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THE DELTA CONJECTURE AT $q = 1$

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ABSTRACT. We use a weight-preserving, sign-reversing involution to find a combinatorial expansion of $\Delta_{e_k} e_n$ at $q = 1$ in terms of the elementary symmetric function basis. We then use a weight-preserving bijection to prove the Delta Conjecture of Haglund, Remmel, and Wilson at $q = 1$. The method of proof provides a variety of structures which can compute the inner product of $\Delta_{e_k} e_n|_{q=1}$ with any symmetric function.

1. INTRODUCTION

In recent developments, modified Macdonald polynomials and their specializations have had an increasing role in areas such as knot theory, Hilbert schemes, Cherednik algebras, the Elliptic Hall algebra, and algebraic combinatorics. For example, after noticing that coefficients of the superpolynomials for $(n, n+1)$ torus knots can be computed from the occurrence of a hook Schur function in what we later define as $DH_n[X; q, t] = \nabla e_n$ [8], [14], Gorsky and Neguţ showed that the superpolynomial of (m, n) torus knots is given by the coefficients of hook Schur functions in a symmetric function $Q_{m,n}(-1)^n$. These operators $Q_{m,n}$ on symmetric functions [9] are a concrete realization of elements in the Elliptic Hall algebra of an elliptic curve over a finite field, which was introduced with generators and relations in [6] . As operators on symmetric functions, these elements are generated by multiplication of symmetric functions (e_1) and applications of the delta operator (Δ_{e_1}) which we define below. Many of these connections were made possible by geometrical means, specifically since Mark Haiman's pivotal work on the Hilbert scheme gave Macdonald polynomials a geometrical setting. So even as we speak of the delta eigenoperators on the modified Macdonald basis, one can also give a Hilbert scheme interpretation, namely that these are operators on the Grothendieck group of torus-equivariant coherent sheaves on $Hilb^n(\mathbb{C}^2)$ induced by certain functors [19]. For instance, the operator ∇ corresponds to a twist by $\mathcal{O}(1) \otimes -$. We will therefore conclude our introduction with a brief history leading to our result.

The modified Macdonald polynomial was introduced by Garsia and Haiman in [2] to prove the Macdonald q, t-Kostka conjectures. The idea was to prove that the Macdonald q, t-Kostka polynomials indexed by partitions of n (with suitable modifications) give the distributions of irreducibles in a bigraded S_n-module. The result would mean they are Schur positive, which means that we have the expansion

$$\widetilde{\mathrm{H}}_\mu[X; q, t] = \sum_{\lambda \vdash n} s_\lambda[X] \widetilde{K}_{\lambda,\mu}(q, t)$$

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with $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. This led Garsia and Haiman to study the S_n-module of diagonal harmonics and conjecture a formula for its Frobenius characteristic $DH_n[X;q,t]$ [1]. The expansion of this polynomial in terms of the modified Macdonald basis led to the introduction by Bergeron and Garsia [11] of the operator ∇ and the birth of the expression

$$DH_n[X;q,t] = \nabla e_n.$$

The final proof of these identities, outlined by Claudio Procesi in 1994, required algebraic geometrical tools and was finally carried out by Mark Haiman [18] around 2001. Many years of progress in this subject led to the formulation by Haglund et al. [17] of the Shuffle Conjecture which gives a combinatorial expression for ∇e_n in terms of parking functions. This conjecture was then refined to its compositional form in [16]. Soon after, Mark Haiman's work on the Hilbert Scheme attracted the attention of algebraic geometers to this subject, culminating to the formulation by Gorsky and Neguţ [8] of an infinite family of Shuffle Conjectures called the Rational Shuffle Conjectures. They give a combinatorial interpretation of $Q_{m,n}(-1)^n$, one for each co-prime pair of integers (m,n), the original case corresponding to when $m = n+1$.

Work from the 1990's [10] introduced a family of eigenoperators Δ_f of the modified Macdonald polynomials generalizing ∇. This led to the discovery of the Schur positivity of $\Delta_{e_k} e_n$ for $1 \leq k \leq n$, and was soon followed by the formulation of the so-called Delta Conjecture [15] which gives a combinatorial formula for $\Delta_{e_k} e_n$, extending the classical Shuffle Conjecture. Recent papers by Erik Carlsson, Anton Mellit [7] then by Anton Mellit [4] give a proof of the Compositional Shuffle Conjecture and the Rational Shuffle Conjectures. Since the Shuffle Conjecture is a special case of the Delta Conjecture, one may ask if the methods of Carlsson and Mellit can be manipulated to give an expression for $\Delta_{e_k} e_n$. However, at the present moment, it is unclear how this can be done.

In this paper, we prove the Delta Conjecture with the specialization $q = 1$. Our main tool is the use of a weight-preserving, sign-reversing involution. The symmetric function $\Delta_{e_k} e_n\big|_{q=1}$ is expanded into a formal power series in t involving an infinite number of terms. The sign-reversing involution is used to cancel all the negative signs and output a finite number of fixed points yielding a polynomial in t with positive integer coefficients. We will see that the weight-preserving property of this involution will leave fixed points whose inequality relations strongly reflect the properties of Dyck paths. This leads to a bijection between the predicted combinatorial side and the symmetric function side of the Delta Conjecture at $q = 1$.

By these methods, we are able to compute other cases of the Delta operator at $q = 1$, such as $\Delta_{h_k} e_n$. However, the importance of our work lies in introducing a new way, namely the sign-reversing involution method, for proving the Delta Conjecture and, in particular, the Shuffle Conjecture. Using similar steps, the rational function $\Delta_{e_k} e_n$ may be expressed as a formal power series in q and t given by a certain set of objects. In principle, one should be able to find a sign-reversing involution of these objects, providing a combinatorial expansion of $\Delta_{e_k} e_n$.

Our main result is a proof of the identity

$$\Delta_{e_k} e_n\Big|_{q=1} = \sum_{\lambda \vdash n} e_\lambda \sum_{s \in M_k^\lambda} t^{\rho(s)}, \tag{1}$$

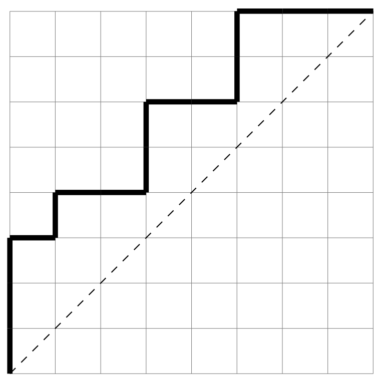
where M_k^λ is the set of all sequences

$$s = ((a_1, b_1), (a_2, b_2), \ldots, (a_{k+1}, b_{k+1}))$$

composed of nonnegative integers satisfying:

(1) $a_1 = 0$,
(2) $(b_1, \ldots, b_{k+1})$ is a rearrangement of $\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0^{k+1-\ell(\lambda)}$ (for $\ell(\lambda) \leq k+1$),
(3) $a_{i+1} < a_i + b_i$ and
(4) $\rho(s) = a_1 + \cdots + a_{k+1}$.

Let D_n be the collection of Dyck paths in the $n \times n$ square. That is, $D \in D_n$ is a lattice path from $(0, 0)$ to (n, n) consisting of north and east steps which stay weakly above the main diagonal $y = x$. Reading from bottom to top, let α_i be the number of lattice cells in the i^{th} row of D found between the path and the main diagonal. We call $(\alpha_1, \ldots, \alpha_n)$ the area sequence of D, and let $\text{area}(D)$ be the sum $\alpha_1 + \cdots + \alpha_n$. We denote by $\lambda(D)$ the partition of n which is made by the lengths of vertical segments in D. To exhibit these definitions, we provide the picture for the Dyck path whose area sequence is $(0, 1, 2, 2, 1, 2, 1, 2)$:



In this case, $\lambda(D) = (3, 2, 2, 1)$ since the vertical segments have lengths $3, 1, 2, 2$.

