)$ be the closure of $\mathrm{Span}\big\{\varepsilon_k \otimes x \mid k \in \mathbb{Z},\ x \in X\big\}$ in the Bochner space $L^p(\Omega_0; X)$. Thus, for any finite family $(x_k)_{k\in\mathbb{Z}}$ of elements of X, we have

$$\Big\|\sum_{k\in\mathbb{Z}} \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_p(X)} = \bigg(\int_{\Omega_0} \Big\|\sum_{k\in\mathbb{Z}} \varepsilon_k(\omega)\, x_k\Big\|_X^p\, d\omega\bigg)^{\frac{1}{p}}.$$

We simply write $\mathrm{Rad}(X) = \mathrm{Rad}_2(X)$. By Kahane's inequalities (see e.g. [DJT95, Theorem 11.1]), the Banach spaces $\mathrm{Rad}(X)$ and $\mathrm{Rad}_p(X)$ are canonically isomorphic.

Let $T\colon X \to X$ be a Ritt operator and let $\alpha > 0$. For any $x \in X$ and any $k \geqslant 0$, consider the element $x_k = (k+1)^{\alpha-\frac{1}{2}} T^k(\mathrm{Id} - T)^\alpha x$ of X. If the series $\sum_{k\geqslant 0} \varepsilon_k \otimes x_k$ converges in $L^2(\Omega_0; X)$, then we set

$$\|x\|_{T,\alpha} = \Big\|\sum_{k=0}^{\infty} (k+1)^{\alpha-\frac{1}{2}}\, \varepsilon_k \otimes T^k(\mathrm{Id} - T)^\alpha x\Big\|_{\mathrm{Rad}(X)}.$$

We set $\|x\|_{T,\alpha} = \infty$ otherwise. These 'square functions' $\|\ \|_{T,\alpha}$ were introduced in [ALM14], to which we refer for more information (see also [LM14]).

In the sequel we consider Banach spaces with finite cotype. We refer the reader e.g. to [DJT95] for information on cotype. We note that if a Banach space X is UMD, then X and X^* have finite cotype.

We also note that if $X = X_{\mathbb{R}} \oplus iX_{\mathbb{R}}$ is an ordered Banach space (see the last part of Section 2), then $L^p(\Omega'; X)$ is the complexification of the real space $L^p(\Omega'; X_{\mathbb{R}})$. Moreover the latter has a natural order defined by writing for $f \in L^p(\Omega'; X_{\mathbb{R}})$ that $f \geqslant 0$ when $f(\omega) \geqslant 0$ for almost every $\omega \in \Omega'$. This makes $L^p(\Omega'; X)$ an ordered Banach space.

Theorem 4.1. *Assume that X is a reflexive Banach space and that X and X^* have finite cotype. Let $T\colon X \to X$ be a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. Let $1 < p < \infty$. Then there exist a measure space Ω', an isometric isomorphism $U\colon L^p(\Omega'; X) \to L^p(\Omega'; X)$ together with two bounded operators $J\colon X \to L^p(\Omega'; X)$ and $Q\colon L^p(\Omega'; X) \to X$ such that*

$$T^n = QU^nJ, \qquad \text{for all } n \geqslant 0.$$

Moreover:

(a) *If X is an ordered Banach space, then the map U can be chosen to be a positive operator.*
(b) *If X is a closed subspace of an L^p-space, then the Bochner space $L^p(\Omega'; X)$ is also a closed subspace of an L^p-space and the map U can be chosen to be the restriction of a positive isometric isomorphism on an L^p-space.*
(c) *If X is an SQ_p-space, then the Bochner space $L^p(\Omega'; X)$ is also an SQ_p-space, and the map U can be chosen to be a compression of a positive isometric isomorphism on an L^p-space.*

Proof. We start with the construction of U, which is a universal operator (it does not depend on T), and we check its properties listed in (a), (b) and (c). First we define $\mathfrak{u}\colon L^p(\Omega_0) \to L^p(\Omega_0)$ as the pullback of the coordinate right shift. That is, for any $f \in L^p(\Omega_0)$ and any $(\omega_k)_{k\in\mathbb{Z}} \in \Omega_0$,

$$\mathfrak{u}(f)\big((\omega_k)_k\big) = f\big((\omega_{k-1})_k\big).$$

Then $\mathfrak{u}$ is a positive isometric isomorphism. Hence one can extend $\mathfrak{u} \otimes \mathrm{Id}_X$ to an isometric isomorphism $\mathfrak{U}$ on the Bochner space $L^p(\Omega_0; X)$.

Note that for any $k \in \mathbb{Z}$, one has $\mathfrak{u}(\varepsilon_k) = \varepsilon_{k-1}$. Consequently, for any element $\sum_{k\in\mathbb{Z}} \varepsilon_k \otimes x_k$ in $\mathrm{Rad}_p(X)$, we have

$$\mathfrak{U}\Big(\sum_{k\in\mathbb{Z}} \varepsilon_k \otimes x_k\Big) = \sum_{k\in\mathbb{Z}} \varepsilon_{k-1} \otimes x_k = \sum_{k\in\mathbb{Z}} \varepsilon_k \otimes x_{k+1}. \tag{4.1}$$

The ℓ^p-direct sum $X \oplus_p L^p(\Omega_0; X)$ is canonically isometrically isomorphic to a Bochner space $L^p(\Omega'; X)$. We let U be the isometric isomorphism $\mathrm{Id}_X \oplus \mathfrak{U}$ on the space $X \oplus_p L^p(\Omega_0; X) = L^p(\Omega'; X)$.

If X is an ordered Banach space, then U is clearly a positive operator. If X is a closed subspace of an L^p-space $L^p(\Omega)$, then U is the restriction of the positive isometry $\mathrm{Id}_{L^p(\Omega)} \oplus (\mathfrak{u}\overline{\otimes}\mathrm{Id}_{L^p(\Omega)})$ on the L^p-space $L^p(\Omega) \oplus_p L^p(\Omega_0; L^p(\Omega)) = L^p(\Omega' \times \Omega)$. If $X = E/F$ for some closed subspaces $F \subset E \subset L^p(\Omega)$, then $L^p(\Omega'; F) \subset L^p(\Omega'; E)$ are closed subspaces of $L^p(\Omega'; L^p(\Omega)) = L^p(\Omega' \times \Omega)$ and, by [DF93, Proposition 7.4], we have an isometric isomorphism

$$L^p(\Omega'; X) = L^p(\Omega'; E/F) = L^p(\Omega'; E)/L^p(\Omega'; F).$$

Hence the Bochner space $L^p(\Omega', X)$ is a quotient of a closed subspace of $L^p(\Omega' \times \Omega)$. Moreover U is the compression of the positive isometric isomorphism $\mathrm{Id}_{L^p(\Omega)} \oplus (\mathfrak{u} \otimes \mathrm{Id}_{L^p(\Omega)})\colon L^p(\Omega' \times \Omega) \to L^p(\Omega' \times \Omega)$ with respect to the subspaces $L^p(\Omega', F) \subset L^p(\Omega', E) \subset L^p(\Omega' \times \Omega)$.

We now show that U is a dilation of T. Since the Banach space X has finite cotype, we obtain from the assumption and [LM14, Theorem 6.4] that the operator T admits a quadratic $H^\infty(B_{\gamma'})$ functional calculus for some $\gamma' \in (\gamma, \frac{\pi}{2})$. That is, there exists a positive constant C such that for any integer $n \geqslant 0$, for any $\phi_0, \dots, \phi_n$ in $H_0^\infty(B_{\gamma'})$ and for any $x \in X$,

$$\Big\| \sum_{k=0}^n \varepsilon_k \otimes \phi_k(T)x \Big\|_{\mathrm{Rad}(X)} \leqslant C\|x\|_X \sup\Big\{ \Big(\sum_{k=0}^n |\phi_k(z)|^2 \Big)^{\frac12} \ : \ z \in B_{\gamma'} \Big\}.$$

Let us apply this estimate with $\phi_k(z) = z^k(1-z)^{\frac12}$. For any $z \in B_{\gamma'}$, we have

$$\sum_{k=0}^\infty |\phi_k(z)|^2 = \sum_{k=0}^\infty |z|^{2k}|1-z| = \frac{|1-z|}{1-|z|^2} \leqslant \frac{|1-z|}{1-|z|} \leqslant C_{\gamma'},$$

by (2.8). We deduce that for any $n \geqslant 1$,

$$\Big\| \sum_{k=0}^n \varepsilon_k \otimes T^k(\mathrm{Id}-T)^{\frac12} x \Big\|_{\mathrm{Rad}(X)} \leqslant CC_\gamma \|x\|_X.$$

Since X has finite cotype, this uniform estimate implies the convergence of the series $\sum_k \varepsilon_k \otimes T^k(\mathrm{Id}-T)^{\frac12}x$ (see [Kwa74]). Then we deduce (with $C' = CC_\gamma$) the following square function estimate:

$$\|x\|_{T,\frac12} \leqslant C'\|x\|_X, \qquad \text{for all } x \in X.$$

Similarly, there is a positive constant C'' such that

$$\|y\|_{T^*,\frac12} \leqslant C''\|y\|_{X^*}, \qquad \text{for all } y \in X^*.$$

Since T is power bounded and X is reflexive, the Mean Ergodic Theorem (see e.g. [Kre85, Subsection 2.1.1]) ensures that we have direct sum decompositions

$$X = \mathrm{Ker}\,(\mathrm{Id}-T) \oplus \overline{\mathrm{Ran}(\mathrm{Id}-T)} \qquad \text{and} \qquad X^* = \mathrm{Ker}\,(\mathrm{Id}-T^*) \oplus \overline{\mathrm{Ran}(\mathrm{Id}-T^*)}.$$

Furthermore for any $x_0 \in \mathrm{Ker}\,(\mathrm{Id}-T), x_1 \in \overline{\mathrm{Ran}(\mathrm{Id}-T)}, y_0 \in \mathrm{Ker}\,(\mathrm{Id}-T^*)$ and $y_1 \in \overline{\mathrm{Ran}(\mathrm{Id}-T^*)}$, we have $\langle x_0, y_1\rangle = \langle x_1, y_0\rangle = 0$, hence

$$\langle x_0 + x_1, y_0 + y_1\rangle \ = \ \langle x_0, y_0\rangle + \langle x_1, y_1\rangle. \tag{4.2}$$

