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VANISHING AND INJECTIVITY THEOREMS FOR HODGE MODULES

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ABSTRACT. We prove a surjectivity theorem for the Deligne canonical extension of a polarizable variation of Hodge structure with quasi-unipotent monodromy at infinity along the lines of Esnault-Viehweg. We deduce from it several injectivity theorems and vanishing theorems for pure Hodge modules. We also give an inductive proof of Kawamata-Viehweg vanishing for the lowest graded piece of the Hodge filtration of a pure Hodge module using mixed Hodge modules of nearby cycles.

1. INTRODUCTION

When X is a smooth projective variety, the famous Kodaira-Nakano vanishing theorem says that sufficiently high cohomology groups vanish when Ω_X^p is twisted by any ample line bundle. Saito proved a more general vanishing theorem [12] using his theory of Hodge modules.

Theorem 1.1 (Kodaira-Saito vanishing theorem). *Let X be a complex projective variety with an ample line bundle L, and M a mixed Hodge module on X. Then*

$$H^i(X, \mathrm{Gr}_k^F \mathrm{DR}(\mathrm{M}) \otimes L) = 0, \ i > 0.$$

A detailed discussion of the proof of this theorem can be found in [10]; another proof following the approach of Esnault-Viehweg can be found in [15]. If X is smooth and of dimension n, taking $M = \mathbb{Q}_X^H[n] := (\omega_X, F_\bullet, \mathbb{Q}_X[n])$, the pure Hodge module corresponding to the trivial variation of Hodge structure on X, we have $\mathrm{Gr}_{-p}^F \mathrm{DR}(M) = \Omega_X^p[n-p]$, and so Saito's result implies Kodaira-Nakano vanishing. For arbitrary M, let $S(M)$ be the lowest graded piece of the Hodge filtration. In particular,

$$H^i(X, S(M) \otimes L) = 0, \ i > 0,$$

which specializes to Kodaira vanishing as well.

On the other hand, Kodaira vanishing can be generalized by replacing ample divisors by nef and big divisors (or even $\mathbb{Q}$-divisors). This generalization is the so-called Kawamata-Viehweg vanishing theorem. In [10], M. Popa proved a version of Kawamata-Viehweg vanishing for some special Hodge modules, and he suggested that a better result would be true. In this paper we remove the extra hypothesis in [10] and prove a Kawamata-Viehweg type statement for pure Hodge modules in full generality.

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Theorem 1.2. *Let X be a complex projective variety, L a nef and big line bundle, and M a polarizable pure Hodge module with strict support X. Then*

$$H^i(X, S(M) \otimes L) = 0,\ i > 0.$$

Note that this specializes to the Kawamata-Viehweg vanishing theorem [9, Theorem 4.3.1] when $M = \mathbb{Q}^H_X[n]$, the trivial Hodge module on smooth X, but also to Kollár's vanishing theorem for nef and big line bundles when M is the j-th cohomology of the push-forward of the trivial Hodge module via a surjective morphism of projective varieties $f : Y \longrightarrow X$ with Y smooth; cf. [10, Section 9]. Notice that the condition of strict support is needed to avoid null locus of L.

This result can be approached in two different ways. We first provide an inductive approach, a strategy similar to Kawamata's original method (see also [9, §4]), based in this setting on an adjunction type formula that involves the nearby cycle functor and mixed Hodge modules. This is presented in Section 5.

At the same time as this proof was completed, J. Suh proved a Nakano type vanishing for the Deligne canonical extension of a polarizable variation of Hodge structure in [17], which in particular implies the same Kawamata-Viehweg type result. (Note that Suh also proves other types of vanishing statements as well, that apply to all graded quotients in the Hodge filtration of the de Rham complex.) His idea is based on the Esnault-Viehweg [3] approach to vanishing theorems. Following J. Suh's proof and C. Schnell's Esnault-Viehweg type proof of Theorem 1.1 in [15], we extend this to a version of the injectivity theorem of Kollár and Esnault-Viehweg for the Deligne canonical extensions of certain polarizable variations of Hodge structures. This is presented as a surjectivity statement below and proved in Section 6.

Theorem 1.3. *Let X be a smooth projective variety with a line bundle L, D a reduced simple normal crossings divisor, and $\mathbb{V} = (\mathcal{V}, F^\bullet, \mathbb{V}_{\mathbb{Q}})$ a variation of Hodge structure defined on $U = X \setminus D$ with quasi-unipotent local monodromies. Assume*

$$L^N = \mathcal{O}_X(D')$$

for some $N \gg 0$ and an effective divisor D' supported on D. If E is an effective divisor supported on $\mathrm{Supp}(D')$*, then for all i, the natural map induced by E,*

$$H^i(X, \mathrm{Gr}_F^{\mathrm{first}}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}) \otimes L^{-1}(-E)) \longrightarrow H^i(X, \mathrm{Gr}_F^{\mathrm{first}}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}) \otimes L^{-1})$$

is surjective, where $\tilde{\mathcal{V}}$ is the Deligne canonical extension of $\mathcal{V}$, and $\mathrm{Gr}_F^{\mathrm{first}}\mathrm{DR}_{(X,D)}$ is the first non-zero graded piece of the logarithmic de Rham complex for $\tilde{\mathcal{V}}$ (see Section 4).

An injectivity theorem for pure Hodge modules with strict support X follows from it.

Theorem 1.4. *Let X be a complex projective variety, E an effective divisor, and M a polarizable pure Hodge module with strict support X. If a line bundle L is either nef and big, or, semi-ample and satisfying $H^0(X, L^m(-E)) \neq 0$ for some $m > 0$, then the natural map*

$$H^i(X, S(M) \otimes L) \longrightarrow H^i(X, S(M) \otimes L(E))$$

is injective for all i.

Replacing E by a sufficiently large multiple of an ample divisor in Theorem 1.4 and applying Serre vanishing provide the second proof of Theorem 1.2. See also Corollary 7.4. If L is perturbed by a sufficiently small effective $\mathbb{Q}$-Cartier divisor, a version of this injectivity theorem still holds (Theorem 7.5), extending Kollár's $\mathbb{Q}$-version of the injectivity theorem for ω_X [8, Theorem 10.13]. We also obtain a version of Kawamata-Viehweg vanishing for $\mathbb{Q}$-divisors which contains the original version for ω_X. See Remark 7.7 for an explanation for both points.

It is worth mentioning that, just like in Theorem 1.1, the statement given here works on arbitrary (not necessarily smooth) projective varieties X, due to the theory of Hodge modules on singular spaces; see Section 3.

In Sections 2, 3 and 4, we briefly review $\mathscr{D}$-modules and Hodge modules. Some necessary theorems are presented for later use. In Section 5, we present the inductive proof of the Kawamata-Viehweg vanishing for pure Hodge modules using nearby cycles. Section 6 is dedicated to the proof of the injectivity theorem in the normal crossings case. Section 7 deals with applications.

2. $\mathscr{D}$-modules and the de Rham complex

In this section, we will recall some terminologies of $\mathscr{D}$-modules which are essentially used in the theory of Hodge modules.