The Delta Conjecture [15], at $q = 1$, may be restated as

$$\Delta_{e_k} e_n \Big|_{q=1} = \sum_{D \in D_n} t^{\text{area}(D)} H_{n-k}(D) e_{\lambda(D)},$$

where if D has area sequence $(\alpha_1, \ldots, \alpha_n)$, then

$$H_{n-k}(D) = (1 + w) \prod_{\alpha_{i-1} = \alpha_i - 1} \left(1 + \frac{w}{t^{\alpha_i}}\right) \Big|_{w^{n-k}}.$$

The identity in (1) proves the Delta Conjecture at $q = 1$ since we will bijectively show

$$\sum_{s \in M_k^\lambda} t^{\rho(s)} = \sum_{\substack{D \in D_n \\ \lambda(D) = \lambda}} t^{\text{area}(D)} H_{n-k}(D). \tag{2}$$

The order of our exposition is as follows: We begin with some definitions and manipulate a symmetric function expansion of $\Delta_{e_k} e_n$. It will become essential to use an addition formula for the forgotten basis of symmetric functions $\{f_\mu\}_\mu$ and include for completeness a section with these computations. The expansion of

$\Delta_{e_k} e_n$ appears as a formal power series with alternating signs, but we find (1) with a sign-reversing involution. We will finish with a bijection, establishing (2).

2. Definitions and formulations

For readers who are not familiar with plethystic notation and other symmetric function amenities, we refer the reader to [3]. However, the following tool will be useful:

Recall the involution ω on symmetric functions is given by

$$\omega(p_n) = (-1)^{n-1} p_n.$$

Letting F be a symmetric function and E any expression, we will compute the transformation of $F[E]$ by ω by setting

$$\omega F[E] = F[-\epsilon E]\Big|_{\epsilon=-1},$$

where ϵ is treated as a variable then evaluated at -1.

Following Macdonald's section on integral forms on page 364 of [13], we have the equality

$$J_\mu[X;1,t] = \prod_{i=1}^{\ell(\mu')} (t:t)_{\mu_i'} e_{\mu_i'}[X],$$

where

$$(q:t)_k = (1-q)(1-qt)\cdots(1-qt^{k-1}).$$

The modified Macdonald polynomials of [2] arise from J_μ by setting

$$\tilde{H}_\mu[X;q,t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-1/t}; q, 1/t\right],$$

where $n(\mu) = \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i = \prod_{i=1}^{\ell(\mu')} \binom{\mu_i'}{2}$. Thus, we have

$$\begin{aligned}
\tilde{H}_\mu[X;1,t] &= t^{n(\mu)} J_\mu\left[\frac{X}{1-1/t}; 1, 1/t\right] = t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} (1/t:1/t)_{\mu_i'} e_{\mu_i'}\left[\frac{X}{1-1/t}\right] \\
&= t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} t^{-(1+2+\cdots+\mu_i')} (-1)^{\mu_i'} (t:t)_{\mu_i'} e_{\mu_i'}\left[-t\frac{X}{1-t}\right] \\
&= t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} t^{-(1+2+\cdots+\mu_i')} (-1)^{\mu_i'} (t:t)_{\mu_i'} (-t)^{\mu_i'} h_{\mu_i'}\left[\frac{X}{1-t}\right] \\
&= t^{n(\mu)} \prod_{i=1}^{\ell(\mu')} t^{-\binom{\mu_i'}{2}} (t:t)_{\mu_i'} h_{\mu_i'}\left[\frac{X}{1-t}\right],
\end{aligned}$$

where we used that for any expression E,

$$e_{\mu_i'}[-tE] = h_{\mu_i'}[-\epsilon(-t)E]\Big|_{\epsilon=-1} = (\epsilon t)^{\mu_i'} h_{\mu_i'}[E]\Big|_{\epsilon=-1} = (-t)^{\mu_i'} h_{\mu_i'}[E].$$

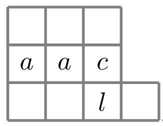
We get the equality

$$\tilde{H}_\mu[X;1,t] = \prod_{i=1}^{\ell(\mu')} (t:t)_{\mu_i'} h_{\mu_i'}\left[\frac{X}{1-t}\right].$$

The Delta operator, first seen in [10] and discussed by Haglund [12] and Wilson, Haglund, Remmel [15], is defined for any symmetric function f as the eigenoperator on the modified Macdonald polynomials given by

$$\Delta_f \tilde{H}_\mu[X;q,t] = f[B_\mu(q,t)]\tilde{H}_\mu[X;q,t],$$

where $B_\mu(q,t) = \sum_{c\in\mu} t^{\mathrm{coleg}_\mu(c)} q^{\mathrm{coarm}_\mu(c)}$. The coleg and coarm of a cell c is given by the number of cells below c and the number of cells to the left of c, respectively, in the French diagram of μ. A quick example will better illustrate these quantities, so below we provide the partition $(4,3,3)$ with a cell c selected:



We filled the cells corresponding to the coarm of c with a and those corresponding to the coleg of c with an l. In this case, we would have $\mathrm{coarm}(c) = 2$ and $\mathrm{coleg}(c) = 1$.

Specializing at $q = 1$, it follows that

$$\Delta_f h_{\mu'}\left[\frac{X}{1-t}\right]\Big|_{q=1} = f[B_\mu(1,t)]h_{\mu'}\left[\frac{X}{1-t}\right].$$

Switching the roles of μ and μ' and using that

$$B_{\mu'}(1,t) = \sum_{c\in\mu'} t^{\mathrm{coleg}_{\mu'}(c)} = \sum_{i=1}^{\ell(\mu)} [\mu_i]_t,$$

we get

$$\Delta_f h_\mu\left[\frac{X}{1-t}\right]\Big|_{q=1} = f\left[\sum_{i=1}^{\ell(\mu)}[\mu_i]_t\right] h_\mu\left[\frac{X}{1-t}\right].$$

This means that $\Delta_f e_n$ can be computed from an expansion of e_n in terms of $\left\{h_\mu\left[\frac{X}{1-t}\right]\right\}_\mu$. To this end, we note that for any expressions X, Y,

$$e_n[XY] = h_n[X(-\epsilon Y)]\Big|_{\epsilon=-1} = \sum_{\mu\vdash n} h_\mu[X]m_\mu[-\epsilon Y]\Big|_{\epsilon=-1} = \sum_{\mu\vdash n} h_\mu[X]f_\mu[Y],$$

where f_μ is the forgotten symmetric function indexed by μ. Thus

$$e_n[X] = e_n\left[\frac{X}{1-t}(1-t)\right] = \sum_{\mu\vdash n} h_\mu\left[\frac{X}{1-t}\right] f_\mu[1-t].$$

For any symmetric function $F \in \Lambda^n$, Haglund's Identity [12] gives the remarkable equality

$$\langle \Delta_{e_k} e_n, F\rangle = \left\langle \Delta_{\omega(F)} e_{k+1}, s_{k+1}\right\rangle.$$

It therefore follows that the coefficient of e_λ in the expansion of $\Delta_{e_k} e_n$ at $q = 1$ equals

$$\begin{aligned}\langle \Delta_{e_k} e_n, f_\lambda\rangle\Big|_{q=1} &= \left\langle \Delta_{\omega(f_\lambda)} e_{k+1}, s_{k+1}\right\rangle\Big|_{q=1} = \langle \Delta_{m_\lambda} e_{k+1}, s_{k+1}\rangle\Big|_{q=1} \\ &= \left\langle \sum_{\mu\vdash k+1} h_\mu\left[\frac{X}{1-t}\right] f_\mu[1-t] m_\lambda\left[\sum [\mu_i]_t\right], s_{k+1}\right\rangle.\end{aligned}$$

However, with an application of the Cauchy formula together with $\langle h_\lambda, h_{k+1}\rangle = 1$, we get

$$\left\langle h_\mu\left[\frac{X}{1-t}\right], s_{k+1}\right\rangle = h_\mu\left[\frac{1}{1-t}\right].$$

This gives the beginning stage of our desired coefficient:

$$\langle \Delta_{e_k} e_n, f_\lambda\rangle\Big|_{q=1} = \sum_{\mu\vdash k+1} h_\mu\left[\frac{1}{1-t}\right] f_\mu[1-t] m_\lambda\left[\sum[\mu_i]_t\right].$$

We note here that if $\ell(\lambda) > k+1$, then $m_\lambda\left[\sum[\mu_i]_t\right] = 0$. So we safely assume λ is a partition whose length is no more than $k+1$.

