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ISOTROPIC MEASURES AND STRONGER FORMS OF THE REVERSE ISOPERIMETRIC INEQUALITY

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ABSTRACT. The reverse isoperimetric inequality, due to Keith Ball, states that if K is an n-dimensional convex body, then there is an affine image $\tilde{K}$ of K for which $S(\tilde{K})^n/V(\tilde{K})^{n-1}$ is bounded from above by the corresponding expression for a regular n-dimensional simplex, where S and V denote the surface area and volume functional. It was shown by Franck Barthe that the upper bound is attained only if K is a simplex. The discussion of the equality case is based on the equality case in the geometric form of the Brascamp-Lieb inequality. The present paper establishes stability versions of the reverse isoperimetric inequality and of the corresponding inequality for isotropic measures.

CONTENTS

1. INTRODUCTION

The isoperimetric inequality states that a Euclidean ball has the smallest surface area among convex bodies (compact convex sets with non-empty interiors) of given volume in Euclidean space $\mathbb{R}^n$ with scalar product $\langle \cdot\,, \cdot \rangle$ and norm $\| \cdot \|$, and that Euclidean balls are the only minimizers. In fact, the isoperimetric inequality holds

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for general Borel sets if the distributional perimeter is used to measure surface area (see [19], [42, Notes for Section 7.1] and the references cited there). In the following, we restrict our considerations to convex sets, since this is required for the study of the reverse isoperimetric inequality and its stability improvement, which is the main topic of the present investigation.

Let B^n be the Euclidean unit ball centred at the origin. Denoting by $S(K)$ the surface area and by $V(K)$ the volume of a convex body K in $\mathbb{R}^n$, the isoperimetric inequality can be expressed by the inequality

$$\frac{S(B^n)^n}{V(B^n)^{n-1}} \leq \frac{S(K)^n}{V(K)^{n-1}}, \tag{1}$$

where equality holds if and only if K is a Euclidean ball. Since surface area and volume are continuous functionals on the space of convex bodies (with respect to the Hausdorff metric) and the extremal bodies of the inequality (1) are precisely the Euclidean balls, the following question arises naturally. Suppose that a convex body K in $\mathbb{R}^n$ satisfies

$$\frac{S(K)^n}{V(K)^{n-1}} \leq (1+\varepsilon)\frac{S(B^n)^n}{V(B^n)^{n-1}}$$

for some $\varepsilon \geq 0$. Does it follow that K is δ-close to a Euclidean ball? A satisfactory answer to this question requires that the distance $\operatorname{dist}(K)$ of K from a Euclidean ball is measured in a suitable way and that δ is given as an explicit function of ε. For instance, the distance function $\operatorname{dist}(\cdot)$ should have the same scaling and motion invariance as the isoperimetric problem, and δ could be of the form $\gamma \cdot \varepsilon^\alpha$ with constants $\gamma, \alpha > 0$ which may depend on the dimension. The problem can also be stated in the following form. Let again K be a convex body in $\mathbb{R}^n$ and assume that $\operatorname{dist}(K) \geq \varepsilon$ for some $\varepsilon \geq 0$. Does it follow that

$$\frac{S(K)^n}{V(K)^{n-1}} \geq (1+f(\varepsilon))\frac{S(B^n)^n}{V(B^n)^{n-1}},$$

where $f : [0,\infty) \to [0,\infty)$ is a continuous and increasing function with $f(0) = 0$? In other words, is it true that

$$\frac{S(K)^n}{V(K)^{n-1}} \geq (1+f(\operatorname{dist}(K)))\frac{S(B^n)^n}{V(B^n)^{n-1}}$$

with an explicitly given function f? Any such inequality provides a strengthening of the classical isoperimetric inequality and is called a stability result related to (1).

Although results of this type can be traced back to work of Minkowski and Bonnesen, a systematic exploration is much more recent. Introductory surveys on geometric stability results were given by H. Groemer [24, 25], an up-to-date coverage of various aspects (including applications) of the topic is provided throughout R. Schneider's book [42]. More specifically, stability results for the isoperimetric problem (based on the Hausdorff distance) have been found, for instance, by Groemer and Schneider [26]. As a recent breakthrough, N. Fusco, F. Maggi, A. Pratelli [19] obtained an optimal stability version of the isoperimetric inequality in terms of the volume difference (for general Borel sets), and A. Figalli, F. Maggi, A. Pratelli [18] established a sharp quantitative version of the anisotropic isoperimetric inequality (for sets of finite perimeter and finite volume) and from this they deduced a sharp quantitative version of the Brunn-Minkowski inequality for convex sets (for the latter, see also [17]).

The isoperimetric ratio $S(K)^n/V(K)^{n-1}$ is unbounded from above, if K ranges over all convex bodies. In fact, simple examples show that K can have arbitrarily small volume and still surface area equal to a prescribed positive value. In order to avoid this type of situation, it is a well-known strategy (see, for instance, F. Behrend [9]) to consider the affine invariant ratio

$$\mathrm{ir}(K) := \inf\left\{\frac{S(\Phi K)^n}{V(\Phi K)^{n-1}} : \Phi \in \mathrm{GL}(n)\right\},$$

where the abbreviation 'ir' should be reminiscent of **i**soperimetric (or **i**nvariant) **r**atio. By definition the functional ir is constant on the affine equivalence class of convex bodies. The infimum is attained and the unique minimizer can be characterized, as shown by C. M. Petty [41] (see also A. Giannopoulos, M. Papadimitrakis [22]). In fact, K minimizes the isoperimetric ratio within its affine equivalence class if and only if the suitably normalized area measure of K is isotropic (as defined below). As a simple consequence, the regular simplex minimizes the isoperimetric ratio within the class of simplices. Since the new functional 'ir' is affine invariant and upper semi-continuous, it attains its maximum on the space of convex bodies. In the Euclidean plane, W. Gustin [31] showed that $\mathrm{ir}(K) \le \mathrm{ir}(T^2)$ with equality if and only if K is a triangle; here T^2 denotes a regular triangle circumscribed to B^2 (that is, $B^2 \subset T^2$ and all facets (i.e., edges) of T^2 touch B^2). An extension of such a result to higher dimensions turned out to be a formidable problem which resisted its solution until K. M. Ball [1, 2] established reverse forms of the isoperimetric inequality. To state one of his main results, note that

$$V(T^n) = \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} \qquad \text{and} \qquad S(T^n) = nV(T^n),$$

where T^n is a regular simplex in $\mathbb{R}^n$ circumscribed to B^n (hence B^n is the inball of T^n).

Theorem A (K. M. Ball)**.** *For any convex body K in $\mathbb{R}^n$, there exists some $\Phi \in \mathrm{GL}(n)$ such that*

$$\frac{S(\Phi K)^n}{V(\Phi K)^{n-1}} \le \frac{S(T^n)^n}{V(T^n)^{n-1}}.$$

It was proved by F. Barthe [5] that equality holds in Theorem A only if K is a simplex.

The main objective of this paper is to establish a stability version of K. M. Ball's reverse isoperimetric inequality. Following [17–19], we define an affine invariant distance of convex bodies K and M based on the volume difference. For this, let $\alpha = V(K)^{-1/n}$, $\beta = V(M)^{-1/n}$, and then define

$$\delta_{\mathrm{vol}}(K, M) := \min\left\{V\left(\Phi(\alpha K)\Delta(x + \beta M)\right) : \Phi \in \mathrm{SL}(n), x \in \mathbb{R}^n\right\}.$$

We observe that $\delta_{\mathrm{vol}}(\cdot,\cdot)$ induces a metric on the affine equivalence classes of convex bodies.

A crucial tool in geometric analysis, and in particular in the proof of the reverse isoperimetric inequality by K. M. Ball, is the John ellipsoid of a convex body K in $\mathbb{R}^n$. This is the unique ellipsoid of maximal volume contained in K. Obviously, there is an affine image of K, whose John ellipsoid is the Euclidean unit ball B^n. Below (see (2) and (3)), we list some properties of the John ellipsoid. For thorough discussions of the properties of the John ellipsoid, and of convex bodies in general, see K. M. Ball [3], P. M. Gruber [27] or R. Schneider [42].

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, whose John ellipsoid is a Euclidean ball, and let $\varepsilon \in [0,1)$. If $\delta_{\mathrm{vol}}(K, T^n) \geq \varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}} \leq (1-\gamma\varepsilon^4)\frac{S(T^n)^n}{V(T^n)^{n-1}},$$

where one may choose $\gamma = n^{-250n}$. In particular, for any convex body K in $\mathbb{R}^n$, $\delta_{\mathrm{vol}}(K, T^n) \geq \varepsilon$ implies that

$$\mathrm{ir}(K) \leq (1-\gamma\varepsilon^4)\ \mathrm{ir}(T^n).$$

Considering a convex body K which is obtained from T^n by cutting off regular simplices of height ε at the vertices of T^n and slabs of width ε^{n-1} parallel to the facets of T^n, one can see that the stability order (the exponent of ε) in the first part of Theorem 1.1 must be at least 1. The origin of the ε^4 term (instead of a preferable ε term) is due to the application of the crucial Proposition 7.1 in the proof of Theorem 1.1, whereas the second main ingredient (Lemma 8.1 (ii)) is of optimal stability order. A main step in the proof of Proposition 7.1 is Lemma 7.2, where a stability estimate for the arithmetic-geometric means inequality is used with a quadratic deviation estimate. This quadratic estimate produces two factors of the order ε^2, one of which arises from a deviation estimate for a transportation map.

In the plane, we obtain a result of optimal stability order.

Theorem 1.2. *Let K be a convex body in $\mathbb{R}^2$, whose John ellipsoid is a Euclidean ball, and let $\varepsilon \in [0,1)$. If $\delta_{\mathrm{vol}}(K, T^2) \geq \varepsilon$, then*

$$\frac{S(K)^2}{V(K)} \leq (1-\gamma\varepsilon)\frac{S(T^2)^2}{V(T^2)},$$

where one may choose $\gamma = 2^{-10}3^{-1}$. In particular, for any convex body K in $\mathbb{R}^2$, $\delta_{\mathrm{vol}}(K, T^2) \geq \varepsilon$ implies that

$$\mathrm{ir}(K) \leq (1-\gamma\varepsilon)\ \mathrm{ir}(T^2).$$

Another affine invariant distance between convex bodies is the (generalized) Banach–Mazur distance $\delta_{\mathrm{BM}}(K, M)$, of convex bodies K and M, which is defined by

$$\delta_{\mathrm{BM}}(K,M) := \ln\min\{\lambda \geq 1 :\ K-x \subset \Phi(M-y) \subset \lambda(K-x) \text{ for } \Phi \in \mathrm{GL}(n), x, y \in \mathbb{R}^n\}.$$

Again, $\delta_{\mathrm{BM}}(\cdot,\cdot)$ induces a metric on the affine equivalence classes of convex bodies. The two metrics are related to each other. It is not difficult to see that $\delta_{\mathrm{vol}} \leq 2n^2\delta_{\mathrm{BM}}$ (see Section 8). In the reverse direction, we have $\delta_{\mathrm{BM}} \leq \gamma\ (\delta_{\mathrm{vol}})^{1/n}$, where γ depends on the dimension n (see [11, Section 5]), and the exponent $1/n$ cannot be replaced by anything larger than $2/(n+1)$ as can be seen from the example of a ball from which a cap is cut off.

Theorem 1.3. *Let K be a convex body in $\mathbb{R}^n$ whose John ellipsoid is a Euclidean ball, and let $\varepsilon \in [0,1)$. If $\delta_{\mathrm{BM}}(K, T^n) \geq \varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}} \leq \left(1-\gamma\varepsilon^{\max\{4,n\}}\right)\frac{S(T^n)^n}{V(T^n)^{n-1}},$$

where one may choose $\gamma = n^{-250n}$. In particular, for any convex body K in $\mathbb{R}^n$, $\delta_{\mathrm{BM}}(K, T^n) \geq \varepsilon$ implies that

$$\mathrm{ir}(K) \leq \left(1-\gamma\varepsilon^{\max\{4,n\}}\right)\mathrm{ir}(T^n).$$

Cutting off regular simplices of edge length ε at the corners of T^n, we see that the error in the first part of Theorem 1.3 can be of order ε^{n-1}. The exponent $\max\{4, n\}$ in the stability estimate is due to the application of Proposition 7.1, which contributes the exponent 4 (as described before), and of Lemma 8.1 (i), which leads at least to an exponent n.

In the plane, the aforementioned approach due to W. Gustin can be used to establish a stability result of optimal order.

Theorem 1.4. *Let K be a convex body in $\mathbb{R}^2$, and let $\varepsilon \in [0, 1)$. If $\delta_{\mathrm{BM}}(K, T^2) \geq \varepsilon$, then*

$$\mathrm{ir}(K) \leq (1 - \gamma\varepsilon)\ \mathrm{ir}(T^2),$$

where we can choose $\gamma = 2^{-3}3^{-2}$.

As pointed out before, we have $\delta_{\mathrm{vol}} \leq 8\,\delta_{\mathrm{BM}}$ (see Lemma 8.2). Hence, Theorem 1.4 implies that if K is a convex body in $\mathbb{R}^2$, $\varepsilon \in [0, 1)$ and if $\delta_{\mathrm{vol}}(K, T^2) \geq \varepsilon$, then $\mathrm{ir}(K) \leq (1 - \gamma\varepsilon)\ \mathrm{ir}(T^2)$, where we can choose $\gamma = 2^{-6}3^{-2}$. With a slightly larger constant γ, the same conclusion is contained in the second part of Theorem 1.2.

As mentioned before, the proof of the reverse isoperimetric inequality by K. M. Ball [1, 2] is based on a volume estimate for convex bodies whose John ellipsoid is the unit ball B^n. Let S^{n-1} denote the Euclidean unit sphere. According to a classical theorem of F. John [32] (see also K. M. Ball [3]), B^n is the ellipsoid of maximal volume inside a convex body K if and only if $B^n \subset K$ and there exist $u_1, \ldots, u_k \in S^{n-1} \cap \partial K$ and $c_1, \ldots, c_k > 0$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n, \tag{2}$$

$$\sum_{i=1}^{k} c_i u_i = 0, \tag{3}$$

where Id_n denotes the $n \times n$ identity matrix and ∂K is the boundary of K.

In the following, all measures are assumed to be Borel measures. Let us call a Borel measure μ on the unit sphere S^{n-1} isotropic if

$$\int_{S^{n-1}} u \otimes u \, d\mu(u) = \mathrm{Id}_n;$$

see [20], [21], [39]. In this case, equating traces of both sides we obtain that

$$\mu(S^{n-1}) = n. \tag{4}$$

(This is the reason why an isotropic measure as defined here is called a normalized isotropic measure in [42, pp. 595-596].) If, in addition, μ is centred, that is to say, if

$$\int_{S^{n-1}} u \, d\mu(u) = 0,$$

then the origin 0 is an interior point of the convex hull of the support $\operatorname{supp}\mu$ of μ, and hence

$$Z(\mu) := \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 1 \text{ for } u \in \operatorname{supp}\mu\}$$

is a convex body. It should be observed that $Z(\mu)$ only depends on the support of the measure μ. However, we only consider $Z(\mu)$ for centred and isotropic measures μ, which provides further information than just the support of μ.

