TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 11, November 2017, Pages 7589–7629
http://dx.doi.org/10.1090/tran/6855
Article electronically published on March 6, 2017

SOME NEW EXAMPLES OF UNIVERSAL HYPERCYCLIC OPERATORS IN THE SENSE OF GLASNER AND WEISS

SOPHIE GRIVAUX

Abstract. A bounded operator A on a real or complex separable infinite-dimensional Banach space Z is *universal* in the sense of Glasner and Weiss if for every invertible ergodic measure-preserving transformation T of a standard Lebesgue probability space $(X, \mathcal{B}, \mu)$, there exists an A-invariant probability measure ν on Z with full support such that the two dynamical systems $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$ are isomorphic. We present a general and simple criterion for an operator to be universal, which allows us to characterize universal operators among unilateral or bilateral weighted shifts on ℓ_p or c_0, to show the existence of universal operators on a large class of Banach spaces and to give a criterion for universality in terms of unimodular eigenvectors. We also obtain similar results for operators which are universal for all ergodic systems (not only for invertible ones) and study necessary conditions for an operator on a Hilbert space to be universal.

1. Introduction and main results

Let G be a topological group and Z a real or complex separable infinite-dimensional Banach space. We denote by $\mathcal{B}(Z)$ the set of bounded linear operators on Z. Let $S : G \to Z$, $g \mapsto S_g$ be a representation of the group G by bounded operators on Z. If $\mathcal{B}_Z$ denotes the Borel σ-field of Z and ν is a Borel probability measure on Z which is S-invariant (i.e. ν is S_g-invariant for every g in G), then S naturally defines a probability-preserving action of the group G on the probability space $(Z, \mathcal{B}_Z, \nu)$. Recall that the measure ν is said to have full support if $\nu(U) > 0$ for any non-empty open subset U of Z.

Glasner and Weiss introduced in the paper [10] the following notion of a universal representation:

Definition 1.1 ([10])**.** The representation $S = (S_g)_{g \in G}$ of the group G on the Banach space Z is said to be *universal* if for every ergodic probability-preserving free action $T = (T_g)_{g \in G}$ of G on a standard Lebesgue probability space $(X, \mathcal{B}, \mu)$, there exists a Borel probability measure ν on Z with full support which is S-invariant and such that the two actions of T and S of G on $(X, \mathcal{B}, \nu)$ and $(Z, \mathcal{B}_Z, \nu)$ respectively are isomorphic.

Received by the editors October 12, 2014 and, in revised form, October 3, 2015 and October 13, 2015.

2010 *Mathematics Subject Classification.* Primary 47A16, 37A35, 47A35, 47B35, 47B37.

Key words and phrases. Universal hypercyclic operators, ergodic theory of linear dynamical systems, frequently hypercyclic operators, isomorphisms of dynamical systems.

This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01).

Recall that $(T_g)_{g\in G}$ is *free* if for any element $g \in G$ different from the identity, $\mu(\{x \in X;\, T_g x = x\}) = 0$, and *ergodic* if the following holds true: if $A \in \mathcal{B}$ is such that $T_g^{-1}(A) = A$ for every $g \in G$, then $\mu(A)(1-\mu(A)) = 0$.

A universal representation of G thus simultaneously models every possible free ergodic action of G on a probability space. The existence of a universal representation is shown in [10] for a large class of groups G, including all countable discrete groups and all locally compact, second countable, compactly generated groups.

When $G = \mathbb{Z}$, the main result of [10] states that there exists a bounded invertible operator S on H which is *universal* in the following sense: for every invertible ergodic probability-preserving transformation T of a standard Lebesgue probability space $(X, \mathcal{B}, \mu)$, there exists an S-invariant probability measure ν on H with full support such that the two dynamical systems $(X, \mathcal{B}, \mu; T)$ and $(H, \mathcal{B}_H, \nu; S)$ are isomorphic. Observe that any invertible ergodic probability-preserving transformation T of $(X, \mathcal{B}, \mu)$ acts freely on $(X, \mathcal{B}, \mu)$: for any $n \in \mathbb{Z}$, the set $\{x \in X;\, T^n x = x\}$ is T-invariant, and the ergodicity of T implies that it is of μ-measure zero. This definition of a universal operator is thus coherent with Definition 1.1.

Any of the systems $(H, \mathcal{B}_H, \nu; S)$ is what is called a *linear dynamical system*, i.e. a system given by the action of a bounded linear operator A on an infinite-dimensional separable Banach space Z. These systems can be studied from both the topological point of view and the ergodic point of view (when one endows the Banach space Z with an A-invariant probability measure), and we refer the reader to the two books [4] and [11] for more on this particular class of dynamical systems.

The result of [10], when specialized to the case where $G = \mathbb{Z}$, thus says that *any* invertible ergodic probability-preserving dynamical system can be represented as a linear dynamical system where the underlying space is a Hilbert space, and, moreover, the same operator S on H can serve as a model for any such dynamical system. Given the apparent rigidity entailed by linearity, the universality result of [10] in the case where $G = \mathbb{Z}$ may seem rather surprising. It is worth pointing out here that a topological version of this result had been obtained previously by Feldman in [8]: there exists a bounded operator A on the Hilbert space $\ell_2(\mathbb{N})$ which has the following property: whenever φ is a continuous self-map of a compact metrizable space K, there exists a compact subset L of $\ell_2(\mathbb{N})$ which is A-invariant and a homeomorphism $\Phi : K \to L$ such that $\varphi = \Phi^{-1} \circ A \circ \Phi$. The proof of this topological result is rather straightforward, but it already gives a hint at the richness of the class of linear dynamical systems.

A bounded operator A acting on the Banach space Z is said to be *hypercyclic* if it admits a vector $z \in Z$ whose orbit $\{A^n z;\ n \geq 0\}$ is dense in Z, and *frequently hypercyclic* if there exists a vector $z \in Z$ such that for every non-empty open subset V of Z, the set $\{n \geq 0;\ A^n z \in V\}$ has positive lower density. If A admits an invariant probability measure with full support with respect to which it is ergodic, then Birkhoff's ergodic theorem is easily seen to imply that almost all vectors of Z are frequently hypercyclic for A. Thus any universal operator is frequently hypercyclic. Let us now say a few words about the construction of universal operators of [10].

The universal operators constructed in [10] are shift operators on certain weighted ℓ_p-spaces of sequences on $\mathbb{Z}$ for $1 < p < +\infty$ or, equivalently, weighted shift operators on $\ell_p(\mathbb{Z})$. The proof uses in a crucial way an ergodic theorem for certain random walks of Jones, Rosenblatt, and Tempelman [12]. This theorem states in particular that whenever η is a symmetric strictly aperiodic probability measure

on $\mathbb{Z}$, the following holds true: for any probability-preserving dynamical system $(X, \mathcal{B}, \mu; T)$ and any function $f \in L^p(X, \mathcal{B}, \mu)$, $1 < p < +\infty$, the powers $A_\eta^n f$ of the random walk operator on $\mathbb{Z}$ defined by

$$A_\eta f(x) = \sum_{k \in \mathbb{Z}} f(T^k x)\, \eta(k)$$

converge for almost every $x \in X$ to the projection $P_{\mathcal{J}} f$ of f onto the subspace $\mathcal{J}$ of $L^p(X, \mathcal{B}, \mu)$ consisting of T-invariant functions. This ergodic theorem can be applied for instance starting from the measure $\eta = (\delta_{-1} + \delta_0 + \delta_1)/3$ on $\mathbb{Z}$. If $(p_n)_{n \geq 1}$ is a sequence of positive real numbers such that $\sum_{n \geq 1} p_n = 1$ and $\sup\,(p_n/p_{n+1}) < +\infty$, the weights considered in [10] are defined by setting $w_k := \sum_{n \geq 1} p_n\, \eta^{*n}(k)$ for every $k \in \mathbb{Z}$. If S is the shift operator defined on

$$\ell_p(\mathbb{Z}, w) := \{\xi = (\xi_k)_{k \in \mathbb{Z}};\ \sum_{k \in \mathbb{Z}} |\xi_k|^p\, w_k < +\infty\}$$

by setting $S\xi = (\xi_{k+1})_{k \in \mathbb{Z}}$ for each $\xi \in \ell_p(\mathbb{Z}, w)$, then S is shown in [10] to be bounded, and the ergodic theorem of [12] is then used to prove that for any function $f \in L^{2p}(X, \mathcal{B}, \mu)$ the sequence $(f(T^k x))_{k \in \mathbb{Z}}$ belongs to $\ell_p(\mathbb{Z}, w)$ for μ-almost every $x \in X$. Setting

$$\begin{aligned} \Phi_f : (X, \mathcal{B}, \mu) &\longrightarrow (\ell_p(\mathbb{Z}, w), \mathcal{B}_{\ell_p(\mathbb{Z}, w)}, \nu_f) \\ x &\longmapsto (f(T^k x))_{k \in \mathbb{Z}} \end{aligned}$$

where ν_f is the measure on $\ell_p(\mathbb{Z}, w)$ defined by $\nu_f(B) = \mu(\Phi_f^{-1}(B))$ for any Borel subset B of $\ell_p(\mathbb{Z}, w)$, it is easy to check that Φ_f intertwines the actions of T on $(X, \mathcal{B}, \mu)$ and of S on $\ell_p(\mathbb{Z}, w)$. The last (and most difficult) step of the proof of [10] is then to construct a function f such that Φ_f is an isomorphism of dynamical systems and ν_f has full support.

Our aim in this paper is to present an alternative construction of universal operators, which is elementary in the sense that it avoids the use of an ergodic theorem such as the one of [12]. It is also more flexible than the construction of [10], yields some rather simple criteria for universality, and allows us to show the existence of universal operators on a large class of Banach spaces. Moreover, this construction makes it possible to exhibit operators which are universal for *all* ergodic dynamical systems, not only for invertible ones. As we will often need to make a distinction between these two notions, we introduce the following definition:

Definition 1.2. Let A be a bounded operator on a real or complex Banach space Z.

- We say that A is *universal for invertible ergodic systems* if for every invertible ergodic dynamical system $(X, \mathcal{B}, \mu; T)$ on a standard Lebesgue probability space there exists a probability measure ν on Z with full support which is A-invariant and such that the dynamical systems $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$ are isomorphic.
- We say that A is *universal for ergodic systems* if the same property holds true for all ergodic dynamical systems $(X, \mathcal{B}, \mu; T)$ on a standard Lebesgue probability space.

Universal operators in the sense of Glasner and Weiss are universal for invertible ergodic systems. When we use simply the term "universal operator" in the rest of the paper, we will mean an operator which is universal either for all ergodic

systems or just for invertible ones. Before stating our main results, we introduce the following intuitive notation: suppose that A is a bounded operator on a real or complex separable Banach space Z, and suppose that $(z_n)_{n\in\mathbb{Z}}$ is a sequence of vectors of Z such that, for every $n \in \mathbb{Z}$, $Az_n = z_{n+1}$. We then write $z_n = A^n z_0$ for every $n \in \mathbb{Z}$.

Our first result consists of a general and simple criterion for an operator to be universal for invertible ergodic systems.

Theorem 1.3. *Let A be a bounded operator on a real or complex separable Banach space Z. Suppose that there exists a sequence $(z_n)_{n\in\mathbb{Z}}$ of vectors of Z such that, for every $n \in \mathbb{Z}$, $Az_n = z_{n+1}$, and such that the following three properties hold true:*

(a) *the vector z_0 is bicyclic, i.e. $\overline{\operatorname{span}}\,[A^{-n}z_0;\ n \in \mathbb{Z}]=Z$;*
(b) *there exists a finite subset F of $\mathbb{Z}$ such that $\overline{\operatorname{span}}\,[A^{-n}z_0;\ n \in \mathbb{Z}\setminus F] \neq Z$;*
(c) *the series $\sum_{n\in\mathbb{Z}} A^{-n}z_0$ is unconditionally convergent in Z.*

Then A is universal for invertible ergodic systems.

There is a very similar criterion which implies that an operator is universal for all ergodic systems:

Theorem 1.4. *Let A be a bounded operator on a real or complex separable Banach space Z. If A satisfies the assumptions of Theorem* 1.3*, and if moreover the sequence $(z_n)_{n\in\mathbb{Z}}$ is such that $A^r z_0 = 0$ for some $r \in \mathbb{Z}$ (or, equivalently, such that $z_0 = 0$), then A is universal for ergodic systems.*

We have already mentioned that a universal operator is necessarily frequently hypercyclic. An operator satisfying the assumptions of either Theorem 1.3 or Theorem 1.4 is easily seen to satisfy the Frequent Hypercyclicity Criterion of [7] (see also [4] or [11]), and so is in particular frequently hypercyclic and chaotic.

The proofs of Theorems 1.3 and 1.4 largely rely on the ideas of [10], but some extra work is needed, in particular in order to cope with the condition (b) in both theorems. The proofs would be simpler if we assumed that $F = \{0\}$ (which is what happens in some of the examples, in particular in those of [10]), but the generality of assumption (b) is needed in several of the examples given in Section 4.

The proofs of Theorems 1.3 and 1.4 are presented in Section 2, as well as two generalizations of these results (Theorems 2.5 and 2.6) in which assumption (a) is relaxed. The next two sections are devoted to applications and examples. In Section 3, we characterize universal operators (both for ergodic systems and for invertible ergodic systems) among unilateral or bilateral weighted backward shifts on the spaces $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$. Recall that if $(e_n)_{n\geq 0}$ denotes the canonical basis of $\ell_p(\mathbb{N})$, or $c_0(\mathbb{N})$, and $(w_n)_{n\geq 1}$ is a bounded sequence of non-zero complex numbers, the weighted backward shift B_w is defined on $\ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$ by setting $B_w e_0 = 0$ and $B_w e_n = w_n e_{n-1}$ for every $n \geq 1$. In the same way, if $(f_n)_{n\in\mathbb{Z}}$ is the canonical basis of $\ell_p(\mathbb{Z})$ or $c_0(\mathbb{Z})$, and $(w_n)_{n\in\mathbb{Z}}$ is again a bounded sequence of non-zero complex numbers, the bilateral weighted shift S_w on $\ell_p(\mathbb{Z})$ or $c_0(\mathbb{Z})$ is defined by setting $S_w e_n = w_n e_{n-1}$ for every $n \in \mathbb{Z}$. Here is the characterization of universal weighted shifts which can be obtained thanks to Theorems 1.3 and 1.4:

Theorem 1.5. *With the notation above, the unilateral backward weighted shift B_w is universal for (invertible) ergodic systems on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, if and only if*

the series

$$\sum_{n\geq 1} \frac{1}{|w_1 \dots w_n|^p}$$

is convergent. It is universal for (invertible) ergodic systems on $c_0(\mathbb{N})$ *if and only if* $|w_1 \dots w_n| \longrightarrow 0$ *as* $n \longrightarrow +\infty$.

In the same way, the bilateral backward weighted shift S_w *is universal for (invertible) ergodic systems on* $\ell_p(\mathbb{Z})$, $1 \leq p < +\infty$, *if and only if the series*

$$\sum_{n\geq 1} \frac{1}{|w_1 \dots w_n|^p} + \sum_{n\geq 1} |w_0 \dots w_{-(n-1)}|^p$$

is convergent. It is universal for (invertible) ergodic systems on $c_0(\mathbb{Z})$ *if and only if* $|w_1 \dots w_n| \longrightarrow 0$ *and* $|w_0 \dots w_{-(n-1)}| \longrightarrow +\infty$ *as* $n \longrightarrow +\infty$.

This result shows in particular the existence of universal operators for ergodic systems living on any of the spaces $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$. The existence of universal operators for invertible ergodic systems on $\ell_p(\mathbb{N})$, $1 < p < +\infty$, is already proved in [10]. A natural question, asked in [10], is to determine which Banach (or Fréchet) spaces support a universal operator. As a universal operator is necessarily frequently hypercyclic and some Banach spaces (like the hereditarily indecomposable spaces, for instance) do not support frequently hypercyclic operators, it follows that not all Banach spaces support a universal operator. But, as a consequence of Theorem 1.3, we obtain the existence of such operators on Banach spaces with a sufficiently rich structure.

Theorem 1.6. *Let* Z *be a separable infinite-dimensional Banach space containing a complemented copy of a space with a subsymmetric basis. Then* Z *supports an operator which is universal for all ergodic systems.*

This result implies for example that any separable Banach space containing a complemented copy of one of the spaces $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$, supports a universal operator. This is the case for all spaces $L^p(\Omega, \mu)$, where (Ω, μ) is a σ-finite measured space.

If A is a bounded operator on a complex infinite-dimensional separable Hilbert space H, it is known (see [2] or [4, Ch. 5]) that A admits an invariant measure with respect to which it is ergodic, and which additionally has full support and admits a second-order moment, if and only if its unimodular eigenvectors are *perfectly spanning*: this means that there exists a continuous probability measure σ on the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C},\ |\lambda| = 1\}$ such that for any Borel subset B of $\mathbb{T}$ with $\sigma(B) = 1$,

$$\overline{\operatorname{span}}\left[\ker(A - \lambda),\ \lambda \in B\right] = H.$$

In this case, the unimodular eigenvectors of A are said to be *σ-spanning*. An eigenvectorfield E of A is a map $E : \mathbb{T} \longrightarrow Z$ such that $A\,E(\lambda) = \lambda\,E(\lambda)$ for every $\lambda \in \mathbb{T}$. We will often be dealing in the rest of the paper with eigenvectorfields E belonging to $L^2(\mathbb{T}, \sigma; Z)$ where σ is a certain probability measure on $\mathbb{T}$: this means that $E : \mathbb{T} \longrightarrow Z$ is σ-measurable, with

$$\int_{\mathbb{T}} ||E(\lambda)||^2 d\sigma(\lambda) < +\infty$$

and $A\,E(\lambda) = \lambda\,E(\lambda)$ σ-almost everywhere. When we write simply that E belongs to $L^2(\mathbb{T}; Z)$, this means that σ is assumed to be the normalized Lebesgue measure $d\lambda$ on $\mathbb{T}$.

As a universal operator on H is necessarily ergodic with respect to a certain invariant measure with full support (although this measure is not required to have a second-order moment), it is natural to look for conditions involving the unimodular eigenvectors of the operator A which imply its universality. This is done in Section 4, where we prove the following two general results.

Theorem 1.7. *Let A be a bounded operator on a complex Banach space Z. Suppose that there exists an eigenvectorfield $E \in L^2(\mathbb{T}; Z)$ for A such that*

(i) *whenever B is a Borel subset of $\mathbb{T}$ of full Lebesgue measure, $\overline{span}\,[E(\lambda),\ \lambda \in B] = Z$;*
(ii) *there exists a non-zero functional $z_0^* \in Z^*$ and a trigonometric polynomial p such that $\langle z_0^*, E(\lambda)\rangle = p(\lambda)$ almost everywhere on $\mathbb{T}$;*
(iii) *if we set for every $n \in \mathbb{Z}$*

$$\widehat{E}(n) = \int_{\mathbb{T}} \lambda^{-n} E(\lambda)\, d\lambda,$$

then the series $\sum_{n\in\mathbb{Z}} \widehat{E}(n)$ is unconditionally convergent.

Then the operator A is universal for invertible ergodic systems. If moreover we have $\widehat{E}(-r) = 0$ for some integer $r \in \mathbb{Z}$ (so that $\widehat{E}(-n) = 0$ for every $n \geq r$), then A is universal for ergodic systems.

Remark that if E is sufficiently smooth (of class $\mathcal{C}^1$ for instance), then the sequence of Fourier coefficients $(\widehat{E}(n))_{n\in\mathbb{Z}}$ goes to zero sufficiently rapidly for the series $\sum_{n\in\mathbb{Z}} ||\widehat{E}(n)||$ to be convergent. Hence the assumption (iii) is automatically satisfied in this case. If E is analytic in a neighborhood of $\mathbb{T}$, it can be renormalized in such a way that assumption (ii) is also satisfied, and this yields

Theorem 1.8. *Suppose that $A \in \mathcal{B}(Z)$ admits an eigenvectorfield E which is analytic in a neighborhood of $\mathbb{T}$ and that $\overline{\operatorname{span}}\left[E(\lambda);\ \lambda \in \mathbb{T}\right] = Z$. Then A is universal for invertible ergodic systems. If E is analytic in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$ and $\overline{\operatorname{span}}\left[E(\lambda);\ \lambda \in \mathbb{T}\right] = Z$, then A is universal for ergodic systems.*

Several applications of these two theorems are given in Section 4, in particular to adjoints of multipliers on $H^2(\mathbb{D})$ (Example 4.2) and to the rather unexpected case of a Kalish-type operator on $L^2(\mathbb{T})$ (Example 4.4).

In Section 5 we try to exhibit some necessary conditions for an operator to be universal. In this generality, and with the present definition of universality, this seems to be delicate. But if we restrict ourselves to operators acting on a Hilbert space, and if we additionally require in the definition of universality that the measure ν admits a moment of order 2 (see Definition 5.1), then we obtain:

Theorem 1.9. *Suppose that $A \in \mathcal{B}(H)$ is universal for ergodic systems in the modified sense presented above. Then the unimodular eigenvalues of A form a subset of $\mathbb{T}$ of Lebesgue measure* 1.

It is a rather puzzling fact that we do not know whether Theorem 1.9 can be extended to operators which are universal for invertible ergodic systems only. This point is discussed in Section 5, as well as some open questions.

2. A criterion for universality: Proofs of Theorems 1.3 and 1.4

The proofs of Theorems 1.3 and 1.4 being very similar, we concentrate on the proof of Theorem 1.3 and will indicate briefly afterwards the modifications needed for proving Theorem 1.4.