3. An addition formula for the forgotten

We wish to find the value of $f_\mu[1-t]$ for $\mu \vdash k+1$. For the remainder, we avoid repeatedly writing $k+1$ and we set $m = k+1$. One can follow the results from Remmel and Eğecioğlu [20], but for completion, we give a separate account.

Suppose we have expressions X, Y, and Z. Then

$$e_m[X(Y+Z)] = \sum_{\mu\vdash m} h_\mu[X] f_\mu[Y+Z].$$

On the other hand,

$$e_m[XY+XZ] = \sum_{i=0}^{m} e_i[XY]e_{m-i}[YZ] = \sum_{i=0}^{m}\sum_{\alpha\vdash i}\sum_{\beta\vdash m-i} h_\alpha[X]f_\alpha[Y]h_\beta[X]f_\beta[Z].$$

For $h_\alpha[X]h_\beta[X] = h_\mu[X]$, we require that the parts of α and β together create μ. Therefore, allowing the empty partition and letting (α,β) be the partition obtained from the union of parts from α and β, we can write

$$f_\mu[Y+Z] = \sum_{(\alpha,\beta)=\mu} f_\alpha[Y]f_\beta[Z].$$

We note that

$$e_m[X(1)] = \sum_{\mu\vdash m} h_\mu[X]f_\mu[1],$$

meaning that $f_\mu[1]$ is the coefficient of h_μ in the Jacobi-Trudi determinant:

$$\det\begin{bmatrix} h_1 & h_2 & \cdots & h_m \\ 1 & h_1 & \cdots & h_{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & h_1 \end{bmatrix}.$$

Though this may be computed using brick tabloids [20], we give a chance to another amazing way of computing this coefficient:

In proving a formula for the number of partitions in a given shape, Sergel Leven and Amdeberhan [21] noticed that for any matrix whose elements below the sub-diagonal are 0, every term in the determinant is uniquely determined by a selection of elements along the sub-diagonal. For example, suppose we select the 1's in rows $2, 3, 5, 6, 7$ in expanding e_8. We get the following picture:

$$\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ \underline{1} & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ 0 & \underline{1} & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0 & 0 & 1 & h_1 & h_2 & h_3 & h_4 & h_5 \\ 0 & 0 & 0 & \underline{1} & h_1 & h_2 & h_3 & h_4 \\ 0 & 0 & 0 & 0 & \underline{1} & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & h_1 & h_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \boxed{h_1} \end{bmatrix}$$

Proceeding from left to right, we first see that columns 1 and 2 have an entry selected. Column 3 has no 1 selected, so it is then forced to take h_3. Now all of the first three rows have an element selected. The next column with no 1 selected is 7. The three previous 1's are selected, so we must take h_4. Lastly, we take h_1 at the bottom. Each block forms a cycle in the permutation that selects these elements in the determinant. We can therefore break up the determinant into squares of sizes $\mu_1, \dots, \mu_{\ell(\mu)}$ along the main diagonal, each contributing a sign of $(-1)^{\mu_i - 1}$ and a factor of h_{μ_i}.

From this, it follows that we can compute that $f_\mu[1]$ equals

$$f_\mu[1] = (-1)^{m-\ell(\mu)} |R(\mu)|$$

where $R(\mu)$ is the collection of rearrangements of the parts $\mu_1, \dots, \mu_{\ell(\mu)}$. We apply this to the summation expansion

$$\begin{aligned} f_\mu[1-t] &= \sum_{(\alpha,\beta)=\mu} f_\alpha[1] f_\beta[-t] \\ &= \sum_{(\alpha,\beta)=\mu} (-t)^\beta f_\alpha[1] m_\beta[1]. \end{aligned}$$

Since $m_\beta[1]$ is 1 if β has only one part and 0 otherwise, we can write

$$f_\mu[1-t] = f_\mu[1] + \sum_{i \in \{\mu_1, \dots, \mu_n\}} (-t)^i f_{\mu-(i)}[1],$$

where $\mu - (i)$ is the partition obtained from μ by removing a part of size i. We will denote the inclusion in the sum as $i \in \{\mu\}$. Our observation from the Jacobi-Trudi determinant then gives

$$\begin{aligned} f_\mu[1-t] &= (-1)^{m-\ell(\mu)} |R(\mu)| + \sum_{i \in \{\mu\}} t^i (-1)^i (-1)^{(m-i)-(\ell(\mu)-1)} |R(\mu - (i))| \\ &= (-1)^{m-\ell(\mu)} \left(|R(\mu)| - \sum_{i \in \{\mu\}} t^i |R(\mu - (i))| \right). \end{aligned}$$

4. An expansion for $h_\mu \left[\frac{1}{1-t}\right] f_\mu[1-t]$

Since

$$h_r \left[\frac{1}{1-t}\right] = \frac{1}{(1-t)(1-t^2)\cdots(1-t^r)},$$

we can think of this factor, which we denote by G_r, as the generating function of all partitions whose largest row is less than or equal to r. In other words,

$$G_r = h_r\left[\frac{1}{1-t}\right] = \sum_{\lambda:\lambda_1\le r} t^{|\lambda|}.$$

Using the expression from the previous section, we get

$$h_\mu\left[\frac{1}{1-t}\right] f_\mu[1-t] = (-1)^{m-\ell(\mu)} G_{\mu_1}\cdots G_{\mu_{\ell(\mu)}}\left(|R(\mu)| - \sum_{i\in\{\mu\}} t^i |R(\mu-(i))|\right).$$

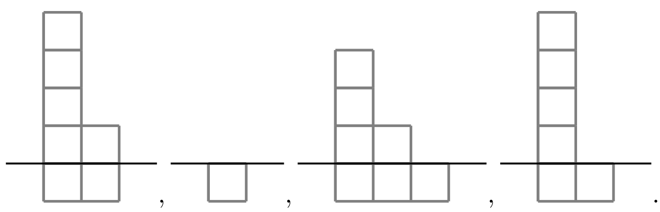
Proposition 1.

$$h_\mu\left[\frac{1}{1-t}\right] f_\mu[1-t] = (-1)^{m-\ell(\mu)} \sum_{i\in\{\mu\}} G_{i-1}\frac{G_{\mu_1}\cdots G_{\mu_{\ell(\mu)}}}{G_i}|R(\mu-(i))|.$$

Proof. To begin introducing the combinatorial objects which we will use to expand the main symmetric function identity, we will now give an over-embellished method of describing this particular sum. Let L_μ denote the objects formed by

(1) selecting a rearrangement $(\mu_{\alpha_1},\ldots,\mu_{\alpha_{\ell(\mu)}})$ of $\mu_1,\ldots,\mu_{\ell(\mu)}$,
(2) choosing a sequence $(\nu^1,\ldots,\nu^{\ell(\mu)})$ of partitions satisfying $\nu^i_1 \le \mu_{\alpha_i}$, and
(3) drawing the sequence of Young diagrams $(\nu^1,\ldots,\nu^{\ell(\mu)})$, where ν^i is drawn above a row of size μ_{α_i}.

