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LINEAR INVISCID DAMPING FOR MONOTONE SHEAR FLOWS

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ABSTRACT. In this article, we prove linear stability, scattering and inviscid damping with optimal decay rates for the linearized 2D Euler equations around a large class of strictly monotone shear flows, $(U(y), 0)$, in a periodic channel under Sobolev perturbations. Here, we consider the settings of both an infinite periodic channel of period L, $\mathbb{T}_L \times \mathbb{R}$, as well as a finite periodic channel, $\mathbb{T}_L \times [0, 1]$, with impermeable walls. The latter setting is shown to not only be technically more challenging, but to exhibit qualitatively different behavior due to boundary effects.

CONTENTS

1. INTRODUCTION

In this article, we study the phenomenon of linear inviscid damping for the 2D incompressible Euler equations in a periodic channel

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= 0, \\ v &= \nabla^{\perp} \Delta^{-1} \omega, \end{aligned} \tag{Euler}$$

written here in vorticity formulation, linearized around strictly monotone shear flow solutions (see Figure 1)

$$\begin{aligned} v &= (U(y), 0), \\ \omega &= -U''(y). \end{aligned}$$

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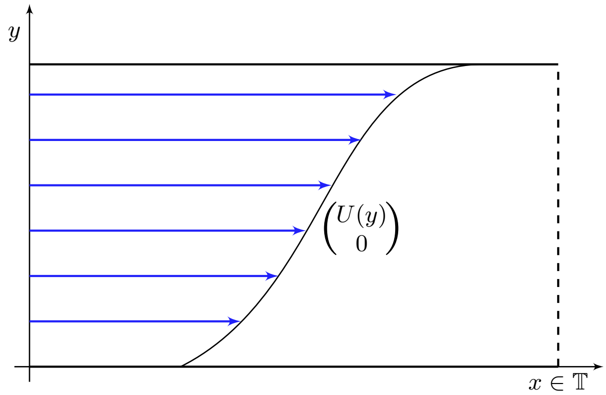

FIGURE 1. An example of a shear flow, $(U(y), 0)$, in a periodic channel.

Here, we consider both the common setting of an infinite periodic channel of period L, $(x, y) \in \mathbb{T}_L \times \mathbb{R}$, as well as the physically relevant setting of a finite periodic channel, $\mathbb{T}_L \times [0, 1]$, with impermeable walls, i.e. in that setting we require

$$v_2 = 0, \qquad \text{for } y \in \{0, 1\}.$$

The latter setting is shown to be not only technically more challenging but to exhibit qualitatively different behavior due to boundary effects.

1.1. **Motivation and literature.** The motivating example for the flows we consider is given by Couette flow, i.e. the linear shear $U(y) = y$, in an infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$. For this specific flow the linearized Euler equations simplify to the free transport equations:

$$\begin{aligned} &\partial_t \omega + y \partial_x \omega = 0, \\ &\qquad (t, x, y) \in \mathbb{R} \times \mathbb{T}_L \times \mathbb{R}, \end{aligned}$$

and, in particular, can be solved explicitly in both spatial and Fourier variables. In this case, one can hence directly compute that perturbations (ω, v) are damped to a shear flow with algebraic rates:

$$\begin{aligned} \|v_1(t) - \langle v_1 \rangle_x\|_{L^2} &\leq \mathcal{O}(t^{-1}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H_x^{-1} H_y^1}, \\ \|v_2(t)\|_{L^2} &\leq \mathcal{O}(t^{-2}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H_x^{-1} H_y^2} \end{aligned} \tag{1}$$

and that the decay rates and regularity requirements are sharp. This classical and, in view of the Hamiltonian structure of the Euler equations (cf. [3]), at first surprising result was experimentally observed and proven for the linearized equations by Kelvin, [8], and Orr, [16], and is called (linear) inviscid damping and shares similarities with Landau damping in plasma physics.

