TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 10, October 2017, Pages 7067–7083
http://dx.doi.org/10.1090/tran/6856
Article electronically published on March 29, 2017

A VANISHING THEOREM ON FAKE PROJECTIVE PLANES WITH ENOUGH AUTOMORPHISMS

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ABSTRACT. For every fake projective plane X with automorphism group of order 21, we prove that $H^i(X, 2L) = 0$ for all i and for every ample line bundle L with $L^2 = 1$. For every fake projective plane with automorphism group of order 9, we prove the same vanishing for every cubic root (and its twist by a 2-torsion) of the canonical bundle K. As an immediate consequence, there are exceptional sequences of length 3 on such fake projective planes.

A compact complex surface with the same Betti numbers as the complex projective plane $\mathbf{P}^2_{\mathbb{C}}$ is called *a fake projective plane* if it is not isomorphic to $\mathbf{P}^2_{\mathbb{C}}$. The canonical bundle of a fake projective plane is ample. So a fake projective plane is nothing but a surface of general type with $p_g = 0$ and $c_1^2 = 3c_2 = 9$. Furthermore, its universal cover is the unit 2-ball in $\mathbb{C}^2$ by [Au] and [Y] and its fundamental group is a co-compact arithmetic subgroup of $\mathrm{PU}(2, 1)$ by [Kl].

Recently, Prasad and Yeung [PY] classified all possible fundamental groups of fake projective planes. Their proof also shows that the automorphism group of a fake projective plane has order 1, 3, 9, 7, or 21. Then Cartwright and Steger ([CS], [CS2]) have carried out group theoretic enumeration based on a computer to obtain a more precise result: there are exactly 50 distinct fundamental groups, each corresponding to a pair of fake projective planes, complex conjugate to each other. They also have computed the automorphism groups of all fake projective planes X. In particular

$$\mathrm{Aut}(X) \cong \{1\},\ C_3,\ C_3^2 \text{ or } 7:3,$$

where C_n is the cyclic group of order n and $7 : 3$ is the unique non-abelian group of order 21. Among the 50 pairs 34 admit a non-trivial group of automorphisms: 3 pairs have $\mathrm{Aut} \cong 7 : 3$, 3 pairs have $\mathrm{Aut} \cong C_3^2$ and 28 pairs have $\mathrm{Aut} \cong C_3$. For each pair of fake projective planes Cartwright and Steger [CS2] have also computed the torsion group $H_1(X, \mathbb{Z}) = \mathrm{Tor}(H^2(X, \mathbb{Z})) = \mathrm{Tor}(\mathrm{Pic}(X))$, which is the abelianization of the fundamental group. According to their computation a fake projective plane with more than 3 automorphisms has no 3-torsion.

It can be shown (Lemma 1.5) that if a fake projective plane X has no 3-torsion in $H_1(X, \mathbb{Z})$, then the canonical class K_X is divisible by 3 and has a unique cubic root, i.e., a unique line bundle L_0, up to isomorphism, such that $3L_0 \cong K_X$. Its isomorphism class $[L_0]$ is fixed by $\mathrm{Aut}(X)$, since $\mathrm{Aut}(X)$ fixes the canonical class.

Received by the editors February 5, 2015 and, in revised form, October 20, 2015.
2010 *Mathematics Subject Classification.* Primary 14J29, 14F05.
This research was supported by the National Research Foundation of Korea (NRF-2007-0093858).

For a fake projective plane X an ample line bundle L is called an *ample generator* if its isomorphism class $[L]$ generates $\mathrm{Pic}(X)$ modulo torsion, or equivalently if the self-intersection number $L^2 = 1$. Any two ample generators differ by a torsion. We will use additive notation for tensoring line bundles, e.g.

$$L + M := L \otimes M, \ \ mL := L^{\otimes m}, \ \ -L := L^{-1}.$$

Theorem 0.1. *Let X be a fake projective plane with $\mathrm{Aut}(X) \cong 7 : 3$. Then for every ample generator L of $\mathrm{Pic}(X)$ and for any i we have the vanishing*

$$H^i(X, 2L) = 0.$$

Moreover, for any i and for the cubic root L_0 of K_X,

$$H^i(X, 2L_0) = H^i(X, L_0) = 0.$$

The proof uses the elliptic structure of the quotient of X by the order 7 automorphism [K08]. In the course of the proof we also show that the I_9-fibre on the minimal resolution of the order 7 quotient has multiplicity 1, which was not determined in [K08]. This additional information will be useful when one tries to give a geometric construction of such elliptic surfaces and fake projective planes. So far no fake projective plane has ever been constructed geometrically.

Theorem 0.2. *Let X be a fake projective plane with $\mathrm{Aut}(X) \cong C_3^2$. Let L be an ample generator of $\mathrm{Pic}(X)$ such that $\tau^*(2L) \cong 2L$ for all $\tau \in \mathrm{Aut}(X)$ (e.g. the cubic root L_0 and its twist $L_0 + t$ satisfy the condition for any 2-torsion bundle t). Then for any i we have the vanishing*

$$H^i(X, 2L) = 0.$$

Moreover, for any i and for the cubic root L_0 of K_X,

$$H^i(X, 2L_0) = H^i(X, L_0) = 0.$$

The proof uses the structure of the quotient of X by an order 3 automorphism [K08].

In both theorems, the core part is the vanishing $H^0(X, 2L) = 0$. The key idea of proof is that if $H^0(X, 2L) \neq 0$, then $\dim H^0(X, 4L) \geq 4$. On the other hand, the Rieman-Roch yields $\dim H^0(X, 4L) = 3$.

Corollary 0.3. *Let X be a fake projective plane with $\mathrm{Aut}(X) \cong 7 : 3$ or C_3^2. Let L_0 be the unique cubic root of K_X. Then the three line bundles*

$$\mathcal{O}_X, \ -L_0, \ -2L_0$$

form an exceptional sequence on X.

This is equivalent to $H^i(X, 2L_0) = H^i(X, L_0) = 0$ for all i, and hence follows from Theorems 0.1 and 0.2. (Since L_0 is a cubic root of K_X, the latter vanishings are equivalent to the single vanishing $H^0(X, 2L_0) = 0$.) This confirms, for fake projective planes with enough automorphisms, the conjecture raised by Galkin, Katzarkov, Mellit and Shinder [GKMS] that predicts the existence of an exceptional sequence of length 3 on every fake projective plane. Disjoint from our cases, N. Fakhruddin [F] recently has confirmed the conjecture for the case of three 2-adically uniformised fake projective planes.