2.1. **The side-change operator of $\mathscr{D}$-modules.** Let X be a complex manifold and let $\mathscr{D}_X$ be the sheaf of differential operators. It is well known that the category of left $\mathscr{D}$-modules and the category of right $\mathscr{D}$-modules are equivalent via the so-called side-change operation,

$$\mathcal{N} \longrightarrow \mathcal{M} = \mathcal{N} \otimes \omega_X,$$

and its quasi-inverse

$$\mathcal{M} \longrightarrow \mathcal{N} = \mathcal{H}om_{\mathscr{O}_X}(\omega_X, \mathcal{M}),$$

where ω_X is the canonical sheaf. When $\mathscr{D}$-modules are filtered, the correspondence of filtrations under the equivalence is

$$F_p(\mathcal{M}) = F_{p+n}(\mathcal{N}) \otimes \omega_X, \text{ and } F_p(\mathcal{N}) = F_{p-n}(\mathcal{M}) \otimes \omega_X^{-1},$$

where $n = \dim X$. In this paper, we use conventions: increasing filtrations (decreasing filtrations respectively) are denoted by $F_\bullet$ ($F^\bullet$ respectively), and the associated graded objects by Gr^F (Gr_F respectively).

2.2. **The de Rham functor.** The de Rham functor is defined to be

$$\mathrm{DR}(\mathcal{N}) := [\mathcal{N} \longrightarrow \Omega^1_X \otimes \mathcal{N} \longrightarrow \cdots \longrightarrow \Omega^n_X \otimes \mathcal{N}][n],$$

for left $\mathscr{D}$-modules, where Ω^k_X is the sheaf of holomorphic k-forms on X. Also

$$\mathrm{DR}(\mathcal{M}) := [\bigwedge^n \mathscr{T}_X \otimes \mathcal{M} \longrightarrow \bigwedge^{n-1} \mathscr{T}_X \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \mathcal{M}][n],$$

for right $\mathscr{D}$-modules, where $\mathscr{T}_X$ is the sheaf of vector fields on X. Both of the above complexes are concentrated in degree $-n, \ldots, -1, 0$. $\mathrm{DR}(\mathcal{N})$ ($\mathrm{DR}(\mathcal{M})$ respectively) is called the de Rham complex of $\mathcal{N}$ ($\mathcal{M}$ respectively). Clearly the de Rham functor is compatible with the side-change operation, i.e., $\mathrm{DR}(\mathcal{N})$ is canonically isomorphic to $\mathrm{DR}(\mathcal{M})$ provided $\mathcal{M} = \mathcal{N} \otimes \omega_X$, because of the canonical isomorphism

$$\bigwedge^{n-k} \mathscr{T}_X \otimes \omega_X \simeq \Omega^k.$$

If the right $\mathscr{D}$-module $\mathcal{M}$ is filtered, then $\mathrm{DR}(\mathcal{M})$ is filtered naturally by

$$F_p\mathrm{DR}(\mathcal{M}) := [\bigwedge^n \mathscr{T}_X \otimes F_{p-n}\mathcal{M} \longrightarrow \bigwedge^{n-1} \mathscr{T}_X \otimes F_{p-n+1}\mathcal{M} \longrightarrow \cdots \longrightarrow F_p\mathcal{M}][n].$$

Similarly, for filtered left $\mathscr{D}$-module $\mathcal{N}$,

$$F_p\mathrm{DR}(\mathcal{N}) := [F_p\mathcal{N} \longrightarrow \Omega^1_X \otimes F_{p+1}\mathcal{N} \longrightarrow \cdots \longrightarrow \Omega^n_X \otimes F_{p+n}\mathcal{N}][n].$$

Obviously, the filtered de Rham functor is also compatible with the side-change operation. Hence, it is not necessary to distinguish the right version and the left version of the (filtered) de Rham functor and (filtered) de Rham complexes. The associated graded complexes for the filtration above are

$$\mathrm{Gr}^F_p\mathrm{DR}(\mathcal{M}) := [\bigwedge^n \mathscr{T}_X \otimes \mathrm{Gr}^F_{p-n}\mathcal{M} \longrightarrow \bigwedge^{n-1} \mathscr{T}_X \otimes \mathrm{Gr}^F_{p-n+1}\mathcal{M} \longrightarrow \cdots \longrightarrow \mathrm{Gr}^F_p\mathcal{M}][n],$$

which are complexes of $\mathscr{O}_X$-modules.

3. Hodge modules

This section will be devoted to briefly recalling Morihiko Saito's theory of Hodge modules. We will only mention basic information of Hodge modules and important theorems that will be needed later on. The main two references are Saito's original papers [11] and [12]. Another useful reference is the recent survey [16].

Varieties are assumed to be irreducible throughout this paper.

3.1. **Pure Hodge modules.** Let X be a smooth complex variety or complex manifold. A variation of Hodge structure $\mathbb{V}$ of weight l is

$$\mathbb{V} = (\mathcal{V}, F^\bullet, \mathbb{V}_\mathbb{Q}),$$

consisting of

- a $\mathbb{Q}$-local system $\mathbb{V}_\mathbb{Q}$;
- a finite decreasing filtration $F^\bullet$ of the vector bundle $\mathcal{V} = \mathbb{V}_\mathbb{Q} \otimes \mathscr{O}_X$ by subbundles, satisfying
- $\forall x \in X$, $\mathbb{V}_x = (\mathcal{V}_x, F^\bullet_x, \mathbb{V}_{\mathbb{Q},x})$ is a Hodge structure of weight l;
- the filtration $F^\bullet$ satisfies the Griffiths transversality condition, namely for the induced connection ∇,
$$\nabla(F^p) \subset \Omega^1 \otimes F^{p-1}.$$

Additionally, a polarization of a variation Hodge structure $\mathbb{V}$ of weight l is a morphism

$$Q : \mathbb{V}_\mathbb{Q} \otimes \mathbb{V}_\mathbb{Q} \longrightarrow \mathbb{Q}(-l),$$

such that Q induces a polarization of $\mathbb{V}_x$ for every $x \in X$, where $\mathbb{Q}(-l) = (2\pi i)^{-l}\mathbb{Q}$.

Note. If we let $F_p\mathcal{V} = F^{-p}\mathcal{V}$, then the last requirement means exactly that the new filtration $F_\bullet$ is good for left $\mathscr{D}$-module $\mathcal{V}$. From now on, $F_\bullet\mathcal{V}$ of a variation of Hodge structure always means this induced increasing filtration.

Saito's theory generalizes variations of Hodge structure by allowing filtered holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structure instead. Here a filtered holonomic $\mathscr{D}$-module with $\mathbb{Q}$-structure is a triple

$$M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$$

where $(\mathcal{M}, F_\bullet\mathcal{M})$ is a holonomic right $\mathscr{D}$-module with a good filtration, and K is a $\mathbb{Q}$-perverse sheaf, satisfying

$$\mathrm{DR}(\mathcal{M}) \simeq \mathbb{C} \otimes_{\mathbb{Q}} K,$$

by Riemann-Hilbert correspondence. From now on, all $\mathscr{D}$-modules will mean right $\mathscr{D}$-modules unless stated explicitly. We will also use the expression "a filtered left $\mathscr{D}$-module underlying a Hodge module (pure or mixed)" if it is so after side-change.

Saito constructed an abelian category $\mathrm{HM}(X, l)^p$ of polarizable pure Hodge modules of weight l for smooth complex variety X (or complex manifold) in [11] which is a semi-simple full subcategory of the category of filtered holonomic $\mathscr{D}$-modules with $\mathbb{Q}$-structures, i.e.,

$$\mathrm{HM}(X, l)^p = \bigoplus_{Z \subseteq X} \mathrm{HM}^Z(X, l)^p,$$

a direct sum over all subvarieties of X. Here $\mathrm{HM}^Z(X, l)^p$ is by definition of objects strictly supported on Z. Such objects correspond one-to-one to variations of Hodge structures generically defined on the smooth locus of Z. See [12, Theorem 3.21].