Going beyond the explicitly solvable (and in this sense trivial) setting of linearized Couette flow, has, however, remained open until recently:

- In [5], Bouchet and Morita give heuristic results suggesting that linear damping and stability results should also hold for general monotone shear flows. However, their methods are nonrigorous and lack necessary regularity, stability and error estimates, as discussed in [20]. In particular, even supposing their asymptotic computations were valid, they do not yield the above decay rates.

- In [19], Stepin uses a spectral approach to study the stability of monotone shear flows. Under the assumption that the associated Rayleigh boundary value problem possesses no eigenvalues, he obtains an asymptotic description of the stream function and nonoptimal decay rates.
- Lin and Zeng, [11], use the explicit solution of linearized Couette flow to establish linear damping also in a finite periodic channel. Furthermore, they show the existence of nontrivial stationary solutions to the full 2D Euler equations in arbitrarily small H^s neighborhoods of Couette flow for any $s < \frac{3}{2}$. As a consequence, nonlinear inviscid damping cannot hold in such low regularity.
- Recently, following the work of Villani and Mouhot, [15], on nonlinear Landau damping, Masmoudi and Bedrossian, [4], have proven nonlinear inviscid damping for small Gevrey perturbations to Couette flow in an infinite periodic channel.

1.2. **Strategy and main results.** As the main results of this article, we, for the first time, rigorously prove linear inviscid damping for a general class of monotone shear flows. Here, in addition to the common setting of an infinite periodic channel of period L, $\mathbb{T}_L \times \mathbb{R}$, we also prove linear inviscid damping in the physically relevant setting of a finite periodic channel, $\mathbb{T}_L \times [0,1]$, with impermeable walls. As we show in section 5, in the latter case, boundary effects asymptotically lead to the formation of (logarithmic) singularities of derivatives of solutions. Stability results are thus limited to fractional Sobolev spaces $H^s, s \leq \frac{3}{2}$, unless one restricts to perturbations, ω_0, with vanishing Dirichlet data, $\omega_0|_{y=0,1}$. As damping with the optimal algebraic rates, (1), only requires stability in H^2, in this article we limit ourselves to establishing stability in H^1 for general perturbations and stability in H^2 for perturbations with vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$.

Our strategy to prove linear inviscid damping is described in Figure 2.

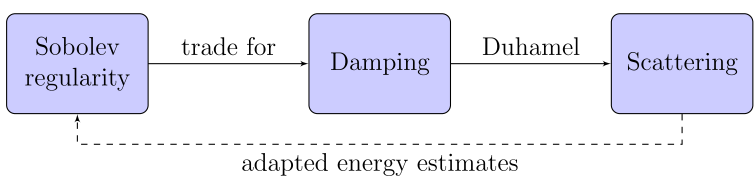

FIGURE 2. Sobolev regularity can be used to obtain damping and scattering in less regular spaces. However, in order to close the argument any loss of regularity has to be avoided.

As a first step, in section 3, we show that linear inviscid damping, like Landau damping, is fundamentally a problem of regularity. For this purpose, we consider the linearized 2D Euler equations

$$\begin{aligned}\partial_t \omega + U(y)\partial_x \omega &= U''(y)v_2 = U''(y)\partial_x \phi, \\ \Delta \phi &= \omega\end{aligned} \tag{2}$$

as a perturbation to the underlying transport equation

$$\partial_t f + U(y)\partial_x f = 0$$

and consider coordinates moving with the flow:

$$W(t,x,y) := \omega(t, x - tU(y), y).$$

In analogy to conventions in dispersive equations we call W the *scattered vorticity* (with respect to the underlying flow).

