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TAME CIRCLE ACTIONS

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Abstract. In this paper, we consider Sjamaar's holomorphic slice theorem, the birational equivalence theorem of Guillemin and Sternberg, and a number of important standard constructions that work for Hamiltonian circle actions in both the symplectic category and the Kähler category: reduction, cutting, and blow-up. In each case, we show that the theory extends to Hamiltonian circle actions on complex manifolds with tamed symplectic forms. (At least, the theory extends if the fixed points are isolated.)

Our main motivation for this paper is that the first author needs the machinery that we develop here to construct a non-Hamiltonian symplectic circle action on a closed, connected six-dimensional symplectic manifold with exactly 32 fixed points; this answers an open question in symplectic geometry. However, we also believe that the setting we work in is intrinsically interesting and elucidates the key role played by the following fact: the moment image of $e^t \cdot x$ increases as $t \in \mathbb{R}$ increases.

1. Introduction

In this paper, we consider Sjamaar's holomorphic slice theorem [16], the birational equivalence theorem of Guillemin and Sternberg [9], and a number of important standard constructions that work for Hamiltonian circle actions in both the symplectic category and the Kähler category: reduction, cutting [13], and blow-up. In each case, we show that the theory extends to Hamiltonian circle actions on complex manifolds with tamed symplectic forms. (At least, the theory extends if the fixed points are isolated.)

Our main motivation for this paper is the following question, which appears in McDuff and Salamon [15] and is often referred to as the "McDuff conjecture": *Does there exist a non-Hamiltonian symplectic circle action with isolated fixed points on a closed, connected symplectic manifold?* The first author needs the machinery that we develop here to answer this question by constructing a non-Hamiltonian symplectic circle action on a closed, connected six-dimensional symplectic manifold with exactly 32 isolated fixed points in [17]. Propositions 3.1, 6.1, and 7.9 play a key role in "adding" the fixed points and analysing the resulting manifold in that paper.

Because of this motivation, we focus on the case that the fixed points are isolated, sometimes allow orbifolds with isolated $\mathbb{Z}_2$-singularities, and work with a slight

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generalisation of tamed forms. To explain concretely, we introduce some notation. Let $\mathbb{C}^\times$ act holomorphically on a complex manifold (M, J). Let ξ_M denote the vector field on M induced by the restricted action of $\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{S}^1 \subset \mathbb{C}^\times$. Let $M^{\mathbb{S}^1}$ be the set of points fixed by this action, and let $\Omega^k(M)^{\mathbb{S}^1}$ be the set of $\mathbb{S}^1$-invariant k-forms. Consider a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ and assume that $\Psi\colon M \to \mathbb{R}$ is a moment map, i.e., $\xi_M \lrcorner \omega = -d\Psi$. Recall that J **tames** ω if $\omega(v, J(v)) > 0$ for all non-zero tangent vectors v. In this paper, we also work with the following significantly weaker condition: We say that the action **tames** ω if $\omega(\xi_M, J(\xi_M)) > 0$ on $M \smallsetminus M^{\mathbb{S}^1}$.

We believe that this setting is intrinsically interesting and hope that it will prove to be a fruitful source of examples and counter-examples. To see why, note that if $\omega \in \Omega^2(M)$ is a Kähler form (and M is compact), then the logarithm of the Duistermaat-Heckman function μ is a concave function on the moment map image $\Psi(M)$ [7]; in particular, μ has no strict local minima on the interior $\Psi(M)^\circ$. If ω is tamed by J instead, then the Duistermaat-Heckman function need not be log-concave; nevertheless, if $\dim_{\mathbb{R}} M = 6$, then μ cannot have a strict local minimum at certain $a \in \mathbb{R}$, e.g., if $\mathbb{S}^1 \subset \mathbb{C}^\times$ acts freely on $\Psi^{-1}(a)$. In contrast, if ω is merely tamed by the action, then the Duistermaat-Heckman function can have strict local minima at such values. Thus, for some of the key pieces that the first author used to construct the non-Hamiltonian example, the symplectic form was tamed by the action but was not (and could not be) Kähler, or even tamed by J [17].

Since $\omega(\cdot, J(\cdot))$ may no longer be a metric for tame symplectic forms, some of the proofs that work for Kähler manifolds do not work for this larger class without significant modification. However, in the Kähler case the gradient flow $\nabla\Psi$ is equal to $-J(\xi_M)$, which is the vector field induced by the $\mathbb{R}$-action given by $(t, x) \mapsto e^t \cdot x$ for all $t \in \mathbb{R}$ and $x \in M$. Hence, the function $t \mapsto \Psi(e^t \cdot x)$ is increasing. This remains true in our setting; we will use it repeatedly throughout this paper.

Lemma 1.1. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon M \to \mathbb{R}$. Then the function $t \mapsto \Psi(e^t \cdot x)$ is strictly increasing for all $x \in M \smallsetminus M^{\mathbb{S}^1}$.*

Proof. Given $x \in M \smallsetminus M^{\mathbb{S}^1}$, let $\gamma_x(t) := \Psi(e^t \cdot x)$. Then for all $t \in \mathbb{R}$,

$$\dot\gamma_x(t) = d\Psi(-J(\xi_M))\Big|_{e^t\cdot x} = \omega(\xi_M, J(\xi_M))\Big|_{e^t\cdot x} > 0.$$

□

2. Tame local normal form

The goal of this section is to prove a $\mathbb{C}^\times$-equivariant holomorphic Bochner linearisation theorem in our setting of a symplectic structure tamed by the action. (More precisely, we develop a $\mathbb{C}^\times$-equivariant holomorphic local normal form for a neighbourhood of a finite number of fixed points in the same moment fibre.) Our proof is adapted from Sjamaar's proof of the holomorphic slice theorem for Kähler actions [16, section 1]; see also [10].

Proposition 2.1. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon M \to \mathbb{R}$. Given $\{p_1, \ldots, p_k\} \in M^{\mathbb{S}^1} \cap \Psi^{-1}(0)$, there exists a $\mathbb{C}^\times$-invariant neighbourhood* *of* $\{p_1,\dots,p_k\}$ *in* M *which is* $\mathbb{C}^\times$*-equivariantly biholomorphic to a neighbourhood of* $\coprod_{j=1}^k\{0\}$ *in* $\coprod_{j=1}^k\mathbb{C}^n$*, where* $\mathbb{C}^\times$ *acts linearly on each* $\mathbb{C}^n$*.*

To prove this, we need a holomorphic version of the Bochner linearisation theorem for compact group actions on complex manifolds. While this is well-known (*cf.*, for example, [5]), we include the proof for completeness. Since we consider actions on orbifolds in Proposition 5.8, we will prove the orbifold version. Here, if a group G acts on a complex orbifold (M,J), we say that the action **respects** J if every $g\in G$ induces an automorphism of (M,J).

Lemma 2.2. *Let a compact Lie group* G *act on a complex orbifold* (M,J)*; assume that the action respects* J*. Given* $p\in M^G$*, there exists a* G*-equivariant biholomorphism from a* G*-invariant neighbourhood of* $0\in T_pM$ *to a neighbourhood of* $p\in M$*.*

Proof. Let Γ be the (orbifold) isotropy group of p. There exists an extension $\widetilde{G}$ of G by Γ, an action of $\widetilde{G}$ on an open set $\widetilde{U}\subset\mathbb{C}^n$ that respects the complex structure and fixes $\widetilde{p}\in\widetilde{U}$, and a G-equivariant biholomorphism from $\widetilde{U}/\Gamma$ to M that sends $[\widetilde{p}]$ to p. (See, for example, [14].) Let $\psi:\widetilde{U}\to T_{\widetilde{p}}\widetilde{U}$ be any holomorphic map whose differential at $\widetilde{p}$ is the identity map on $T_{\widetilde{p}}\widetilde{U}$. Since $\widetilde{G}$ is compact, we can average ψ to obtain a $\widetilde{G}$-equivariant map $\bar{\psi}\colon\widetilde{U}\to T_{\widetilde{p}}\widetilde{U}$ such that $d\bar{\psi}|_{\widetilde{p}}$ is equal to the identity, defined by

$$\bar{\psi}(q):=\int_{\widetilde{G}}g_*\psi(g^{-1}\cdot q)dg$$

for all $q\in\widetilde{U}$, where dg is the Haar measure on $\widetilde{G}$. Since the action respects the complex structure, the map $\bar{\psi}$ is holomorphic. By the inverse function theorem, we can invert $\bar{\psi}$ on a neighbourhood of $\widetilde{p}$ to construct the required biholomorphism. □

Let (M_1,J_1) and (M_2,J_2) be complex manifolds with holomorphic $\mathbb{C}^\times$-actions, A be an $\mathbb{S}^1$-invariant open subset of M_1, and $\varphi\colon A\to M_2$ be an $\mathbb{S}^1$-equivariant holomorphic map. Then φ sends the vector $J_1(\xi_{M_1})\big|_x$ to the vector $J_2(\xi_{M_2})\big|_{\varphi(x)}$ for all $x\in A$. Hence, if (t_-,t_+) is the connected component of $\{t\in\mathbb{R}\mid e^t\cdot x\in A\}$ containing 0, then

$$\varphi(e^t\cdot x)=e^t\cdot\varphi(x)\text{ for all }t\in(t_-,t_+).\tag{2.1}$$

However, (2.1) need not hold for all $t\in\mathbb{R}$ such that $e^t\cdot x\in A$. This motivates the following definition.

Definition 2.3. Let (M,J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action. Then a subset $A\subseteq M$ is **orbitally convex** (with respect to the $\mathbb{C}^\times$-action) if it is $\mathbb{S}^1$-invariant and the set $\{t\in\mathbb{R}\mid e^t\cdot x\in A\}$ is connected for all $x\in A$.

The following proposition, which is Proposition 1.4 in [16], is a consequence of the discussion above.

Proposition 2.4. *Let* (M_1,J_1) *and* (M_2,J_2) *be complex manifolds with holomorphic* $\mathbb{C}^\times$*-actions. Assume that* $A\subseteq M_1$ *is an orbitally convex open set and* $\varphi\colon A\to M_2$ *is an* $\mathbb{S}^1$*-equivariant holomorphic map. Then,* φ *extends to a* $\mathbb{C}^\times$*-equivariant holomorphic map* $\widetilde{\varphi}\colon\mathbb{C}^\times\cdot A\to M_2$*. Consequently, if* $\varphi(A)$ *is open and orbitally convex in* M_2 *and* $\varphi\colon A\to\varphi(A)$ *is biholomorphic, then* $\widetilde{\varphi}$ *is biholomorphic onto the open set* $\mathbb{C}^\times\cdot\varphi(A)$*.*

Remark 2.5. The proof of Proposition 2.4 in [16] uses the definition stated here in Definition 2.3. Note, however, that $A \subset M$ may not be orbitally convex even if the intersection $\{e^t \cdot x \mid t \in \mathbb{R}\} \cap A$ is connected for all $x \in A$. To see this, let $1 \in \mathbb{Z}$ act on $\mathbb{C}^\times \subset \mathbb{C}$ by $z \mapsto 2z$, and let $M := \mathbb{C}^\times/\mathbb{Z}$ with the natural $\mathbb{C}^\times$-action. Let $B \subset M$ be the image of $\mathbb{S}^1 \subset \mathbb{C}^\times$, and let $A = M \smallsetminus B$. Then $\{e^t \cdot x \mid t \in \mathbb{R}\} \cap A$ is connected for all $x \in M$, but A is not orbitally convex.

To complete the proof of Proposition 2.1, we show that we can choose the neighbourhoods in Lemma 2.2 to be orbitally convex. For this, we need the following lemma.

Lemma 2.6. *Let $\mathbb{C}^\times$ act linearly on $\mathbb{C}^n$, and let J be the standard complex structure on $\mathbb{C}^n$. Let U be an $\mathbb{S}^1$-invariant neighbourhood of 0 in $\mathbb{C}^n$, with a symplectic form $\omega \in \Omega^2(U)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon U \to \mathbb{R}$ sending 0 to 0. There exist $\delta > 0$ and an orbitally convex neighbourhood $V \subset U$ of 0 so that the following holds: Given $z \in V$, if $(t_-, t_+) = \{t \in \mathbb{R} \mid e^t \cdot z \in V\}$, then*

1. *either $t_+ = \infty$ or there exists $t \in (t_-, t_+)$ with $\Psi(e^t \cdot z) > \delta$; and*
2. *either $t_- = -\infty$ or there exists $t \in (t_-, t_+)$ with $\Psi(e^t \cdot z) < -\delta$.*

Proof. We may assume $\mathbb{C}^n = \mathbb{C}^{k_-} \times \mathbb{C}^{k_+} \times \mathbb{C}^l$, where $\mathbb{C}^\times$ acts on $\mathbb{C}^{k_-}$ with negative weights $(-\alpha_1^-, \dots, -\alpha_{k_-}^-) \in (-\mathbb{N})^{k_-}$; that is, $\lambda \cdot (z_1, \dots, z_{k_-}) = (\lambda^{-\alpha_1^-} z_1, \dots, \lambda^{-\alpha_{k_-}^-} z_{k_-})$; on $\mathbb{C}^{k_+}$ with positive weights $(\alpha_1^+, \dots, \alpha_{k_+}^+) \in \mathbb{N}^{k_+}$; and on $\mathbb{C}^l$ trivially.

Define continuous $\mathbb{S}^1$-invariant functions $N_\pm\colon \mathbb{C}^{k_\pm} \to \mathbb{R}$ by

$$N_\pm(z) = \left(\sum_{j=1}^{k_\pm} |z_j|^{2/\alpha_j^\pm}\right)^{\frac{1}{2}},$$

and define $N\colon \mathbb{C}^{k_-} \times \mathbb{C}^{k_+} \times \mathbb{C}^l \to \mathbb{R}$ by $N(z_-, z_+, w) = N_-(z_-)N_+(z_+)$. Note that

$$N_\pm(e^t \cdot z_\pm) = e^{\pm t} N_\pm(z_\pm), \text{ and so} \tag{2.2}$$

$$N(e^t \cdot (z_-, z_+, w)) = N(z_-, z_+, w) \tag{2.3}$$

for all $z_\pm \in \mathbb{C}^{k_\pm}$, $w \in \mathbb{C}^l$, and $t \in \mathbb{R}$.

Given $\varepsilon > 0$, define

$$D_\varepsilon^\pm = \{z \in \mathbb{C}^{k_\pm} \mid N_\pm(z) < \varepsilon\} \text{ and } S_\varepsilon^\pm = \{z \in \mathbb{C}^{k_\pm} \mid N_\pm(z) = \varepsilon\} \subsetneq \overline{D_\varepsilon^\pm}.$$

If $\varepsilon < 1$, then $|z_j| < 1$ for all $z \in D_\varepsilon^\pm$ and $j \in \{1, \dots, k_\pm\}$, and so $N_\pm(z) \geq \left(\sum_j |z_j|^2\right)^{\frac{1}{2}} = |z|$. Thus there exists $\varepsilon > 0$ and a compact, connected neighbourhood K of $0 \in \mathbb{C}^l$ such that $\overline{D_\varepsilon^-} \times \overline{D_\varepsilon^+} \times K \subset U$.