If Y is a subvariety of X, Hodge modules on Y can be defined as

$$\mathrm{HM}(Y, l)^p := \mathrm{HM}_Y(X, l)^p,$$

where the right-hand side is the category of polarizable pure Hodge modules of weight l on X supported on Y. In fact, this definition is intrinsic. See [16, §14] for details.

3.2. **Mixed Hodge modules.** Denote the category of graded polarizable mixed Hodge modules by $\mathrm{MHM}(X)^p$. Mixed Hodge modules have many good properties, one of which is that $\mathrm{MHM}(X)^p$ is stable by j_*j^{-1} (see [12, Proposition 2.11]), where j is the open embedding

$$j : X \setminus D \longrightarrow X$$

for some effective divisor D. This is called localizations of mixed Hodge modules in [10]. At the level of perverse sheaves, this functor is precisely

$$Rj_*j^{-1}(K).$$

If $M \in \mathrm{MHM}(X)^p$, it is hard to compute $j_*j^{-1}M$ explicitly in general, even for $j_*j^{-1}\mathbb{Q}^H_X[n]$. But in some special situations, for instance D is a smooth irreducible divisor, $j_*j^{-1}\mathbb{Q}^H_X[n]$ is understood very well ([10, §5]). We will calculate some special examples of mixed Hodge modules of this type in the next section.

3.3. **Direct image functor and the stability theorem.** Let $f : X \longrightarrow Y$ be a projective morphism of smooth complex varieties (or complex manifolds), and let M be a Hodge module (pure or mixed) on X. The direct image of M is

$$f_*M := (f_+(\mathcal{M}, F_\bullet\mathcal{M}), Rf_*K),$$

i.e., combining the direct image of the filtered $\mathscr{D}$-module and the direct image of the perverse sheaf. Detailed discussion about direct image functors can be found in [16, §26 and §28]. One of the main results of M. Saito's theory is the following theorem about the direct image functor.

Theorem 3.1 (Stability theorem and decomposition theorem [11])**.** *Let $f : X \longrightarrow Y$ be a projective morphism between complex manifolds, and let*

$$M = (\mathcal{M}, F_\bullet\mathcal{M}, K) \in \mathrm{HM}^Z(X, l)^p,$$

for some subvariety Z of X. Then,

(1) *$f_+(\mathcal{M}, F_\bullet\mathcal{M})$ is strict, and $\mathcal{H}^i f_* M \in \mathrm{HM}(Y, n+i)^p$;*
(2) *$f_* M \simeq \bigoplus \mathcal{H}^i f_* M[-i]$ (non-canonical).*

Remark 3.2. When $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a graded-polarizable weakly mixed Hodge module M, the stability theorem is still true (see [12, Theorem 2.14]). But since $f_* M$ may not be of pure weights, the decomposition theorem doesn't hold for weakly mixed Hodge modules or even mixed Hodge modules.

The strictness of the direct image functor implies the following easy but useful corollary:

Corollary 3.3 ([11, §2.3])**.** *Let $f : X \longrightarrow \bullet$ be the constant morphism from a smooth projective variety, with $(\mathcal{M}, F_\bullet\mathcal{M})$ a filtered holonomic $\mathscr{D}$-module underlying some graded-polarizable weakly mixed Hodge module. Then the Hodge-de Rham spectral sequence*

$$E_1^{p,q} = H^{p+q}(X, \mathrm{Gr}^F_{-p}\mathrm{DR}(\mathcal{M})) \implies H^{p+q}(X, \mathrm{DR}(\mathcal{M}))$$

degenerates at E_1.

Definition 3.4. Suppose $M = (\mathcal{M}, F_\bullet\mathcal{M}, K)$ is a pure Hodge module on a variety X. Set

$$p(M) := \min\{p \mid F_p\mathcal{M} \neq 0\},$$

and

$$S(M) := F_{p(M)}\mathcal{M} = \mathrm{Gr}^F_{p(M)}\mathrm{DR}(\mathcal{M}).$$

$S(M)$ will play a similar role as the canonical sheaf does in vanishing theorems. From the decomposition theorem, Saito also proved,

Proposition 3.5 ([11, §2.3.7])**.** *Under the setting of Theorem* 3.1*,*

$$Rf_*\mathrm{Gr}^F_p\mathrm{DR}(\mathcal{M}) \simeq \bigoplus_i \mathrm{Gr}^F_p\mathrm{DR}(\mathcal{M}_i)[-i].$$

In particular,

$$Rf_* S(M) \simeq \bigoplus_i F_{p(M)}\mathcal{M}_i[-i],$$

where $(\mathcal{M}_i, F_\bullet\mathcal{M}_i) := \mathcal{H}^i f_+(\mathcal{M}, F_\bullet\mathcal{M})$.

3.4. **V-filtrations and the torsion-freeness of $S(M)$.** The V-filtration is fundamentally important in Saito's theory of Hodge modules. We use it in this section to recall the torsion-freeness of $S(M)$ due to Saito.

Let X be a smooth complex variety or complex manifold of dimension n and let X_0 be a smooth divisor of X with local defining equation t. $\mathscr{D}_X$ is filtered by

$$V_i\mathscr{D}_X = \{P \in \mathscr{D}_X | P \cdot \mathcal{I}^j_{X_0} \subseteq \mathcal{I}^{j-i}_{X_0}\},$$

where $\mathcal{I}_{X_0}$ is the ideal sheaf of X_0 with the convention that $\mathcal{I}^j_{X_0} = \mathscr{O}_X$ for $j \leq 0$.

Definition 3.6 (V-filtration)**.** A V-filtration of a coherent $\mathscr{D}$-module $\mathcal{M}$ along X_0 is an increasing filtration $V_\alpha\mathcal{M}$ indexed by rational number α discretely, so that:

- The filtration is exhaustive, and each $V_\alpha\mathcal{M}$ is a coherent $V_0\mathscr{D}_X$-module.
- $V_\alpha\mathcal{M}\cdot V_i\mathscr{D}_X \subseteq V_{\alpha+i}\mathcal{M}$ for every $\alpha\in\mathbb{Q}$ and $i\in\mathbb{Z}$; and
$$V_\alpha\mathcal{M}\cdot t = V_{\alpha-1}\mathcal{M},\ \text{for } \alpha<0.$$
- The action of $t\partial_t-\alpha$ on $Gr_\alpha^V\mathcal{M}$ is nilpotent for any α, where $Gr_\alpha^V\mathcal{M} := V_\alpha\mathcal{M}/V_{<\alpha}\mathcal{M}$ and $V_{<\alpha}\mathcal{M}=\bigcup_{\beta<\alpha}V_\beta\mathcal{M}$.

In fact, the morphism of the action t in the second requirement of the above definition is an isomorphism ([11, Lemme 3.1.4]), i.e.,
$$t: V_\alpha\mathcal{M}\xrightarrow{\simeq} V_{\alpha-1}\mathcal{M}$$
is an isomorphism for all $\alpha<0$.