2.1. **General pattern of the proof of Theorem 1.3.** Let A be a bounded operator on the infinite-dimensional separable Banach space Z satisfying the assumptions of Theorem 1.3. We will suppose in the rest of the proof that Z is a complex Banach space, but the proof obviously holds true for real spaces as well. Let $(X, \mathcal{B}, \mu; T)$ be an invertible ergodic dynamical system on a standard probability space. For any complex-valued function $f \in L^\infty(X, \mathcal{B}, \mu)$, let Φ_f be the map from $(X, \mathcal{B}, \mu)$ into Z defined by setting

$$\Phi_f(x) = \sum_{k\in\mathbb{Z}} f(T^k x)\, A^{-k} z_0.$$

Since f is essentially bounded and the series $\sum_{k\in\mathbb{Z}} A^{-k} z_0$ is unconditionally convergent, $\Phi_f(x)$ is well defined for μ-almost every $x \in X$. Also we have

$$\Phi_f(Tx) = \sum_{k\in\mathbb{Z}} f(T^{k+1}x)\, A^{-k} z_0 = \sum_{k\in\mathbb{Z}} f(T^k x)\, A^{-k+1} z_0 = A\, \Phi_f(x)$$

since $A^{-k+1} z_0 = z_{-k+1} = A z_{-k} = A.A^{-k} z_0$.

If we denote by $\mathcal{B}_Z$ the Borel σ-field of Z and by ν_f the Borel probability measure on Z which is the image of μ under the map Φ_f (i.e. $\nu_f(B) = \mu(\Phi_f^{-1}(B))$ for every Borel subset B of Z), it then follows that $\Phi_f : (X, \mathcal{B}, \mu; T) \longrightarrow (Z, \mathcal{B}_Z, \nu_f; A)$ is a factor map. This first argument is rather similar to the one employed in [10], but the map Φ_f is defined differently in [10], and for any $f \in L^4(X, \mathcal{B}, \mu)$, thanks to the ergodic theorem of [12]. The goal in [10] is then to construct a function $f \in L^4(X, \mathcal{B}, \mu)$ as a limit of certain finitely-valued functions $f_n \in L^\infty(X, \mathcal{B}, \mu)$, in such a way that Φ_f becomes an isomorphism of dynamical systems and the measure ν_f has full support. As Φ_f is not necessarily well defined here when $f \in L^4(X, \mathcal{B}, \mu)$ (or $f \in L^p(X, \mathcal{B}, \mu)$, $1 < p < +\infty$), we will construct, in the same spirit as in [10], a sequence of finitely-valued functions $f_n \in L^\infty(X, \mathcal{B}, \mu)$ such that Φ_{f_n} converges in $L^2(X, \mathcal{B}, \mu; Z)$ to a certain function $\Phi \in L^2(X, \mathcal{B}, \mu; Z)$ which will be an isomorphism between the two systems $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$, where ν is the image of μ under the map Φ.

Let $(Q_j)_{j\geq 0}$ be a sequence of Borel subsets of X which is dense in $(\mathcal{B}, \mu)$ (i.e. for every $\varepsilon > 0$ and every $B \in \mathcal{B}$, there exists a $j \geq 0$ such that $\mu(Q_j \triangle B) < \varepsilon$) with $Q_0 = X$. Moreover, we suppose that for any $i \geq 0$, the set $J_i = \{j \geq i \,;\, Q_j = Q_i\}$ is infinite. Since the span of the vectors $A^k z_0$, $k \in \mathbb{Z}$, is dense in Z, there exists a sequence $(u_n)_{n\geq 1}$ of vectors of Z of the form

$$u_n = \sum_{|k|\leq d_n} a_k^{(n)}\, A^{-k} z_0, \quad a_k^{(n)} \in \mathbb{C}, \quad \max_{|k|\leq d_n} |a_k^{(n)}| > 0$$

which is dense in Z. For each $n \geq 1$, let r_n be a positive number such that the open balls $U_n = B(u_n, r_n)$ centered at u_n and of radius r_n form a basis of the topology of Z. We set $U_0 = Z$. Lastly, by assumption (b) of Theorem 1.3 there exists a finite subset F of $\mathbb{Z}$ and a non-zero functional $z_0^* \in Z^*$ such that $\langle z_0^*, A^{-n} z_0\rangle = 0$ for all $n \in \mathbb{Z} \setminus F$ and $\langle z_0^*, A^{-n} z_0\rangle \neq 0$ for all $n \in F$ (we may have to modify the initial set F to obtain this property). If we replace the vector z_0 by the vector $z_0' = A^{-p} z_0$,

then $\langle z_0^*, A^{-n} z_0' \rangle \neq 0$ if and only if $n \in F' = F - p$. If we choose $p \in F$, then $\langle z_0^*, z_0' \rangle \neq 0$. So, replacing z_0 by z_0' and F by F', we can suppose that $0 \in F$ and that z_0 and z_0^* are such that $c_n = \langle z_0^*, A^{-n} z_0 \rangle$ is non-zero if and only if $n \in F$. We let $d = \max |F|$.

2.2. **Construction of the functions f_n, $n \geq 0$.** We are now ready to start the construction of the functions f_n. This construction is very much inspired from that of [10], but many technical details need to be adjusted to the present situation. For any $z \in \mathbb{C}$ and $r > 0$, $D(z, r)$ denotes the open disk centered at z of radius r.

We construct by induction

- a sequence $(f_n)_{n \geq 0}$ of functions of $L^\infty(X, \mathcal{B}, \mu; \mathbb{C})$;
- sequences $(\alpha_n)_{n \geq 0}$, $(\beta_n)_{n \geq 0}$, $(\gamma_n)_{n \geq 0}$, $(\delta_n)_{n \geq 0}$, and $(\eta_n)_{n \geq 0}$ of positive real numbers, decreasing to zero extremely fast;
- for each $n \geq 0$, families $(D_{i,0}^{(n)})_{0 \leq i \leq n}$ and $(D_{i,1}^{(n)})_{0 \leq i \leq n}$, $(E_{i,0}^{(n)})_{0 \leq i \leq n}$ and $(E_{i,1}^{(n)})_{0 \leq i \leq n}$, $(F_{i,0}^{(n)})_{0 \leq i \leq n}$ and $(F_{i,1}^{(n)})_{0 \leq i \leq n}$ of Borel subsets of $\mathbb{C}$;
- for each $n \geq 0$, families $(G_i^{(n)})_{0 \leq i \leq n}$ and $(H_i^{(n)})_{0 \leq i \leq n}$ of μ-measurable subsets of X, and two measurable subsets B_n and C_n of X

such that:

(1) The sets $E_{i,0}^{(n)}$ and $E_{i,1}^{(n)}$, $0 \leq i \leq n$, are finite, and the range of f_n is equal to
$$\left(\bigcup_{i=0}^{n} E_{i,0}^{(n)} \right) \cup \left(\bigcup_{i=0}^{n} E_{i,1}^{(n)} \right);$$
moreover, for every $i \in \{0, \ldots, n\}$, $E_{i,0}^{(n)} \cap E_{i,1}^{(n)} = \varnothing$.

(2) We have
  (2a) $\mu(C_n) > 1 - \eta_n$;
  (2b) if we set, for every $i \in \{0, \ldots, n\}$,
$$D_{i,0}^{(n)} = \Big\{ \sum_{p \in F} c_p \, f_n(T^p x);\ x \in C_n \text{ and } f_n(x) \in E_{i,0}^{(n)} \Big\}$$
and
$$D_{i,1}^{(n)} = \Big\{ \sum_{p \in F} c_p \, f_n(T^p x);\ x \in C_n \text{ and } f_n(x) \in E_{i,1}^{(n)} \Big\},$$
then $D_{i,0}^{(n)} \cap D_{i,1}^{(n)} = \varnothing$;
  (2c) if we set, for every $i \in \{0, \ldots, n\}$,
$$F_{i,0}^{(n)} = D_{i,0}^{(n)} + D(0, \beta_n) \quad \text{and} \quad F_{i,1}^{(n)} = D_{i,1}^{(n)} + D(0, \beta_n),$$
then $F_{i,0}^{(n)} \cap F_{i,1}^{(n)} = \varnothing$.

(3) For every $i \in \{0, \ldots, n\}$,
  (3a) $\mu(H_i^{(n)}) < \alpha_i (1 - 2^{-n})$;
  (3b) $H_i^{(n-1)} \subseteq H_i^{(n)}$ for every $i \in \{0, \ldots, n-1\}$;
  (3c) for every $x \in Q_i \setminus H_i^{(n)}$, $f_n(x) \in E_{i,0}^{(n)}$,
  and
  for every $x \in (X \setminus Q_i) \setminus H_i^{(n)}$, $f_n(x) \in E_{i,1}^{(n)}$.

(4) We have
  (4a) $\mu(B_n) > 1 - \eta_n$;
  (4b) for every $x \in B_n$, $|f_n(x) - f_{n-1}(x)| < \gamma_n$;
  (4c) for every $x \in B_n$, $||\Phi_{f_n}(x) - \Phi_{f_{n-1}}(x)|| < \gamma_n$.
(5) We have
  (5a) $||f_n - f_{n-1}||_{L^2(X,\mathcal{B},\mu)} < 2^{-n}$;
  (5b) $||\Phi_{f_n} - \Phi_{f_{n-1}}||_{L^2(X,\mathcal{B},\mu;Z)} < 2^{-n}$.
(6) For every $i \in \{0, \dots, n\}$,
  (6a) $\mu(G_i^{(n)}) \geq \delta_i(1 + 2^{-n})$;
  (6b) $G_i^{(n)} \subseteq G_i^{(n-1)}$ for every $i \in \{0, \dots, n-1\}$;
  (6c) $\Phi_{f_n}(x) \in U_i$ for every $x \in G_i^{(n)}$.

We start the construction by setting (recall that $Q_0 = X$ and $U_0 = Z$): $E_{0,0}^{(0)} = \{0\}$, $E_{0,1}^{(0)} = \varnothing$, $B_0 = C_0 = X$, $\alpha_0 = \beta_0 = \gamma_0 = \delta_0 = \eta_0 = 1/8$, $G_0^{(0)} = X$, $H_0^{(0)} = \varnothing$ and $f_0 = 0$.

Suppose now that the construction has been carried out until step n. At step $n+1$, we start by introducing

- an integer $N \geq 1$, which will be chosen very large at the end of the construction;
- two positive numbers η and γ, independent of each other, which will be chosen very small at the end of the construction.

As T is invertible and ergodic, there exists a measurable subset E of X such that $\mu(E) > 0$, $\mu\left(\bigcup_{|k|\leq N} T^k E\right) < \eta$ and the sets $T^k E$, $|k| \leq N$, are pairwise disjoint.

Recall that

$$u_{n+1} = \sum_{|k|\leq d_{n+1}} a_k^{(n+1)} A^{-k} z_0 \quad \text{and} \quad U_{n+1} = B(u_{n+1}, r_{n+1}).$$

We suppose that $N \geq d_{n+1}$.

Step 1. We first define an auxiliary function g_{n+1} on X in the following way:

$$g_{n+1}(x) = \begin{cases} a_k^{(n+1)} & \text{if } x \in T^k E, \quad |k| \leq d_{n+1}, \\ 0 & \text{if } x \in T^k E, \quad d_{n+1} < |k| \leq N, \\ f_n(x) & \text{if } x \notin \bigcup\limits_{|k|\leq N} T^k E. \end{cases}$$

The function g_{n+1} thus defined is finite-valued and it coincides with f_n on the set

$$B = X \setminus \bigcup_{|k|\leq N} T^k E,$$

which has μ-measure larger than $1 - \eta$. The range of g_{n+1} is equal to

$$\mathrm{Ran}(f_n \restriction_B) \cup \{0\} \cup \{a_k^{(n+1)};\ |k| \leq d_{n+1}\},$$

and we write this finite set as $\{c_l^{(n+1)};\ 0 \leq l \leq l_{n+1}\}$, with all numbers $c_l^{(n+1)}$ distinct.

By the Rokhlin Lemma, we can choose a subset $E' \in \mathcal{B}$ of X and an integer $M \geq d$ such that the sets $T^k E'$, $|k| \leq M$, are pairwise disjoint, and

$$\mu\Big(\bigcup_{|k|\leq M-d} T^k E'\Big) > 1-\eta.$$

Step 2. We state and prove in this step a simple abstract lemma which will be used in the forthcoming Steps 3 and 4 in order to approximate certain finite families of scalars (like the family $(c_l^{(n+1)})_{0\leq l\leq l_{n+1}}$) by other families of scalars with further additional properties.

Lemma 2.1. *Let $r \geq 1$ and let $\boldsymbol{d} = (d_1, \ldots, d_r)$ be an r-tuple of positive integers. We denote by $E_{\boldsymbol{d}}$ the subset of $\mathbb{Z}^r$ defined by*

$$E_{\boldsymbol{d}} = \{\boldsymbol{u} = (u_1, \ldots, u_r);\ 0 \leq u_i \leq d_i \text{ for every } i = 1, \ldots, r\}.$$

For every $i = 1, \ldots, r$, let λ_i be a map from F into $\{0, \ldots, d_i\}$. We denote by $\boldsymbol{\lambda}$ the r-tuple of maps $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ from F into $\{0, \ldots, d_1\} \times \ldots \times \{0, \ldots, d_r\}$. For any such $\boldsymbol{\lambda}$, let $\sigma_{\boldsymbol{\lambda}}$ be the functional on the vector space of functions from $E_{\boldsymbol{d}}$ into $\mathbb{C}$, identified with $\mathbb{C}^{\#E_{\boldsymbol{d}}}$, defined by

$$\begin{aligned} \sigma_{\boldsymbol{\lambda}} :\quad & \mathbb{C}^{\#E_{\boldsymbol{d}}} \longrightarrow \mathbb{C} \\ & \big(\gamma_{\boldsymbol{u}}\big)_{\boldsymbol{u}\in E_{\boldsymbol{d}}} \longmapsto \sum_{p\in F} c_p\, \gamma_{\boldsymbol{\lambda}(p)}. \end{aligned}$$

There exists a dense subset of $\mathbb{C}^{\#E_{\boldsymbol{d}}}$ consisting of elements $(\gamma_{\boldsymbol{u}})_{\boldsymbol{u}\in E_{\boldsymbol{d}}}$ with the following property:

$$\sigma_{\boldsymbol{\lambda}}((\gamma_{\boldsymbol{u}})_{\boldsymbol{u}\in E_{\boldsymbol{d}}}) \neq \sigma_{\boldsymbol{\lambda}'}((\gamma_{\boldsymbol{u}})_{\boldsymbol{u}\in E_{\boldsymbol{d}}})$$

for all maps $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ such that $\boldsymbol{\lambda}(F) \neq \boldsymbol{\lambda}'(F)$.

Proof. Let us first observe that if $\boldsymbol{\lambda}(F) \neq \boldsymbol{\lambda}'(F)$, $\sigma_{\boldsymbol{\lambda}} \neq \sigma_{\boldsymbol{\lambda}'}$. This follows from the fact that all coefficients c_p, $p \in F$, are distinct. Then let

$$\boldsymbol{\Sigma} = \{(\boldsymbol{\lambda}, \boldsymbol{\lambda}');\ \boldsymbol{\lambda}(F) \neq \boldsymbol{\lambda}'(F)\}.$$

For each $(\boldsymbol{\lambda}, \boldsymbol{\lambda}') \in \boldsymbol{\Sigma}$, the kernel $\ker(\sigma_{\boldsymbol{\lambda}} - \sigma_{\boldsymbol{\lambda}'})$ is different from the whole space $\mathbb{C}^{\#E_{\boldsymbol{d}}}$. The set $\boldsymbol{\Sigma}$ being finite, the Baire Category Theorem yields that

$$\bigcup_{(\boldsymbol{\lambda},\boldsymbol{\lambda}')\in\boldsymbol{\Sigma}} (\sigma_{\boldsymbol{\lambda}} - \sigma_{\boldsymbol{\lambda}'})^{-1}(\mathbb{C}^*)$$

is dense in $\mathbb{C}^{\#E_{\boldsymbol{d}}}$, which proves our claim. □

Step 3. We define a second auxiliary function h_{n+1} on X by setting

$$h_{n+1}(x) = \begin{cases} c_{l,\,k}^{(n+1)} & \text{if } g_{n+1}(x) = c_l^{(n+1)} \text{ and } x \in T^k E' \text{ for some } |k| \leq M, \\ c_l^{(n+1)} & \text{if } g_{n+1}(x) = c_l^{(n+1)} \text{ and } x \notin \bigcup_{|k|\leq M} T^k E', \end{cases}$$

where for every $l \in \{0, \ldots, l_{n+1}\}$, $c_{l,\,k}^{(n)}$ is so close to $c_l^{(n+1)}$ for each $|k| \leq M$ that

$$||h_{n+1} - g_{n+1}||_\infty \leq \frac{\gamma}{2},$$

all the numbers $c_{l,\,k}^{(n+1)}$, $l \in \{0, \ldots, l_{n+1}\}$, $|k| \leq M$, and $c_l^{(n+1)}$, $l \in \{0, \ldots, l_{n+1}\}$, are distinct, and, moreover, the numbers $c_{l,\,k}^{(n+1)}$, $l \in \{0, \ldots, l_{n+1}\}$, $|k| \leq M$, have the following property.

Whenever τ, τ' are two maps from F into $\{0,\dots,l_{n+1}\}$ and k, k' are two integers with $|k|$, $|k'| \le M-d$, we have

$$\sum_{p\in F} c_p\, c^{(n+1)}_{\tau(p),\,k+p} \neq \sum_{p\in F} c_p\, c^{(n+1)}_{\tau'(p),\,k'+p}$$

as soon as there exists a $p\in F$ such that $(\tau(p),k+p)\neq(\tau'(p),k'+p)$.

Observe that since $|k|$, $|k'|\le M-d$ and $d=\max|F|$, $|k+p|$, $|k'+p|\le M$ for every $p\in F$, so that the quantities $c^{(n+1)}_{\tau(p),\,k+p}$ and $c^{(n+1)}_{\tau'(p),\,k'+p}$ in the expression above are well defined.

That the scalars $c^{(n+1)}_{l,\,k}$ can indeed be chosen so as to satisfy these properties is a consequence of Lemma 2.1. Denote by Σ the set of all 4-tuples (τ,τ',k,k'), where τ,τ' are maps from F into $\{0,\dots,l_{n+1}\}$ and k,k' are integers with $|k|$, $|k'|\le M-d$, such that there exists a $p\in F$ with $(\tau(p),k+p)\neq(\tau'(p),k'+p)$. For any map $\tau: F\longrightarrow\{0,\dots,l_{n+1}\}$ and any integer k with $|k|\le M-d$, let

$$\begin{aligned}\boldsymbol{\lambda}_{\tau,\,k}\ :\qquad & F\longrightarrow\{0,\dots,l_{n+1}\}\times\{-M,\dots,M\}\\ & p\longmapsto(\tau(p),p+k).\end{aligned}$$

Let us check that if (τ,τ',k,k') belongs to Σ, $\boldsymbol{\lambda}_{\tau,\,k}(F)\neq\boldsymbol{\lambda}_{\tau',\,k'}(F)$. If $\boldsymbol{\lambda}_{\tau,\,k}(F)=\boldsymbol{\lambda}_{\tau',\,k'}(F)$, then

$$\big\{(\tau(p),\,k+p);\ p\in F\big\}=\big\{(\tau'(p),k'+p);\ p\in F\big\},$$

so that $k+F=k'+F$. As the set F is finite, $k=k'$, and thus for every $p\in F$ there exists a $p'\in F$ such that $(\tau(p),k+p)=(\tau'(p'),k+p')$. So $p=p'$ and $\tau(p)=\tau'(p)$. Thus $(\tau(p),k+p)=(\tau'(p),k+p)$ for every $p\in F$, which is contrary to our assumption. So $\boldsymbol{\lambda}_{\tau,\,k}(F)\neq\boldsymbol{\lambda}_{\tau',\,k'}(F)$ as soon as (τ,τ',k,k') belongs to Σ.

Applying Lemma 2.1, it follows from the observation above that we can choose a family of scalars $\big(c^{(n+1)}_{l,\,k}\big)_{0\le l\le l_{n+1},\,|k|\le M}$ such that

$$\big|c^{(n+1)}_{l,\,k}-c^{(n+1)}_{l}\big|<\frac{\gamma}{2}\qquad\text{for every } l\in\{0,\dots,l_{n+1}\}\text{ and }|k|\le M,$$

all the numbers $c^{n+1}_{l,\,k}$ and $c^{(n+1)}_{l}$ are distinct, and

$$\big(\sigma_{\boldsymbol{\lambda}_{\tau,\,k}}-\sigma_{\boldsymbol{\lambda}_{\tau',\,k'}}\big)\big(\big(c^{(n+1)}_{l,\,k}\big)_{0\le l\le l_{n+1},\,|k|\le M}\big)\neq 0\quad\text{for every }(\tau,\,\tau',\,k,\,k')\in\Sigma,$$

i.e.

$$\sum_{p\in F} c_p\, c^{(n+1)}_{\tau(p),\,k+p}\neq\sum_{p\in F} c_p\, c^{(n+1)}_{\tau'(p),\,k'+p}\quad\text{for every }(\tau,\,\tau',\,k,\,k')\in\Sigma.$$

Now that the function h_{n+1} has been defined, we observe that it is finite-valued, and we write its range as

$$\{b^{(n+1)}_j;\ 0\le j\le j_{n+1}\}$$

where all the numbers $b^{(n+1)}_j$ are distinct. We also set

$$C_{n+1}=\bigcup_{|k|\le M-d}T^kE'.$$

By our assumptions on M and E', $\mu(C_{n+1})>1-\eta$.