The crucial statement leading to the reverse isoperimetric inequality is the following.

Theorem B. *If μ is a centred, isotropic measure on S^{n-1}, then*

$$V(Z(\mu)) \leq V(T^n). \tag{5}$$

Equality holds if and only if $Z(\mu)$ is a regular simplex circumscribed to B^n.

For a discrete measure μ, the inequality (5) is due to K. M. Ball [1, 2]. The equality case was settled by F. Barthe [5]. The case of an arbitrary centred, isotropic measure was treated by F. Barthe [6] and E. Lutwak, D. Yang, G. Zhang [40], where [40] also characterized the equality case. The measures on S^{n-1} which have an isotropic linear image are characterized by K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang [12], building on work of E. A. Carlen, and D. Cordero-Erausquin [15], J. Bennett, A. Carbery, M. Christ and T. Tao [10] and B. Klartag [35]. We note that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by R. Kannan, L. Lovász and M. Simonovits [33] as well as in the analysis of Bourgain's hyperplane conjecture (slicing problem); see, for instance, F. Barthe and D. Cordero-Erausquin [7], O. Guédon and E. Milman [30], B. Klartag [34] and B. Klartag and E. Milman [36], S. Brazitikos, A. Giannopoulos, P. Valettas and B.-H. Vritsiou [14].

To state a stability version of Theorem B, we use the "spherical" Hausdorff distance of compact sets $X, Y \subset S^{n-1}$, which is defined by

$$\delta_H(X,Y) := \max\left\{\max_{x\in X}\min_{y\in Y}\angle(x,y), \max_{y\in Y}\min_{x\in X}\angle(x,y)\right\},$$

where $\angle(x,y)$ denotes the geodesic distance of x, y on S^{n-1}. In addition, for $x \in S^{n-1}$, we write $\delta[x]$ to denote the Dirac measure on S^{n-1} supported on $\{x\}$, that is, if $A \subset S^{n-1}$ is a measurable set, then $\delta[x](A) = 1$ if $x \in A$ and zero otherwise. If S is a regular simplex circumscribed to B^n with contact points $v_0, \dots, v_n \in S^{n-1}$, then we set

$$\mu_S = \sum_{i=0}^{n} \frac{n}{n+1}\,\delta[v_i].$$

For the total mass of μ_S we obtain $\mu_S(S^{n-1}) = n$ as for μ in (4).

Theorem 1.5. *Let μ be a centred, isotropic measure on S^{n-1}, $n \geq 3$, and let $\varepsilon \in [0,1)$. If*

$$V(Z(\mu)) \geq (1-\varepsilon)V(T^n),$$

then there exists a regular simplex S circumscribed to B^n such that

$$\delta_H(\operatorname{supp}\mu, \operatorname{supp}\mu_S) \leq \gamma\varepsilon^{1/4},$$

where one may choose $\gamma = n^{70n}$.

Each of the corresponding $n+1$ spherical balls of radius $n^{70n}\varepsilon^{1/4}$ has μ-measure of order $\frac{n}{n+1} + O(\varepsilon^{1/4})$, and hence the Kantorovich-Monge-Rubinstein (or the Wasserstein distance) of μ from μ_S is $O(\varepsilon^{1/4})$ where the implied constant in $O(\cdot)$ depends only on n (see Section 10).

Again we obtain a result of optimal order for $n = 2$.

Theorem 1.6. *Let μ be a centred, isotropic measure on S^1. If*

$$V(Z(\mu)) \geq (1-\varepsilon)V(T^2)$$

for some $\varepsilon \in [0,1)$, then there exists a regular triangle S circumscribed to B^2 such that

$$\delta_H(\operatorname{supp}\mu, \operatorname{supp}\mu_S) \leq 32\varepsilon.$$

We note that the proof of Theorem B is based on the rank one case of the geometric Brascamp-Lieb inequality. While we do not actually use the Brascamp-Lieb inequality, an essential tool in our approach is the proof provided by F. Barthe [4], which is based on mass transportation. Therefore, it is instructive to review the argument from [4], which is done in Section 2. At the end of that section, we outline the arguments leading to Theorem 1.1, Theorem 1.3 and Theorem 1.5 and roughly describe the structure of the paper.

2. A brief review of the Brascamp-Lieb inequality

The rank one geometric Brascamp-Lieb inequality, identified by K. Ball [1] as an essential case of the rank one Brascamp-Lieb inequality, due to H. J. Brascamp, E. H. Lieb [13], reads as follows. If $u_1, \dots, u_k \in S^{n-1}$ are distinct unit vectors and $c_1, \dots, c_k > 0$ satisfy

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n,$$

and $f_1, \dots, f_k$ are non-negative measurable functions on $\mathbb{R}$, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i\rangle)^{c_i}\, dx \leq \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i\right)^{c_i}. \tag{6}$$

According to F. Barthe [5], if equality holds in (6) and none of the functions f_i is identically zero or a scaled version of a Gaussian, then $k = n$ and $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$. Conversely, equality holds in (6) if each f_i is a scaled version of the same centred Gaussian, or if $k = n$ and $u_1, \dots, u_n$ form an orthonormal basis.

A thorough discussion of the rank one Brascamp-Lieb inequality can be found in E. Carlen, D. Cordero-Erausquin [15]. The higher rank case, due to E. H. Lieb [37], is reproved and further explored by F. Barthe [5] (including a discussion of the equality case), and is again carefully analysed by J. Bennett, A. Carbery, M. Christ, T. Tao [10]. In particular, see F. Barthe, D. Cordero-Erausquin, M. Ledoux, B. Maurey [8] for an enlightening review of the relevant literature and an approach via Markov semigroups in a quite general framework.

F. Barthe [4,5] provides a concise proof of (6) based on mass transportation (see also K. M. Ball [3]). We sketch the main ideas of this approach, since this will be the starting point for subsequent refinements.

We assume that each of the functions f_i is a positive and continuous probability density. Let $g(t) = e^{-\pi t^2}$ be the Gaussian density. For $i = 1, \dots, k$, we consider the transportation map $T_i : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{-\infty}^{t} f_i(s)\, ds = \int_{-\infty}^{T_i(t)} g(s)\, ds.$$

It is easy to see that T_i is bijective, differentiable and

$$f_i(t) = g(T_i(t)) \cdot T_i'(t), \qquad t \in \mathbb{R}. \tag{7}$$

To these transportation maps, we associate the transformation $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\Theta(x) := \sum_{i=1}^{k} c_i T_i(\langle u_i, x\rangle)\, u_i, \qquad x \in \mathbb{R}^n,$$

which satisfies

$$d\Theta(x) = \sum_{i=1}^{k} c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i.$$

In this case, $d\Theta$ is positive definite and $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [4]). We will need the following two estimates due to K. M. Ball [1]:

(i) For any $t_1, \dots, t_k > 0$, we have

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i};$$

(see also Lemma 4.1 below).

(ii) If $z = \sum_{i=1}^{k} c_i \theta_i u_i$ for $\theta_1, \dots, \theta_k \in \mathbb{R}$, then

$$\|z\|^2 \leq \sum_{i=1}^{k} c_i \theta_i^2. \tag{8}$$

Therefore, using first (7), and then (i) and (ii), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle u_i, x\rangle)^{c_i}\, dx &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} g(T_i(\langle u_i, x\rangle))^{c_i}\right) \left(\prod_{i=1}^{k} T_i'(\langle u_i, x\rangle)^{c_i}\right) dx \\
&\leq \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} e^{-\pi c_i T_i(\langle u_i, x\rangle)^2}\right) \det\left(\sum_{i=1}^{k} c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i\right) dx \\
&\leq \int_{\mathbb{R}^n} e^{-\pi \|\Theta(x)\|^2} \det\left(d\Theta(x)\right) dx \\
&\leq \int_{\mathbb{R}^n} e^{-\pi \|y\|^2}\, dy = 1.
\end{aligned}$$

The stability version of (i) (with $v_i = \sqrt{c_i} u_i$), which is provided in Lemma 4.3, is an essential tool in proving a stability version of the Brascamp-Lieb inequality leading to Theorem 1.5.

Let us briefly discuss how K. M. Ball [1] used the Brascamp-Lieb inequality to prove the discrete version of Theorem B, since this type of argument is hidden in the proof of Proposition 7.1 which is crucial for our approach. Let μ be a discrete, centred, isotropic measure on S^{n-1} with $\operatorname{supp} \mu = \{u_1, \dots, u_k\} \subset S^{n-1}$ and $c_i = \mu(\{u_i\})$. We embed $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\}$ into $\mathbb{R}^{n+1}$ and write e_{n+1} for the unit vector in $\mathbb{R}^{n+1}$ orthogonal to $\mathbb{R}^n$. Then

$$\tilde{u}_i := -\sqrt{\frac{n}{n+1}}\, u_i + \sqrt{\frac{1}{n+1}}\, e_{n+1} \in S^n \quad \text{for } i = 1, \dots, k,$$

satisfies

$$\sum_{i=1}^{k} \tilde{c}_i \tilde{u}_i \otimes \tilde{u}_i = \mathrm{Id}_{n+1}, \qquad \text{where } \tilde{c}_i := \tfrac{n+1}{n}\, c_i \text{ for } i = 1, \dots, k.$$

Now the Brascamp-Lieb inequality is applied to the sequences $\tilde{u}_1, \dots, \tilde{u}_k \in S^n$ and $\tilde{c}_1, \dots, \tilde{c}_k > 0$ and where each of the functions f_i is the exponential density, that is, $f_i(t) = e^{-t}$ if $t \geq 0$ and $f_i(t) = 0$ otherwise, for $i = 1, \dots, k$. For the open convex cone $C = \{y \in \mathbb{R}^{n+1} : \langle y, \tilde{u}_i \rangle > 0,\ i = 1, \dots, k\}$, the formulas (31) and (32) in Section 7 yield

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} f_i(\langle y, \tilde{u}_i \rangle)^{\tilde{c}_i}\, dy = \int_C \exp\left(-\sum_{i=1}^{k} \tilde{c}_i \langle y, \tilde{u}_i \rangle \right) dy = V(Z(\mu)) V(T^n)^{-1}.$$

Since the Brascamp-Lieb inequality implies that this expression is at most 1, we conclude Theorem B.

Equality in Theorem B leads to equality in the Brascamp-Lieb inequality, and hence $k = n + 1$ and $\tilde{u}_1, \dots, \tilde{u}_{n+1}$ form an orthonormal basis in $\mathbb{R}^{n+1}$. In turn, $u_1, \dots, u_{n+1}$ are the vertices of a regular simplex.

To obtain a stability version of Theorem B, we need a stability version of the Brascamp-Lieb inequality in the special case we use. For example, we strengthen (i) in Section 4, and estimate derivatives of the corresponding transportation map in Section 6. The estimates in Section 6 are very specific for our particular choice of the functions f_i, and no method is known to the authors that could lead to a stability version of the Brascamp-Lieb inequality (6) in general.

The overall structure of the paper is as follows. Sections 3, 4 and 5 provide various important analytic and geometric estimates concerning John's theorem, related to discrete, isotropic measures and geometric stability results for polytopes close to a regular simplex. In Section 6, we provide auxiliary estimates for the transportation map between the exponential and the Gaussian distribution. After these preparations, we establish in Section 7 the crucial statement, Proposition 7.1, on which Theorem 1.1, Theorem 1.3 and Theorem 1.5 are based. Then, Section 8 contains the proofs of Theorem 1.1 and Theorem 1.3. In Section 9, we derive Theoren 1.4, whose proof is independent of the remaining results. Then, we extend Proposition 7.1 to general centred, isotropic measures in Section 10, which proves Theorem 1.5. Finally, we establish Theorem 1.6 in Section 11 and Theorem 1.2 in Section 12.

3. Some consequences of John's condition

According to the classical theorem of F. John [32], if B^n is the ellipsoid of maximal volume inside a convex body K, then there exist $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ and $c_1, \dots, c_k > 0$ such that (2) and (3) are satisfied. Equating the traces on the two sides of (2) we obtain

$$\sum_{i=1}^{k} c_i = n. \tag{9}$$

In addition, we may assume that

$$n + 1 \leq k \leq n(n+3)/2,$$

where the lower bound on k follows from (2) and (3) and the upper bound on k is implied by the proof of John's theorem [32] (see also P. M. Gruber, F. E. Schuster [28]). We note that (2) is equivalent to

$$\sum_{i=1}^{k} c_i \langle x, u_i \rangle^2 = \|x\|^2 \quad \text{for all } x \in \mathbb{R}^n.$$

Applying this to $x = u_i$ shows that

$$c_i \leq 1 \quad \text{for } i = 1, \dots, k. \tag{10}$$

In this section, we discuss properties that only use (2). This can be written as

$$\sum_{i=1}^{k} v_i \otimes v_i = \mathrm{Id}_n \tag{11}$$

with $v_i := \sqrt{c_i}\, u_i$. We note that (11) is equivalent to

$$\sum_{i=1}^{k} \langle x, v_i \rangle^2 = \|x\|^2 \quad \text{for all } x \in \mathbb{R}^n. \tag{12}$$

Moreover, whenever (11) (or (12)) holds for some vectors $v_1, \dots, v_k \in \mathbb{R}^n$, it follows that $\langle v_i, v_i \rangle^2 \leq \|v_i\|^2$, and thus $\|v_i\| \leq 1$ for $i = 1, \dots, k$.

Given $v_1, \dots, v_k \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_k > 0$, we consider the $n \times k$ matrix

$$U := [\sqrt{\lambda_1}\, v_1, \dots, \sqrt{\lambda_k}\, v_k].$$

According to the Cauchy-Binet formula, we have

$$\det\left(\sum_{i=1}^{k} \lambda_i v_i \otimes v_i\right) = \det\left(UU^\top\right) = \sum_{1 \leq i_1 < \dots < i_n \leq k} \det[\sqrt{\lambda_{i_1}}\, v_{i_1}, \dots, \sqrt{\lambda_{i_n}}\, v_{i_n}]^2. \tag{13}$$

It has been pointed out by K. M. Ball that the special case $\lambda_1 = \dots = \lambda_k = 1$ yields the following estimate.

Lemma 3.1. *If $v_1, \dots, v_k \in \mathbb{R}^n$ satisfy $\sum_{i=1}^{k} v_i \otimes v_i = \mathrm{Id}_n$, then there exist $1 \leq i_1 < \dots < i_n \leq k$ such that*

$$\det[v_{i_1}, \dots, v_{i_n}]^2 \geq \binom{k}{n}^{-1}.$$

For non-zero vectors v and w, we write $\angle(v, w)$ to denote their angle, that is, the geodesic distance of the unit vectors $\|v\|^{-1} v$ and $\|w\|^{-1} w$ on the unit sphere.