Proposition 3.7. *Let X be a complex variety, and M a pure Hodge module strictly supported on X. Then $S(M)$ is torsion-free.*

Proof. Since torsion-freeness is local, we can assume $X\hookrightarrow Y$ embedded into a complex manifold Y, and the underlying filtered $\mathscr{D}$-module is $(\mathcal{M},F_\bullet\mathcal{M})$ on Y. Suppose $\mathcal{T}$ is the torsion submodule of $S(M)$, and $\mathcal{T}$ is annihilated by $\tilde{f}$ which can be lifted to a holomorphic function $f: Y\longrightarrow\mathbb{C}$ on Y. From [11, §3.1]
$$S(M)=V_{<0}\mathcal{M}_f\cap j_*j^*F_{p(M)}\mathcal{M}_f.$$
And the action of $\tilde{f}$ on $S(M)$ is exactly the action of t on $F_{p(M)}\mathcal{M}_f$. But since t acts on V_α injectively for $\alpha<0$, $\mathcal{T}$ must be 0. $\square$

Because of the compatibility of the V-filtration and the Hodge filtration, Saito also proved the following theorem about $p(M)$.

Proposition 3.8 ([13, Proposition 2.6])**.** *Suppose $f: X\longrightarrow Y$ is a projective morphism of complex manifolds with $M\in HM^Z(X,l)^p$ for some subvariety Z. If M' is a direct summand of $\mathcal{H}^jf_*M$ and its strict support $Z'\neq f(Z)$, then*
$$p(M')>p(M).$$

Combining Proposition 3.5, Proposition 3.7 and Proposition 3.8, one obtains

Corollary 3.9. *Let $f: X\longrightarrow Y$ be a surjective projective morphism of complex varieties, and let M be a polarizable pure Hodge module strictly supported on X. Then $R^if_*S(M)$ is torsion-free for all i.*

This corollary was first conjectured by Kollár as a generalization of his theorem about higher direct images of dualizing sheaves [7], and was proved by M. Saito in [13].

Corollary 3.10. *Assume $f: X\longrightarrow Y$ is a birational morphism of complex projective varieties, and $M\in HM^X(X,l)^p$. Then*
$$R^if_*S(M)=\begin{cases} S(M'), & i=0,\\ 0, & i<0,\end{cases}$$
where M' is the direct summand of $\mathcal{H}^0f_*M$ strictly supported on Y.

Proof. The fact that $R^0f_*S(M)=S(M')$ follows from Proposition 3.8. Now $R^if_*S(M)=0$ for $i>0$ because $R^if_*S(M)$ is both torsion and torsion-free. $\square$

4. Hodge modules and the Deligne extension

Let X be a complex smooth projective variety, and let $D = \sum_{i=1}^{r} D_i$ be a reduced simple normal crossings divisor with irreducible components D_i. Set $U = X \setminus D$, and

$$j : U \hookrightarrow X,$$

the open embedding.

Assume that $\mathbb{V} = (\mathcal{V}, F^\bullet\mathcal{V}, \mathbb{V}_\mathbb{Q})$ is a polarizable variation of Hodge structure on U. Then it is well known that $\mathcal{V}$ extends to vector bundles with flat logarithmic connections along D. This extension is unique if the eigenvalues of the residue along each D_i are required to be in a fixed strip of $\mathbb{C}$ of length 1 ([5, Theorem 5.2.17]). Since all the local monodromies are assumed to be quasi-unipotent, the eigenvalues are in fact rational numbers. The Deligne canonical extension is the extension with eigenvalues in $[0, 1)$, denoted by $\tilde{\mathcal{V}}$. $\tilde{\mathcal{V}}$ is filtered by

$$F_p\tilde{\mathcal{V}} := \tilde{\mathcal{V}} \cap j_*(F_p\mathcal{V}), \tag{4.1}$$

where $F_p\mathcal{V} = F^{-p}\mathcal{V}$. Because of Schmid's theorem on nilpotent orbits, filtrations of all Deligne extensions are locally free (see [12, §3.b]). Hence we have the filtered logarithmic de Rham complex for $\tilde{\mathcal{V}}$, similar to the filtered de Rham complex,

$$\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}, F_\bullet) := [\tilde{\mathcal{V}} \longrightarrow \Omega^1(\log D) \otimes \tilde{\mathcal{V}} \longrightarrow \cdots \longrightarrow \Omega^n(\log D) \otimes \tilde{\mathcal{V}}][n],$$

with filtration

$$\begin{aligned} &F_p\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}, F_\bullet) \\ &\quad := [F_p\tilde{\mathcal{V}} \longrightarrow \Omega^1(\log D) \otimes F_{p+1}\tilde{\mathcal{V}} \longrightarrow \cdots \longrightarrow \Omega^n(\log D) \otimes F_{p+n}\tilde{\mathcal{V}}][n], \end{aligned}$$

where $n = \dim X$.

Then $\tilde{\mathcal{V}}(*D) = \tilde{\mathcal{V}} \otimes \mathscr{O}_X(*D)$ is the regular meromorphic connection extending $\mathcal{V}$, which is a left regular holonomic $\mathscr{D}$-module a priori. According to [12, §3.b], $\tilde{\mathcal{V}}(*D)$ can be filtered by

$$F_p\tilde{\mathcal{V}}(*D) = \sum F_{p-i}\mathscr{D}_X \cdot F_i\tilde{\mathcal{V}}_1,$$

where $\tilde{\mathcal{V}}_1$ is another extension of $\mathcal{V}$ with eigenvalues in $[-1, 0)$.

One denotes the pure Hodge module of strict support X corresponding to $\mathbb{V}$ by M. Since $\mathbb{V}$ is defined outside of a normal crossings divisor, M can be described explicitly in terms of a Deligne extension (see [12, Theorem 3.20]). In particular,

$$S(M) = \omega_X \otimes F_{p(M)+n}\tilde{\mathcal{V}}_2,$$

where $\tilde{\mathcal{V}}_2$ is another Deligne extension with eigenvalues in $(-1, 0]$. Hence $S(M)$ is locally free. Saito proved that the filtered left $\mathscr{D}$-module $(\tilde{\mathcal{V}}(*D), F_\bullet\tilde{\mathcal{V}}(*D))$ underlies $j_*j^{-1}M$. This point is clear at least for $\tilde{\mathcal{V}}(*D)$ without filtration. It is the underlying left $\mathscr{D}$-module of $j_*j^{-1}M$ because $\mathrm{DR}(\tilde{\mathcal{V}}(*D)) \simeq Rj_*\mathbb{V}_\mathbb{C}$ (see [5, Theorem 5.2.24]). Furthermore, by [12, Proposition 3.11 (3.11.4)], there is a filtered quasi-isomorphism

$$\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}, F_\bullet\tilde{\mathcal{V}}) \simeq \mathrm{DR}(\tilde{\mathcal{V}}(*D), F_\bullet\tilde{\mathcal{V}}(*D)). \tag{4.2}$$

This filtered quasi-isomorphism will be used repeatedly.

Since $\mathbb{V}$ is finitely filtered by $F_\bullet\mathcal{V}$, it is also useful to define (following [13])

$$q(M) = q(\mathbb{V}) := \max\{p \mid \mathrm{Gr}^F_p(\mathbb{V}) \neq 0\}.$$

For any pure Hodge module N strictly supported on X, besides $S(N)$ we define (following [13])

$$Q_X(N) := \mathbb{D}(S_X(N^*)) = R\mathcal{H}om_{\mathscr{O}_X}(S_X(N^*), \omega_X[n]), \tag{4.3}$$

where N^* is the pure Hodge module extension of the dual of the variation of Hodge structure defined on a Zariski open dense subset, and $\mathbb{D}$ is the Grothendieck duality functor. When X is a singular projective variety, $Q_X(N)$ is defined to be

$$Q_X(N) = R\mathcal{H}om_{\mathscr{O}_Y}(S_X(N^*), \omega_Y[\mathrm{dim}Y])$$

for some smooth Y containing X as a subvariety. This definition is independent of Y because of the well-known fact that the push-forward functor and the Grothendieck duality functor commute.