Step 4. We construct in this step complex numbers $b_{j,0}^{(n+1)}$ and $b_{j,1}^{(n+1)}$, $0 \leq j \leq j_{n+1}$, which are such that all the numbers $b_{j,0}^{(n+1)}$ and $b_{j,1}^{(n+1)}$ are distinct, and both $b_{j,0}^{(n+1)}$ and $b_{j,1}^{(n+1)}$ are so close to $b_j^{(n+1)}$ for each $j \in \{0, \dots, j_{n+1}\}$ that

$$\sup_{j \in \{0,\dots,j_{n+1}\}} \Big(|b_{j,0}^{(n+1)} - b_j^{(n+1)}| + |b_{j,1}^{(n+1)} - b_j^{(n+1)}| \Big) < \frac{\gamma}{2}\cdot$$

Moreover, if $b_j^{(n+1)} = c_{l,k}^{(n+1)}$ for some $l \in \{0, \dots, l_{n+1}\}$ and $|k| \leq M$, we write

$$b_{j,0}^{(n+1)} = c_{l,k,0}^{(n+1)} \quad \text{and} \quad b_{j,1}^{(n+1)} = c_{l,k,1}^{(n+1)},$$

and we require that the following holds true:

For any maps $\theta, \theta' : F \longrightarrow \{0,1\}$, $\tau, \tau' : F \longrightarrow \{0, \dots, l_{n+1}\}$ and any integers k, k' with $|k|, |k'| \leq M - d$,

$$\sum_{p \in F} c_p \, c_{\tau(p),\, k+p,\, \theta(p)}^{(n+1)} \neq \sum_{p \in F} c_p \, c_{\tau'(p),\, k'+p,\, \theta'(p)}^{(n+1)}$$

as soon as there exists a $p \in F$ such that $(\tau(p), k+p, \theta(p)) \neq (\tau'(p), k'+p, \theta'(p))$.

The proof of the existence of such numbers again relies on Lemma 2.1. Denote by $\mathcal{F}$ the set of all 6-tuples $(\tau, \tau', k, k', \theta, \theta')$, where τ, τ' are maps from F into $\{0, \dots, l_{n+1}\}$, θ, θ' are maps from F into $\{0,1\}$, and k, k' are integers with $|k|, |k'| \leq M - d$, such that there exists a $p \in F$ with $(\tau(p), k+p, \theta(p)) \neq (\tau'(p), k'+p, \theta'(p))$. For any maps $\tau : F \longrightarrow \{0, \dots, l_{n+1}\}$, $\theta : F \longrightarrow \{0,1\}$, and any integer k with $|k| \leq M - d$, let

$$\begin{aligned} \boldsymbol{\lambda}_{\tau,k,\theta} : \qquad & F \longrightarrow \{0, \dots, l_{n+1}\} \times \{-M, \dots, M\} \times \{0,1\} \\ & p \longmapsto (\tau(p), k+p, \theta(p)). \end{aligned}$$

We claim that if $(\tau,\, \tau',\, k,\, k',\, \theta,\, \theta')$ belongs to $\mathcal{F}$, then $\boldsymbol{\lambda}_{\tau,k,\theta}(F) \neq \boldsymbol{\lambda}_{\tau',k',\theta'}(F)$. Indeed, if these two sets were equal, we would have

$$\big\{ \big(\tau(p),\, k+p,\, \theta(p)\big);\ p \in F \big\} = \big\{ \big(\tau'(p),\, k'+p,\, \theta'(p)\big);\ p \in F \big\}\cdot$$

Hence $k + F = k' + F$, so that $k = k'$. Thus for every $p \in F$ there exists $p' \in F$ such that $\big(\tau(p),\, k+p,\, \theta(p)\big) = \big(\tau'(p'),\, k+p',\, \theta'(p')\big)$. Necessarily, $p = p'$, so that $\tau(p) = \tau'(p)$ and $\theta(p) = \theta'(p)$. Hence $\big(\tau(p),\, k+p,\, \theta(p)\big) = \big(\tau'(p),\, k+p,\, \theta'(p)\big)$ for every $p \in F$, and this contradicts our initial assumption. So $\boldsymbol{\lambda}_{\tau,k,\theta}(F) \neq \boldsymbol{\lambda}_{\tau',k',\theta'}(F)$. It thus follows from Lemma 2.1 that numbers $c_{l,k,0}^{(n+1)}$ and $c_{l,k,1}^{(n+1)}$ can be chosen as close to $c_{l,k}^{(n+1)}$ as we wish, all distinct, and such that

$$\sum_{p \in F} c_p \, c_{\tau(p),\, k+p,\, \theta(p)}^{(n+1)} \neq \sum_{p \in F} c_p \, c_{\tau'(p),\, k'+p,\, \theta'(p)}^{(n+1)} \qquad \text{for every } (\tau,\, \tau',\, k,\, k',\, \theta,\, \theta') \in \mathcal{F}.$$

This defines $b_{j,0}^{(n+1)}$ and $b_{j,1}^{(n+1)}$ when $b_j^{(n+1)} = c_{l,k}^{(n+1)}$ for some $l \in \{0, \dots, l_{n+1}\}$ and $|k| \leq M$. It is then easy to define the numbers $b_{j,0}^{(n+1)}$ and $b_{j,1}^{(n+1)}$ for the remaining indices in such a way that they are sufficiently close to $b_j^{(n+1)}$, distinct, and distinct from all the numbers $c_{l,k,0}^{(n+1)}$ and $c_{l,k,1}^{(n+1)}$.

Step 5. We can now define the function f_{n+1} on X by setting

$$f_{n+1}(x) = \begin{cases} b_{j,0}^{(n+1)} & \text{if } h_{n+1}(x) = b_j^{(n+1)} \text{ and } x \in Q_{n+1}, \\ b_{j,1}^{(n+1)} & \text{if } h_{n+1}(x) = b_j^{(n+1)} \text{ and } x \in X \setminus Q_{n+1}. \end{cases}$$

Obviously

$$||h_{n+1} - f_{n+1}||_\infty < \frac{\gamma}{2}\cdot$$

If x belongs to C_{n+1}, then there exists an integer k with $|k| \leq M - d$ such that $x \in T^k E'$. Hence $T^p x \in T^{p+k} E'$ for every $p \in F$ and $|k+p| \leq M$. It follows that there exists a map $\tau : F \longrightarrow \{0, \dots, l_{n+1}\}$ such that

$$h_{n+1}(T^p x) = c^{(n+1)}_{\tau(p),\, k+p} \quad \text{for every } p \in F.$$

By the definition of the function f_{n+1}, there exists a map $\theta : F \longrightarrow \{0,1\}$ such that

$$f_{n+1}\big(T^p x\big) = c^{(n+1)}_{\tau(p),\, k+p,\, \theta(p)} \quad \text{for every } p \in F.$$

This map θ satisfies $\theta(0) = 0$ if $x \in Q_{n+1}$ and $\theta(0) = 1$ if $x \in X \setminus Q_{n+1}$. We have

$$\sum_{p \in F} c_p f_{n+1}\big(T^p x\big) = \sum_{p \in F} c_p\, c^{(n+1)}_{\tau(p),\, k+p,\, \theta(p)}\cdot$$

Step 6. For every $i \in \{0, \dots, n\}$, let

$$\begin{aligned} J^{(n+1)}_{i,\,0} = \Big\{ & j \in \{0, \dots, j_{n+1}\};\ \text{there exists } l \in \{0, \dots, l_{n+1}\} \text{ such that } c^{(n+1)}_l \in E^{(n)}_{i,\,0} \\ & \text{with either} \\ & b^{(n+1)}_j = c^{(n+1)}_l \text{ or } b^{(n+1)}_j = c^{(n+1)}_{l,\,k} \text{ for some } |k| \leq M \Big\} \\ J^{(n+1)}_{i,\,1} = \Big\{ & j \in \{0, \dots, j_{n+1}\};\ \text{there exists } l \in \{0, \dots, l_{n+1}\} \text{ such that } c^{(n+1)}_l \in E^{(n)}_{i,\,1} \\ & \text{with either} \\ & b^{(n+1)}_j = c^{(n+1)}_l \text{ or } b^{(n+1)}_j = c^{(n+1)}_{l,\,k} \text{ for some } |k| \leq M \Big\}. \end{aligned}$$

Set for $i \in \{0, \dots, n\}$:

$$\begin{aligned} E^{(n+1)}_{i,\,0} &= \big\{ b^{(n+1)}_{j,\,0};\ j \in J^{(n+1)}_{i,0} \big\} \cup \big\{ b^{(n+1)}_{j,\,1};\ j \in J^{(n+1)}_{i,0} \big\}, \\ E^{(n+1)}_{i,\,1} &= \big\{ b^{(n+1)}_{j,\,0};\ j \in J^{(n+1)}_{i,1} \big\} \cup \big\{ b^{(n+1)}_{j,\,1};\ j \in J^{(n+1)}_{i,1} \big\}, \\ E^{(n+1)}_{n+1,\,0} &= \big\{ b^{(n+1)}_{j,\,0};\ j \in \{0, \dots, j_{n+1}\} \big\}, \\ E^{(n+1)}_{n+1,\,1} &= \big\{ b^{(n+1)}_{j,\,1};\ j \in \{0, \dots, j_{n+1}\} \big\}. \end{aligned}$$

Step 7. With these definitions, let us check that property (1) holds true. Of course, all the sets $E^{(n+1)}_{i,\,0}$ and $E^{(n+1)}_{i,\,1}$ are finite and

$$\operatorname{ran}(f_{n+1}) = \Big(\bigcup_{i=0}^{n+1} E^{(n+1)}_{i,\,0} \Big) \cup \Big(\bigcup_{i=0}^{n+1} E^{(n+1)}_{i,\,1} \Big) = E^{(n+1)}_{n+1,\,0} \cup E^{(n+1)}_{n+1,\,1}.$$

All the numbers $b^{(n+1)}_{j,\,0}$ and $b^{(n+1)}_{j,\,1}$, $j \in \{0, \dots, j_{n+1}\}$, are distinct, and for every index $i \in \{0, \dots, n\}$, $J^{(n+1)}_{i,\,0} \cap J^{(n+1)}_{i,\,1} = \varnothing$ (because $E^{(n)}_{i,\,0} \cap E^{(n)}_{i,\,1} = \varnothing$). So $E^{(n+1)}_{i,\,0} \cap E^{(n+1)}_{i,\,1} = \varnothing$ for all $i \in \{0, \dots, n\}$. Also clearly $E^{(n+1)}_{n+1,\,0} \cap E^{(n+1)}_{n+1,\,1} = \varnothing$. So property (1) holds true.

Step 8. In order to check property (2), let us fix $i \in \{0, \dots, n+1\}$, and x, $y \in C_{n+1}$ such that $f_{n+1}(x) \in E_{i,\,0}^{(n+1)}$ and $f_{n+1}(y) \in E_{i,\,1}^{(n+1)}$. By Step 5 above, there exist maps τ, $\tau' : F \longrightarrow \{0, \dots, l_{n+1}\}$, integers k, k' with $|k|$, $|k'| \leq M - d$, and maps θ, $\theta' : F \longrightarrow \{0, 1\}$ such that

$$f_{n+1}\big(T^p x\big) = c^{(n+1)}_{\tau(p),\,k+p,\,\theta(p)} \quad \text{and} \quad f_{n+1}\big(T^p y\big) = c^{(n+1)}_{\tau'(p),\,k'+p,\,\theta'(p)} \quad \text{for every } p \in F.$$

Recall that $0 \in F$. Since $E_{i,\,0}^{(n+1)} \cap E_{i,\,1}^{(n+1)} = \varnothing$, $f_{n+1}(x) \neq f_{n+1}(y)$, so that $(\tau(0),\, k,\, \theta(0)) \neq (\tau'(0),\, k',\, \theta'(0))$. Hence $(\tau,\, \tau',\, k,\, k',\, \theta,\, \theta')$ belongs to $\mathcal{F}$, and

$$\sum_{p \in F} c_p\, c^{(n+1)}_{\tau(p),\,k+p,\,\theta(p)} \neq \sum_{p \in F} c_p\, c^{(n+1)}_{\tau'(p),\,k'+p,\,\theta'(p)},$$

i.e.

$$\sum_{p \in F} c_p\, f_{n+1}\big(T^p x\big) \neq \sum_{p \in F} c_p\, f_{n+1}\big(T^p y\big).$$

Thus $D_{i,\,0}^{(n+1)} \cap D_{i,\,1}^{(n+1)} = \varnothing$ for every $i \in \{0, \dots, n+1\}$. Once η_{n+1} is fixed (and this will be done only later on in the construction), one can choose $\eta < \eta_{n+1}$, and then β_{n+1} so small that properties (2a), (2b), and (2c) hold true.

Step 9. Our next step is to define the sets $H_i^{(n+1)}$ for $i \in \{0, \dots, n+1\}$ and to prove property (3). We set

$$H_i^{(n+1)} = H_i^{(n)} \cup \Big(\bigcup_{|k| \leq N} T^k E \Big) \quad \text{for every } i \in \{0, \dots, n\},$$

$$H_{n+1}^{(n+1)} = \varnothing.$$

Then for every $i \in \{0, \dots, n\}$, $\mu(H_i^{(n+1)}) \leq \mu(H_i^{(n)}) + \eta < \alpha_i(1 - 2^{-n}) + \eta$. So, if η is chosen sufficiently small,

$$\mu(H_i^{(n+1)}) < \alpha_i(1 - 2^{-(n+1)}).$$

Also, $\mu(H_{n+1}^{(n+1)}) = 0 < \alpha_{n+1}(1 - 2^{-(n+1)})$ whatever the value of α_{n+1}. So (3a) holds. As (3b) is obvious, it remains to check (3c).

Let $i \in \{0, \dots, n\}$ and $x \in Q_i \setminus H_i^{(n+1)}$. Then $x \in X \setminus \big(\bigcup_{|k| \leq N} T^k E\big)$ so that

$$g_{n+1}(x) = f_n(x) = c_l^{(n+1)} \text{ for some } l \in \{0, \dots, l_{n+1}\}.$$

Also $x \in Q_i \setminus H_i^{(n)}$, so $f_n(x) \in E_{i\,,0}^{(n)}$ by the induction assumption, that is, $c_l^{(n+1)} \in E_{i,\,0}^{(n)}$. We have either

$$h_{n+1}(x) = c_l^{(n+1)} \quad \text{or} \quad h_{n+1}(x) = c_{l,\,k}^{(n+1)} \text{ for some } |k| \leq M.$$

If we write $h_{n+1}(x) = b_j^{(n+1)}$ for some $j \in \{0, \dots, j_{n+1}\}$, then j belongs to $J_{i,\,0}^{(n+1)}$. So $b_{j,\,0}^{(n+1)}$ and $b_{j,\,1}^{(n+1)}$ belong to $E_{i,\,0}^{(n+1)}$. Since $f_{n+1}(x)$ is equal to either $b_{j,\,0}^{(n+1)}$ or $b_{j,\,1}^{(n+1)}$, it follows that $f_{n+1}(x)$ belongs to $E_{i,\,0}^{(n+1)}$. In the same way, if $x \in \big(X \setminus Q_i\big) \setminus H_i^{(n+1)}$, then $f_{n+1}(x)$ belongs to $E_{i,\,1}^{(n+1)}$.

Now let $i = n+1$. Let $x \in Q_{n+1}$, and let $j \in \{0, \dots, j_{n+1}\}$ be such that $h_{n+1}(x) = b_j^{(n+1)}$. Then $f_{n+1}(x) = b_{j,\,0}^{(n+1)}$, and so by definition of the set $E_{n+1,\,0}^{(n+1)}$,

$f_{n+1}(x)$ belongs to $E^{(n+1)}_{n+1,\,0}$. Similarly, if $x \in X \setminus Q_{n+1}$, then $f_{n+1}(x)$ belongs to $E^{(n+1)}_{n+1,\,1}$. This proves property (3c).

Step 10. We now have to check properties (4) and (5). We have $g_{n+1}(x) = f_n(x)$ for every $x \in B$ and $\mu(B) > 1 - \eta$ (the set B has been defined in Step 1). If $\gamma > 0$ is an arbitrarily small positive number, the numbers $c^{(n+1)}_{l,\,k}$, $b^{(n+1)}_{j,\,0}$ and $b^{(n+1)}_{j,\,1}$ have been chosen so close to $c^{(n+1)}_l$ and $b^{(n+1)}_j$ respectively that $||f_{n+1} - g_{n+1}||_\infty < \gamma$, so that in particular

$$|f_{n+1}(x) - f_n(x)| < \gamma \quad \text{for every } x \in B.$$

Moreover,

$$\begin{aligned} ||f_{n+1} - f_n||_\infty &\leq ||f_{n+1}||_\infty + ||f_n||_\infty \\ &\leq ||g_{n+1}||_\infty + \gamma + ||f_n||_\infty \\ &\leq ||f_n||_\infty + \max_{|k|\leq d_{n+1}} |a^{(n+1)}_k| + \gamma + ||f_n||_\infty \\ &\leq 2\big(||f_n||_\infty + \max_{|k|\leq d_{n+1}} |a^{(n+1)}_k|\big) \end{aligned}$$

if $\gamma < \max_{|k|\leq d_{n+1}} |a^{(n+1)}_k|$. The quantity on the right-hand side depends only on the construction until step n and on the vector u_{n+1}, but not on the rest of the construction at step $n+1$. In particular, it does not depend on γ or on η. We have

$$\int_X ||f_{n+1}(x) - f_n(x)||^2 d\mu(x) < \gamma^2 + \mu(X \setminus B)\, ||f_{n+1} - f_n||^2_\infty \leq \gamma^2 + \eta\, ||f_{n+1} - f_n||^2_\infty.$$

If both γ and η are chosen sufficiently small, we can ensure that for instance

$$||f_{n+1} - f_n||_{L^2(X,\mathcal{B},\mu)} < 2^{-(n+1)}.$$

This proves (5a).

Let us now estimate, for $x \in X$,

$$||\Phi_{f_{n+1}}(x) - \Phi_{f_n}(x)|| = \Big\|\sum_{k\in\mathbb{Z}} \big(f_{n+1}(T^k x) - f_n(T^k x)\big) A^{-k} z_0\Big\|.$$

The series $\sum_{k\in\mathbb{Z}} A^{-k} z_0$ being unconditionally convergent in Z, there exists for every $\rho > 0$ a positive integer k_ρ such that, for every bounded sequence $(a_k)_{k\in\mathbb{Z}}$ of complex numbers,

$$\Big\| \sum_{|k|\geq k_\rho} a_k\, A^{-k} z_0 \Big\| \leq \rho \sup_{|k|\geq k_\rho} |a_k|.$$

So

$$\begin{aligned} ||\Phi_{f_{n+1}}(x) - \Phi_{f_n}(x)|| &\leq \sup_{|k|<k_\rho} \big|f_{n+1}(T^k x) - f_n(T^k x)\big| \,.\, \sum_{|k|<k_\rho} ||A^{-k} z_0|| \\ &\quad + \rho \,.\, \sup_{|k|\geq k_\rho} \big|f_{n+1}(T^k x) - f_n(T^k x)\big| \\ &\leq C_\rho \sup_{|k|<k_\rho} \big|f_{n+1}(T^k x) - f_n(T^k x)\big| + \rho\, ||f_{n+1} - f_n||_\infty, \end{aligned}$$

where $C_\rho = \sum_{|k|<k_\rho} ||A^{-k} z_0||$. We have seen already that $||f_{n+1} - f_n||_\infty$ does not depend on the quantities introduced at step $n+1$ of the construction, so let us fix

$\rho = \rho_{n+1} > 0$ so small that $\rho_{n+1}||f_{n+1} - f_n||_\infty < \gamma$. Then $k_{n+1} = k_{\rho_{n+1}}$ depends on the construction until step n, but not on η. We set

$$B_{n+1} = \bigcap_{|k|<k_{n+1}} T^{-k}B.$$

Then $\mu(B_{n+1}) > 1 - (2k_{n+1} - 1)\eta$ which can be made as close to 1 as we wish provided η is small enough. We also have

$$\sup_{|k|<k_{n+1}} \left|f_{n+1}(T^k x) - f_n(T^k x)\right| < \gamma \quad \text{for every } x \in B_{n+1}.$$

It follows that for every $x \in B_{n+1}$, $||\Phi_{f_{n+1}}(x) - \Phi_{f_n}(x)|| \leq C_{\rho_{n+1}} . \gamma + \gamma$. So if γ' is any positive number, we can ensure by taking γ sufficiently small that

$$||\Phi_{f_{n+1}}(x) - \Phi_{f_n}(x)|| \leq \gamma' \quad \text{for every } x \in B_{n+1}.$$

Thus if γ_{n+1} and η_{n+1} are any fixed positive numbers, taking γ and η sufficiently small yields that properties (4a), (4b) and (4c) are true.

Also, we have

$$\begin{aligned}\int_X &||\Phi_{f_{n+1}}(x) - \Phi_{f_n}(x)||^2 d\mu(x)\\ &< \gamma'^2 + \mu(X \setminus B_{n+1}) \left(C_{\rho_{n+1}} + \rho_{n+1}\right)^2 . ||f_{n+1} - f_n||_\infty^2\\ &\leq \gamma'^2 + (2k_{n+1} - 1)\, \eta \left(C_{\rho_{n+1}} + \rho_{n+1}\right) . ||f_{n+1} - f_n||_\infty^2.\end{aligned}$$

Since η can be chosen as small as we wish compared to k_{n+1}, ρ_{n+1}, and $||f_{n+1} - f_n||_\infty$, we can make the bound above as small as we wish provided γ and η are sufficiently small. So we can in particular ensure that

$$||\Phi_{f_{n+1}} - \Phi_{f_n}||_{L^2(X,\mathcal{B},\mu;Z)} < 2^{-(n+1)},$$

which is (5b).

Step 11. It remains to construct the sets $G_i^{(n+1)}$, $i \in \{0, \dots, n+1\}$, in such a way that property (6) holds true.