Let p' be the conjugate of p. Note that since X is reflexive we have an isometric isomorphism

$$L^p(\Omega; X)^* \ = \ L^{p'}(\Omega; X^*). \tag{4.3}$$

From the above square function estimates, we may define a bounded linear map

$$\begin{aligned} J_1\colon X = \mathrm{Ker}\,(\mathrm{Id}-T) \oplus \overline{\mathrm{Ran}(\mathrm{Id}-T)} &\to X \oplus_p L^p(\Omega_0; X) \\ x_0 + x_1 &\mapsto \Big(x_0, \sum_{k=0}^\infty \varepsilon_k \otimes T^k(\mathrm{Id}-T)^{\frac12} x_1 \Big) \end{aligned}$$

and a similar $J_2\colon X^* \to X^* \oplus_{p'} L^{p'}(\Omega_0; X)$. Consider $x_0 \in \operatorname{Ker}(\mathrm{Id} - T)$, $x_1 \in \overline{\operatorname{Ran}(\mathrm{Id} - T)}$, $y_0 \in \operatorname{Ker}(\mathrm{Id} - T^*)$ and $y_1 \in \overline{\operatorname{Ran}(\mathrm{Id} - T^*)}$. For any $n \geqslant 0$, we have

$$U^n J_1(x_0 + x_1) = \Big(x_0, \sum_{k=-n}^{\infty} \varepsilon_k \otimes T^{k+n}(\mathrm{Id} - T)^{\frac{1}{2}} x_1\Big),$$

by (4.1). Hence

$$\begin{aligned}
&\big\langle U^n J_1(x_0 + x_1), J_2(y_0 + y_1)\big\rangle \\
&\quad = \Big\langle \Big(x_0, \sum_{k=-n}^{\infty} \varepsilon_k \otimes T^{k+n}(\mathrm{Id} - T)^{\frac{1}{2}} x_1\Big), \Big(y_0, \sum_{k=0}^{\infty} \varepsilon_k \otimes T^{*n}(\mathrm{Id} - T^*)^{\frac{1}{2}} y_1\Big)\Big\rangle \\
&\quad = \langle x_0, y_0\rangle + \sum_{k=0}^{\infty} \big\langle T^{k+n}(\mathrm{Id} - T)^{\frac{1}{2}} x_1, T^{*n}(\mathrm{Id} - T^*)^{\frac{1}{2}} y_1\big\rangle.
\end{aligned}$$

As in the proof of [ALM14, Theorem 4.8] one shows, using the fact that $x_1 \in \overline{\operatorname{Ran}(\mathrm{Id} - T)}$ and $y_1 \in \overline{\operatorname{Ran}(\mathrm{Id} - T^*)}$, that we have

$$\sum_{k=0}^{\infty} \big\langle T^{k+n}(\mathrm{Id} - T)^{\frac{1}{2}} x_1, T^{*n}(\mathrm{Id} - T^*)^{\frac{1}{2}} y_1\big\rangle = \big\langle (\mathrm{Id} + T)^{-1} T^n x_1, y_1\big\rangle.$$

Now we introduce the bounded operator

$$\begin{aligned}
\Theta\colon X = \operatorname{Ker}(\mathrm{Id} - T) \oplus \overline{\operatorname{Ran}(\mathrm{Id} - T)} &\to \operatorname{Ker}(\mathrm{Id} - T) \oplus \overline{\operatorname{Ran}(\mathrm{Id} - T)} = X \\
x_0 + x_1 &\mapsto x_0 + (\mathrm{Id} + T) x_1.
\end{aligned}$$

Then it follows from the results above that

$$\big\langle U^n J_1 \Theta(x_0 + x_1), J_2(y_0 + y_1)\big\rangle = \langle x_0, y_0\rangle + \langle T^n x_1, y_1\rangle.$$

Using (4.3), we define $Q = J_2^*\colon X \oplus_p L^p(\Omega_0; X) \to X$. Then letting $J = J_1 \Theta$, the above identity and (4.2) yield $T^n = Q U^n J$ for any $n \geqslant 0$. □

We now pass to semigroups. Roughly speaking, Rademacher averages used in the proof of Theorem 4.1 will be replaced by deterministic stochastic integration with respect to some Brownian motion. We need some preliminaries on second quantization and on the vector valued Gaussian spaces $\gamma(\cdot\,, X)$ introduced by Kalton-Weis in [KW14] (a first version of this paper was circulated in 2001). These γ-spaces were introduced in order to define abstract square functions in the context of H^∞ functional calculus. Since then, they were used in various other directions; see in particular [LM10] and [vNW05, vNVW13]. Note that in the present paper we need to mix real valued Gaussian variables and complex Banach spaces.

We start with a little background on Gaussian Hilbert spaces and second quantization. For more systematic discussions on this topic we refer to the books [Jan97] and [Sim74]. Let $H_{\mathbb{R}}$ be a real Hilbert space. A Gaussian random process indexed by $H_{\mathbb{R}}$ is a probability space $\widehat{\Omega}$ together with a linear isometry

$$W\colon H_{\mathbb{R}} \longrightarrow L^2_{\mathbb{R}}(\widehat{\Omega}) \tag{4.4}$$

satisfying the following two properties.

(i) Each $W(h)$ is a Gaussian random variable.
(ii) The linear span of the products $W(h_1)W(h_2)\cdots W(h_m)$, with $m \geqslant 0$ and $h_1, \ldots, h_m$ in $H_{\mathbb{R}}$, is dense in the real Hilbert space $L^2_{\mathbb{R}}(\widehat{\Omega})$.

Here we make the convention that the empty product, correspondong to $m = 0$ in (ii), is the constant function 1. Each product $W(h_1)W(h_2)\cdots W(h_m)$ belongs to $L^p_{\mathbb{R}}(\widehat{\Omega})$ for any $1 \leqslant p < \infty$ and their linear span is also dense in $L^p_{\mathbb{R}}(\widehat{\Omega})$.

Let $T\colon H_{\mathbb{R}} \to H_{\mathbb{R}}$ be a contraction. The 'second quantization of T' is a positive operator $\Gamma(T)\colon L^1(\widehat{\Omega}) \to L^1(\widehat{\Omega})$ such that $\Gamma(T)(1) = 1$,

$$\Gamma(T)W(h) = W\big(T(h)\big), \qquad \text{for all } h \in H_{\mathbb{R}}, \tag{4.5}$$

and for any $1 \leqslant p < \infty$, $\Gamma(T)$ restricts to a contraction

$$\Gamma_p(T)\colon L^p(\widehat{\Omega}) \longrightarrow L^p(\widehat{\Omega}).$$

Furthermore, the second quantization functor Γ satisfies the following.

Lemma 4.2. *Let $1 \leqslant p < \infty$.*

(i) *For any two contractions $T_1, T_2\colon H_{\mathbb{R}} \to H_{\mathbb{R}}$, we have*

$$\Gamma_p(T_1T_2) = \Gamma_p(T_1)\Gamma_p(T_2).$$

(ii) *If $(T_t)_{t\geqslant 0}$ is a C_0-semigroup of contractions on $H_{\mathbb{R}}$, then $\big(\Gamma_p(T_t)\big)_{t\geqslant 0}$ is a C_0-semigroup of contractions on $L^p(\widehat{\Omega})$.*

These properties can be found in [Jan97, Chapter 4], except the assertion (ii) from Lemma 4.2. In the latter statement, the semigroup property follows from (i), and strong continuity of $t \mapsto \Gamma_p(T_t)$ follows from the argument in [Jan97, Theorem 4.20]. Note that in this construction, T acts on the *real* Banach space $H_{\mathbb{R}}$, whereas $\Gamma_p(T)$ acts on the *complex* Banach space $L^p(\widehat{\Omega})$.

In the sequel we let H denote the standard complexification of $H_{\mathbb{R}}$. For convenience we keep the notation W to denote the complexification $W\colon H \to L^2(\widehat{\Omega})$ of (4.4). This is an isometry. Let $(g_n)_{n\geqslant 1}$ be an independent sequence of real valued standard Gaussian variables on some probability space.

Let X be a complex Banach space and let H^* denote the dual space of H. We identify the algebraic tensor product $H \otimes X$ with the subspace of $\mathcal{B}(H^*, X)$ of all bounded finite rank operators in the usual way. Namely for any $h_1, \dots, h_n$ in H and any $x_1, \dots, x_n$ in X, we identify the element $\sum_{k=1}^n h_k \otimes x_k$ with the operator $u\colon H^* \to X$ defined by $u(\xi) = \sum_{k=1}^n \xi(h_k)\, x_k$ for any $\xi \in H^*$. For any $u \in H \otimes X$, there exists a finite orthonormal family $(e_1, \dots, e_n)$ in the real space $H_{\mathbb{R}}$ and a finite family $(x_1, \dots, x_n)$ of X such that $u = \sum_{k=1}^n e_k \otimes x_k$. Then for any $1 \leqslant p < \infty$, we set

$$\|u\|_{\gamma^p(H^*,X)} = \left(\mathbb{E}\Big\|\sum_{k=1}^n g_k \otimes x_k\Big\|_X^p\right)^{\frac{1}{p}}.$$

By [DJT95, Corollary 12.17] and its proof, this definition does not depend on the e_k's and x_k's representing u. We let $\gamma^p(H^*, X)$ be the completion of $H \otimes X$ with respect to this norm. The identity mapping on $H \otimes X$ extends to an injective and contractive embedding of $\gamma^p(H^*, X)$ into $\mathcal{B}(H^*, X)$. We may thus identify $\gamma^p(H^*, X)$ with a linear subspace in $\mathcal{B}(H^*, X)$. Note that by the Khintchine-Kahane inequalities, $\gamma^p(H^*, X)$ does not depend on p as a linear space.

For any orthonormal family $(e_1, \dots, e_n)$ of $H_{\mathbb{R}}$, the n-tuple $(W(e_1), \dots, W(e_n))$ is an orthonormal family as well and all linear combinations of the $W(e_i)$ are Gaussian. Hence $(W(e_1), \dots, W(e_n))$ is an independent family of real valued standard

Gaussian variables. Therefore the operator $W \otimes \mathrm{Id}_X \colon H \otimes X \to L^p(\widehat{\Omega}) \otimes X$ extends to an isometry

$$W_{p,X} \colon \gamma^p(H^*, X) \longrightarrow L^p(\widehat{\Omega}; X).$$

We record for further use the following tensor extension property (see [KW14, Proposition 4.3] and its proof).

Lemma 4.3. *Let $T \colon H \to H$ be a real operator. Then for any $1 \leqslant p < \infty$, $T \otimes \mathrm{Id}_X \colon H \otimes X \to H \otimes X$ extends to a bounded operator*

$$M_{p,T} \colon \gamma^p(H^*, X) \longrightarrow \gamma^p(H^*, X),$$

with $\|M_{p,T}\| = \|T\|$. Furthermore, if we regard $u \in \gamma^p(H^*, X)$ as an operator from H^* into X, then $M_{p,T}(u) = u \circ T^*$.

We now consider the special case when $H = L^2(\Omega)$ for some measure space (Ω, μ) and $H_{\mathbb{R}} = L^2_{\mathbb{R}}(\Omega)$. We identity $L^2(\Omega)^*$ with $L^2(\Omega)$ through the standard duality pairing

$$\langle h', h \rangle = \int_\Omega h'(s) h(s)\, d\mu(s),$$

so that the above construction leads to $\gamma^p(L^2(\Omega), X)$ and yields an isometric embedding of that space into $L^p(\widehat{\Omega}; X)$.

Let $f \colon \Omega \to X$ be a measurable function which is weakly L^2, that is, $x^* \circ f \in L^2(\Omega)$ for any $x^* \in X^*$. Following [KW14, Section 4] we can define a bounded operator $u^f \colon L^2(\Omega) \to X$ by

$$\big\langle x^*, u^f(h) \big\rangle_{X^*,X} = \int_\Omega \big\langle x^*, f(s) \big\rangle_{X^*,X} h(s)\, d\mu(s), \qquad x^* \in X^*.$$

Then we let $\gamma^p(\Omega, X)$ denote the space of all functions f such that $u^f \in \gamma^p(L^2(\Omega), X)$, equipped with the induced norm (that is, $\|f\|_{\gamma^p(\Omega,X)} := \|u_f\|_{\gamma^p(L^2(\Omega),X)}$). Note that $\gamma^p(\Omega, X)$ contains all finite rank operators. More precisely for any $h \in L^2(\Omega)$ and any $x \in X$, $h \otimes x = u^f$, where $f \colon \Omega \to X$ is defined by $f(s) = h(s)x$.