For an ample line bundle M on a fake projective plane X, M^2 is a square integer. When $M^2 \geq 9$, $H^0(X, M) \neq 0$ if and only if $M \not\cong K_X$. This follows from the Riemann-Roch and the Kodaira vanishing theorems. When $M^2 \leq 4$, $H^0(X, M)$ may not vanish, though no example of non-vanishing has been known. If it does not vanish, then it gives an effective curve of small degree. The non-vanishing of $H^0(X, M)$ is equivalent to the existence of certain automorphic form on the 2-ball.

Notation.

- K_Y: the canonical class of Y
- $b_i(Y)$: the i-th Betti number of Y
- $e(Y)$: the topological Euler number of Y
- $q(X) := \dim H^1(X, \mathcal{O}_X)$, the irregularity of a surface X
- $p_g(X) := \dim H^2(X, \mathcal{O}_X)$, the geometric genus of a surface X
- curves of type $[n_1, n_2, \ldots, n_l]$: a string of smooth rational curves of self-intersection $-n_1, -n_2, \ldots, -n_l$
- $\sim$: numerical equivalence between divisors on a (singular) variety
- $[D]$: the linear equivalence class of a divisor D
- $[L]$: the isomorphism class of a line bundle L

1. Preliminaries

Lemma 1.1. *Let X be a fake projective plane. Assume that $H^0(X, 2L) = 0$ for any ample generator $L \in \mathrm{Pic}(X)$. Then*

$$H^1(X, 2L) = H^2(X, 2L) = 0$$

for any ample generator L.

Proof. The vanishing $H^0(X, 2L) = 0$ implies $H^0(X, L) = 0$. Since $K - 2L$ is an ample generator,

$$H^2(X, 2L) = H^0(X, K - 2L) = 0.$$

Finally by the Riemann-Roch,

$$H^1(X, 2L) = 0.$$

□

Lemma 1.2. *Let X be a fake projective plane such that K_X is divisible by 3. Let L_0 be an ample generator such that $K_X \cong 3L_0$. Assume that $H^0(X, 2L_0) = 0$. Then for all i,*

$$H^i(X, 2L_0) = H^i(X, L_0) = 0.$$

Proof. In this case $K - 2L_0 = L_0$, thus $H^2(X, 2L_0) = H^0(X, L_0) = 0$. Then the vanishing of H^1 follows from the Riemann-Roch. □

Lemma 1.3. *Let X be a fake projective plane. Then for any ample generator $L \in \mathrm{Pic}(X)$ and any torsion $t \in \mathrm{Pic}(X)$,*

$$h^0(X, 4L + t) = 3.$$

Proof. Since $4L + t - K_X$ is ample, Kodaira's vanishing theorem gives

$$H^1(X, 4L + t) = H^2(X, 4L + t) = 0.$$

Thus the claim follows from the Riemann-Roch. □

Lemma 1.4. *Let X be a fake projective plane. Assume that X has 2-torsions only. Then for any ample generator L and for any automorphism $\sigma \in \mathrm{Aut}(X)$,*

$$\sigma^*(2L) = 2L.$$

Proof. Since L is ample, so is $\sigma^*(L)$. Since

$$(\sigma^*(L))^2 = L^2 = 1,$$

$\sigma^*(L)$ is an ample generator. Thus $\sigma^*(L) - L$ is a torsion, and hence

$$\sigma^*(L) = L + s$$

for some 2-torsion $s \in \mathrm{Pic}(X)$. Finally $2(L+s) = 2L$. □

Lemma 1.5. *Let X be a fake projective plane.*

(1) *If $H_1(X, \mathbb{Z})$ does not contain a 3-torsion, then the canonical class K_X is divisible by 3 and there is a unique cubic root of K_X.*

(2) *If there is a line bundle L_0 such that $K_X \cong 3L_0$, then for any automorphism $\sigma \in \mathrm{Aut}(X)$,*

$$\sigma^*(L_0) = L_0 \quad (\textit{modulo a } 3\textit{-torsion}).$$

Proof. (1) Since $p_g(X) = q(X) = 0$, the long exact sequence induced by the exponential sequence gives $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$. By the universal coefficient theorem, $\mathrm{Tor}(H^2(X, \mathbb{Z})) = \mathrm{Tor}(H_1(X, \mathbb{Z}))$. So X does not admit a 3-torsion line bundle. Let L be an ample line bundle with $L^2 = 1$. Then

$$K_X = 3L + t$$

for some torsion line bundle t. Since the order of t is coprime to 3, one can write $t = 3t'$. This proves that K_X is divisible by 3. The second conclusion follows from the fact that two cubic roots of K_X differ by a 3-torsion.

(2) Write $\sigma^*(L_0) = L_0 + s$ for some torsion $s \in \mathrm{Pic}(X)$. Since σ^* preserves $K_X = 3L_0$, we see that $3s = 0$. □

Remark 1.6. (1) By a result of Kollár ([Ko], p. 96) the 3-divisibility of K_X is equivalent to the liftability of the fundamental group to $\mathrm{SU}(2,1)$. Except for pairs of fake projective planes the fundamental groups lift to $\mathrm{SU}(2,1)$ ([PY], Section 10.4, [CS], [CS2]). In the notation of [CS], these exceptional 4 pairs are the 3 pairs in the class $(\mathcal{C}_{18}, p = 3, \{2\})$ whose automorphism groups are of order 3, and the one in the class $(\mathcal{C}_{18}, p = 3, \{2I\})$ whose automorphism group is trivial.

(2) There are fake projective planes with a 3-torsion and with canonical class divisible by 3 [CS2]. The canonical class of such a surface has multiple cubic roots.

1.1. **Quotients of fake projective planes.** Let X be a fake projective plane with a non-trivial group G acting on it. In [K08], all possible structures of the quotient surface X/G and its minimal resolution were classified:

(1) If $G = C_3$, then X/G is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 3$.

(2) If $G = C_3^2$, then X/G is a $\mathbb{Q}$-homology projective plane with 4 singular points of type $\frac{1}{3}(1,2)$ and its minimal resolution is a minimal surface of general type with $p_g = 0$ and $K^2 = 1$.

(3) If $G = C_7$, then X/G is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{7}(1,5)$ and its minimal resolution is a $(2,3)$-, $(2,4)$-, or $(3,3)$-elliptic surface.

(4) If $G = 7 : 3$, then X/G is a $\mathbb{Q}$-homology projective plane with 4 singular points, where three of them are of type $\frac{1}{3}(1,2)$ and one of them is of type $\frac{1}{7}(1,5)$, and its minimal resolution is a $(2,3)$-, $(2,4)$-, or $(3,3)$-elliptic surface.