We know that for every $i \in \{0, \dots, n\}$ and every $x \in G_i^{(n)}$, $\Phi_{f_n}(x)$ belongs to U_i, i.e. $||\Phi_{f_n}(x) - u_i|| < r_i$, and that $\mu(G_i^{(n)}) \geq \delta_i(1 + 2^{-n})$. Let $0 < \kappa < r_i$ be so small that for every $i \in \{0, \dots, n\}$,

$$\mu\big(\big\{x \in G_i^{(n)} \ ; \ ||\Phi_{f_n}(x) - u_i|| < r_i - \kappa\big\}\big) \geq \delta_i\big(1 + \frac{3}{4}2^{-n}\big).$$

This number κ only depends on the construction until step n. We set for $i \in \{0, \dots, n\}$,

$$G_i^{(n+1)} = \big\{x \in G_i^{(n)} \ ; \ ||\Phi_{f_n}(x) - u_i|| < r_i - \kappa\big\} \cap B_{n+1}$$

and $G_{n+1}^{(n+1)} = E$. Then obviously $G_i^{(n+1)} \subseteq G_i^{(n)}$ for $i \in \{0, \dots, n\}$ and

$$\mu(G_i^{(n+1)}) \geq \delta_i\,(1 + \frac{3}{4}2^{-n}) - (2k_{n+1} - 1)\,\eta \geq \delta_i\,(1 + 2^{-(n+1)})$$

if η is small enough. Also $\mu(G_{n+1}^{(n+1)}) = \mu(E) \geq \delta_{n+1}(1 + 2^{-(n+1)})$ if δ_{n+1} is small enough. So properties (6a) and (6b) are true.

If $i \in \{0, \dots, n\}$ and $x \in G_i^{(n+1)}$, then $||\Phi_{f_n}(x) - u_i|| < r_i - \kappa$. Also $x \in B_{n+1}$ so that $||\Phi_{f_{n+1}}(x) - \Phi_{f_n}(x)|| < \gamma'$. If γ' is chosen less than κ (which is possible since κ depends only on the construction until step n), we have $||\Phi_{f_{n+1}}(x) - u_i|| < r_i$,

i.e. $\Phi_{f_{n+1}}(x) \in U_i$. If $i = n+1$ and $x \in G_{n+1}^{(n+1)} = E$, then $T^k x \in T^k E$ for every $k \in \mathbb{Z}$. Hence

$$\Phi_{g_{n+1}}(x) = \sum_{|k| \le d_{n+1}} a_k^{(n+1)} A^{-k} z_0 + \sum_{|k|>N} g_{n+1}(T^k x) A^{-k} z_0$$

by the definition of the function g_{n+1}. Thus

$$||\Phi_{g_{n+1}}(x) - u_{n+1}|| \le \big|\big| \sum_{|k|>N} g_{n+1}(T^k x) A^{-k} z_0 \big|\big|.$$

Now let $\rho > 0$ be such that $\rho\, ||g_{n+1}||_\infty < r_{n+1}/2$. Since

$$||g_{n+1}||_\infty \le ||f_n||_\infty + \max_{|k| \le d_{n+1}} |a_k^{(n)}|,$$

the number ρ only depends on the construction until step n, and we can choose N so large that $N > k_\rho$. Then we have for every $x \in E$,

$$||\Phi_{g_{n+1}}(x) - u_{n+1}|| \le \rho \sup_{|k|>N} |g_{n+1}(T^k x)| \le \rho\, ||g_{n+1}||_\infty < \frac{r_{n+1}}{2}.$$

Recall that there exists a positive constant C such that

$$\big|\big| \sum_{k \in \mathbb{Z}} a_k A^{-k} z_0 \big|\big| \le C \sup_{k \in \mathbb{Z}} |a_k|$$

for all bounded sequences $(a_k)_{k \in \mathbb{Z}}$ of complex numbers. Since $||f_{n+1} - g_{n+1}||_\infty < \gamma$, we can assume by taking γ sufficiently small that $||f_{n+1} - g_{n+1}||_\infty < r_{n+1}/(2C)$. Then we have, for every $x \in X$,

$$||\Phi_{f_{n+1}}(x) - \Phi_{g_{n+1}}(x)|| \le C\, ||f_{n+1} - g_{n+1}||_\infty < \frac{r_{n+1}}{2}.$$

Hence $||\Phi_{f_{n+1}}(x) - u_{n+1}|| < r_{n+1}$ for every $x \in E$, i.e. $\Phi_{n+1}(x)$ belongs to U_{n+1} for every $x \in E = G_{n+1}^{(n+1)}$. Thus property (6c) is satisfied.

This finishes the construction by induction of the functions f_n.

2.3. **Construction of the isomorphism Φ and proof of Theorem 1.3.** By property (5b), the sequence $(\Phi_{f_n})_{n \ge 0}$ converges in $L^2(X, \mathcal{B}, \mu; Z)$ to a function Φ which belongs to $L^2(X, \mathcal{B}, \mu; Z)$. Our aim now is to prove that the probability measure ν on Z defined by $\nu(B) = \mu(\Phi^{-1}(B))$ for any Borel subset B of Z has full support, and that Φ is an isomorphism between the two dynamical systems $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$.

Observe that the property (5a) of the sequence of functions $(f_n)_{n \ge 0}$ implies that $(f_n)_{n \ge 0}$ converges in $L^2(X, \mathcal{B}, \mu)$ to a certain function $f \in L^2(X, \mathcal{B}, \mu)$. Obviously, f does not belong to $L^\infty(X, \mathcal{B}, \mu)$, and thus it makes no sense to speak of the map Φ_f (this is different from what happens in the construction of [10]). But a link between Φ and f can be obtained thanks to the assumption (b) of Theorem 1.3: for μ-almost every $x \in X$ and every $n \ge 0$ we have $\langle z_0^*, \Phi_{f_n}(x) \rangle = \sum_{p \in F} c_p\, f_n(T^p x)$, where the set F is finite. Since there exists a strictly increasing sequence $(n_k)_{k \ge 0}$ of integers such that $\Phi_{f_{n_k}} \longrightarrow \Phi$ and $f_{n_k} \longrightarrow f$ μ-almost everywhere, it follows that

$$\langle z_0^*, \Phi(x) \rangle = \sum_{p \in F} c_p\, f(T^p x) \quad \mu\text{-almost everywhere.}$$

• **Let us first show that ν has full support.** By property (4c) we have that for every $n \geq 0$ and every $x \in \widetilde{B}_n = \bigcap_{k \geq n} B_k$, $||\Phi_{f_k}(x) - \Phi_{f_{k-1}}(x)|| < \gamma_k$ for every $k \geq n$. Observe that

$$\mu(\widetilde{B}_n) \geq 1 - \sum_{k \geq n} \eta_k > 1 - 2\eta_n$$

if the sequence $(\eta_n)_{n \geq 0}$ decreases sufficiently fast, so that $\mu(\widetilde{B}_n) \to 1$ as $n \to +\infty$. For every $n \geq 0$ and every $x \in \widetilde{B}_n$ we have in particular that

$$\big|\big|\Phi_{f_k}(x) - \Phi_{f_n}(x)\big|\big| \leq \sum_{j=n+1}^{k} \big|\big|\Phi_{f_j}(x) - \Phi_{f_{j-1}}(x)\big|\big| \leq \sum_{j \geq n+1} \gamma_j < 2\gamma_{n+1}$$

if the sequence $(\gamma_k)_{k \geq 0}$ is sufficiently rapidly decreasing. As $(\Phi_{f_k})_{k \geq 0}$ converges to Φ in $L^2(X, \mathcal{B}, \mu; Z)$, there exists a subsequence of $(\Phi_{f_k})_{k \geq 0}$ which converges to Φ μ-almost everywhere on X and thus we have

$$\text{for } \mu\text{-almost every } x \in \widetilde{B}_n, \quad ||\Phi(x) - \Phi_{f_n}(x)|| \leq 2\gamma_{n+1}.$$

Let us now fix $i \geq 0$. By property (6c), we have that, for every $n \geq i$ and every $x \in G_i^{(n)}$, $||\Phi_{f_n}(x) - u_i|| < r_i$. Let

$$G_i = \bigcap_{n \geq i} G_i^{(n)}.$$

As the sequence of sets $\big(G_i^{(n)}\big)_{n \geq i}$ is decreasing by (6b), we have that $\mu(G_i) = \lim_{n \to +\infty} \mu(G_i^{(n)})$ so that $\mu(G_i) \geq \delta_i > 0$ by property (6a). Now let $n \geq i$ be so large that $\mu(G_i \cap \widetilde{B}_n) > 0$. If $x \in G_i \cap \widetilde{B}_n$, then

$$||\Phi(x) - u_i|| \leq ||\Phi(x) - \Phi_{f_n}(x)|| + ||\Phi_{f_n}(x) - u_i|| < 2\gamma_{n+1} + r_i.$$

Hence $\nu\big(B(u_i, r_i + 2\gamma_{n+1})\big) > 0$ for all n sufficiently large, so that it follows in particular that $\nu\big(B(u_i, 2r_i)\big) > 0$. This being true for all $i \geq 0$, the measure ν has full support.

Let us observe at this point that the measure ν admits a moment of order 2. Indeed

$$\int_Z ||z||^2 \, d\nu(z) = \int_X ||\Phi(x)||^2 \, d\mu(x) < +\infty$$

since $\Phi \in L^2(X, \mathcal{B}, \mu; Z)$.

• **It remains to show that Φ is an isomorphism between $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$.** For this it suffices to prove (see for instance [13, pp. 59–60]) that the two transformations T and A are conjugated via the map Φ. First of all, we need to check that $\Phi(Tx) = A\Phi(x)$ for μ-almost every $x \in X$. Since Φ_{f_n} tends to Φ in $L^2(X, \mathcal{B}, \mu; Z)$ as n tends to infinity, there exists a subsequence $\big(\Phi_{f_{n_k}}\big)_{k \geq 0}$ of $\big(\Phi_{f_n}\big)$ which tends to Φ μ-almost everywhere. As $\Phi_{f_n}(Tx) = A\Phi_{f_n}(x)$ μ-almost everywhere for each $n \geq 0$, it follows that $\Phi(Tx) = A\Phi(x)$ μ-almost everywhere.

The second point is to check that for every subset Q of X, $Q \in \mathcal{B}$, there exists a subset B of Z, $B \in \mathcal{B}_Z$, such that $\mu\big(Q \triangle \Phi^{-1}(B)\big) = 0$. So let $Q \in \mathcal{B}$. Suppose that we are able to exhibit a Borel subset C of $\mathbb{C}$ such that $Q = \big\{x \in X;\ \sum_{p \in F} c_p f(T^p x) \in C\big\}$ up to a set of measure zero. Setting $B = \big\{z \in Z;\ \langle z_0^*, z\rangle \in C\big\}$ and remembering that $\langle z_0^*, \Phi(x)\rangle = \sum_{p \in F} c_p f(T^p x)$ μ-almost

everywhere, we obtain that B is a Borel subset of Z such that $\Phi^{-1}(B) = Q$ (up to a set of μ-measure zero). So it suffices to find C with the property above.

Let us first introduce some notation: for each $n \geq 0$ we define the function F_n on X by setting $F_n(x) = \sum_{p\in F} c_p\, f_n(T^p x)$ and the function F by setting $F(x) = \sum_{p\in F} c_p\, f(T^p x)$. For every $i \geq 0$, let

$$C_{i,0} = \bigcap_{n\geq i} F_{i,0}^{(n)} \quad \text{and} \quad C_{i,1} = \bigcap_{n\geq i} F_{i,1}^{(n)}.$$

These are Borel subsets of $\mathbb{C}$. Recall that $J_i = \{j \geq i;\ Q_j = Q_i\}$ is supposed to be infinite. Let

$$\Gamma_{i,0} = \bigcup_{j\geq 0} \bigcap_{\substack{k\geq j\\ k\in J_i}} C_{k,0} \quad \text{and} \quad \Gamma_{i,1} = \bigcup_{j\geq 0} \bigcap_{\substack{k\geq j\\ k\in J_i}} C_{k,1}.$$

These two sets are Borel in $\mathbb{C}$. Moreover $\Gamma_{i,0} \cap \Gamma_{i,1} = \varnothing$. Indeed, if it is not the case, we have

$$\Big(\bigcap_{\substack{k\geq j_1\\ k\in J_i}} C_{k,0}\Big) \cap \Big(\bigcap_{\substack{k\geq j_2\\ k\in J_i}} C_{k,1}\Big) \neq \varnothing$$

for some integers j_1 and j_2. In particular, if $k \in J_i$ is such that $k \geq \max(j_1, j_2)$ (and such a k does exist because J_i is infinite), $C_{k,0} \cap C_{k,1} \neq \varnothing$. So for all $n \geq k$, $F_{k,0}^{(n)} \cap F_{k,1}^{(n)} \neq \varnothing$, which contradicts (2c).

For every $i \geq 0$, we consider the subsets of X:

$$\begin{aligned}
D_i &= \bigcap_{p\in F} \Big(\bigcap_{k\geq i} T^{-p} B_k\Big) \cap \Big(\bigcap_{k\geq i} C_k\Big),\\
H_i &= \bigcup_{n\geq i} H_i^{(n)},\\
\Omega_i &= \bigcap_{k\geq i} \big(D_k \setminus H_k\big),\\
\Omega &= \bigcup_{i\geq 0} \Omega_i.
\end{aligned}$$

We have $\mu(D_i) \geq 1 - (2d+2)\sum_{k\geq i} \eta_k \geq 1 - 4(d+1)\eta_i$ if the sequence $(\eta_i)_{i\geq 0}$ decreases sufficiently fast. Also, (3a) and (3b) imply that $\mu(H_i) \leq \alpha_i$ so that

$$\mu(\Omega_i) \geq 1 - \sum_{k\geq i} \big(\mu(X\setminus D_k) + \mu(H_k)\big) \geq 1 - 4(d+1)\sum_{k\geq i} \eta_k - \sum_{k\geq i} \alpha_k \geq 1 - 8(d+1)(\alpha_i + \eta_i)$$

if the sequences $(\eta_i)_{i\geq 0}$ and $(\alpha_i)_{i\geq 0}$ decrease to zero sufficiently fast. Thus $(\Omega_i)_{i\geq 0}$ is an increasing sequence of sets such that $\mu(\Omega_i) \longrightarrow 1$ as $i \longrightarrow +\infty$. So $\mu(\Omega) = 1$.

For every $n \geq 0$, every $x \in D_n$, every $p \in F$, and every $k \geq n$, by property (4b), $|f_{k+1}(T^p x) - f_k(T^p x)| < \gamma_{k+1}$. Hence $|F_{k+1}(x) - F_k(x)| \leq (\sup_{p\in F} |c_p|)\, \gamma_{k+1}$. Since $(F_k)_{k\geq 0}$ converges to F in $L^2(X, \mathcal{B}, \mu)$, there exists a subsequence $(F_{k_j})_{j\geq 0}$ of $(F_k)_{k\geq 0}$ which converges to F μ-almost everywhere. Hence for (μ-almost) every $x \in D_n$ we have

$$|F(x) - F_n(x)| \leq \sup_{p\in F} |c_p| \sum_{k\geq n} \gamma_{k+1} < \beta_n$$

if $\gamma_{n+1}, \gamma_{n+2}, \ldots$ are chosen at steps $n+1$, $n+2, \ldots$ sufficiently small with respect to β_n. Thus $F(x) \in D(F_n(x), \beta_n)$ for every $x \in D_n$.

After these preliminaries, our aim now is to show that for every $i \geq 0$, $F^{-1}(\Gamma_{i,0}) = Q_i$ and $F^{-1}(\Gamma_{i,1}) = X \setminus Q_i$. So, let us fix $i \geq 0$ and $k \in J_i$. Suppose that $x \in Q_i \cap \Omega_k$. Then for every $n \geq k$, $x \in Q_i \cap (D_k \setminus H_k) \cap (D_n \setminus H_n)$ so that $x \in Q_i \cap \big(D_n \setminus H_k^{(n)}\big) = Q_k \cap (D_n \setminus H_k^{(n)})$. By property (3c), $f_n(x) \in E_{k,0}^{(n)}$. Since $x \in D_n \subseteq C_n$, this implies that $F_n(x) \in D_{k,0}^{(n)}$. It follows that for every $n \geq k$, $F(x) \in F_{k,0}^{(n)}$, so that $F(x) \in \bigcap_{n \geq k} F_{k,0}^{(n)} = C_{k,0}$. We have thus proved that if $k \in J_i$ and $x \in Q_i \cap \Omega_k$, then $F(x) \in C_{k,0}$.

Now let $j \geq 0$. If $k \geq j$ and $k \in J_i$, then, since $\Omega_j \subseteq \Omega_k$, we have $Q_i \cap \Omega_j \subseteq Q_i \cap \Omega_k$. It follows that if $x \in Q_i \cap \Omega_j$, $F(x) \in C_{k,0}$ for every $k \in J_i$, $k \geq j$, and so

$$F(x) \in \bigcap_{\substack{k \geq j \\ k \in J_i}} C_{k,0}.$$

Suppose now that $x \in Q_i \cap \Omega$: there exists a $j \geq 0$ such that $x \in Q_i \cap \Omega_j$. Hence

$$F(x) \in \bigcup_{j \geq 0} \bigcap_{\substack{k \geq j \\ k \in J_i}} C_{k,0} = \Gamma_{i,0}.$$

In exactly the same way, if $x \in (X \setminus Q_i) \cap \Omega$, then $F(x) \in \Gamma_{i,1}$. Since $\Gamma_{i,0} \cap \Gamma_{i,1} = \varnothing$ and $\mu(\Omega) = 1$, we have proved that

$$F^{-1}(\Gamma_{i,0}) = Q_i \quad \text{and} \quad F^{-1}(\Gamma_{i,1}) = X \setminus Q_i$$

up to sets of μ-measure zero. The proof is now nearly finished. Let

$$\mathcal{Q} = \big\{Q \in \mathcal{B};\ \text{there exists a Borel subset } C \text{ of } \mathbb{C} \text{ such that } \mu(Q \vartriangle F^{-1}(C)) = 0\big\}.$$

Then $\mathcal{Q}$ is a σ-algebra which contains all the sets Q_i, $i \geq 0$, and these sets generate $\mathcal{B}$. Thus $\mathcal{Q} = \mathcal{B}$. This finishes the proof that Φ is an isomorphism of dynamical systems, and Theorem 1.3 is proved.

2.4. **Proof of Theorem 1.4.** The proof of Theorem 1.4 is similar in spirit, but some technical points need to be adjusted. We briefly list the most important ones. If $(X, \mathcal{B}, \mu; T)$ is an ergodic dynamical system which is not necessarily invertible, then the maps Φ_f, $f \in L^\infty(X, \mathcal{B}, \mu)$, are defined as $\Phi_f(x) = \sum_{k \geq 1} f(T^k x)\, A^{-k+r} z_0$ for μ-almost every $x \in X$, where $r \in \mathbb{Z}$ is such that $A^r z_0 = 0$. Then

$$\Phi_f(Tx) = \sum_{k \in \mathbb{Z}} f(T^k x)\, A^{-k+1+r} z_0 = \sum_{k \geq 1} f(T^k x)\, A^{-k+1+r} z_0$$

since $A^r z_0 = 0$. Hence $\Phi_f(Tx) = A\, \Phi_f(x)$ μ-almost everywhere. As in the proof of Theorem 1.3, we can suppose by shifting the sequence $(z_n)_{n \in \mathbb{Z}}$ that $0 \in F$ and that there exists a non-zero functional $z_0^* \in Z^*$ such that $c_n = \langle z_0^*, A^{-n+r} z_0 \rangle$ is non-zero if and only if $n \in F$. The vectors u_n are defined as

$$u_n = \sum_{k=0}^{d_n} a_k^{(n)}\, A^{-k+r} z_0, \quad a_k^{(n)} \in \mathbb{C}.$$

The set E and the integer N are chosen in such a way that $\mu(E) > 0$, $\mu\big(\bigcup_{k=0}^{N} T^{-k}E\big) < \eta$, and the sets $T^{-k}E$ are pairwise disjoint. This entails a modification of the definition of the function g_{n+1}:

$$g_{n+1}(x) = \begin{cases} a_k^{(n+1)} & \text{if } x \in T^{-N+k}E, \quad 0 \le k \le d_{n+1}, \\ 0 & \text{if } x \in T^{-N+k}E, \quad d_{n+1} < k \le N, \\ f_n(x) & \text{if } x \in X \setminus \bigcup_{k=0}^{N} T^{-k}E. \end{cases}$$

The set E' and the integer M are chosen so that $\mu\big(\bigcup_{k=0}^{M-d} T^{-k}E'\big) < 1 - \eta$ and the sets $T^{-k}E'$, $0 \le k \le M$, are pairwise disjoint (where $d = \max|F|$). The set $G_{n+1}^{(n+1)}$ needs to be defined as $G_{n+1}^{(n+1)} = T^{-N}E$. Up to these modifications, the proof is the same, and we leave the details to the reader.

2.5. **Some remarks.** Now we have carried out the proofs of these two results, a few remarks are in order.

Remark 2.2. We have seen during the proofs of Theorem 1.3 and 1.4 that the measures ν constructed on Z admit a second-order moment. Thus operators satisfying the assumptions of these theorems are 2-universal for (invertible) ergodic systems in the sense of Definition 5.1 (see Section 5). It is not difficult to check that one can ensure that all the measures ν actually admit moments of all orders.

Remark 2.3. If A is an invertible operator on Z, then it cannot be universal for all ergodic systems: indeed, suppose that $(X, \mathcal{B}, \mu; T)$ is a dynamical system which is isomorphic to $(Z, \mathcal{B}_Z, \nu; A)$ for some A-invariant measure ν with full support. Then ν is A^{-1}-invariant too. If $\Phi : X \longrightarrow Z$ is an isomorphism between $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$, then set $S = \Phi^{-1}A^{-1}\Phi$. The measure μ is S-invariant, and since $T = \Phi^{-1}A\Phi$, we have $TS = ST = \mathrm{id}_X$, so that T is invertible. Thus an invertible operator may only be universal for invertible ergodic systems.