By [14, Lemma 2.4], under this normal crossings assumption,

$$Q_X(M) \simeq \frac{\tilde{\mathcal{V}}}{F_{q(M)-1}\tilde{\mathcal{V}}}[n].$$

Hence one obtains[1]

$$Q_X(M) \simeq \mathrm{Gr}^F_{q(M)}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}, F_\bullet) \simeq \mathrm{Gr}^F_{q(M)}\mathrm{DR}(\tilde{\mathcal{V}}(*D), F_\bullet). \tag{4.4}$$

The second quasi-isomorphism is from the filtered quasi-isomorphism (4.2).

5. An inductive proof of Kawamata-Viehweg type vanishing theorem

In this section, we prove the Kawamata-Viehweg type vanishing theorem inductively by a method similar to Kawamata's original approach (see [9, the proof of Theorem 4.3.1]). It will be proved alternatively by the injectivity theorem in Section 7. See the first half of Corollary 7.4.

Theorem 5.1. *Let X be a complex projective variety with polarizable pure Hodge module M strictly supported on X and let L be a nef and big line bundle on X. Then*

$$H^i(X, S(M) \otimes L) = 0,\ i > 0.$$

Proof. The proof will be divided into two steps. Step 1 is a Norimatsu type statement (see [9, Lemma 4.3.5]); Step 2 reduces the general statement to the case of Step 1.

Step 1. In this step, we make the following assumptions:

- X is smooth;
- M is extended from a polarizable variation of Hodge structure $\mathbb{V} = (\mathcal{V}, F^\bullet\mathcal{V}, \mathbb{V}_\mathbb{Q})$ on U, with $D = X \setminus U$ a normal crossings divisor;
- the local monodromy of $\mathbb{V}$ along each irreducible component of D is unipotent;
- $L \simeq \mathscr{O}_X(A + E)$ with A an ample divisor and $E = \sum_{i=1}^t E_i$ a reduced simple normal crossings divisor, and $\mathrm{Supp}(E) \subset D$.

[1] We learned this point from J. Suh.

Denote the Deligne canonical extension of $\mathcal{V}$ by $\tilde{\mathcal{V}}$, as in Section 4. Then the third assumption means the residue of $\tilde{\mathcal{V}}$ along each irreducible component of D is nilpotent. As explained in Section 4,

$$Q(M) = \frac{\tilde{\mathcal{V}}}{F_{q(M)-1}\tilde{\mathcal{V}}}[\dim X] = \mathrm{Gr}^F_{q(M)}\tilde{\mathcal{V}}[\dim X].$$

Set $Q' = Q(M)[-\dim X]$; it is a locally free sheaf.

Under these assumptions, we show that

$$H^i(X, Q'(-A-E)) = 0, \ i < \dim X.$$

We use induction on t, the number of components of E, and on dimension. The case $t = 0$ is Kodaira-Saito vanishing because of the filtered quasi-isomorphism (4.2). Assuming the result known for E with $\leq k$ components, consider the short exact sequence,

$$0 \longrightarrow \mathscr{O}_X(-A - \sum_{i=1}^{k+1} E_i) \longrightarrow \mathscr{O}_X(-A - \sum_{i=1}^{k} E_i) \longrightarrow \mathscr{O}_{E_{k+1}}(-A - \sum_{i=1}^{k} E_i) \longrightarrow 0.$$

Tensoring with Q' gives,

$$\begin{aligned} 0 \longrightarrow\ & Q'(-A - \sum_{i=1}^{k+1} E_i) \\ \longrightarrow\ & Q'(-A - \sum_{i=1}^{k} E_i) \longrightarrow Q'|_{E_{k+1}} \otimes \mathscr{O}_{E_{k+1}}(-A - \sum_{i=1}^{k} E_i) \longrightarrow 0. \end{aligned} \tag{5.2}$$

By the inductive assumption,

$$H^i(X, Q'(-A - \sum_{i=1}^{k} E_i)) = 0, \ i < \dim X. \tag{5.3}$$

Now $\tilde{\mathcal{V}}|_{E_{k+1}}$ is not a Deligne canonical extension of some variation of Hodge structure defined on some open set of E_{k+1}. However, the residue along E_{k+1} gives rise to a finite increasing filtration $W_\bullet$ for $\tilde{\mathcal{V}}|_{E_{k+1}}$, the monodromy weight filtration. Moreover, $\mathrm{Gr}^W_l(\tilde{\mathcal{V}}|_{E_{k+1}})$ for any l, is the Deligne canonical extension of some polarizable variation of Hodge structure defined on $E_{k+1} \setminus \sum_{i=1}^k E_i$, following from the SL_2-orbit theorem [2]. The filtration on $\mathrm{Gr}^W_l(\tilde{\mathcal{V}}|_{E_{k+1}})$ induced from the variation of Hodge structure is precisely the filtration induced from $\tilde{\mathcal{V}}$, i.e.,

$$F_p\mathrm{Gr}^W_l(\tilde{\mathcal{V}}|_{E_{k+1}}) = \frac{F_p\tilde{\mathcal{V}}|_{E_{k+1}} \cap W_l}{F_p\tilde{\mathcal{V}}|_{E_{k+1}} \cap W_{l-1}}.$$

Namely $\tilde{\mathcal{V}}|_{E_{k+1}}$ is the Deligne canonical extension of an admissible variation of mixed Hodge structure because of the unipotency assumption of the monodromy. See [1, Theorem 3.20],[2] or [4, Proposition 4.3].

By the inductive assumption again,

$$H^i(E_{k+1}, \mathrm{Gr}^F_{q(M)}\mathrm{Gr}^W_l\tilde{\mathcal{V}}|_{E_{k+1}} \otimes \mathscr{O}_{E_{k+1}}(-A - \sum_{i=1}^{k} E_i)) = 0,$$

[2]This process is called the graded nearby-cycle functor in [1].

for $i < \dim X - 1$ and all l. By a simple calculation, there are short exact sequences for any p and l,

$$0 \longrightarrow \frac{F_p\tilde{\mathcal{V}}|_{E_{k+1}} \cap W_{l-1}}{F_{p-1}\tilde{\mathcal{V}}|_{E_{k+1}} \cap W_{l-1}} \longrightarrow \frac{F_p\tilde{\mathcal{V}}|_{E_{k+1}} \cap W_l}{F_{p-1}\tilde{\mathcal{V}}|_{E_{k+1}} \cap W_l} \longrightarrow \mathrm{Gr}^F_p \mathrm{Gr}^W_l \tilde{\mathcal{V}}|_{E_{k+1}} \longrightarrow 0.$$

Using the long exact sequences of cohomology associated to the above short exact sequences, one sees that

$$H^i(E_{k+1}, Q'|_{E_{k+1}} \otimes \mathscr{O}_{E_{k+1}}(-A - \sum_{i=1}^{k} E_i)) = 0,\ i < \dim X - 1. \tag{5.4}$$