If $z \in \overline{D_\varepsilon^\pm}$, then $\lim_{t \to \mp\infty} e^t \cdot z = 0$. Moreover, $e^t \cdot z \in D_\varepsilon^\pm$ for all $t \in \mp(0, \infty)$ by (2.2). Additionally, $\Psi(0, 0, w) = 0$ for any $w \in K$; therefore, by Lemma 1.1, $\Psi(z_-, 0, w) < 0$ for every non-zero $z_- \in \overline{D_\varepsilon^-}$, and $\Psi(0, z_+, w) > 0$ for every non-zero $z_+ \in \overline{D_\varepsilon^+}$. Hence, since $S_\varepsilon^\pm$ and K are compact, there exist $\delta > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that

$$\Psi(S_\varepsilon^- \times D_{\varepsilon'}^+ \times K) \subsetneq (-\infty, -\delta) \text{ and } \Psi(D_{\varepsilon'}^- \times S_\varepsilon^+ \times K) \subsetneq (\delta, \infty). \tag{2.4}$$

Define $V := \{(z_-, z_+, w) \in D_\varepsilon^- \times D_\varepsilon^+ \times K \mid N(z_-, z_+) < \varepsilon\varepsilon'\} \subset U$. Then V is an orbitally convex neighbourhood of 0 by (2.2) and (2.3). Fix $(z_-, z_+, w) \in V$. If $z_+ = 0$, then $N(e^t \cdot (z_-, z_+, w)) = 0$ for all $t \in \mathbb{R}$, and so $e^t \cdot (z_-, z_+, w)$ lies in V for all

$t \geq 0$. On the other hand, if $z_+ \neq 0$, then $t_+ := \ln \frac{\varepsilon}{N_+(z_+)} > 0$ because $N_+(z_+) < \varepsilon$. A straightforward calculation using (2.2) shows that $e^t \cdot (z_-, z_+, w) \in D^-_\varepsilon \times D^+_\varepsilon \times K$ for all $t \in [0, t_+)$. By (2.3), this implies that $e^t \cdot (z_-, z_+, w) \in V$ for all $t \in [0, t_+)$. Since $e^{t_+} \cdot (z_-, z_+, w) \in D^-_{\varepsilon'} \times S^+_\varepsilon \times K$, claim (1) follows from (2.4). The proof of claim (2) is nearly identical. □

Proof of Proposition 2.1. By Lemma 2.2, there exists an $\mathbb{S}^1$-equivariant biholomorphism φ from an $\mathbb{S}^1$-invariant neighbourhood U of $\coprod_j \{0\}$ in $\coprod_j \mathbb{C}^n$ to a neighbourhood of $\{p_1, \dots, p_k\}$ in M, where $\mathbb{C}^\times$ acts linearly on each $\mathbb{C}^n$. Let δ be the positive real and $V \subset U$ be the orbitally convex neighbourhood of $\coprod_j \{0\}$ given by repeatedly applying Lemma 2.6. Fix $x \in \varphi(V)$. Assume that $\varphi(z) = e^{t'} \cdot x$, where $z \in V$ and $t' \in \mathbb{R}$, and let $(t_-, t_+) := \{t \in \mathbb{R} \mid e^t \cdot z \in V\}$. Then, since V is orbitally convex, equation (2.1) (alternatively, Proposition 2.4) implies that $\varphi(e^t \cdot z) = e^{t+t'} \cdot x$ for all $t \in (t_-, t_+)$. In particular, $e^t \cdot x \in \varphi(V)$ for all $t \in (t' + t_-, t' + t_+)$. Moreover, by Lemma 2.6,

(1) either $t_+ = \infty$ or there exists $t \in (t' + t_-, t' + t_+)$ with $\Psi(e^t \cdot x) > \delta$; and
(2) either $t_- = -\infty$ or there exists $t \in (t' + t_-, t' + t_+)$ with $\Psi(e^t \cdot x) < -\delta$.

Finally, by Lemma 1.1, the function $t \mapsto \Psi(e^t \cdot x)$ is increasing. Therefore, $\varphi(V)$ is open and orbitally convex. Thus, by Proposition 2.4, $\varphi|_V$ extends to a $\mathbb{C}^\times$-equivariant biholomorphism from $\mathbb{C}^\times \cdot V$ to $\mathbb{C}^\times \cdot \varphi(V)$. □

Since the function $t \mapsto \Psi(e^t \cdot x)$ is increasing, claims (1) and (2) above imply that $\Psi(e^t \cdot x) \notin (-\delta, \delta)$ if $t \notin (t' + t_-, t' + t_+)$. Since also $e^t \cdot x \in \varphi(V)$ for all $t \in (t' + t_-, t' + t_+)$, the above proof has the following corollary.

Corollary 2.7. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon M \to \mathbb{R}$. Given $p \in M^{\mathbb{S}^1} \cap \Psi^{-1}(0)$ and an $\mathbb{S}^1$-invariant neighbourhood U of p, there exists $\delta > 0$ and an $\mathbb{S}^1$-invariant open neighbourhood $V \subseteq U$ of p such that $\mathbb{C}^\times \cdot V \cap \Psi^{-1}(-\delta, \delta) \subset V$.*

3. Tame reduction

It is well-known that the reduced spaces of Kähler manifolds naturally inherit Kähler structures [8, 11, 12]. The goal of this section is to generalise this fact to our setting; in particular, the reduced spaces of complex manifolds with tamed symplectic forms naturally inherit complex structures that tame the reduced symplectic form.

Proposition 3.1. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon M \to \mathbb{R}$. Given a regular value a of Ψ, let $U_a := \mathbb{C}^\times \cdot \Psi^{-1}(a)$. Then the following hold.*

(1) *The quotient $U_a/\mathbb{C}^\times$ is naturally a complex orbifold, and U_a is a holomorphic $\mathbb{C}^\times$-bundle over $U_a/\mathbb{C}^\times$.*
(2) *There is a complex structure J_a on the reduced space $M/\!/_a\, \mathbb{S}^1$ so that the inclusion $\Psi^{-1}(a) \hookrightarrow U_a$ induces a biholomorphism $M/\!/_a\, \mathbb{S}^1 \to U_a/\mathbb{C}^\times$.*

(3) *For all $x \in \Psi^{-1}(a)$, the natural map*

$$\{X \in T_x M \mid \omega(\xi_M, X) = \omega(\xi_M, J(X)) = 0\} \to T_{[x]}(M/\!/_a \mathbb{S}^1)$$

is an isomorphism of complex and symplectic vector spaces. Consequently, if J tames ω at x, then J_a tames the reduced symplectic form at $[x]$.

Remark 3.2. In the situation of Proposition 3.1, if (M, J, ω) is Kähler, then the reduced space $M/\!/_a \mathbb{S}^1$ is Kähler by claim (3). Thus, the standard theorem for Kähler manifolds is a special case. We have written our proofs to show that the analogous statement holds whenever applicable in this paper: cutting (Proposition 4.1), blow-ups (Proposition 5.1 and Proposition 5.8), and adding fixed points (Proposition 6.1). Here, we rely on the final claims in Lemmas 5.6, 5.7, and 6.4.

Remark 3.3. A **symplectic toric orbifold** is a triple (M, ω, Φ), where (M, ω) is a $2n$-dimensional compact, connected symplectic orbifold and $\Phi\colon M \to \mathbb{R}^n$ is a moment map for an effective $(\mathbb{S}^1)^n$-action. The **moment polytope** $\Delta := \Phi(M)$ is a convex rational simple polytope. Given a facet $F \subset \Delta$ with interior F°, there exists a natural number k_F so that $\mathbb{Z}_{k_F}$ is the orbifold isotropy subgroup of each point in $\Phi^{-1}(F^\circ)$. Symplectic toric orbifolds are classified (up to $(\mathbb{S}^1)^n$-equivariant symplectomorphism) by their moment polytopes (up to translation) and these natural numbers. Moreover, the stabiliser of $x \in M$ is the connected subgroup $H \subset (\mathbb{S}^1)^n$ with Lie algebra $\mathfrak{h}$, where $\Phi(x) + \mathfrak{h}^0$ is the minimal affine plane that contains a face of Δ containing $\Phi(x)$; in particular, $x \in M^{(\mathbb{S}^1)^n}$ exactly if $\Phi(x) \in \Delta$ is a vertex.

Hence, we can visualise the constructions in this paper by considering the case that the circle $\mathbb{S}^1 \times \{1\}^{n-1} \subset (\mathbb{S}^1)^n$ acts on a toric manifold (M, ω, Φ), and so the $\mathbb{S}^1$-moment map Ψ is the first component of Φ. Since all symplectic manifolds with Hamiltonian circle actions are locally isomorphic to toric manifolds, this gives valuable insight into the underlying symplectic geometry in the general case.

For example, in the situation described above, $a \in \mathbb{R}$ is a regular value of Ψ exactly if no vertex of Δ lies in $\{a\} \times \mathbb{R}^{n-1}$. In this case, the reduced space $M/\!/_a \mathbb{S}^1$ is a symplectic toric orbifold with moment polytope $\Delta \cap \big(\{a\} \times \mathbb{R}^{n-1}\big)$. See Remarks 4.2, 5.9, 6.2, and 7.10 for further discussion.

To prove Proposition 3.1, we will need the following important technical lemma, which only depends on Lemma 1.1 and the fact that Ψ is equivariant.

Lemma 3.4. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon M \to \mathbb{R}$. Define*

$$\mathcal{U} := \{(x, s) \in \big(M \smallsetminus M^{\mathbb{S}^1}\big) \times \mathbb{R} \mid s \in \Psi(\mathbb{C}^\times \cdot x)\}.$$

Then for all $(x, s) \in \mathcal{U}$, there exists a unique $f(x, s) \in \mathbb{R}$ such that

$$\Psi(e^{f(x,s)} \cdot x) = s;$$

moreover, $\mathcal{U}$ is open and $f\colon \mathcal{U} \to \mathbb{R}$ is a smooth $\mathbb{S}^1$-invariant function.

Proof. Given $(x, s) \in \mathcal{U}$, let $\gamma_x(t) := \Psi(e^t \cdot x)$ for all $t \in \mathbb{R}$. The moment map is $\mathbb{S}^1$-invariant; hence, by definition, there exists $f(x, s) \in \mathbb{R}$ such that $\gamma_x(f(x, s)) = s$. Moreover, by Lemma 1.1, γ_x is strictly increasing, and so $f(x, s)$ is unique. By the implicit function theorem, there exist open neighbourhoods V of $(x,s)\in\mathcal{U}$ and W of $f(x,s)\in\mathbb{R}$, and a smooth function $\widetilde{f}\colon V\to W$ such that

$$\big\{\big((y,u),\widetilde{f}(y,u)\big)\;\big|\;(y,u)\in V\big\}=\{(y,u,t)\in V\times W\mid \Psi(e^t\cdot y)=u\}.$$

Therefore $\mathcal{U}$ is open and f is smooth. Finally, since Ψ is $\mathbb{S}^1$-invariant, the function f is as well. (Here, $\mathbb{S}^1$ acts trivially on $\mathbb{R}$.) □

Proof of Proposition 3.1. By Lemma 3.4, the $\mathbb{C}^\times$-invariant set

$$U_a:=\mathbb{C}^\times\cdot\Psi^{-1}(a)=\{x\in M\smallsetminus M^{\mathbb{S}^1}\mid a\in\Psi(\mathbb{C}^\times\cdot x)\}$$

is open and there exists a smooth function $f\colon U_a\to\mathbb{R}$ such that

$$\Psi(e^{f(x)}\cdot x)=a$$

for all $x\in U_a$.

Define $\Theta\colon\mathbb{C}^\times\times M\to M\times M$ by $\Theta(e^{u+iv},x)=(e^{u+iv}\cdot x,x)$ for all $u,v\in\mathbb{R}$ and $x\in M$. Given a compact set $L\subseteq M\times M$, there exists a closed interval $[s,t]\subset\mathbb{R}$ so that $f(x_1)\in[s,t]$ and $f(x_2)\in[s,t]$ for all $(x_1,x_2)\in L$. Since Ψ is $\mathbb{S}^1$-invariant,

$$f(e^{u+iv}\cdot x)=f(x)-u$$

for all $u,v\in\mathbb{R}$ and $x\in U_a$. Thus, for all $(e^{u+iv},x)\in\Theta^{-1}(L)$, we have $f(x)-u\in[s,t]$ and $f(x)\in[s,t]$, and so $u\in[s-t,t-s]$. Thus $\Theta^{-1}(L)$ is contained in a compact set. Since Θ is continuous, this implies that $\Theta^{-1}(L)$ is compact, that is, the $\mathbb{C}^\times$-action on U_a is proper.

Since the action is proper, the stabiliser Γ of x is finite for all $x\in U_a$. Moreover, by the slice theorem there is a $\mathbb{C}^\times$-invariant neighbourhood of the orbit $\mathbb{C}^\times\cdot x$ that is $\mathbb{C}^\times$-equivariantly biholomorphic to the associated bundle $\mathbb{C}^\times\times_\Gamma D$, where D is a disc in the normal space to $\mathbb{C}^\times\cdot x$ at x. The quotient map $D\to D/\Gamma$ is an orbifold chart on $U_a/\mathbb{C}^\times$ near $[x]$; hence, $U_a/\mathbb{C}^\times$ is a complex orbifold and the quotient map $U_a\to U_a/\mathbb{C}^\times$ is holomorphic. Since the action of Γ on D lifts to the the diagonal Γ-action on the product $\mathbb{C}^\times\times D$, this implies that $U_a\to U_a/\mathbb{C}^\times$ is a holomorphic $\mathbb{C}^\times$-bundle; *cf.* [5, Corollaries B.31 and B.32].

The natural inclusion map $\Psi^{-1}(a)\hookrightarrow U_a$ descends to a well-defined smooth map $i\colon M/\!/_a\,\mathbb{S}^1\to U_a/\mathbb{C}^\times$. Similarly, the map $U_a\to\Psi^{-1}(a)$ defined by $x\mapsto e^{f(x)}\cdot x$ descends to a smooth map $g\colon U_a/\mathbb{C}^\times\to M/\!/_a\,\mathbb{S}^1$. Moreover, these induced maps are inverses of each other. Under the resulting identification, the symplectic quotient $M/\!/_a\,\mathbb{S}^1$ inherits a natural complex structure from $U_a/\mathbb{C}^\times$.

Fix $x\in\Psi^{-1}(a)$. By the preceding paragraph, the natural map $T_{[x]}(M/\!/_a\,\mathbb{S}^1)\to T_{[x]}(U_a/\mathbb{C}^\times)$ is an isomorphism of complex vector spaces. Since $\omega(\xi_M,J(\xi_M))>0$ on $\Psi^{-1}(a)$, we can represent every vector in $T_{[x]}(M/\!/_a\,\mathbb{S}^1)$ by a unique vector in the J-invariant subspace $\{X\in T_xM\mid\omega(\xi_M,X)=\omega(\xi_M,J(X))=0\}$ of T_xM. The third claim follows. □

4. Tame cutting

Next, we show that symplectic cutting, developed by Lerman in [13], also works in our setting; see also [2] for a discussion of the Kähler case.

Proposition 4.1. *Let (M,J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega\in\Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon M\to\mathbb{R}$. Assume that $0\in\mathbb{R}$ is a regular value of Ψ. Then there exists a complex orbifold $(M_{\mathrm{cut}},J_{\mathrm{cut}})$ with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega_{\mathrm{cut}}\in\Omega^2(M_{\mathrm{cut}})^{\mathbb{S}^1}$*

tamed by the action, and a moment map $\Psi_{\mathrm{cut}}\colon M_{\mathrm{cut}} \to \mathbb{R}$ so that the following hold.