We will need the following duality result. Note that if $1 < p < \infty$ and p' denotes its conjugate, then $L^{p'}(\widehat{\Omega}; X^*)$ is a subspace of the dual space of $L^p(\widehat{\Omega}; X)$, so that it makes sense to consider the duality pairing $\langle \cdot\,,\,\cdot \rangle_{L^{p'}(\widehat{\Omega};X^*),L^p(\widehat{\Omega};X)}$.

Lemma 4.4. *Let $1 < p, p' < \infty$ be conjugate numbers. Let $f \in \gamma^p(\Omega, X)$ and $g \in \gamma^{p'}(\Omega, X^*)$. Then $s \mapsto \langle g(s), f(s) \rangle_{X^*,X}$ belongs to $L^1(\Omega)$ and we have*

$$\int_\Omega \langle g(s), f(s) \rangle_{X^*,X}\, d\mu(s) \;=\; \big\langle W_{p',X^*}(u^g), W_{p,X}(u^f) \big\rangle_{L^{p'}(\widehat{\Omega};X^*),L^p(\widehat{\Omega};X)}.$$

Proof. The integrability of $\langle g(\cdot), f(\cdot) \rangle$ is established in [KW14, Corollary 5.5]. Moreover it is shown in that paper that for any $f \in \gamma^p(\Omega, X)$ and $g \in \gamma^{p'}(\Omega, X^*)$, the composition operator $u^{g*}u^f \colon L^2(\Omega) \to L^2(\Omega)$ is trace class and

$$\mathrm{tr}\big(u^{g*}u^f\big) = \int_\Omega \langle g(s), f(s) \rangle_{X^*,X}\, d\mu(s)\,.$$

Moreover

$$\big|\mathrm{tr}\big(u^{g*}u^f\big)\big| \leqslant \|f\|_{\gamma^p(\Omega,X)} \|g\|_{\gamma^{p'}(\Omega,X^*)}. \tag{4.6}$$

It therefore suffices to show that

$$\mathrm{tr}\big(u^{g*}u^f\big) \;=\; \big\langle W_{p',X^*}(u^g), W_{p,X}(u^f) \big\rangle_{L^{p'}(\widehat{\Omega};X^*),L^p(\widehat{\Omega};X)}. \tag{4.7}$$

Let $(u_k)_k$ and $(u'_k)_k$ be sequences in $L^2(\Omega)\otimes X$ and $L^2(\Omega)\otimes X^*$ respectively, such that $u_k \to u^f$ in $\gamma^p(L^2(\Omega), X)$ and $u'_k \to u^g$ in $\gamma^{p'}(L^2(\Omega), X^*)$, when $k\to\infty$. Then by (4.6), $\mathrm{tr}\big({u'_k}^* u_k\big) \to \mathrm{tr}\big(u^{g*}u_f\big)$. By the continuity of W_{p',X^*} and $W_{p,X}$ we also have that $\big\langle W_{p',X^*}(u'_k), W_{p,X}(u_k)\big\rangle \to \big\langle W_{p',X^*}(u^g), W_{p,X}(u^f)\big\rangle$. Hence it suffices to show (4.7) in the finite rank case. By linearity, we are reduced to checking this identity when $u = u^f$ and $u' = u^g$ are rank one.

To proceed we let $h, h' \in L^2(\Omega)$, $x \in X, x^* \in X^*$ and consider $u = h\otimes x$ and $u' = h'\otimes x^*$. On the one hand, $u'^* u = \langle x^*, x\rangle h\otimes h'$, hence

$$\mathrm{tr}(u'^* u) = \langle x^*, x\rangle \int_\Omega h(s)h'(s)\, d\mu(s)\,.$$

On the other hand, by the definition of $W_{p,X}$ and W_{p',X^*}, we have

$$\langle W_{p',X^*}(u'), W_{p,X}(u)\rangle = \langle x^*, x\rangle \int_{\widehat{\Omega}} W(h')W(h)\,.$$

Since $W\colon L^2(\Omega)\to L^2(\widehat{\Omega})$ is an isometry, the right-hand side is equal to

$$\langle x^*, x\rangle \int_\Omega h'h\,.$$

This shows (4.7) in that special case, and hence for any $f\in\gamma^p(\Omega, X)$ and $g\in\gamma^{p'}(\Omega, X^*)$ by the preceding reasoning. □

Theorem 4.5. *Assume that X is a reflexive Banach space and that X and X^* have finite cotype. Let A be a sectorial operator which admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta\in(0,\frac{\pi}{2})$. Let $1<p<\infty$. Then there exists a measure space Ω', a C_0-group $(U_t)_{t\in\mathbb{R}}$ of isometries on the Banach space $L^p(\Omega';X)$ together with two bounded operators $J\colon X\to L^p(\Omega';X)$ and $Q\colon L^p(\Omega';X)\to X$ such that*

$$e^{-tA} = QU_tJ, \qquad \text{for all } t\geqslant 0.$$

Moreover:

(a) *If X is an ordered Banach space, then the maps U_t can be chosen to be positive operators.*
(b) *If $X\subset L^p(\Omega)$ is a closed subspace of an L^p-space, then $(U_t)_{t\in\mathbb{R}}$ is the restriction of a C_0-group $(V_t)_{t\in\mathbb{R}}$ of positive isometries on $L^p(\Omega'\times\Omega)$.*
(c) *If X is an SQ_p-space, then there exists a measure space Ω, a C_0-group $(V_t)_{t\in\mathbb{R}}$ of positive isometries on $L^p(\Omega'\times\Omega)$ and two closed subspaces $F\subset E\subset L^p(\Omega'\times\Omega)$ which are invariant under each V_t, such that $L^p(\Omega';X) = E/F$ and for any $t\in\mathbb{R}$, U_t is the compression of V_t to E/F.*

Proof. The scheme of proof is similar to the one of Theorem 4.1. We start with the definition of $(U_t)_{t\in\mathbb{R}}$, which is a universal C_0-group. We apply the preceding construction to the case when $\Omega=\mathbb{R}$, equipped with the Lebesgue measure. Note that $W(\mathbf{1}_{[0,t]})_{t\geqslant 0}$ is a Brownian motion. For any $t\in\mathbb{R}$, we let $\tau_t\colon L^2(\mathbb{R})\to L^2(\mathbb{R})$ denote the shift operator defined by

$$\tau_t(h) = h(\cdot + t), \qquad h\in L^2(\mathbb{R}).$$

This is a real operator and $(\tau_t)_{t\in\mathbb{R}}$ is a C_0-group of isometries on the real Hilbert space $L^2_{\mathbb{R}}(\mathbb{R})$. Hence by Lemma 4.2, $\big(\Gamma_p(\tau_t)\big)_{t\in\mathbb{R}}$ is a C_0-group of positive isometries on $L^p(\widehat{\Omega})$. By positivity, $\Gamma_p(\tau_t)\otimes \mathrm{Id}_X$ extends to an isometry $V_t\colon L^p(\widehat{\Omega};X)\to L^p(\widehat{\Omega};X)$ for any $t\in\mathbb{R}$. Furthermore, $(V_t)_{t\in\mathbb{R}}$ is a C_0-group on that space.

From (4.5) we see that $\Gamma_p(\tau_t)W = W\tau_t$ for any t. Taking tensor extensions and applying Lemma 4.3, we obtain that

$$\forall\, t \in \mathbb{R}, \qquad V_t W_{p,X} = W_{p,X} M_{p,\tau_t}.$$

Suppose that $f \in \gamma^p(\mathbb{R}, X)$. Using the property $\tau_t^* = \tau_{-t}$ it is easy to see that $f(\cdot + t)$ also belongs to $\gamma^p(\mathbb{R}, X)$ and that $M_{p,\tau_t}(u^f) = u^{f(\cdot+t)}$. Combining with the above identity, we deduce that

$$\forall\, t \in \mathbb{R},\ \forall\, f \in \gamma^p(\mathbb{R}, X), \qquad V_t W_{p,X}(u^f) = W_{p,X}(u^{f(\cdot+t)}). \tag{4.8}$$

We set $U_t = \mathrm{Id}_X \oplus V_t \colon X \oplus_p L^p(\widehat{\Omega}; X) \to X \oplus_p L^p(\widehat{\Omega}; X)$ for any $t \in \mathbb{R}$. Then $(U_t)_{t\in\mathbb{R}}$ is a C_0-group of isometries, and $X \oplus_p L^p(\widehat{\Omega}; X)$ can be canonically identified with a space $L^p(\Omega'; X)$. The arguments to show that $(U_t)_{t\in\mathbb{R}}$ satisfies the 'moreover part' of the statement are similar to the ones in the proof of Theorem 4.1.

In the sequel we write $T_s = e^{-sA}$ for any $s \geqslant 0$. Let $\theta' \in (\theta, \frac{\pi}{2})$. Since X has finite cotype and A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, there exists, for any $\varphi \in H_0^\infty(\Sigma_{\theta'})$, a positive constant C such that for any $x \in X$, the function $s \mapsto \varphi(sA)x$ belongs to $\gamma^p\big(L^2(\mathbb{R}_+^*, dt/t), X\big)$ and

$$\big\| s \mapsto \varphi(sA)x \big\|_{\gamma^p(\mathbb{R}_+^*, X)} \leqslant C\|x\|_X.$$

This fundamental result is due to Kalton-Weis (see [KW14, Section 7]). In the latter paper only the case $p = 2$ is treated. However since $\gamma^p(\cdot\,, X)$ and $\gamma^2(\cdot\,, X)$ are the same space with equivalent norms, this special case implies the general case. A proof can also be obtained by mimicking the one of [JLMX06, Theorem 7.6], which corresponds to the case when X is a noncommutative L^p-space. In the case when X is an L^p-space, this estimate reduces to [CDMY96, Section 6].

We apply this result with the function φ defined by $\varphi(z) = z^{\frac{1}{2}} e^{-z}$, which belongs to $H_0^\infty(\Sigma_{\theta'})$ for any $\theta' < \frac{\pi}{2}$. In this case, $\varphi(sA)x = s^{\frac{1}{2}} A^{\frac{1}{2}} T_s x$. Let $\chi = \chi_{(0,\infty)}$ on $\mathbb{R}$. It is easy to check that $\|s \mapsto \varphi(sA)x\|_{\gamma^p(\mathbb{R}_+^*, X)} = \|s \mapsto \chi(s) A^{\frac{1}{2}} T_s x\|_{\gamma^p(\mathbb{R}, X)}$. It therefore follows from above that we have an estimate

$$\big\| s \mapsto \chi(s) A^{\frac{1}{2}} T_s x \big\|_{\gamma^p(\mathbb{R}, X)} \leqslant C\|x\|_X, \qquad x \in X. \tag{4.9}$$

We have a similar estimate on the dual space X^*,

$$\big\| s \mapsto \chi(s) A^{*\frac{1}{2}} T_s^* y \big\|_{\gamma^{p'}(\mathbb{R}, X^*)} \leqslant C\|y\|_{X^*}, \qquad y \in X^*.$$

Since X is reflexive, we have direct sum decompositions

$$X = \mathrm{Ker}\,(A) \oplus \overline{\mathrm{Ran}(A)} \qquad \text{and} \qquad X^* = \mathrm{Ker}\,(A^*) \oplus \overline{\mathrm{Ran}(A^*)};$$

see e.g. [Haa06, Proposition 2.1.1].