Lemma 3.2. *Let $v_1, \dots, v_k \in \mathbb{R}^n \setminus \{0\}$ satisfy $\sum_{i=1}^{k} v_i \otimes v_i = \mathrm{Id}_n$, and let $0 < \eta < 1/(3\sqrt{k})$. Assume for any $i \in \{1, \dots, k\}$ that $\|v_i\| \leq \eta$ or there is some $j \in \{1, \dots, n\}$ with $\angle(v_i, v_j) \leq \eta$. Then there exists an orthonormal basis $w_1, \dots, w_n$ such that $\angle(v_i, w_i) < 3\sqrt{k}\,\eta$ for $i = 1, \dots, n$.*

Proof. For $i = 1, \dots, n$, let $u_i = v_i / \|v_i\|$. We partition the index set $\{1, \dots, k\}$ into sets $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_n$ such that $i \in \mathcal{V}_i$ for $i = 1, \dots, n$, and in such a way that if $j \in \mathcal{V}_0$, then $\|v_j\| \leq \eta$, and if $j \in \mathcal{V}_i$ for some $i \in \{1, \dots, n\}$, then $\angle(v_i, v_j) \leq \eta$. Observe that $\mathcal{V}_0$ is possibly empty. For $i = 1, \dots, n$, (12) yields

$$1 = \|u_i\|^2 \geq \sum_{j \in \mathcal{V}_i} \langle u_i, v_j \rangle^2 \geq \sum_{j \in \mathcal{V}_i} \|v_j\|^2 \cos^2 \eta,$$

and hence

$$\sum_{j\in\mathcal{V}_i} \|v_j\|^2 \le (\cos\eta)^{-2}. \tag{14}$$

For $i = 1,\dots,n$, let $\tilde{w}_i \in S^{n-1}$ be orthogonal to v_j, $j \in \{1,\dots,n\}\setminus\{i\}$, and satisfy $\langle \tilde{w}_i, v_i\rangle \ge 0$. In addition, let $\alpha_i \le \pi/2$ be the minimal angle of $\tilde{w}_i$ and any v_j with $j \in \mathcal{V}_i$, and hence

$$\angle(\tilde{w}_i, v_i) \le \alpha_i + \eta. \tag{15}$$

To bound α_i from above, for $i = 1,\dots,n$, we observe that $|\langle \tilde{w}_i, v_j\rangle| \le \eta$ if $j \in \mathcal{V}_0$. Moreover, if $j \in \mathcal{V}_i$, then $\langle \tilde{w}_i, v_j\rangle \le \|v_j\|\cos\alpha_i$, and if $j \in \mathcal{V}_l$ for some $l \in \{1,\dots,n\}\setminus\{i\}$, then $\angle(\tilde{w}_i, v_j) \ge (\pi/2) - \eta$ and therefore $\langle \tilde{w}_i, v_j\rangle \le \|v_j\|\sin\eta$. Using these facts and (14), we deduce

$$\sum_{j\in\mathcal{V}_0} \langle \tilde{w}_i, v_j\rangle^2 \le (k-n)\eta^2 \le \frac{(k-n)\sin^2\eta}{\cos^2\eta},$$

$$\sum_{j\in\mathcal{V}_l} \langle \tilde{w}_i, v_j\rangle^2 \le \sin^2\eta \sum_{j\in\mathcal{V}_l} \|v_j\|^2 \le \frac{\sin^2\eta}{\cos^2\eta}, \qquad \text{for } l \in \{1,\dots,n\}\setminus\{i\},$$

$$\sum_{j\in\mathcal{V}_i} \langle \tilde{w}_i, v_j\rangle^2 \le \cos^2\alpha_i \sum_{j\in\mathcal{V}_i} \|v_j\|^2 \le \frac{\cos^2\alpha_i}{\cos^2\eta},$$

where the sum for $\mathcal{V}_0$ is set to be zero if $\mathcal{V}_0$ is empty. We conclude by (12) that

$$1 = \|\tilde{w}_i\|^2 \le \frac{(k-n)\sin^2\eta}{\cos^2\eta} + \frac{(n-1)\sin^2\eta}{\cos^2\eta} + \frac{\cos^2\alpha_i}{\cos^2\eta},$$

and hence

$$\sin^2\alpha_i = 1 - \cos^2\alpha_i \le 1 - \cos^2\eta + (k-1)\sin^2\eta = k\sin^2\eta.$$

Moreover, for $\eta < 1/(3\sqrt{k})$, we have

$$\frac{\sin(2\sqrt{k}\,\eta)}{\sqrt{k}\sin(\eta)} \ge \frac{\sin(2\sqrt{k}\,\eta)}{\sqrt{k}\,\eta} \ge 2\frac{\sin(2/3)}{2/3} \ge 1.$$

Therefore, (15) and $\eta < 1/(3\sqrt{k})$ yield

$$\angle(\tilde{w}_i, v_i) \le \alpha_i + \eta \le 2\sqrt{k}\,\eta + \eta < 3\sqrt{k}\,\eta, \qquad i = 1,\dots,n.$$

Recalling the definition of $\tilde{w}_1,\dots,\tilde{w}_n$, this shows that $v_1,\dots,v_n$ are linearly independent.

We define $w_1 = u_1$, and for $i = 2,\dots,n$ we let w_i be the unit vector in $\operatorname{lin}\{v_1,\dots,v_i\}$ which is orthogonal to $v_1,\dots,v_{i-1}$ and satisfies $\langle w_i, v_i\rangle > 0$. Writing L_i for the orthogonal complement of $\operatorname{lin}\{v_1,\dots,v_{i-1}\}$, we have $\tilde{w}_i \in L_i$. Since w_i is parallel to the orthogonal projection of v_i to L_i, we conclude that $\angle(w_i, v_i) \le \angle(\tilde{w}_i, v_i) < 3\sqrt{k}\,\eta$. □

4. Analytic stability estimates

To calculate the optimal constant in the Brascamp-Lieb inequality (6), the following statement has been proved by K. M. Ball [1]; see F. Barthe [5, Proposition 9] for a simple argument.

Lemma 4.1 (K. M. Ball)**.** *If* $v_1, \dots, v_k \in \mathbb{R}^n$ *satisfy* $\sum_{i=1}^k v_i \otimes v_i = \mathrm{Id}_n$ *and if* $t_1, \dots, t_k > 0$*, then*

$$\det\left(\sum_{i=1}^k t_i v_i \otimes v_i\right) \geq \prod_{i=1}^k t_i^{\langle v_i, v_i\rangle}.$$

Remark. E. Lutwak, D. Yang, G. Zhang [39] generalized Lemma 4.1 for any isotropic measure μ on S^{n-1} and for any positive continuous function t on $\operatorname{supp}\mu$ in the form

$$\det\left(\int_{S^{n-1}} t(u)\ u \otimes u\, d\mu(u)\right) \geq \exp\left(\int_{S^{n-1}} \ln t(u)\, d\mu(u)\right),$$

where equality holds if and only if the quantity $t(v_1)\cdots t(v_n)$ is constant for linearly independent $v_1, \dots, v_n \in \operatorname{supp}\mu$. Actually Lemma 4.1 is the case when $\operatorname{supp}\mu = \{u_1, \dots, u_k\}$, and $v_i = \sqrt{c_i}\, u_i$ for $c_i = \mu(\{u_i\})$. We do not need this generalized version in the present paper.

In Lemma 4.3, we prove a (stronger) stability version of Lemma 4.1 by replacing the arithmetic-geometric mean inequality with the following stability version in the argument of [5].

Lemma 4.2. *If* ν *is a probability measure and* f *is a measurable function which is bounded from above and from below by positive constants, then*

$$\frac{\int f\, d\nu}{\exp\left\{\int \ln f\, d\nu\right\}} \geq 1 + \frac{1}{2} \int \left(\frac{\sqrt{f}}{\sqrt{\int f\, d\nu}} - 1\right)^2 d\nu.$$

Proof. We note that for $a, b \geq 0$, we have

$$\frac{a+b}{2} - \sqrt{a}\,\sqrt{b} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2. \tag{16}$$

Here we choose $b = 1$ and

$$a = \frac{f}{\int f\, d\nu}.$$

Integrating (16) with this choice of a, b against ν, we get

$$1 - \frac{\int \sqrt{f}\, d\nu}{\sqrt{\int f\, d\nu}} = \frac{1}{2} \int \left(\frac{\sqrt{f}}{\sqrt{\int f\, d\nu}} - 1\right)^2 d\nu.$$

Since $1 - x \geq 1 - \sqrt{x}$ for $x \in [0,1]$, we obtain

$$1 - \frac{\left(\int \sqrt{f}\, d\nu\right)^2}{\int f\, d\nu} \geq \frac{1}{2} \int \left(\frac{\sqrt{f}}{\sqrt{\int f\, d\nu}} - 1\right)^2 d\nu.$$

Jensen's inequality yields

$$\left(\int \sqrt{f}\, d\nu\right)^2 \geq \exp\left\{\int \ln f\, d\nu\right\},$$

and hence we conclude Lemma 4.2 by observing that $(d/c) - 1 \geq 1 - (c/d)$ for any $c, d > 0$. $\square$

Lemma 4.3. *Let $k \geq n+1$, $t_1, \dots, t_k > 0$, and let $v_1, \dots, v_k \in \mathbb{R}^n$ be vectors which satisfy $\sum_{i=1}^k v_i \otimes v_i = \mathrm{Id}_n$. Then*

$$\det\left(\sum_{i=1}^k t_i v_i \otimes v_i\right) \geq \theta^* \prod_{i=1}^k t_i^{\langle v_i, v_i\rangle}$$

where

$$\theta^* = 1 + \frac{1}{2} \sum_{1 \leq i_1 < \dots < i_n \leq k} \det[v_{i_1}, \dots, v_{i_n}]^2 \left(\frac{\sqrt{t_{i_1} \cdots t_{i_n}}}{t_0} - 1\right)^2,$$

$$t_0 = \sqrt{\sum_{1 \leq i_1 < \dots < i_n \leq k} t_{i_1} \cdots t_{i_n} \det[v_{i_1}, \dots, v_{i_n}]^2}.$$

Proof. In this argument, I always denotes some subset of $\{1, \dots, k\}$ of cardinality n. For $I = \{i_1, \dots, i_n\}$, we define

$$d_I := \det[v_{i_1}, \dots, v_{i_n}]^2 \qquad \text{and} \qquad t_I := t_{i_1} \cdots t_{i_n}.$$

From $\sum_{i=1}^k v_i \otimes v_i = \mathrm{Id}_n$ and (13) we obtain

$$\sum_I d_I = 1 \qquad \text{and} \qquad \det\left(\sum_{i=1}^k t_i v_i \otimes v_i\right) = \sum_I t_I d_I,$$

where the summations extend over all sets $I \subset \{1, \dots, k\}$ of cardinality n. It follows that the discrete measure μ on the n element subsets of $\{1, \dots, k\}$ defined by $\mu(\{I\}) = d_I$ is a probability measure. According to Lemma 4.2, writing $t_0 = \sqrt{\sum_I t_I d_I}$, we deduce that

$$\det\left(\sum_{i=1}^k t_i v_i \otimes v_i\right) = \sum_I t_I d_I \geq \left(1 + \frac{1}{2} \sum_I d_I \left(\frac{\sqrt{t_I}}{t_0} - 1\right)^2\right) \prod_I t_I^{d_I}. \tag{17}$$

The factor t_i is used in $\prod_I t_I^{d_I}$ exactly $\sum_{I,\, i \in I} d_I$ times. Moreover, (13) applied to the vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$ implies

$$\begin{aligned}\sum_{I,\, i \in I} d_I = \sum_I d_I - \sum_{I,\, i \notin I} d_I &= 1 - \det\left(\sum_{j \neq i} v_j \otimes v_j\right) \\ &= 1 - \det\left(\mathrm{Id}_n - v_i \otimes v_i\right) = \langle v_i, v_i \rangle.\end{aligned}$$

Substituting this into (17) yields the lemma. □

Proof. To estimate from below (in the proof of Lemma 7.2) the factor θ^* in Lemma 4.3, we use the following observation. □

Lemma 4.4. *If $a, b, x > 0$, then*

$$(xa - 1)^2 + (xb - 1)^2 \geq \frac{(a^2 - b^2)^2}{2(a^2 + b^2)^2}.$$

Proof. Differentiating $f(x) = (xa - 1)^2 + (xb - 1)^2$ for fixed a, b with respect to x shows that f attains its minimum at $x = \frac{a+b}{a^2+b^2}$. Thus

$$(xa - 1)^2 + (xb - 1)^2 \geq \frac{(a-b)^2}{a^2 + b^2} = \frac{(a^2 - b^2)^2}{(a^2 + b^2)(a+b)^2} \geq \frac{(a^2 - b^2)^2}{2(a^2 + b^2)^2}. \qquad \square$$

5. Polytopes close to a regular simplex

We prove two quantitative statements about the approximation of a polytope by a simplex. First, we provide a lemma which will allow us to put a given orthonormal basis into a more convenient position by a small rotation.

Lemma 5.1. *Let $e \in S^{n-1}$, and let $\tau \in (0, 1/(2n))$. If $w_1, \dots, w_n$ is an orthonormal basis of $\mathbb{R}^n$ such that*

$$\frac{1}{\sqrt{n}} - \tau < \langle e, w_i \rangle < \frac{1}{\sqrt{n}} + \tau \qquad \text{for } i = 1, \dots, n,$$

then there exists an orthonormal basis $\tilde{w}_1, \dots, \tilde{w}_n$ such that $\langle e, \tilde{w}_i \rangle = \frac{1}{\sqrt{n}}$ and $\angle(w_i, \tilde{w}_i) < n\tau$ for $i = 1, \dots, n$.