(1) $\Psi_{\mathrm{cut}}(M_{\mathrm{cut}}) \subseteq (-\infty, 0]$.
(2) *A neighbourhood of the fixed component $\Psi_{cut}^{-1}(0)$ is $\mathbb{C}^\times$-equivariantly biholomorphic to the holomorphic line bundle $U_0 \times_{\mathbb{C}^\times} \mathbb{C}$, where $U_0 := \mathbb{C}^\times \cdot \Psi^{-1}(0)$, $\mathbb{C}^\times$ acts diagonally on $U_0 \times \mathbb{C}$, and $\mathbb{C}^\times$ acts on $U_0 \times_{\mathbb{C}^\times} \mathbb{C}$ by $\lambda \cdot [x, z] = [\lambda \cdot x, z]$.*
(3) *There exists an $\mathbb{S}^1$-equivariant symplectomorphism from $\Psi^{-1}(-\infty, 0)$ to $\Psi_{\mathrm{cut}}^{-1}(-\infty, 0)$ that induces a biholomorphism between the reduced spaces at all regular $s \in (-\infty, 0)$.*
(4) *If J tames ω near $\Psi^{-1}(s)$ for some $s \in \mathbb{R}$, then J_{cut} tames ω_{cut} near $\Psi_{\mathrm{cut}}^{-1}(s)$.*
(5) *If Ψ is proper, then Ψ_{cut} is proper.*

Proof. Let J' be the product complex structure on $M' = M \times \mathbb{C}$. The diagonal $\mathbb{C}^\times$-action on M' is holomorphic, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication. Consider the symplectic form $\omega' = \omega + i dz \wedge d\overline{z}/2 \in \Omega^2(M')^{\mathbb{S}^1}$. If $(x, z) \in M' \smallsetminus (M')^{\mathbb{S}^1}$, then either $x \notin M^{\mathbb{S}^1}$ or $z \neq 0$. In either case, the fact that $\xi_{M'} = \xi_M + \xi_{\mathbb{C}}$ implies that

$$\omega'(\xi_{M'}, J'(\xi_{M'}))\big|_{(x,z)} = \omega(\xi_M, J(\xi_M))\big|_x + |z|^2 > 0,$$

and so ω' is tamed by the action. The function $\Psi'\colon M' \to \mathbb{R}$ sending (x, z) to $\Psi(x) + |z|^2/2$ is a moment map. Since 0 is a regular value of Ψ, it is also a regular value of Ψ'.

Define $U_0' := \mathbb{C}^\times \cdot (\Psi')^{-1}(0) \subseteq M'$. By Proposition 3.1, the quotient $U_0'/\mathbb{C}^\times$ is naturally a complex orbifold, and U_0' is a holomorphic $\mathbb{C}^\times$-bundle over $U_0'/\mathbb{C}^\times$. Moreover, the reduced space

$$M_{\mathrm{cut}} := M' /\!/_0\, \mathbb{S}^1 = \big\{(x, z) \in M \times \mathbb{C} \,\big|\, \Psi(x) + |z|^2/2 = 0\big\}/\mathbb{S}^1 \tag{4.1}$$

inherits a symplectic structure ω_{cut} and a complex structure J_{cut} so that the inclusion $(\Psi')^{-1}(0) \hookrightarrow U_0'$ induces a biholomorphism $M_{\mathrm{cut}} \to U_0'/\mathbb{C}^\times$. Finally, for all $(x, z) \in (\Psi')^{-1}(0)$, the natural map

$$\{X' \in T_{(x,z)}M' \mid \omega'(\xi_{M'}, X') = \omega'(\xi_{M'}, J(X')) = 0\} \to T_{[x,z]}(M_{\mathrm{cut}}) \tag{4.2}$$

is an isomorphism of complex and symplectic vector spaces.

Since the holomorphic $\mathbb{C}^\times$-action on M' given by $\lambda \cdot (x, z) = (\lambda \cdot x, z)$ commutes with the diagonal action, it descends to a holomorphic $\mathbb{C}^\times$-action on M_{cut}. Moreover, ω' is invariant under the associated $\mathbb{S}^1$-action on M', and so $\omega_{\mathrm{cut}} \in \Omega^2(M_{\mathrm{cut}})^{\mathbb{S}^1}$. The function $\Psi_{\mathrm{cut}}\colon M_{\mathrm{cut}} \to \mathbb{R}$ defined by

$$\Psi_{\mathrm{cut}}([x, z]) = \Psi(x) \tag{4.3}$$

is a moment map for the $\mathbb{S}^1$-action on M_{cut}. The vector field Ξ on $M' \smallsetminus (M')^{\mathbb{S}^1}$ given by

$$\Xi := \xi_M - \frac{\omega(\xi_M, J(\xi_M))}{\omega(\xi_M, J(\xi_M)) + |z|^2}\,\xi_{M'} = \frac{|z|^2\xi_M - \omega(\xi_M, J(\xi_M))\xi_{\mathbb{C}}}{\omega(\xi_M, J(\xi_M)) + |z|^2} \tag{4.4}$$

descends to $\xi_{M_{\mathrm{cut}}}$ on M_{cut} and satisfies $\omega'(\xi_{M'}, \Xi) = \omega'(\xi_{M'}, J'(\Xi)) = 0$. Since (4.2) is an isomorphism of symplectic and complex vector spaces, this implies that

$$\omega_{\mathrm{cut}}(\xi_{M_{\mathrm{cut}}}, J_{\mathrm{cut}}(\xi_{M_{\mathrm{cut}}})) = \omega'(\Xi, J'(\Xi)) = \frac{|z|^2\omega(\xi_M, J(\xi_M))}{\omega(\xi_M, J(\xi_M)) + |z|^2}.$$

In particular, this is positive if $[x,z] \in M_{\mathrm{cut}} \smallsetminus M_{\mathrm{cut}}^{\mathbb{S}^1}$, because then $x \notin M^{\mathbb{S}^1}$ and $z \neq 0$; hence ω_{cut} is tamed by the action.

Claims (1) and (5) are immediate consequences of (4.1) and (4.3).

Fix (x,z) in $U_0 \times \mathbb{C}$. By definition, there exists $t \in \mathbb{R}$ such that $e^t \cdot x \in \Psi^{-1}(0)$. Thus, since 0 is a regular value, Lemma 1.1 implies that the function $s \mapsto \Psi(e^s \cdot x)$ is strictly increasing. Therefore,

$$\lim_{s \to -\infty} \Psi'(e^s \cdot (x,z)) = \lim_{s \to -\infty} \Psi(e^s \cdot x) < 0$$

and

$$\lim_{s \to \infty} \Psi'(e^s \cdot (x,z)) \geq \lim_{s \to \infty} \Psi(e^s \cdot x) > 0.$$

By continuity, this implies that $(x,z) \in U_0'$; therefore, $U_0 \times \mathbb{C} \subset U_0'$. Moreover, by (4.1) and (4.3),

$$\Psi_{\mathrm{cut}}^{-1}(0) = \{(x,0) \in M \times \mathbb{C} \mid \Psi(x) = 0\}/\mathbb{S}^1.$$

Hence, the $\mathbb{C}^\times$-equivariant biholomorphism $M_{\mathrm{cut}} \to U_0'/\mathbb{C}^\times$ maps $\Psi_{\mathrm{cut}}^{-1}(0)$ into $U_0 \times_{\mathbb{C}^\times} \mathbb{C} \subset U_0'/\mathbb{C}^\times$. Since $U_0 \times_{\mathbb{C}^\times} \mathbb{C}$ is open by Lemma 3.4, this proves claim (2).

It is straightforward to check that the map from $\Psi^{-1}(-\infty,0)$ to $\Psi_{\mathrm{cut}}^{-1}(-\infty,0)$ that sends x to $\left[x, \sqrt{-2\Psi(x)}\,\right]$ for all $x \in \Psi^{-1}(-\infty,0)$ is an $\mathbb{S}^1$-equivariant symplectomorphism that intertwines the moment maps. Given a regular $s < 0$, it restricts to an $\mathbb{S}^1$-equivariant diffeomorphism from $\Psi^{-1}(s)$ to $\Psi_{\mathrm{cut}}^{-1}(s)$, and so induces a diffeomorphism from $M/\!/_s\,\mathbb{S}^1$ to $M_{\mathrm{cut}}/\!/_s\,\mathbb{S}^1$. Under the identification $TM' \cong TM \times T\mathbb{C}$, it sends $X \in T_x(\Psi^{-1}(s))$ to $(X,0) \in T_{[x,\sqrt{-2s}]}(\Psi_{\mathrm{cut}}^{-1}(s))$. Let Ξ be defined by (4.4). Since (4.2) is an isomorphism, Proposition 3.1 implies that the natural map from

$$\begin{aligned}(4.5)\quad \{X' \in T_{(x,\sqrt{-2s})}M' \mid \omega'(\xi_{M'}, X') &= \omega'(\xi_{M'}, J'(X')) \\ &= \omega'(\Xi, X') = \omega'(\Xi, J'(X')) = 0\}\end{aligned}$$

to $T_{[x,\sqrt{-2s}]}(M_{\mathrm{cut}}/\!/_s\,\mathbb{S}^1)$ is an isomorphism of complex and symplectic vector spaces. Under the identification $TM' \cong TM \times T\mathbb{C}$, the vector space in (4.5) can be rewritten as

$$\{(X,0) \in T_{(x,\sqrt{-2s})}M' \mid \omega(\xi_M, X) = \omega(\xi_M, J(X)) = 0\}.$$

Therefore, claim (3) follows from part (3) of Proposition 3.1.

Finally, fix $s \in \mathbb{R}$ and assume that J tames ω near $\Psi^{-1}(s)$. If $[x,z] \in \Psi_{\mathrm{cut}}^{-1}(s)$, then $x \in \Psi^{-1}(a)$ by (4.3). Hence, J tames ω near x, and so J' tames ω' near (x,z). Thus J_{cut} tames ω_{cut} near $[x,z]$ by (4.2). This proves claim (4). □

Remark 4.2. Let the circle $\mathbb{S}^1 \times \{1\}^{n-1} \subset (\mathbb{S}^1)^n$ act on a symplectic toric manifold (M, ω, Φ) with moment polytope Δ, as described in Remark 3.3, satisfying the assumptions of Proposition 4.1. In this case, the cut space M_{cut} is a symplectic toric orbifold with moment polytope

$$\Delta_{\mathrm{cut}} := \Delta \cap \{x \in \mathbb{R}^n \mid x_1 \leq 0\};$$

moreover, the fixed component $\Psi_{\mathrm{cut}}^{-1}(0)$ maps to the new facet $\Delta \cap \left(\{0\} \times \mathbb{R}^{n-1}\right)$.

Remark 4.3. Let (M,J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi\colon M \to \mathbb{R}$. Then the **reversed** $\mathbb{C}^\times$-action on M, given by $(\lambda, x) \mapsto \lambda^{-1} \cdot x$, is also holomorphic. Since $-\xi_M$ is the induced vector field for the associated $\mathbb{S}^1 \subset \mathbb{C}^\times$-action, the symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ is tamed by the reversed action, and $\Psi' := -\Psi$

is a moment map for it. The reduced space $(\Psi')^{-1}(s)/\mathbb{S}^1$ is naturally biholomorphically symplectomorphic to the reduced space $\Psi^{-1}(-s)/\mathbb{S}^1$ for all $s \in \mathbb{R}$. However, the Euler classes of the bundles $\mathbb{C}^\times \cdot (\Psi')^{-1}(s) \to (\Psi')^{-1}(s)/\mathbb{S}^1$ and $\mathbb{C}^\times \cdot \Psi^{-1}(-s) \to \Psi^{-1}(-s)/\mathbb{S}^1$ are additive inverses, as are the weights of the original and reversed actions at each fixed point.

Thus, by reversing the action, applying Proposition 4.1, and reversing the action again, we see that Proposition 4.1 still holds with the following modifications: Replace $(-\infty, 0]$ by $[0,\infty)$ in claim (1); replace the diagonal action by the antidiagonal action in claim (2); and replace $(-\infty, 0)$ by $(0,\infty)$ in claim (3). See Remarks 6.3 and 7.11 for further applications.

Remark 4.4. Cutting can be used to compactify manifolds. Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map $\Psi\colon M \to \mathbb{R}$. If $a < b$ are regular values, then by applying Proposition 4.1 twice (once modified as in Remark 4.3), we get a compact complex manifold (M', J') with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega' \in \Omega^2(M')^{\mathbb{S}^1}$ tamed by the action, and a moment map $\Psi'\colon M' \to \mathbb{R}$, so that $\Psi'(M') \subseteq [a, b]$ and the appropriate analogues of claims (2)–(4) hold.

5. Tame blow-ups

It is well-known that the blow-ups of Kähler manifolds admit Kähler forms. In this section, we generalise blow-ups to our setting. In particular, we first show that the blow-up at one point of a complex manifold with a tamed symplectic form admits a tamed symplectic form, and then extend this claim to the blow-up of an isolated $\mathbb{Z}_2$-singularity in a complex orbifold.

Proposition 5.1. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action (everywhere) and tamed by J on $W \subseteq M$, and a moment map $\Psi\colon M \to \mathbb{R}$. Let $(\widehat{M}, \widehat{J})$ be the complex blow-up of M at $p \in M^{\mathbb{S}^1} \cap W$. For sufficiently small $t > 0$, there exist a symplectic form $\widehat{\omega} \in \Omega^2(\widehat{M})^{\mathbb{S}^1}$ tamed by the action (everywhere) and tamed by $\widehat{J}$ on $q^{-1}(W)$, and a moment map $\widehat{\Psi}\colon \widehat{M} \to \mathbb{R}$ such that*

$$[\widehat{\omega}] = q^*[\omega] - t\mathcal{E},$$

where $q\colon \widehat{M} \to M$ is the blow-down map and $\mathcal{E}$ is the Poincaré dual of the exceptional divisor $q^{-1}(p)$. Moreover, given a neighbourhood V of p, we may assume that $\widehat{\omega} = q^*\omega$ and $\widehat{\Psi} = q^*\Psi$ on $\widehat{M} \smallsetminus q^{-1}(V)$.

Proof. We may assume that W is open. By Lemma 2.2, there exists an $\mathbb{S}^1$-equivariant biholomorphism from an $\mathbb{S}^1$-invariant neighbourhood of $0 \in \mathbb{C}^n$ to a neighbourhood $U \subseteq W \cap V$ of p, where $\mathbb{S}^1$ acts on $\mathbb{C}^n$ with weights $(\alpha_1, \dots, \alpha_n)$. We identify these neighbourhoods, and also identify $q^{-1}(U)$ with a neighbourhood of the exceptional divisor E in

$$\widehat{\mathbb{C}^n} := (\mathbb{C}^n \smallsetminus \{0\}) \times_{\mathbb{C}^\times} \mathbb{C},$$

where $\mathbb{C}^\times$ acts on $(\mathbb{C}^n \smallsetminus \{0\}) \times \mathbb{C}$ by

$$\lambda \cdot (z_1, \dots, z_n; u) = (\lambda z_1, \dots, \lambda z_n; \lambda^{-1}u).$$

In these coordinates, the blow-down map $q\colon \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ sends $[z_1, \dots, z_n; u]$ to $(uz_1, \dots, uz_n)$. Define $\pi\colon \widehat{\mathbb{C}^n} \to \mathbb{CP}^{n-1}$ by $\pi\big([z_1, \dots, z_n; u]\big) = [z_1, \dots, z_n]$.