Here a $\mathbb{Q}$-homology projective plane is a normal projective surface with the same Betti numbers as $\mathbb{P}^2_{\mathbb{C}}$ (cf. [HK1], [HK2]). A fake projective plane is a non-singular $\mathbb{Q}$-homology projective plane, hence every quotient is again a $\mathbb{Q}$-homology projective plane. An (a,b)-elliptic surface is a relatively minimal elliptic surface over $\mathbb{P}^1$ with $c_2 = 12$ having two multiple fibres of multiplicity a and b respectively. It has Kodaira dimension 1 if and only if $a \geq 2, b \geq 2, a+b \geq 5$. It is an Enriques surface iff $a = b = 2$, and it is rational iff $a = 1$ or $b = 1$. All (a,b)-elliptic surfaces have $p_g = q = 0$, and by [D] its fundamental group is the cyclic group C_d, where d is the greatest common divisor of a and b. A $(2,3)$-elliptic surface is called a Dolgachev surface.

1.2. **Normal surfaces with quotient singularities.** Let S be a normal projective surface with quotient singularities and

$$f : S' \to S$$

be a minimal resolution of S. The free part of the second cohomology group of S',

$$H^2(S', \mathbb{Z})_{free} := H^2(S', \mathbb{Z})/(torsion),$$

becomes a unimodular lattice. For a quotient singular point $p \in S$, let

$$\mathcal{R}_p \subset H^2(S', \mathbb{Z})_{free}$$

be the sublattice spanned by the numerical classes of the components of $f^{-1}(p)$. It is negative definite, and its discriminant group

$$\mathrm{disc}(\mathcal{R}_p) := \mathrm{Hom}(\mathcal{R}_p, \mathbb{Z})/\mathcal{R}_p$$

is isomorphic to the abelianization $G_p/[G_p, G_p]$ of the local fundamental group G_p. In particular, the absolute value $|\det(\mathcal{R}_p)|$ of the determinant of the intersection matrix of $\mathcal{R}_p$ is equal to the order $|G_p/[G_p, G_p]|$. Define

$$\mathcal{R} := \bigoplus_{p \in Sing(S)} \mathcal{R}_p \subset H^2(S', \mathbb{Z})_{free}.$$

Quotient singularities are log-terminal, thus the adjunction formula can be written as

$$K_{S'} \sim f^* K_S - \sum_{p \in Sing(S)} \mathcal{D}_p,$$

where $\mathcal{D}_p = \sum (a_j A_j)$ is an effective $\mathbb{Q}$-divisor with $0 \leq a_j < 1$ supported on $f^{-1}(p) = \bigcup A_j$ for each singular point p. This implies that

$$K_S^2 = K_{S'}^2 - \sum_p \mathcal{D}_p^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'}.$$

The coefficients a_j of $\mathcal{D}_p$ can be computed by solving the system of equations

$$\mathcal{D}_p A_j = -K_{S'} A_j = 2 + A_j^2$$

given by the adjunction formula for each exceptional curve $A_j \subset f^{-1}(p)$.

Let $[n_1, n_2, \ldots, n_l]$ denote a Hirzebruch-Jung continued fraction, i.e.,

$$[n_1, n_2, \ldots, n_l] = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}}.$$

For a fixed Hirzebruch-Jung continued fraction $w = [n_1, n_2, \ldots, n_l]$, we define

(1) $|w| = q$, the order of the cyclic singularity corresponding to w, i.e., $w = \dfrac{q}{q_1}$ with $1 \le q_1 < q$, $\gcd(q, q_1) = 1$. Note that $|w|$ is the absolute value of the determinant of the intersection matrix corresponding to w. We also define
(2) $u_j := |[n_1, n_2, \ldots, n_{j-1}]|$ $(2 \le j \le l+1)$, $\quad u_0 = 0,\ u_1 = 1$,
(3) $v_j := |[n_{j+1}, n_{j+2}, \ldots, n_l]|$ $(0 \le j \le l-1)$, $\quad v_l = 1,\ v_{l+1} = 0$.
Note that $u_{l+1} = v_0 = |[n_1, n_2, \ldots, n_l]| = |w|$.

For a cyclic singularity p, the coefficients of $\mathcal{D}_p$ can be expressed in terms of v_j and u_j.

Lemma 1.7 ([HK2], Lemma 3.1)**.** *Let $p \in S$ be a cyclic singular point of order q. Assume that $f^{-1}(p)$ has l components $A_1, \ldots, A_l$ with $A_i^2 = -n_i$ forming a string of smooth rational curves $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$. Then*

(1) $\mathcal{D}_p = \sum\limits_{j=1}^{l} \big(1 - \dfrac{v_j + u_j}{q}\big) A_j$,

(2) $\mathcal{D}_p K_{S'} = -\mathcal{D}_p^2 = \sum\limits_{j=1}^{l} \big(1 - \dfrac{v_j + u_j}{q}\big)(n_j - 2)$.

In the rest of this section, we assume that S is a $\mathbb{Q}$-homology projective plane with cyclic singularities. Then $p_g(S') = q(S') = 0$, and thus

$$\mathrm{Pic}(S') \cong H^2(S', \mathbb{Z}).$$

Denote by

$$\bar{\mathcal{R}} \subset \mathrm{Pic}(S')_{free} := \mathrm{Pic}(S')/(torsion)$$

the primitive closure of $\mathcal{R} \subset \mathrm{Pic}(S')_{free}$.

Lemma 1.8 ([HK2], Lemma 3.7)**.** *Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities and $f : S' \to S$ be a minimal resolution. Assume that K_S is not numerically trivial. Then the following hold true:*

(1) *$D := |\det(\mathcal{R})| K_S^2$ is a non-zero square number.*
(2) *$\mathrm{disc}(\bar{\mathcal{R}})$ is a cyclic group of order $|\det(\bar{\mathcal{R}})| = \frac{|\det(\mathcal{R})|}{c^2}$, where c is the order of $\bar{\mathcal{R}}/\mathcal{R}$.*
(3) *Define*

$$D' := |\det(\bar{\mathcal{R}})| K_S^2 = \frac{D}{c^2}.$$

Then $\mathrm{Pic}(S')_{free}$ is generated over $\mathbb{Z}$ by the numerical equivalence classes of exceptional curves, an element $T \in \mathrm{Pic}(S')_{free}$ giving a generator of $\bar{\mathcal{R}}/\mathcal{R}$ and a $\mathbb{Q}$-divisor of the form

$$M = \frac{1}{\sqrt{D'}} f^* K_S + z,$$

where z is a generator of $\mathrm{disc}(\bar{\mathcal{R}})$, *and hence of the form* $z = \sum_{p\in Sing(S)} b_p e_p$ *for some integers* b_p, *where* e_p *is a generator of* $\mathrm{disc}(\mathcal{R}_p)$.

(4) *For each singular point* p, *denote by* $A_{1,p}, A_{2,p}, \ldots, A_{l_p,p}$ *the exceptional curves of* f *at* p *and by* q_p *the order of the local fundamental group at* p. *Then every element* $E \in \mathrm{Pic}(S')_{free}$ *can be written uniquely as*

$$E \sim mM + \sum_{p\in Sing(S)} \sum_{i=1}^{l_p} a_{i,p} A_{i,p} \tag{1.1}$$

for some integer m *and some* $a_{i,p} \in \frac{1}{c}\mathbb{Z}$ *for all* i, p.