Proof. For $i = 1, \dots, n$, let

$$\langle e, w_i \rangle = \frac{1}{\sqrt{n}} + \alpha_i, \quad \text{and hence } |\alpha_i| < \tau.$$

It follows that

$$1 = \|e\|^2 = \sum_{i=1}^{n} \left(\frac{1}{\sqrt{n}} + \alpha_i \right)^2 < 1 + \frac{2}{\sqrt{n}} \left(\sum_{i=1}^{n} \alpha_i \right) + n\tau^2,$$

which in turn yields that

$$\left\langle e, \sum_{i=1}^{n} w_i \right\rangle = \sqrt{n} + \sum_{i=1}^{n} \alpha_i > \sqrt{n} - \frac{n\sqrt{n}}{2} \tau^2 \geq \left\| \sum_{i=1}^{n} w_i \right\| \cos(n\tau),$$

since $\cos(n\tau) \leq 1 - \frac{1}{2} n\tau^2$ for $\tau \in (0, 1/(2n))$ and $n \geq 2$. In particular, we conclude that $\angle(e, \sum_{i=1}^{n} w_i) < n\tau$. We define $\tilde{w}_i = \Phi(w_i)$ for $i = 1, \dots, n$, where Φ is the orthogonal transformation, which rotates $\sum_{i=1}^{n} w_i$ into $\sqrt{n}\, e$ via their acute angle in the two-dimensional linear subspace L containing them, and keeps all vectors in $L^\perp$ fixed. Then $\langle e, \tilde{w}_i \rangle = \langle \Phi^{-1}(e), w_i \rangle = \sqrt{n}^{-1} \langle \sum_{j=1}^{n} w_j, w_i \rangle = 1/\sqrt{n}$ for $i = 1, \dots, n$. □

For convex bodies containing the origin in their interiors, we introduce a very specific scaling and rotation invariant distance from regular simplices whose centroid is the origin. If K is a convex body with $0 \in \operatorname{int} K$, then we define

$$d(K) := \ln \min\{\lambda \geq 1 : \ sT^n \subset \Phi K \subset \lambda s T^n \quad \text{for } s > 0 \text{ and } \Phi \in \mathrm{O}(n)\}.$$

Clearly, $d(K) = 0$ if and only if K is a regular simplex with centroid at the origin.

In the proof of the next lemma, for a set $M \subset \mathbb{R}^n$ we denote by $M^\circ =$ the polar set (see [42, p. 32]).

Lemma 5.2. *Let Z be a polytope, and let S be a regular simplex circumscribed to B^n. Assume that the facets of Z and S touch B^n at $u_1, \dots, u_k$ and $w_1, \dots, w_{n+1}$, respectively. Fix $\eta \in (0, 1/(2n))$. If $\delta_H(\{u_1, \dots, u_k\}, \{w_1, \dots, w_{n+1}\}) \leq \eta$, then*

$$(1 - n\eta) S \subset Z \subset (1 + 2n\eta) S.$$

In particular, $d(Z) < 3n\eta$.

Proof. For $u \in S^{n-1}$, let $H^-(u) := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq 1\}$. Then

$$Z = \bigcap_{i=1}^{k} H^-(u_i) \qquad \text{and} \qquad S = \bigcap_{j=1}^{n+1} H^-(w_j)$$

are polytopes containing B^n. Since Z and S are bounded, we obtain $Z^\circ = \operatorname{conv}\{u_1, \dots, u_k\}$ and $S^\circ = \operatorname{conv}\{w_1, \dots, w_{n+1}\}$; see, e.g., [42, Lemma 2.4.5], where conv denotes the convex hull operator. By assumption, we have $Z^\circ \subset S^\circ + \eta B^n$ and $S^\circ \subset Z^\circ + \eta B^n$.

Since $n^{-1}B^n \subset S^\circ$, we get $n^{-1}B^n \subset Z^\circ + \eta B^n$ and therefore $(n^{-1} - \eta)B^n \subset Z^\circ$. Hence

$$S^\circ \subset Z^\circ + \frac{\eta}{n^{-1} - \eta} Z^\circ \subset (1 + 2n\eta) Z^\circ,$$

since $\eta < 1/(2n)$, which implies that $Z \subset (1 + 2n\eta)S$.

Moreover, from $B^n \subset nS^\circ$, we obtain

$$Z^\circ \subset S^\circ + \eta B^n \subset S^\circ + n\eta S^\circ = (1 + n\eta)S^\circ,$$

which yields $Z \supset (1 + n\eta)^{-1} S \supset (1 - n\eta)S$.

The remaining assertion follows from $\ln((1 + 2x)/(1 - x)) < 3x$ for $x \in (0, 1/2)$. □

Lemma 5.3. *Let Z be a polytope, and let S be a regular simplex circumscribed to B^n. Fix $\gamma = 2^{n+4} n^{2n+2}$ and $\eta \in (0, \gamma^{-1})$. Assume that the facets of Z and S touch B^n at $u_1, \dots, u_k$ and $w_1, \dots, w_{n+1}$, respectively. If $\angle(u_i, w_i) \le \eta$ for $i = 1, \dots, n+1$ and $\angle(u_k, w_i) \ge \gamma\eta$ for $i = 1, \dots, n+1$, then*

$$V(Z) \le \left(1 - \frac{\min_{i=1,\dots,n+1} \angle(u_k, w_i)}{2^{n+2} n^{2n}}\right) V(S).$$

Proof. Let $H^+ := \{x \in \mathbb{R}^n : \langle x, u_k \rangle \ge 1\}$, and let F_i be the facet of S touching B^n at w_i. We may assume that $\angle(u_k, w_1) \le \angle(u_k, w_i)$ for $i \ge 2$, and hence $\langle u_k, w_1 \rangle > 0$.

First, we estimate $V(S \cap H^+)$. Let z be the closest point of $H^+ \cap F_1$ to w_1. In particular, we have $\|z - w_1\| \le 1$, while F_1 contains the $(n-1)$-ball of radius $\sqrt{\frac{n+1}{n-1}} > 1 + \frac{1}{n}$ centred at w_1. Thus $F_1 \cap H^+$ contains a regular $(n-1)$-simplex of height $\frac{1}{n}$, and in turn a congruent copy of $\frac{1}{2n^2} F_1$. In addition, the distance of w_1 from any F_i, $i \ge 2$, is $1 + \frac{1}{n}$, thus the distance of z from F_i is at least

$$\frac{1/n}{\|z - w_1\| + (1/n)} \left(1 + \frac{1}{n}\right) > \frac{h}{2n^2},$$

where $h = n + 1$ is the height of S. We deduce that $H^+ \cap S$ contains a point whose distance from F_1 is at least $\frac{h}{2n^2} \sin \angle(u_k, w_1)$, and hence

$$V(S \cap H^+) \ge \left(\frac{1}{2n^2}\right)^{n-1} \frac{\angle(u_k, w_1)}{4n^2} V(S) = \frac{\angle(u_k, w_1)}{2^{n+1} n^{2n}} V(S).$$

Let Z_0 be the simplex whose facets touch B^n at $u_1, \dots, u_{n+1}$. Then

$$(1 - n\eta)S \subset Z_0 \subset (1 + 2n\eta)S$$

by Lemma 5.2. Since $S \cap H^+ \subset (Z_0 \cap H^+) \cup (S \setminus [(1 - n\eta)S])$, it follows that

$$\begin{aligned} V(Z_0 \cap H^+) &\ge V(S \cap H^+) - (V(S) - V((1 - n\eta)S)) \\ &\ge \frac{\angle(u_k, w_1)}{2^{n+1} n^{2n}} V(S) - n^2 \eta\, V(S). \end{aligned}$$

Since $(1+2n\eta)^n \le 1+3n^2\eta$ for $\eta < 1/\gamma$, we get

$$\begin{aligned} V(Z) &\le V(Z_0) - V(Z_0 \cap H^+) \\ &\le V((1+2n\eta)S) - \left(\frac{\angle(u_k,w_1)}{2^{n+1}n^{2n}} - n^2\eta\right) V(S) \\ &\le \left(1 + 4n^2\eta - \frac{\angle(u_k,w_1)}{2^{n+1}n^{2n}}\right) V(S) \\ &\le \left(1 - \frac{\angle(u_k,w_1)}{2^{n+2}n^{2n}}\right) V(S), \end{aligned}$$

which completes the proof. □

6. The transportation map

The argument of F. Barthe [4] uses the transportation map $\varphi : (0,\infty) \to \mathbb{R}$ between the exponential and the standard Gaussian density, and hence

$$1 - e^{-t} = \int_0^t e^{-s}\,ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\varphi(t)} e^{-s^2}\,ds. \tag{18}$$

Clearly, φ is strictly increasing and $\varphi(\ln 2) = 0$.

Lemma 6.1. *If $t \ge 4$, then $\sqrt{2} < \varphi(t) < \sqrt{t}$, $\frac{1}{3\sqrt{t}} < \varphi'(t) < 1$ and $\varphi''(t) < -\frac{1}{12t^{3/2}}$.*

Proof. The definition (18) of φ can be written in the form

$$e^{-t} = \frac{1}{\sqrt{\pi}} \int_{\varphi(t)}^{\infty} e^{-s^2}\,ds. \tag{19}$$

According to the Gordon-Mill inequality (or Mill's ratio, see R. D. Gordon [23], L. Dümbgen [16, (2)], or by a straightforward direct argument), if $z > 0$, then

$$\frac{e^{-z^2}}{2\sqrt{\pi}z} \cdot \frac{2z^2}{2z^2+1} < \frac{1}{\sqrt{\pi}} \int_z^{\infty} e^{-s^2}\,ds < \frac{e^{-z^2}}{2\sqrt{\pi}z}. \tag{20}$$

We deduce from the left-hand side of (20) that

$$e^{-4} < \frac{1}{\sqrt{\pi}} \int_{\sqrt{2}}^{\infty} e^{-s^2}\,ds,$$

which in turn implies $\varphi(4) > \sqrt{2}$ by (19). From (19) and the right-hand side of (20), we deduce that $\varphi(t) < \sqrt{t}$ for $t > 4$.

We turn to the estimation of derivatives. Differentiating (19), we get

$$e^{-t} = \frac{e^{-\varphi(t)^2}\varphi'(t)}{\sqrt{\pi}}, \qquad t > 0. \tag{21}$$

In particular, this shows that $\varphi'(t) > 0$ for $t > 0$. Equation (21) combined with the right-hand side of (20) leads to

$$2\varphi(t)\varphi'(t) < 1 \quad \text{for } t > \ln 2. \tag{22}$$

Taking the logarithm of (21), we deduce the formula

$$-t = -\ln\sqrt{\pi} - \varphi(t)^2 + \ln\varphi'(t), \qquad t > 0, \tag{23}$$

and differentiating this implies

$$\varphi''(t) = \varphi'(t)(2\varphi(t)\varphi'(t) - 1). \tag{24}$$

Therefore $\varphi''(t) < 0$ follows on the one hand from $\varphi'(t) > 0$, and on the other hand from $\varphi(t) \le 0$ if $t \le \ln 2$, and from (22) if $t > \ln 2$. Thus $\varphi'(t) < \varphi'(\ln 2) = \sqrt{\pi}/2 < 1$ by (21) for $t > \ln 2$.

We also estimate φ'' in terms of φ. To this end, we use an improved version of the right-hand side of the Gordon-Mill inequality (20) (see L. Dümbgen [16, (2)], or by a simple direct argument); namely

$$\frac{1}{\sqrt{\pi}} \int_z^\infty e^{-s^2}\, ds < \frac{e^{-z^2}}{2\sqrt{\pi} z} \cdot \frac{2z^2+2}{2z^2+3}, \qquad z > 0.$$

We deduce from this and the left-hand side of (20) that if $z \ge \sqrt{2}$, then

$$\frac{e^{-z^2}}{3\sqrt{\pi} z} < \frac{1}{\sqrt{\pi}} \int_z^\infty e^{-s^2}\, ds < \frac{e^{-z^2}}{2\sqrt{\pi} z} \left(1 - \frac{1}{4z^2}\right).$$

If $t > 4$, then $\varphi(t) > \sqrt{2}$, thus

$$\frac{1}{3\varphi(t)} < \varphi'(t) = \sqrt{\pi} e^{\varphi(t)^2 - t} < \frac{1}{2\varphi(t)} \left(1 - \frac{1}{4\varphi(t)^2}\right). \tag{25}$$

In particular, $\varphi'(t) > \frac{1}{3\sqrt{t}}$, and combining (22) and (25) yields

$$\varphi''(t) = \varphi'(t)(2\varphi(t)\varphi'(t) - 1) < -\frac{\varphi'(t)}{4\varphi(t)^2} < \frac{-1}{12\varphi(t)^3} \quad \text{for } t > 4, \tag{26}$$

which completes the argument. □

7. Circumscribed polytopes

F. Barthe [4] proves the Brascamp-Lieb inequality for functions in one variable in full generality. This section is based on K. M. Ball's [3] interpretation of F. Barthe's argument in the special case needed for the geometric application. Since our stability argument uses in an essential way that the Brascamp-Lieb inequality is required only for the exponential density function, we do not separate the statement of the Brascamp-Lieb inequality.

Proposition 7.1 is the main ingredient for the proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.5. We recall that if K is a convex body with $0 \in \operatorname{int} K$, then $d(K)$ is the minimal λ such that there exists a regular simplex S whose centroid is the origin and $S \subset K \subset e^\lambda S$.

In the following, we use the abbreviation $N := n(n+3)/2$. In this section, we consider the case $n \ge 3$, although (with slightly different constants) the proof extends also to the case $n = 2$. In the plane, however, we can argue in a different way to obtain results of optimal order. For this reason we defer the two-dimensional case to Section 11.

Proposition 7.1. *Let μ be a discrete, centred, isotropic measure on S^{n-1}. Let $n \ge 3$. Assume that the cardinality of* $\operatorname{supp}\mu$ *is at most $N+1$, and let $\tau \in (0, n^{-240n})$. If*

$$V(Z(\mu)) > (1-\tau)V(T^n),$$

then there exists a regular simplex S circumscribed to B^n such that

$$\delta_H(\operatorname{supp}\mu, \operatorname{supp}\mu_S) < n^{60n}\tau^{1/4} \quad \textit{and} \quad d(Z(\mu)) < n^{60n}\tau^{1/4}.$$

Before we prove Proposition 7.1, we first set up the corresponding notions following K. M. Ball [1], [2], and then prove the preparatory statement Lemma 7.2.

Let $\operatorname{supp}\mu = \{u_1, \dots, u_k\}$, and let $c_i = \mu(\{u_i\})$. Then $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n$, $\sum_{i=1}^k c_i u_i = 0$ and $k \le N+1$.

As before we embed $\mathbb{R}^n$ into $\mathbb{R}^n \times \{0\} = \mathbb{R}^{n+1}$ and write e_{n+1} for the unit vector in $\mathbb{R}^{n+1}$ orthogonal to $\mathbb{R}^n$. We define

$$\tilde{u}_i := -\sqrt{\frac{n}{n+1}}\, u_i + \sqrt{\frac{1}{n+1}}\, e_{n+1} \in S^n \quad \text{and} \quad \tilde{c}_i := \frac{n+1}{n}\, c_i \quad \text{for } i = 1, \dots, k,$$

and hence

$$\sum_{i=1}^k \tilde{c}_i\ \tilde{u}_i \otimes \tilde{u}_i = \mathrm{Id}_{n+1},$$

$$\sum_{i=1}^k \tilde{c}_i \tilde{u}_i = \sqrt{n+1}\, e_{n+1}, \tag{27}$$

$$\sum_{i=1}^k \tilde{c}_i = n+1. \tag{28}$$

We observe that if $Z(\mu)$ is a regular simplex circumscribed to B^n, then $k = n+1$ and $\tilde{u}_1, \dots, \tilde{u}_{n+1}$ are an orthonormal basis of $\mathbb{R}^{n+1}$.