Since the restriction $q\colon \widehat{M} \smallsetminus E \to M \smallsetminus \{p\}$ is $\mathbb{C}^\times$-equivariant and biholomorphic, the closed form $q^*\omega \in \Omega^2(\widehat{M})^{\mathbb{S}^1}$ is symplectic on $\widehat{M} \smallsetminus E$, tamed by the action on $\widehat{M} \smallsetminus E$, and tamed on $q^{-1}(W) \smallsetminus E$. Since q is holomorphic and $\ker(\pi_*|_m) \cap \ker(q_*|_m) = \{0\}$,

$$q^*\omega(X, \widehat{J}(X)) \geq 0$$

for all $m \in E$ and $X \in T_m\widehat{M}$, with equality impossible if $\pi_* X = 0$ and $X \neq 0$. Finally, since $q\colon \widehat{M} \to M$ is equivariant, $\xi_{\widehat{M}} \lrcorner\, q^*\omega = -dq^*\Psi$.

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth function which is 1 on a neighbourhood of 0 and such that $z \mapsto \rho(|z|^2)$ has compact support in U. Define $f\colon \mathbb{R}^+ \to \mathbb{R}$ by $f(t) = \frac{1}{2\pi}\rho(t)\ln t$ (*cf.* [9, Section 5]). As shown in [9, Section 4], on the complement of E,

$$\pi^*(\Omega) = q^*\Big(\frac{i}{2\pi}\partial\overline{\partial}\ln\big(|z|^2\big)\Big),$$

where Ω is the Fubini-Study form on $\mathbb{CP}^{n-1}$. Therefore, there exists a closed real form $\eta \in \Omega^{1,1}(\widehat{M})^{\mathbb{S}^1}$ with support in U equal to $q^*(i\partial\overline{\partial}(f(|z|^2)))$ on $q^{-1}(U) \smallsetminus E$ and equal to $\pi^*(\Omega)$ near E. Define $\Phi : \widehat{M} \to \mathbb{R}$ with support in U by

$$\Phi(z) = q^*\left(\sum_{j=1}^n \alpha_j |z_j|^2 f'(|z|^2)\right)$$

on $q^{-1}(U) \smallsetminus E$ and by

$$\pi^*\left(\frac{\sum_{j=1}^n \alpha_j |z_j|^2}{2\pi|z|^2}\right) \tag{5.1}$$

near E. Then $\xi_{\widehat{M}} \lrcorner\, \eta = -d\Phi$ by a straightforward calculation on $U \smallsetminus \{0\}$. The restriction of $[\eta] \in H^2(\widehat{M})$ to E is the positive generator of $H^2(E;\mathbb{Z}) \cong \mathbb{Z}$, which is the negative of the Euler class of the normal bundle to E in $\widehat{M}$. Moreover, since η is supported in a tubular neighbourhood of E, the restriction of $[\eta]$ to $\widehat{M} \smallsetminus E$ vanishes. Hence, $[\eta] = -\mathcal{E}$.

Therefore, for all $t \in \mathbb{R}$,

$$[q^*\omega + t\eta] = q^*[\omega] - t\mathcal{E} \text{ and } \xi_{\widehat{M}} \lrcorner\, (q^*\omega + t\eta) = -d(q^*\Psi + t\Phi).$$

It remains to show that $q^*\omega + t\eta \in \Omega^2(\widehat{M})^{\mathbb{S}^1}$ is symplectic, is tamed by the action (everywhere), and is tamed on $q^{-1}(W)$, for all sufficiently small $t > 0$. We will do this by looking at three regions.

(1) On a neighbourhood of the exceptional divisor, $\eta = \pi^*(\Omega)$. Since π is holomorphic and Ω is Kähler, this implies that $\eta(X, \widehat{J}(X)) \geq 0$ for all $X \in T_m\widehat{M}$, with equality exactly if $\pi_*(X) = 0$. Moreover, by the second paragraph of this proof, $q^*\omega(X, \widehat{J}(X)) \geq 0$ with equality impossible if $\pi_* X = 0$ and $X \neq 0$. Therefore, for all $t > 0$,

$$\big(q^*\omega + t\eta\big)(X, \widehat{J}(X)\big) > 0$$

for all non-zero $X \in T_m\widehat{M}$, that is, $q^*\omega + t\eta$ is tamed.

(2) On the complement of $q^{-1}(\operatorname{supp}(\rho)) \subseteq \widehat{M}$, the form η vanishes; hence, by the second paragraph, $q^*\omega + t\eta$ is symplectic, is tamed by the action (everywhere), and is tamed on $q^{-1}(W)$, for all t.

(3) The complement of the open sets considered in (1) and (2) is compact. Since $q^*\omega$ is tamed on this set, $q^*\omega + t\eta$ is also tamed for all sufficiently small t. □

Remark 5.2. In Proposition 5.1 (or Proposition 5.8 below), if Ψ is proper, then we may choose $\widehat{\omega}$ and $\widehat{\Psi}$ so that $\widehat{\Psi}$ is proper. To see this, let V be a neighbourhood with compact closure.

The exact same argument shows that Proposition 5.1 still holds if (M, J) is a complex *orbifold*, as long as we blow up at a smooth point $p \in M$. In order to prove Proposition 6.1, we need to extend that argument to the blow-up of a complex orbifold at an isolated $\mathbb{Z}_2$-singularity. First, we recall the definition of blow-up in this case.

Definition 5.3. Let $\mathbb{Z}_2$ act diagonally on $\mathbb{C}^n$. The **blow-up** of $\mathbb{C}^n/\mathbb{Z}_2$ at $[0]$ is

$$\widehat{\mathbb{C}^n/\mathbb{Z}_2} := (\mathbb{C}^n \smallsetminus \{0\}) \times_{\mathbb{C}^\times} \mathbb{C},$$

where $\mathbb{C}^\times$ acts on $(\mathbb{C}^n \smallsetminus \{0\}) \times \mathbb{C}$ by

$$\lambda \cdot (z_1, \dots, z_n; u) = (\lambda z_1, \dots, \lambda z_n; \lambda^{-2} u).$$

The **blow-down map** $q\colon \widehat{\mathbb{C}^n/\mathbb{Z}_2} \to \mathbb{C}^n/\mathbb{Z}_2$ is given by

$$q([z_1, \dots, z_n; u]) := [\sqrt{u} z_1, \dots, \sqrt{u} z_n].$$

Unfortunately, although the map q is continuous and the pullback q^*f is holomorphic for every holomorphic function $f : \mathbb{C}^n/\mathbb{Z}_2 \to \mathbb{C}$, the blow-down map q is not smooth. For example, $[w] \mapsto |w|^2$ is a smooth function on $\mathbb{C}^n/\mathbb{Z}_2$, but its pullback $[z; u] \mapsto |u||z|^2$ is not a smooth function on $\widehat{\mathbb{C}^n/\mathbb{Z}_2}$. (Note that when $n = 1$, $\widehat{\mathbb{C}/\mathbb{Z}_2} \cong \mathbb{C}$ and $q(u) = [\sqrt{u}\,]$.) However, the exceptional divisor $E := q^{-1}([0])$ is biholomorphic to $\mathbb{CP}^{n-1}$, and q restricts to a biholomorphism from $\widehat{\mathbb{C}^n/\mathbb{Z}_2} \smallsetminus E$ to $\mathbb{C}^n/\mathbb{Z}_2 \smallsetminus \{[0]\}$. Thus, there is a well-defined blow-up of a complex orbifold at any isolated $\mathbb{Z}_2$-singularity.

Since the blow-down map is not smooth, we need to modify the symplectic form locally before pulling it back to the blow-up. We will do this in two stages, using the following criterion for Kähler forms, which we adapted from [9, Lemma 5.3].

Lemma 5.4. *Given $n > 1$ and a smooth function $f\colon \mathbb{R} \to \mathbb{R}$, the form $\omega = \frac{i}{2}\partial\overline{\partial} f\big(|z|^2\big) \in \Omega^{1,1}(\mathbb{C}^n)$ is Kähler at $z_0 \in \mathbb{C}^n$ exactly if*

$$f'\big(|z_0|^2\big) > 0 \quad \textit{and} \quad f'\big(|z_0|^2\big) + |z_0|^2 f''\big(|z_0|^2\big) > 0.$$

Proof. Define $g\colon \mathbb{C}^n \to \mathbb{R}$ by $g(z) = f\big(|z|^2\big)$. The form $\omega = \frac{i}{2}\partial\overline{\partial} g$ is Kähler exactly if the eigenvalues of the Hermitian matrix

$$\left[\frac{\partial^2 g}{\partial z_j\, \partial \overline{z}_k}\right] = \Big[f'\big(|z|^2\big)\delta_{jk} + f''\big(|z|^2\big)\overline{z}_j z_k\Big]$$

are all positive. Since the matrix $[\overline{z}_j z_k]$ is of rank 1 and has eigenvalues 0 and $|z|^2$, the matrix above has eigenvalues $f'\big(|z|^2\big)$ and $f'\big(|z|^2\big) + |z|^2 f''\big(|z|^2\big)$. □

Remark 5.5. Similarly, given a smooth function $f\colon \mathbb{R} \to \mathbb{R}$, the form $\omega = \frac{i}{2}\partial\overline{\partial} f\big(|z|^2\big) \in \Omega^{1,1}(\mathbb{C})$ is Kähler at $z_0 \in \mathbb{C}$ exactly if $f'\big(|z_0|^2\big) + |z_0|^2 f''\big(|z_0|^2\big) > 0$.

The next lemma translates [16, Theorem 1.10] to our setting.

Lemma 5.6. *Let G be a closed subgroup of* $\mathrm{U}(n)$. *Let U be a G-invariant neighbourhood of $0 \in \mathbb{C}^n$, and let $\omega \in \Omega^2(U)^G$ be a tamed symplectic form. Then there exists $\nu \in \Omega^1(U)^G$ with compact support such that $\widetilde{\omega} := \omega - d\nu$ is a tamed symplectic form and is constant in a neighbourhood of* 0. *Moreover, if $\omega \in \Omega^{1,1}(U)^G$ is Kähler, then we may also choose ν so that $\widetilde{\omega}$ is Kähler.*

Proof. We may assume that U is a ball centred at 0. Pick a G-invariant smooth function $\rho\colon \mathbb{C}^n \to [0,1]$ so that ρ has compact support K in U and is 1 on a neighbourhood of 0. Given $\lambda > 0$, define $\rho_\lambda\colon \mathbb{C}^n \to [0,1]$ by $\rho_\lambda(x) = \rho(\lambda x)$; the support of ρ_λ is $\frac{1}{\lambda}K$.

Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Let $\omega' \in \Omega^2(U)^G$ be the unique constant form satisfying $\omega'|_0 = \omega|_0$. By the Poincaré Lemma (and averaging), there exists $\mu = \sum_i f_i dx_i \in \Omega^1(U)^G$ such that $d\mu = \omega - \omega'$. Since $d\mu|_0 = 0$, we may assume that $f_i(0) = \frac{\partial f_i}{\partial x_j}(0) = 0$ for all i, j. Therefore, by the Mean Value Theorem, there exists $C > 0$ so that $\frac{\partial f_i}{\partial x_j}(x) \leq C|x|$ and $f_i(x) \leq C|x|^2$ for all $x \in K$ and all i, j. Since tameness is an open condition, a straightforward calculation in coordinates shows that $\omega - d(\rho_\lambda\, \mu)$ is tamed for sufficiently large λ. Let $\nu = \rho_\lambda\, \mu \in \Omega^1(U)^G$.

If $\omega \in \Omega^{1,1}(U)$, then by the Poincaré Lemma for d and ∂, there exists a smooth G-invariant potential function $h\colon U \to \mathbb{R}$ such that $\omega - \omega' = \frac{i}{2}\partial\overline{\partial}(\rho_\lambda h)$. Since $\omega|_0 = \omega'|_0$, we may assume that $h(0) = 0$, $\partial h|_0 = 0$, and $\overline{\partial} h|_0 = 0$. By a calculation similar to the one above, if we let $\nu := \frac{i}{4}\big(\overline{\partial}(\rho_\lambda h) - \partial(\rho_\lambda h)\big) \in \Omega^1(U)^G$, then $\widetilde{\omega} := \omega - d\nu$ is Kähler for sufficiently large λ. □

Lemma 5.7. *Let G be a closed subgroup of* $\mathrm{U}(n)$. *Let U be a G-invariant neighbourhood of $0 \in \mathbb{C}^n$, and let $\omega \in \Omega^2(U)^G$ be a tamed symplectic form. Given a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ so that the function $z \mapsto f(|z|^2)$ is strictly plurisubharmonic on $\mathbb{C}^n \smallsetminus \{0\}$, there exists $\nu \in \Omega^1(U)^G$ with compact support such that $\widetilde{\omega} := \omega - d\nu$ satisfies the following:*

1. *$\widetilde{\omega}$ is a tamed symplectic form on $U \smallsetminus \{0\}$; and*
2. *$\widetilde{\omega}^{1,1} = \frac{i}{2}\partial\overline{\partial} f\big(|z|^2\big)$, and $\widetilde{\omega}^{2,0}$ and $\widetilde{\omega}^{0,2}$ are constant, on a neighbourhood of $0 \in U$.*

Moreover, if $\omega \in \Omega^{1,1}(U)^G$ is Kähler, then we may also choose ν so that $\widetilde{\omega}$ is Kähler on $U \smallsetminus \{0\}$.

Proof. By Lemma 5.6, we may assume that ω is constant. Write $\omega = \omega^{1,1} + \omega^{2,0} + \omega^{0,2}$, where $\omega^{j,k} \in \Omega^{j,k}(\mathbb{C}^n)$ for all j, k. By a linear change of variables, we may further assume that $\omega^{1,1} = \frac{i}{2}\sum_j dz_j \wedge d\overline{z}_j = \frac{i}{2}\partial\overline{\partial}|z|^2$ on U. Finally, U contains a closed ball B_r of radius $r > 0$ centred at 0.

Assume first that $n > 1$. By assumption, the form $\frac{i}{2}\partial\overline{\partial} f\big(|z|^2\big)$ is Kähler on $\mathbb{C}^n \smallsetminus \{0\}$. By Lemma 5.4, this implies that $f'(t)$ and $tf''(t) + f'(t)$ are positive for all $t > 0$. Hence, there exists a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(t) = tf'(t)$ in a neighbourhood of 0, $h(t) = t$ for all $t > r$, and $h(t)$ and $h'(t)$ are positive for all $t > 0$. Let $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ be the smooth function with $\widetilde{f}'(t) = h(t)/t$ and $\widetilde{f}(0) = f(0)$. Then $\widetilde{f}(t) = f(t)$ near 0, $\widetilde{f}'(t) = 1$ for all $t > r$, and $\widetilde{f}'(t)$ and $t\widetilde{f}''(t) + \widetilde{f}'(t)$ are positive for all $t > 0$. Hence, Lemma 5.4 implies that the form $\frac{i}{2}\partial\overline{\partial}\widetilde{f}\big(|z|^2\big)$ is Kähler on $\mathbb{C}^n \smallsetminus \{0\}$. By Remark 5.5, a similar argument applies if $n = 1$.