Using (4.9) one can define a bounded linear map

$$\begin{aligned} J_1 \colon X = \mathrm{Ker}\,(A) \oplus \overline{\mathrm{Ran}(A)} &\longrightarrow X \oplus_p L^p(\widehat{\Omega}; X) \\ x_0 + x_1 &\mapsto \Big(x_0, W_{p,X}\big(s \mapsto \chi(s) A^{\frac{1}{2}} T_s x_1\big)\Big). \end{aligned}$$

Analogously, one can also define

$$\begin{aligned} J_2 \colon X^* = \mathrm{Ker}\,(A^*) \oplus \overline{\mathrm{Ran}(A^*)} &\longrightarrow X^* \oplus_{p'} L^{p'}(\widehat{\Omega}; X^*) \\ y_0 + y_1 &\mapsto \Big(\frac{y_0}{2}, W_{p',X^*}\big(s \mapsto \chi(s) A^{*\frac{1}{2}} T_s^* y_1\big)\Big). \end{aligned}$$

For any $t \in \mathbb{R}$, we have

$$\begin{aligned} U_t J_1(x_0 + x_1) &= \Big(x_0, V_t W_{p,X}\big(s \mapsto \chi(s) A^{\frac{1}{2}} T_s x_1\big)\Big) \\ &= \Big(x_0, W_{p,X}\big(s \mapsto \chi(t+s) A^{\frac{1}{2}} T_{t+s} x_1\big)\Big) \end{aligned}$$

by (4.8). Hence

$$\begin{aligned} &\big\langle U_t J_1(x_0 + x_1), J_2(y_0 + y_1)\big\rangle \\ &\quad = \frac{1}{2}\langle x_0, y_0\rangle + \Big\langle W_{p,X}\big(s \mapsto \chi(t+s) A^{\frac{1}{2}} T_{t+s} x_1\big), W_{p',X^*}\big(s \mapsto \chi(s) A^{*\frac{1}{2}} T_s^* y_1\big)\Big\rangle \\ &\quad = \frac{1}{2}\langle x_0, y_0\rangle + \int_0^\infty \big\langle A^{\frac{1}{2}} T_{t+s} x_1, A^{*\frac{1}{2}} T_s^* y_1\big\rangle \, ds \\ &\quad = \frac{1}{2}\langle x_0, y_0\rangle + \int_0^\infty \big\langle A T_{2s} T_t x_1, y_1\big\rangle \, ds, \end{aligned}$$

by Lemma 4.4. For any $z \in \overline{\mathrm{Ran}(A)}$, we have $\int_0^\infty A T_s z = z$. Applying this identity to $z = T_t x_1$, we deduce that

$$\big\langle U_t J_1(x_0 + x_1), J_2(y_0 + y_1)\big\rangle = \frac{1}{2}\big(\langle x_0, y_0\rangle + \langle T_t x_1, y_1\rangle\big).$$

Since X is reflexive, we can apply (4.3). Hence the above identity shows that $2 J_2^* U_t J_1 = T_t$, which concludes the proof. □

We conclude this section with an application to the problem of renorming C_0-semigroups. It is well-known that any bounded C_0-semigroup $(T_t)_{t \geqslant 0}$ on an arbitrary Banach space $(X, \|\cdot\|_X)$ becomes contractive for the equivalent norm $\|x\|' := \sup_{t \geqslant 0} \|T_t x\|_X$. However, this renorming may destroy regularity properties of the original norm. For example by a classical result of Packel [Pac69] there exist bounded C_0-semigroups on Hilbert spaces which are not contractive for any equivalent Hilbert space norm (equivalently, which are not similar to a contractive semigroup). The third author showed that among the bounded analytic semigroups on Hilbert space, those whose negative generator admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$ are exactly those which are contractive for an equivalent Hilbert space norm; see [LM98b] and [LM07, Theorem 4.2]. We prove a partial analogue of this result for uniformly convex renormings.

Corollary 4.6. *Let $(T_t)_{t \geqslant 0}$ be a bounded analytic C_0-semigroup on a uniformly convex Banach space X. Suppose that its negative generator admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$. Then there exists an equivalent uniformly convex norm on X for which $(T_t)_{t \geqslant 0}$ is contractive.*

Proof. We fix some $1 < p < \infty$. The uniform convexity ensures that X is reflexive and that X and X^* have finite cotype (see e.g. [LT79, Theorem 1.e.16]). Hence, by Theorem 4.5, there exist a measure space Ω', a C_0-group of isometries $(U_t)_{t \in \mathbb{R}}$ on the Bochner space $L^p(\Omega'; X)$ together with two bounded operators $J \colon X \to L^p(\Omega'; X)$ and $Q \colon L^p(\Omega'; X) \to X$ such that

$$T_t = Q U_t J, \qquad \text{for all } t \geqslant 0.$$

Then according to Proposition 2.4, there exist closed subspaces $F \subset E \subset L^p(\Omega'; X)$ that are invariant under each operator U_t and an isomorphism $S \colon X \to E/F$ such

that

$$T_t = S^{-1}\widetilde{U}_t S, \qquad \text{for all } t \geqslant 0,$$

where $(\widetilde{U}_t)_{t\geqslant 0}$ is the induced contractive semigroup on the quotient space E/F.

According to [Fig76, Fig80] (see also [Pis11, Lemma 4.4]), the space $L^p(\Omega';X)$ is uniformly convex. We deduce that E/F is uniformly convex as well. Now let

$$\|x\|' := \|Sx\|_{E/F}, \qquad \text{for all } x \in X.$$

Then $\|\cdot\|'$ is a uniformly convex norm on X for which $(T_t)_{t\geqslant 0}$ is contractive, and $\|\cdot\|'$ is equivalent to the original norm. □

Using Theorem 4.1 in the place of Theorem 4.5, we obtain the following analogue for Ritt operators.

Corollary 4.7. *Let T be a Ritt operator on a uniformly convex Banach space X. Suppose that T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. Then there exists an equivalent uniformly convex norm on X for which T is contractive.*

In the above two corollaries, uniform convexity could be replaced by any Banach space property which is preserved by passing from X to $L^p(X)$ and by passing to subspaces and quotients. In particular this applies to the class of SQ_p-spaces. More results for this class will be given in Corollaries 5.4 and 5.9.

5. Characterization results

In this section we consider Ritt operators or bounded analytic semigroups and characterize bounded H^∞ functional calculus for them on various classes of Banach spaces.

We start with general UMD spaces. The idea expressed by the next statement (and Theorem 5.6 below) is that any Ritt operator or bounded analytic semigroup with a bounded H^∞ functional calculus can be dilated into a contractive one with a bounded H^∞ functional calculus. Next, more precise results will be given for operators or semigroups acting on some specific classes of UMD spaces.

Theorem 5.1. *Let $T\colon X \to X$ be a Ritt operator on a UMD Banach space X and let $1 < p < \infty$. The following conditions are equivalent.*

(i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*

(ii) *There exist a measure space Ω', a contractive Ritt operator $R\colon L^p(\Omega';X) \to L^p(\Omega';X)$ which admits a bounded $H^\infty(B_{\gamma'})$ functional calculus for some $\gamma' \in (0, \frac{\pi}{2})$, and two bounded operators $J\colon X \to L^p(\Omega';X)$ and $Q\colon L^p(\Omega';X) \to X$ such that*

$$T^n = QR^nJ, \qquad \textit{for all } n \geqslant 0.$$

If moreover X is an ordered UMD Banach space, then the operator R in (ii) *can be chosen to be positive.*

Proof. The implication '(ii)⇒(i)' is easy. Indeed if (ii) holds, then we have $\phi(T) = Q\phi(R)J$ for any $\phi \in \mathcal{P}$ and there is a constant $K \geqslant 0$ such that $\|\phi(R)\| \leqslant K\|\phi\|_{H^\infty(B_{\gamma'})}$ for any such ϕ. Consequently,

$$\|\phi(T)\| \leqslant K\|J\|\|Q\|\|\phi\|_{H^\infty(B_{\gamma'})}, \qquad \text{for all } \phi \in \mathcal{P}.$$

By Lemma 2.2, this shows that T has a bounded $H^\infty(B_{\gamma'})$ functional calculus.

Assume (i). Then it follows from Proposition 3.2 that for a sufficiently small $\alpha > 1$, the fractional power T_α of T has a bounded $H^\infty(B_{\gamma''})$ functional calculus for some $\gamma'' \in (0, \frac{\pi}{2})$. Since X is UMD, it is reflexive and X and X^* have finite cotype. Hence we can apply Theorem 4.1 to the operator $T_\alpha \colon X \to X$. We obtain that there exist an isometric isomorphism $U \colon L^p(\Omega'; X) \to L^p(\Omega'; X)$ and bounded operators $J \colon X \to L^p(\Omega'; X)$ and $Q \colon L^p(\Omega'; X) \to X$ such that

$$T_\alpha^n = QU^n J, \qquad \text{for all } n \geqslant 0. \tag{5.1}$$

This implies that for any polynomial $\phi \in \mathcal{P}$, we have

$$\phi(T_\alpha) = Q\phi(U)J. \tag{5.2}$$

Let $\beta = \frac{1}{\alpha}$. Then we have $\beta \in (0, 1)$. By Theorem 3.1, the fractional power U_β is a contractive Ritt operator. Moreover, by Theorem 3.3, U_β has a bounded $H^\infty(B_{\gamma'})$ functional calculus for some $\gamma' \in (0, \frac{\pi}{2})$.

Consider the polynomials $P_m(z) = \sum_{k=1}^m a_{\beta,k} z^k$, where the $a_{\beta,k}$'s are the coefficients in the series expansion of $1 - (1-z)^\beta$ as given by (3.2). Let $n \geqslant 0$ be an integer. Using equality (5.2) with the polynomial $\phi = P_m^n$, we see that

$$\big(P_m(T_\alpha)\big)^n = P_m^n(T_\alpha) = QP_m^n(U)J = Q\big(P_m(U)\big)^n J.$$

Taking the limit when $m \to \infty$ on both sides yields

$$T^n = QU_\beta^n J,$$

by (3.3). We deduce (ii) by setting $R = U_\beta$.

If X is ordered, U can be chosen to be positive by part (a) of Theorem 4.1. Then R is positive by part (c) of Theorem 3.1.

□

We now focus on special classes of UMD spaces. We start with L^p-spaces.

Theorem 5.2. *Let Ω be a measure space and $1 < p < \infty$. Let $T \colon L^p(\Omega) \to L^p(\Omega)$ be a Ritt operator. The following conditions are equivalent.*

(i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*
(ii) *There exist a measure space Ω', a contractive and positive Ritt operator $R \colon L^p(\Omega') \to L^p(\Omega')$ together with two bounded operators $J \colon L^p(\Omega) \to L^p(\Omega')$ and $Q \colon L^p(\Omega') \to L^p(\Omega)$ such that*

$$T^n = QR^n J \qquad \textit{for all } n \geqslant 0.$$

Proof. By [LMX12, Theorem 3.3] (see also [LM14, Theorem 8.3]), a positive contractive Ritt operator on an L^p-space admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$. Moreover $L^p(\Omega'; X)$ is an L^p-space whenever X is an L^p-space. Hence the result is a special case of Theorem 5.1. □

If we take part (b) of Theorem 4.1 into account in the proof of Theorem 5.1, we obtain the following special case.