(5) E *is supported on* $f^{-1}(Sing(S))$ *if and only if* $m = 0$. *Moreover, if* E *is effective (modulo a torsion) and not supported on* $f^{-1}(Sing(S))$, *then* $m > 0$ *when* K_S *is ample and* $m < 0$ *when* $-K_S$ *is ample.*

Let E be a divisor on S'. Then by Lemma 1.8(4), the numerical equivalence class of E can be written as the form (1.1). Then one can express the intersection numbers $EK_{S'}$ and E^2 in terms of the intersection numbers $EA_{j,p}$.

Proposition 1.9 ([HK2], Proposition 4.2)**.** *Assume that* K_S *is not numerically trivial. Let* E *be a divisor on* S'. *Write the numerical equivalence class of* E *as the form* (1.1). *Then the following hold true:*

(1) $EK_{S'} = \frac{m}{\sqrt{D'}}K_S^2 - \sum_p \sum_{j=1}^{l_p} \big(1 - \frac{v_{j,p}+u_{j,p}}{q_p}\big) EA_{j,p}$.

(2) $E^2 = \frac{m^2}{D'}K_S^2 - \sum_p \sum_{j=1}^{l_p} \Big(\sum_{k=1}^{j} \frac{v_{j,p}u_{k,p}}{q_p}(EA_{k,p}) + \sum_{k=j+1}^{l_p} \frac{v_{k,p}u_{j,p}}{q_p}(EA_{k,p})\Big) EA_{j,p}$.

If, for each $p \in Sing(S)$, E *has a non-zero intersection number with at most* 2 *components of* $f^{-1}(p)$, *i.e.,* $EA_{j,p} = 0$ *for* $j \neq s_p, t_p$ *for some* s_p *and* t_p *with* $1 \leq s_p < t_p \leq l_p$, *then*

$$E^2 = \frac{m^2}{D'}K_S^2 - \sum_p \Big(\frac{v_{s_p}u_{s_p}}{q_p}(EA_{s_p})^2 + \frac{v_{t_p}u_{t_p}}{q_p}(EA_{t_p})^2 + \frac{2v_{t_p}u_{s_p}}{q_p}(EA_{s_p})(EA_{t_p})\Big).$$

Proposition 1.10 ([K11], Proposition 2.4)**.** *Let* S *be a* $\mathbb{Q}$*-homology projective plane with* 3 *singular points* p_1, p_2, p_3 *of type* $\frac{1}{7}(1,5) = [2,2,3]$. *Let* A_{i1}, A_{i2}, A_{i3} *be the three components of* $f^{-1}(p_i)$ *of type* $[2,2,3]$. *Then the following hold true:*

(1) $K_S^2 = \frac{9}{7}$, $D = 3^2 7^2$, $D' = 3^2$.

(2) *If* S' *contains a* (-2)*-curve* E *not contracted by* $f : S' \to S$, *then*
   (a) $\sum_{i=1}^3 (A_{i1}E + 2A_{i2}E)$ *is divisible by* 3,
   (b) $\sum_{i=1}^3 (A_{i1}E + A_{i2}E + A_{i3}E) \geq 3$,

2. Proof of Theorem 0.1

The existence of a fake projective plane with Aut $\cong 7:3$ was first proved in [K06] as a degree 21 cover of Ishida surface which is a $(2,3)$-elliptic surface. Mumford surface [M] is also a degree 21 cover of Ishida surface, but not Galois. In terms of Prasad and Yeung [PY], Mumford surface and Keum surface belong to the *same class* in the sense that their fundamental groups are contained in the same maximal arithmetic subgroup of $\mathrm{PU}(2,1)$. The ball quotient by this maximal subgroup has 4 singular points, 3 of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{7}(1,5)$, and its minimal resolution is Ishida surface [I]. Ishida surface is a Dolgachev surface, a simply connected surface of Kodaira dimension 1 with $p_g = q = 0$.

Throughout this section, X will denote a fake projective plane with $\mathrm{Aut}(X) \cong 7:3$. Let σ_7 and σ_3 be automorphisms of X, of order 7 and 3, respectively. Then

$$\mathrm{Aut}(X) = \langle \sigma_7,\ \sigma_3 \rangle.$$

According to the explicit computation of Cartwright and Steger [CS2], there are 6 such surfaces (3 pairs from 3 different classes), and these pairs are distinguished by the torsion group

$$H_1(X,\mathbb{Z}) = C_2^3,\ C_2^4 \text{ or } C_2^6.$$

In the following proof of Theorem 0.1, we will use the structure of the quotient of X by an order 7 automorphism. The proof is split into the 3 cases: the minimal resolution of the quotient is a $(2,3)$-, $(2,4)$-, or $(3,3)$-elliptic surface.

2.1. **The case of a $(2,3)$-elliptic surface.** First we refine the result of [K08] and [K12] in the $(2,3)$-elliptic surface case.

Theorem 2.1. *Let Z be a $\mathbb{Q}$-homology projective plane with 4 singular points, 3 of type $\frac{1}{3}(1,2)$ and one of type $\frac{1}{7}(1,5)$. Assume that its minimal resolution $\tilde{Z}$ is a $(2,3)$-elliptic surface. Then the following hold true:*

(1) *The triple cover Y of Z branched at the 3 singular points of type $\frac{1}{3}(1,2)$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{7}(1,5)$. The degree 7 cover X of Y branched at the 3 singular points is a fake projective plane.*
(2) *The elliptic fibration on $\tilde{Z}$ has 4 I_3-fibres whose union contains the 8 exceptional (-2)-curves.*
(3) *The minimal resolution $\tilde{Y}$ of Y is a $(2,3)$-elliptic surface, where every fibre of the elliptic fibration on $\tilde{Z}$ does not split.*
(4) *The elliptic fibration on $\tilde{Y}$ has 4 singular fibres, one of type I_9 and 3 of type I_1, and each fibre has the same multiplicity as the corresponding fibre on $\tilde{Z}$.*
(5) *The I_9-fibre on $\tilde{Y}$ has multiplicity 1.*

Proof. The first 4 assertions are contained in [K08], Corollary 4.12, and [K12], Theorem 0.5. We need to prove the last assertion.

Let $y_1, y_2, y_3 \in Y$ be the 3 singular points and

$$A_{i1},\ A_{i2},\ A_{i3} \subset \tilde{Y}$$

be the 3 exceptional curves of type $[2,2,3]$ lying over y_i. Let

$$A_{11}, A_{12}, B_1, A_{21}, A_{22}, B_2, A_{31}, A_{32}, B_3$$

be the 9 components of the I_9-fibre F_0 in a circular order. We need to prove that F_0 is non-multiple. Suppose that F_0 is multiple. Then its multiplicity is 2 or 3.