Next we consider the open cone

$$C := \{y \in \mathbb{R}^{n+1} : \langle y, \tilde{u}_i \rangle > 0,\ i = 1, \dots, k\} \tag{29}$$

$$= \{x + r e_{n+1} \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ r > 0,\ \langle x, u_i \rangle < r/\sqrt{n},\ i = 1, \dots, k\} \tag{30}$$

and the map $\Theta : C \to \mathbb{R}^{n+1}$ defined by

$$\Theta(y) := \sum_{i=1}^k \tilde{c}_i\ \varphi(\langle y, \tilde{u}_i \rangle)\ \tilde{u}_i,$$

where $\langle y, \tilde{u}_i \rangle > 0$ by (29). In particular, the differential of Θ is

$$d\Theta(y) = \sum_{i=1}^k \tilde{c}_i\ \varphi'(\langle y, \tilde{u}_i \rangle)\ \tilde{u}_i \otimes \tilde{u}_i.$$

The map $d\Theta(y) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is positive definite since φ' is positive and

$$\langle z, d\Theta(y) z \rangle = \sum_{i=1}^k \tilde{c}_i\ \varphi'(\langle y, \tilde{u}_i \rangle)\ \langle z, \tilde{u}_i \rangle^2.$$

It follows that Θ is injective.

From (30) we conclude that the section $\{y \in C : \langle y, e_{n+1} \rangle = r\}$ of C for $r > 0$ is a translate of $\operatorname{int}((r/\sqrt{n}) Z(\mu))$. Therefore

$$\begin{aligned} \int_C e^{-\langle y, \sqrt{n+1}\, e_{n+1} \rangle}\, dy &= \int_0^\infty \int_{\frac{r}{\sqrt{n}} Z} e^{-\sqrt{n+1}\ r}\, dx\, dr \\ &= V(Z(\mu)) \int_0^\infty \left(\frac{r}{\sqrt{n}}\right)^n e^{-\sqrt{n+1}\ r} dr \\ &= V(Z(\mu)) V(T^n)^{-1}. \end{aligned} \tag{31}$$

By first applying (27), then (23), and finally (28), we deduce that

(32)
$$\int_C e^{-\langle y,\sqrt{n+1}\,e_{n+1}\rangle}\,dy = \int_C \exp\left(-\sum_{i=1}^k \tilde{c}_i\langle y,\tilde{u}_i\rangle\right)\,dy$$
$$= \int_C \exp\left(\sum_{i=1}^k \tilde{c}_i(-\ln\sqrt{\pi}-\varphi(\langle y,\tilde{u}_i\rangle)^2+\ln\varphi'(\langle y,\tilde{u}_i\rangle)\right)\,dy$$
$$= \pi^{-\frac{n+1}{2}}\int_C \exp\left(-\sum_{i=1}^k \tilde{c}_i\varphi(\langle y,\tilde{u}_i\rangle)^2\right)\prod_{i=1}^k \varphi'(\langle y,\tilde{u}_i\rangle)^{\tilde{c}_i}\,dy. \tag{33}$$

For each fixed $y\in C$, we estimate the product of the two terms in (33) after the integral sign.

To estimate the first term in (33), we apply (8) with $\theta_i=\varphi(\langle y,\tilde{u}_i\rangle)$, and hence the definition of Θ yields

$$\exp\left(-\sum_{i=1}^k \tilde{c}_i\varphi(\langle y,\tilde{u}_i\rangle)^2\right)\le \exp\left(-\|\Theta(y)\|^2\right). \tag{34}$$

To estimate the second term, we apply Lemma 4.3 with $v_i=\sqrt{\tilde{c}_i}\;\tilde{u}_i$ and $t_i=\varphi'(\langle y,\tilde{u}_i\rangle)$, and write $\theta(y)$ and $t_0(y)$ to denote the corresponding $\theta^*\ge 1$ and t_0. In particular,

$$\theta(y)=1+\frac{1}{2}\sum_{1\le i_1<\ldots<i_{n+1}\le k}\tilde{c}_{i_1}\cdots\tilde{c}_{i_{n+1}}\det[\tilde{u}_{i_1},\ldots,\tilde{u}_{i_{n+1}}]^2$$
$$\times\left(\frac{\sqrt{\varphi'(\langle y,\tilde{u}_{i_1}\rangle)\cdots\varphi'(\langle y,\tilde{u}_{i_{n+1}}\rangle)}}{t_0(y)}-1\right)^2, \tag{35}$$

and Lemma 4.3 yields

$$\prod_{i=1}^k \varphi'(\langle y,\tilde{u}_i\rangle)^{\tilde{c}_i}\le \theta(y)^{-1}\det\left(d\Theta(y)\right). \tag{36}$$

We conclude that

$$V(Z(\mu))\le \frac{V(T^n)}{\pi^{\frac{n+1}{2}}}\int_C \theta(y)^{-1}e^{-\|\Theta(y)\|^2}\det\left(d\Theta(y)\right)\,dy \tag{37}$$
$$\le \frac{V(T^n)}{\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^{n+1}} e^{-\|z\|^2}\,dz=\frac{n^{n/2}(n+1)^{(n+1)/2}}{n!}=V(T^n). \tag{38}$$

According to Lemma 3.1, used for $v_i=\sqrt{\tilde{c}_i}\;\tilde{u}_i$, $i=1,\ldots,k$, we may assume that

$$\tilde{c}_1\cdots\tilde{c}_{n+1}\det[\tilde{u}_1,\ldots,\tilde{u}_{n+1}]^2\ge\binom{k}{n+1}^{-1}. \tag{39}$$

Then, in particular, the vectors $\tilde{u}_1,\ldots,\tilde{u}_{n+1}$ are linearly independent. Since each factor on the left-hand side of (39) is at most 1 (compare (10)), the product of the remaining factors is at least $\binom{k}{n+1}^{-1}$. For Lemma 7.2, we define

$$\varepsilon:=n^{60n}\tau^{1/4}<1 \qquad\text{and}\qquad \omega:=\frac{1}{3^5n^54^{n+1}n^{2n}}. \tag{40}$$

In the following lemma, we adopt the assumptions and the notation from above.

Lemma 7.2. *Let the assumptions of Proposition* 7.1 *be satisfied. If* $i \in \{1, \dots, k\}$, *then* $\tilde{c}_i \le \omega^2 \varepsilon^2$ *or* $\angle(\tilde{u}_i, \tilde{u}_j) \le \omega\varepsilon$ *for some* $j \in \{1, \dots, n+1\}$.

Proof. If $i \in \{1, \dots, n+1\}$, we can choose $j = i$ and then have $\angle(\tilde{u}_i, \tilde{u}_i) = 0$. Thus it remains to consider the cases where $i \in \{n+2, \dots, k\}$. For this, we proceed by contradiction and hence assume that there is some $i \in \{n+2, \dots, k\}$ such that $\tilde{c}_i > \omega^2\varepsilon^2$ and $\angle(\tilde{u}_i, \tilde{u}_j) > \omega\varepsilon$ for all $j \in \{1, \dots, n+1\}$. Under this assumption, we will identify a subset Ξ of C with reasonably large volume such that

$$\theta(y) \ge 1 + \gamma_0\, \varepsilon^4 \quad \text{for } y \in \Xi, \tag{41}$$

where $\gamma_0 := n^{-18n-78}$ depends on n (see (47)). From this we will then deduce a contradiction.

Since $\tilde{u}_1, \dots, \tilde{u}_{n+1}$ are linearly independent, there are uniquely determined $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{R}$ such that

$$\tilde{u}_i = \lambda_1 \tilde{u}_1 + \ldots + \lambda_{n+1} \tilde{u}_{n+1}. \tag{42}$$

We adjust the indices of $\tilde{u}_1, \dots, \tilde{u}_{n+1}$ so that

$$\lambda_1 \ge \ldots \ge \lambda_{n+1}.$$

Since $\langle \tilde{u}_j, e_{n+1} \rangle = 1/\sqrt{n+1}$ for $j = 1, \dots, k$, we have $\lambda_1 + \ldots + \lambda_{n+1} = 1$, and thus we obtain $\lambda_1 \ge \frac{1}{n+1}$. Combining $\tilde{c}_1 \le 1$, (39), and (42), we thus conclude that

$$\tilde{c}_2 \dots \tilde{c}_{n+1} \tilde{c}_i \det[\tilde{u}_2, \dots, \tilde{u}_{n+1}, \tilde{u}_i]^2 \ge \frac{\omega^2 \varepsilon^2}{(n+1)^2} \binom{k}{n+1}^{-1} > \omega_0 \varepsilon^2, \tag{43}$$

where we define $\omega_0 := n^{-10n-30}$. The inequality on the right-hand side is confirmed by an elementary calculation, which is based on $k \le N+1$ and $n! \ge \sqrt{2\pi n}\, (n/e)^n$.

Next we construct the set Ξ for which (41) is satisfied. The open convex cone

$$C_0 := \left\{ y \in \mathbb{R}^{n+1} : \langle y, e_{n+1} \rangle > \|y\| \, \frac{n}{\sqrt{n^2+1}} \right\}$$

satisfies $C_0 \subset C$. In fact, if $y = x + r e_{n+1} \in C_0$ with $x \in \mathbb{R}^n$ and $r > 0$, then

$$r > \sqrt{\|x\|^2 + r^2} \, \frac{n}{\sqrt{n^2+1}}.$$

But this is equivalent to $\|x\| < r/n$, which in turn implies that $\langle x, u_i \rangle < r/\sqrt{n}$ for $i = 1, \dots, k$, hence $y \in C$.

Writing α and β to denote the acute angles with $\cos\alpha = \langle \tilde{u}_j, e_{n+1} \rangle = \frac{1}{\sqrt{n+1}}$, $j = 1, \dots, k$, and $\cos\beta = \frac{n}{\sqrt{n^2+1}}$, we have $\alpha - \beta < \angle(y, \tilde{u}_j) < \alpha + \beta$ for $y \in C_0$ and $j = 1, \dots, k$. For $y \in C_0$ and $j = 1, \dots, k$, we deduce that

$$\cos\angle(y, \tilde{u}_j) < \cos(\alpha - \beta) = \frac{1}{\sqrt{n+1}} \frac{n}{\sqrt{n^2+1}} + \sqrt{\frac{n}{n+1}} \sqrt{\frac{1}{n^2+1}} < \frac{2}{\sqrt{n}}$$

and

$$\cos\angle(y, \tilde{u}_j) > \cos(\alpha + \beta) = \frac{n - \sqrt{n}}{\sqrt{n+1}\sqrt{n^2+1}} > \frac{1}{5\sqrt{n}}.$$

Hence, for $y \in C_0$ and $j = 1, \dots, k$, we have

$$\frac{1}{5\sqrt{n}} \|y\| < \langle y, \tilde{u}_j \rangle < \frac{2}{\sqrt{n}} \|y\|. \tag{44}$$

We also observe that the section $\{y \in C_0 : \langle y, e_{n+1}\rangle = t\}$ is an $(n-1)$-ball of radius t/n for $t > 0$. Now we are ready to define

$$\Xi := \left\{ y \in C_0 : 20\sqrt{n} < \langle y, e_{n+1}\rangle < 40\sqrt{n} \text{ and } \langle y, \tilde{u}_i - \tilde{u}_1\rangle > \frac{\omega\varepsilon}{\sqrt{n}} \right\}.$$

The assumption implies that $\|\tilde{u}_i - \tilde{u}_1\| > \omega\varepsilon/2$. Let $y \in \mathbb{R}^{n+1}$ be such that $\langle y, e_{n+1}\rangle = 20\sqrt{n}$ and $\langle y, (\tilde{u}_i - \tilde{u}_1)/\|\tilde{u}_i - \tilde{u}_1\|\rangle > 2/\sqrt{n}$. Then $\langle y, \tilde{u}_i - \tilde{u}_1\rangle > \omega\varepsilon/\sqrt{n}$. Hence $\{y \in C_0 : \langle y, e_{n+1}\rangle = 20\sqrt{n}\}$ contains an n-ball of diameter $18/\sqrt{n}$ and thus an n-simplex of circumradius $9/\sqrt{n}$. Therefore

$$V(\Xi) > \frac{1}{2} 20\sqrt{n} V\left(\frac{9}{\sqrt{n}} \frac{1}{n} T_n\right) \geq \frac{10\ n^2}{n^{3n/2}} V(T^n), \tag{45}$$

where $V(\Xi)$ denotes the $(n+1)$-dimensional volume of Ξ. Using (44), we also get

$$4 < \langle y, \tilde{u}_j\rangle < 120 \quad \text{for } y \in \Xi \text{ and } j = 1, \dots, k. \tag{46}$$

For $y \in \Xi$, we estimate $\theta(y)$ from below using the n-tuples $(1, \dots, n+1)$ and $(2, \dots, n+1, i)$ of indices in (35) (note that in addition to (43) we also have $\binom{k}{n+1}^{-1} \geq \omega_0$). We deduce by first applying (39), (43) and Lemma 4.4, secondly $\varphi'(\langle y, \tilde{u}_j\rangle) < 1$ for $j = 1, \dots, k$ (see Lemma 6.1), and thirdly by $\langle y, \tilde{u}_i - \tilde{u}_1\rangle > \frac{\omega\varepsilon}{\sqrt{n}}$ and $\varphi''(t) < -12^{-4}$ for $4 < t < 120$ (see Lemma 6.1) that

$$\begin{aligned}
\theta(y) &\geq 1 + \frac{1}{2} \frac{(\varphi'(\langle y, \tilde{u}_1\rangle) - \varphi'(\langle y, \tilde{u}_i\rangle))^2}{2(\varphi'(\langle y, \tilde{u}_1\rangle) + \varphi'(\langle y, \tilde{u}_i\rangle))^2}\ \omega_0 \varepsilon^2 \\
&> 1 + \frac{(\varphi'(\langle y, \tilde{u}_1\rangle) - \varphi'(\langle y, \tilde{u}_i\rangle))^2}{16}\ \omega_0 \varepsilon^2 \\
&> 1 + \frac{\omega^2 \omega_0}{16\ n\ 12^8}\ \varepsilon^4 > 1 + n^{-18n-78}\ \varepsilon^4.
\end{aligned} \tag{47}$$

According to (46) and Lemma 6.1, if $y \in \Xi$ and $j = 1, \dots, k$, then $\varphi(\langle y, \tilde{u}_j\rangle)^2 < 120$ and $\varphi'(\langle y, \tilde{u}_j\rangle) > \frac{1}{33}$. It follows from (34) and (36), taking into account (28), that

$$\begin{aligned}
e^{-\|\Theta(y)\|^2} \det(d\Theta(y)) &\geq \exp\left(-\sum_{j=1}^{k} \tilde{c}_j \varphi(\langle y, \tilde{u}_j\rangle)^2\right) \prod_{j=1}^{k} \varphi'(\langle y, \tilde{u}_j\rangle)^{\tilde{c}_j} \\
&\geq e^{-120(n+1)}\, 33^{-(n+1)} \geq e^{-124(n+1)} \geq e^{-186n}.
\end{aligned} \tag{48}$$