Assuming W to be regular uniformly in time, i.e.

$$\|W(t)\|_{L^2_x H^2_y} < C < \infty,$$

it has been shown in the author's master's thesis, [20], that damping estimates of the form (1) can be extended to general, strictly monotone shear flows (Theorem 3.1):

Theorem (Generalization of [11, Theorem 3]; [20])**.** *Let Ω be either the infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$, or the finite periodic channel, $\mathbb{T}_L \times [0,1]$. Let ω be a solution to the linearized Euler equations,* (2)*, around a strictly monotone shear flow $U(y)$, on the domain Ω. Suppose further that $\frac{1}{U'} \in W^{2,\infty}(\Omega)$. Then the following statements hold:*

(1) *If $W(t) \in H^{-1}_x H^1_y(\Omega)$ for all times, then*

$$\|v(t) - \langle v\rangle_x\|_{L^2(\Omega)} = \mathcal{O}(t^{-1})\|W(t) - \langle W\rangle_x\|_{H^{-1}_x H^1_y(\Omega)}, \quad \text{as } t \to \pm\infty.$$

(2) *If $W(t) \in H^{-1}_x H^2_y(\Omega)$ for all times, then*

$$\|v_2(t)\|_{L^2(\Omega)} = \mathcal{O}(t^{-2})\|W(t) - \langle W\rangle_x\|_{H^{-1}_x H^2_y(\Omega)}, \quad \text{as } t \to \pm\infty.$$

As a consequence of sufficiently rapid decay of v_2, we observe that the right-hand-side $U''v_2$ in (2) is an integrable perturbation:

Corollary (Scattering)**.** *Let $U'' \in L^\infty$ and let $W(t) \in L^2$ be a solution of* (2) *such that, for some $\epsilon > 0$,*

$$\|v_2(t)\|_{L^2} = \mathcal{O}(t^{-1-\epsilon}),$$

as $t \to \infty$. Then, there exists $W^\infty \in L^2$ such that

$$W(t) \xrightarrow{L^2} W^\infty,$$

as $t \to \infty$, and

$$\|W(t) - W^\infty\|_{L^2} = \mathcal{O}(t^{-\epsilon}).$$

In section 6, this scattering result is further extended to arbitrary L^2 initial data. We stress that the *higher regularity* of W is necessary in Theorem 3.1, as the underlying shear,

$$e^{tS} : (x,y) \mapsto (x - tU(y), y),$$

is an L^2-isometry and hence

$$W = e^{-tS}\omega \mapsto V := e^{-tS}v = e^{-tS}\nabla^\perp \Delta e^{tS} W,$$

when considered as an operator from L^2 to L^2, has a time-independent operator norm. In order to prove the desired stability result for W,

$$\|W(t) - \langle W\rangle_x\|_{H^{-1}_x H^2_y} \lesssim \|\omega_0 - \langle \omega_0\rangle_x\|_{H^{-1}_x H^2_y},$$

we thus have to invest considerable technical effort to use finer properties of the dynamics.

As the first main result of this article, in section 4 we establish stability of the linearized Euler equations around regular, strictly monotone shear flows in an infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, for arbitrarily high Sobolev norms (Theorem 4.1).

Theorem 1.1 (Sobolev stability for the infinite periodic channel)**.** *Let $s \in \mathbb{N}_0$ and let $U'(U^{-1}(\cdot)), U''(U^{-1}(\cdot)) \in W^{s+1,\infty}(\mathbb{R})$ and suppose that there exists $c > 0$ such that*

$$0 < c < U' < c^{-1} < \infty.$$

Suppose further that

$$L\|U''(U^{-1}(\cdot))\|_{W^{s+1,\infty}}$$

is sufficiently small. Then for all $m \in \mathbb{Z}$ and $\omega_0 \in H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})$, the solution W of the linearized Euler equations in scattering formulation, (21)*, with initial datum ω_0 satisfies*

$$\|W(t)\|_{H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})} \lesssim \|\omega_0\|_{H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})}.$$

The proof of this theorem is broken down into several steps, which form the subsections of section 4.

When considering a finite channel instead, we show that such a stability result cannot hold in arbitrary Sobolev spaces, but rather that in general boundary derivatives of W asymptotically develop (logarithmic) singularities at the boundary. Thus, stability in Sobolev spaces $H^s, s > \frac{3}{2}$, is not possible unless one restricts to perturbations, ω_0, with vanishing Dirichlet data, $\omega_0|_{y=0,1}$. The stability result in H^2 under such perturbations is then given by Theorem 5.1.