(1) Suppose that the I_9-fibre F_0 on $\tilde{Y}$ has multiplicity 3.

Since a general fibre is numerically equivalent to $6K_{\tilde{Y}}$, we see that

$$A_{11} + A_{12} + B_1 + A_{21} + A_{22} + B_2 + A_{31} + A_{32} + B_3 = F_0 \sim 2K_{\tilde{Y}};$$

hence by pushing forward to Y,

$$B_1' + B_2' + B_3' \sim 2K_Y,$$

where $B_i' \subset Y$ is the image of B_i. The induced action of σ_3 on $\tilde{Y}$ has order 3 and rotates the I_9-fibre. Let

$$\pi : X \to X/\langle\sigma_7\rangle = Y$$

be the quotient map. Then

$$\pi^*B_1' + \pi^*B_2' + \pi^*B_3' \sim 2\pi^*K_Y \equiv 2K_X.$$

The 3 curves π^*B_1', π^*B_2', π^*B_3' are rotated by σ_3 but fixed by σ_7. Let

$$L \in \mathrm{Pic}(X)$$

be a fixed ample generator. Then $2K_X \sim 6L$, and hence

$$\pi^*B_1' \equiv 2L + t$$

for some 2-torsion t. Here we use that X has 2-torsions only. By Lemma 1.4

$$\pi^*B_1' = \sigma_7^*(\pi^*B_1') \equiv \sigma_7^*(2L) + \sigma_7^*(t) = 2L + \sigma_7^*(t),$$

and hence

$$t = \sigma_7^*(t).$$

We know that

$$\pi_1(Y) \cong \pi_1(\tilde{Y}) = \{1\},$$

in particular, X cannot have a σ_7^*-invariant torsion. Thus $t = 0$ and

$$\pi^*B_1' \equiv \pi^*B_2' \equiv \pi^*B_3' \equiv 2L.$$

Next we need to determine the intersection numbers

$$k_{ij} := A_{i3}B_j$$

for $i, j = 1, 2, 3$. Since A_{i3} is a (-3)-curve, it is a 6-section; thus

$$2 = A_{i3}F_0 = k_{i1} + k_{i2} + k_{i3} + 1,$$

so $k_{i1} + k_{i2} + k_{i3} = 1$ for each i. Applying Proposition 1.9(1) to $E = B_j$, we get

$$0 = B_jK_{\tilde{Y}} = \frac{m_j}{3}\frac{9}{7} - \frac{3}{7}(k_{1j} + k_{2j} + k_{3j} + 1),$$

so

$$m_j = k_{1j} + k_{2j} + k_{3j} + 1,$$

where m_j is the leading coefficient of B_j in the form (1.1). Now by Proposition 1.9(2),

$$-2 = B_j^2 = \frac{m_j^2}{9}\frac{9}{7} - \frac{1}{7}\{3(k_{1j}^2 + k_{2j}^2 + k_{3j}^2) + 11 + 4k_{jj} + 2k_{j+1,j}\},$$

so

$$m_j^2 + 14 = 3(k_{1j}^2 + k_{2j}^2 + k_{3j}^2) + 11 + 4k_{jj} + 2k_{j+1,j},$$

where $k_{43} = k_{13}$. It is easy to check that this system has a unique solution:

$$(k_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, y_j, $y_{j+1} \in B_j'$ and $y_{j+2} \notin B_j'$ for $j = 1, 2, 3$, where $y_{k+3} = y_k$. Let $x_j = \pi^{-1}(y_j) \in X$ be the fixed points of σ_7. Then for $j = 1, 2, 3$,

$$x_j,\, x_{j+1} \in \pi^*B_j', \qquad x_{j+2} \notin \pi^*B_j',$$

where $x_{k+3} = x_k$. Now let $g_j \in H^0(X, 2L)$ be a section giving the divisor $\pi^* B'_j$. Then the 4 sections

$$g_1^2,\ g_2^2,\ g_1 g_2,\ g_3^2 \in H^0(X, 4L)$$

are linearly independent, which can be easily checked by evaluating any given linear dependence at each x_j. This implies $h^0(X, 4L) \geq 4$, contradicting Lemma 1.3.

(2) Suppose that the I_9-fibre F_0 on $\tilde{Y}$ has multiplicity 2.

In this case,

$$A_{11} + A_{12} + B_1 + A_{21} + A_{22} + B_2 + A_{31} + A_{32} + B_3 = F_0 \sim 3K_{\tilde{Y}},$$

and hence

$$B'_1 + B'_2 + B'_3 \sim 3K_Y$$

and

$$\pi^* B'_1 \sim \pi^* B'_2 \sim \pi^* B'_3 \sim 3L.$$

Write

$$\pi^* B'_1 \equiv 3L_0 + t,$$

where L_0 is an ample generator such that $K_X = 3L_0$ and t is a torsion. We know that $\pi^* B'_1$ is σ_7^*-invariant. Since L_0 is σ_7^*-invariant by Lemma 1.5, so is t. Since $Y = X/\langle \sigma_7 \rangle$ is simply connected, X cannot have a σ_7^*-invariant torsion. Thus $t = 0$. Then

$$\pi^* B'_1 \equiv 3L_0 = K_X,$$

and hence K_X is effective, contradicting $p_g(X) = 0$. This completes the proof of Theorem 2.1. □

Proof of Theorem 0.1. By Lemma 1.1, it is enough to show that $H^0(X, 2L) = 0$. Suppose that

$$H^0(X, 2L) \neq 0.$$

By Lemma 1.4, $\sigma_7^*(2L) \cong 2L$, i.e. σ_7^* acts on the projective space $\mathbf{P}H^0(X, 2L)$. Every finite order automorphism of a projective space has a fixed point, so there is a σ_7^*-invariant curve

$$C \in |2L|,$$

possibly reducible. Then there is a curve

$$C' \subset Y = X/\langle \sigma_7 \rangle$$

such that $C = \pi^* C'$. Since $C'^2 = 4/7$, we see that

$$C' \sim \frac{2}{3} K_Y.$$

Let $\tilde{C} \subset \tilde{Y}$ be the proper transform of C' under the resolution

$$f : \tilde{Y} \to Y.$$

Then

$$\tilde{C} \sim f^* C' - \sum_{k=1}^{3} \mathcal{C}_k \sim \frac{2}{3} f^* K_Y - \sum_{k=1}^{3} \mathcal{C}_k,$$

where $\mathcal{C}_k$ is a $\mathbb{Q}$-divisor supported on $A_{k1} \cup A_{k2} \cup A_{k3}$. For any exceptional curve A_{ij} we have $A_{ij} f^* C' = 0$; hence

$$0 \leq A_{ij} \tilde{C} = -A_{ij} \sum_{k=1}^{3} \mathcal{C}_k = -A_{ij} \mathcal{C}_i.$$