Using the long exact sequence of cohomology associated to the short exact sequence (5.2), together with (5.3) and (5.4), we get

$$H^i(X, Q'(-A - \sum_{i=1}^{k+1} E_i)) = 0,\ i < \dim X.$$

Step 2. Since the Leray spectral sequence degenerates at E_1 because of Corollary 3.10, by passing to a log-resolution (see also the proof of Theorem 7.1), it suffices to assume that X is smooth, M is extended from a polarizable variation of Hodge structure $\mathbb{V}$ defined on U with $D = X \backslash U = \sum D_i$ a reduced simple normal crossings divisor, and $L^m = \mathscr{O}(A + E)$ with A an ample divisor and E an effective divisor with support contained in D. Denote

$$E = \sum_{i=1}^{t} \alpha_i D_i,\ \alpha = \alpha_1 \cdot \dots \cdot \alpha_t,\ \text{and}\ \alpha'_i = \alpha/\alpha_i,$$

with $\alpha_i > 0$. Using a Kawamata covering [6], we can construct a finite flat covering

$$f : Y \longrightarrow X,$$

such that f^*D still has simple normal crossings support,

$$f^*D_i = m\alpha'_i D'_i,$$

for $i = 1, \dots, t$ and $E' = \sum_{i=1}^{t} D'_i$ is a reduced simple normal crossings divisor. Furthermore, the monodromy along each component of f^*D of $\mathbb{V}' := f_1^*\mathbb{V}$ is unipotent, since m may be as divisible as needed. Here $f_1 = f|_{f^{-1}U}$ and the pull-back is just the non-characteristic pull-back for filtered left $\mathscr{D}$-modules. Hence $\mathbb{V}'$ is also a polarizable variation of Hodge structure on $f^{-1}U$.

Now put $L' = f^*L$ and $A' = f^*(A)$, so that

$$(L')^m \simeq \mathscr{O}_Y(A' + m\alpha E').$$

Hence

$$(L'(-E'))^{m\alpha} \simeq (L')^{m(\alpha-1)}(A').$$

Since L' is nef, one knows that the line bundle on the right-hand side is also ample. Therefore

$$L' \simeq \mathscr{O}_Y(H + E'),$$

for some ample divisor H on Y.

It is obvious that $\mathbb{V}$ is a direct summand of $f_{1,*}\mathbb{V}'$.[3] Hence so is M of $\mathcal{H}^0 f_* M'$ where M' is the pure Hodge module corresponding to $\mathbb{V}'$. Therefore, $S(M)$ is also

[3]Here the push-forward is the Hodge module direct image of $\mathbb{V}$ after side-change.

a direct summand of $f_*(S(M'))$. Since f is finite, it is sufficient to prove vanishing for $S(M')$ and L' on Y. By Serre duality (see (4.3)), it is equivalent to prove

$$H^i(Y, Q(M'^*)[-n] \otimes (L')^{-1}) = 0,\ i < n,$$

where M'^* is the pure Hodge module extended from the dual of $\mathbb{V}'$ and $n = \dim Y$. Consequently, the proof is done by Step 1. $\square$

6. Injectivity in the normal crossings case

Let X be a smooth complex projective variety, and $D = \sum_{i=1}^r D_i$ a reduced simple normal crossings divisor with irreducible components D_i. Assume M is the polarizable pure Hodge module extending a polarizable variation of Hodge structure $\mathbb{V} = (\mathcal{V}, F_\bullet\mathcal{V}, \mathbb{V}_\mathbb{Q})$ defined on $U = X \setminus D$. Then one uses the notation introduced in Section 4.

The result in this section owes a lot to the approach of J. Suh to the Kawamata-Viehweg vanishing theorem for Hodge modules in [17]. The proof closely follows his argument and a similar argument of C. Schnell in [15] with a slight refinement that leads to a more general injectivity statement. All of these arguments follow in turn the general strategy of Esnault-Viehweg [3] towards vanishing and injectivity theorems.

Let L be a line bundle on X. Assume

$$L^N \simeq \mathcal{O}_X(D')$$

for some N large enough, such that

$$D' = \sum_{i=1}^r \alpha_i D_i, \quad 0 \le \alpha_i \ll N,$$

and $D' \neq 0$.

Theorem 6.1. *Let $E = \sum_{i=1}^r \mu_i D_i$ be an effective divisor such that* $\mathrm{Supp}(E) \subset \mathrm{Supp}(D')$. *Then for all i, the natural map induced by E*

$$H^i(X, \mathrm{Gr}^F_{q(M)}\mathrm{DR}(\tilde{\mathcal{V}}(*D)) \otimes L^{-1}(-E)) \longrightarrow H^i(X, \mathrm{Gr}^F_{q(M)}\mathrm{DR}(\tilde{\mathcal{V}}(*D)) \otimes L^{-1})$$

is surjective.

Proof. Set

$$L^{(i)^{-1}} := L^{\otimes -i}(\lfloor \frac{iD'}{N} \rfloor).$$

By [3, Theorem 3.2], one knows that each $L^{(i)^{-1}}$ has a flat logarithmic connection and its residue along D_i is just multiplication by $\{\frac{i\alpha_i}{N}\}$ (the fractional part of $\frac{i\alpha_i}{N}$).

Let $\pi : Y \longrightarrow X$ be the cyclic cover obtained by taking the N-th root of D'. See [3, §3] for the construction of cyclic covers. Set $\pi' = \pi|_{U' = \pi^{-1}(X\setminus D)}$. So π' is étale. Since the relative de Rham complex is trivial, we get a variation of Hodge structure (the Gauss-Manin connection of π'),

$$\mathbb{V}_1 := (\pi'_*\mathscr{O}_{U'}, F_\bullet\pi'_*\mathscr{O}_{U'}, \pi'_*\mathbb{Q}_{U'}),$$

with trivial filtration. Since $\pi'_*\mathscr{O}_{U'} = \bigoplus_{i=0}^{N-1} L^{(i)^{-1}}|_{U'}$, we have a new variation of Hodge structure

$$\mathbb{V} \otimes \mathbb{V}_1 = \left(\mathcal{V} \otimes \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)^{-1}}|_U, F^\bullet, \mathbb{V}_\mathbb{Q} \otimes \pi'_*\mathbb{Q}_{U'} \right),$$

with

$$F^p\left(\mathcal{V}\otimes\bigoplus_{i=0}^{N-1}\mathcal{L}^{(i)^{-1}}|_U\right)=F^p\mathcal{V}\otimes\bigoplus_{i=0}^{N-1}\mathcal{L}^{(i)^{-1}}|_U.$$

Then we obtain a mixed Hodge module $j_*j^{-1}M_1$, where M_1 is the pure Hodge module extending $\mathbb{V}\otimes\mathbb{V}_1$. The filtered left $\mathscr{D}$-module $\mathcal{N}$,

$$\mathcal{N}:=\tilde{\mathcal{V}}(*D)\otimes\bigoplus_{i=0}^{N-1}L^{(i)^{-1}}(*D),$$

underlies $j_*j^{-1}M_1$. Hence the filtered left $\mathscr{D}$-module $\tilde{\mathcal{V}}(*D)\otimes L^{(1)^{-1}}(*D)$ is a direct summand of $\mathcal{N}$. We also know $\tilde{\mathcal{V}}\otimes L^{(1)^{-1}}$ is the Deligne canonical extension of $\mathcal{V}\otimes L^{(1)^{-1}}|_U$ because of the assumption $\alpha_i\ll N$. Therefore, by filtered quasi-isomorphism (4.2) we have a filtered quasi-isomorphism

$$\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{(1)^{-1}},F_\bullet)\simeq\mathrm{DR}(\tilde{\mathcal{V}}(*D)\otimes L^{(1)^{-1}}(*D),F_\bullet). \tag{6.2}$$