Theorem 1.2 (H^2 stability for the finite periodic channel)**.** *Let W be a solution of the linearized Euler equations in scattering formulation* (59)*. Let further $U'(U^{-1}(\cdot)), U''(U^{-1}()\cdot) \in W^{3,\infty}([0,1])$ and suppose that there exists $c > 0$ such that*

$$0 < c < U' < c^{-1} < \infty.$$

Suppose further that

$$\|U''(U^{-1}(\cdot))\|_{W^{1,\infty}} L$$

is sufficiently small.

Then, for any $m \in \mathbb{Z}$ and any $\omega_0 \in H^m_x H^2_y(\mathbb{T}_L \times [0,1])$ with $\omega_0|_{y=0,1} = 0$ and for any time t,

$$\|W(t)\|_{H^m_x H^2_y(\mathbb{T}_L \times [0,1])} \lesssim \|\omega_0\|_{H^m_x H^2_y(\mathbb{T}_L \times [0,1])}.$$

In section 6, we combine the stability and damping results to prove linear inviscid damping with the optimal rates in both an infinite periodic channel and a finite periodic channel in Theorem 6.1.

Theorem (Linear inviscid damping for monotone shear flows)**.** *Let ω be a solution to the linearized Euler equation*

$$\partial_t \omega + U(y)\partial_x \omega = U'' v_2,$$

with initial data $\omega_0 \in H^2_y L^2_x$ on either the infinite channel, $\mathbb{T}_L \times \mathbb{R}$, or on the finite channel, $\mathbb{T}_L \times [0,1]$, where $\mathbb{T}_L = [0,L]/\sim$.

Suppose there exists $c > 0$ such that

$$c < |U'| < c^{-1},$$

and that U'' and L are such that

$$L \left\| U''(U^{-1}(\cdot)) \right\|_{W^{3,\infty}}$$

is sufficiently small. In the case of a finite channel, additionally assume that ω_0 vanishes on the boundary

$$\omega_0(x, 0) \equiv 0 \equiv \omega_0(x, 1).$$

Then there exist asymptotic profiles $W^\infty(x, y)$ and $v^\infty(y)$ such that

$$\text{(Stability)} \qquad \|\omega(t, x - tU(y), y)\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y},$$

$$\text{(Damping)} \qquad \|v(t) - (v^\infty, 0)\|_{L^2_{xy}} = \mathcal{O}(t^{-1}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{\dot{H}^{-1}_x H^1_y},$$

$$\|v_2(t)\|_{L^2} = \mathcal{O}(t^{-2}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{\dot{H}^{-1}_x H^2_y},$$

$$\text{(Scattering)} \qquad \omega(t, x - tU(y), y) \xrightarrow{L^2} W^\infty,$$

$$\|\omega(t, x - tU(y), y) - W^\infty\|_{L^2_{xy}} = \mathcal{O}(t^{-1}) \|\omega_0\|_{\dot{H}^{-1}_x H^2_y},$$

as $t \to \pm\infty$.

As a consequence of Theorem 6.1 and the stability results of section 4.2 and section 5.1, we also obtain scattering for general L^2 initial data in Corollary 6.1.

Corollary (L^2 scattering)**.** *Let U, L be as in Theorem* 6.1 *and let $\omega_0 \in L^2$. Then there exists $W_\infty \in L^2$ such that*

$$W(t, x, y) \xrightarrow{L^2} W_\infty, \text{ as } t \to \infty.$$

1.3. **Outline of the article.** We conclude this introduction with a short overview of the organization of the article:

- In section 2, we consider linearized Couette flow on the infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, as a motivating example, which allows explicit solutions in physical as well as Fourier space. In particular, the damping mechanism and the regularity requirements are most transparent in this setting.
- In section 3, the damping results are generalized to smooth strictly monotone shear flows under the *assumption* of controlling the Sobolev regularity of the perturbation $W(t, x, y) := \omega(t, x - tU(y), y)$. Linear inviscid damping is hence shown to fundamentally be a problem of regularity and stability, as is also the case for Landau damping. This section is in part based on the author's master's thesis, [20], and generalizes previous results by [11] and [5].
- In section 4, we establish stability for the case of an infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$, in any Sobolev norm $H^s_y H^m_x(\mathbb{R} \times \mathbb{T}_L), s, m \in \mathbb{N}_0$, provided $L\|U''(U^{-1}(\cdot))\|_{W^{s,\infty}(\mathbb{R})}$ is sufficiently small. In particular, instead of imposing assumptions on the period L, we could restrict to shear flows that are close to affine.
- In section 5, we treat the case of a finite periodic channel, $\mathbb{T}_L \times [0, 1]$, with impermeable walls. Here, we show that boundary effects cannot be neglected and that for perturbations, ω_0, with nontrivial Dirichlet data, ω_0, $\partial_y W$ asymptotically develops logarithmic singularities at the boundary.

While H^1 stability results can be established for general perturbations, the H^2 stability results hence necessarily have to restrict to perturbations with vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$.
- In the final section 6, we conclude our proof of linear inviscid damping with optimal decay rates for monotone shear flows in an infinite periodic channel and a finite periodic channel. In the case of an infinite periodic channel, we also discuss consistency with the nonlinear equation, following an argument of [5].

2. Couette flow

The linearized Euler equations around Couette flow, $U(y) = y$, in an infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, are given by

$$\begin{aligned} \partial_t \omega + y\partial_x \omega &= 0, \\ v &= \nabla^\perp \Delta^{-1} \omega, \\ (t, x, v) &\in \mathbb{R} \times \mathbb{T} \times \mathbb{R}. \end{aligned}$$

We note that the first equation is (up to a change of notation) identical to free transport. The equations can hence be explicitly solved using the method of characteristics:

$$\omega(t, x, y) = \omega_0(x - ty, y).$$

As an example of the behavior of solutions, consider the case ω_0 being the indicator function of a box, depicted in Figure 3.

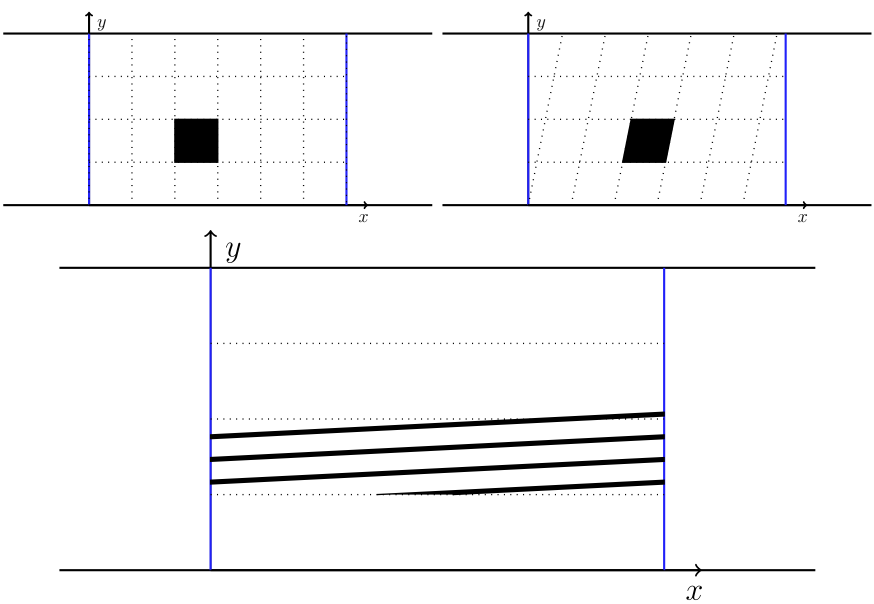

Figure 3. The vorticity is sheared by the flow.

From this one observes two opposing behaviors:

- Rapid oscillations in y damp anti-derivatives such as the velocity $v = \nabla^\perp \Delta\omega$ towards averaged quantities with a rate depending on the regularity of the initial data ω_0.
- The evolution loses regularity in y as time increases. For this reason the mechanism is sometimes called "violent relaxation" [15].