If A is a universal operator for ergodic systems and $(X, \mathcal{B}, \mu; T)$ is an invertible ergodic system, there exists a probability measure ν on Z such that $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$ are isomorphic. Hence A is invertible as a measure-preserving transformation of $(Z, \mathcal{B}_Z, \nu)$, although, as observed above, it is not invertible as a topological map from Z into itself. There is no contradiction in this: there indeed exists a Borel subset M of Z with $A(M) = M$ and $\nu(M) = 1$ such that $A : M \longrightarrow M$ is an invertible measure-preserving transformation, i.e. there exists a map $S : M \longrightarrow M$ which is measurable and measure-preserving such that $AS(x) = SA(x) = x$ for every $x \in M$. But M being distinct from the whole space X, and S not necessarily uniformly continuous on M, there is no reason why S should extend into a topological inverse of A on Z.

Remark 2.4. The ergodicity of the system $(X, \mathcal{B}, \mu; T)$ has been used in the proofs of Theorems 1.3 and 1.4 only via the Rokhlin Lemma. So we could have supposed instead that the system was aperiodic, and the operators A of Theorems 1.3 and 1.4 are in fact universal for (invertible) aperiodic systems.

We finish this section with a generalization of Theorems 1.3 and 1.4, in which assumption (a) is relaxed. This generalization is useful for proving the universality of some operators (see Section 4) for which the (bi)cyclicity assumption (a) is not easily seen to hold true.

Theorem 2.5. *Let $A \in \mathcal{B}(Z)$ be a bounded operator on Z. Suppose that there exist sequences $\left(z_n^{(\iota)}\right)_{n\in\mathbb{Z}}$ of vectors of Z, $\iota \in I$, where either $I = \{0, \dots, N\}$ for some integer $N \geq 0$ or $I = \mathbb{Z}^+$, such that $Az_n^{(\iota)} = z_{n+1}^{(\iota)}$ for every $\iota \in I$ and $n \in \mathbb{Z}$. Suppose that the following three properties hold true:*

(a') $\overline{\operatorname{span}}\left[A^{-n}z_0^{(\iota)};\ n \in \mathbb{Z},\ \iota \in I\right] = Z$;

(b') *there exists a finite subset F of $\mathbb{Z}$ such that*

$$\overline{\operatorname{span}}\left[\{A^{-n}z_0^{(0)};\ n \in \mathbb{Z} \setminus F\} \cup \{A^{-n}z_0^{(\iota)};\ n \in \mathbb{Z},\ \iota \in I \setminus \{0\}\}\right] \neq Z;$$

(c') *the series $\sum_{n\in\mathbb{Z}} A^{-n}z_0^{(\iota)}$, $\iota \in I$, are unconditionally convergent in Z.*

Then A is universal for invertible ergodic systems.

The analogous statement for universality for ergodic systems is

Theorem 2.6. *Let $A \in \mathcal{B}(Z)$ be a bounded operator on Z. With the notation of Theorem 2.5, suppose that there exist sequences $\left(z_n^{(\iota)}\right)_{n\in\mathbb{Z}}$, $\iota \in I$, of vectors of Z and integers $r^{(\iota)}$, $\iota \in I$, where either $I = \{0, \dots, N\}$ for some integer $N \geq 0$ or $I = \mathbb{Z}^+$, such that $Az_n^{(\iota)} = z_{n+1}^{(\iota)}$ for every $n \in \mathbb{Z}$ and $A^{r^{(\iota)}}z_0^{(\iota)} = 0$, for every $\iota \in I$.*

If assumptions (a'), (b') *and* (c') *of Theorem* 2.5 *hold true, A is universal for ergodic systems.*

Proof. The proofs are almost the same as those of Theorems 1.3 and 1.4, with the additional complication that several functions $f^{(\iota)} \in L^2(X, \mathcal{B}, \mu)$, $\iota \in I$, have to be considered, and several sequences $(f_n^{(\iota)})_{n\geq\iota}$ of functions of $L^\infty(X, \mathcal{B}, \mu)$ introduced. We sketch here the proof of Theorem 2.5, and leave the proof of Theorem 2.6 to the reader.

Let $(u_n)_{n\geq 0}$ be a sequence of vectors of Z which is dense in Z and has the following property: there exist for each $n \geq 0$ an integer d_n and complex coefficients $a_k^{(\iota,\, n)}$, $k \in \{0, \dots, n+1\}$, $\iota \in \{0, \dots, n\}$, such that

$$u_n = \sum_{\iota=0}^{n} \sum_{k=0}^{d_n} a_k^{(\iota,\, n)} A^{-n} z_0^{(\iota)}.$$

Let $(r_n)_{n\geq 0}$ be a decreasing sequence of positive radii such that the balls $U_n = B(u_n, r_n)$, $n \geq 0$, form a basis of the topology of Z. Also let z_0^* be a non-zero functional and F a finite subset of $\mathbb{Z}$ such that $\langle z_0^*, A^{-n}z_0^{(\iota)}\rangle = 0$ for every $\iota \in I \setminus \{0\}$ and every $n \in \mathbb{Z}$, $c_n = \langle z_0^*, A^{-n}z_0^{(0)}\rangle = 0$ for every $n \in \mathbb{Z} \setminus F$, and $c_n = \langle z_0^*, A^{-n}z_0^{(0)}\rangle$ is non-zero for every $n \in F$ (the set F can be assumed to contain 0).

Then we construct sequences $\left(f_n^{(\iota)}\right)_{n\geq\iota}$, $\iota \in I$, as in the proof of Theorem 1.3. At each step n we construct the functions $f_n^{(\iota)}$ for $\iota \in \{0, \dots, n\}$: for each such ι, the open set used in the construction is $U_n^{(\iota)} = B(u_n^{(\iota)}, 2^{-(\iota+1)}r_n)$, where

$$u_n^{(\iota)} = \sum_{k=0}^{d_n} a_k^{(\iota,\, n)} A^{-n} z_0^{(\iota)}$$

if $\iota \in \{0, \dots, n\}$, $u_n^{(\iota)} = 0$ otherwise (if I is finite, $\iota \in \{0, \dots, n\}$ means that $\iota \in \{0, \dots, \min(|I|, n)\}$). Hence $u_n = \sum_{\iota\in I} u_n^{(\iota)}$ for every $n \geq 0$.

We carry out the construction in such a way that properties (1)–(6) hold true for $f_n^{(0)}$, and properties (4)–(6) hold true for $f_n^{(\iota)}$, $\iota \in \{1, \dots, n\}$. The important

points in this construction, when compared to the proof of Theorem 1.3, are the following:

– The initial functions $f_\iota^{(\iota)}$, $\iota \in I$, are such that $||f_\iota^{(\iota)}||_{L^\infty(X,\mathcal{B},\mu)} < 2^{-(\iota+1)}\frac{r^{(\iota)}}{C_\iota}$, so that

$$||\Phi_\iota^{(\iota)}||_{L^2(X,\mathcal{B},\mu;Z)} \leq C_\iota ||f_\iota^{(\iota)}||_{L^\infty(X,\mathcal{B},\mu)} < 2^{-(\iota+1)}r^{(\iota)},$$

where C_ι is a positive constant such that

$$\left|\left|\sum_{k\in\mathbb{Z}} a_k^{(\iota)} A^{-k} z_0^{(\iota)}\right|\right| \leq C_\iota \sup_{k\in\mathbb{Z}} |a_k^{(\iota)}|$$

for every $(a_k^\iota)_{k\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z})$.

– At a given step of the construction, the sets E and E' and the parameters γ and η can be chosen independently of the index $\iota \in \{0,\dots,n\}$. It follows that the sets B_n and $G_i^{(n)}$, which may a priori depend on the index ι, can be constructed so as not to depend on it, as well as the parameters γ_n, η_n, and δ_n (the other parameters in the construction, as well as the sets C_n, $D_{i,0}^{(n)}$, $D_{i,1}^{(n)}$, $F_{i,0}^{(n)}$, $F_{i,1}^{(n)}$, and $H_i^{(n)}$ are involved only for the index $\iota = 0$). More precisely, Step 11 in the proof of the construction has to be modified as follows: we know that for every $i \in \{0,\dots,n\}$ and every $x \in G_i^{(n)}$, we have

$$||\Phi_{f_n^{(\iota)}} - u_i^{(\iota)}|| < 2^{-(\iota+1)}r_i \quad \text{for every } \iota \in \{0,\dots,n\}.$$

Let κ be so small that for every $i \in \{0,\dots,n\}$,

$$\mu\big(\big\{x \in G_i^{(n)} \text{ ; for every } \iota \in \{0,\dots,n\},$$
$$||\Phi_{f_n^{(\iota)}}(x) - u_i^{(\iota)}|| < 2^{-(\iota+1)}(r_i - \kappa)\big\}\big) \geq \delta_i\big(1 + \frac{3}{4}2^{-n}\big).$$

We set for $i \in \{0,\dots,n\}$,

$$G_i^{(n+1)} = \big\{x \in G_i^{(n)} \text{ ; for every } \iota \in \{0,\dots,n\},$$
$$||\Phi_{f_n^{(\iota)}}(x) - u_i^{(\iota)}|| < 2^{-(\iota+1)}(r_i - \kappa)\big\} \cap B_{n+1}$$

and $G_{n+1}^{(n+1)} = E$. Properties (6a) and (6b) are clearly true. It remains to prove property (6c). It is not difficult to see that for every $i \in \{0,\dots,n\}$ and every $\iota \in \{0,\dots,n\}$, $\Phi_{f_{n+1}^{(\iota)}}(x) \in U_i^{(\iota)}$ for every $x \in G_i^{(n+1)}$. We then have to check that for every $i \in \{0,\dots,n\}$ and every $x \in G_i^{(n+1)}$, $\Phi_{f_{n+1}^{(n+1)}}(x) \in U_i^{(n+1)}$, i.e. that

$$||\Phi_{f_{n+1}^{(n+1)}}(x) - u_i^{(n+1)}|| < 2^{-(n+2)}r_i.$$

But $u_i^{(n+1)} = 0$ and

$$||\Phi_{f_{n+1}^{(n+1)}}||_{L^\infty(X,\mathcal{B},\mu)} < 2^{-(n+2)}r_{n+1} < 2^{-(n+2)}r_i,$$

so we do have that $\Phi_{f_{n+1}^{(n+1)}}(x) \in U_i^{(n+1)}$. The last item in the proof of property (6c) is to show that $\Phi_{f_{n+1}^{(\iota)}}(x) \in U_{n+1}^{(\iota)}$ for every $x \in G_{n+1}^{(n+1)} = E$ and every $\iota \in \{0,\dots,n\}$, and here the proof is again exactly the same.

We thus have

$$||f_n^{(\iota)} - f_{n-1}^{(\iota)}||_{L^2(X,\mathcal{B},\mu)} < 2^{-n} \quad \text{and} \quad ||\Phi_{f_n}^{(\iota)} - \Phi_{f_{n-1}}^{(\iota)}||_{L^2(X,\mathcal{B},\mu;Z)} < 2^{-n}$$

for every $n \geq 1$ and $\iota \in \{0, \dots, n\}$ and

$$||\Phi_{f_n^{(\iota)}}(x) - u_n^{(\iota)}|| < 2^{-(\iota+1)} r_n$$

for every $x \in G_i^{(n)}$, $i \in \{0, \dots, n\}$, $\iota \in \{0, \dots, n\}$. It follows that the sequence $\big(f_n^{(\iota)}\big)_{n \geq \iota}$ converges in $L^2(X, \mathcal{B}, \mu)$ to a function $f^{(\iota)}$ which satisfies

$$\begin{aligned} ||f^{(\iota)}||_{L^2(X,\mathcal{B},\mu)} &\leq ||f_\iota^{(\iota)}||_{L^2(X,\mathcal{B},\mu)} + \sum_{n \geq \iota} ||f_{n+1}^{(\iota)} - f_n^{(\iota)}||_{L^2(X,\mathcal{B},\mu)} \\ &< 2^{-(\iota+1)} + \sum_{n \geq \iota} 2^{-(n+1)} \leq 2^{-(\iota-1)}. \end{aligned}$$

In the same way, $\big(\Phi_{f_n}^{(\iota)}\big)_{n \geq \iota}$ converges in $L^2(X, \mathcal{B}, \mu; Z)$ to a function $\Phi^{(\iota)}$ which satisfies $||\Phi^{(\iota)}||_{L^2(X,\mathcal{B},\mu;Z)} < 2^{-(\iota-1)}$. In particular, the series $\sum_{\iota \in I} ||\Phi^{(\iota)}||_{L^2(X,\mathcal{B},\mu;Z)}$ is convergent.

We now consider the map $\Phi : X \longrightarrow Z$ defined as

$$\Phi = \sum_{\iota \in I} \Phi^{(\iota)}.$$

Since

$$\int_X \sum_{\iota \in I} ||\Phi^{(\iota)}(x)||\, d\mu(x) \leq \sum_{\iota \in I} \Big(\int_X ||\Phi^{(\iota)}(x)||^2 d\mu(x) \Big)^{1/2} \leq \sum_{\iota \in I} 2^{-(\iota-1)} < +\infty,$$

$\Phi(x)$ is defined μ-almost everywhere.

If ν is the measure defined on Z by setting $\nu(B) = \mu(\Phi^{-1}(B))$ for every $B \in \mathcal{B}_Z$, then it is clear that $\Phi : (X, \mathcal{B}, \mu; T) \longrightarrow (Z, \mathcal{B}_Z, \nu; A)$ is a factor map. It is an isomorphism of dynamical systems for the same reason as in the proof of Theorem 1.3: if Q is any Borel subset of X, we know that there exists a Borel subset C of $\mathbb{C}$ such that

$$\big\{x \in X;\ \sum_{p \in F} c_p f^{(0)}(T^p x) \in C\big\} = Q.$$

Now let $B = \big\{z \in Z;\ \langle z_0^*, z\rangle \in C\big\}$. We have

$$\Phi^{-1}(B) = \big\{x \in X;\ \langle z_0^*, \Phi(x)\rangle = \sum_{p \in F} c_p\, f^{(0)}(T^p x) \in C\big\} = Q.$$

So Φ is an isomorphism between $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$. It remains to prove that the measure ν has full support: for every $i \geq 0$, every $n \geq i$, every $\iota \in \{0, \dots, n\}$, and every $x \in G_i = \bigcap_{n \geq i} G_i^{(n)}$, $||\Phi_{f_n^{(\iota)}} - u_n^{(\iota)}|| < 2^{-(\iota+1)} r_i$. We also know that for every $k \geq 0$, every $\iota \in \{0, \dots, k\}$ and every $x \in B_k$,

$$||\Phi_{f_k^{(\iota)}}(x) - \Phi_{f_{k-1}^{(\iota)}}(x)|| < \gamma_k.$$

So, if $x \in G_i \cap \big(\bigcap_{k \geq n} B_k\big)$ and $\iota \in \{0, \dots, n\}$, then

$$\begin{aligned} ||\Phi_{f^{(\iota)}}(x) - u_i^{(\iota)}|| &\leq \sum_{k \geq n+1} ||\Phi_{f_k^{(\iota)}}(x) - \Phi_{f_{k-1}^{(\iota)}}(x)|| + ||\Phi_{f_n^{(\iota)}}(x) - u_i^{(\iota)}|| \\ &\leq \sum_{k \geq n+1} \gamma_k + 2^{-(\iota+1)} r_i < \gamma_n + 2^{-(\iota+1)} r_i. \end{aligned}$$

Let us now fix $i \geq 0$. Let $(n_\iota)_{\iota \in I}$ be an increasing (finite or infinite) sequence of integers such that $\gamma_{n_\iota} < 2^{-(\iota+1)} r_i$ and $\mu\big(G_i \cap \big(\bigcap_{k \geq n_0} B_k\big)\big) > 0$. If $x \in G_i \cap \big(\bigcap_{k \geq n_0} B_k\big)$, then $x \in G_i \cap \big(\bigcap_{k \geq n_\iota} B_k\big)$ for every $\iota \in I$, so that

$$||\Phi_{f^{(\iota)}}(x) - u_i^{(\iota)}|| < 2^{-\iota} r_i \quad \text{for every } \iota \in I.$$

Hence

$$||\sum_{\iota \in I} \Phi_{f^{(\iota)}}(x) - \sum_{\iota \in I} u_i^{(\iota)}|| < 2r_i \quad \text{for every } x \in G_i \cap \big(\bigcap_{k \geq n_0} B_k\big),$$

i.e. $\Phi(x) \in B(u_i, 2r_i)$. This being true for every $i \geq 0$, it follows that the measure ν has full support. □

3. Universal unilateral and bilateral weighted shifts

The easiest class of operators to which the criteria of Theorems 1.3 and 1.4 can be applied is the class of weighted shifts on $\ell_p(\mathbb{N})$ or $\ell_p(\mathbb{Z})$, $1 \leq p < \infty$, or on $c_0(\mathbb{N})$ or $c_0(\mathbb{Z})$. We first give the proof of Theorem 1.5.

3.1. **Proof of Theorem 1.5.** Suppose that B_w is a unilateral weighted shift with weights $(w_n)_{n \geq 1}$ on one of the spaces $\ell_p(\mathbb{N})$, $1 \leq p < \infty$, or $c_0(\mathbb{N})$, with $B_w e_0 = 0$ and $B_w e_n = w_n e_{n-1}$ for every $n \geq 1$. Let us set $z_0 = e_0$ and $z_{-n} = 1/(w_1 \dots w_n)\, e_n$ for every $n \geq 1$. Then

$$B_w z_{-n} = \frac{1}{w_1 \dots w_n} w_n e_{n-1} = z_{-(n-1)} \quad \text{for every } n \geq 1, \text{ and } B_w z_0 = 0.$$

If we write $z_{-n} = B_w^{-n} e_0$ for every $n \geq 1$, then the vectors $B_w^{-n} z_0$, $n \geq 0$, span a dense subset of $\ell_p(\mathbb{N})$ (or $c_0(\mathbb{N})$). Also, the linear span of the vectors $B_w^{-n} z_0$, $n \geq 1$, is not dense, so that assumption (b) of Theorem 1.4 always holds true with $F = \{0\}$.

The series $\sum_{n \geq 0} B_w^{-n} e_0$ is unconditionally convergent in $\ell_p(\mathbb{N})$ (resp. $c_0(\mathbb{N})$) if and only if the series $\sum_{n \geq 1} 1/|w_1 \dots w_n|^p$ is convergent (resp. if and only if $|w_1 \dots w_n| \longrightarrow +\infty$ as $n \longrightarrow +\infty$). So if this last condition is satisfied, B_w is universal on $\ell_p(\mathbb{N})$ (resp. $c_0(\mathbb{N})$) for ergodic systems.

This condition is also necessary for B_w to be universal for either ergodic or invertible ergodic systems, but the arguments are different depending on whether B_w acts on $\ell_p(\mathbb{N})$ for some $1 \leq p < +\infty$ or on $c_0(\mathbb{N})$.

– If B_w acts on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, and is universal for ergodic systems, then B_w is necessarily frequently hypercyclic, and by the characterization of frequently hypercyclic weighted shifts on $\ell_p(\mathbb{N})$ of [6], the series $\sum_{n \geq 1} 1/|w_1 \dots w_n|^p$ is convergent.

– If B_w acts on $c_0(\mathbb{N})$, then the same argument does not apply since the condition $|w_1 \dots w_n| \longrightarrow +\infty$ as $n \longrightarrow +\infty$ does not characterize frequently hypercyclic backward weighted shifts on $c_0(\mathbb{N})$ (see [3] and [6] for details). But if B_w is universal for (invertible) ergodic systems, then it is necessarily strongly mixing with respect to some invariant measure with full support, hence in particular topologically mixing. So, for every $\varepsilon > 0$, there exists an integer n_ε such that, for every $n \geq n_\varepsilon$, there exists a vector $x^{(n,\varepsilon)} \in c_0(\mathbb{N})$ with $||x^{(n,\varepsilon)}||_\infty < \varepsilon$ such that $||B_w^n x^{(n,\varepsilon)} - e_0||_\infty < 1/2$. In particular, $|x_n^{(n,\varepsilon)} w_1 \dots w_n - 1| < 1/2$, so that $|x_n^{(n,\varepsilon)} w_1 \dots w_n| > 1/2$. Thus $\varepsilon |w_1 \dots w_n| > 1/2$ for every $n \geq n_\varepsilon$. It follows that $|w_1 \dots w_n| \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

The arguments for bilateral weighted shifts are exactly the same and we leave them to the reader.

3.2. **Proof of Theorem 1.6.** The proof of Theorem 1.6 relies on the following simple idea: suppose that Z is a Banach space admitting a biorthogonal system $(u_n, u_n^*)_{n\geq 0}$ having the following property:

There exists a bounded sequence $(\omega_n)_{n\geq 1}$ of non-zero weights such that B_ω defined by $B_\omega u_n = \omega_n u_{n-1}$ for every $n \geq 1$ and $B_\omega u_0 = 0$ is a bounded operator on Z and

$$\sum_{n\geq 1} \frac{||u_n||}{|\omega_1 \dots \omega_n|} < +\infty.$$

Then B_ω is a universal operator for ergodic systems on Z.

The proof of this statement is similar to that of Theorem 1.5: we set $z_0 = u_0$ and $z_{-n} = (1/|\omega_1 \dots \omega_n|)\, u_n$. Then $\overline{\text{span}}\left[z_{-n};\ n \geq 0\right] = Z$, $\langle u_0^*, z_{-n}\rangle = 0$ for every $n \geq 1$ so that $\overline{\text{span}}\left[z_{-n}; n \geq 1\right] \neq Z$, and lastly the series $\sum_{n\geq 0} z_{-n}$ is unconditionally convergent since $\sum_{n\geq 0} ||z_{-n}|| < +\infty$. So Theorem 1.4 applies.