By Corollary 3.3, we see that the spectral sequence

$$E_1^{p,q}=H^{p+q}(X,\mathrm{Gr}^F_{-p}\mathrm{DR}(\mathcal{N},F_\bullet))\implies H^{p+q}(X,\mathrm{DR}(\mathcal{N}))$$

degenerates at E_1. Hence as a direct summand, we know that the spectral sequence (since $L^{(1)^{-1}}=L^{-1}$)

$$E_1^{p,q}=H^{p+q}(X,\mathrm{Gr}^F_{-p}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1},F_\bullet))\implies H^{p+q}(X,\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1},F_\bullet))$$

degenerates at E_1. From the filtration of $\mathbb{V}\otimes\mathbb{V}_1$ and the definition of the filtration on the Deligne canonical extension (equation (4.1)), it is clear that

$$\mathrm{Gr}^F_p\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1},F_\bullet)=\mathrm{Gr}^F_p\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}},F_\bullet)\otimes L^{-1}. \tag{6.3}$$

For any integer $a\le 0$, all the eigenvalues of residue of $\tilde{\mathcal{V}}\otimes L^{-1}(aD_i)$ along D_i (irreducible components of E) are strictly positive ([3, Lemma 2.7]). Therefore, by repeatedly applying [3, Lemma 2.10] for the components of E, we see that the natural map

$$\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1}(-E))\longrightarrow\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1}) \tag{6.4}$$

is a quasi-isomorphism (without filtrations). If one puts the trivial filtration on $L^{-1}(-E)$, we get another spectral sequence

$$\begin{aligned}E_1^{p,q}=H^{p+q}(X,\mathrm{Gr}^F_{-p}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1}(-E),F_\bullet))&\\ \implies H^{p+q}(X,\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1}(-E),F_\bullet)).&\end{aligned}$$

Clearly as before,

$$\mathrm{Gr}^F_p\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}}\otimes L^{-1}(-E),F_\bullet)=\mathrm{Gr}^F_p\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}},F_\bullet)\otimes L^{-1}(-E). \tag{6.5}$$

Now this spectral sequence may not degenerate at E_1. However, one has the following diagram:

$$\begin{tikzcd} H^{p-q(M)}(X,\mathcal{A}) & H^{p-q(M)}(X,\mathcal{C}) \arrow[l, "\alpha"'] \\ H^{p-q(M)}(X,\mathcal{B}) \arrow[u, "\beta"] & H^{p-q(M)}(X,\mathcal{D}), \arrow[l] \arrow[u, "\gamma"'] \end{tikzcd}$$

where $\mathcal{A} = \mathrm{Gr}^F_{q(M)}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}} \otimes L^{-1}, F_\bullet)$, $\mathcal{B} = \mathrm{Gr}^F_{q(M)}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}} \otimes L^{-1}(-E), F_\bullet)$, $\mathcal{C} = \mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}} \otimes L^{-1})$ and $\mathcal{D} = \mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}} \otimes L^{-1}(-E))$.

Observe that α is surjective because of E_1 degeneracy. Also, γ is an isomorphism because of the quasi-isomorphism (6.4). Hence,

$$\beta : H^i(X, \mathrm{Gr}^F_{q(M)}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}} \otimes L^{-1}(-E), F_\bullet)) \longrightarrow H^i(X, \mathrm{Gr}^F_{q(M)}\mathrm{DR}_{(X,D)}(\tilde{\mathcal{V}} \otimes L^{-1}, F_\bullet))$$

is surjective for all i. Combining this with the filtered quasi-isomorphism (6.2) and the isomorphisms (6.3) and (6.5), the proof is finished. □

Remark 6.6. In the above proof, the condition $0 < \alpha_i \ll N$ is needed because it ensures that $\tilde{\mathcal{V}} \otimes L^{-1}$ is the Deligne canonical extension. Therefore, Theorem 6.1 is still true if we require that

$$\alpha_i + \lambda_{D_i} < 1$$

instead, where λ_{D_i} is the maximal eigenvalue of the residue of $\tilde{\mathcal{V}}$ along D_i.

By (4.4),

$$H^b(X, \mathrm{Gr}^F_{q(M)}\mathrm{DR}(\tilde{\mathcal{V}}(*D), F_\bullet) \otimes L^{-1}) = H^b(X, Q(M) \otimes L^{-1}).$$

By Grothendieck-Serre duality,

$$H^i(X, Q(M) \otimes L^{-1}) = H^{-i}(X, S(M^*) \otimes L)^*.$$

Here M^* is the pure Hodge module corresponding to the dual of the variation of Hodge structure $\mathbb{V}$. Therefore, the following corollary has been proved by replacing M by M^* in Theorem 6.1.

Corollary 6.7. *Under the assumption of Theorem* 6.1*,*

$$H^i(X, S_X(M) \otimes L) \longrightarrow H^i(X, S_X(M) \otimes L(E))$$

is injective for all i*.*

7. Injectivity in the general case

In this section we prove Theorem 1.4 from the introduction, and further extensions. From now on, divisors always mean Cartier divisors. When X is smooth, Cartier and Weil divisors will not be distinguished.

Theorem 7.1. *Let X be a complex projective variety, and let M be a pure Hodge module with strict support X. If L is a semi-ample line bundle and E an effective divisor with $H^0(X, L^v(-E)) \neq 0$ for some $v > 0$, then the natural map*

$$H^i(X, S_X(M) \otimes L) \longrightarrow H^i(X, S_X(M) \otimes L(E))$$

is injective for all i*.*

Proof. By assumption, write

$$L^v \simeq \mathcal{O}_X(E + C)$$

for an effective divisor C. Take a log-resolution of $E + C +$ singular locus of M:

$$f : X_1 \longrightarrow X.$$

Here the singular locus of M is the complement of the largest open subset where M is a variation of Hodge structure. Set $E_1 = f^*E$, $C_1 = f^*C$. Since f is birational, the variation of Hodge structure also extends to a polarizable pure Hodge module strictly supported on Y, called M_1, and the singular locus of M_1 is just the exceptional divisor of f, which is a simple normal crossing divisor. Clearly, $L_1 := f^*(L)$ is still semi-ample. Hence, L_1^u is base-point free for some u large enough. Pick a general divisor D_1 in $|L_1^u|$ so that it is transverse to the exceptional divisor of f. Then

$$L_1^{u+v} \simeq \mathcal{O}_X(D_1 + E_1 + C_1),$$

and $D_1 + E_1 + C_1$ is a divisor with simple normal crossings support. So the assumption of Corollary 6.7 has been fulfilled.

By Corollary 3.10,

$$R^i f_* S(M_1) = \begin{cases} S(M), & i = 0, \\ 0, & i > 0. \end{cases}$$

Hence by the degeneracy of the Leray spectral sequence and the projection formula,

$$H^b(X_1, S(M_1) \otimes f^*L) = H^b(X, S(M) \otimes L).$$

The proof is done by Corollary 6.7. □

Similarly, we also get an injectivity theorem for nef and big line bundles.