Corollary 5.3. *Let $1 < p < \infty$, let X be a closed subspace of an L^p-space and let T be a Ritt operator on X. The following conditions are equivalent.*

(i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*

(ii) *There exist a measure space Ω', a contractive and positive Ritt operator $R\colon L^p(\Omega') \to L^p(\Omega')$, an R-invariant subspace $E \subset L^p(\Omega')$, together with two operators $J\colon X \to E$ and $Q\colon E \to X$ such that*

$$T^n = QR^nJ, \qquad \text{for all } n \geqslant 0.$$

For the class of quotients of subspaces of L^p, we have the following.

Corollary 5.4. *Let $1 < p < \infty$, let X be an SQ_p-space and let T be a Ritt operator on X. The following conditions are equivalent.*

(i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.*

(ii) *There exist a measure space Ω', a contractive and positive Ritt operator $R\colon L^p(\Omega') \to L^p(\Omega')$, two R-invariant subspaces $F \subset E \subset L^p(\Omega')$ and an isomorphism $S\colon X \to E/F$ such that*

$$T^n = S^{-1}\widetilde{R}^nS, \qquad \text{for all } n \geqslant 0,$$

where $\widetilde{R}\colon E/F \to E/F$ is the compression of R to E/F.

Proof. To prove the implication '(ii) $\Rightarrow$ (i)', it suffices to use [LMX12, Theorem 3.3] as in the proof of Theorem 5.2, together with the fact that the boundedness of the functional calculus is preserved by passing to an invariant subspace, by factorizing through an invariant subspace and by similarity transforms.

To prove the converse, assume (i) and apply the proof of Theorem 5.1 to this special case. Applying part (c) of Theorem 4.1, we can assume that the operator $U\colon L^p(\Omega'; X) \to L^p(\Omega'; X)$ satisfying (5.1) is a compression of a positive isometric isomorphism $V\colon L^p(\Omega'') \to L^p(\Omega'')$. By Proposition 2.4, there exists a Banach space Y which is a quotient of two U-invariant subspaces of $L^p(\Omega'; X)$, as well as an isomorphism $S\colon X \to Y$ such that

$$T_\alpha^n = S^{-1}W^nS, \qquad \text{for all } n \geqslant 0,$$

where $W\colon Y \to Y$ is the resulting compression of U. Then we may write $Y = E/F$ for some V-invariant subspaces $F \subset E \subset L^p(\Omega'')$, in such a way that W is a compression of V.

Let $q\colon E \to E/F$ be the canonical quotient map. For any polynomial ϕ, we have

$$\phi(W)q = q\phi(V)_{|E}.$$

Arguing as in the proof of Theorem 5.1 we deduce that

$$W_\beta q = qV_{\beta|E};$$

that is, W_β is the compression of V_β, and

$$T^n = S^{-1}W_\beta^nS, \qquad \text{for all } n \geqslant 0.$$

We deduce the result with $R = V_\beta$ and $\widetilde{R} = W_\beta$. □

Remark 5.5. If X is a Banach lattice, then $L^p(\Omega'; X)$ also is a Banach lattice. So Theorem 5.1 shows that any Ritt operator on a UMD Banach lattice with a bounded $H^\infty(B_\gamma)$ functional calculus can be dilated into a positive contractive Ritt operator acting on a bigger UMD Banach lattice and admitting a bounded $H^\infty(B_{\gamma'})$ functional calculus for some $\gamma' < \frac{\pi}{2}$.

Likewise if X is a noncommutative L^p-space with $1 < p < \infty$, then X is UMD [BGM89, Theorem 6.1] and $L^p(\Omega'; X)$ is a noncommutative L^p-space. Consequently

any Ritt operator on a noncommutative L^p-space with a bounded $H^\infty(B_\gamma)$ functional calculus can be dilated into a positive contractive Ritt operator acting on a bigger noncommutative L^p-space and admitting a bounded $H^\infty(B_{\gamma'})$ functional calculus for some $\gamma' < \frac{\pi}{2}$.

We now turn to bounded analytic semigroups. The results below improve and extend some of the main results by the second named author in [Fac14].

Theorem 5.6. *Let A be a sectorial operator on a UMD Banach space X and let $1 < p < \infty$. The following conditions are equivalent.*

(i) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$.*

(ii) *There exist a measure space Ω', a sectorial operator B of type $< \frac{\pi}{2}$ on $L^p(\Omega'; X)$ which admits a bounded $H^\infty(\Sigma_{\theta'})$ functional calculus for some $\theta' \in (0, \frac{\pi}{2})$, and two bounded operators $J\colon X \to L^p(\Omega'; X)$ and $Q\colon L^p(\Omega'; X) \to X$ such that*

$$e^{-tA} = Qe^{-tB}J, \qquad \text{for all } t \geqslant 0,$$

and

$$\|e^{-tB}\| \leqslant 1, \qquad \text{for all } t \geqslant 0.$$

If moreover X is an ordered UMD Banach space, then the sectorial operator B in (ii) *can be chosen so that e^{-tB} is positive for any $t \geqslant 0$.*

Proof. The proof of '(ii) $\Rightarrow$ (i)' is similar to the one for Theorem 5.1, so we omit it.

Assume (i). According to Lemma 2.1 we can find $\alpha > 1$ and $\theta'' \in (0, \frac{\pi}{2})$ such that the fractional power A^α has a bounded $H^\infty(\Sigma_{\theta''})$ functional calculus. Then it follows from Theorem 4.5 applied to the operator A^α that there exist a measure space Ω', two bounded operators $J\colon X \to L^p(\Omega'; X)$ and $Q\colon L^p(\Omega'; X) \to X$ and a C_0-group $(U_t)_{t\in\mathbb{R}}$ of isometries (positive if X is ordered) such that

$$e^{-tA^\alpha} = QU_tJ, \qquad \text{for all } t \geqslant 0. \tag{5.3}$$

We denote by C the negative generator of $(U_t)_{t\geqslant 0}$, so that we can write $U_t = e^{-tC}$ for any $t \geqslant 0$. Since X is UMD, the operator C admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus for any $\omega > \frac{\pi}{2}$ [HP98]. Let $\beta = \frac{1}{\alpha}$. Since this real number belongs to $(0, 1)$, we deduce from Lemma 2.1 that the operator C^β is sectorial of type $< \frac{\pi}{2}$ and admits a bounded $H^\infty(\Sigma_{\theta'})$ functional calculus for some $\theta' \in (0, \frac{\pi}{2})$.

We now use subordination. By [Yos80, IX,11], for any $t > 0$, there exists a nonnegative function $f_{t,\beta} \in L^1(\mathbb{R}_+)$ with $\int_0^\infty f_{t,\beta}(s)\, ds = 1$, such that the semigroup $(e^{-tC^\beta})_{t\geqslant 0}$ generated by $-C^\beta$ is given in the strong sense by

$$e^{-tC^\beta} = \int_0^\infty f_{t,\beta}(s) U_s\, ds.$$

This explicit formula shows that e^{-tC^β} is contractive for all $t \geqslant 0$ since each U_s is contractive (and positive if X is ordered). Likewise we have

$$e^{-tA} = e^{-t(A^\alpha)^\beta} = \int_0^\infty f_{t,\beta}(s) e^{-sA^\alpha}\, ds$$

for any $t > 0$. These identities together with (5.3) show that for any $t > 0$,

$$\begin{aligned} e^{-tA} &= \int_0^\infty f_{t,\beta}(s) Q U_s J \, ds \\ &= Q\bigg(\int_0^\infty f_{t,\beta}(s) U_s \, ds\bigg) J \;=\; Q e^{-tC^\beta} J. \end{aligned}$$

We conclude by taking $B = C^\beta$. □

When we restrict to L^p-spaces, we obtain the following result.

Theorem 5.7. *Let Ω be a measure space and $1 < p < \infty$. Let A be a sectorial operator on $L^p(\Omega)$. The following conditions are equivalent.*

(i) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$.*
(ii) *There exist a measure space Ω', a sectorial operator B of type $< \frac{\pi}{2}$ on $L^p(\Omega')$, and two bounded operators $J\colon L^p(\Omega) \to L^p(\Omega')$ and $Q\colon L^p(\Omega') \to L^p(\Omega)$ such that*

$$e^{-tA} = Q e^{-tB} J, \qquad \text{for all } t \geqslant 0,$$

and

$$e^{-tB} \text{ is a positive contraction, for all } t \geqslant 0.$$

Proof. (i) ⇒ (ii) Since $L^p(\Omega'; X)$ is an L^p-space whenever X is an L^p-space, this implication is a special case of Theorem 5.6.

(ii) ⇒ (i) By [Wei01, Remark 4.c], the operator B admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$. Then the dilation assumption implies that A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus. □

The next theorem is obtained by combining Theorem 5.6 for subspaces of L^p together with part (b) of Theorem 4.5.

Corollary 5.8. *Let $1 < p < \infty$, let X be a closed subspace of an L^p-space and let A be a sectorial operator on X. The following conditions are equivalent.*

(i) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$.*
(ii) *There exist a measure space Ω', a sectorial operator B of type $< \frac{\pi}{2}$ on $L^p(\Omega')$, a subspace $E \subset L^p(\Omega')$ which is $(e^{-tB})_{t \geqslant 0}$-invariant, and two bounded operators $J\colon X \to E$ and $Q\colon E \to X$ such that*

$$e^{-tA} = Q e^{-tB} J, \qquad \text{for all } t \geqslant 0,$$

and

$$e^{-tB}\colon L^p(\Omega') \to L^p(\Omega') \text{ is a positive contraction, for all } t \geqslant 0.$$

We finally consider semigroups acting on quotients of subspaces of L^p. Arguing as in the proof of Corollary 5.4, we recover the following result of the second named author [Fac14].

Corollary 5.9. *Let $1 < p < \infty$, let X be an SQ_p-space and let A be a sectorial operator on X. The following conditions are equivalent.*

(i) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$.*

(ii) *There exist a measure space Ω', a sectorial operator B of type $< \frac{\pi}{2}$ on $L^p(\Omega')$, two subspaces $F \subset E \subset L^p(\Omega')$ which are $(e^{-tB})_{t\geqslant 0}$-invariant, and an isomorphism $S\colon X \to E/F$ such that*

$$e^{-tA} = S^{-1}\widetilde{e^{-tB}}S, \qquad \text{for all } t \geqslant 0,$$

where $\widetilde{e^{-tB}}\colon E/F \to E/F$ is the compression of e^{-tB} to E/F, and

$$e^{-tB}\colon L^p(\Omega') \to L^p(\Omega') \text{ is a positive contraction, for all } t \geqslant 0.$$

Comments similar to the ones in Remark 5.5 apply to the sectorial setting.

Remark 5.10. Let Ω' be a measure space, let $1 < p < \infty$, let $E \subset L^p(\Omega')$ be a closed subspace and let $(R_t)_{t\geqslant 0}$ be a bounded analytic semigroup with generator $-B$. Assume that each $R_t\colon E \to E$ is contractively regular (in the sense of [Pis94]). According to [LMS01, Corollary 3.2], B admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$ (this can also be derived from the dilation result [Fac15, Theorem 4.2.11]). However we do not know if B admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0, \frac{\pi}{2})$. If such a result were true, it would be an analogue of Weis's Theorem [Wei01, Remark 4.c] for subspaces of L^p and would lead to a more precise form of Corollary 5.8.