Suppose that Z can be decomposed as a topological sum $Z = E \oplus Y$ where E has a subsymmetric basis $(e_n)_{n\geq 0}$ (i.e. the basis $(e_n)_{n\geq 0}$ is unconditional and equivalent to each of its subsequences). Let $(e_n^*)_{n\geq 0}$ denote the family of biorthogonal functionals on E, which we extend to Z by setting $\langle e_n^*, y\rangle = 0$ for every $y \in Y$. Also let $(y_n, y_n^*)_{n\geq 0}$ be a bounded biorthogonal system for Y, where each y_n^* is extended to Z by setting $\langle y_n^*, e\rangle = 0$ for every $e \in E$. We denote by P_E and P_Y the projections of Z onto E and Y respectively, associated to the decomposition $Z = E \oplus Y$, and we let $M = \max(||P_E||, ||P_Y||)$. Since the basis $(e_n)_{n\geq 0}$ is subsymmetric, the operator B_α defined by $B_\alpha e_n = \alpha_n e_{n-1}$ for every $n \geq 1$ and $B_\alpha e_0 = 0$ is a bounded operator on E for any bounded sequence of weights $(\alpha_n)_{n\geq 1}$. Let $(n_k)_{k\geq 0}$ be a strictly increasing sequence of integers with $n_0 = 0$. Consider the biorthogonal system (u_n, u_n^*) of Z defined by setting

$$u_n = \begin{cases} y_k & \text{if } n = n_k \text{ for some } k \geq 0, \\ e_{n-k-1} & \text{if } n \in \{n_k + 1, \dots, n_{k+1} - 1\} \text{ for some } k \geq 0, \end{cases}$$

$$u_n^* = \begin{cases} y_k^* & \text{if } n = n_k \text{ for some } k \geq 0, \\ e_{n-k-1}^* & \text{if } n \in \{n_k + 1, \dots, n_{k+1} - 1\} \text{ for some } k \geq 0. \end{cases}$$

With this definition, $\{u_n;\ n_k < n < n_{k+1}\} = \{e_n;\ n_k - k \leq n < n_{k+1} - (k+1)\}$ and $\{u_n;\ n \in \{n_k;\ k \geq 0\}\} = \{y_n;\ n \geq 0\}$, so that $(u_n, u_n^*)_{n\geq 0}$ is indeed a bounded biorthogonal system of Z.

We define an operator A on Z by setting for every $z \in Z$,

$$Az = \sum_{k\geq 1} u_{n_k}^*(z)\, w_k u_{n_k - 1} + 2\sum_{k\geq 0} \Big(\sum_{n=n_k+2}^{n_{k+1}-1} u_n^*(z)\, u_{n-1} \Big) + \sum_{k\geq 0} u_{n_k+1}^*(z)\, w_k' u_{n_k}$$

where the weights w_k and w_k' are defined by

$$w_k = \frac{2^{-k}}{||y_k^*||} \quad \text{and} \quad w_k' = \frac{2^{-k}}{||y_k||}, \quad k \geq 0.$$

Observe that these weights do not depend on the sequence $(n_k)_{k\geq 0}$.

This operator is a backward weighted shift with respect to the biorthogonal system $(u_n, u_n^*)_{n\geq 0}$: $Au_n = \omega_n u_{n-1}$ where

$$\omega_n = \begin{cases} w_k & \text{if } n = n_k \text{ for some } k \geq 1, \\ 2 & \text{if } n \in \{n_k+2, \dots, n_{k+1}-1\} \text{ for some } k \geq 0, \\ w'_k & \text{if } n = n_k + 1 \text{ for some } k \geq 1. \end{cases}$$

Let us first check that A is bounded. We have for every $z \in Z$,

$$Az = \sum_{k\geq 1} y_k^*(z)\, w_k e_{n_{k-1}-k} + \sum_{k\geq 0} \left(\sum_{n=n_k+2}^{n_{k+1}-1} e_{n-k-1}^*(z)\, 2\, e_{n-k-2} \right) + \sum_{k\geq 0} e_{n_k-k}^*(z)\, w'_k y_k.$$

Since the basis $(e_n)_{n\geq 0}$ is subsymmetric, there exists a positive constant C such that for every $z \in Z$,

$$\left\| \sum_{k\geq 0} \left(\sum_{n=n_k+2}^{n_{k+1}-1} e_{n-k-1}^*(z)\, e_{n-k-2} \right) \right\| \leq C||z||.$$

Hence

$$||Az|| \leq ||z|| \sum_{k\geq 1} ||y_k^*||\, w_k ||e_{n_{k-1}-k}|| + 2C||z|| + ||z|| \sum_{k\geq 0} ||e_{n_k-k}^*||\, w'_k ||y_k||.$$

Since $\sup_{n\geq 0} ||e_n||$ and $\sup_{n\geq 0} ||e_n^*||$ are finite, the conditions on the weights w_k and w'_k imply that A is bounded.

In order to show that A is universal for ergodic systems, it remains to choose the sequence $(n_k)_{k\geq 0}$ in such a way that the series

$$\sum_{n\geq 1} \frac{||u_n||}{\omega_1 \dots \omega_n}$$

is convergent. We have

$$\sum_{n\geq 1} \frac{||u_n||}{\omega_1 \dots \omega_n} = \sum_{k\geq 0} \sum_{n=n_k+1}^{n_{k+1}} \frac{||u_n||}{\omega_1 \dots \omega_n}$$
$$= \sum_{k\geq 0} \frac{1}{w_1 \dots w_k\, w'_1 \dots w'_k\, 2^{n_k-2k}} \left(\sum_{n=n_k+1}^{n_{k+1}-1} \frac{||u_n||}{2^{n-(n_k+1)}} + \frac{||u_{n_{k+1}}||}{2^{(n_{k+1}-1)-(n_k+1)} w_{k+1}} \right).$$

If we write $C_k = \max\{||u_n||;\ n_k + 1 \leq n \leq n_{k+1}\}$, this yields

$$\sum_{n\geq 1} \frac{||u_n||}{\omega_1 \dots \omega_n} \leq \sum_{k\geq 0} \frac{C_k}{w_1 \dots w_k\, w'_1 \dots w'_k\, 2^{n_k-2k}} \left(2 + \frac{1}{2^{n_{k+1}-n_k-2}\, w_{k+1}} \right).$$

Since the weights w_k and w'_k are defined independently of the sequence $(n_k)_{k\geq 0}$, we can choose this sequence growing so fast that

$$2^{n_k} > \max\left(\frac{2^{n_{k-1}+2}}{w_k}, \frac{2^{3k}\, C_k}{w_1 \dots w_k\, w'_1 \dots w'_k} \right) \quad \text{for every } k \geq 1.$$

Then

$$\sum_{n\geq 1} \frac{||u_n||}{\omega_1 \dots \omega_n} \leq \sum_{k\geq 0} 3 \,.\, 2^{-k},$$

from which it follows that the series

$$\sum_{n\geq 1} \frac{||u_n||}{\omega_1 \dots \omega_n}$$

is convergent. This completes the proof of Theorem 1.6.

4. Unimodular eigenvectors and universality

We begin this section with the proof of Theorem 1.7, which gives a straightforward criterion in terms of unimodular eigenvectors for an operator to be universal for (invertible) ergodic systems.

4.1. **Proof of Theorem 1.7.** Let A be a bounded operator on the complex separable infinite-dimensional Banach space Z, admitting a unimodular eigenvectorfield E satisfying assumptions (i), (ii), and (iii) of Theorem 1.7. Set, for each $n \in \mathbb{Z}$, $z_n = \widehat{E}(-n)$. Then $Az_n = z_{n+1}$ for every $n \in \mathbb{Z}$. The vectors z_n, $n \in \mathbb{Z}$, span a dense subspace of Z. Indeed, suppose that $z^* \in Z^*$ is such that $\langle z^*, z_n\rangle = 0$ for every $n \in \mathbb{Z}$. Then the function $\langle z^*, E(\,.\,)\rangle$ is zero almost everywhere, and by (i) it follows that $z^* = 0$. So z_0 is bicyclic for A. Now let F be the spectrum of the polynomial q defined by $q(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \widehat{p}(-n)e^{in\theta}$. For every $n \in \mathbb{Z} \setminus F$, we have

$$\langle z_0^*, z_n\rangle = \int_{\mathbb{T}} \lambda^n \,\langle z_0^*, E(\lambda)\rangle \, d\lambda = \widehat{q}(n) = 0.$$

Since z_0^* is non-zero, it follows that the linear span of the vectors z_n, $n \in \mathbb{Z}\setminus F$, is not dense in Z. Lastly, assumption (iii) of Theorem 1.7 states that the series $\sum_{n\in\mathbb{Z}} z_n$ is unconditionally convergent. So the hypotheses of Theorem 1.3 are satisfied, and A is universal for invertible ergodic systems. If $\widehat{E}(-r) = 0$ for some integer $r \in \mathbb{Z}$, then $A^r z_0 = 0$, and Theorem 1.4 applies.

Theorem 1.8 can now be obtained as a consequence of Theorem 1.7.

4.2. **Proof of Theorem 1.8.** Suppose that $A \in \mathcal{B}(Z)$ admits a unimodular eigenvectorfield E which is analytic in a neighborhood Ω of $\mathbb{T}$, and such that the vectors $E(\lambda)$, $\lambda \in \mathbb{T}$, span a dense subspace of Z. Then the restriction of E to $\mathbb{T}$ is σ-spanning for any measure σ with infinite support. In particular, it is $d\lambda$-spanning, where $d\lambda$ is the normalized Lebesgue measure on $\mathbb{T}$.

Let z_0^* be any non-zero element of Z^*. The function φ defined on Ω by setting

$$\varphi(\lambda) = \langle z_0^*, E(\lambda)\rangle, \quad \lambda \in \mathbb{T},$$

is analytic on Ω and not identically zero since the span of the vectors $E(\lambda)$, $\lambda \in \mathbb{T}$, is dense in Z. Thus φ admits only finitely many zeroes $z_1, \dots, z_r$ on $\mathbb{T}$, with respective multiplicities $d_1, \dots, d_r$. Let $p(z) = \prod_{j=1}^r (z - z_j)^{d_j}$. There exists a function ψ which is analytic on Ω and does not vanish on a neighborhood Ω' of $\mathbb{T}$ such that $\varphi(z) = p(z)\psi(z)$ for every $z \in \Omega'$.

Consider the eigenvectorfield $F : \Omega' \longrightarrow Z$ defined by $F(\lambda) = E(\lambda)/\psi(\lambda)$. The span of the vectors $F(\lambda)$, $\lambda \in \mathbb{T}$, is dense in Z. Moreover, for every $\lambda \in \mathbb{T}$,

$$\langle z_0^*, F(\lambda)\rangle = \frac{1}{\psi(\lambda)} \,\langle z_0^*, E(\lambda)\rangle = p(\lambda).$$

Lastly, since F is analytic on a neighborhood of $\mathbb{T}$, there exists $a \in (0,1)$ such that

$$||\widehat{F}(n)|| = \mathrm{O}(a^{|n|}) \quad \text{as } |n| \longrightarrow +\infty.$$

Hence the series $\sum_{n\in\mathbb{Z}} \widehat{F}(n)$ is unconditionally convergent. The assumptions of Theorem 1.7 are thus satisfied, and A is universal for invertible ergodic systems.

If E is analytic in a neighborhood of $\overline{\mathbb{D}}$, the same kind of reasoning applies: if $z_1, \dots, z_r$ are the zeroes of $\varphi(\lambda) = \langle z_0^*, E(\lambda)\rangle$ on $\overline{\mathbb{D}}$ with multiplicities $d_1, \dots, d_r$, and if $p(z) = \prod_{j=1}^r (z - z_j)^{d_j}$, then $\varphi(z) = p(z)\psi(z)$ on a neighborhood of $\overline{\mathbb{D}}$, where ψ is an analytic function on this neighborhood which does not vanish. If we consider again the eigenvectorfield F defined by $F(\lambda) = E(\lambda)/\psi(\lambda)$, then F is analytic in a neighborhood of $\overline{\mathbb{D}}$ so that $\widehat{F}(n) = 0$ for every $n < 0$. So Theorem 1.7 applies again and A is universal for ergodic systems.

Let us mention here that, using Theorems 2.5 and 2.6, Theorems 1.7 and 1.8 can be generalized to the case where A admits several eigenvectorfields.

4.3. **Examples and applications.** We present in this section several examples of operators which can be shown to be universal thanks to Theorems 1.7 and 1.8. First, one can easily retrieve the universality of some of the weighted shifts considered in Theorem 1.5 above.

Example 4.1. Let αB, $|\alpha| > 1$, be a multiple of the unweighted backward shift on $\ell_p(\mathbb{N})$, $1 \le p < +\infty$, or $c_0(\mathbb{N})$. Then αB admits an eigenvectorfield E defined on the disk of radius $|\alpha|$ by

$$E(\lambda) = \sum_{n\ge 0} \left(\frac{\lambda}{\alpha}\right)^{n-1} e_n.$$

So E is analytic on $\mathbb{D}(0, |\alpha|)$, and it is easy to check that the eigenvectors $E(\lambda)$, $\lambda \in \mathbb{T}$, span a dense subspace of $\ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$. Thus Theorem 1.8 applies, and αB is universal for ergodic systems. The same argument applies for instance to the weighted shift S_w on $\ell_p(\mathbb{Z})$ or $c_0(\mathbb{Z})$, where the weight w is given by $w_n = 2$ if $n \ge 1$ and $w_n = 1/2$ if $n \le 0$: S_w is universal for invertible ergodic systems.

Our second class of examples is given by adjoints of multipliers M_φ^* on $H^2(\mathbb{D})$. This is a natural class of operators to consider here, since their dynamical properties (hypercyclicity, frequent hypercyclicity, ergodicity) are rather well understood. See for instance [4] for details.

Example 4.2. Denoting by $\mathbb{D}$ the open unit disk, let $\varphi : \mathbb{D} \longrightarrow \mathbb{C}$ be an analytic map belonging to $H^\infty(\mathbb{D})$, and consider the adjoint M_φ^* of the multiplier M_φ defined on the Hardy space $H^2(\mathbb{D})$ by setting $M_\varphi f = \varphi f$ for every $f \in H^2(\mathbb{D})$. If $\mathbb{T} \subseteq \varphi(\mathbb{D})$, then M_φ^* is universal for invertible ergodic systems.

Proof of Example 4.2. Suppose that the analytic map $\varphi : \mathbb{D} \longrightarrow \mathbb{C}$ is such that $\varphi(\mathbb{D})$ contains the unit circle. Since φ is open, $K = \varphi^{-1}(\mathbb{T})$ is a compact subset of $\mathbb{D}$, and there exists $\rho \in (0,1)$ such that $K \subseteq D(0,\rho)$. The derivative φ' of φ can vanish only finitely many times on K, and we denote by $z_1, \dots, z_r$ the distinct zeroes of φ' on K, with respective multiplicities $m_1, \dots, m_r$. There exists for every $j \in \{1, \dots, r\}$ a disk $D(z_j, \varepsilon_j)$ with $\varepsilon_j > 0$ and two holomorphic functions ψ_j and σ_j on $D(z_j, \varepsilon_j)$ which do not vanish here such that for every $z \in D(z_j, \varepsilon_j)$,

$$\varphi(z) = \lambda_j + (z - z_j)^{d_j}\psi_j(z) \quad \text{and} \quad \varphi'(z) = (z - z_j)^{m_j}\sigma_j(z)$$

where $d_j = m_j + 1$ and $\lambda_j = \varphi(z_j)$. Also, there exists a holomorphic function β_j on $D(z_j, \varepsilon_j)$ such that $\psi_j = \beta_j^{d_j}$. If we set $\alpha_j(z) = (z - z_j)\beta_j(z)$ for $z \in D(z_j, \varepsilon_j)$,

we can assume that α_j is a biholomorphism from a certain open neighborhood V_j of z_j contained in $D(z_j, \varepsilon_j)$ onto an open disk $D(0, \delta_j)$. Let

$$\Omega_{j,0} = D(0, \delta_j^{d_j}) \setminus [0, \delta_j^{d_j})$$

and let $\gamma_{j,0}$ be a holomorphic determination of the d_j-th root of z on $\Omega_{j,0}$: $\gamma_{j,0}(z)^{d_j} = z$ for every $z \in \Omega_{j,0}$. We also set $V_{j,0} = \{z \in V_j;\ \alpha_j(z)^{d_j} \in \Omega_{j,0}\}$, $U_{j,0} = \lambda_j + \Omega_{j,0}$ and $U_j = D(\lambda_j, \delta_j^{d_j})$. Observe that $U_{j,0} \cap \mathbb{T}$ contains a set of the form $\Gamma_j \setminus \{\lambda_j\}$, where Γ_j is an open subarc of $\mathbb{T}$ containing the point λ_j.

We now claim that φ is a biholomorphism from $V_{j,0}$ onto $U_{j,0}$. Let us first check that $\varphi(V_{j,0}) = U_{j,0}$: $z \in V_j$ belongs to $V_{j,0}$ if and only if $\alpha_j(z)^{d_j} = \varphi(z) - \lambda_j$ belongs to $\Omega_{j,0}$, i.e. if and only if $\varphi(z)$ belongs to $U_{j,0}$. Let us set, for $z \in U_{j,0}$,

$$\varphi_{j,0}^{-1}(z) = \alpha_j^{-1} \circ \gamma_{j,0}\,(z - \lambda_j).$$

This definition makes sense: if $z \in U_{j,0}$, then $z - \lambda_j \in \Omega_{j,0}$, so that $\gamma_{j,0}(z - \lambda_j) \in D(0, \delta_j)$, and α_j is a biholomorphism from V_j onto $D(0, \delta_j)$. This function $\varphi_{j,0}^{-1}$ is thus well defined and holomorphic on $U_{j,0}$, and we have for every $z \in U_{j,0}$,

$$\varphi\big(\varphi_{j,0}^{-1}(z)\big) = \lambda_j + \alpha_j\big(\varphi_{j,0}^{-1}(z)\big)^{d_j} = \lambda_j + \big(\gamma_{j,0}(z - \lambda_j)\big)^{d_j} = \lambda_j + z - \lambda_j = z$$

since $z - \lambda_j$ belongs to $\Omega_{j,0}$ and $\gamma_{j,0}(z)^{d_j} = z$ for every $z \in \Omega_{j,0}$. It follows from this that $\varphi : V_{j,0} \longrightarrow U_{j,0}$ is a biholomorphism, the inverse of which is $\varphi_{j,0}^{-1}$. Restricting the sets $V_{j,0}$ and $U_{j,0}$, we can and do assume that, for every $j \in \{1, \dots, r\}$, $U_{j,0} \cap \mathbb{T} = \Gamma_j \setminus \{\lambda_j\}$, where Γ_j is an open subarc of $\mathbb{T}$ containing the point λ_j, and that the arcs Γ_j, $j \in \{1, \dots, r\}$, do not intersect.

If $z \in K$ is such that $\varphi(z) \notin \{\lambda_1, \dots, \lambda_r\}$ and $\varphi'(z) \neq 0$, then φ is a biholomorphism in a neighborhood of z. For every such z, let V_z be an open neighborhood of z, and U_z a disk centered at $\varphi(z)$ of radius $\rho_z > 0$ such that $\varphi : V_z \longrightarrow U_z$ is a biholomorphism, φ' does not vanish on V_z, and the closure of the set U_z contains none of the points λ_j, $j \in \{1, \dots, r\}$. The disks U_z, $z \in \varphi^{-1}(\mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\})$, and U_j, $j \in \{1, \dots, r\}$, form an open covering of $\mathbb{T}$ (remember our assumption that $\mathbb{T} \subseteq \varphi(\mathbb{D})$), so one can extract from it a finite covering of the form $U_{\xi_1}, \dots, U_{\xi_s}, U_1, \dots, U_r$, $s \geq 1$. Denote by $\big(\Omega_l\big)_{l \in \Lambda}$ the finite family of open subsets of $\mathbb{D}$ consisting of the sets U_{ξ_i}, $1 \leq i \leq s$, and $U_{j,0}$, $1 \leq j \leq r$. For every $l \in \Lambda$, $\Omega_l' = \Omega_l \cap \mathbb{T}$ is either an open subarc of $\mathbb{T}$ whose closure is contained in $\mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$ (when $\Omega_l = U_{\xi_i}$ for some $i \in \{1, \dots, s\}$) or an open subarc minus one point (when $\Omega_l = U_{j,0}$ for some $j \in \{1, \dots, r\}$, in which case $\Omega_l' = \Gamma_j \setminus \{\lambda_j\}$). We have

$$\bigcup_{l \in \Lambda} \Omega_l' \cap \mathbb{T} = \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}.$$

Observe that for every $j \in \{1, \dots, r\}$, there is a unique index $l \in \Lambda$ such that Ω_l' contains the point λ_j in its closure. Writing this index as l_j, we have $\Omega_{l_j} = U_{j,0}$ and $\Omega_{l_j}' = \Omega_{l_j} \cap \mathbb{T} = \Gamma_j \setminus \{\lambda_j\}$. For every $l \in \Lambda$, we denote by φ_l^{-1} the inverse of φ defined on the set Ω_l.

Let $\big(v_l\big)_{l \in \Lambda}$ be a finite $\mathcal{C}^1$-partition of the unity associated to the finite covering $\big(\Omega_l'\big)_{l \in \Lambda}$ of $\mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$, with v_l supported on Ω_l' for every $l \in \Lambda$. If $\Omega_l' = U_{\xi_i} \cap \mathbb{T}$ for some $i \in \{1, \dots, s\}$, the function v_l, which is supported on Ω_l', obviously extends into a $\mathcal{C}^1$ function on the whole circle $\mathbb{T}$ by setting $v_l(\lambda_j) = 0$ for every $j \in \{1, \dots, r\}$. If $\Omega_l' = \Omega_{l_j}'$ for some $j \in \{1, \dots, r\}$, the observation above shows that there exists

an open subarc Γ'_j of Γ_j, containing λ_j, such that the only index $l \in \Lambda$ for which v_l does not identically vanish on $\Gamma'_j \setminus \{\lambda_j\}$ is l_j. Hence v_{l_j} is equal to 1 on $\Gamma'_j \setminus \{\lambda_j\}$, and it follows that v_{l_j} can be extended into a $\mathcal{C}^1$ function on the whole circle $\mathbb{T}$ by setting $v_{l_j}(\lambda_j) = 1$. We can thus assume that all the functions v_l, $l \in \Lambda$, are defined and of class $\mathcal{C}^1$ on $\mathbb{T}$.