For example, given the partition $\mu = (3,2,2,1)$ and rearrangement $(2,1,3,2)$ we could find in L_μ the sequence $((2,1,1,1),(0),(2,1,1),(1,1,1,1))$, which we depict by



For $T \in L_\mu$ with partitions $\nu^1,\ldots,\nu^{\ell(\mu)}$, let $w(T) = |\nu^1| + \cdots + |\nu^{\ell(\mu)}|$. For instance, the weight of our example is $5+0+4+4 = 13$. Then

$$G_{\mu_1}\cdots G_{\mu_{\ell(\mu)}}|R(\mu)| = \sum_{T\in L_\mu} t^{w(T)} = \sum_{i\in\{\mu\}} G_i \sum_{S\in L_{\mu-(i)}} t^{w(S)},$$

where the last equality separates the sum in terms of which part of μ begins the rearrangement $(\mu_{\alpha_1} = i)$. On the other hand,

$$\sum_{i\in\{\mu\}} t^i G_{\mu_1}\cdots G_{\mu_{\ell(\mu)}}|R(\mu-(i))| = \sum_{i\in\{\mu\}} t^i G_i \sum_{S\in L_{\mu-(i)}} t^{w(S)}.$$

But t^iG_i can be thought of as the generating function of all partitions whose biggest part equals i. So we have $G_i - t^iG_i = G_{i-1}$, giving the proposition

$$h_\mu\left[\frac{1}{1-t}\right] f_\mu[1-t] = (-1)^{m-\ell(\mu)} \sum_{i\in\{\mu\}} G_{i-1} \sum_{S\in L_{\mu-(i)}} t^{w(S)}.$$

□

Let $\widehat{\mathrm{L}}_\mu$ be the subset of objects from L_μ with the restriction that the first partition in our sequence, say ν^1 corresponding to the part μ_{α_1}, satisfies $\nu^1_1 \leq \mu_{\alpha_1} - 1$ instead of μ_{α_1}. Then it follows from the proposition that

$$h_\mu\left[\frac{1}{1-t}\right] f_\mu[1-t] = (-1)^{m-\ell(\mu)} \sum_{T\in\widehat{\mathrm{L}}_\mu} t^{w(T)}.$$

5. The summation terms

We now describe the product

$$h_\mu\left[\frac{1}{1-t}\right] f_\mu[1-t] m_\lambda\left[\sum [\mu_i]_t\right]$$

by labeling the objects in $\widehat{\mathrm{L}}_\mu$. Let $\widetilde{\mathrm{L}}_\mu$ be the set of objects formed in the following way:

(1) Select an element in $\widehat{\mathrm{L}}_\mu$. This is to say, select a rearrangement $\mu_{\alpha_1}, \ldots, \mu_{\alpha_{\ell(\mu)}}$ of μ and a sequence of partitions $\nu^1, \ldots, \nu^{\ell(\mu)}$ such that $\nu^i_1 \leq \mu_{\alpha_i}$ with further condition $\nu^1_1 < \mu_{\alpha_1}$. We draw each partition ν^i over a row of length μ_{α_i}. We refer to this row as μ_{α_i} itself.
(2) Place each of $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$ in a cell of $\mu_1, \ldots, \mu_{\ell(\mu)}$ such that each cell contains at most 1 entry. Fill the remaining cells with a 0.
(3) If λ_i is placed in a cell with j cells to the left, then we say that it contributes $j \cdot \lambda_i$.

For $T \in \widetilde{\mathrm{L}}_\mu$, we define the weight $p(T)$ by

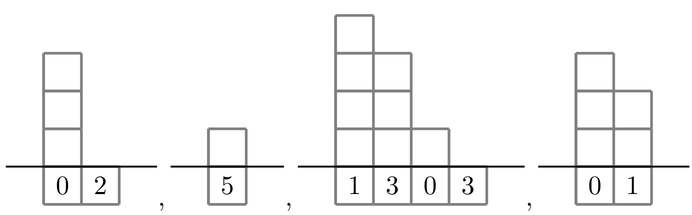
$$p(T) = w(T) + \sum_{\lambda_i} (\text{the contribution of } \lambda_i).$$

For example, let us select the partition $\mu = (4,2,2,1)$ and $\lambda = (5,3,3,2,1,1)$. Then we can select a rearrangement of μ: $(2,1,4,2)$. Then select partitions $(\nu^1, \nu^2, \nu^3, \nu^4)$ so that the largest part of ν^1 is at most $2-1$, the largest part of ν^2 is at most 1, the largest part in ν^3 is at most 4 and the largest part in ν^4 is at most 2. Here we choose

$$((1,1,1),(1),(3,2,2,1),(2,2,1)).$$

We then place the parts of λ in the cells corresponding to $\mu_1, \ldots, \mu_{\ell(\mu)}$. One such selection is given by

$S =$



We get $w(S) = 17$ since the sum of sizes of the partitions is 17. The contributions from the fillings (from left to right respectively) is given by

$$0\cdot(0) + 1\cdot(2),\ 0\cdot(5),\ 0\cdot(1) + 1\cdot(3) + 2\cdot(0) + 3\cdot(3),\ 0\cdot(0) + 1\cdot(1)$$

or $2, 0, 12, 1$. Therefore $p(S) = 17 + (2 + 0 + 12 + 1) = 32$. We say that the sign associated to $T \in \widetilde{\mathrm{L}}_\mu$ is $(-1)^{m-\ell(\mu)}$. So in this case, the sign is given by $(-1)^{9-4} = -1$.

We then get, for $\mu \vdash m$,

Proposition 2.

$$h_\mu \left[\frac{1}{1-t}\right] f_\mu[1-t] m_\lambda \left[\sum [\mu_i]_t\right] = (-1)^{m-\ell(\mu)} \sum_{T \in \widetilde{\mathrm{L}}_\mu} t^{p(T)}.$$

Proof. We have

$$m_\lambda(x_1, \dots, x_m) = \sum_{(\lambda_{i_1}, \dots, \lambda_{i_m})} x_1^{\lambda_{i_1}} \cdots x_m^{\lambda_{i_m}},$$

where the sum is over all rearrangements $(\lambda_{i_1}, \dots, \lambda_{i_m})$ of $\lambda_1, \dots, \lambda_m = \lambda, 0^{m-\ell(\lambda)}$. We can interpret this as labeling $x_1, \dots, x_m$ with $\lambda_1, \dots, \lambda_m$. Now we substitute $(x_1, x_2, \cdots, x_{\mu_1}) = (1, t, \dots, t^{\mu_1 - 1})$, then $(x_{\mu_1+1}, \dots, x_{\mu_1+\mu_2}) = (1, t, \dots, t^{\mu_2-1})$, and so on. For example, suppose we are computing $m_{3,2,1,1}[[3]_t + [2]_t + [2]_t]$. Then a monomial is given by a rearrangement of $3, 2, 1, 1, 0, 0, 0$. Selecting the rearrangement $(0, 1, 0, 2, 3, 0, 1)$ corresponds to taking the monomial

$$x_1^0 x_2^1 x_3^0 x_4^2 x_5^3 x_6^0 x_7^1.$$

Writing this rearrangement in the cells of $\mu_1 = 3, \mu_2 = 2$, and $\mu_3 = 2$, we get the filling

0	1	0
1	t	t^2

2	3
1	t

0	1
1	t

We wrote under each cell the term which it represents in $[\mu_1]_t + [\mu_2]_t + [\mu_3]_t$. Reading from left to right, we get the term

$$1^0 t^1 (t^2)^0 1^2 t^3 1^0 t^1.$$

This means $m_\lambda[\sum [\mu_i]_t]$ can be interpreted as the sum over fillings of $\mu_1, \dots, \mu_{\ell(\mu)}$ by $\lambda_1, \dots, \lambda_m$. If μ_i is filled by $(\lambda_{i_1}, \dots, \lambda_{i_{\mu_i}})$, the filling of μ_i will contribute

$$(1)^{\lambda_{i_1}} (t)^{\lambda_{i_2}} \cdots (t^{\mu_i - 1})^{\lambda_{i_{\mu_i}}}$$

to the term in the sum. This corresponds to saying that if λ_i is placed in a cell with j cells to the left, then it contributes to the term by $(t^j)^{\lambda_i}$.

The sum over the contributions of λ_i gives exactly the power of t contributed by a monomial in the expansion of m_λ. □

We have the essential formulation

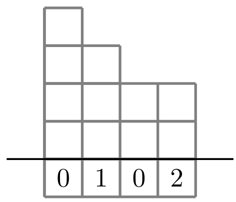
$$\langle \Delta_{e_k} e_n, f_\lambda \rangle \Big|_{q=1} = \sum_{\mu \vdash k+1} (-1)^{k+1-\ell(\mu)} \sum_{T \in \widetilde{\mathrm{L}}_\mu} t^{p(T)}.$$

We now use a sign-reversing involution to reduce this series to a positive polynomial.