Define $g\colon \mathbb{C}^n \to \mathbb{R}$ by $g(z) := |z|^2 - \widetilde{f}(|z|^2)$, and let $\nu := \frac{i}{4}(\overline{\partial} g - \partial g) \in \Omega^1(\mathbb{C}^n)^{\mathrm{U}(n)}$. Define $\widetilde{\omega} := \omega - d\nu \in \Omega^2(U)^G$. Since $\widetilde{f}'(t) = 1$ for all $t > r$, the support of ν is contained in $B_r \subset U$. Since $d\nu = \frac{i}{2}\partial\overline{\partial} g \in \Omega^{1,1}(\mathbb{C}^n)^{\mathrm{U}(n)}$, we have

$$\widetilde{\omega}^{1,1} = \frac{i}{2}\partial\overline{\partial}\widetilde{f}\big(|z|^2\big), \quad \widetilde{\omega}^{0,2} = \omega^{0,2}, \quad \text{and} \quad \widetilde{\omega}^{2,0} = \omega^{2,0}.$$

This shows that $\widetilde{\omega}^{1,1}$ is Kähler on $U \smallsetminus \{0\}$ and proves claim (2). If J is the standard complex structure on $\mathbb{C}^n$, then

$$\omega^{2,0}(X, JX) = \omega^{0,2}(X, JX) = 0$$

for all vectors $X \in TU$. This proves claim (1). The last claim follows immediately. □

We are now ready to extend Proposition 5.1 to the blow-up of a complex orbifold at an isolated $\mathbb{Z}_2$-singularity.

Proposition 5.8. *Let (M, J) be a complex orbifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action (everywhere) and tamed by J on $W \subseteq M$, and a moment map $\Psi\colon M \to \mathbb{R}$. Let $\big(\widehat{M}, \widehat{J}\big)$ be the complex blow-up of M at an isolated $\mathbb{Z}_2$-singularity $p \in M^{\mathbb{S}^1} \cap W$. For sufficiently small $t > 0$, there exist a symplectic form $\widehat{\omega} \in \Omega^2(\widehat{M})^{\mathbb{S}^1}$ tamed by the action (everywhere) and tamed by $\widehat{J}$ on $q^{-1}(W)$, and a moment map $\widehat{\Psi}\colon \widehat{M} \to \mathbb{R}$ such that*

$$[\widehat{\omega}] = q^*[\omega] - \frac{t}{2}\mathcal{E},$$

where $q : \widehat{M} \to M$ the blow-down map and $\mathcal{E}$ is the Poincaré dual of the exceptional divisor $q^{-1}(p)$. Moreover, given a neighbourhood V of p, we may assume that $\widehat{\omega} = q^*\omega$ and $\widehat{\Psi} = q^*\Psi$ on $\widehat{M} \smallsetminus q^{-1}(V)$.

Proof. We may assume that W is open. By Lemma 2.2, there exists an $\mathbb{S}^1$-equivariant biholomorphism from an $\mathbb{S}^1$-invariant neighbourhood of $[0] \in \mathbb{C}^n/\mathbb{Z}_2$ to a neighbourhood $U \subseteq W \cap V$ of p, where $\mathbb{Z}_2$ acts diagonally on $\mathbb{C}^n$ and $\mathbb{S}^1$ acts with weights $(\alpha_1, \dots, \alpha_n)$. We identify these neighbourhoods and also identify $q^{-1}(U)$ with a neighbourhood of the exceptional divisor E in $\widehat{\mathbb{C}^n/\mathbb{Z}_2} := (\mathbb{C}^n \smallsetminus \{0\}) \times_{\mathbb{C}^\times} \mathbb{C}$. (See Definition 5.3.) Define $\pi\colon \widehat{\mathbb{C}^n/\mathbb{Z}_2} \to \mathbb{CP}^{n-1}$ by $\pi([z_1, \dots, z_n; u]) = [z_1, \dots, z_n]$.

The function $z \mapsto |z|^4$ is strictly plurisubharmonic on $\mathbb{C}^n \smallsetminus \{0\}$. Hence, by Lemma 5.7 there exists a closed $\mathbb{S}^1$-invariant form $\widetilde{\omega} \in [\omega] \in H^2(M)$ so that $\widetilde{\omega} = \omega$ on $M \smallsetminus U$; moreover $\widetilde{\omega}$ satisfies the following:

(1) $\widetilde{\omega}$ is tamed on $U \smallsetminus \{p\}$; and
(2) $\widetilde{\omega}^{1,1} = \frac{i}{2}\partial\overline{\partial}(|z|^4)$, and $\widetilde{\omega}^{2,0}$ and $\widetilde{\omega}^{0,2}$ are constant, on a neighbourhood of p.

Let $\widetilde{\Psi} : M \to \mathbb{R}$ be the smooth function satisfying $\xi_M \lrcorner \widetilde{\omega} = -d\widetilde{\Psi}$ so that $\widetilde{\Psi} = \Psi$ on $M \smallsetminus U$. Since the restriction $q\colon \widehat{M} \smallsetminus E \to M \smallsetminus \{p\}$ is $\mathbb{C}^\times$-equivariant and biholomorphic, the form $q^*\widetilde{\omega} \in \Omega^2(\widehat{M} \smallsetminus E)^{\mathbb{S}^1}$ is symplectic, tamed on $q^{-1}(W) \smallsetminus E$, and satisfies $q^*\widetilde{\omega}(\xi_{\widehat{M}}, \widehat{J}(\xi_{\widehat{M}})) > 0$ on $\widehat{M} \smallsetminus \big(\widehat{M}^{\mathbb{S}^1} \cup E\big)$. A straightforward calculation in local coordinates shows that (2) implies that there exists a unique closed form on $\widehat{M}$ that restricts to $q^*\widetilde{\omega}$ on $\widehat{M} \smallsetminus E$; by a slight abuse of notation, we will call this form $q^*\widetilde{\omega} \in \Omega^2(\widehat{M})^{\mathbb{S}^1}$. Moreover,

$$(q^*\widetilde{\omega})^{1,1} = \frac{i}{2}\partial\overline{\partial}\big(|u|^2|z|^4\big)$$

on a neighbourhood of E. Hence, another straightforward calculation implies that for all $m \in E$ and $X \in T_m\widehat{M}$,

$$q^*\widetilde{\omega}(X, \widehat{J}(X)) \geq 0,$$

with equality impossible if $\pi_* X = 0$ and $X \neq 0$. Finally, since $q : \widehat{M} \smallsetminus E \to M \smallsetminus \{p\}$ is smooth and equivariant, $\xi_{\widehat{M}} \lrcorner\, q^*\widetilde{\omega} = -dq^*\widetilde{\Psi}$ on $\widehat{M} \smallsetminus E$. Since $q^*\widetilde{\Psi}$ is continuous and $\widehat{M} \smallsetminus E$ is dense in $\widehat{M}$, this implies that $\xi_{\widehat{M}} \lrcorner\, q^*\widetilde{\omega} = -dq^*\widetilde{\Psi}$ on $\widehat{M}$.

The remainder of the proof is nearly identical to the proof of Proposition 5.1. The main distinction is that here the Euler class to the normal bundle of E in $\widehat{M}$ is twice the (negative) generator of $H^2(E;\mathbb{Z})$. Thus, if we construct η as in the proof of Proposition 5.1, then $[\eta] = -\frac{1}{2}\mathcal{E}$. Hence, $[\widetilde{\omega} + t\eta] = q^*[\omega] - \frac{t}{2}\mathcal{E}$. □

Remark 5.9. Let (M, ω, Φ) be a symplectic toric orbifold with moment polytope Δ, as described in Remark 3.3. Let $\eta_1, \dots, \eta_n \in \mathbb{Z}^n$ be the primitive outward normals to the facets that intersect at a vertex $v \in \Delta$, and assume the natural number associated to each of these n facets is 1. The preimage $\Phi^{-1}(v)$ is smooth exactly if $\eta_1, \dots, \eta_n$ form a basis for $\mathbb{Z}^n$. In contrast, it is an isolated $\mathbb{Z}_2$-singularity exactly if $\frac{1}{2}(\eta_1 + \cdots + \eta_n) \in \mathbb{Z}^n$ and $\eta_1, \dots, \eta_n$ generate a sublattice of $\mathbb{Z}^n$ of index 2. In the former case, for sufficiently small $t > 0$, the moment polytope of the blow-up $(\widehat{M}, \widehat{\omega})$ of M at $\Phi^{-1}(v)$ with $[\widehat{\omega}] = q^*[\omega] - t\mathcal{E}$ is

$$\widehat{\Delta} = \Delta \cap \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \langle \eta_i, x\rangle \leq \sum_{i=1}^n \langle \eta_i, v\rangle - \frac{t}{2\pi} \right\}.$$

The same claim holds in the latter case except now $[\widehat{\omega}] = q^*[\omega] - \frac{t}{2}\mathcal{E}$.

6. Adding fixed points to tame actions

We are now ready to build the specific machinery that the first author needs to construct a non-Hamiltonian symplectic circle action with isolated fixed points on a closed, connected symplectic manifold in [17].

Proposition 6.1. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action (everywhere) and tamed by J near $\Psi^{-1}(0)$, and a proper moment map $\Psi\colon M \to \mathbb{R}$. Assume that the $\mathbb{S}^1$-action on $\Psi^{-1}(0)$ is free except for k orbits with stabilisers $\mathbb{Z}_2$. Then for sufficiently small $\varepsilon > 0$ there exist a complex manifold $\big(\widetilde{M}, \widetilde{J}\big)$ with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\widetilde{\omega} \in \Omega^2\big(\widetilde{M}\big)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map $\widetilde{\Psi}\colon \widetilde{M} \to \mathbb{R}$ so that the following hold:*

1. *$\widetilde{\Psi}^{-1}(-\varepsilon, 0]$ contains exactly k fixed points; each lies in $\widetilde{\Psi}^{-1}(0)$ and has weights $\{-2, 1, \dots, 1\}$.*
2. *There is an $\mathbb{S}^1$-equivariant symplectomorphism from $\widetilde{\Psi}^{-1}(-\infty, -\varepsilon/2)$ to $\Psi^{-1}(-\infty, -\varepsilon/2)$ that induces a biholomorphism from $\widetilde{M}/\!/_s\,\mathbb{S}^1$ to $M/\!/_s\,\mathbb{S}^1$ for all regular $s \in (-\infty, -\varepsilon/2)$.*
3. *$\widetilde{\omega}$ is tamed on $\widetilde{\Psi}^{-1}(-\varepsilon, \varepsilon)$.*

Proof. Let $(M_{\text{cut}}, J_{\text{cut}})$ be the complex orbifold with holomorphic $\mathbb{C}^\times$-action, symplectic form $\omega_{\text{cut}} \in \Omega^2(M_{\text{cut}})^{\mathbb{S}^1}$ tamed by the action, and proper moment map

$\Psi_{\mathrm{cut}}\colon M_{\mathrm{cut}} \to \mathbb{R}$ described in Proposition 4.1. Then $\Psi_{\mathrm{cut}}(M_{\mathrm{cut}}) \subseteq (-\infty, 0]$. Additionally, a neighbourhood of the fixed component $F := \Psi_{\mathrm{cut}}^{-1}(0)$ in M_{cut} is $\mathbb{C}^\times$-equivariantly biholomorphic to the complex line bundle $U_0 \times_{\mathbb{C}^\times} \mathbb{C} \xrightarrow{\pi} U_0/\mathbb{C}^\times$, where $U_0 := \mathbb{C}^\times \cdot \Psi^{-1}(0)$, $\mathbb{C}^\times$ acts diagonally on $U_0 \times \mathbb{C}$, and $\mathbb{C}^\times$ acts on $U_0 \times_{\mathbb{C}^\times} \mathbb{C}$ by $\lambda\cdot[y,z] = [\lambda\cdot y, z] = [y, \lambda^{-1}z]$ for all $\lambda \in \mathbb{C}^\times$ and $[y,z] \in U_0 \times_{\mathbb{C}^\times} \mathbb{C}$. We identify these manifolds. In particular, F is diffeomorphic to $M/\!/_0\, \mathbb{S}^1 = U_0/\mathbb{C}^\times$. Moreover, there exists an $\mathbb{S}^1$-equivariant symplectomorphism from $\Psi^{-1}(-\infty,0)$ to $\Psi_{\mathrm{cut}}^{-1}(-\infty,0)$ that intertwines the moment maps and induces a biholomorphism between the reduced spaces at all regular $s < 0$. Hence, the orbifold M_{cut} is smooth except at isolated $\mathbb{Z}_2$-singularities $p_1, \dots, p_k \in F$. Finally, J_{cut} tames ω_{cut} on a neighbourhood $W \subset U_0 \times_{\mathbb{C}^\times} \mathbb{C}$ of F.

There exists $\varepsilon > 0$ so that $\Psi_{\mathrm{cut}}^{-1}(-\varepsilon, 0] \subseteq W$ and $\Psi_{\mathrm{cut}}^{-1}(-\varepsilon, 0)$ contains no fixed points. Since $\Psi_{\mathrm{cut}}(M_{\mathrm{cut}}) \subseteq (-\infty, 0]$, we may also assume that $\Psi_{\mathrm{cut}}^{-1}(-\infty, -\varepsilon/2) \neq \emptyset$.

Let $Q\colon U_0 \times_{\mathbb{C}^\times} \mathbb{C} \to \mathbb{R}$ be the quadratic form associated to a Hermitian metric on the line bundle $U_0 \times_{\mathbb{C}^\times} \mathbb{C}$, and let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth function which is 1 on a neighbourhood of 0 and such that the support of $x \mapsto \rho(Q(x))$ is contained in $\Psi_{\mathrm{cut}}^{-1}(-\varepsilon/2, 0]$. Define $h\colon U_0 \times_{\mathbb{C}^\times} (\mathbb{C} \smallsetminus \{0\}) \to \mathbb{R}$ by $h(x) := \rho(Q(x)) \ln Q(x)$. Since the closed real form $i\partial\overline{\partial} \ln Q \in \Omega^{1,1}(U_0 \times_{\mathbb{C}^\times} (\mathbb{C} \smallsetminus \{0\}))$ is basic, there exists a closed real form $\Omega \in \Omega^{1,1}(U_0/\mathbb{C}^\times)$ such that $\pi^*(\Omega) = i\partial\overline{\partial} \ln Q$. Therefore, there exists a closed real form $\beta \in \Omega^{1,1}(M_{\mathrm{cut}})$ with $\operatorname{supp}(\beta) \subset \Psi_{\mathrm{cut}}^{-1}(-\varepsilon/2, 0]$ that is equal to $i\partial\overline{\partial} h$ on $U_0 \times_{\mathbb{C}^\times} (\mathbb{C} \smallsetminus \{0\})$ and equal to $\pi^*(\Omega)$ near the zero section $F = U_0/\mathbb{C}^\times$. Similarly, there exists a smooth function $\chi\colon M_{\mathrm{cut}} \to \mathbb{R}$ with $\operatorname{supp}(\chi) \subset \Psi_{\mathrm{cut}}^{-1}(-\varepsilon/2, 0]$ that is equal to $i\xi_{M_{\mathrm{cut}}} \lrcorner \left(\overline{\partial} h - \partial h\right)/2$ on $U_0 \times_{\mathbb{C}^\times} (\mathbb{C} \smallsetminus \{0\})$ and equal to 1 near the zero section $F = U_0/\mathbb{C}^\times$. Then $\xi_{M_{\mathrm{cut}}} \lrcorner \beta = -d\chi$ by a straightforward calculation. Moreover, by Lemma 6.4 below, we may assume that β vanishes (and so χ is 1) on pairwise disjoint neighbourhoods $V_j \subset \Psi_{\mathrm{cut}}^{-1}(-\varepsilon/2, 0]$ of p_j for all j. Since the support of β is compact, there exists $\delta > 0$ such that for all $t' \in (-\delta, \delta)$ the form $\omega_{\mathrm{cut}} + t'\beta$ is symplectic, tamed by the action (everywhere), and tamed on W.