An essentially equivalent question is whether any contractively regular Ritt operator $R\colon E \to E$ admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \frac{\pi}{2})$.

6. Representations of amenable groups

Let G be a locally compact group, let X be a Banach space and let $\pi\colon G \to B(X)$ be a representation. We say that π is continuous when $t \mapsto \pi(t)x$ is continuous for any $x \in X$. Then π is said to be bounded when

$$\|\pi\| := \sup\big\{\|\pi(t)\| \,:\, t \in G\big\} < \infty\,.$$

A famous theorem of Dixmier asserts that if G is amenable and $X = H$ is a Hilbert space, then any bounded continuous representation $\pi\colon G \to B(H)$ is similar to a unitary representation; that is, there exists an isomorphism $S \in B(H)$ such that $S\pi(t)S^{-1}$ is a unitary for any $t \in G$ [Dix50].

The aim of this section is to establish Banach space analogues of that result. In the Banach space context, the role of 'unitary representations' will be played by 'isometric representations', that is, representations π such that $\pi(t)$ is an isometry for any t. Note that this holds true if and only if $\pi(t)$ is a contraction for any $t \in G$.

In the case when $G = \mathbb{Z}$ or $G = \mathbb{R}$, the results we obtain give an alternate route to prove some of the results from Section 5; see Remark 6.4 for details.

We will use ultraproducts of Banach spaces. We pay special attention to the case when these spaces are ordered ones. We recall that if $(X_j)_{j\in I}$ is a family of Banach spaces indexed by an arbitrary set I and $\mathcal{U}$ is an ultrafilter on I, then the ultraproduct $(X_j)^{\mathcal{U}}$ is defined as the quotient space $\ell^\infty(I; X_j)/N_{\mathcal{U}}$, where $\ell^\infty(I; X_j)$ is the space of all bounded families $(x_j)_{j\in I}$ with $x_j \in X_j$, equipped with the sup norm, and $N_{\mathcal{U}}$ is the subspace of all such families for which $\lim_{\mathcal{U}} \|x_j\|_{X_j} = 0$. Thus any element z of $(X_j)^{\mathcal{U}}$ is a class of bounded families $(x_j)_{j\in I}$ modulo $N_{\mathcal{U}}$. Further if $(x_j)_{j\in I}$ is any representative of z, then

$$\|z\| = \lim_{\mathcal{U}} \|x_j\|_{X_j}.$$

We refer the reader to [KS01] for general information on this construction.

When all spaces X_j are equal to a single space X, the associated ultraproduct is called an ultrapower and is denoted by $X^{\mathcal{U}}$.

Assume that each X_j is the complexification of a real Banach space $X_{j,\mathbb{R}}$. Let $(X_j)^{\mathcal{U}}_{\mathbb{R}}$ be the subset of all elements of $(X_j)^{\mathcal{U}}$ which have a representative $(x_j)_{j\in I}$, with $x_j \in X_{j,\mathbb{R}}$ for any $j \in I$. Clearly $(X_j)^{\mathcal{U}}_{\mathbb{R}}$ is a real subspace of $(X_j)^{\mathcal{U}}$. Consider z^1, z^2 in $(X_j)^{\mathcal{U}}_{\mathbb{R}}$ and let $x^1_j, x^2_j \in X_{j,\mathbb{R}}$ such that $(x^1_j)_{j\in I}$ and $(x^2_j)_{j\in I}$ are representatives of z^1 and z^2, respectively. Applying (2.12) to each X_j and passing to the limit along $\mathcal{U}$, we deduce that $\lim_{\mathcal{U}} \|x^1_j\|_{X_j} \leqslant \lim_{\mathcal{U}} \|x^1_j + ix^2_j\|$, which means that

$$\|z^1\| \leqslant \|z^1 + iz^2\|.$$

This implies that $(X_j)^{\mathcal{U}}_{\mathbb{R}} \cap i(X_j)^{\mathcal{U}}_{\mathbb{R}} = \{0\}$, and hence $(X_j)^{\mathcal{U}}$ is the real direct sum of $(X_j)^{\mathcal{U}}_{\mathbb{R}}$ and $i(X_j)^{\mathcal{U}}_{\mathbb{R}}$. Moreover $(X_j)^{\mathcal{U}}_{\mathbb{R}}$ is closed. Likewise, we have $\|z^2\| \leqslant \|z^1 + iz^2\|$. Hence $(X_j)^{\mathcal{U}}$ is the complexification of $(X_j)^{\mathcal{U}}_{\mathbb{R}}$.

Assume now that each X_j is a Riesz-normed space. We may define an order on $(X_j)^{\mathcal{U}}_{\mathbb{R}}$ by asserting that $z \geqslant 0$ when it has a representative $(x_j)_{j\in I}$, with $x_j \geqslant 0$ for any $j \in I$. Then the corresponding positive cone $\mathcal{C} = \{z \geqslant 0\}$ is closed (see e.g. the argument in [Hei81, p. 224]).

Let $z, z' \in (X_j)^{\mathcal{U}}_{\mathbb{R}}$, with $-z' \leqslant z \leqslant z'$. Since $z' + z \geqslant 0$ and $z' - z \geqslant 0$ one can find, for each $j \in I$, $x_j \geqslant 0$ and $y_j \geqslant 0$ in X_j such that $(x_j)_{j\in I}$ and $(y_j)_{j\in I}$ are representatives of $z' + z$ and $z' - z$, respectively. Then $\big(\frac{1}{2}(y_j - x_j)\big)_{j\in I}$ and $\big(\frac{1}{2}(y_j + x_j)\big)_{j\in I}$ are representatives of z and z', respectively. We have

$$-(y_j + x_j) \leqslant (y_j - x_j) \leqslant (y_j + x_j), \qquad j \in I.$$

Since each X_j is absolutely monotone, this implies that $\|y_j - x_j\| \leqslant \|y_j + x_j\|$ for any $j \in I$. Passing to the limit along $\mathcal{U}$, we deduce that $\|z\| \leqslant \|z'\|$. This shows that $\mathcal{C}$ is proper and that $(X_j)^{\mathcal{U}}$ is an absolutely monotone ordered space.

Let $z \in (X_j)^{\mathcal{U}}_{\mathbb{R}}$, with representative $(x_j)_{j\in I}$, and let $\epsilon > 0$. Since each X_j is Riesz-normed, one can find $y_j \in X_{j,\mathbb{R}}$ such that $-y_j \leqslant x_j \leqslant y_j$ and $\|y_j\| \leqslant (1+\epsilon)\|x_j\|$, for any $j \in I$. Let $w \in (X_j)^{\mathcal{U}}$ be the class of $(y_j)_{j\in I}$ (which is a bounded family). Then $-w \leqslant z \leqslant w$ and passing to the limit, we have $\|w\| \leqslant (1+\epsilon)\|z\|$. This shows that $(X_j)^{\mathcal{U}}$ is a Riesz-normed space.

For further use we note that for any $1 < p < \infty$, the ultraproduct of a family of L^p-spaces is an L^p-space; see e.g. [KS01, Example 2.20].

Let X be a Banach space. For any $1 < p < \infty$ we denote by $\mathcal{U}(p; X)$ the class of Banach spaces which are ultraproducts of families of the form $(L^p(\Omega_j; X))_{j\in I}$ for some arbitrary measure spaces Ω_j.

Let G be a locally compact group, endowed with a fixed right Haar measure simply denoted by dt. For a measurable set $E \subset G$, we let $|E|$ denote its Haar measure. Then we have $|Es| = |E|$ for any $s \in G$. Let $E \triangle F$ denote the symmetric difference of two subsets of G. A net $(E_i)_{i\in I}$ of measurable subsets of G is called a *Følner net* if $0 < |E_i| < \infty$ for any $i \in I$ and

$$\lim_i \frac{|E_i s \triangle E_i|}{|E_i|} = 0, \qquad \text{for all } s \in G. \tag{6.1}$$

The existence of a Følner net is equivalent to the amenability of G (see [Pat88] for details and other equivalent definitions).

Let $E \subset G$ be a measurable set such that $0 < |E| < \infty$ and let $s \in G$. Consider $F_1 = E \setminus (Es \cap E)$ and $F_2 = Es \setminus (Es \cap E)$. Since $|Es| = |E|$, we have $|F_1| = |F_2|$.

Moreover $Es\triangle E$ is the disjoint union of F_1 and F_2, hence $|Es\triangle E| = 2|F_1|$. Since E is the disjoint union of $Es \cap E$ and F_1, we also have $|E| = |Es \cap E| + |F_1|$. Altogether, we obtain that

$$|Es \cap E| + \tfrac{1}{2}|Es\triangle E| = |E|.$$

Consequently

$$\frac{|Es \cap E|}{|E|} = 1 - \frac{1}{2}\frac{|Es\triangle E|}{|E|}.$$

Thus if $(E_i)_{i\in I}$ is a Følner net on G, then

$$\lim_i \frac{|E_i s \cap E_i|}{|E_i|} = 1, \qquad \text{for all } s \in G. \tag{6.2}$$

In the sequel we let

$$\kappa_X \colon X \hookrightarrow X^{**}$$

denote the canonical embedding of X into its bidual.

Theorem 6.1. *Let $\pi\colon G \to \mathcal{B}(X)$ be a bounded continuous representation of an amenable locally compact group G on a Banach space X. Suppose $1 < p < \infty$.*

(1) *There exist a Banach space Y in the class $\mathcal{U}(p;X)$, an isometric representation $\widehat{\pi}\colon G \to \mathcal{B}(Y)$ and two bounded operators $J\colon X \to Y$ and $Q\colon Y \to X^{**}$ such that $\|J\|\|Q\| \leqslant \|\pi\|^2$ and*

$$\kappa_X \pi(t) = Q\widehat{\pi}(t)J, \qquad \text{for all } t \in G.$$

(2) *For any $x \in X$, the map $t \mapsto \widehat{\pi}(t)J(x)$ from G into Y is continuous.*

Moreover, if X is a Riesz-normed space, then Y is a Riesz-normed space as well and the representation $\widehat{\pi}$ can be chosen such that $\widehat{\pi}(t)$ is a positive operator on Y for any $t \in G$.

Proof. Let $E \subset G$ be a measurable set with $0 < |E| < \infty$. We let $j_E\colon X \to L^p(G;X)$ be the linear map given by

$$j_E(x) = \frac{1}{|E|^{\frac{1}{p}}}\chi_E\, \pi(\cdot)x$$

for any $x \in X$. This is well-defined; indeed, the continuity of π shows that $j_E(x)$ is measurable and we have

$$\|j_E(x)\|^p_{L^p(G;X)} = \frac{1}{|E|}\int_E \|\pi(t)x\|^p_X\, dt \leqslant \|\pi\|^p\|x\|^p_X.$$

This shows that j_E is bounded with $\|j_E\| \leqslant \|\pi\|$. Let p' be the conjugate of p. Then similarly we define $q_E\colon L^p(G;X) \to X$ by

$$q_E(f) = \frac{1}{|E|^{\frac{1}{p'}}}\int_E \pi(t^{-1})\big(f(t)\big)\, dt$$

for any $f \in L^p(G;X)$. Using Hölder's inequality we see that

$$\begin{aligned}\big\|q_E(f)\big\|_X &\leqslant \frac{1}{|E|^{\frac{1}{p'}}}\int_E \big\|\pi(t^{-1})\big(f(t)\big)\big\|_X\, dt \leqslant \frac{\|\pi\|}{|E|^{\frac{1}{p'}}}\int_E \big\|f(t)\big\|_X\, dt \\ &\leqslant \|\pi\|\Big(\int_E \big\|f(t)\big\|^p_X\, dt\Big)^{\frac{1}{p}} = \|\pi\|\|f\|_{L^p(G;X)}.\end{aligned} \tag{6.3}$$

This shows that $\|q_E\| \leqslant \|\pi\|$.