Recall that $\gamma_0 = n^{-18n-78}$ and observe that (47) implies that

$$1 - \theta(y)^{-1} \geq \frac{\gamma_0 \varepsilon^4}{1 + \gamma_0 \varepsilon^4} \geq \frac{1}{2} \gamma_0 \varepsilon^4. \tag{49}$$

Now we use (45), (48) and (49), and argue as for (37) and (38), to obtain

$$\begin{aligned}
V(Z(\mu)) &\leq \frac{V(T^n)}{\pi^{\frac{n+1}{2}}} \int_C e^{-\|\Theta(y)\|^2} \det(d\Theta(y))\ dy \\
&\quad - \frac{V(T^n)}{\pi^{\frac{n+1}{2}}} \int_C \left(1 - \theta(y)^{-1}\right) e^{-\|\Theta(y)\|^2} \det(d\Theta(y))\ dy \\
&\leq V(T^n) - \frac{V(T^n)}{\pi^{\frac{n+1}{2}}} \int_\Xi \left(1 - \theta(y)^{-1}\right) e^{-\|\Theta(y)\|^2} \det(d\Theta(y))\ dy \\
&\leq V(T^n) \left[1 - \frac{1}{\pi^{\frac{n+1}{2}}} \int_\Xi \tfrac{1}{2} \gamma_0 \varepsilon^4 e^{-186n}\, dy\right]
\end{aligned}$$

$$\begin{aligned}
&\le V(T^n)\left[1-\frac{V(\Xi)}{2\pi^{\frac{n+1}{2}}}\gamma_0\varepsilon^4 e^{-186n}\right]\\
&\le V(T^n)\left[1-\frac{5n^2V(T^n)}{n^{\frac{3n}{2}}\pi^{\frac{n+1}{2}}}\gamma_0\varepsilon^4 e^{-186n}\right]\\
&\le \left(1-n^{-240n}\varepsilon^4\right)V(T^n)=(1-\tau)V(T^n),
\end{aligned}$$

where we used (40) in the last step. This contradicts the assumptions of Proposition 7.1, and hence proves Lemma 7.2. □

Proof of Proposition 7.1. For $i=1,\dots,k$, we define $\tilde v_i := \sqrt{\tilde c_i}\tilde u_i \in \mathbb{R}^{n+1}$, hence $\|\tilde v_i\|=\sqrt{\tilde c_i}$. Lemma 7.2 ensures that the assumptions for the application of Lemma 3.2 are satisfied for $\tilde v_1,\dots,\tilde v_k$ in $\mathbb{R}^{n+1}$ with $\eta=\omega\varepsilon<1/(3\sqrt{k})$. Hence, by Lemma 3.2 there is an orthonormal basis $\bar w_1,\dots,\bar w_{n+1}$ of $\mathbb{R}^{n+1}$ such that $\angle(\tilde v_i,\bar w_i)<3\sqrt{k}\omega\varepsilon$ for $i=1,\dots,n+1$. Writing $\alpha_i=\angle(e_{n+1},\bar w_i)$ and $\beta_i=\angle(e_{n+1},\tilde v_i)=\angle(e_{n+1},\tilde u_i)$, we get

$$\left|\langle e_{n+1},\bar w_i\rangle-\frac{1}{\sqrt{n+1}}\right|=|\cos\alpha_i-\cos\beta_i|\le|\alpha_i-\beta_i|\le\angle(\bar w_i,\tilde v_i)<3\sqrt{k}\omega\varepsilon.$$

Since $3\sqrt{k}\omega\varepsilon<1/(2(n+1))$, we can apply Lemma 5.1, which yields the existence of an orthonormal basis $\tilde w_1,\dots,\tilde w_{n+1}$ in $\mathbb{R}^{n+1}$ such that $\langle e_{n+1},\tilde w_i\rangle=1/\sqrt{n+1}$ and $\angle(\tilde w_i,\bar w_i)\le(n+1)3\sqrt{k}\omega\varepsilon$. But then

$$\angle(\tilde w_i,\tilde u_i)\le\angle(\tilde w_i,\bar w_i)+\angle(\bar w_i,\tilde u_i)\le 3(n+1)\sqrt{k}\omega\varepsilon+3\sqrt{k}\omega\varepsilon\le 8n^2\omega\varepsilon.$$

For $i=1,\dots,n+1$, we define

$$w_i=\sqrt{\frac{n+1}{n}}\left(-\tilde w_i+\sqrt{\frac{1}{n+1}}\,e_{n+1}\right)\in\mathbb{R}^n,$$

and hence there exists a regular simplex S whose facets touch B^n at $w_1,\dots,w_{n+1}$. Subsequently, we use that

$$1-\frac{1}{2}t^2<\cos t<1-\frac{3}{8}t^2\quad\text{for } t\in(0,1).$$

Since

$$1-\langle w_i,u_i\rangle=\frac{n+1}{n}\left(1-\langle\tilde w_i,\tilde u_i\rangle\right)\le\frac{n+1}{n}\,\frac{1}{2}(8n^2\,\omega\varepsilon)^2\le 48n^4\omega^2\varepsilon^2,$$

we deduce that $\angle(w_i,u_i)<12n^2\omega\varepsilon$ for $i=1,\dots,n+1$.

We observe that $\gamma=2^{n+4}n^{2n+2}$ from Lemma 5.3 and $\omega=(3^5n^5 4^{n+1}n^{2n})^{-1}$ satisfy

$$\frac{1}{9n2^{n-2}}\le 12\gamma n^2\omega\le\frac{1}{9n},\tag{50}$$

and claim that

$$\delta_H(\operatorname{supp}\mu,\operatorname{supp}\mu_S)<12\gamma n^2\omega\varepsilon\le\frac{1}{9n}\varepsilon=\frac{1}{9n}n^{60n}\tau^{1/4}.\tag{51}$$

Let us suppose that contrary to (51), there exists some $i\in\{n+2,\dots,k\}$ such that $\angle(u_i,w_j)\ge 12\gamma n^2\omega\varepsilon$ for $j=1,\dots,n+1$. To apply Lemma 5.3, we note that $\varepsilon<1$ and (50) yield that $12n^2\omega\varepsilon<\gamma^{-1}$. Since $\varepsilon=n^{60n}\tau^{1/4}>n^{240n}\tau$, we conclude from (50) that

$$V(Z(\mu))\le\left(1-\frac{12\gamma n^2\omega\varepsilon}{2^{n+2}n^{2n}}\right)V(T^n)<(1-\tau)V(T^n).$$

This contradicts the condition on μ, and hence implies (51). Finally, combining (51) and Lemma 5.2 yields $d(Z(\mu)) < n^{60n}\tau^{1/4}$. $\square$

8. Proofs of Theorems 1.1 and 1.3

We assume that B^n is the ellipsoid of maximal volume inside the convex body K in $\mathbb{R}^n$, and hence there exist $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ and $c_1, \dots, c_k > 0$ such that $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n$ and $\sum_{i=1}^k c_i u_i = 0$, where

$$n+1 \le k \le n(n+3)/2. \tag{52}$$

We write Z to denote the circumscribed polytope whose facets touch B^n at $u_1, \dots, u_k$; namely,

$$Z = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \le 1,\ i = 1, \dots, k\}.$$

For any $x \in \partial K$, let u_x denote an exterior unit normal at x, which is unique (almost everywhere) and measurable with respect to the $(n-1)$-dimensional Hausdorff-measure on ∂K. We note that

$$V(K) = \int_{\partial K} \frac{\langle x, u_x \rangle}{n}\, dx \ge \frac{S(K)}{n}. \tag{53}$$

It follows from (53) and (37) (that is, Theorem B) that

$$\frac{S(K)^n}{V(K)^{n-1}} \le n^n V(K) \le n^n V(Z) \le n^n V(T^n) = \frac{S(T^n)^n}{V(T^n)^{n-1}}. \tag{54}$$

Lemma 8.1. *Let $\varepsilon \in (0,1)$.*

(i) *If $d(Z) \le \varepsilon/(4n^2)$ and $\delta_{\mathrm{BM}}(K, T^n) \ge \varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}} \le \left(1 - \frac{1}{e^2}\left(\frac{\varepsilon}{e}\right)^n\right) \frac{S(T^n)^n}{V(T^n)^{n-1}}.$$

(ii) *If $d(Z) \le \varepsilon/(4n^2)$ and $\delta_{\mathrm{vol}}(K, T^n) \ge \varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}} \le \left(1 - \frac{\varepsilon}{8}\right) \frac{S(T^n)^n}{V(T^n)^{n-1}}.$$

Proof. Let $\gamma := 1/(4n^2)$ so that $4n\gamma\varepsilon \le \frac{1}{n}$. Suppose that $d(Z) \le \varepsilon/(4n^2)$. Then we can assume that

$$e^{-\gamma\varepsilon}\, T^n \subset Z \subset e^{\gamma\varepsilon}\, T^n. \tag{55}$$

For the proof of (ii), we first choose $\lambda > 0$ such that $V(T^n) = V(\lambda Z)$. Then (55) yields that $e^{-\gamma\varepsilon} \le \lambda \le e^{\gamma\varepsilon}$. Therefore, again by (55) we obtain

$$\begin{aligned}
\delta_{\mathrm{vol}}(Z, T^n) &\le \frac{V((\lambda Z)\Delta T^n)}{V(T^n)} \le \lambda^n e^{n\gamma\varepsilon} - \lambda^n e^{-n\gamma\varepsilon} \\
&\le 2n\gamma\varepsilon\lambda^n e^{n\gamma\varepsilon} \le 2n\gamma\varepsilon e^{2n\gamma\varepsilon} \le 2n\gamma\varepsilon(1 + 4n\gamma\varepsilon) \\
&\le 4n\gamma\varepsilon \le \varepsilon/2,
\end{aligned}$$

where we used that $e^t \le 1 + 2t$ for $0 \le t \le 1/2$.

Let $\eta, \nu \ge 0$ satisfy $V(K) = V(\eta Z)$ and $V(Z) = (1+\nu)V(K)$, and hence $\eta = (1+\nu)^{-1/n}$. It follows from $\delta_{\mathrm{vol}}(K, T^n) \ge \varepsilon$, $\delta_{\mathrm{vol}}(Z, T^n) \le \varepsilon/2$ and $\eta \le 1$ that

$$\varepsilon/2 \le \delta_{\mathrm{vol}}(Z, K) \le \frac{V((\eta Z)\Delta K)}{V(K)} \le \frac{2V(Z \backslash K)}{V(K)} = 2\nu,$$

and hence (54) yields that

$$\frac{S(K)^n}{V(K)^{n-1}} \le n^n V(K) = n^n (1+\nu)^{-1} V(Z) \le (1+\nu)^{-1} \frac{S(T^n)^n}{V(T^n)^{n-1}}$$
$$\le \left(1+\frac{\varepsilon}{4}\right)^{-1} \frac{S(T^n)^n}{V(T^n)^{n-1}} \le \left(1-\frac{\varepsilon}{8}\right) \frac{S(T^n)^n}{V(T^n)^{n-1}}.$$

We turn to (i). It follows from $\delta_{\mathrm{BM}}(K, T^n) \ge \varepsilon$ and (55) that there is a vertex v of T^n such that

$$e^{\gamma\varepsilon-\varepsilon} v \notin \operatorname{int} K.$$

In particular, there exists a half-space H^+ containing $e^{\gamma\varepsilon-\varepsilon}v$, and disjoint from $\operatorname{int} K$. Since $p = e^{\gamma\varepsilon-\varepsilon}v$ is the centroid of the simplex $p + \lambda T^n \subset e^{-\gamma\varepsilon}T^n$ for $\lambda := e^{-\gamma\varepsilon} - e^{\gamma\varepsilon-\varepsilon}$, a result by B. Grünbaum [29, p. 1260, (iii)] yields that

$$V(H^+ \cap (p+\lambda T^n)) \ge \frac{1}{e} V(p+\lambda T^n) = \frac{\lambda^n}{e} V(T^n).$$

Therefore, as $p + \lambda T^n \subset e^{-\gamma\varepsilon}T^n \subset Z$ by (55) and with the preceding choice of λ, we deduce that

$$V(Z \setminus K) \ge V(H^+ \cap (p+\lambda T^n)) \ge \frac{\lambda^n}{e} V(T^n) = \frac{e^{-n\gamma\varepsilon}(1-e^{2\gamma\varepsilon-\varepsilon})^n}{e} V(T^n)$$
$$\ge \frac{1}{e^2}\left(\frac{\varepsilon}{e}\right)^n V(T^n).$$

Hence, by (54) we get

$$V(K) + \frac{1}{e^2}\left(\frac{\varepsilon}{e}\right)^n V(T^n) \le V(Z) \le V(T^n),$$

and therefore

$$V(K) \le \left(1 - \frac{1}{e^2}\left(\frac{\varepsilon}{e}\right)^n\right) V(T^n).$$

Now the proof can be completed as in the previous case by using once again (54). □

Proofs of Theorems 1.1 *and* 1.3. If $d(Z) > \varepsilon/(4n^2)$, then Proposition 7.1 can be applied by (52), and this implies that

$$V(Z) \le (1 - 4^{-4} n^{-248n} \varepsilon^4) V(T^n) \le (1 - n^{-250n}\varepsilon^4) V(T^n).$$

In turn, we conclude Theorem 1.3 and Theorem 1.1 by (54).

If $d(Z) \le \varepsilon/(4n^2)$, then Lemma 8.1 (i) yields Theorem 1.3, and Lemma 8.1 (ii) implies Theorem 1.1. □

For the sake of completeness we provide the following fact, which is mentioned in the introduction.

Lemma 8.2. *Let K, M be convex bodies in $\mathbb{R}^n$. Then $\delta_{\mathrm{vol}}(K, M) \le 2n^2 \delta_{\mathrm{BM}}(K, M)$ and $\delta_{\mathrm{BM}}(K, M) \le \gamma\ \delta_{\mathrm{vol}}(K, M)^{\frac{1}{n}}$, where γ is a constant which depends on n.*

Proof. The assertions follow from [11, Section 5]. Since the first assertion is used explicitly (in the introduction) and the definitions of the distances used here differ from those given in [11], we outline the short argument for the first inequality.