Theorem 7.2. *Let X be a complex projective variety, and let M be a pure Hodge module with strict support X. If L is a nef and big line bundle and E an effective divisor, then*

$$H^i(X, S(M) \otimes L) \longrightarrow H^i(X, S(M) \otimes L(E))$$

is injective for all i.

Proof. Since L is nef and big, we can write

$$L^v \simeq \mathcal{O}_X(A + D + E)$$

for an ample divisor A, an effective divisor D and some $v > 0$. By taking a log-resolution for $D + E +$ singular locus of M, it is enough to assume that X is smooth and the singular locus of M is a simple normal crossings divisor containing $\mathrm{Supp}(D + E)$. Choosing N large enough, then

$$L^N \simeq L^{N-v}(A) \otimes \mathcal{O}_X(D + E).$$

Since L is nef, $H := L^{N-v}(A)$ is ample. Hence

$$L^{mN} \simeq H^m \otimes \mathcal{O}_X(mD + mE) \simeq \mathcal{O}_X(D_1 + mD + mE)$$

for some sufficiently general $D_1 \in |H^m|$, so that all the transversality conditions are satisfied. The statement of the theorem follows from Corollary 6.7. □

Remark 7.3 (Injectivity for nef and abundant line bundles). After a series of blowing-ups, nef and abundant ($\kappa(L) = v(L)$) line bundles can be reduced to semi-ample ones. Therefore, Theorem 7.2 is also true if L is only nef and abundant. (More details can be found in [3, §5].)

Choosing E as a multiple of a very ample divisor, by Serre vanishing we obtain another proof of Theorem 5.1.

Corollary 7.4. *If X is a complex projective variety, L a nef and big line bundle, and M a pure Hodge module with strict support X, then*

$$H^i(X, S(M) \otimes L) = 0, \; i > 0.$$

More generally, if L is nef only, then

$$H^i(X, S(M)\otimes L) = 0,\ i > n-\kappa(L).$$

Proof. First, $\kappa(L) \leq \kappa(f^*L)$ for any birational morphism $f : Y \longrightarrow X$. Hence, it is enough to assume X is smooth, $S_X(M)$ is locally free and $\kappa(L) < n$. Choose a general hyperplane section H of a very ample line bundle, such that

$$H^b(X, S(M)\otimes L(H)) = 0,\ b > 0,$$

and H is non-characteristic for M. Therefore, we get a short exact sequence,

$$0 \longrightarrow S(M)\otimes L \longrightarrow S_X(M)\otimes L(H) \longrightarrow S(M)|_H \otimes \mathcal{O}_H(H)\otimes L|_H \longrightarrow 0.$$

Hence, by passing to the long exact sequence of cohomology,

$$H^{i-1}(H, S(M)|_H\otimes \mathcal{O}_H(H)\otimes L|_H) = H^i(X, S(M)\otimes L).$$

Since $\kappa(L|_H) \geq \kappa(L)$ and $S(M)|_H \otimes \mathcal{O}_H(H) \simeq S(i^*(M))$, both groups vanish for $i > n-\kappa(L)$ by induction on dimension. $\square$

Since the eigenvalues of the Deligne canonical extension lie in $[0,1)$, the injectivity theorem is still true if the nef and big line bundle is perturbed by an effective $\mathbb{Q}$-divisor of sufficiently small coefficients in the following sense.

Theorem 7.5. *Let X be a projective variety with a polarizable pure Hodge module M strictly supported on X, and let N be a nef and big $\mathbb{Q}$-divisor, D and B two effective divisors. There exists an $\varepsilon = \varepsilon(M,N,D,B) > 0$ such that if a line bundle $L \sim_{\mathbb{Q}} N + \varepsilon_1 D$ for $0 < \varepsilon_1 < \varepsilon$, then the natural map*

$$H^i(X, S(M)\otimes L) \longrightarrow H^i(X, S(M)\otimes L(B))$$

is injective for all i.

Proof. Since N is nef and big, by Kodaira's lemma,

$$nN \sim A + C + B$$

for some $0 < n \in \mathbb{Z}$, A an ample divisor and C an effective divisor. Take a log-resolution for $C + B + D +$ singular locus of M,

$$f : Y \longrightarrow X.$$

Then $A_1 := f^*pA - E$ is ample for some $p \ll 0$ and E an effective divisor with simple normal crossings support. Hence,

$$f^*npN \sim A_1 + E + pf^*C + pf^*B.$$

Write $f^*D = \sum d_i D_i'$. Then we can assume U is Zariski open in Y and $Y \setminus U$ is a simple normal crossings divisor containing the support of all effective divisors that appear, and there is a variation of Hodge structure $\mathbb{V}$ on U which corresponds to M. Take ε to be

$$\varepsilon = \min\{\frac{1-\lambda_{D_i'}}{d_i}\},$$

where $\lambda_{D_i'}$ is the maximal eigenvalue of the residue of the Deligne canonical extension of $\mathbb{V}$ along D_i'. If a line bundle $L \sim_{\mathbb{Q}} N + \varepsilon_1 D$ for $0 < \varepsilon_1 < \varepsilon$, then

$$f^*L^k \sim \frac{k}{np}(A_1 + E + pf^*C + pf^*B) + k\varepsilon_1 f^*D,$$

where k is some big enough multiple of np. Since $f^*L^l(-\varepsilon_1 lD)$ is a nef line bundle for $l \ll 0$ and sufficiently divisible,

$$f^*L^{k+l} \sim A_2 + \frac{k}{np}(E + pf^*C + pf^*B) + (k+l)\varepsilon_1 f^*D,$$

for some other ample divisor A_2. Hence

$$f^*L^{(k+l)m} \sim H + \frac{km}{np}(E + pf^*C + pf^*B) + (k+l)m\varepsilon_1 f^*D,$$

where $H \in |\mathscr{O}(mA_2)|$ is sufficiently general. The statement then follows from Corollary 6.7 and Remark 6.6. $\square$

Note that ε in the statement depends on the choice of a resolution, but can be made effective once one has been fixed; see also Remark 7.7 below. A similar argument works for the following Kawamata-Viehweg type vanishing for $\mathbb{Q}$-divisors.

Theorem 7.6. *Let X be a projective variety with a pure Hodge module M strictly supported on X, and let N be a nef and big $\mathbb{Q}$-divisor and D an effective divisor. There exists an $\varepsilon = \varepsilon(M, N, D) > 0$ such that if a line bundle $L \sim_{\mathbb{Q}} N + \varepsilon_1 D$ for $0 < \varepsilon_1 < \varepsilon$, then*

$$H^i(X, S(M) \otimes L) = 0, \ i > 0.$$

Remark 7.7. If X is a smooth projective variety of dimension n, $M = \mathbb{Q}_X^H[n]$ (or more generally for any smooth M, i.e., M corresponding to a polarizable variation of Hodge structure defined on X), and D is a reduced simple normal crossings divisor, then since there is no residue under these assumptions, $\varepsilon = 1$ in both Theorem 7.5 and Theorem 7.6 by [9, Theorem 9.4.17(i)]. Hence Theorem 7.6 reduces to the original Kawamata-Viehweg vanishing for $\mathbb{Q}$-divisors [9, Theorem 9.1.18], or a special case of [10, Theorem 11.1]; Theorem 7.5 specializes to the injectivity theorem for $\mathbb{Q}$-divisors [8, Theorem 10.13].

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