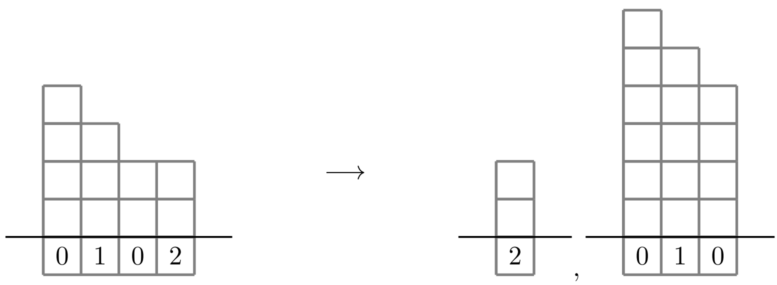
6. The involution

The involution follows from the following observation:

Given a partition ν over a row of size $\mu_i > 1$ and a labeling of μ_i whose last cell is labeled by c, we can create a pair of partitions ν^1, ν^2 over rows of sizes 1 and $\mu_i - 1$ respectively, by letting the last column in ν be ν^1, and letting ν^2 be the partition obtained from ν by removing the last column and adding c rows of length $\mu_i - 1$. We then add c to the cell under ν^1. To better describe this long-winded process, we will walk through a particular example. Suppose we have the partition $(4, 4, 2, 1)$ above a row of length $\mu_i = 4$. And suppose we have filled μ_i with $(0, 1, 0, 2)$ so that the picture looks as follows:



In this case $c = 2$. We then make the last column (ν^1 over c) the first part of our image and add $c = 2$ rows of length 3 to the partition over $(0, 1, 0)$:



This operation preserves the weight of the object since the contribution of c in the pre-image is equal to $c \cdot (\mu_i - 1)$, which is given by the c rows of length $\mu_i - 1$ that were added in the second partition. We note that given a column ν^1 over a cell filled with c and partition ν^2 over a row of length $\mu_i - 1$, we can combine the two, inverting this procedure, provided that we can remove c full rows from ν_2 and add a column of size $|\nu^1|$ on the right. This condition is described by the inequality

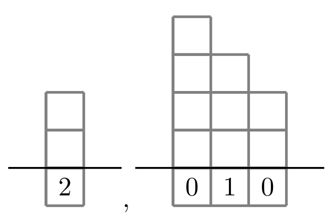
$$|\nu^1| \leq (\text{ the number of rows in } \nu^2 \text{ of size } \mu_i - 1) - c.$$

We instead write the inequality as

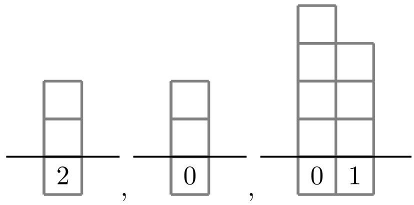
$$c + |\nu^1| \leq (\text{ the number of rows in } \nu^2 \text{ of size } \mu_i - 1).$$

For example,

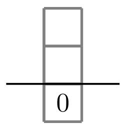
$\nu^1, \nu^2 =$



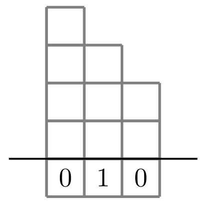
cannot be combined since the number of rows in ν^2 of length 3 is 2, and $c + |\nu^1| = 2 + 2 = 4 > 2$. However, we can expand the second partition because the length of the row underneath it is greater than 1. We get



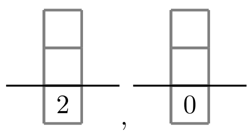
The crucial observation is that the number of full rows in



and



is 2 for both. Therefore, if we couldn't combine ν^1 and ν^2, then we cannot combine the pair



Proposition 3. *We have the following:*

(1) *Given a diagram with partition ν and image pair ν^1, ν^2, the number of full rows of ν is equal to the number of full rows of ν^1. Thus, if ν^0, ν cannot be combined, then ν^0, ν^1 cannot be combined.*

(2) *Suppose we have a sequence ν^1, ν^2, ν^3 such that ν^1 cannot combine with ν^2, but ν^2 can combine with ν^3 to form ν. Then ν^1 cannot combine with ν.*

Proof. (1) This statement is obvious, since ν^1 is the rightmost column of ν.

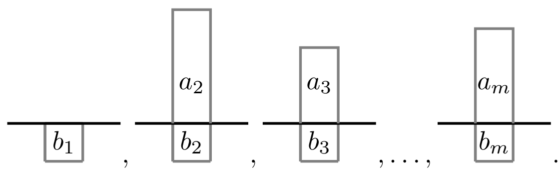
(2) The number of full rows in ν equals the number of full rows of ν^2, so that the inequality which determines whether we can combine ν^1 and ν^2 is still not satisfied between ν^1 and ν.

□

Thus, we get the following sign-reversing involution: Given $(\nu^1, \ldots, \nu^{\ell(\mu)}) \in \widetilde{\mathrm{L}}_\mu$ corresponding to the rearrangement $(\mu_{\alpha_1}, \ldots, \mu_{\alpha_{\ell(\mu)}})$ proceed from $i = 1$ to $i = \ell(\mu)$ until we find either the first ν^i which can combine with ν^{i+1} or the first ν^i for which $\mu_{\alpha_i} > 1$. In the first case we combine ν^i and ν^{i+1}. In the second case, we separate as described above. If no such ν^i is found, then leave the object fixed.

Our proposition above ensures that if a partition is separated, then reapplying the map will locate the pair which was created, since no preceding partitions were combined or separated. Likewise, if a pair was combined, then the map will locate the new partition which was formed. In either case the length of μ which underlies the sequence of partitions changes by exactly 1. Therefore, the sign associated to $(\nu^1, \ldots, \nu^{\ell(\mu)})$ will change parity. It remains to find the sequences which are fixed points.

Suppose $(\nu^1, \dots, \nu^{\ell(\mu)})$ corresponding to the rearrangement $(\mu_{\alpha_1}, \dots, \mu_{\alpha_{\ell(\mu)}})$ is a fixed point. If $\mu_i > 1$ for some i, then the object has a pair in the sign-reversing involution, so we must have $\mu_i = 1$ for all i. Therefore every fixed object is an element of $\widetilde{\mathrm{L}}_{(1,\dots,1)}$ and has associated sign $(-1)^{(k+1)-(k+1)} = 1$. This means that we are looking for columns of lengths $(a_1, \dots, a_{k+1})$ and some underlying labels $(b_1, \dots, b_{k+1})$ respectively such that no two columns can be combined. In pictures, we have a sequence which looks like



We first see that $a_1 = 0$, since it is an element of $\widetilde{\mathrm{L}}_{(1,\dots,1)}$. The condition that no two can be combined can be written as the inequality

$$a_{i+1} < a_i + b_i.$$

We therefore define a new set of objects M_k^λ given by the set of all sequences

$$s = ((a_1, b_1), (a_2, b_2), \dots, (a_{k+1}, b_{k+1}))$$

with the conditions that $a_1 = 0$; $(b_1, \dots, b_{k+1})$ is a rearrangement of $\lambda_1, \dots, \lambda_{\ell(\lambda)}$, $0^{k+1-\ell(\lambda)}$ (where there are $k+1-\ell(\lambda)$ zeroes added to λ) and $a_{i+1} < a_i + b_i$. For such a sequence s define its weight by

$$\rho(s) = a_1 + a_2 + \cdots + a_{k+1}.$$

This proves the main result:

Theorem 1.

$$\langle \Delta_{e_k} e_n, f_\lambda \rangle \Big|_{q=1} = \sum_{s \in M_k^\lambda} t^{\rho(s)}.$$

7. The bijection

To prove the Delta Conjecture [15] at $q = 1$, we must show

Theorem 2.

$$\sum_{s \in M_k^\lambda} t^{\rho(s)} = \sum_{\substack{D \in D_n \\ \lambda(D) = \lambda}} t^{\mathrm{area}(D)} H_{n-k}(D).$$