Due to the distinct role of the average, we briefly pause to discuss its behavior. The average in x is a function of y and t only and satisfies

$$\partial_t \langle \omega \rangle_x + y \langle \partial_x \omega \rangle_x = 0.$$

By periodicity $\langle \partial_x w \rangle_x = 0$, and thus

$$\langle \omega \rangle_x(y) = \langle \omega_0 \rangle_x(y)$$

is conserved by the evolution.

Incorporating the average of the initial perturbation into the underlying shear flow $U \mapsto U(y) + \int^y \langle \omega_0 \rangle_x dy'$ or using the linearity of the equation, we may thus without loss of generality assume that our perturbation satisfies

$$\langle \omega \rangle_x = \langle \omega_0 \rangle_x \equiv 0.$$

Remark 1. The same reduction can be used for the linearized equation for general shear flows $U(y)$, as

$$\langle U''(y) v_2 \rangle_x = U''(y) \langle \partial_x \phi \rangle_x = 0.$$

In the nonlinear setting one would also like to remove this average; however it is not conserved anymore. Therefore, one has to scatter around a shear profile changing in time, which introduces considerable technical difficulties (see [4]).

With the average set to zero, the above heuristic example suggests that positive Sobolev norms in y blow up as $t \to \pm\infty$ while negative Sobolev norms tend to zero.

In order to obtain a more quantitative description, it is useful to restrict to the whole space setting $\mathbb{T} \times \mathbb{R}$, where a Fourier transform is available. After a Fourier transform in x and y, which in the sequel is denoted by $\tilde{\cdot}$, our equation is given by

$$\partial_t \tilde{\omega} + k \partial_\eta \tilde{\omega} = 0,$$
$$\tilde{v} = \begin{pmatrix} -i\eta \\ ik \end{pmatrix} \frac{1}{k^2 + \eta^2} \tilde{\omega}$$

(see Figure 4). So we again obtain a transport equation, which we may solve using the method of characteristics:

$$\tilde{\omega}(t, k, \eta) = \tilde{\omega}_0(k, \eta + kt).$$

Remark 2. When considering linear Landau damping, to compute the force field one is only interested in an average of the density, which in our notation would be the case $\eta = 0$. In that case, high regularity of ω_0 directly translates into a high decay speed of $\tilde{\omega}_0(k, kt)$. In particular, analytic regularity allows one to deduce exponential decay, [13], [15].In the case of the Euler equations, however, the velocity field depends on all η and a main difficulty arises in the control of $\eta \approx -kt$.

Remark 3.
- While neither Rayleigh's nor Fjørtoft's theorems are applicable, since $(y)'' = 0$, Couette flow is linearly stable in L^p for all $p \in [0, \infty]$ as the above change of variables is a volume preserving diffeomorphism.

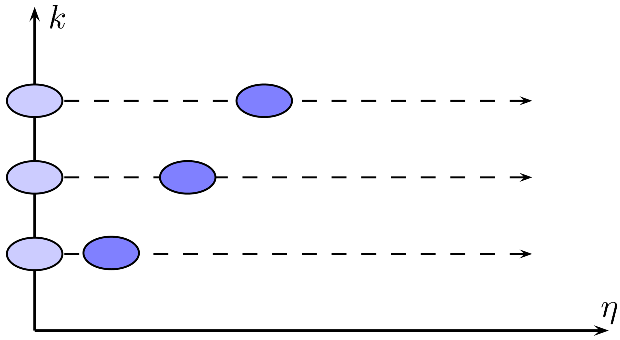

FIGURE 4. Transport in Fourier space.

- L^p stability remains true in the nonlinear setting, as the conservation of
$$\int f(\Omega)dx$$
for $\Omega = 1+\omega$ also implies the conservation of $\|\omega\|_{L^p}$ by choosing $f(t) = |t-1|^p$. For further results in this direction consult [18], [6], [11], [9].
- Despite its simplicity Couette flow has been of significant interest in physical research due to being (up to symmetries) the only shear flow that also is a stationary solution of the Navier-Stokes equation as well as being a sufficient model for naturally occurring flows in pipes, channels or similar simple geometries.