Since δ_{vol} and δ_{BM} are translation invariant in both arguments, we can assume that $0 \in K, M$ and $K \subset M \subset e^{\delta}K$, where $\delta := \delta_{\mathrm{BM}}$, and therefore $V(K) \leq V(M) \leq e^{n\delta}V(K)$ or

$$1 \leq \left(\frac{V(M)}{V(K)}\right)^{\frac{1}{n}} \leq e^{\delta}.$$

Thus we conclude that

$$e^{-\delta}K_0 \subset M_0 \subset e^{\delta}K_0,$$

where $K_0 := V(K)^{-\frac{1}{n}}K$ and $M_0 := V(M)^{-\frac{1}{n}}M$. But then

$$V(K_0 \Delta M_0) \leq V((e^{\delta}K_0) \setminus K_0) + V((e^{\delta}M_0) \setminus M_0) \leq 2\left(e^{\delta} - 1\right) \leq 2\delta e^{\delta}.$$

Now the assertion follows since $\delta_{\mathrm{BM}}(K, M) \leq \ln(n^2)$. The latter follows from the triangle inequality, since as a consequence of John's theorem, we have $\delta_{\mathrm{BM}}(K, B^n) \leq \ln(n)$ and $\delta_{\mathrm{BM}}(M, B^n) \leq \ln(n)$ (see, for instance, [42, p. 657], where (however) the multiplicative version of the (generalized) Banach–Mazur distance is used). □

9. Proof of Theorem 1.4

Throughout the proof, we have $n = 2$. The argument is based on [31], which we briefly recall. For a convex body K in $\mathbb{R}^n$ and $u \in S^{n-1}$, we write $H^-(K, u)$ for the supporting half-space of K which contains K and has exterior unit normal u, and $H(K, u)$ for its bounding hyperplane.

For the proof, we assume that

$$\mathrm{ir}(K) \geq (1 - \varepsilon)\mathrm{ir}(T^2). \tag{56}$$

Let $\mathrm{IR}(K) := S(K)^2/V(K)$ for a convex body K in $\mathbb{R}^2$. Then $\mathrm{ir}(T^2) = \mathrm{IR}(T^2)$. Let T_1 be a triangle of maximal area contained in K. We can assume that T_1 is a regular triangle centred at 0 with height 1, whose vertices are denoted by p_1, p_2, p_3. Let $u_1, u_2, u_3 \in S^1$ denote the exterior normal vectors of the edges of T_1. Then the lines $H(T_1, -u_i)$, $i = 1, 2, 3$, pass through the vertices of T_1 and bound a regular triangle T_2 of height 2 which contains K. Choose $q_i \in K \cap H(K, u_i)$ and let $x_i \in [0, 1]$ be the distance of q_i from $H(T_1, u_i)$ for $i = 1, 2, 3$. Then

$$\begin{aligned} T_1 \subset P_1 &:= \mathrm{conv}\{p_1, p_2, p_3, q_1, q_2, q_3\} \subset K \subset \bigcap_{i=1}^{3} H^-(K, u_i) \cap \bigcap_{i=1}^{3} H^-(K, -u_i) \\ &=: P_2 \subset T_2. \end{aligned}$$

Let $x := (x_1 + x_2 + x_3)/3 \in [0, 1]$. Elementary geometric arguments show (see [31]) that

$$S(P_2) = (1 + x)S(T_2) \qquad \text{and} \qquad V(P_1) = (1 + 3x)V(T_2),$$

and therefore

$$\mathrm{ir}(K) \leq \mathrm{IR}(K) \leq \frac{S(P_2)^2}{V(P_1)} \leq \left(1 - \frac{x(1 - x)}{1 + 3x}\right)\mathrm{ir}(T^2).$$

From (56) we conclude that $(1 + 3x)^{-1}x(1 - x) \leq \varepsilon$, and thus $x(1 - x) \leq 4\varepsilon$.

If $x \leq 1/2$, then $x \leq 8\varepsilon$ and thus $x_i \leq 24\varepsilon$ for $i = 1, 2, 3$. If $x \geq 1/2$, then in fact $x \geq 1 - 8\varepsilon$ and hence $x_i \geq 1 - 24\varepsilon$ for $i = 1, 2, 3$. In the first case, we conclude that

$$T_1 \subset K \subset P_2 \subset (1 + 72\varepsilon)T_1,$$

which implies

$$\delta_{\mathrm{BM}}(K, T^2) \leq \ln(1 + 72\varepsilon) \leq 72\varepsilon.$$

In the second case, we find a regular triangle T centred at 0 and homothetic to T_2 such that $T \subset K \subset T_2$ whose edges have distance at least $(2/3) - 24(2/3)\sqrt{3}\,\varepsilon$ from 0. This shows that

$$\delta_{\mathrm{BM}}(K, T^2) \leq \ln\left(\frac{1}{1 - 24\sqrt{3}\varepsilon}\right) \leq 72\varepsilon$$

for $\varepsilon \leq 1/72$. This completes the proof in both cases.

10. Isotropic measures: Proof of Theorem 1.5

Our proof of Theorem 1.5 will be based on Proposition 7.1. For this reason we have to ensure that we can switch from a centred, isotropic measure μ on S^{n-1} to a discrete, centred, isotropic measure on S^{n-1} with support contained in the support of μ and whose support has bounded cardinality. That this can indeed be achieved is shown by the following lemma.

Recall that $N = n(n+3)/2$.

Lemma 10.1. *Let μ be a centred, isotropic measure on S^{n-1}. Then there exists a discrete, centred, isotropic measure μ_0 on S^{n-1} such that* $\operatorname{supp} \mu_0 \subset \operatorname{supp} \mu$ *and the cardinality of* $\operatorname{supp} \mu_0$ *is at most $N+1$.*

Proof. We consider the map $F : \operatorname{supp} \mu \to \mathbb{R}^N$ given by $F(u) := (u \otimes u, u)$. Here we interpret $u \otimes u$ as the upper triangular part (including the main diagonal) of the symmetric matrix $u \otimes u$, and thus we identify the vectors $(u \otimes u, u)$ with vectors in $\mathbb{R}^N$. Since $\operatorname{supp} \mu \subset S^{n-1}$ is compact and F is continuous, the image set $F(\operatorname{supp} \mu) \subset \mathbb{R}^N$ is compact as well. Then also the convex hull of this image set, $\operatorname{conv}(F(\operatorname{supp} \mu)) \subset \mathbb{R}^N$ is compact. The probability measure $\bar{\mu} := \mu/n$ has the same support as μ and satisfies

$$\left(\int_{S^{n-1}} u \otimes u \, d\bar{\mu}(u), \int_{S^{n-1}} u \, d\bar{\mu}(u)\right) = \left(\frac{1}{n}\mathrm{Id}_n, 0\right) \in \mathbb{R}^N.$$

Let $\mathcal{D}_l$ be a decomposition of S^{n-1} into finitely many disjoint Borel sets of diameter at most $1/l$, $l \in \mathbb{N}$. We put $\mathcal{D}_l^* := \{\Delta \in \mathcal{D}_l : \Delta \cap \operatorname{supp} \bar{\mu} \neq \emptyset\}$. For $\Delta \in \mathcal{D}_l^*$, we fix some $v_\Delta \in \Delta \cap \operatorname{supp} \bar{\mu}$. Then

$$\bar{\mu}_l := \sum_{\Delta \in \mathcal{D}_l^*} \bar{\mu}(\Delta)\delta[v_\Delta]$$

is a discrete probability measure on S^{n-1} and $\operatorname{supp} \bar{\mu}_l \subset \operatorname{supp} \bar{\mu}$. Moreover, $\bar{\mu}_l \to \bar{\mu}$ in the weak topology as $l \to \infty$. Therefore, we conclude that

$$\sum_{\Delta \in \mathcal{D}_l^*} \bar{\mu}(\Delta)\,(v_\Delta \otimes v_\Delta, v_\Delta) = \left(\int_{S^{n-1}} v \otimes v \, d\bar{\mu}_l(v), \int_{S^{n-1}} v \, d\bar{\mu}_l(v)\right) \to \left(\frac{1}{n}\mathrm{Id}_n, 0\right)$$

in $\mathbb{R}^N$ as $l \to \infty$. This shows that

$$\left(\frac{1}{n}\mathrm{Id}_n, 0\right) \in \operatorname{cl} \operatorname{conv}(F(\operatorname{supp} \bar{\mu})) = \operatorname{conv}(F(\operatorname{supp} \bar{\mu})).$$

By Carathéodory's theorem (see, e.g., [42, Theorem 1.1.4]) there exist $k \leq N+1$ vectors $u_1, \ldots, u_k \in \operatorname{supp} \bar{\mu} \subset S^{n-1}$ such that

$$\left(\frac{1}{n}\mathrm{Id}_n, 0\right) \in \operatorname{conv}(F(\{u_1, \ldots, u_k\})),$$

that is, there exist $\alpha_1, \ldots, \alpha_k \geq 0$ with $\alpha_1 + \ldots + \alpha_k = 1$ such that

$$\left(\frac{1}{n}\mathrm{Id}_n, 0\right) = \sum_{i=1}^{k} \alpha_i F(u_i) = \sum_{i=1}^{k} \alpha_i (u_i \otimes u_i, u_i).$$

This shows that with $c_i := n\alpha_i$ for $i = 1, \ldots, k$ the measure

$$\mu_0 := \sum_{i=1}^{k} c_i \delta[u_i]$$

satisfies all requirements. □

For the proof of Theorem 1.5 we can assume that $\varepsilon \in (0, n^{-268n})$, since otherwise $n^{70n}\varepsilon^{\frac{1}{4}} \geq n^{3n}$ and the assertion is trivial. For the given measure μ there is a measure μ_0 as described in Lemma 10.1. Combined with the assumption of Theorem 1.5 this yields that

$$(1-\varepsilon)V(T^n) \leq V(Z(\mu)) \leq V(Z(\mu_0)).$$

Hence we can apply Proposition 7.1 and obtain a regular simplex S circumscribed to B^n with contact points $w_1, \ldots, w_{n+1}$ and such that

$$\delta_H(\text{supp } \mu_0, \text{supp } \mu_S) \leq n^{60n}\varepsilon^{\frac{1}{4}}. \tag{57}$$

If supp $\mu_0 =$ supp μ, the proof is finished. Hence, let $u^* \in \text{supp}(\mu) \setminus \text{supp}(\mu_0)$ and let Z^* be the polytope circumscribed to B^n with contact points $\text{supp}(\mu_0) \cup \{u^*\}$. Then we have

$$(1-\varepsilon)V(T^n) \leq V(Z(\mu)) \leq V(Z^*).$$

Let $\eta := n^{60n}\varepsilon^{\frac{1}{4}} < \gamma^{-1} = (2^{n+4}n^{2n+2})^{-1}$. From (57) we conclude that we can assume that supp $\mu_0 = \{u_1, \ldots, u_k\}$, $k \geq n+1$, with $\angle(u_i, w_i) \leq \eta$ for $i = 1, \ldots, n+1$. Assume that $\angle(u^*, w_i) \geq \gamma\eta$ for $i = 1, \ldots, n+1$. Then Lemma 5.3 implies that

$$(1-\varepsilon)V(T^n) \leq V(Z^*) \leq \left(1 - \frac{\gamma\eta}{2^{n+2}n^{2n}}\right) V(T^n),$$

and therefore $\gamma\eta \leq 2^{n+2}n^{2n}\varepsilon$, which contradicts $\varepsilon \leq 1$. This shows that $\angle(u^*, w_i) < \gamma\eta$ for some $i \in \{1, \ldots, n+1\}$. Since $\gamma\eta \leq n^{67n}\varepsilon^{\frac{1}{4}}$, it finally follows that $\delta_H(\text{supp } \mu, \text{supp } \mu_S) \leq n^{67n}\varepsilon^{\frac{1}{4}}$, which proves the theorem. □

Finally, we justify the remark following Theorem 1.5 by establishing the next lemma. For $w \in S^{n-1}$ and $\varepsilon \geq 0$, we consider $U(w, \varepsilon) := \{u \in S^{n-1} : \angle(u, w) \leq \varepsilon\}$, that is, the closed spherical (geodesic) ball with centre w and radius ε.

Lemma 10.2. *Let S be a regular simplex circumscribed to B^n with contact points $w_1, \ldots, w_{n+1} \in S^{n-1}$, let μ be a centred, isotropic Borel measure on S^{n-1}, and let $\varepsilon \in (0, 1/2)$. If $\delta_H(\text{supp } \mu, \text{supp } \mu_S) \leq \varepsilon$, then*

$$\left|\mu(U(w_i, \varepsilon)) - \frac{n}{n+1}\right| \leq 2n\varepsilon, \qquad i = 1, \ldots, n+1.$$

Proof. Let the map $G : S^{n-1} \to S^n$ be defined by

$$G(u) := -\sqrt{\frac{n}{n+1}}\, u + \sqrt{\frac{1}{n+1}}\, e_{n+1}.$$

Since μ is centred and isotropic, we obtain

$$\mathrm{Id}_{n+1} = \frac{n+1}{n} \int_{S^{n-1}} G(u) \otimes G(u)\, d\mu(u).$$

By assumption, $\operatorname{supp} \mu \subset \bigcup_{i=1}^{n+1} U(w_i, \varepsilon)$ and the union is disjoint. For $u \in U(w_i, \varepsilon)$ and $x \in S^n$, using the triangle and the Cauchy-Schwarz inequality as well as the fact that $G(u), G(w_i)$ and x are unit vectors, we get

$$\|\langle G(u), x\rangle G(u) - \langle G(w_i), x\rangle G(w_i)\| \le 2\|G(u) - G(w_i)\| \le 2\|u - w_i\| \le 2\varepsilon.$$

Hence, for any $x \in S^n$,

$$\begin{aligned}
&\left\| x - \frac{n+1}{n} \sum_{i=1}^{n+1} \mu(U(w_i, \varepsilon)) \langle G(w_i), x\rangle G(w_i) \right\| \\
&\quad = \frac{n+1}{n} \left\| \int_{S^{n-1}} \langle G(u), x\rangle G(u)\, d\mu(u) - \sum_{i=1}^{n+1} \mu(U(w_i, \varepsilon)) \langle G(w_i), x\rangle G(w_i) \right\| \\
&\quad \le \frac{n+1}{n} \sum_{i=1}^{n+1} \int_{U(w_i,\varepsilon)} \|\langle G(u), x\rangle G(u) - \langle G(w_i), x\rangle G(w_i)\|\, d\mu(u) \\
&\quad \le \frac{n+1}{n} 2\varepsilon \sum_{i=1}^{n+1} \mu(U(w_i, \varepsilon)) = 2(n+1)\varepsilon.
\end{aligned}$$

The special choice $x = G(w_i)$, for some $i \in \{1, \dots, n+1\}$, together with the fact that $G(w_1), \dots, G(w_{n+1})$ is an orthonormal basis of $\mathbb{R}^{n+1}$ then yields

$$|1 - ((n+1)/n)\mu(U(w_i, \varepsilon))| \le 2(n+1)\varepsilon,$$

from which the assertion follows. □

Let the assumptions of Lemma 10.2 be satisfied. Furthermore, let $f : S^{n-1} \to \mathbb{R}$ be Lipschitz with Lipschitz constant $\|f\|_L$. Here the definition of the Lipschitz constant is based on the geodesic distance on S^{n-1}. Since μ and μ_S have the same total measure n, we can replace f by $f - f(e_1)$ in the following estimation, and therefore we can assume that the sup norm $\|f\|_\infty$ of f satisfies $\|f\|_\infty \le 4\|f\|_L$. Thus, we get

$$\begin{aligned}
\left| \int_{S^{n-1}} f\, d\mu - \int_{S^{n-1}} f\, d\mu_S \right| &\le \sum_{i=1}^{n+1} \int_{U(w_i,\varepsilon)} |f - f(w_i)|\, d\mu + \sum_{i=1}^{n+1} |f(w_i)| 2n\varepsilon \\
&\le \|f\|_L \varepsilon n + \|f\|_\infty 2n(n+1)\varepsilon \\
&\le 13 n^2 \varepsilon \|f\|_L,
\end{aligned}$$

which yields the asserted bound for the Wasserstein distance $d_W(\mu, \mu_S)$.