We know that

$$K_{\tilde{Y}} \sim f^*K_Y - \frac{1}{7}(A_{11}+2A_{12}+3A_{13}) - \frac{1}{7}(A_{21}+2A_{22}+3A_{23}) - \frac{1}{7}(A_{31}+2A_{32}+3A_{33}).$$

It follows that

$$K_{\tilde{Y}}\tilde{C} = (f^*K_Y)(f^*C') + \sum_{i=1}^{3} \frac{1}{7}(A_{i1}+2A_{i2}+3A_{i3})\mathcal{C}_i \le (f^*K_Y)(f^*C') = K_Y C' = \frac{6}{7}.$$

Since $\tilde{Y}$ is a relatively minimal elliptic surface with Kodaira dimension 1, $K_{\tilde{Y}}$ is numerically effective. Thus $K_{\tilde{Y}}\tilde{C}$ is a non-negative integer, and hence

$$K_{\tilde{Y}}\tilde{C} = 0.$$

Thus the curve $\tilde{C}$ is contained in a union of fibres.

On the other hand, since the I_9-fibre has multiplicity 1,

$$B'_1 + B'_2 + B'_3 \sim 6K_Y$$

and

$$\pi^*B'_1 + \pi^*B'_2 + \pi^*B'_3 \sim 6\pi^*K_Y \equiv 6K_X \sim 18L,$$

and hence

$$\pi^*B'_1 \sim \pi^*B'_2 \sim \pi^*B'_3 \sim 6L.$$

If F_1 is the reduced curve of the fibre with multiplicity 2 or 3, then it is irreducible and the image $F'_1 \subset Y$ satisfies $F'_1 \sim 3K_Y$ or $2K_Y$, and hence

$$\pi^*F'_1 \sim 9L \quad \text{or} \quad 6L.$$

Thus no union of irreducible components of fibres can be equal to the curve $\tilde{C}$. The proof is completed.

Remark 2.2. One can determine the intersection numbers

$$k_{ij} := A_{i3}B_j,$$

where $A_{11}, A_{12}, B_1, A_{21}, A_{22}, B_2, A_{31}, A_{32}, B_3$ are the 9 components of the I_9-fibre F_0 as above. Since the I_9-fibre has multiplicity 1, we get

$$k_{i1} + k_{i2} + k_{i3} = 5,$$

$$m_j = k_{1j} + k_{2j} + k_{3j} + 1,$$

$$m_j^2 + 14 = 3(k_{1j}^2 + k_{2j}^2 + k_{3j}^2) + 11 + 4k_{jj} + 2k_{j+1,j}.$$

There are many solutions to this system. Among them two are symmetric with respect to the order 3 rotation $(A_{13}, B_1) \to (A_{23}, B_2) \to (A_{33}, B_3) \to (A_{13}, B_1)$:

$$(k_{ij}) = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 3 \\ 3 & 1 & 1 \end{pmatrix}.$$

Each of the two cases leads to a fake projective plane as shown in [K06].

2.2. **The case of a** $(2,4)$**-elliptic surface.** First we refine the result of [K08] and [K11] in the $(2,4)$-elliptic surface case.

Theorem 2.3. *Let Z be a $\mathbb{Q}$-homology projective plane with 4 singular points, 3 of type $\frac{1}{3}(1,2)$ and 1 of type $\frac{1}{7}(1,5)$. Assume that its minimal resolution $\tilde{Z}$ is a $(2,4)$-elliptic surface. Then the following hold true:*

(1) *The triple cover Y of Z branched at the 3 singular points of type $\frac{1}{3}(1,2)$ is a $\mathbb{Q}$-homology projective plane with 3 singular points of type $\frac{1}{7}(1,5)$. The degree 7 cover X of Y branched at the 3 singular points is a fake projective plane.*

(2) *The elliptic fibration on $\tilde{Z}$ has 4 I_3-fibres whose union contains the 8 exceptional (-2)-curves.*

(3) *The minimal resolution $\tilde{Y}$ of Y is a $(2,4)$-elliptic surface, where every fibre of the elliptic fibration on $\tilde{Z}$ does not split.*

(4) *The elliptic fibration on $\tilde{Y}$ has 4 singular fibres, 1 of type I_9 and 3 of type I_1, and each fibre has the same multiplicity as the corresponding fibre on $\tilde{Z}$.*

(5) *The I_9-fibre on $\tilde{Y}$ has multiplicity* 1.

Proof. The first 4 assertions are contained in [K08], Corollary 4.12, and [K11], Theorem 1.1 and 4.1. We need to prove the assertion (5).

For a $(2,4)$-elliptic surface the general fibre F is numerically equivalent to $4K$. Thus any (-3)-curve is a 4-section, since it has intersection number 1 with the canonical class K.

If the I_9-fibre on $\tilde{Y}$ has multiplicity 4, the (-3)-curve A_{i3}, being a 4-section, intersects only one component of the I_9-fibre, namely A_{i2}. Thus the curve B_1 does not meet A_{i3} for any i, and hence

$$\sum_{i=1}^{3}(A_{i1}B_1 + A_{i2}B_1 + A_{i3}B_1) = 2,$$

contradicting Proposition 1.10.

Suppose that the I_9-fibre on $\tilde{Y}$ has multiplicity 2. This case can be ruled out by a similar argument as in the $(2,3)$-elliptic surface case. Since a general fibre is numerically equivalent to $4K_{\tilde{Y}}$, we see that

$$A_{11} + A_{12} + B_1 + A_{21} + A_{22} + B_2 + A_{31} + A_{32} + B_3 = F_0 \sim 2K_{\tilde{Y}},$$

and hence

$$\pi^*B_1' + \pi^*B_2' + \pi^*B_3' \sim 2\pi^*K_Y \equiv 2K_X.$$

The 3 curves π^*B_1', π^*B_2', π^*B_3' are rotated by the order 3 automorphism σ_3 but are fixed by the order 7 automorphism σ_7. Let L be a fixed ample generator of $\mathrm{Pic}(X)$. Then

$$\pi^*B_1' \equiv 2L + t$$

for some σ_7^*-invariant torsion t. Since π^*B_2' is also σ_7^*-invariant and

$$\pi^*B_2' = \sigma_3^*(\pi^*B_1') \equiv \sigma_3^*(2L) + \sigma_3^*(t) = 2L + \sigma_3^*(t),$$

we see that $\sigma_3^*(t)$ is also σ_7^*-invariant. We know that

$$\pi_1(Y) \cong \pi_1(\tilde{Y}) \cong C_2;$$

in particular, X may have at most one σ-invariant torsion, and in that case it must be a 2-torsion. Since both t and $\sigma_3^*(t)$ are σ_7^*-invariant, either t is trivial or a σ_3^*-invariant 2-torsion. It follows that