Let $\left(\widetilde{M}, \widetilde{J}\right)$ be the complex blow-up of M_{cut} at $p_1, \dots, p_k$, let $q\colon \widetilde{M} \to M_{\mathrm{cut}}$ be the blow-down map, and let $\mathcal{E}_j$ be the Poincaré dual to the exceptional divisor $E_j := q^{-1}(p_j)$ for all j. By Proposition 5.8, for each j and for sufficiently small $t_j > 0$ there exist a symplectic form $\widetilde{\omega}_j \in \Omega^2\left(\widetilde{M} \smallsetminus \bigcup_{l \neq j} E_l\right)^{\mathbb{S}^1}$ that is tamed by the action (everywhere) and tamed on $q^{-1}(W)$, and a moment map $\widetilde{\Psi}_j : \widetilde{M} \smallsetminus \bigcup_{l \neq j} E_l \to \mathbb{R}$ such that

$$[\widetilde{\omega}_j] = q^*[\omega_{\mathrm{cut}}] - \frac{t_j}{2}\mathcal{E}_j \in H^2\Big(\widetilde{M} \smallsetminus \bigcup_{l \neq j} E_l\Big). \tag{6.1}$$

Moreover, $\widetilde{\omega}_j = q^*\omega_{\mathrm{cut}}$ and $\widetilde{\Psi}_j = q^*\Psi_{\mathrm{cut}}$ on the complement of a closed subset of $q^{-1}(V_j)$. Given $t' \in (-\delta, \delta)$, we can glue these forms together to construct a symplectic form $\widetilde{\omega} \in \Omega^2\big(\widetilde{M}\big)^{\mathbb{S}^1}$ and moment map $\widetilde{\Psi} : \widetilde{M} \to \mathbb{R}$ that are equal to $q^*(\omega_{\mathrm{cut}} + t'\beta)$ and $q^*(\Psi_{\mathrm{cut}} + t'\chi)$, respectively, on $\widetilde{M} \smallsetminus \bigcup_j q^{-1}(V_j)$ and equal to $\widetilde{\omega}_j$ and $\widetilde{\Psi}_j + t'$ on $q^{-1}(V_j)$ for each j. By construction, $\widetilde{\omega}$ is tamed by the action (everywhere) and tamed by J on W. Moreover, $\widetilde{\omega} = q^*\omega_{\mathrm{cut}}$ and $\widetilde{\Psi} = q^*\Psi_{\mathrm{cut}}$ on $\widetilde{M} \smallsetminus (q^*\Psi_{\mathrm{cut}})^{-1}(-\varepsilon/2, 0]$. Since q^* and Ψ_{cut} are proper, and since $[-\varepsilon/2, 0] \supset (-\varepsilon/2, 0]$ is compact, this implies that the moment map $\widetilde{\Psi}$ is proper.

By Lemma 2.2, for each $j \in \{1,\dots,k\}$ there exists an $\mathbb{S}^1$-equivariant biholomorphism from an $\mathbb{S}^1$-invariant neighbourhood of $[0] \in \mathbb{C}^n/\mathbb{Z}_2$ to a neighbourhood of $p_j \in M$, where $\mathbb{Z}_2$ acts diagonally on $\mathbb{C}^n$ and $\mathbb{S}^1$ acts with weights $(-1,0,\dots,0)$. In the coordinates described in Definition 5.3, the corresponding $\mathbb{S}^1$-action on the blow-up $\widehat{\mathbb{C}^n/\mathbb{Z}_2}$ is given by

$$\lambda \cdot [z_1,\dots,z_n;u] = [\lambda^{-1}z_1, z_2,\dots,z_n;u] = [z_1,\lambda z_2,\dots,\lambda z_n;\lambda^{-2}u]$$

for all $\lambda \in \mathbb{S}^1$ and $[z_1,\dots,z_n;u] \in \widehat{\mathbb{C}^n/\mathbb{Z}_2}$. In particular, $[z_1,\dots,z_n;u] \in \widehat{\mathbb{C}^n/\mathbb{Z}_2}$ is fixed exactly if either $z_2 = \cdots = z_n = u = 0$ or $z_1 = 0$. Therefore, $q^{-1}(F) \cap M^{\mathbb{S}^1}$ consists of k isolated fixed points $\{\widetilde{p_j} \in E_j\}_{j=1}^k$, each with weights $\{-2,1,\dots,1\}$, and a component $\widetilde{F}$ that is the blow-up of F at $p_1,\dots,p_k$. By (6.1), the restriction of $[\widetilde{\omega}]$ to E_j is t_j times the positive generator $[\Omega] \in H^2(\mathbb{CP}^{n-1};\mathbb{Z})$; hence,

$$\widetilde{\Psi}\big(\widetilde{F}\big) = \widetilde{\Psi}\big(E_j \cap \widetilde{F}\big) = \widetilde{\Psi}(\widetilde{p}_j) + \frac{t_j}{2\pi}.$$

(Compare with (5.1).) Additionally, $\widetilde{\Psi}(\widetilde{F}) = \Psi_{\text{cut}}(F) + t'\chi(F) = t'$. We can fix $t' \in (0,\delta)$ so that $t_j = 2\pi t'$ is sufficiently small for all j. Then $\widetilde{\Psi}(\widetilde{p}_j) = 0$ for all j and $\widetilde{\Psi}(\widetilde{F}) > 0$. Moreover, if $F' \subset \widetilde{M}$ is any other fixed component, then $q^*\Psi_{\text{cut}}(F') \le -\varepsilon$ and so $\widetilde{\Psi}(F') = q^*\Psi_{\text{cut}}(F') \le -\varepsilon$. This proves claim (1).

Since $\widetilde{\omega} = q^*\omega_{\text{cut}}$ and $\widetilde{\Psi} = q^*\Psi_{\text{cut}}$ on $\widetilde{M} \smallsetminus (q^*\Psi_{\text{cut}})^{-1}(-\varepsilon/2,0]$, the blow-down map q restricts to an $\mathbb{S}^1$-equivariant biholomorphic symplectomorphism from $(q^*\Psi_{\text{cut}})^{-1}(-\infty,-\varepsilon/2)$ to $\Psi_{\text{cut}}^{-1}(-\infty,-\varepsilon/2)$ that intertwines $\widetilde{\Psi}$ and Ψ_{cut}. Moreover,

$$\begin{aligned}\widetilde{\Psi}^{-1}(-\infty,-\varepsilon/2) = {} & (q^*\Psi_{\text{cut}})^{-1}(-\infty,-\varepsilon/2) \\ & \times \amalg\big(\widetilde{\Psi}^{-1}(-\infty,-\varepsilon/2) \cap (q^*\Psi_{\text{cut}})^{-1}(-\varepsilon/2,\infty)\big).\end{aligned} \tag{6.2}$$

Since $\widetilde{\Psi}$ is a proper moment map, the preimage $\widetilde{\Psi}^{-1}(-\infty,-\varepsilon/2)$ is connected. Additionally, we chose ε so that $(q^*\Psi_{\text{cut}})^{-1}(-\infty,-\varepsilon/2) \neq \emptyset$. Hence, (6.2) implies that

$$\widetilde{\Psi}^{-1}(-\infty,-\varepsilon/2) = (q^*\Psi_{\text{cut}})^{-1}(-\infty,-\varepsilon/2).$$

Together with the first paragraph, this proves claim (2).

Finally, since $\widetilde{\Psi} = q^*\Psi_{\text{cut}}$ on $\widetilde{M} \smallsetminus (q^*\Psi_{\text{cut}})^{-1}(-\varepsilon/2,0]$, we have inclusions

$$\widetilde{\Psi}^{-1}(-\varepsilon,\infty) \subset (q^*\Psi_{\text{cut}})^{-1}(-\varepsilon,\infty) = (q^*\Psi_{\text{cut}})^{-1}(-\varepsilon,0] \subset q^{-1}(W).$$

Since $\widetilde{\omega}$ is tamed on $q^{-1}(W)$, this proves claim (3). □

Remark 6.2. Let the circle $\mathbb{S}^1 \times \{1\}^{n-1} \subset (\mathbb{S}^1)^n$ act on a symplectic toric manifold (M,ω,Φ) with moment polytope $\Delta = \Phi(M)$, as described in Remark 3.3, satisfying the assumptions of Proposition 6.1. If M is cut at sufficiently small $\varepsilon > 0$, then each of the k new fixed points in M_{cut} with orbifold isotropy $\mathbb{Z}_2$ corresponds to a vertex on the facet $\Delta \cap \big(\{\varepsilon\} \times \mathbb{R}^{n-1}\big) \subset \Delta_{\text{cut}}$ (see Remark 4.2). If M_{cut} is blown up at these fixed points by $2\pi\varepsilon$, the resulting moment polytope $\widetilde{\Delta}$ agrees with Δ on $\{x \in \mathbb{R}^n \mid x_1 < 0\}$, has k vertices in $\{0\} \times \mathbb{R}^{n-1}$, and a new facet $\widetilde{\Delta} \cap \big(\{\varepsilon\} \times \mathbb{R}^{n-1}\big)$ that is the blow-up of $\Delta \cap \big(\{\varepsilon\} \times \mathbb{R}^{n-1}\big)$ at k vertices (see Remark 5.9).

Remark 6.3. By reversing the action, as described in Remark 4.3, we see that Proposition 6.1 still holds with the following modifications: Replace $(-\varepsilon,0]$ by $[0,\varepsilon)$ and $\{-2,1,\dots,1\}$ by $\{2,-1,\dots,-1\}$ in claim (1), and replace each $(-\infty,-\varepsilon/2)$ by $(\varepsilon/2,\infty)$ in claim (2).

We conclude with the lemma that we used in the proof of Proposition 6.1.

Lemma 6.4. *Let G be a closed subgroup of $\mathrm{U}(n)$. Let U be a G-invariant neighbourhood of $0 \in \mathbb{C}^n$, and let $\beta \in \Omega^2(U)^G$ be closed. Then there exists $\nu \in \Omega^1(U)^G$ with compact support such that $\widetilde{\beta} := \beta - d\nu$ vanishes in a neighbourhood of 0. Finally, if $\beta \in \Omega^{1,1}(U)^G$, then we may also choose ν so that $\widetilde{\beta} \in \Omega^{1,1}(U)^G$.*

Proof. By the Poincaré Lemma (and averaging), after possibly shrinking U there exists $\mu \in \Omega^1(U)^G$ such that $d\mu = \beta$. Pick a G-invariant smooth function $\rho\colon U \to [0,1]$ with compact support that is 1 on a neighbourhood of 0. Let $\nu = \rho\mu \in \Omega^1(U)^G$. If $\beta \in \Omega^{1,1}(U)$, then by the Poincaré Lemma for d and ∂, there exists a smooth G-invariant potential function $h\colon U \to \mathbb{R}$ such that $\beta = \frac{i}{2}\partial\overline{\partial}h$. Pick ρ as before, and let $\nu := \frac{i}{4}\big(\overline{\partial}(\rho h) - \partial(\rho h)\big)$. □

7. Tame birational equivalence

In this section we study how, in our setting, the reduced space changes as we vary the value at which we reduce. Our first goal is to prove the proposition below, which shows that the birational equivalence theorem of Guillemin and Sternberg also holds for tamed symplectic forms [9] (if the fixed points are isolated); see also [3] for the Kähler case. We then use the fact that most of our tools work for arbitrary weights to prove Proposition 7.9, which implies the tame analogue of (a special case of) a theorem of Godinho [6]. This will allow us to analyse manifolds with fixed points that we construct using Proposition 6.1.

Proposition 7.1. *Let (M,J) be a complex manifold such that $\dim_{\mathbb{C}}(M) > 1$ with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map $\Psi\colon M \to \mathbb{R}$. Fix $a_- < 0 < a_+$ so that $\Psi^{-1}(a_-,a_+)$ contains exactly k fixed points; each lies in $\Psi^{-1}(0)$ and has weights $\{-1,1,\dots,1\}$. Then there exists a complex orbifold (X,I) and $\kappa,\eta \in H^2(X;\mathbb{R})$ so that:*

- *for all $s \in (a_-,0)$, the reduced space $M/\!/_s\,\mathbb{S}^1$ is biholomorphically symplectomorphic to (X,I,σ_s), where $\sigma_s \in \Omega^2(X)$ satisfies $[\sigma_s] = \kappa - 2\pi s\,\eta$; and*
- *for all $s \in (0,a_+)$, the reduced space $M/\!/_s\,\mathbb{S}^1$ is biholomorphically symplectomorphic to $(\widehat{X},\widehat{I},\widehat{\sigma}_s)$, where $\widehat{\sigma}_s \in \Omega^2(\widehat{X})$ satisfies*

$$[\widehat{\sigma}_s] = q^*\kappa - 2\pi s\, q^*\eta - 2\pi s\sum_{j=1}^{k}\mathcal{E}_j.$$

Here, $\widehat{X}$ is the blow-up of X at smooth points $x_1,\dots,x_k \in X$, the map $q\colon \widehat{X} \to X$ is the blow-down map, and $\mathcal{E}_j$ is the Poincaré dual of the exceptional divisor $q^{-1}(x_j)$ for all j.

Moreover, under the identifications above, the Euler class of the holomorphic principal $\mathbb{C}^\times$-bundle $\mathbb{C}^\times \cdot \Psi^{-1}(s) \to M/\!/_s\,\mathbb{S}^1$ is η for all $s \in (a_-,0)$, and $q^*\eta + \sum \mathcal{E}_j$ for all $s \in (0,a_+)$.

This statement is particularly nice when there are no fixed points in $\Psi^{-1}(a_-,a_+)$. (Note that the $\dim_{\mathbb{C}}(M) = 1$ case is trivial.)

Corollary 7.2. *Let (M,J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map*

$\Psi\colon M \to \mathbb{R}$. If $\Psi^{-1}(a_-, a_+)$ contains no fixed points, then there exists a complex orbifold (X, I) and $\kappa, \eta \in H^2(X)$, so that the reduced space $M/\!/_s\,\mathbb{S}^1$ is biholomorphically symplectomorphic to (X, I, σ_s), where $\sigma_s \in \Omega^2(X)$ satisfies $[\sigma_s] = \kappa - 2\pi s\,\eta$, for all $s \in (a_-, a_+)$. Moreover, under the identifications above, the Euler class of the holomorphic principal $\mathbb{C}^\times$-bundle $\mathbb{C}^\times \cdot \Psi^{-1}(s) \to M/\!/_s\,\mathbb{S}^1$ is η for all $s \in (a_-, a_+)$.

By Proposition 3.1, the reduced space $M/\!/_s\,\mathbb{S}^1$ is isomorphic to $\big(\mathbb{C}^\times \cdot \Psi^{-1}(s)\big)/\mathbb{C}^\times$ for all regular $s \in \mathbb{R}$. Hence, the first step in proving Proposition 7.1 is analysing $\mathbb{C}^\times \cdot \Psi^{-1}(s)$ for $s \in \mathbb{R}$, or equivalently, $\Psi(\mathbb{C}^\times \cdot x)$ for $x \in M$; we accomplish this in the next two lemmas.

Lemma 7.3. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map $\Psi\colon M \to \mathbb{R}$. Given $a \in \mathbb{R}$, the following hold for all $x \in M$:*

(1) *If $\Psi(x) < a$, then $a \in \Psi(\mathbb{C}^\times \cdot x)$ exactly if*
$$\overline{\{e^t \cdot x \mid t \geq 0\}} \cap \Psi^{-1}[\Psi(x), a] \cap M^{\mathbb{S}^1} = \emptyset.$$
(2) *If $\Psi(x) > a$, then $a \in \Psi(\mathbb{C}^\times \cdot x)$ exactly if*
$$\overline{\{e^t \cdot x \mid t \leq 0\}} \cap \Psi^{-1}[a, \Psi(x)] \cap M^{\mathbb{S}^1} = \emptyset.$$

Proof. By symmetry, it suffices to consider $x \in M$ with $\Psi(x) < a$. If $x \in M^{\mathbb{S}^1}$, then – since the action is holomorphic – x is also fixed by $\mathbb{C}^\times$. Hence, $a \notin \Psi(\mathbb{C}^\times \cdot x)$ and the equality displayed in claim (1) does not hold. Thus, we may assume that $x \notin M^{\mathbb{S}^1}$.