  It is also frequently studied in the context of turbulence (for an introduction see [7]). Associated with it is the so-called "Sommerfeld paradox": The flow is a linearly stable solution of the Navier-Stokes equation for all Reynold's numbers $Re > 0$, but as seen by numerical and physical experiments it becomes turbulent when Re is large.

While all L^p norms are conserved, Sobolev norms involving y change in time. Using the characterization in Fourier variables and the explicit solution we compute

$$\begin{aligned}\|\omega\|^2_{H^{s_1}_x H^{s_2}_y} &= \sum \int \langle k\rangle^{2s_1} \langle \eta\rangle^{2s_2} |\omega_0(k, \eta + kt)|^2 dk d\eta \\ &= \sum \int \langle k\rangle^{2s_1} \langle \eta - kt\rangle^{2s_2} |\omega_0(k,\eta)|^2 dk d\eta.\end{aligned}$$

We thus heuristically observe that $\|\omega\|_{H^s_y} \sim \langle t\rangle^s$, i.e. positive Sobolev norms grow as t increases, while negative Sobolev norms tend to zero.

However, these estimates are only asymptotic and not uniform. Consider for example an initial datum ω_0 highly concentrated at $(k_0, -k_0 c)$ for some k_0 and $c \gg 0$. The vorticity ω will then in turn be concentrated at $(k_0, k_0(t-c))$, which in particular implies that for $0 < t < c$ any negative Sobolev norm of ω is in fact *increasing* and even though it is decreasing for $t > c$, it will only be small for $t \gg 2c$.

Therefore, to obtain uniform estimates, L^2 control of ω_0 is not sufficient as it is invariant under translation in Fourier space and it is necessary to invest additional regularity to penalize Fourier modes with
$$\eta \approx kt.$$

A more precise theorem concerning the decay properties of the velocity field depending on the regularity of the initial datum is given by Lin and Zeng.

Theorem 2.1 (Damping for Couette flow [11, Theorem 3]). *Let ω_0 be initial data such that $\int \omega_0(x,y)dx = 0$ and let ω, v be the corresponding solution. Then:*

(1) *if $\omega_0 \in H^{-1}_x L^2_y$, then $\|v\|_{L^2} \to 0$,*
(2) *if $\omega_0 \in H^{-1}_x H^1_y$, then $\|v\|_{L^2} = \mathcal{O}(t^{-1})$,*
(3) *if $\omega_0 \in H^{-1}_x H^2_y$, then $\|v_2\|_{L^2} = \mathcal{O}(t^{-2})$.*

Remark 4. The original proof of Lin and Zeng also handles the case of a bounded domain and is generalized in section 3 to general monotone shear flows. For this section, we prefer a more direct and easier proof using explicit calculations in Fourier variables.

Proof. Define $V_2(t,x,y) = v_2(t, x-ty, y)$. Then $\|V_2\|_{L^2} = \|v_2\|_{L^2}$, and it is thus sufficient to consider V_2. By the previous calculations,

$$\tilde{V}_2 = \frac{-k^2}{k^2 + (\eta - kt)^2} \frac{\tilde{\omega}_0}{ik}.$$

We note that the first factor is uniformly bounded by 1 and converges point-wise to 0 and that $\frac{\omega_0}{ik} \in L^2$, since by assumption $\omega_0 \in H^{-1}L^2$. Splitting

$$\tilde{V}_2 = \frac{-k^2}{k^2 + (\eta - kt)^2}(1_{|\eta| \leq R} \frac{\tilde{\omega}_0}{ik}) + \frac{-k^2}{k^2 + (\eta - kt)^2}(1_{|\eta| > R} \frac{\tilde{\omega}_0}{ik}),$$

for R sufficiently large, the L^2 norm of the second term is smaller than ϵ, while for fixed R the multiplier is supported in the compact set B_R and decays to zero uniformly as $t \to \pm\infty$. Taking an appropriate diagonal sequence in (R,t) then yields the first result for v_2.