$$\pi^* B_1' \equiv \pi^* B_2' \equiv \pi^* B_3' \equiv 2L + t.$$

Furthermore, the intersection numbers $k_{ij} := A_{i3}B_j = \delta_{ij}$. This follows from the exactly same computation as in Theorem 2.1, since A_{i3} is a 4-section and $2 = A_{i3}F_0 = k_{i1} + k_{i2} + k_{i3} + 1$. The rest is ditto, except that we choose a section $g_j \in H^0(X, 2L + t)$ that gives the divisor $\pi^* B_j'$. Since $2t$ is trivial, the 4 sections

$$g_1^2,\ g_2^2,\ g_1 g_2,\ g_3^2 \in H^0(X, 4L).$$

□

Proof of Theorem 0.1. In this case the proof is simpler than the previous case. If there is a σ_7^*-invariant curve $C \in |2L|$, then the same argument shows that the corresponding curve $\tilde{C} \subset \tilde{Y}$ is contained in a union of fibres. On the other hand, the I_9-fibre has multiplicity 1 and hence

$$B_1' + B_2' + B_3' \sim 4K_Y,$$
$$\pi^* B_1' + \pi^* B_2' + \pi^* B_3' \sim 4\pi^* K_Y \equiv 4K_X \sim 12L,$$

and hence

$$\pi^* B_1' \sim \pi^* B_2' \sim \pi^* B_3' \sim 4L.$$

If F_1 is the reduced curve of the fibre with multiplicity 2 or 4, then it is irreducible and the image $F_1' \subset Y$ satisfies $F_1' \sim 2K_Y$ or K_Y, and hence

$$\pi^* F_1' \sim 6L \ \text{ or } \ 3L.$$

Thus no union of irreducible components of fibres can be equal to the curve $\tilde{C}$. The proof is completed.

Remark 2.4. In [K11], Remark 3.3, the intersection numbers

$$k_{ij} := A_{i3}B_j$$

were computed, where $A_{11}, A_{12}, B_1, A_{21}, A_{22}, B_2, A_{31}, A_{32}, B_3$ are the 9 components of the I_9-fibre F_0 as before. Using the same notation as before, we get

$$k_{i1} + k_{i2} + k_{i3} = 3,$$
$$m_j = k_{1j} + k_{2j} + k_{3j} + 1,$$
$$m_j^2 + 14 = 3(k_{1j}^2 + k_{2j}^2 + k_{3j}^2) + 11 + 4k_{jj} + 2k_{j+1,j}.$$

There are many solutions to this system. Among them two are symmetric with respect to the order 3 rotation $(A_{13} : B_1) \to (A_{23}, B_2) \to (A_{33}, B_3) \to (A_{13}, B_1)$:

$$(k_{ij}) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Each of the two cases, if it exists, leads to a fake projective plane ([K11], Section 5). According to the computer based computation by Cartwright and Steger [CS2] there is exactly one pair of fake projective planes, complex conjugate to each other, with $\mathrm{Aut}(X) = 7 : 3$ such that the order 7 quotient has fundamental group of order 2. Thus at least 1 of the 2 cases for (k_{ij}) occurs. If both occur, then the corresponding fake projective planes must be complex conjugate to each other.

2.3. **The case of a** $(3,3)$**-elliptic surface.** Cartwright and Steger have computed the fundamental groups of all quotients of fake projective planes [CS2]. According to their group theoretic computation based on a computer, the quotient of a fake projective plane by an order 7 automorphism, if it exists, has fundamental group either trivial or cyclic of order 2. This eliminates the possibility of a $(3,3)$-elliptic surface case. However we will show that our argument works even in this case, instead of referring to their work.

First we note that all statements of Theorem 2.3 hold, even if $(2,4)$ is replaced by $(3,3)$. Indeed, for a $(3,3)$-elliptic surface the general fibre F is numerically equivalent to $3K$, thus any (-3)-curve is a 3-section, since it has intersection number 1 with K. Then by Proposition 1.10 the I_9-fibre on $\tilde{Y}$ cannot have multiplicity 3, hence 1.

If there is a σ_7^*-invariant curve $C \in |2L|$, then the same argument shows that the corresponding curve $\tilde{C} \subset \tilde{Y}$ is contained in a union of fibres. Since a general fibre is numerically equivalent to $3K_{\tilde{Y}}$, we see that

$$A_{11} + A_{12} + B_1 + A_{21} + A_{22} + B_2 + A_{31} + A_{32} + B_3 = F_0 \sim 3K_{\tilde{Y}},$$

$$B_1' + B_2' + B_3' \sim 3K_Y,$$

$$\pi^* B_1' + \pi^* B_2' + \pi^* B_3' \sim 3\pi^* K_Y \equiv 3K_X,$$

and hence

$$\pi^* B_1' \sim \pi^* B_2' \sim \pi^* B_3' \sim K_X \sim 3L.$$

If F_1 is the reduced curve of the fibre with multiplicity 3, then it is irreducible and $F_1 \sim K_{\tilde{Y}}$, so the image $F_1' \subset Y$ satisfies $F_1' \sim K_Y$, and hence

$$\pi^* F_1' \sim K_X \sim 3L.$$

Thus no union of irreducible components of fibres can be equal to the curve $\tilde{C}$.

3. Proof of Theorem 0.2

Throughout this section X will denote a fake projective plane with $\mathrm{Aut}(X) \cong C_3^2$.

According to the computation of Cartwright and Steger [CS2], X has

$$H_1(X, \mathbb{Z}) = C_7,\ C_{14} \text{ or } C_2^2 \times C_{13}.$$

In particular X has no 3-torsion, and hence has a unique cubic root of K_X.

Let L be an ample generator of $\mathrm{Pic}(X)$ such that $\tau^*(2L) \cong 2L$ for all $\tau \in \mathrm{Aut}(X)$. Such an L exists, e.g. a cubic root of K_X (Lemma 1.5).