Assume first that
$$p \in \overline{\{e^t \cdot x \mid t \geq 0\}} \cap \Psi^{-1}[\Psi(x), a] \cap M^{\mathbb{S}^1} \neq \emptyset.$$
Since $p \in M^{\mathbb{S}^1}$, $e^t \cdot x \neq p$ for any $t \in \mathbb{R}$. Hence, since $p \in \overline{\{e^t \cdot x \mid t \geq 0\}}$, there exists a sequence of $t_i \in \mathbb{R}$ with
$$\lim_{i\to\infty} t_i = \infty \text{ and } \lim_{i\to\infty} e^{t_i} \cdot x = p.$$
Moreover, by Lemma 1.1, the map $t \mapsto \Psi(e^t \cdot x)$ is strictly increasing, and so $\Psi(e^t \cdot x) < \Psi(p) \leq a$ for all $t \in \mathbb{R}$. Therefore, since Ψ is $\mathbb{S}^1$-invariant, $a \notin \Psi(\mathbb{C}^\times \cdot x)$.

So assume instead that
$$\overline{\{e^t \cdot x \mid t \geq 0\}} \cap \Psi^{-1}[\Psi(x), a] \cap M^{\mathbb{S}^1} = \emptyset.$$
If $a \notin \Psi(\mathbb{C}^\times \cdot x)$, then since the map $t \mapsto \Psi(e^t \cdot x)$ is increasing, $\Psi(e^t \cdot x) \in [\Psi(x), a)$ for all $t \geq 0$. Hence, since Ψ is proper, the set $K := \overline{\{e^t \cdot x \mid t \geq 0\}}$ is a compact set that contains no fixed points. Thus, Lemma 1.1 implies that there exists $\varepsilon > 0$ so that $\frac{d}{dt}\Big|_{t=0} \Psi(e^t \cdot y) > \varepsilon$ for all $y \in K$. Since $e^s \cdot (e^t \cdot y) = e^{st} \cdot y$, this implies that $\frac{d}{dt}\Psi(e^t \cdot x) > \varepsilon$ for all $t \geq 0$. Thus, $\Psi(e^t \cdot x) > \Psi(x) + t\varepsilon$ for all $t > 0$. Since this gives a contradiction, $a \in \Psi(\mathbb{C}^\times \cdot x)$. □

Lemma 7.4. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map $\Psi\colon M \to \mathbb{R}$. Assume that $M^{\mathbb{S}^1} \cap \Psi^{-1}(a_-, a_+) = \{p_1, \ldots, p_k\} \subset \Psi^{-1}(0)$. There exists a $\mathbb{C}^\times$-invariant neighbourhood $W \subset M$ of $\{p_1, \ldots, p_k\}$ which is $\mathbb{C}^\times$-equivariantly biholomorphic to a neighbourhood of $\coprod_{j=1}^k \{0\}$ in $\coprod_{j=1}^k \mathbb{C}^{n_j^-} \times \mathbb{C}^{n_j^+}$, where $\mathbb{C}^\times$*

acts linearly on each $\mathbb{C}^{n_j^-}$ *(respectively,* $\mathbb{C}^{n_j^+}$*) with negative (respectively, positive) weights. If we identify these neighbourhoods, then*

$$\mathbb{C}^\times \cdot \Psi^{-1}(s) = \begin{cases} U_- := \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+) \smallsetminus \left(\coprod_{j=1}^k \left(\{0\} \times \mathbb{C}^{n_j^+}\right)\right) & \text{if } s \in (a_-, 0), \\ U_+ := \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+) \smallsetminus \left(\coprod_{j=1}^k \left(\mathbb{C}^{n_j^-} \times \{0\}\right)\right) & \text{if } s \in (0, a_+), \\ (U_- \cap U_+) \cup \{p_1, \dots, p_k\} & \text{if } s = 0, \end{cases}$$

where $\coprod_{j=1}^k \left(\{0\} \times \mathbb{C}^{n_j^+}\right) \subset W$ *and* $\coprod_{j=1}^k \left(\mathbb{C}^{n_j^-} \times \{0\}\right) \subset W$.

Proof. The first claim is an immediate consequence of Proposition 2.1.

For each j, the connected component W_j of W containing p_j is open and $\mathbb{C}^\times$-invariant. Hence, for any $x \in M$, if $p_j \in \overline{\{e^t \cdot x \mid t \in \mathbb{R}\}}$, then $x \in W_j$ and so

$$p_j \in \overline{\{e^t \cdot x \mid t \geq 0\}} \text{ exactly if } x \in \mathbb{C}^{n_j^-} \times \{0\} \subset W_j.$$

Since $p_1, \dots, p_k$ are the only fixed points in $\Psi^{-1}(a_-, a_+)$, this implies that

$$\overline{\{e^t \cdot x \mid t \geq 0\}} \cap \Psi^{-1}(a_-, a_+) \cap M^{\mathbb{S}^1} \neq \emptyset \text{ exactly if } x \in \coprod_{j=1}^k \left(\mathbb{C}^{n_j^-} \times \{0\}\right) \subset W.$$

By a similar argument,

$$\overline{\{e^t \cdot x \mid t \leq 0\}} \cap \Psi^{-1}(a_-, a_+) \cap M^{\mathbb{S}^1} \neq \emptyset \text{ exactly if } x \in \coprod_{j=1}^k \left(\{0\} \times \mathbb{C}^{n_j^+}\right) \subset W.$$

Therefore, since $M^{\mathbb{S}^1} \cap \Psi^{-1}(a_-, a_+) = \{p_1, \dots, p_k\} \subset \Psi^{-1}(0)$, the claim follows from Lemma 7.3. □

If we remove the claims about the cohomology classes $[\sigma_s]$ and $[\widehat{\sigma}_s]$ from Proposition 7.1, then the revised statement follows easily from Proposition 3.1 and Lemma 7.4 above. (For details, see the proof of Proposition 7.1 later in this section.) Since these claims are formally identical to statements in [9, Theorem 13.2], it may seem that we should complete our proof by simply quoting their results. Unfortunately, the blow-down maps in our paper and their paper are not identical. Thus for completeness we include a proof of these claims, which relies on the next two lemmas.

Lemma 7.5. *Let* $\mathbb{S}^1$ *act on a symplectic manifold* (M, ω) *with moment map* $\Psi\colon M \to \mathbb{R}$. *Assume that* $M^{\mathbb{S}^1} = \{p_1, \dots, p_k\} \subset \Psi^{-1}(0)$. *Then there exists* $\overline{\kappa} \in H^2\big(M/\mathbb{S}^1\big)$ *so that*

$$[\omega_s] = i_s^*(\overline{\kappa}) - 2\pi s\, \eta_s$$

for every regular value $s \in \mathbb{R}$, *where* $\omega_s \in \Omega^2(M/\!/_s\, \mathbb{S}^1)$ *is the reduced symplectic form,* $i_s\colon M/\!/_s\, \mathbb{S}^1 \to M/\mathbb{S}^1$ *is the natural inclusion, and* $\eta_s \in H^2(M/\!/_s\, \mathbb{S}^1)$ *is the Euler class of the circle bundle* $\Psi^{-1}(s) \to M/\!/_s\, \mathbb{S}^1$.

Proof. Let $\theta \in \Omega^1\big(M \smallsetminus M^{\mathbb{S}^1}\big)^{\mathbb{S}^1}$ be a connection one-form and $\Omega \in \Omega^2\big(\big(M \smallsetminus M^{\mathbb{S}^1}\big)/\mathbb{S}^1\big)$ the associated curvature form. Since $\omega - d(\Psi\theta) \in \Omega^2\big(M \smallsetminus M^{\mathbb{S}^1}\big)^{\mathbb{S}^1}$ is closed and basic, it is the pull-back of a closed form $\overline{\omega} \in \Omega^2\big(\big(M \smallsetminus M^{\mathbb{S}^1}\big)/\mathbb{S}^1\big)$. Moreover, $i_s^*([\overline{\omega}]) = [\omega_s] + s[\,\Omega|_{M/\!/_s\, \mathbb{S}^1}] = [\omega_s] + 2\pi s\, \eta_s$ for all regular $s \in \mathbb{R}$.

By Lemma 6.4 and the Bochner linearisation theorem (*cf.* Lemma 2.2), there exists $\nu \in \Omega^1(M)^{\mathbb{S}^1}$ such that $\omega' := \omega - d\nu$ vanishes on a neighbourhood of

$\{p_1,\dots,p_k\}$; let $\Psi' = \Psi - \xi_M \lrcorner \nu$. Since $\xi_M \lrcorner \omega' = -d\Psi'$ and $\Psi'(p_j) = \Psi(p_j) = 0$ for all j, the function Ψ' also vanishes on a neighbourhood of $\{p_1,\dots,p_k\}$. Therefore, $\omega' - d(\Psi'\theta) \in \Omega^2(M)^{\mathbb{S}^1}$ is well-defined. Hence, it is the pull-back of a closed form $\overline{\omega}' \in \Omega^2(M/\mathbb{S}^1)$; let $\overline{\kappa} = [\overline{\omega}'] \in H^2(M/\mathbb{S}^1)$. Since $\omega' - d(\Psi'\theta)$ and $\omega - d(\Psi\theta)$ differ by the differential of the basic form $\nu - (\xi_M \lrcorner \nu)\theta \in \Omega^1\big(M \smallsetminus M^{\mathbb{S}^1}\big)^{\mathbb{S}^1}$, $i_s^*(\overline{\kappa}) = i_s^*([\overline{\omega}])$ for all regular $s \in \mathbb{R}$. □

Remark 7.6. Alternately, we could replace the second paragraph of the proof above with the following more sophisticated argument, which works for arbitrary $M^{\mathbb{S}^1} \subset \Psi^{-1}(0)$:

In the de Rham model of equivariant cohomology, $\omega + \Psi$ represents an equivariant cohomology class on M; moreover, $[\omega + \Psi]$ maps to $[\overline{\omega}]$ under the natural isomorphism $H^*_{\mathbb{S}^1}\big(M \smallsetminus M^{\mathbb{S}^1}\big) \xrightarrow{\cong} H^*\big(\big(M \smallsetminus M^{\mathbb{S}^1}\big)/\mathbb{S}^1\big)$ [1], [5]. Since $\Psi\big(M^{\mathbb{S}^1}\big) = 0$, the restriction $[\omega + \Psi]\big|_{M^{\mathbb{S}^1}} \in H^2_{\mathbb{S}^1}\big(M^{\mathbb{S}^1}\big)$ is in the image of the natural inclusion $H^*\big(M^{\mathbb{S}^1}\big) \hookrightarrow H^*_{\mathbb{S}^1}\big(M^{\mathbb{S}^1}\big)$. Hence, by the Leray spectral sequence for the natural map $M \times_{\mathbb{S}^1} E\mathbb{S}^1 \to M/\mathbb{S}^1$, there exists a class $\overline{\kappa} \in H^*(M/\mathbb{S}^1)$ that maps to $[\omega + \Psi]$ under the pull-back $H^*(M/\mathbb{S}^1) \to H^*_{\mathbb{S}^1}(M)$. By construction, $i_s^*(\overline{\kappa}) = i_s^*([\overline{\omega}])$ for all regular $s \in \mathbb{R}$.

Lemma 7.7. *Let (M, J) be a complex manifold with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map $\Psi\colon M \to \mathbb{R}$. Assume that $M^{\mathbb{S}^1} \cap \Psi^{-1}(a_-, a_+) = \{p_1,\dots,p_k\} \subset \Psi^{-1}(0)$ for some $a_- < 0 < a_+$. Then for all $x \in \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+)$ and $\tau \in [0,1]$, there exists a unique $G(x,\tau) \in M$ such that*

$$G(x,\tau) \in \overline{\{e^t \cdot x \mid t \in \mathbb{R}\}} \text{ and } \Psi(G(x,\tau)) = \tau\,\Psi(x);$$

moreover, $G\colon \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+) \times [0,1] \to M$ is a continuous $\mathbb{S}^1$-equivariant map.

Proof. Define

$$\mathcal{U} := \{(x,s) \in \big(M \smallsetminus M^{\mathbb{S}^1}\big) \times \mathbb{R} \mid s \in \Psi(\mathbb{C}^\times \cdot x)\}.$$

By Lemma 3.4, for all $(x,s) \in \mathcal{U}$, there exists a unique $f(x,s) \in \mathbb{R}$ such that

$$\Psi(e^{f(x,s)} \cdot x) = s;$$

moreover, $\mathcal{U}$ is open, and $f\colon \mathcal{U} \to \mathbb{R}$ is a smooth $\mathbb{S}^1$-invariant function. Moreover, Lemma 7.4 implies that, for $x \in \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+) \smallsetminus \{p_1,\dots,p_k\}$ and $s \in (a_-, a_+)$,

$$(x,s) \notin \mathcal{U} \Leftrightarrow \Big(x \in \coprod_{j=1}^k \big(\mathbb{C}^{n_j^-} \times \{0\}\big) \text{ and } s \geq 0\Big) \text{ or } \Big(x \in \coprod_{j=1}^k \big(\{0\} \times \mathbb{C}^{n_j^+}\big) \text{ and } s \leq 0\Big).$$

Therefore, for all $x \in \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+)$ and $\tau \in [0,1]$, there exists a unique $G(x,\tau) \in M$ such that $G(x,\tau) \in \overline{\{e^t \cdot x \mid t \in \mathbb{R}\}}$ and $\Psi(G(x,\tau)) = \tau\,\Psi(x)$:

$$G(x,\tau) = \begin{cases} p_j & \text{if } x = p_j \text{ for some } j, \\ p_j & \text{if } x \in \big(\big(\mathbb{C}^{n_j^-} \times \{0\}\big) \cup \big(\{0\} \times \mathbb{C}^{n_j^+}\big)\big) \\ & \text{for some } j \text{ and } \tau = 0, \\ e^{f(x,\tau\,\Psi(x))} \cdot x & \text{otherwise.} \end{cases}$$

Clearly, $G\colon \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+) \times [0,1] \to M$ is an $\mathbb{S}^1$-equivariant map that is continuous on the complement of $G^{-1}(\{p_1,\dots,p_k\})$. Let $U \subseteq \Psi^{-1}(a_-, a_+)$ be an

$\mathbb{S}^1$-invariant neighbourhood of p_j. By Corollary 2.7, there exists $\delta > 0$ and an $\mathbb{S}^1$-invariant open neighbourhood $V \subseteq U$ of p_j such that $\mathbb{C}^\times \cdot V \cap \Psi^{-1}(-\delta, \delta) \subset V$. Then

$$\big\{(x,\tau) \in (\mathbb{C}^\times \cdot V) \times [0,1] \,\big|\, |\tau\,\Psi(x)| < \delta\big\}$$

is an open subset of $G^{-1}(U)$ that contains $G^{-1}(p_j)$. Therefore, G is continuous. □

This has an immediate corollary.