Proof. We first write the second sum as

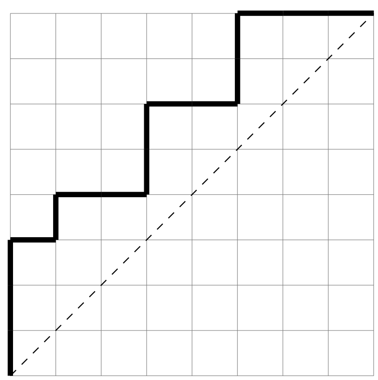
$$\sum_{\substack{D \in D_n \\ \lambda(D) = \lambda}} t^{\mathrm{area}(D)} H_{n-k}(D) = \sum_{\overline{D} \in D_{n-k}^\lambda} t^{\mathrm{area}(\overline{D})},$$

where the objects in D_{n-k}^λ are created in the following way:

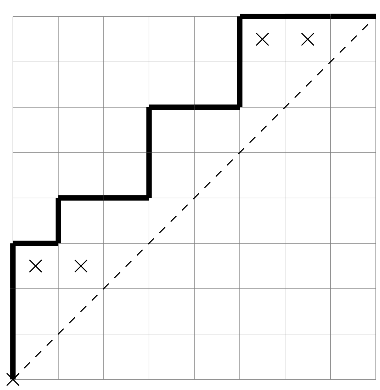
(1) Select $D \in D_n$ such that $\lambda(D) = \lambda$. Suppose its area sequence is $\alpha_1, \dots, \alpha_n$, and define the 0 row with area $\alpha_0 = 0$ to be the origin.
(2) Select $n-k$ distinct rows $i_1, \dots, i_{n-k} \in \{0, 2, \dots, n\}$ such that either $i_j = 0$ or $\alpha_{i_j - 1} = \alpha_{i_j} - 1$.

(3) For all j, draw an $\times$ at each cell in row i_j that contributes to the area of D. If $i_j = 0$, then mark the origin with an $\times$. After doing this for all j, call this new object $\overline{D}$.
(4) Define the area of $\overline{D}$ by $\alpha_1 + \cdots + \alpha_n - (\alpha_{i_1} + \cdots + \alpha_{i_{n-k}})$. That is, count all the cells with no $\times$ which contribute to the area of D.

Here is an example: Suppose D has area sequence $(0,1,2,2,1,2,1,2)$ so that the Dyck path is given by



We select rows $0, 3$, and 8 to get the following labeled Dyck path in $D_3^{(3,2,2,1)}$:



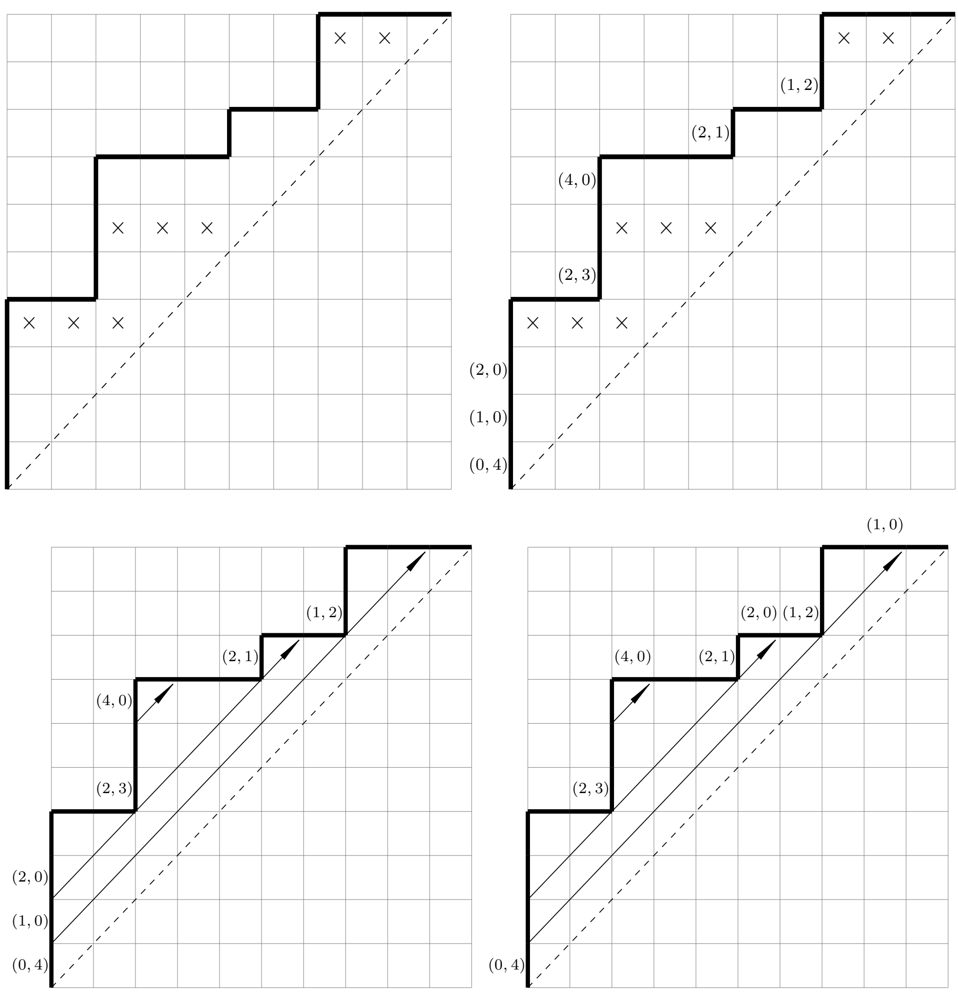
Reading the rows bottom to top, the area of this object would be $0+1+0+2+1+2+1+0 = 7$.

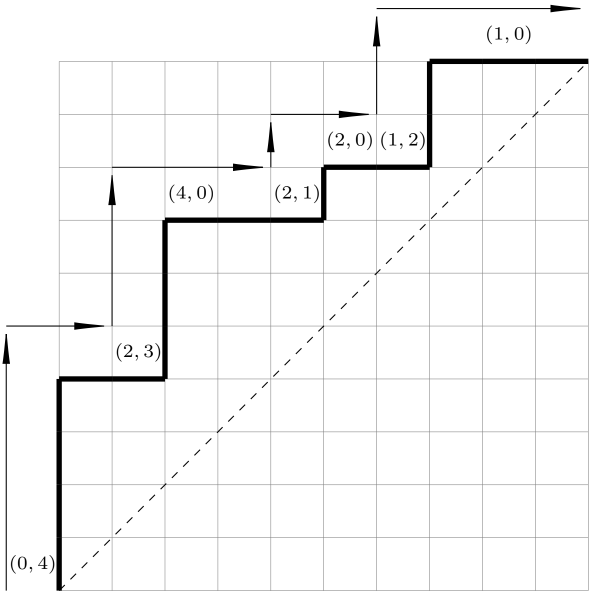
The weight-preserving bijection $\phi : D^{\lambda}_{n-k} \to M^{\lambda}_{k}$ is given by the following steps:

(1) To the left of every north step which begins a vertical segment, say of length b, we write (a, b), where a is the area contribution of this row.
(2) To the left of every remaining north step whose row is not labeled, write $(c, 0)$, where c is the area contribution of this row.

(3) For each label $(c, 0)$, draw a north-east diagonal line from the beginning of its north step to the first start of an east step, and write $(c, 0)$ above the east step which precedes it. (This east step exists since Dyck paths must always return to the main diagonal. So the only way the Dyck path can cross diagonal c is with two consecutive east steps. This means every pair of consecutive north steps can be associated to a unique pair of consecutive east steps.)
(4) Follow the Dyck path from bottom to top writing the pairs in the order in which they appear.
(5) If the origin is not labeled, include a $(0, 0)$ at the end.