Pick two automorphisms σ and σ' such that

$$\mathrm{Aut}(X) = \langle \sigma,\ \sigma' \rangle \cong C_3^2.$$

By Lemma 1.2, it is enough to show that $H^0(X, 2L) = 0$. Suppose that

$$H^0(X, 2L) \neq 0.$$

Let $x_1,\ x_2,\ x_3 \in X$ be the 3 fixed points of σ,

$$\mathrm{Fix}(\sigma) = \{x_1,\ x_2,\ x_3\} \subset X.$$

Then σ' rotates x_1, x_2, x_3 and fixes 3 points, different from the x_i's (see subsection 1.1 or [K08]). Note that $\mathrm{Aut}(X)$ acts on the projective space

$$\mathbf{P}H^0(X, 2L).$$

An automorphism of finite order of a projective space has a fixed point; thus there is a curve, possibly reducible,

$$C \in |2L| \text{ with } \sigma^*(C) = C.$$

Let

$$\pi : X \to X/\langle\sigma\rangle = Y$$

be the quotient map. Then there is a curve

$$C' \subset Y = X/\langle\sigma\rangle$$

such that $C = \pi^* C'$. Since $C'^2 = 4/3$, we see that

$$C' \sim \frac{2}{3} K_Y.$$

Let $y_i \in Y$ be the image of x_i. Then y_i is a singular point of type $\frac{1}{3}(1,2)$. Note that

$$C \equiv \sigma'^*(C) \equiv \sigma'^{2*}(C) \equiv 2L.$$

Claim. C passes through exactly 2 of the 3 points $x_1, x_2, x_3 \in X$.

This is equivalent to saying that C' passes through exactly 2 of the 3 singular points $y_1, y_2, y_3 \in Y$. To see this, let $A_{i1}, A_{i2} \subset \tilde{Y}$ be the exceptional (-2)-curves over y_i. Let $\tilde{C} \subset \tilde{Y}$ be the proper transform of C' under the resolution

$$f : \tilde{Y} \to Y.$$

Then

$$\tilde{C} \sim f^* C' - \sum_{k=1}^{3} \mathcal{C}_k,$$

where $\mathcal{C}_k$ is a $\mathbb{Q}$-divisor supported on $A_{k1} \cup A_{k2}$, which can be computed as follows:

$$A_{k1}\tilde{C} = a,\ A_{k2}\tilde{C} = b \Longleftrightarrow \mathcal{C}_k = \frac{(2a+b)}{3} A_{k1} + \frac{(a+2b)}{3} A_{k2}.$$

In this case

$$\mathcal{C}_k^2 = -\frac{2}{3}(a^2+b^2+ab) = -\frac{2}{3},\ -\frac{6}{3},\ -\frac{8}{3},\ -\frac{14}{3}, \ldots.$$

It follows that

$$\mathcal{C}_k^2 = -\frac{2}{3} \Longleftrightarrow a+b = A_{k1}\tilde{C} + A_{k2}\tilde{C} = 1 \Longleftrightarrow \mathrm{mult}_{x_k} C = 1.$$

We compute

$$\tilde{C}^2 = C'^2 + \sum_{k=1}^{3} \mathcal{C}_k^2 = \frac{4}{3} + \sum_{k=1}^{3} \mathcal{C}_k^2.$$

Note that C passes through x_k if and only if $\mathcal{C}_k \neq 0$. It is easy to check that $\frac{4}{3} - \frac{2}{3}(a^2+b^2+ab)$ is not an integer for any integer $a, b \geq 0$. Since $\tilde{C}^2$ is an integer, at least 2 of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ are non-zero, i.e. C passes through at least 2 of the 3 points x_1, x_2, x_3. Suppose $x_1, x_2, x_3 \in C$. If C has multiplicity ≥ 2 at x_3, then the curve $\sigma'^*(C)$ has multiplicity ≥ 2 at x_2, and hence

$$4 = \sigma'^*(C)C \geq \sum_{k=1}^{3} \mathrm{mult}_{x_k}\sigma'^*(C) \cdot \mathrm{mult}_{x_k} C \geq 1 + 2 + 2,$$

which is absurd. If C has multiplicity 1 at each x_k, then $\tilde{C}^2 = \frac{4}{3} - \frac{2}{3} \cdot 3 = -\frac{2}{3}$, not an integer. This proves the Claim.

Now we may assume that $x_1,\ x_2 \in C,\ x_3 \notin C$. Then

$$x_3,\ x_1 \in \sigma'^*(C),\ x_2 \notin \sigma'^*(C),$$

$$x_2,\ x_3 \in \sigma'^{2*}(C),\ x_1 \notin \sigma'^{2*}(C).$$

This proves that $\dim H^0(X, 2L) \geq 3$, hence $h^0(X, 4L) \geq 4$, contradicting Lemma 1.3. Indeed, if $g_1, g_2, g_3 \in H^0(X, 2L)$ are sections giving the divisor C, $\sigma'^*(C)$, $\sigma'^{2*}(C)$, respectively, then the 4 sections

$$g_1^2,\ g_2^2,\ g_1 g_2,\ g_3^2 \in H^0(X, 4L)$$

are linearly independent.

4. Exceptional sequences on a fake projective plane

Let $D^b(coh(W))$ denote the bounded derived category of coherent sheaves on a smooth variety W. It is a triangulated category.

An object E in a triangulated category is called exceptional if $\mathrm{Hom}(E, E[i]) = \mathbb{C}$ if $i = 0$, and $= 0$ otherwise.

A sequence $E_1, \ldots, E_n$ of exceptional objects is called an exceptional sequence if $\mathrm{Hom}(E_j, E_k[i]) = 0$ for any $j > k$, any i.

When W is a smooth surface with $p_g = q = 0$, every line bundle is an exceptional object in $D^b(coh(W))$.

Let X be a fake projective plane and L be an ample generator of $\mathrm{Pic}(X)$. The three line bundles

$$2L,\ L,\ \mathcal{O}_X$$

form an exceptional sequence if and only if $H^j(X, 2L) = H^j(X, L) = 0$ for all j. Thus Corollary 0.3 follows from Theorems 0.1 and 0.2.

Write

$$D^b(coh(X)) = \langle 2L,\ L,\ \mathcal{O}_X,\ \mathcal{A} \rangle,$$

where $\mathcal{A}$ is the orthogonal complement of the admissible triangulated subcategory generated by $2L,\ L,\ \mathcal{O}_X$. Then the Hochschild homology

$$HH_*(\mathcal{A}) = 0.$$

This can be read from the Hodge numbers. In fact, the Hochschild homology of X is the direct sum of Hodge spaces $H^{p,q}(X)$, and its total dimension is the sum of all Hodge numbers. The latter is equal to the topological Euler number $c_2(X)$, as a fake projective plane has Betti numbers $b_1(X) = b_3(X) = 0$.

The Grothendieck group $K_0(X)$ has filtration

$$K_0(X) = F^0 K_0(X) \supset F^1 K_0(X) \supset F^2 K_0(X)$$

with

$$F^0 K_0(X)/F^1 K_0(X) \cong CH^0(X) \cong \mathbb{Z},$$

$$F^1 K_0(X)/F^2 K_0(X) \cong \mathrm{Pic}(X),$$

$$F^2 K_0(X) \cong CH^2(X).$$

If the Bloch conjecture holds for X, i.e. if $CH^2(X) \cong \mathbb{Z}$, then $K_0(\mathcal{A})$ is finite.

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