Corollary 7.8. *Assume that we are in the situation of Lemma 7.7. Fix a non-zero $s \in (a_-, a_+)$, and let $U_s := \mathbb{C}^\times \cdot \Psi^{-1}(s)$. For every $[x] \in U_s/\mathbb{C}^\times$, there exists a unique $\mathbb{S}^1$-orbit*

$$h_s([x]) \in \Psi^{-1}(0)/\mathbb{S}^1 \cap \big(\overline{\mathbb{C}^\times \cdot x}\big)/\mathbb{S}^1;$$

moreover, $h_s \colon U_s/\mathbb{C}^\times \to \Psi^{-1}(a_-, a_+)/\mathbb{S}^1$ is a continuous map. Additionally, if $j_s \colon M/\!/_s\,\mathbb{S}^1 \to U_s/\mathbb{C}^\times$ is induced by the inclusion map, then the composition $h_s \circ j_s \colon M/\!/_s\,\mathbb{S}^1 \to \Psi^{-1}(a_-, a_+)/\mathbb{S}^1$ is isotopic to the inclusion map.

We now prove our main proposition.

Proof of Proposition 7.1. We may assume without loss of generality that $0 \in (a_-, a_+)$ (this is by hypothesis if there are fixed points). By Lemma 7.4, there exists a $\mathbb{C}^\times$-invariant neighbourhood W of $\Psi^{-1}(0) \cap M^{\mathbb{S}^1}$ which is $\mathbb{C}^\times$-equivariantly biholomorphic to a neighbourhood of $\coprod_{j=1}^k \{0\}$ in $\coprod_{j=1}^k \mathbb{C}^n$, where in each case $\mathbb{C}^\times$ acts on $\mathbb{C}^n$ by

$$\lambda \cdot (z_1, \dots, z_n) = (\lambda^{-1} z_1, \lambda z_2, \dots, \lambda z_n) \tag{7.1}$$

for all $\lambda \in \mathbb{C}^\times$ and $z \in \mathbb{C}^n$. Moreover, if we identify these neighbourhoods, then

$$\mathbb{C}^\times \cdot \Psi^{-1}(s) = \begin{cases} U_- := \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+) \smallsetminus \Big(\coprod_{j=1}^k \big(\{0\} \times \mathbb{C}^{n-1}\big)\Big) & \text{if } s \in (a_-, 0), \\ U_+ := \mathbb{C}^\times \cdot \Psi^{-1}(a_-, a_+) \smallsetminus \Big(\coprod_{j=1}^k \big(\mathbb{C} \times \{0\}\big)\Big) & \text{if } s \in (0, a_+). \end{cases}$$

By Proposition 3.1, the quotients $U_\pm/\mathbb{C}^\times$ are naturally complex orbifolds, and $U_\pm$ is a holomorphic $\mathbb{C}^\times$-bundle over $U_\pm/\mathbb{C}^\times$ with Euler class $\eta_\pm \in H^2(U_\pm/\mathbb{C}^\times)$. Moreover, the inclusion $\Psi^{-1}(s_\pm) \hookrightarrow U_\pm$ induces a biholomorphism $j_{s_\pm} \colon M/\!/_{s_\pm}\,\mathbb{S}^1 \to U_\pm/\mathbb{C}^\times$ for each $s_- \in (a_-, 0)$ and $s_+ \in (0, a_+)$. Thus, the Euler class of the $\mathbb{S}^1$-bundle $\Psi^{-1}(s_\pm) \to M/\!/_{s_\pm}\,\mathbb{S}^1$ is $j^*_{s_\pm}(\eta_\pm)$ for each $s_- \in (a_-, 0)$ and $s_+ \in (0, a_+)$. By Corollary 7.8, there exist continuous maps $h_\pm \colon U_\pm/\mathbb{C}^\times \to \Psi^{-1}(a_-, a_+)/\mathbb{S}^1$ such that $h_\pm([x]) \in \Psi^{-1}(0)/\mathbb{S}^1 \cap \big(\overline{\mathbb{C}^\times \cdot x}\big)/\mathbb{S}^1$ for all $[x] \in U_\pm/\mathbb{C}^\times$. Moreover, the composition $h_\pm \circ j_{s_\pm} \colon M/\!/_{s_\pm}\,\mathbb{S}^1 \to \Psi^{-1}(a_-, a_+)/\mathbb{S}^1$ is isotopic to the natural inclusion for each $s_- \in (a_-, 0)$ and $s_+ \in (0, a_+)$. Therefore, applying Lemma 7.5 to $\Psi^{-1}(a_-, a_+)$, there exists $\overline{\kappa} \in H^2(\Psi^{-1}(a_-, a_+)/\mathbb{S}^1)$ so that

$$[\omega_{s_\pm}] = j^*_{s_\pm}\big(h^*_\pm(\overline{\kappa}) - 2\pi s_\pm\,\eta_\pm\big)$$

for each $s_- \in (a_-, 0)$ and $s_+ \in (0, a_+)$, where $\omega_{s_\pm} \in \Omega^2(M/\!/_{s_\pm}\,\mathbb{S}^1)$ is the reduced symplectic form.

The quotient $\mathbb{C} \times_{\mathbb{C}^\times} (\mathbb{C}^{n-1} \smallsetminus \{0\})$ is the blow-up of $\mathbb{C}^{n-1} \cong (\mathbb{C} \smallsetminus \{0\}) \times_{\mathbb{C}^\times} \mathbb{C}^{n-1}$ at 0; the blow-down map sends $[u; z_1, \dots, z_{n-1}]$ to $[1; uz_1, \dots, uz_{n-1}]$. Therefore, $U_+/\mathbb{C}^\times$ is the blow-up of $U_-/\mathbb{C}^\times$ at the smooth points

$$\{x_1, \dots, x_k\} := \coprod_j \big((\mathbb{C} \smallsetminus \{0\}) \times \{0\}\big)/\mathbb{C}^\times \subset W/\mathbb{C}^\times;$$

moreover, $h_- \circ q = h_+$, where $q\colon U_+/\mathbb{C}^\times \to U_-/\mathbb{C}^\times$ is the blow-down map. (To see this, note that $h_\pm([x]) = [0] \in \mathbb{C}^n/\mathbb{S}^1$ for all non-zero $x \in (\mathbb{C}\times\{0\})\cup(\{0\}\times\mathbb{C}^{n-1}) \subset \mathbb{C}^n \subset W$. Hence, $h_+^*(\overline{\kappa}) = q^*(h_-^*(\overline{\kappa}))$.)

Finally, we complete the proof by using an argument from [9] to show that

$$\eta_+ = q^*(\eta_-) + \sum_j \mathcal{E}_j, \tag{7.2}$$

where $\mathcal{E}_j$ is the Poincaré dual of the exceptional divisor $E_j := q^{-1}(x_j)$ for each j. Define a section of the $\mathbb{C}^\times$-bundle

$$\big(\mathbb{C}\smallsetminus\{0\}\big)\times\big(\mathbb{C}^{n-1}\smallsetminus\{0\}\big) \to \big(\mathbb{C}\smallsetminus\{0\}\big)\times_{\mathbb{C}^\times}\big(\mathbb{C}^{n-1}\smallsetminus\{0\}\big)$$

by $\sigma[u;z_1,\dots,z_{n-1}] = (1;uz_1,\dots,uz_{n-1})$. Let L be the $\mathbb{C}^\times$-bundle over $U_+/\mathbb{C}^\times$ constructed by using σ to glue together the $\mathbb{C}^\times$-bundle $W\cap U_+ \to (W\cap U_+)/\mathbb{C}^\times$ and the trivial $\mathbb{C}^\times$-bundle over $U_+/\mathbb{C}^\times \smallsetminus \coprod_j E_j$. On the one hand, σ does not extend to a section of the $\mathbb{C}^\times$-bundle

$$\mathbb{C}\times\big(\mathbb{C}^{n-1}\smallsetminus\{0\}\big) \to \mathbb{C}\times_{\mathbb{C}^\times}\big(\mathbb{C}^{n-1}\smallsetminus\{0\}\big)$$

associated to $W\cap U_+$. Instead, it extends to a holomorphic section of the associated $\mathbb{C}$-bundle that is transverse to the zero section, and vanishes exactly on the exceptional divisor. Hence, the Euler class of L is $\sum_j \mathcal{E}_j$. On the other hand, σ does extend to a section of the $\mathbb{C}^\times$-bundle

$$\big(\mathbb{C}\smallsetminus\{0\}\big)\times\mathbb{C}^{n-1} \to \big(\mathbb{C}\smallsetminus\{0\}\big)\times_{\mathbb{C}^\times}\mathbb{C}^{n-1}$$

associated to $W\cap U_-$. Since the Euler class of the tensor product of line bundles is the sum of their Euler classes, (7.2) follows immediately. □

Finally, we generalise Proposition 7.1 so that it applies to manifolds constructed using Proposition 6.1.

Proposition 7.9. *Let (M,J) be a complex manifold such that* $\dim_{\mathbb{C}} M > 1$ *with a holomorphic $\mathbb{C}^\times$-action, a symplectic form $\omega \in \Omega^2(M)^{\mathbb{S}^1}$ tamed by the action, and a proper moment map $\Psi\colon M \to \mathbb{R}$. Fix $a_- < 0 < a_+$ so that $\Psi^{-1}(a_-,a_+)$ contains exactly k fixed points; each lies in $\Psi^{-1}(0)$ and has weights $\{-2,1,\dots,1\}$. Then there exists a complex orbifold (X,I) and classes $\kappa,\eta \in H^2(X;\mathbb{R})$ so that*

- *for all $s \in (a_-,0)$, the reduced space $M/\!/_s\,\mathbb{S}^1$ is biholomorphically symplectomorphic to (X,I,σ_s), where $\sigma_s \in \Omega^2(X)$ satisfies $[\sigma_s] = \kappa - 2\pi s\,\eta$; and*
- *for all $s \in (0,a_+)$, the reduced space $M/\!/_s\,\mathbb{S}^1$ is biholomorphically symplectomorphic to $(\widehat{X},\widehat{I},\widehat{\sigma}_s)$, where $\widehat{\sigma}_s \in \Omega^2(\widehat{X})$ satisfies*

$$[\widehat{\sigma}_s] = q^*\kappa - 2\pi s\, q^*\eta - \pi s\sum_{j=1}^{k}\mathcal{E}_j.$$

Here, $\widehat{X}$ is the blow-up of X at isolated $\mathbb{Z}_2$-singularities $x_1,\dots,x_k \in X$, the map $q\colon \widehat{X} \to X$ is the blow-down map, and $\mathcal{E}_j$ is the Poincaré dual of the exceptional divisor $q^{-1}(x_j)$ for all j.

Moreover, under the identifications above, the Euler class of the holomorphic principal $\mathbb{C}^\times$-bundle $\mathbb{C}^\times\cdot\Psi^{-1}(s) \to M/\!/_s\,\mathbb{S}^1$ is η for all $s \in (a_-,0)$, and $q^*\eta + \sum\mathcal{E}_j/2$ for all $s \in (0,a_+)$.

Proof. The first paragraph of the proof of Proposition 7.1 applies without change, except that (7.1) should be replaced by

$$\lambda \cdot (z_1, \dots, z_n) = (\lambda^{-2} z_1, \lambda z_2, \dots, \lambda z_n).$$

By Definition 5.3, $U_+/\mathbb{C}^\times$ is the blow-up of $U_-/\mathbb{C}^\times$ at the isolated $\mathbb{Z}_2$-singularities

$$\{x_1, \dots, x_k\} := \coprod_j \big((\mathbb{C} \smallsetminus \{0\}) \times \{0\}\big)/\mathbb{C}^\times;$$

moreover, $h_- \circ q = h_+$, where $q\colon U_+/\mathbb{C}^\times \to U_-/\mathbb{C}^\times$ is the blow-down map.

In analogy with the proof of Proposition 7.1, this reduces the argument to showing that

$$2\eta_+ = 2q^*(\eta_-) + \sum_j \mathcal{E}_j, \tag{7.3}$$

where $\mathcal{E}_j$ is the Poincaré dual of the exceptional divisor $E_j := q^{-1}(x_j)$. Define a section of the $\mathbb{C}^\times$-bundle

$$\big(\mathbb{C} \smallsetminus \{0\}\big) \times_{\mathbb{Z}_2} \big(\mathbb{C}^{n-1} \smallsetminus \{0\}\big) \to \big(\mathbb{C} \smallsetminus \{0\}\big) \times_{\mathbb{C}^\times} \big(\mathbb{C}^{n-1} \smallsetminus \{0\}\big)$$

by $\sigma[u; z_1, \dots, z_{n-1}] = [1; \sqrt{u} z_1, \dots, \sqrt{u} z_{n-1}]$, and construct a $\mathbb{C}^\times$-bundle L over $U_+/\mathbb{C}^\times$ by using σ to glue together the $\mathbb{C}^\times$-bundle $(W \cap U_+)/\mathbb{Z}_2 \to (W \cap U_+)/\mathbb{C}^\times$ and the trivial bundle over $U_+/\mathbb{C}^\times \smallsetminus \coprod_j E_j$. The Euler class of L is $\sum_j \mathcal{E}_j$ because σ extends to a holomorphic section of the $\mathbb{C}$-bundle associated to the $\mathbb{C}^\times$-bundle

$$\mathbb{C} \times_{\mathbb{Z}_2} \big(\mathbb{C}^{n-1} \smallsetminus \{0\}\big) \to \mathbb{C} \times_{\mathbb{C}^\times} \big(\mathbb{C}^{n-1} \smallsetminus \{0\}\big)$$

that is transverse to the zero section and vanishes exactly on the exceptional divisor. Since σ extends to a section of the $\mathbb{C}^\times$-bundle

$$\big(\mathbb{C} \smallsetminus \{0\}\big) \times_{\mathbb{Z}_2} \mathbb{C}^{n-1} \to \big(\mathbb{C} \smallsetminus \{0\}\big) \times_{\mathbb{C}^\times} \mathbb{C}^{n-1},$$

and since the Euler class of $U_\pm/\mathbb{Z}^2 \to U_\pm/\mathbb{C}^\times$ is twice the Euler class of $U_\pm \to U_\pm/\mathbb{C}^\times$, this proves (7.3). ☐

Remark 7.10. Let the circle $\mathbb{S}^1 \times \{1\}^{n-1} \subset (\mathbb{S}^1)^n$ act on a symplectic toric manifold (M, ω, Φ) with moment polytope Δ, as described in Remark 3.3, satisfying the assumptions of Proposition 7.1 (or Proposition 7.9). It is straightforward to see that, as we vary $s_- \in (a_-, 0)$, the moment polytopes $\Delta \cap \big(\{s_-\} \times \mathbb{R}^{n-1}\big)$ of the reduced spaces $M/\!/_{s_-} \mathbb{S}^1$ all have the same facets, but the position of these facets varies linearly in s_-. The identical process occurs as we vary the moment polytopes of the reduced spaces over $s_+ \in (0, a_+)$. However, the latter polytopes have k extra facets, corresponding to the blow-up at the k fixed points. (See Remarks 3.3 and 5.9.)

Remark 7.11. By reversing the action, as described in Remark 4.3, we see that Proposition 7.1 (respectively, Proposition 7.9) still holds with the following modifications: Replace $\{-1, 1, \dots, 1\}$ by $\{1, -1, \dots, -1\}$ (respectively, replace $\{-2, 1, \dots, 1\}$ by $\{2, -1, \dots, -1\}$); replace each $(0, a_+)$ by $(a_-, 0)$ and each $(a_-, 0)$ by $(0, a_+)$; and replace each $\sum_j \mathcal{E}_j$ by $-\sum_j \mathcal{E}_j$.

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