For any $s \in G$ let $\tau_s \colon L^p(G;X) \to L^p(G;X)$ be the isometry given by the right regular vector valued representation: $\big(\tau_s(f)\big)(t) = f(ts)$ for any $f \in L^p(G;X)$ and any $t \in G$. Let $x \in X$ and $s \in G$. For E as above, we have

$$\big(\tau_s j_E(x)\big)(t) = \frac{1}{|E|^{\frac{1}{p}}}\chi_E(ts)\pi(ts)x = \frac{1}{|E|^{\frac{1}{p}}}\chi_{Es^{-1}}(t)\pi(t)\pi(s)x \tag{6.4}$$

for all $t \in G$. Applying q_E to both sides of the above equality we obtain

$$q_E\tau_s j_E(x) = \frac{1}{|E|}\int_E \chi_{Es^{-1}}(t)\pi(s)x\,dt \;=\; \frac{|Es^{-1}\cap E|}{|E|}\,\pi(s)x. \tag{6.5}$$

Now let $(E_i)_{i\in I}$ be a Følner net on G and form, for each $i \in I$, the operators j_{E_i} and q_{E_i} as above. Let $\mathcal{U}$ be an ultrafilter on I refining the filter generated by the order of I. Then let $Y = L^p(G;X)^{\mathcal{U}}$ be the ultrapower of $L^p(G;X)$ with respect to $\mathcal{U}$. For any $x \in X$, the family $(j_{E_i}(x))_i$ is bounded, and we may therefore define $J \colon X \to Y$ by $J(x) = (j_{E_i}(x))_i^\bullet$, the class of the family $(j_{E_i}(x))_i$ in the ultrapower Y. Then we have $\|J\| \leqslant \|\pi\|$.

Let $(f_i)_{i\in I}$ belong to $\ell^\infty(I;L^p(G;X))$ and let $K = \sup_i \|f_i\|$. We have

$$\big\|\kappa_X q_{E_i}(f_i)\big\|_{X^{**}} \;=\; \|q_{E_i}(f_i)\|_X \leqslant K\|\pi\|.$$

Since bounded sets of X^{**} are w^*-compact, we deduce the existence of the weak* limit $w^* - \lim_{\mathcal{U}} \kappa_X q_{E_i}(f_i)$ in X^{**}. Furthermore this limit does only depend on the class of $(f_i)_{i\in I}$ in Y. Then we may define a map

$$\begin{array}{rccc} Q\colon & Y & \longrightarrow & X^{**} \\ & (f_i)^\bullet & \longmapsto & w^* - \lim_{\mathcal{U}} \kappa_X q_{E_i}(f_i); \end{array}$$

this map is linear and by the above estimates, we have $\|Q\| \leqslant \|\pi\|$.

Next for any $s \in G$ we denote by $\widehat{\pi}(s) \colon Y \to Y$ the map induced by the operators τ_s. That is, for any $(f_i)_{i\in I}$ in $\ell^\infty(I;L^p(G;X))$,

$$\widehat{\pi}(s)\big((f_i)_i^\bullet\big) \;=\; \big((\tau_s(f_i))_i^\bullet\big).$$

It is clear that $\widehat{\pi} \colon G \to \mathcal{B}(Y)$ is an isometric (a priori noncontinuous) representation.

If X is Riesz-normed, then $L^p(G;X)$ is Riesz-normed and then the ultrapower Y is a Riesz-normed space, as explained before the statement of Theorem 6.1. In this case, τ_s is positive and $\widehat{\pi}(s)$ is positive for any $s \in G$.

Let $x \in X$, $\eta \in X^*$ and $s \in G$. Recall that $\mathcal{U}$ refines the order of I. Then by (6.5) and the Følner condition (6.2), we have

$$\begin{aligned} \big\langle \eta, Q\widehat{\pi}(s)Jx\big\rangle_{X^*,X^{**}} &= \lim_{\mathcal{U}} \big\langle \eta, q_{E_i}\tau_s j_{E_i}(x)\big\rangle_{X^*,X} \\ &= \lim_{\mathcal{U}} \frac{|E_i s^{-1}\cap E_i|}{|E_i|}\,\big\langle \eta, \pi(s)x\big\rangle_{X^*,X} \\ &= \big\langle \eta, \pi(s)x\big\rangle_{X^*,X}. \end{aligned}$$

This shows the factorization property in part (1).

Let us now prove part (2). Let $x \in X$. We fix $s \in G$. Consider as before a measurable set $E \subset G$ with $0 < |E| < \infty$. We have seen in (6.4) that

$$\tau_s j_E(x) = \frac{1}{|E|^{\frac{1}{p}}}\chi_{Es^{-1}}\pi(\cdot)\pi(s)x.$$

Hence,

$$\begin{aligned}\tau_s j_E(x) - j_E(x) &= \chi_{Es^{-1}} \frac{\pi(\cdot)\pi(s)x}{|E|^{\frac{1}{p}}} - \chi_E \frac{\pi(\cdot)x}{|E|^{\frac{1}{p}}} \\ &= \frac{\big(\chi_{Es^{-1}} - \chi_E\big)\pi(\cdot)x}{|E|^{\frac{1}{p}}} + \frac{\chi_{Es^{-1}}\pi(\cdot)\big(\pi(s)x - x\big)}{|E|^{\frac{1}{p}}}.\end{aligned}$$

We estimate the norm of each term in $L^p(G;X)$. On the one hand we have

$$\begin{aligned}\left\| \frac{\big(\chi_{Es^{-1}} - \chi_E\big)\pi(\cdot)x}{|E|^{\frac{1}{p}}} \right\|^p_{L^p(G;X)} &= \frac{1}{|E|} \int_G \big|\chi_{Es^{-1}}(t) - \chi_E(t)\big| \|\pi(t)x\|^p_X \, dt \\ &\leqslant \|\pi\|^p \|x\|^p_X \frac{|Es^{-1} \triangle E|}{|E|}.\end{aligned}$$

On the other hand one has

$$\begin{aligned}\left\| \frac{\chi_{Es^{-1}}\pi(\cdot)\big(\pi(s)x - x\big)}{|E|^{\frac{1}{p}}} \right\|^p_{L^p(G;X)} &= \frac{1}{|E|} \int_{Es^{-1}} \big\|\pi(t)\big(\pi(s)x - x\big)\big\|^p_X \, dt \\ &\leqslant \|\pi\|^p \|\pi(s)x - x\|^p_X.\end{aligned}$$

We deduce that

$$\big\|\tau_s j_E(x) - j_E(x)\big\|_{L^p(G;X)} \leqslant \|\pi\| \left(\frac{|Es^{-1} \triangle E|^{\frac{1}{p}}}{|E|^{\frac{1}{p}}} \|x\| + \|\pi(s)x - x\| \right).$$

Applying this inequality to the sets E_i of the Følner net considered in the proof of (1) and using (6.1), we obtain

$$\big\|\widehat{\pi}(s)Jx - Jx\big\|_Y = \lim_{\mathcal{U}} \big\|\tau_s j_{E_i}(x) - j_{E_i}(x)\big\|_{L^p(G;X)} \leqslant \|\pi\| \|\pi(s)x - x\|_X.$$

This estimate and the continuity of $\pi(\cdot)x$ show that $\widehat{\pi}(\cdot)Jx$ is continuous at the origin and hence on G. □

In general, the ultrapower Y introduced in the above proof is too big to expect the representation $\widehat{\pi}$ to be continuous. This defect can be avoided by passing to a suitable subspace. More precisely we have the following corollary (relevant only in the case when G is not a discrete group). Its proof relies on notions and results from the paper [dLG65] concerning possibly discontinuous representations.

Corollary 6.2. *Let $\pi\colon G \to \mathcal{B}(X)$ be a bounded continuous representation of an amenable locally compact group G on a Banach space X. Suppose $1 < p < \infty$.*

(1) *There exist a Banach space Y in the class $\mathcal{U}(p;X)$, a subspace $Z \subset Y$, a continuous isometric representation $\widehat{\pi}\colon G \to \mathcal{B}(Z)$ and two bounded operators $J\colon X \to Z$ and $Q\colon Z \to X^{**}$ such that $\|J\|\|Q\| \leqslant \|\pi\|^2$ and*

$$\kappa_X \pi(t) = Q\widehat{\pi}(t)J, \qquad \text{for all } t \in G.$$

(2) *Assume further that X and X^* are uniformly convex.*
   - (2.i) *Then Z can be chosen to be 1-complemented in Y; i.e. there exists a contractive projection $P\colon Y \to Y$ with range equal to Z.*
   - (2.ii) *Moreover if X is a Riesz-normed space, then Y is a Riesz-normed space, Z is a Riesz-normed subspace of Y, $\widehat{\pi}(t)\colon Z \to Z$ is positive for any $t \in G$, and the contractive projection P is positive.*

Proof. We start from the space Y and the representation $\widehat{\pi}$ constructed in Theorem 6.1. Then we set

$$Z = \big\{ y \in Y \,:\, s \mapsto \widehat{\pi}(s)y \text{ is continuous from } G \text{ into } Y \big\}.$$

It is plain that Z is a closed subspace of Y. For any $s_0 \in G$ and any $y \in Z$, we have $\widehat{\pi}(s)(\widehat{\pi}(s_0)y) = \widehat{\pi}(ss_0)y$ for any $s \in G$. Hence $s \mapsto \widehat{\pi}(s)(\widehat{\pi(s_0)}y)$ is continuous. This shows that Z is π-invariant. We keep the notation $\widehat{\pi}\colon G \to \mathcal{B}(Z)$ for the representation obtained by taking restrictions. By construction, this representation is continuous. By part (2) of Theorem 6.1, Z contains the range of J. Then changing Q into its restriction to Z, we obtain part (1) of the corollary.

Assume now that X and X^* are uniformly convex. As already mentioned in the proof of Corollary 4.6, the Bochner spaces $L^p(G;X)$ and $L^{p'}(G;X^*)$ are uniformly convex. By [KS01, Example 2.17], the ultrapower $Y = L^p(G;X)^{\mathcal{U}}$ is uniformly convex as well. Hence by [KS01, Theorem 2.19], we can isometrically identify the dual of Y with the ultrapower $L^{p'}(G;X^*)^{\mathcal{U}}$. Applying [KS01, Example 2.17] again, this dual space Y^* is also uniformly convex.