It is easier to follow the bijection with pictures, so we illustrate this series of steps for a particular example. We select a Dyck path in $D_3^{(4,3,2,1)}$ and provide an element of $M_7^{(4,3,2,1)}$:

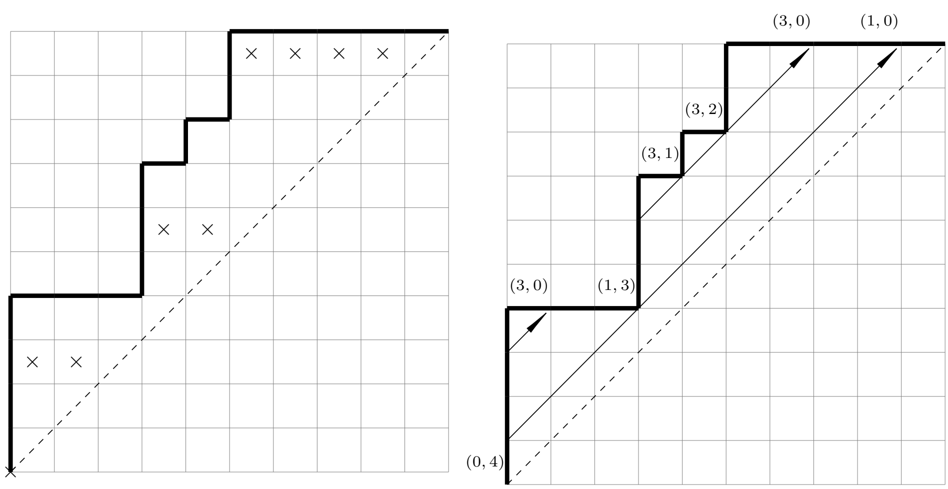




Since the origin is not labeled, we include a $(0,0)$ at the end to get

$$((0,4),(2,3),(4,0),(2,1),(2,0),(1,2),(1,0),(0,0)).$$

For another example, we have



Since the origin is labeled, we do not include a $(0,0)$ at the end to get

$$((0,4),(3,0),(1,3),(3,1),(3,2),(3,0),(1,0)).$$

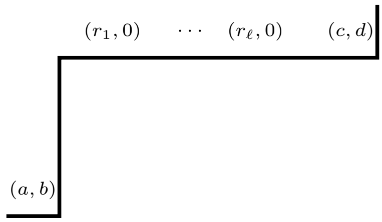
We must show that the image of D^{λ}_{n-k} is in M^{λ}_{k}. It is easy to see that if

$$\phi(\overline{D}) = ((a_1,b_1),\dots,(a_{k+1},b_{k+1})),$$

then by construction $(b_1,\dots,b_{k+1})$ is a rearrangement of $\lambda, 0^{k+1-\ell(\lambda)}$. We also have $a_1 = 0$. It then suffices to show the inequalities are satisfied. Every two consecutive vertical line segments of lengths b and d (beginning at areas a and c respectively) will create a sequence in the image of the form

$$(a, b), (r_1, 0), \ldots, (r_\ell, 0), (c, d).$$

This is described by the following diagram:



We note that the top corner in the first vertical segment of the picture lies on diagonal $a + b$. Likewise $r_1, \ldots, r_\ell, c$ describe the right diagonal of the east steps which they label. Therefore, since we are moving down diagonals as we proceed to the right, we have the inequalities

$$a + b > r_1 > \cdots > r_\ell > c.$$

This shows that $\phi(\overline{D})$ provides a sequence which satisfies the defining inequalities of M_k^λ and therefore ϕ is properly defined to give an element of M_k^λ. The area of the rows in $\overline{D}$ which contribute to the area are precisely the values of $a_1, \ldots, a_{k+1}$. This means that $\text{area}(\overline{D}) = \rho\left(\phi(\overline{D})\right)$.

To prove bijectivity, we need only provide the inverse. We do this first without proof, then show that this operation is indeed valid: Given a sequence $((a_1, b_1), \ldots, (a_{k+1}, b_{k+1})) \in M_k^\lambda$, we do the following:

(1) Proceeding from left to right, for each (a, b) with $b \neq 0$ draw a line segment of length b beginning on diagonal a, so that we form a Dyck path.
(2) For $(r_i, 0)$ between (a, b) and (c, d), $b, d \neq 0$, draw $(r_i, 0)$ over the east step ending in diagonal r_i.
(3) Move all $(r_i, 0)$ south-west, down their diagonal, until we reach an end of a north step. Place $(r_i, 0)$, to the left of the north step which lies above.
(4) Label with $\times$'s the rows which have no pair to the left.
(5) If $(0, 0)$ was not in the list, label the origin.

We are simply doing the exact opposite from the definition of ϕ. However, there are two implicit requirements which we have to check. The first step claims that the pairs (a, b) with $b \neq 0$ when read from left to right describe a Dyck path. We again note that the sequence is composed of portions of the form

$$(a, b), (r_1, 0), \ldots, (r_\ell, 0), (c, d).$$

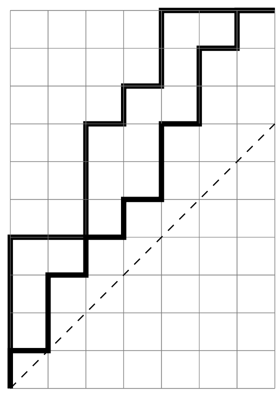
The fact that $a+b > r_1 > \cdots > r_\ell > c$ means that if we were to draw a vertical line segment of length b such that the bottom north step contributes area a, then the diagonal of the top corner of the segment (this diagonal would be $a + b$) appears above diagonal c. So we have to place the segment of length d strictly to the right of the segment of length b. Thus, the first step produces a Dyck path.

The second place where some detail is needed is in step 2, where we place $(r_i, 0)$ between (a, b) and (c, d). Since $r_1 > \cdots > r_\ell$, no two pairs will be placed above the same east step. Moreover, this portion of the Dyck path is horizontal starting from diagonal $a+b$ and ending on diagonal c, so there is indeed a unique east step which ends on diagonal r_i for each i. This completes the bijection. □

7.1. **A second bijection.** The referee observed that the result can be obtained from a different formulation of the Delta Conjecture. As a consequence, it was noted that we get a new and important Schur expansion for $\nabla e_n|_{q=1}$, which we present in the last section. We discuss this bijection here.

Consider the lattice rectangle of height n and length $k+1$, and let $D_{n,k}$ be the set of paths consisting of north and east steps from $(0,0)$ to $(k+1,n)$ which stay weakly above the line $y=x$ in the first k columns and end with an east step. Define the set of leaning stacks $LS_{n,k}$ to be the set of pairs of paths $(P,Q) \in D_{n,k} \times D_{n,k}$ such that P is weakly above Q, and Q has no two consecutive east steps in the first k columns. It should be noted that this definition differs from that in [15]. Let area$((P,Q))$ be the number of lattice cells between P and Q.

For example, we could select the paths

$(R,S) =$ 

The area of this pair is 14. Another formulation for the Delta Conjecture at $q=1$ given in [15] is as follows: Let

$$LS^{\lambda}_{n,k} = \{(P,Q) \in LS_{n,k} : \lambda(P) = \lambda\}.$$

Then

Theorem 3.
$$\Delta_{e_k} e_n \Big|_{q=1} = \sum_{\lambda \vdash n} e_\lambda \sum_{(P,Q)\in LS^{\lambda}_{n,k}} t^{\mathrm{area}((P,Q))}.$$

We note that there is a second way to describe the pairs (P,Q). First, make a sequence $(b_1, b_2, \ldots, b_{k+1})$ where b_i gives the number of north steps in column i of P (where the north steps of column i are on the line $y=i-1$). For example, the pair (R,S) given above has 4 north steps in the first column, the second has 0 and the third has 3. Proceeding in this manner we get the sequence

$$(4,0,3,1,2,0,0).$$

Now create the sequence $(a_1, \ldots, a_{k+1})$ where $a_1 = 0$ and a_i gives the number of lattice cells between P and Q in column $i-1$. The above example has 3 cells in the first column, 1 in the second, 3 in the third, and so on. This will give the sequence

$$(0,3,1,3,3,3,1).$$

Set

$$\psi((P,Q)) = ((a_1,b_1),(a_2,b_2),\ldots,(a_{k+1},b_{k+1})).$$

Continuing our example, we get $\psi((R,S)) = ((0,4),(3,0),(1,3),(3,1),(3,2),(3,0),(1,0))$. This is in fact the last example in the prior section.