For every $z \in \mathbb{D}$, let k_z be the reproducing kernel of the space $H^2(\mathbb{D})$ at the point z:

$$k_z(\xi) = \sum_{n\geq 0} \overline{z}^n \xi^n = \frac{1}{1-\overline{z}\,\xi}, \quad \xi \in \mathbb{D},$$

and k_z is characterized by the property that $\langle f, k_z\rangle = f(z)$ for every $f \in H^2(\mathbb{D})$. We have $M_\varphi^* k_z = \overline{\varphi(z)} k_z$ for every $z \in \mathbb{D}$. We then introduce the polynomial

$$p(z) = \prod_{j=1}^{r} (z-\lambda_j)^2$$

and the maps $F : \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\} \longrightarrow H^2(\mathbb{D})$ and $E : \mathbb{T} \setminus \{\overline{\lambda}_1, \dots, \overline{\lambda}_r\} \longrightarrow H^2(\mathbb{D})$ defined by

$$\forall\, \lambda \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}, \qquad F(\lambda) = \overline{p(\lambda)} \sum_{l\in\Lambda} v_l(\lambda)\, k_{\varphi_l^{-1}(\lambda)}$$

and

$$\forall\, \lambda \in \mathbb{T} \setminus \{\overline{\lambda}_1, \dots, \overline{\lambda}_r\}, \qquad E(\lambda) = F(\overline{\lambda}).$$

Observe that the quantity $k_{\varphi_l^{-1}(\lambda)}$ in the expression of $F(\lambda)$ above only makes sense when λ belongs to Ω'_l. But since v_l is supported on Ω'_l, the function $\lambda \mapsto v_l(\lambda) k_{\varphi_l^{-1}(\lambda)}$ extends into a $\mathcal{C}^\infty$ function on $\mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$ by defining it to be zero outside the set Ω'_l, so that F is a well-defined $\mathcal{C}^\infty$ function on $\mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$.

For every $\lambda \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$ we have

$$M_\varphi^* F(\lambda) = \overline{p(\lambda)} \sum_{l\in\Lambda} v_l(\lambda) \overline{\varphi\big(\varphi_l^{-1}(\lambda)\big)}\, k_{\varphi_l^{-1}(\lambda)}$$

so that $M_\varphi^* F(\lambda) = \overline{\lambda}\, F(\lambda)$. It follows that $E : \mathbb{T} \setminus \{\overline{\lambda}_1, \dots, \overline{\lambda}_r\} \longrightarrow H^2(\mathbb{D})$ is a unimodular eigenvectorfield for M_φ^*.

Our aim is to show that the assumptions of Theorem 1.7 are satisfied. Assumptions (i) and (ii) are easy to check: suppose that $f \in H^2(\mathbb{D})$ is such that $\langle f, E(\lambda)\rangle = 0$ for every $\lambda \in B$ where $B \subseteq \mathbb{T} \setminus \{\overline{\lambda}_1, \dots, \overline{\lambda}_r\}$ is a Borel subset of $\mathbb{T}$ of full Lebesgue measure. Then by continuity

$$\overline{p(\lambda)} \sum_{l\in\Lambda} v_l(\lambda)\, f\big(\varphi_l^{-1}(\lambda)\big) = 0 \quad \text{for every } \lambda \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\},$$

so that

$$\sum_{l\in\Lambda} v_l(\lambda)\, f\big(\varphi_l^{-1}(\lambda)\big) = 0 \quad \text{for every } \lambda \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}.$$

It follows that for every $j \in \{1, \dots, r\}$ and every $\lambda \in \Gamma'_j \setminus \{\lambda_j\}$, $f\big(\varphi_{l_j}^{-1}(\lambda)\big) = 0$. Since $\varphi_{l_j}^{-1}(\Gamma'_j \setminus \{\lambda_j\})$ has accumulation points in $\mathbb{D}$, $f = 0$, and $\operatorname{span}\{E(\lambda);\ \lambda \in B\}$ is dense in $H^2(\mathbb{D})$. Assumption (i) is thus satisfied. Assumption (ii) clearly holds true: if $f \equiv 1$, then $\langle f, E(\lambda)\rangle = \overline{p(\overline{\lambda})} = q(\lambda)$ for every $\lambda \in \mathbb{T} \setminus \{\overline{\lambda}_1, \dots, \overline{\lambda}_r\}$, where $q(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \overline{\hat{p}(n)} e^{in\theta}$. The main difficulty is to check that the series $\sum_{n\in\mathbb{Z}} \widehat{E}(n)$,

or equivalently the series $\sum_{n\in\mathbb{Z}} \widehat{F}(n)$, is unconditionally convergent. Since $H^2(\mathbb{D})$ does not contain a copy of c_0, it suffices to prove that for every $f \in H^2(\mathbb{D})$, the series $\sum_{n\in\mathbb{Z}} |\langle f, \widehat{F}(n)\rangle|$ is convergent.

We are going to show that for every $f \in H^2(\mathbb{D})$, the function ϕ_f defined on $\mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$ by

$$\phi_f(\lambda) = \langle f, F(\lambda)\rangle = \overline{p(\lambda)} \sum_{l\in\Lambda} v_l(\lambda)\, f\big(\varphi_l^{-1}(\lambda)\big)$$

extends into a function of class $\mathcal{C}^1$ on $\mathbb{T}$. Bernstein's Theorem will then imply that the series $\sum_{n\in\mathbb{Z}} |\langle f, \widehat{F}(n)\rangle|$ is convergent.

We have seen that the function F is of class $\mathcal{C}^\infty$ on $\mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$. Since, for every $l \in \Lambda$, $\sup\big\{|\varphi_l^{-1}(\lambda)|;\ \lambda \in \Omega_l'\big\} \le \rho < 1$, the quantity $\sup\big\{\big|\big|k_{\varphi_l^{-1}(\lambda)}\big|\big|;\ \lambda \in \Omega_l'\big\}$ is finite for every $l \in \Lambda$. Now $p(\lambda_j) = 0$ for every $j \in \{1, \dots, r\}$, and since each function v_l, $l \in \Lambda$, is uniformly bounded on $\mathbb{T}$ it follows that F can be extended into a continuous map on $\mathbb{T}$ by setting $F(\lambda_j) = 0$ for every $j \in \{1, \dots, r\}$. So ϕ_f is actually continuous on $\mathbb{T}$ for every $f \in H^2(\mathbb{D})$. Let us now compute the derivative of ϕ_f. Writing, for $0 \le \theta < 2\pi$, $\lambda = e^{i\theta}$ and

$$\phi_f(e^{i\theta}) = \overline{p(e^{i\theta})} \sum_{l\in\Lambda} v_l(e^{i\theta})\, f\big(\varphi_l^{-1}(e^{i\theta})\big)$$

we have

$$\begin{aligned}\frac{d\phi_f}{d\theta}(e^{i\theta}) = &- i\, e^{-i\theta}\, \overline{p'(e^{i\theta})} \sum_{l\in\Lambda} v_l(e^{i\theta})\, f\big(\varphi_l^{-1}(e^{i\theta})\big)\\ &+ \overline{p(e^{i\theta})} \sum_{l\in\Lambda} ie^{i\theta}\, \frac{dv_l}{d\theta}(e^{i\theta})\, f\big(\varphi_l^{-1}(e^{i\theta})\big)\\ &+ \overline{p(e^{i\theta})} \sum_{l\in\Lambda} v_l(e^{i\theta})\, (i\, e^{i\theta})^2 \frac{1}{\varphi'\big(\varphi_l^{-1}(e^{i\theta})\big)}\, f'\big(\varphi_l^{-1}(e^{i\theta})\big)\end{aligned}$$

(the notation $\dfrac{d}{d\theta}$ is used for the derivative of a function on $\mathbb{T}$ with respect to the real variable θ, while the sign ' is used for the complex derivative of a holomorphic function).

The same argument as above, using the facts that Λ is finite, that the functions v_l are of class $\mathcal{C}^1$ on $\mathbb{T}$ (and hence have uniformly bounded derivatives on $\mathbb{T}$), and that $p(\lambda_j) = p'(\lambda_j) = 0$ for every $j \in \{1, \dots, r\}$, shows that the first two terms in this expression tend to 0 as $\lambda = e^{i\theta} \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$ tends to λ_j, $j \in \{1, \dots, r\}$. So it remains to deal with the last term. If $\lambda \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$ tends to λ_j for some $j \in \{\lambda_1, \dots, \lambda_r\}$, we can suppose without loss of generality that λ belongs to Ω_{l_j}'. The third term in the expression above is then equal to

$$\overline{p(\lambda)} \frac{1}{\varphi'\big(\varphi_{l_j}^{-1}(\lambda)\big)} f'(\varphi_{l_j}^{-1}(\lambda)), \quad \text{where } \lambda = e^{i\theta}.$$

As $\sup\big\{\big|f'\big(\varphi_{l_j}^{-1}(\lambda)\big)\big|;\ \lambda \in \Omega_{l_j}'\big\}$ is finite, it suffices to show that

$$\frac{\overline{p(\lambda)}}{\varphi'\big(\varphi_{l_j}^{-1}(\lambda)\big)} \to 0 \quad \text{as } \lambda \to \lambda_j,\ \lambda \in \Omega_{l_j}'.$$

We have seen that for every $z \in D(z_j\varepsilon_j)$, $\varphi'(z) = (z - z_j)^{m_j}\sigma_j(z)$, where σ_j is a holomorphic function which does not vanish on $D(z_j, \varepsilon_j)$. Hence the quantity

$$\sup\Big\{ \frac{|z - z_j|^{m_j}}{|\varphi'(z)|};\ z \in D(z_j, \varepsilon_j) \setminus \{z_j\} \Big\}$$

is finite. So there exists a positive constant C such that

$$\sup\Big\{ \frac{\big|\varphi_{l_j}^{-1}(\lambda) - z_j\big|^{m_j}}{\big|\varphi'\big(\varphi_{l_j}^{-1}(\lambda)\big)\big|};\ \lambda \in \Omega_{l_j} \Big\} \le C.$$

We have

$$p(\lambda) = p\big(\varphi\big(\varphi_{l_j}^{-1}(\lambda)\big)\big) = p\big(\lambda_j + \big(\varphi_{l_j}^{-1}(\lambda) - z_j\big)^{d_j}\psi_j\big(\varphi_{l_j}^{-1}(\lambda)\big)\big)$$

for every $\lambda \in \Omega_{l_j}$, and thus $p(\lambda) = \big(\varphi_{l_j}^{-1}(\lambda) - z_j\big)^{2d_j} q_j\big(\varphi_{l_j}^{-1}(\lambda)\big)$ for every $\lambda \in \Omega_{l_j}$, where q_j is a holomorphic function which is bounded on $\varphi_{l_j}^{-1}(\Omega_{l_j})$. Hence there exists a positive constant C' such that

$$\begin{aligned}\Big|\frac{p(\lambda)}{\varphi'\big(\varphi_{l_j}^{-1}(\lambda)\big)}\Big| &= \Big|\frac{\big(\varphi_{l_j}^{-1}(\lambda) - z_j\big)^{2d_j}}{\big(\varphi_{l_j}^{-1}(\lambda) - z_j\big)^{m_j}}\Big| \cdot \big|q_j\big(\varphi_{l_j}^{-1}(\lambda)\big)\big| \cdot \Big|\frac{\big(\varphi_{l_j}^{-1}(\lambda) - z_j\big)^{m_j}}{\varphi'\big(\varphi_{l_j}^{-1}(\lambda)\big)}\Big| \\ &\le C' \cdot \big|\varphi_{l_j}^{-1}(\lambda) - z_j\big|^{2d_j - m_j} \qquad \text{for every } \lambda \in \Omega_{l_j}.\end{aligned}$$

The right-hand bound tends to 0 as $\varphi_{l_j}^{-1}(\lambda)$ tends to z_j since $2d_j = 2(m_j+1) > m_j$. We now claim that if $\lambda \in \Omega_{l_j}$ tends to λ_j, then $\varphi_{l_j}^{-1}(\lambda)$ tends to z_j. Indeed, since $\Omega_{l_j} = U_{j,0}$, $\varphi_{l_j}^{-1}(\lambda) = \varphi_{j,0}^{-1}(\lambda) = \alpha_j^{-1}\big(\gamma_{j,0}(\lambda - \lambda_j)\big)$. If $\lambda \to \lambda_j$, then $\gamma_{j,0}(\lambda - \lambda_j) \to 0$. Now the map $\alpha_j : D(z_j, \varepsilon_j) \longrightarrow D(0, \delta_j)$ is a biholomorphism such that $\alpha_j(z_j) = 0$. Thus $\alpha_j^{-1}\big(\gamma_{j,0}(\lambda - \lambda_j)\big) \to z_j$ as $\lambda \to \lambda_j$, $\lambda \in \Omega_{l_j}$, and this proves our claim.

So we have proved that if $\lambda \in \Omega_{l_j}$ tends to λ_j, then

$$\Big|\frac{\overline{p(\lambda)}}{\varphi'\big(\varphi_{l_j}^{-1}(\lambda)\big)}\Big| = \Big|\frac{p(\lambda)}{\varphi'\big(\varphi_{l_j}^{-1}(\lambda)\big)}\Big| \quad \text{tends to 0.}$$

As explained above, this shows that the third term in the expression of $\dfrac{d\phi_f}{d\theta}(e^{i\theta})$ given above tends to 0 as $d\big(e^{i\theta}, \{\lambda_1, \dots, \lambda_r\}\big)$ tends to 0 with $e^{i\theta} \in \mathbb{T} \setminus \{\lambda_1, \dots, \lambda_r\}$. So ϕ_f is a map of class $\mathcal{C}^1$ on $\mathbb{T}$, and this finishes the proof of Example 4.2. $\square$

Example 4.3. Using the notation of Example 4.1, the operator αB^2 is universal on $\ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$ for any α with $|\alpha| > 1$.

This can be proved in several ways. One of these is to observe that αB^2 admits two eigenvectorfields E_1 and E_2 which are analytic on $D(0, |\alpha|)$:

$$E_1(\lambda) = \sum_{n \ge 0} \Big(\frac{\lambda}{\alpha}\Big)^n e_{2n+1} \quad \text{and} \quad E_2(\lambda) = \sum_{n \ge 0} \Big(\frac{\lambda}{\alpha}\Big)^n e_{2n}.$$

We have $\overline{\operatorname{span}}\,\big[E_1(\lambda), E_2(\lambda);\ \lambda \in \mathbb{T}\big] = Z$. Using the generalization of Theorem 1.8 following from Theorem 2.6, we obtain that αB^2 is universal for ergodic systems.

If one is interested only in the universality of αB^2 for invertible ergodic systems, one can simply use Theorem 1.7 and the following argument: let φ be a function

of class $\mathcal{C}^\infty$ on $\mathbb{T}$ such that $0 \leq \varphi \leq 1$ and the support of φ is a non-trivial closed subarc Γ of $\mathbb{T}$. Consider the $\mathcal{C}^\infty$ eigenvectorfield of αB^2 defined by

$$E(\lambda) = \varphi(\lambda)\, E_1(\lambda) + (1 - \varphi(\lambda))\, E_2(\lambda), \quad \lambda \in \mathbb{T}.$$

Let us show that $\overline{\operatorname{span}}\big[E(\lambda);\ \lambda \in \mathbb{T}\big] = Z$: if x^* is a functional such that $\langle x^*, E(\lambda)\rangle = 0$ for every $\lambda \in \mathbb{T}$, then $(1 - \varphi(\lambda))\,\langle x^*, E_2(\lambda)\rangle = \langle x^*, E_2(\lambda)\rangle = 0$ for every $\lambda \in \mathbb{T} \setminus \Gamma$. Hence by analyticity of E_2, $\langle x^*, E_2(\lambda)\rangle = 0$ for every $\lambda \in \mathbb{T}$. It follows that $\varphi(\lambda)\,\langle x^*, E_1(\lambda)\rangle = 0$ for every $\lambda \in \Gamma$, and the same argument shows that $\langle x^*, E_1(\lambda)\rangle = 0$ for every $\lambda \in \mathbb{T}$. Since $\overline{\operatorname{span}}\big[E_1(\lambda), E_2(\lambda);\ \lambda \in \mathbb{T}\big] = Z$, $x^* = 0$, and thus $\overline{\operatorname{span}}\big[E(\lambda);\ \lambda \in \mathbb{T}\big] = Z$. Moreover, $\langle e_0^* + e_1^*, E(\lambda)\rangle = 1$ for every $\lambda \in \mathbb{T}$ (where e_i^* is the functional on $\ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$ mapping a vector x of the space on its i-th coordinate). Lastly, E being of class $\mathcal{C}^\infty$ on $\mathbb{T}$, the series

$$\sum_{n\in\mathbb{Z}} ||\widehat{E}(n)||$$

is convergent. So Theorem 1.7 applies. If we consider αB^2 as acting on $\ell_2(\mathbb{N})$, Example 4.2 applies directly since αB^2 is unitarily similar to M_φ^* where $\varphi(z) = \overline{\alpha} z^2$, $z \in \mathbb{D}$.

Many other examples can be obtained along these lines (such as $\bigoplus_{\ell_p} \alpha B$ on the infinite direct sum $\bigoplus_{\ell_p} \ell_p$, $1 \leq p < +\infty$, $|\alpha| > 1, \dots$).

One can observe that all the universal operators presented until now admit eigenvectorfields which are analytic in a neighborhood of some points of $\mathbb{T}$, and in particular have a rather large spectrum. So one may naturally wonder whether this condition is necessary for A to be universal. Our last example, which is rather unexpected, shows that it is not the case. It is to be found within the class of Kalish-type operators on $L^2(\mathbb{T})$ (see [4] for details).

Example 4.4. Let A be the operator defined on $L^2(\mathbb{T})$ by setting for every $f \in L^2(\mathbb{T})$ and every $0 \leq \theta < 2\pi$,

$$Af(e^{i\theta}) = e^{i\theta} f(e^{i\theta}) - \int_0^\theta ie^{it} f(e^{it})dt.$$

Then A is universal for invertible ergodic systems.

It is not difficult to check that for every $\lambda = e^{i\theta} \in \mathbb{T}$, $0 \leq \theta < 2\pi$, $\ker(A - \lambda) = \operatorname{span}\big[E(\lambda)\big]$ where $E(\lambda) = \chi_{(\lambda,1)}$. Here $\chi_{(\lambda,1)}$ denotes the indicator function of the arc $\Gamma_\lambda = \big\{e^{i\tau};\ \theta < \tau < 2\pi\big\}$. The eigenvectorfield E is 1/2-Hölderian on $\mathbb{T}$. Since the spectrum of A coincides with $\mathbb{T}$, A does not admit any eigenvectorfield which is analytic in a neighborhood of some point of $\mathbb{T}$.

Proof of Example 4.4. The most obvious idea is to try to apply Theorem 1.7. Setting

$$y_n(e^{it}) = \int_0^{2\pi} e^{-in\theta} E(e^{i\theta})(e^{it}) \frac{d\theta}{2\pi}$$

for every $n \in \mathbb{Z}$, we have

$$y_0(e^{it}) = \frac{t}{2\pi} \quad \text{and} \quad y_n(e^{it}) = \frac{1 - e^{-int}}{2i\pi n} \quad \text{for every } n \in \mathbb{Z} \setminus \{0\}\cdot$$

The series $\sum_{n\in\mathbb{Z}} y_n$ is obviously not unconditionally convergent in $L^2(\mathbb{T})$, and Theorem 1.7 cannot be applied this way. So we consider instead of E the unimodular eigenvectorfield F defined by $F(\lambda) = (1-\lambda)E(\lambda)$, and we set

$$z_n(e^{i\theta}) = \int_0^{2\pi} e^{-in\theta} F(e^{i\theta})(e^{it})\frac{d\theta}{2\pi} = \int_0^{2\pi} \big(e^{-in\theta} - e^{-i(n-1)\theta}\big)E(e^{i\theta})(e^{it})\frac{d\theta}{2\pi}.$$

Thus $z_n = y_n - y_{n-1}$ for every $n \in \mathbb{Z}$. We have for every $n \in \mathbb{Z}\setminus\{0,1\}$,

$$z_n(e^{it}) = \frac{1}{2i\pi}\left(\frac{e^{-i(n-1)t}}{n-1} - \frac{e^{-int}}{n} - \frac{1}{n(n-1)}\right).$$

Since the series $\sum_{|n|\geq 2} 1/(n(n-1))$ is convergent, and the series $\sum_{|n|\geq 2} e^{-int}/n$ is unconditionally convergent in $L^2(\mathbb{T})$, it follows that the series $\sum_{|n|\geq 2} z_n$ (and hence the series $\sum_{n\in\mathbb{Z}} z_n$) is unconditionally convergent in $L^2(\mathbb{T})$.

Let us now check that the functions z_n, $n \in \mathbb{Z}$, span a dense subspace of $L^2(\mathbb{T})$. Suppose that $f \in L^2(\mathbb{T})$ is such that $\langle f, z_n\rangle = 0$ for every $n \in \mathbb{Z}$. Then

$$\frac{\widehat{f}(n)}{n} = \frac{\widehat{f}(n-1)}{n-1} + \widehat{f}(0)\left(\frac{1}{n} - \frac{1}{n-1}\right) \quad \text{for every } n \notin \{0,1\}.$$

Summing these equalities for $n \geq 2$, we obtain

$$\sum_{n\geq 2}\frac{\widehat{f}(n)}{n} = \widehat{f}(1) + \sum_{n\geq 2}\frac{\widehat{f}(n)}{n} - \widehat{f}(0), \text{ so that } \widehat{f}(1) = \widehat{f}(0).$$

As

$$\widehat{f}(n) = \frac{n}{n-1}\widehat{f}(n-1) - \frac{1}{n-1}\widehat{f}(0)$$

for every $n \geq 2$, $\widehat{f}(2) = 2\widehat{f}(1) - \widehat{f}(0) = \widehat{f}(0)$. By induction $\widehat{f}(n) = \widehat{f}(0)$ for every $n \geq 2$, so that $\widehat{f}(n) = 0$ for every $n \geq 0$. So

$$\widehat{f}(n-1) = \frac{n-1}{n}\widehat{f}(n) \quad \text{for every } n \geq -1,$$

which implies that $\widehat{f}(-n) = n\widehat{f}(-1)$ for every $n \geq 1$. Hence $f = 0$. So $\overline{\text{span}}\,\big[z_n;\ n \in \mathbb{Z}\big] = L^2(\mathbb{T})$ (we have actually proved that $\overline{\text{span}}\,\big[z_n;\ n \in \mathbb{Z}\setminus\{0,1\}\big] = L^2(\mathbb{T})$).