Following [dLG65], let $\mathcal{V}$ be the set of all neighbourhoods V of the identity of G. Then for any such V, consider $\overline{\{\widehat{\pi}(t) : t \in V\}}^{wo}$, the closure being taken in the weak operator topology of $\mathcal{B}(Y)$. Then let $\mathcal{S}$ be the closure in the weak operator topology of the convex hull of $\bigcap_{V\in\mathcal{V}} \overline{\{\widehat{\pi}(t) : t \in V\}}^{wo}$. According to [dLG65, Theorem 3.1] and its proof, the uniform convexity of Y and of Y^* ensure that $\mathcal{S}$ contains a projection $P\colon Y \to Y$ with range equal to Z.

Since $\widehat{\pi}(t)$ is a contraction for any $t \in G$, any element of $\mathcal{S}$ is a contraction. Consequently, P is a contractive projection.

Assume now that X is a Riesz-normed space. We already noticed that the ultrapower Y is a Riesz-normed space and according to Theorem 6.1, $\widehat{\pi}(t)\colon Y \to Y$ is positive for any $t \in G$. We deduce that $\langle P(y), y^*\rangle \geqslant 0$ for any $y \in Y$ and $y^* \in Y^*$. By Hahn-Banach, this implies that P is positive. (In fact, any element of $\mathcal{S}$ is positive.) Since Y is Riesz-normed, this implies that P is real.

Let $y \in Z$ and let $y_1, y_2 \in Y_{\mathbb{R}}$ such that $y = y_1 + iy_2$. Since Z is the range of P, we have $y = P(y_1) + iP(y_2)$. Since P is a real operator, $P(y_1)$ and $P(y_2)$ belong to $Y_{\mathbb{R}}$. Consequently, $P(y_1) = y_1$ and $P(y_2) = y_2$. Thus y_1 and y_2 belong to Z. Define $Z_{\mathbb{R}} = Z \cap Y_{\mathbb{R}}$. The above reasoning shows that $Z = Z_R \oplus iZ_{\mathbb{R}}$, and we immediately deduce that Z is an ordered subspace of Y. It inherits the absolute monotonicity of Y.

Finally consider $y \in Z$ and $\epsilon > 0$. Since Y is Riesz-normed, there exists $y' \in Y$ such that $-y' \leqslant y \leqslant y'$ and $\|y'\| \leqslant (1+\epsilon)\|y\|$. Since $P(y) = y$ and P is a positive contraction, we have both $-P(y') \leqslant y \leqslant P(y')$ and $\|P(y')\| \leqslant (1+\epsilon)\|y\|$. Since $P(y') \in Z$, this shows that Z is a Riesz-normed space. □

We now consider the special case when X is an L^p-space for $1 < p < \infty$. Then any element in $\mathcal{U}(p, X)$ is an L^p-space as well. Moreover the range of a positive contractive projection on an L^p-space is positively and isometrically isomorphic to an L^p-space (see e.g. [Ran01, Theorem 4.10]). We therefore deduce the following corollary.

Corollary 6.3. *Let $1 < p < \infty$, let Ω be a measure space and let $\pi\colon G \to \mathcal{B}(L^p(\Omega))$ be a bounded continuous representation of an amenable locally compact group G. Then there exist a measure space Ω', a continuous isometric representation*

$\pi' \colon G \to \mathcal{B}(L^p(\Omega'))$ such that $\pi'(t)$ is positive for any $t \in G$, and two bounded operators $J \colon L^p(\Omega) \to L^p(\Omega')$ and $Q \colon L^p(\Omega') \to L^p(\Omega)$ such that $\|J\|\|Q\| \leqslant \|\pi\|^2$ and

$$\pi(t) = Q\pi'(t)J, \qquad \textit{for all } t \in G.$$

Remark 6.4. (1) In the case when $G = \mathbb{Z}$, the above corollary means the following: whenever $T \colon L^p(\Omega) \to L^p(\Omega)$ is an invertible operator such that $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$, there exist a measure space Ω', an isometric isomorphism $U \colon L^p(\Omega') \to L^p(\Omega')$ and bounded operators $J \colon L^p(\Omega) \to L^p(\Omega')$ and $Q \colon L^p(\Omega') \to L^p(\Omega)$ such that $T^n = QU^nJ$ for any $n \in \mathbb{Z}$. This result was shown to the first and third authors by Éric Ricard in 2011 as a way to improve [ALM14, Theorem 4.8]. The argument in the proof of the first part of Theorem 6.1 is an extension of Ricard's original argument.

(2) In the case when X is an L^p-space, Theorem 4.1 can be obtained by combining the above result (Corollary 6.3 for $G = \mathbb{Z}$), [LM14, Theorem 6.4] and [ALM14, Theorem 4.8].

Likewise, Theorem 4.5 in the case when X is an L^p-space can be obtained by combining Corollary 6.3 for $G = \mathbb{R}$ and [FW06, Section 5]. Details are left to the reader.

We now derive analogues of Dixmier's Theorem in this Banach space setting.

Theorem 6.5. *Let G be an amenable locally compact group, let X be a reflexive Banach space and let $\pi \colon G \to \mathcal{B}(X)$ be a bounded continuous representation of G on X. Suppose $1 < p < \infty$. Then there exist a Banach space $\widetilde{X}$ which is a quotient of a subspace of an element of $\mathcal{U}(p; X)$ and an isomorphism $S \colon X \to \widetilde{X}$ such that $\|S\|\|S^{-1}\| \leqslant \|\pi\|^2$ and*

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{B}(\widetilde{X}) \\ t & \longmapsto & S\pi(t)S^{-1} \end{array}$$

is an isometric representation of G on $\widetilde{X}$.

Proof. By Theorem 6.1 there exist a space Y in $\mathcal{U}(p; X)$, an isometric representation $\widehat{\pi} \colon G \to \mathcal{B}(Y)$ and two bounded operators $J \colon X \to Y$ and $Q \colon Y \to X$ such that $\|J\|\|Q\| \leqslant \|\pi\|^2$ and

$$\pi(t) = Q\widehat{\pi}(t)J, \qquad \text{for all } t \in G.$$

According to Proposition 2.4, there exist two $\widehat{\pi}$-invariant closed subspaces $F \subset E \subset Y$ and an isomorphism $S \colon X \to E/F$ with $\|S\|\|S^{-1}\| \leqslant \|J\|\|Q\|$ such that the compressed representation $\widetilde{\widehat{\pi}}(t) \colon E/F \to E/F$ of $\widehat{\pi}(t)$ satisfies

$$\pi(t) = S^{-1}\widetilde{\widehat{\pi}}(t)S, \qquad \text{for all } t \in G.$$

Each $\widetilde{\widehat{\pi}}(t)$ is a contraction; hence we obtain the result with $\widetilde{X} = E/F$. □

Note that in the above situation, the representation $t \mapsto S\pi(t)S^{-1}$ is necessarily continuous, although $\widehat{\pi}$ may be discontinuous.

Specializing to L^p-spaces, we obtain the following corollary.

Corollary 6.6. *Let G be an amenable locally compact group, let Ω be a measure space, let $1 < p < \infty$ and let $\pi \colon G \to \mathcal{B}(L^p(\Omega))$ be a bounded continuous representation. Then there exist an SQ_p-space $\widetilde{X}$ and an isomorphism $S \colon X \to \widetilde{X}$ such*

that $\|S\|\|S^{-1}\| \leqslant \|\pi\|^2$ *and*

$$\begin{array}{ccc} G & \longrightarrow & \mathcal{B}(\widetilde{X}) \\ t & \longmapsto & S\pi(t)S^{-1} \end{array}$$

is an isometric representation of G *on* $\widetilde{X}$.

Note that the class of SQ_2-spaces coincide with the class of Hilbert spaces. Hence Dixmier's Theorem corresponds to the case $p = 2$ in the above corollary.

Except when $p = 2$ and X is a Hilbert space, Theorem 6.1 is a stronger (more precise) result than Theorem 6.5.

We conclude this paper with an application to unconditional bases (more generally to unconditional decompositions). Consider a Schauder decomposition $(X_k)_{k\geqslant 1}$ of a Banach space X, and let $(Q_k)_{k\geqslant 1}$ be the sequence of associated projections onto X_k. Thus for any $j \neq k$, we have $X_j \subset \mathrm{Ker}(Q_k)$ and $\mathrm{Ran}(Q_k) = X_k$. Moreover

$$x = \sum_{k=1}^{\infty} Q_k x, \qquad \text{for all } x \in X.$$

The Schauder decomposition is called unconditional if the above series is unconditionally convergent for any $x \in X$. Recall that this holds true if and only if there exists a positive constant C such that

$$\left\| \sum_{k=1}^{n} \omega_k Q_k x \right\|_X \leqslant C \left\| \sum_{k=1}^{n} Q_k x \right\|_X \tag{6.6}$$

for any choices of $\omega_k \in \{-1, 1\}$, $x \in X$ and $n \in \mathbb{N}$. We necessarily have $C \geqslant 1$, and the smallest $C \geqslant 1$ satisfying (6.6) is called the unconditional constant of the decomposition.

Theorem 6.7. *Let* $(X_k)_{k\geqslant 1}$ *be an unconditional decomposition of a reflexive Banach space* X *with unconditional constant* $C \geqslant 1$. *Suppose* $1 < p < \infty$. *Then there exist a Banach space* $\widetilde{X}$ *which is a quotient of a subspace of an element of* $\mathcal{U}(p; X)$ *and an isomorphism* $S\colon X \to \widetilde{X}$ *with* $\|S\|\|S^{-1}\| \leqslant C^2$ *such that the unconditional constant of the unconditional decomposition* $(S(X_k))_{k\geqslant 1}$ *of* $\widetilde{X}$ *is equal to* 1.

Proof. For any $\omega \in \{-1, 1\}^{\mathbb{N}}$, we can consider the bounded linear operator

$$\begin{array}{cccc} \pi(\omega)\colon & X & \longrightarrow & X \\ & x = \sum_k Q_k(x) & \longmapsto & \sum_k \omega_k Q_k(x). \end{array}$$

Then, we obtain a bounded continuous representation $\pi\colon \{-1, 1\}^{\mathbb{N}} \to \mathcal{B}(X)$ of the compact group $\{-1, 1\}^{\mathbb{N}}$, with $\|\pi\| \leqslant C$. By Theorem 6.5, there exist a Banach space $\widetilde{X}$ which is a quotient of a subspace of an element of $\mathcal{U}(p; X)$ and an isomorphism $S\colon X \to \widetilde{X}$ with $\|S\|\|S^{-1}\| \leqslant C^2$ such that

$$\begin{array}{ccc} \{-1, 1\}^{\mathbb{N}} & \longrightarrow & \mathcal{B}(\widetilde{X}) \\ \omega & \longmapsto & S\pi(\omega)S^{-1} \end{array}$$

is an isometric representation of the group $\{-1, 1\}^{\mathbb{N}}$ on the Banach space $\widetilde{X}$. The sequence $(S(X_k))_{k\geqslant 1}$ is an unconditional decomposition of $\widetilde{X}$ whose associated projections are equal to SQ_kS^{-1}. Then it is easy to check that its unconditional constant is equal to 1. $\square$

If $X = H$ is a Hilbert space, then we recover the classical result (see e.g. [Nik02, Theorem 3.1.4]) which says that if $(H_k)_{k\geqslant 1}$ is an unconditional decomposition of H, then there exists an isomorphism $S\colon H \to H$ such that $(S(H_k))_{k\geqslant 1}$ is an orthogonal decomposition.

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