Suppose $\psi((P,Q)) = ((a_1,b_1),\ldots,(a_{k+1},b_{k+1}))$. First note that if $\lambda(P) = \lambda$, then $b_1,\ldots,b_{k+1}$ is a rearrangement of $\lambda, 0^{k+1-\ell(\lambda)}$. The fact that Q has no consecutive east steps in the first k columns means that the number of lattice cells in column $i-1$ plus the number of north steps in column i is larger than the number of lattice cells in column i (for $i \leq k$). In other words:

$$a_i + b_i > a_{i+1}.$$

This gives exactly the definition for elements in M_k^λ. Therefore, ψ defines a bijection between $LS_{n,k}^\lambda$ and M_k^λ. Moreover, since $\operatorname{area}((P,Q)) = \rho(\psi((P,Q)))$, this bijection is weight-preserving. This proves the above expansion.

8. Further comments

Our computation reveals a rich collection of structures which can give the inner product of $\omega(\Delta_{e_k} e_n)|_{q=1}$ in terms of any symmetric function $F = \sum_\lambda c_\lambda m_\lambda$ expanded into monomials.

Theorem 4. *Let F be a symmetric function of degree n. Then*

$$\langle \omega(\Delta_{e_k} e_n), F\rangle \Big|_{q=1} = \sum_{\lambda \vdash n} c_\lambda \sum_{s \in M_k^\lambda} t^{\rho(s)}.$$

This theorem immediately gives that $\Delta_{e_k} e_n|_{q=1}$ is t-positive in terms of the bases $\{s_\lambda\}_\lambda$, $\{m_\lambda\}_\lambda$, $\{e_\lambda\}_\lambda$, and surprisingly even $\{f_\lambda\}_\lambda$. In fact, every monomial $x_1^{b_1}\cdots x_{k+1}^{b_{k+1}}$ in $F(x_1,\ldots,x_{k+1})$ will contribute to $\langle \omega(\Delta_{e_k} e_n), F\rangle|_{q=1}$ by the weighted sum over sequences $s = ((a_1,b_1),\ldots,(a_{k+1},b_{k+1}))$, where the a_i are chosen so that $s \in M_k^{\lambda(b_1,\ldots,b_{k+1})}$.

We give here two particular examples which are important in studying the Delta Conjecture.

8.1. **The Hilbert series.**

In recent work, Wilson [5] was able to compute the Hilbert series $\langle \Delta_{e_k} e_n, p_1^n\rangle$ at $q = 1$. His result can be obtained by our general method. We can represent the symmetric function $p_1^n(x_1,\ldots,x_{k+1})$ by

$$p_1^n(x_1,\ldots,x_{k+1}) = \sum_{B_1,\ldots,B_{k+1}} x_1^{|B_1|}\cdots x_{k+1}^{|B_{k+1}|}$$

where $B_1,\ldots,B_{k+1}$ are disjoint subsets of $\{1,\ldots,n\}$ whose union is the whole set. In other words, we get an ordered set partition of length $k+1$, allowing empty sets.

Let P_k^n be the set of sequences

$$P = ((a_1,B_1),\ldots,(a_{k+1},B_{k+1}))$$

of pairs where $a_1 = 0$, the B_i are disjoint subsets of $\{1,\ldots,n\}$ whose union is all of $\{1,\ldots,n\}$, and

$$a_{i+1} < a_i + |B_i|.$$

Letting $\rho(P) = a_1 + \cdots + a_{k+1}$, we get

$$\langle \Delta_{e_k} e_n, p_1^n\rangle \Big|_{q=1} = \sum_{P \in P_k^n} t^{\rho(P)}.$$

We can use the same bijection as in our previous section where the b_i represent $|B_i|$, but now the sets describe cars along north segments. This means that the set of such sequences is equal to the set of parking functions in the $n \times n$ square with $n-k$ labeled rows, t-enumerated by the area statistic. That is,

$$\sum_{P \in P_k^n} t^{\rho(P)} = \sum_{D \in D_n} H_{n-k}(D) \sum_{D(PF)=D} t^{\mathrm{area}(PF)},$$

where $D(PF)$ is the supporting Dyck path of a parking function PF.

8.2. **A Schur function expansion and diagonal harmonics.** The Delta Conjecture predicts that $\Delta_{e_k} e_n$ is q,t-Schur positive. We mean by this that the coefficient of s_λ in $\Delta_{e_k} e_n$ is a polynomial in q and t with positive integer coefficients. Using the same methods we get the following expansion at $q=1$:

For a semistandard tableau T let

$$c_i(T) = (\text{the number of times } i \text{ occurs in } T).$$

Let S_k^λ be the set of objects formed by

(1) selecting a semistandard filling T of λ with entries in $1,\ldots,k+1$,
(2) setting $a_1 = 0$ and choosing $a_2,\ldots,a_{k+1}$ such that $a_{i+1} < a_i + c_i(T)$, and
(3) writing it all together as $S = ((a_1,\ldots,a_{k+1}),T)$.
(4) Let $p(S) = a_1 + \cdots + a_{k+1}$.

Then

$$\langle \omega(\Delta_{e_k} e_n), s_\lambda \rangle \Big|_{q=1} = \sum_{S \in S_k^\lambda} t^{p(S)}.$$

This gives the expansion

$$\Delta_{e_k} e_n \Big|_{q=1} = \sum_{\lambda \vdash n} s_\lambda \sum_{S \in S_k^{\lambda'}} t^{p(S)}.$$

To look at a specific yet nontrivial case, let us investigate

$$\Delta_{e_n} e_n \Big|_{q=1} = \nabla e_n \Big|_{q=1} = DH_n[X;1,t].$$

To get a Schur basis expansion, let us look at one element $S = ((a_1,\ldots,a_{n+1}),T) \in S_n^\lambda$. Note that

$$s = ((a_1, c_1(T)),\ldots,(a_{n+1}, c_{n+1}(T))) \in M_n^\nu$$

where $\nu = \lambda(c_1(T),\ldots,c_{n+1}(T))$. We can then view s by its image $\psi(s) = (P,Q)$ from the bijection in the last section. These paths are from $(0,0)$ to $(n+1,n)$, and Q has no consecutive east steps in the first n columns. This forces Q to be the path with no consecutive north steps: (north, east)^n east. This means that the terms $a_1,\ldots,a_{n+1}$ are completely determined by P. Therefore, every semistandard tableau T has at most one sequence $(a_1,\ldots,a_{n+1})$ so that $a_{i+1} < a_i + c_i(T)$. This sequence exists so long as $(c_1(T),\ldots,c_{n+1}(T))$ describes a Dyck path, as given by the map ψ. This is to say $c_{n+1} = 0$ and

$$c_1(T) + \cdots + c_i(T) \geq i.$$

Given a partition λ, let SS^λ be the set of semistandard tableaux T of λ with entries in $1,\ldots,n$ so that $c_1(T) + \cdots + c_i(T) \geq i$ for all i. In other words, when read in (weakly) increasing order, the i^{th} smallest integer in T is no greater than i. Define the area of T to be

$$\operatorname{area}(T) = n \cdot c_1(T) + (n-1) \cdot c_2(T) + \cdots + c_n(T) - \binom{n+1}{2}.$$

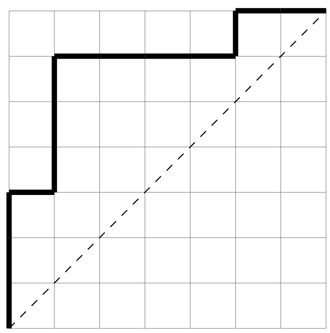
This is another way to write the area of the Dyck path which is given by the sequence $c_1(T), \ldots, c_n(T)$. For an example, let us take the semistandard tableau

$$T = \begin{array}{|c|c|c|c}\hline 2 & 2 & 6 & \\ \hline 1 & 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}.$$

In this case

$$(c_1(T), c_2(T), c_3(T), c_4(T), c_5(T), c_6(T), c_7(T)) = (3, 3, 0, 0, 0, 1, 0).$$

The corresponding Dyck path would be



This would mean the area of T is the area of this Dyck path: 13.

Then

Theorem 5.

$$DH_n[X; 1, t] = \sum_{\lambda \vdash n} s_\lambda \sum_{T \in SS^{\lambda'}} t^{\operatorname{area}(T)}.$$

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