11. Proof of Theorem 1.6

We state the next lemma in general dimensions although we will need it only in the plane. For the reader's convenience, we include a short proof.

Lemma 11.1. *Let μ be a centred and isotropic Borel measure on S^{n-1}. Let $v \in S^{n-1}$ be given. Then there is some $u^* \in \operatorname{supp} \mu$ such that $\langle u^*, v\rangle \ge 1/n$.*

Proof. We fix $v \in S^{n-1}$ and define $S_+ := \{u \in S^{n-1} : \langle u, v\rangle \geq 0\}$ and $S_- := S^{n-1} \setminus S_+$. Since μ is centred and $\langle u, v\rangle \geq -1$, we have

$$-\int_{S_+} \langle u, v\rangle \, d\mu(u) = \int_{S_-} \langle u, v\rangle \, d\mu(u) \geq -\mu(S_-),$$

and hence

$$\mu(S_-) \geq \int_{S_+} \langle u, v\rangle \, d\mu(u). \tag{58}$$

Choose $u^* \in \operatorname{supp} \mu$ such that $\langle u^*, v\rangle = \max\{\langle u, v\rangle : u \in \operatorname{supp} \mu\}$. The maximum exists as $\operatorname{supp} \mu$ is compact. It is also clear (since μ is centred) that $u^* \in S_+$. Then (58) implies

$$\int_{S_+} \langle u, v\rangle^2 \, d\mu(u) \leq \langle u^*, v\rangle \int_{S_+} \langle u, v\rangle \, d\mu(u) \leq \langle u^*, v\rangle \mu(S_-). \tag{59}$$

In addition, we have

$$\begin{aligned} \int_{S_-} \langle u, v\rangle^2 \, d\mu(u) &\leq \int_{S_-} |\langle u, v\rangle| \, d\mu(u) = -\int_{S_-} \langle u, v\rangle \, d\mu(u) = \int_{S_+} \langle u, v\rangle \, d\mu(u) \\ &\leq \langle u^*, v\rangle \mu(S_+). \end{aligned} \tag{60}$$

Using (59), (60), the isotropy of μ and $\mu(S^{n-1}) = n$, we conclude

$$\begin{aligned} 1 &= \int_{S^{n-1}} \langle u, v\rangle^2 \, d\mu(u) = \int_{S_+} \langle u, v\rangle^2 \, d\mu(u) + \int_{S_-} \langle u, v\rangle^2 \, d\mu(u) \\ &\leq \langle u^*, v\rangle \mu(S_-) + \langle u^*, v\rangle \mu(S_+) = \langle u^*, v\rangle \mu(S^{n-1}) = n\langle u^*, v\rangle, \end{aligned}$$

which yields the assertion. □

We say that a non-empty closed subset X of S^1 is proper, if for any $v \in S^1$, there exists some $u \in X$ such that $\langle v, u\rangle \geq \frac{1}{2}$. A closed set $X \subset S^1$ is proper if and only if the angle of two consecutive points of X is at most $2\pi/3$.

For a non-empty closed set $X \subset S^1$, let $d_0(X)$ be the minimum of $\delta_H(X, \sigma)$ where σ runs through the set of contact points of the regular triangles circumscribed to B^2. If X is proper, then clearly $d_0(X) \leq \pi/3$.

Lemma 11.2. *If $X \subset S^1$ is proper, and $d_0(X) \geq \eta$ for $\eta \in (0, \frac{\pi}{6}]$, then there exist $u, v \in X$ such that $\eta \leq \angle(u, v) \leq \frac{2\pi}{3} - \eta$.*

Proof. We prove the lemma by contradiction, thus we suppose that for any $u, v \in X$, we have

$$\text{either } \angle(u, v) < \eta \text{ or } \angle(u, v) > \tfrac{2\pi}{3} - \eta \geq \tfrac{\pi}{2} > 2\eta. \tag{61}$$

The set X has at least four elements since X is proper and $d_0(X) > 0$. Thus there exist $u_1', v_1' \in X$ such that $0 < \angle(u_1', v_1') \leq \frac{\pi}{2}$. We deduce from (61) that $\angle(u_1', v_1') < \eta$. According to (61), there exists $v_1 \in X$ such that $\angle(u_1', v_1)$ is maximal under the conditions $\angle(u_1', v_1) < \eta$ and $v_1' \in \operatorname{pos}\{u_1', v_1\}$. Similarly, there exists $u_1 \in X$ such that $\angle(u_1, v_1)$ is maximal under the conditions $\angle(u_1, v_1) < \eta$ and $u_1' \in \operatorname{pos}\{u_1, v_1\}$.

As X is proper, there exists $u_2 \in X$ such that $\operatorname{lin} v_1$ separates u_1 and u_2, and $\angle(u_2, v_1)$ is minimal under the conditions $\angle(u_2, v_1) \leq \frac{2\pi}{3}$ and that $\operatorname{lin} v_1$ separates u_1 and u_2. We actually have

$$\tfrac{\pi}{2} \leq \tfrac{2\pi}{3} - \eta < \angle(u_2, v_1) \leq \tfrac{2\pi}{3}, \tag{62}$$

since $\angle(u_2, v_1) < \eta$ would imply $\eta \le \angle(u_2, u_1) < 2\eta$, contradicting (61). In particular, we have $X \cap \operatorname{pos}\{u_2, v_1\} = \{u_2, v_1\}$. Similarly, there exists $v_3 \in X$ such that $\operatorname{lin} u_1$ separates v_1 and v_3, and

$$\tfrac{\pi}{2} \le \tfrac{2\pi}{3} - \eta < \angle(v_3, u_1) \le \tfrac{2\pi}{3}, \tag{63}$$

moreover $X \cap \operatorname{pos}\{v_3, u_1\} = \{v_3, u_1\}$. It also follows from (62) and (63) that u_2 and v_3 are not opposite, and the shorter arc of S^1 connecting them does not contain u_1 and v_1.

Finally, let $v_2 \in X \cap \operatorname{pos}\{u_2, v_3\}$ maximize $\angle(v_2, u_2)$ under the condition $\angle(v_2, u_2) < \eta$, and let $u_3 \in X \cap \operatorname{pos}\{u_2, v_3\}$ maximize $\angle(u_3, v_3)$ under the condition $\angle(u_3, v_3) < \eta$. Here possibly $v_2 = u_2$ or $u_3 = v_3$. If there were $w \in X \cap \operatorname{int} \operatorname{pos}\{v_2, u_3\}$, then $\angle(w, v_3) > \frac{\pi}{2}$ and $\angle(w, u_2) > \frac{\pi}{2}$ would follow from (61), which is absurd. Therefore $X \cap \operatorname{pos}\{u_3, v_2\} = \{u_3, v_2\}$, and

$$\tfrac{\pi}{2} \le \tfrac{2\pi}{3} - \eta < \angle(u_3, v_2) \le \tfrac{2\pi}{3}. \tag{64}$$

Now the arcs $S^1 \cap \operatorname{pos}\{u_1, v_2\}$, $S^1 \cap \operatorname{pos}\{u_2, v_3\}$ and $S^1 \cap \operatorname{pos}\{u_3, v_1\}$ cover S^1 by their constructions, thus

$$\angle(u_1, v_2) + \angle(u_2, v_3) + \angle(u_3, v_1) > 2\pi. \tag{65}$$

In particular, one of $\angle(u_1, v_2)$, $\angle(u_2, v_3)$ and $\angle(u_3, v_1)$ is larger than $\frac{2\pi}{3}$ by (65).

If $\angle(u_1, v_2) > \frac{2\pi}{3}$, then we define $p_3 \in S^1$ in such a way that $-p_3$ is the midpoint of the arc $S^1 \cap \operatorname{pos}\{u_1, v_2\}$. For $i = 1, 2$, let $p_i \in S^1$ satisfy $\angle(p_i, p_3) = \frac{2\pi}{3}$ in such a way that p_1 and p_2 lie on the same side of $\operatorname{lin} p_3$ where u_1 and v_2 lie, respectively. In particular, p_1, p_2 and p_3 are vertices of a regular triangle. We deduce using (63) and (64) that

$$p_1, p_2 \in \operatorname{pos}\{u_1, v_2\} \quad \text{and} \quad \angle(u_1, v_2) < \tfrac{2\pi}{3} + 2\eta. \tag{66}$$

For $i = 1, 2$, it follows from (66) that if $w \in S^1 \cap \operatorname{pos}\{u_i, v_i\}$, then $\angle(w, p_i) < \eta$. In addition, (63) and (64) yield that if $w \in S^1 \cap \operatorname{pos}\{u_3, v_3\}$, then $\angle(w, p_1) < \eta$, and hence $d_0(X) < \eta$, which is a contradiction. If $\angle(u_2, v_3) > \frac{2\pi}{3}$ or $\angle(u_3, v_1) > \frac{2\pi}{3}$ in (65), then similar arguments lead to a contradiction, which completes the proof of Lemma 11.2. □

In the following, we use the fact (T) that for $0 \le \beta \le \alpha \le 2\pi/3$ the function

$$F(t) = \tan\left(\frac{\alpha + t}{2}\right) + \tan\left(\frac{\beta - t}{2}\right) = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(t + \frac{\alpha-\beta}{2}\right)}$$

is increasing for $0 \le t \le \min\{\beta, \frac{2\pi}{3} - \alpha\}$.

After these preparations, we turn to the proof of Theorem 1.6.

Proof. It is sufficient to prove that if $\eta \in (0, \frac{\pi}{6}]$, and $d_0(\operatorname{supp} \mu) \ge \eta$, then

$$V(Z(\mu)) \le \left(1 - \frac{\eta}{8}\right) V(T^2). \tag{67}$$

Indeed, if $d_0(\operatorname{supp} \mu) > 32\varepsilon$, then $8\varepsilon < \pi/6$, since $d_0(\operatorname{supp} \mu) \le 2\pi/3$ by Lemma 11.1. But then the preceding claim can be applied with $\eta = 8\varepsilon$.

Now we turn to the proof of the claim. It follows from Lemma 11.2 that there exist $u_1, u_2 \in \operatorname{supp} \mu$ such that

$$\eta \le \angle(u_1, u_2) \le \tfrac{2\pi}{3} - \eta. \tag{68}$$

Since by Lemma 11.1 $\operatorname{supp}\mu$ is proper, there exist $u_3, \dots, u_k \in \operatorname{supp}\mu$, $k \geq 4$, such that $u_1, \dots, u_k$ (in this order) lie on S^1 and form a proper set. Then

$$V(Z(\mu)) \leq \sum_{i=1}^{k} \tan\left(\frac{\alpha_i}{2}\right),$$

where $\alpha_1 = \angle(u_1, u_2) \in [\eta, \frac{2\pi}{3} - \eta]$, $\alpha_i = \angle(u_i, u_{i+1})$ with $u_{k+1} := u_1$ and $0 \leq \alpha_i \leq 2\pi/3$. Applying repeatedly (T) to pairs of the angles $\alpha_2, \dots, \alpha_k$, it follows that

$$\begin{aligned}
\sum_{i=1}^{k} \tan\left(\frac{\alpha_i}{2}\right) &\leq \left(\tan\left(\frac{\alpha_1}{2}\right) + \tan\left(\frac{2\pi - \frac{4\pi}{3} - \alpha_1}{2}\right) + 2\tan\left(\frac{\pi}{3}\right)\right) \\
&= \left(\tan\left(\frac{\alpha_1}{2}\right) + \tan\left(\frac{\pi}{3} - \frac{\alpha_1}{2}\right) + 2\sqrt{3}\right) \\
&\leq \left(\tan\left(\frac{\eta}{2}\right) + \tan\left(\frac{\pi}{3} - \frac{\eta}{2}\right) + 2\sqrt{3}\right) \\
&\leq \left(\frac{\sqrt{3}}{\frac{1}{2} + \cos\left(\eta - \frac{\pi}{3}\right)} + 2\sqrt{3}\right) \leq \left(1 - \frac{\eta}{8}\right) 3\sqrt{3} \\
&= \left(1 - \frac{\eta}{8}\right) V(T^2),
\end{aligned}$$

which proves the assertion. □

12. Proof of Theorem 1.2

Let K be a convex body in $\mathbb{R}^2$ whose John ellipsoid is the Euclidean unit ball. As before (at the beginning of Section 8), the contact points of K and B^2 define a discrete, centred, isotropic measure μ and a polytope $Z = Z(\mu)$ which contains K.

If $V(Z) \geq (1-\varepsilon)V(T^2)$ with some $\varepsilon \in (0,1)$, then Theorem 1.6 implies the existence of a regular simplex S circumscribed to B^2 such that $\delta_H(\operatorname{supp}\mu, \operatorname{supp}\mu_S) \leq 32\,\varepsilon$. Choosing $\eta := 32\,\varepsilon < 1/4$, that is, with $\varepsilon < 1/(4 \cdot 32)$, we see from Lemma 5.2 that $d(Z) < 6 \cdot 32\,\varepsilon$. Hence, if $d(Z) \geq 6 \cdot 32\,\varepsilon$ and $\varepsilon < 1/(4 \cdot 32)$, then $V(Z) < (1-\varepsilon)V(T^2)$, and therefore $S(K)^2/V(K) \leq (1-\varepsilon)\mathrm{ir}(T^2)$. On the other hand, if $d(Z) < 6 \cdot 32\,\varepsilon$ and $\delta_{\mathrm{vol}}(K, T^2) \geq 16 \cdot 6 \cdot 32\,\varepsilon$, then Lemma 8.1 (ii) implies that

$$\frac{S(K)^2}{V(K)} \leq \left(1 - \frac{1}{8}\,16 \cdot 6 \cdot 32\,\varepsilon\right)\mathrm{ir}(T^2) = (1 - 12 \cdot 32\,\varepsilon)\mathrm{ir}(T^2),$$

provided that $16 \cdot 6 \cdot 32\,\varepsilon < 1$. This implies the assertion of the theorem. □

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