Lastly, observe that if we set $f_0(e^{i\theta}) = e^{i\theta}$, then $\langle f_0, z_n\rangle = 0$ for every n with $|n| \geq 2$, $\langle f_0, z_1\rangle = -\langle f_0, z_{-1}\rangle = -1/2i\pi$, and that a simple computation shows that $\langle f_0, z_0\rangle = 0$. So $\langle f_0, F(e^{i\theta})\rangle = \big(e^{i\theta} - e^{-i\theta}\big)/2i\pi$, and all the assumptions of Theorem 1.7 (or, directly, Theorem 1.3) are satisfied. So A is universal for invertible ergodic systems. □

5. Miscellaneous results and comments

We begin this section by investigating necessary conditions for an operator to be universal. We have already seen some such necessary conditions: a universal operator must be frequently hypercyclic and topologically mixing. If it is universal for all ergodic systems, it cannot be invertible. Without any additional assumption, it seems difficult to say more. But looking at the examples of universal operators presented in Sections 3 and 4, we observe that all of them admit continuous unimodular eigenvectorfields and that every $\lambda \in \mathbb{T}$ is an eigenvalue. This is not

completely a coincidence: if $A \in \mathcal{B}(Z)$ satisfies the assumptions of Theorem 1.3, then A admits a continuous unimodular eigenvectorfield defined as

$$E(\lambda) = \sum_{n \in \mathbb{Z}} \lambda^n A^{-n} z_0.$$

If A satisfies the assumptions of Theorem 1.4 with $r = 1$ for instance, then A admits a continuous eigenvectorfield on $\overline{\mathbb{D}}$ defined as $E(\lambda) = \sum_{n \in \mathbb{N}} \lambda^n A^{-n} z_0$. In both cases assumption (b) implies that all $\lambda \in \mathbb{T}$ except possibly finitely many are eigenvectors of A. Indeed, there exists a finite subset F of $\mathbb{Z}$ and a non-zero functional $z_0^* \in Z^*$ such that

$$\langle z_0^*, E(\lambda) \rangle = \sum_{n \in F} \lambda^n \langle z_0^*, A^{-n} z_0 \rangle.$$

The function $\langle z_0^*, E(\,.\,) \rangle$ is a non-zero trigonometric polynomial, so it has only finitely many zeroes on $\mathbb{T}$. This implies that $E(\lambda)$ is non-zero for all $\lambda \in \mathbb{T}$ except possibly finitely many.

So it comes as a natural question to ask whether a universal operator necessarily admits some (or many) unimodular eigenvectors. It is possible to answer this question in the affirmative under two additional assumptions: first, that the operator lives on a Hilbert space and, second, that we add in the definition of the universality the requirement that all the measures ν on Z involved in the definition have a moment of order 2.

Definition 5.1. A bounded operator A on Z is said to be 2-universal for (invertible) ergodic systems if for every (invertible) ergodic dynamical system $(X, \mathcal{B}, \mu; T)$ there exists a Borel probability measure ν on Z which is A-invariant, has full support, has a moment of order 2 (i.e. $\int_Z ||z||^2 d\nu(z) < +\infty$), and is such that the two dynamical systems $(X, \mathcal{B}, \mu; T)$ and $(Z, \mathcal{B}_Z, \nu; A)$ are isomorphic.

We have seen in Section 2 that operators satisfying the assumptions of Theorems 1.3, 1.4, 2.5 or 2.6 are 2-universal. Thus all the universal operators presented in Sections 3 and 4 above are 2-universal. A first necessary condition for an operator on a Hilbert space to be 2-universal is:

Proposition 5.2. *Let H be a complex separable infinite-dimensional Hilbert space, and let A be a 2-universal operator on H. Then A admits a perfectly spanning (and even a $\mathcal{U}_0$-perfectly spanning) set of unimodular eigenvectors.*

Proof. The proof is an easy consequence of some results of [3] and [5] concerning the ergodic theory of linear dynamical systems (see also [4]). Since A is 2-universal, it is weakly mixing with respect to some probability measure ν with full support which has a moment of order 2. Consider the centered Gaussian probability measure m on H whose covariance operator S is given by

$$\langle Sx, y \rangle = \int_H \langle x, z \rangle \overline{\langle y, z \rangle} \, d\nu(z) \quad \text{for every } x, y \in H.$$

Then

$$\int_H \langle x, z \rangle \overline{\langle y, z \rangle} \, dm(z) = \langle Sx, y \rangle = \int_H \langle x, z \rangle \overline{\langle y, z \rangle} \, d\nu(z)$$

for every $x, y \in H$. Hence m is A-invariant and has full support. Moreover, since A is weakly mixing with respect to ν,

$$\frac{1}{N}\sum_{n=0}^{N-1}\left|\int_H \langle x, A^n z\rangle\,\overline{\langle y, z\rangle}\,d\nu(z)\right|^2 = \frac{1}{N}\sum_{n=0}^{N-1}\left|\int_H \langle x, A^n z\rangle\,\overline{\langle y, z\rangle}\,dm(z)\right|^2$$

tends to 0 as N tends to infinity for every $x, y \in H$, and this implies (see [2] or [4] for details) that A is weakly mixing with respect to m. Since it is proved in [3] that any operator on a space of cotype 2 which is weakly mixing with respect to some Gaussian measure with full support has perfectly spanning unimodular eigenvectors, the first part of the result follows. The second part is proved in exactly the same way: A is necessarily strongly mixing with respect to some probability measure on Z with full support and is hence strongly mixing with respect to some Gaussian probability measure on Z with full support. It is proved in [5] that this implies that the unimodular eigenvectors of A are $\mathcal{U}_0$-perfectly spanning (i.e. for any Borel set $B \subseteq \mathbb{T}$ which is a set of extended uniqueneness, $\overline{\operatorname{span}}\big[\ker(T-\lambda);\ \lambda \in \mathbb{T}\setminus B\big] = H$. See [5] for more about these questions). $\square$

If A is supposed to be 2-universal for ergodic systems (and not only for invertible ones), we can moreover prove that the unimodular point spectrum of A is a subset of $\mathbb{T}$ of full Lebesgue measure. These are the contents of Theorem 1.9, which we now prove.

Proof of Theorem 1.9. Let A be a 2-universal operator on H for ergodic systems. Consider the dynamical system T defined on $\big([0,1], \mathcal{B}_{[0,1]}, dx\big)$, where dx is the Lebesgue measure on $[0,1]$, by $Tx = 2x \mod 1$. Then T is strongly mixing and has the following property of decay of correlations: for any $f, g \in L^2\big([0,1]\big)$ and $n \geq 0$, define the n-th correlation between f and g as

$$\mathcal{C}_n(f,g) = \int_0^1 f(T^n x)\,\overline{g(x)}\,dx - \left(\int_0^1 f\right)\cdot\overline{\left(\int_0^1 g\right)}.$$

Then we have for every $f \in L^2\big([0,1]\big)$, $g \in \mathcal{C}^1\big([0,1]\big)$, and $n \geq 0$,

$$|\,\mathcal{C}_n(f,g)| \leq 2^{-n}\,\frac{||f||_2 \cdot ||g'||_\infty}{\sqrt{3}}.$$

Thus the correlations decay exponentially fast provided one of the two functions f and g is sufficiently smooth.

Since A is 2-universal for ergodic systems, there exists an A-invariant measure ν on H with full support and with a moment of order 2 for which there exists an isomorphism Φ between the two dynamical systems $\big([0,1], \mathcal{B}_{[0,1]}, dx; T\big)$ and $\big(H, \mathcal{B}_H, \nu; A\big)$. We denote by $L^2_0([0,1])$ (resp. $L^2_0(H, \mathcal{B}_H, \nu)$) the set of functions $f \in L^2([0,1])$ (resp. $F \in L^2(H, \mathcal{B}_H, \nu)$) such that $\int_0^1 f(x)\,dx = 0$ (resp. $\int_H F(z)\,d\nu(z) = 0$). For all functions $F, G \in L^2_0(H, \mathcal{B}_H, \nu)$ such that $G = g \circ \Phi^{-1}$ for some function $g \in \mathcal{C}^1\big([0,1]\big) \cap L^2_0\big([0,1]\big)$, there exists a positive constant $c(F,G)$ such that if we denote for every $n \geq 0$ by $C_n(F,G)$ the correlation

$$C_n(F,G) = \int_Z F(A^n z)\,\overline{G(z)}\,d\nu(z),$$

then $\big|C_n(F,G)\big| \leq c(F,G)\,2^{-n}$. Since the measure ν has a moment of order 2, one can consider its covariance operator S on H defined as

$$\langle Sx, y\rangle = \int_Z \langle x, z\rangle \overline{\langle y, z\rangle}\, d\nu(z)$$

for every $x, y \in H$. The operator S is self-adjoint, positive, and of trace class. Since ν has full support, S has dense range. Hence there exist an orthonormal basis $(e_l)_{l\geq 1}$ of H and a sequence $(\sigma_l^2)_{l\geq 1}$ of positive numbers with $\sum_{l\geq 1}\sigma_l^2 < +\infty$ such that $Se_l = 2\sigma_l^2 e_l$ for every $l \geq 1$. Also it follows from the orthogonality of the vectors e_l in H that the functions $\langle e_l, .\rangle$ are orthogonal in $L^2(H, \mathcal{B}_H, \nu)$. Let $\mathcal{E} = \overline{\text{span}}\,\big[\langle e_l, .\rangle;\ l \geq 1\big]$, where the closed linear span is taken in $L^2(H, \mathcal{B}_H, \nu)$: $\mathcal{E}$ is a closed subspace of $L^2(H, \mathcal{B}_H, \nu)$ which consists of all functions $F \in \mathcal{E}$ which can be written as a convergent series in $L^2(H, \mathcal{B}_H, \nu)$ of the form

$$F = \sum_{l\geq 1} a_l \langle e_l, .\rangle, \quad \text{where} \quad \sum_{l\geq 1} |a_l|^2\, \sigma_l^2 < +\infty.$$

Remark that the function $\langle x, .\rangle$ belongs to $\mathcal{E}$ for every $x \in H$. We denote by ι the injection operator $\iota : H \longrightarrow \mathcal{E}$ defined by $\iota(x) = \langle x, .\rangle$. If U_A denotes the Koopman operator associated to $(H, \mathcal{B}_H, \nu; A)$, then $U_A(\mathcal{E}) \subseteq \mathcal{E}$. Proceeding as in the proof of [3, Thm. 4.1], we apply the spectral decomposition theorem to U_A, which is an isometry on $L^2(H, \mathcal{B}_H, \nu)$: there exists a finite or countable family $(H_i)_{i\in I}$ of Hilbert spaces, with either $H_i = H^2(\mathbb{T})$ or $H_i = L^2(\mathbb{T}, \sigma_i)$ for some probability measure on $\mathbb{T}$, and an invertible isometry $J : \bigoplus_{i\in I} H_i \longrightarrow L^2(H, \mathcal{B}_H, \nu)$ such that $U_A J = JM$. Here M acts on $\bigoplus_{i\in I} H_i$ as $M = \bigoplus_{i\in I} M_i$, where M_i is the multiplication operator by λ on H_i: $(M_i f_i)(\lambda) = \lambda f_i(\lambda)$ for every $f_i \in H_i$.

Now let $K : \bigoplus_{i\in I} H_i \longrightarrow H$ be the operator defined as $K = \iota^* J$. For every $x \in H$, we have

$$\langle Sx, x\rangle = \int_H |\langle x, z\rangle|^2\, d\nu(z) = ||\iota(x)||^2 = ||K^* x||^2.$$

It follows that K^* is a Hilbert-Schmidt operator, and so there exists for every $i \in I$ a unimodular eigenvectorfield $E_i \in L^2(\mathbb{T}, \sigma_i; H)$ such that $K^* x = \bigoplus_{i\in I} \langle x, E_i(\,.\,)\rangle$ (see [14], [3] or [4] for more details). We thus have for every $x, y \in H$,

$$\langle Sx, y\rangle = \langle K^* x, K^* y\rangle = \sum_{i\in I} \int_{\mathbb{T}} \langle x, E_i(\lambda)\rangle\, \overline{\langle y, E_i(\lambda)\rangle}\, d\sigma_i(\lambda).$$

Now let $G \in L^2_0(H, \mathcal{B}_H, \nu)$ be a function of the form $G = g \circ \Phi^{-1}$ with $g \in L^2_0\big([0,1]\big) \cap \mathcal{C}^1\big([0,1]\big)$, which is such that its orthogonal projection on $\mathcal{E}$, which we denote by F, is non-zero. Such a function G does exist because $\big\{g \circ \Phi^{-1};\ g \in L^2_0\big([0,1]\big) \cap \mathcal{C}^1\big([0,1]\big)\big\}$ is dense in $L^2(H, \mathcal{B}_H, \nu)$. For every $n \geq 0$ we have

$$\mathcal{C}_n(F,G) = \int_Z F(A^n z)\, \overline{G(z)}\, d\nu(z) = \langle U_A^n F, G\rangle = \langle U_A^n F, F\rangle + \langle U_A^n F, G - F\rangle.$$

Since $U_A(\mathcal{E}) \subseteq \mathcal{E}$ and $G - F$ is orthogonal to $\mathcal{E}$, the second term vanishes and

$$\mathcal{C}_n(F,G) = \mathcal{C}_n(F,F) = \int_Z F(A^n z)\, \overline{F(z)}\, d\nu(z) \quad \text{for every } n \geq 0.$$

Hence there exists a positive constant C such that for every $n \geq 0$, $|\mathcal{C}_n(F,F)| \leq C.2^{-n}$. Writing F as $F = \sum_{l\geq 1} a_l \langle e_l, .\rangle$, where $\sum_{l\geq 1} |a_l|\, \sigma_l^2 < +\infty$ and at least

one of the coefficients a_l is non-zero, we can write $\mathcal{C}_n(F,F)$ as

$$\begin{aligned}\mathcal{C}_n(F,F) &= \sum_{k,l\geq 1} a_k\overline{a_l}\,\langle SA^{*n}\,e_k, e_l\rangle \\ &= \sum_{k,l\geq 1} a_k\overline{a_l}\sum_{i\in I}\int_{\mathbb{T}}\langle A^{*n}e_k, E_i(\lambda)\rangle\,\overline{\langle e_l, E_i(\lambda)\rangle}\,d\sigma_i(\lambda) \\ &= \int_{\mathbb{T}}\lambda^n\sum_{i\in I}\Big(\sum_{k,l\geq 1} a_k\overline{a_l}\,\langle e_k, E_i(\lambda)\rangle\,\overline{\langle e_l, E_i(\lambda)\rangle}\Big)\,d\sigma_i(\lambda) \\ &= \int_{\mathbb{T}}\lambda^n\sum_{i\in I}\Big|\sum_{k\geq 1} a_k\,\langle e_k, E_i(\lambda)\rangle\Big|^2\,d\sigma_i(\lambda).\end{aligned}$$

All these computations make sense because

$$\int_{\mathbb{T}}\sum_{i\in I}\Big|\lambda^n\sum_{k\geq 1} a_k\,\langle e_k, E_i(\lambda)\rangle\Big|^2\,d\sigma_i(\lambda) = \int_Z |F(z)|^2 d\nu(z) < +\infty.$$

Let us denote by σ the positive finite measure

$$\sigma = \sum_{i\in I}\Big|\sum_{k\geq 1} a_k\,\langle e_k, E_i(\,.\,)\rangle\Big|^2\sigma_i.$$

Then $|\widehat{\sigma}(n)| \leq C.2^{-n}$ for every $n\geq 0$, and since σ is a positive measure, $\widehat{\sigma}(-n) = \overline{\widehat{\sigma}(n)}$ for every $n\geq 0$, so that $|\widehat{\sigma}(n)| \leq C.2^{-|n|}$ for every $n\in\mathbb{Z}$. Hence there exists a function φ which is analytic in $\mathbb{A}_{1/2} = \big\{\lambda\in\mathbb{C};\ 1/2<|\lambda|<2\big\}$ such that $d\sigma = \varphi\,d\lambda$, where $d\lambda$ is the normalized Lebesgue measure on $\mathbb{T}$. Since $\mathcal{C}_0(F,F) = \widehat{\sigma}(0) > 0$ (recall that the function F is non-zero), the function φ cannot be identically zero. If E is a subset of $\mathbb{T}$ of positive Lebesgue measure, it is thus impossible that $E_i(\lambda) = 0$ for every $\lambda\in E$ and every $i\in I$. So the unimodular point spectrum of A has full Lebesgue measure in $\mathbb{T}$. This proves Theorem 1.9. □

The proof of Theorem 1.9 does not extend to operators which are 2-universal for invertible ergodic systems: the proof uses in a crucial way that the correlations $\mathcal{C}_n(f,g)$ of the system $x\mapsto 2x \mod 1$ on $[0,1]$ decay exponentially fast for *all* $f\in L^2\big([0,1]\big)$ and sufficiently smooth $g\in L^2\big([0,1]\big)$. This system is not invertible, and it seems to be in the nature of things that for an invertible system, the correlations decay exponentially fast only for sufficiently smooth functions f *and* g (see [1] for more on these questions). So the following question remains open:

Question 5.3. If A is a 2-universal operator for invertible ergodic systems on a Hilbert space, is it true that the unimodular point spectrum of A has full Lebesgue measure?

We do not know any example of a 2-universal operator whose unimodular point spectrum is not the whole unit circle.

Question 5.4. If A is a 2-universal operator for (invertible) ergodic systems on a Hilbert space, is it true that the unimodular point spectrum of A is equal to $\mathbb{T}$?

If an affirmative answer to this question could be obtained, it would be a first step towards a characterization of symbols $\varphi\in H^\infty(\mathbb{D})$ such that M_φ^* acting on $H^2(\mathbb{D})$ is 2-universal for invertible ergodic systems.

Question 5.5. Let $\varphi \in H^\infty(\mathbb{D})$. Is it true that $M_\varphi^* \in \mathcal{B}(H^2(\mathbb{D}))$ is 2-universal for invertible ergodic systems if and only if $\mathbb{T} \subseteq \varphi(\mathbb{D})$?

It would also be interesting to obtain a characterization of adjoints of multipliers on $H^2(\mathbb{D})$ which are universal for ergodic systems.

Question 5.6. Let $\varphi \in H^\infty(\mathbb{D})$. If $\overline{\mathbb{D}} \subseteq \varphi(\mathbb{D})$ (where $\overline{\mathbb{D}}$ denotes the closure of the unit disk $\mathbb{D}$), is M_φ^* universal for ergodic systems? Is it true that M_φ^* is 2-universal for ergodic systems if and only if $\overline{\mathbb{D}} \subseteq \varphi(\mathbb{D})$?

Of course things would be simpler if we knew that a universal operator is necessarily 2-universal, but this does not seem easy to prove. It is not even known whether a frequently hypercyclic operator on a reflexive space admits an invariant measure with full support having a moment of order 2, although it is known that it admits invariant measures with full support (see [9]).

Question 5.7. Does there exist a universal operator which is not 2-universal?

This brings us back to questions about the existence of unimodular eigenvectors for universal operators.

Question 5.8. Does there exist universal (or 2-universal) operators admitting no unimodular eigenvalue? What about universal operators on a Hilbert space?

The second half of this question seems hard, again because we do not know whether a frequently hypercyclic operator on a Hilbert space necessarily has some unimodular eigenvalue. The first half of Question 5.8 may be more tractable, and a potential example would be the Kalish-type operator A of Example 4.4 acting on the space $\mathcal{C}_0([0, 2\pi])$ of continuous functions on $[0, 2\pi]$ vanishing at 0. It is proved in [3] that although this operator has no unimodular eigenvalue, it admits a Gaussian invariant measure with full support with respect to which it is strongly mixing. We have seen that A acting on $L^2(\mathbb{T})$ is universal for invertible ergodic systems, so one may naturally wonder about the following:

Question 5.9. Let A be the bounded operator on $\mathcal{C}_0([0, 2\pi])$ defined by setting, for every $f \in \mathcal{C}_0([0, 2\pi])$ and every $\theta \in [0, 2\pi]$,

$$Af(\theta) = e^{i\theta} f(\theta) - \int_0^\theta ie^{it} f(e^{it})\, dt.$$

Is A a universal (or 2-universal) operator for invertible ergodic systems on $\mathcal{C}_0([0, 2\pi])$?

A positive answer to Question 5.9 cannot be obtained via an application of Theorem 1.3, or any variant of it, since we have seen that assumption (c) of Theorem 1.3 for instance implies that A admits a continuous unimodular eigenvectorfield E which is such that $\overline{\operatorname{span}}\big[E(\lambda);\ \lambda \in \mathbb{T}\big] = Z$. So we finish the paper with this last question:

Question 5.10. Does there exist any universal operator for invertible ergodic systems which does not satisfy the assumptions of Theorem 1.3?

Acknowledgement

The author is grateful to the referee for a careful reading of the manuscript and for suggestions which enabled the author to clarify some points in the presentation of the text and to simplify some arguments.

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CNRS, Laboratoire Amiénois de Mathématique Fondamentale et Appliquée, UMR 7352, Université de Picardie Jules Verne, 33 rue Saint-Leu, 80039 Amiens Cedex 1, France

E-mail address: sophie.grivaux@u-picardie.fr