For $V_1(t,x,y) := v_1(t, x-ty, y)$ we proceed analogously with

$$\tilde{V}_1 = \frac{-(\eta - kt)k}{k^2 + (\eta - kt)^2} \frac{\tilde{\omega}_0}{ik}.$$

In order to show the other two claims, we multiply by a factor $1 = \frac{\max(1,|\eta|^j)}{\max(1,|\eta|^j)}, j = 1, 2$, and split it as follows:

$$\tilde{V}_2 = \frac{-k^2}{(k^2 + (\eta - kt)^2)\max(1, |\eta|^j)} \frac{\max(1, |\eta|^j)\tilde{\omega}_0}{ik}.$$

Note that the first factor is still uniformly bounded, but in addition decays uniformly like t^{-j} in the case of V_2. When considering V_1 only a decay of t^{-1} may be obtained in this way, since

$$\frac{-(\eta - kt)}{k^2 + (\eta - kt)^2}$$

only decays with rate t^{-1}. □

Remark 5.
- The decay speed depends on the regularity of ω_0 and can be seen to be sharp in the sense that for each fixed t one can find a worst case ω_0 such that the multiplier is of size 1.
- In contrast to the Vlasov-Poisson equation assuming regularity higher than H^2 on ω_0 does not give any additional decay speed for $\|v_2\|_{L^2}$.

- Lin and Zeng prove this theorem by using a dual formulation of the stream function and integrating by parts. In the simple setting of Couette flow in the whole space this is not necessary as we may calculate explicitly in Fourier variables. However, their method may be generalized to other shear flows where Fourier methods are not available, as we will explore in the following section.
- Lin and Zeng in addition give interpolated inequalities for $\omega_0 \in H^{s_1}H^{s_2}$. Details can be found in [11].

3. Damping under regularity assumptions

In the following, we extend the damping result for Couette flow of section 2 to more general shear flows $(U(y), 0)$. Here, we consider the settings of an infinite channel of period L, $\mathbb{T}_L \times \mathbb{R}$, as well as of a finite periodic channel, $\mathbb{T}_L \times [0, 1]$. The linearized Euler equations around a shear flow $(U(y), 0)$ are given by:

$$\begin{aligned} \partial_t \omega + U(y)\partial_x \omega &= U'' v_2, \\ v_2 &= \partial_x \phi, \\ \Delta \phi &= \omega, \\ v = \nabla^\perp \phi &\in L^2, \\ \partial_x \phi|_{y=0,1} &= 0 \text{ for the case } (x, y) \in \mathbb{T} \times [0, 1]. \end{aligned} \tag{3}$$

In view of the damping results of section 2, we consider the right-hand-side, $U''v_2$, to be a perturbation and introduce the *scattered vorticity*

$$W(t, x, y) := \omega(t, x - tU(y), y). \tag{4}$$

As for Couette flow, taking the x average of the equation, we see that

$$\langle W \rangle_x(t, y) = \langle \omega \rangle_x(t, y) = \langle \omega_0 \rangle_x(y) \tag{5}$$

is independent of time. By linearity and writing

$$\omega_0 = (\omega_0 - \langle \omega_0 \rangle_x) + \langle \omega_0 \rangle_x,$$

in the following without loss of generality we only consider the case $\langle W \rangle_x \equiv 0$.

The results of section 2 for Couette flow show that regularity of W is needed to establish damping results for the velocity field. In this section, we *assume* W to be of regularity comparable to ω_0 also in high Sobolev norms, uniformly in time.

The proof of stability of W and hence control of

$$\|W(t)\|_{L^2_x H^2_y},$$

which is the main result of this article, is obtained in sections 4 and 5 for the setting of an infinite periodic channel and a finite periodic channel, respectively.

Using the regularity, we establish damping results with the same optimal algebraic rates as for Couette flow also for general, strictly monotone shear flows, where the bounds are now in terms of W instead of ω_0. In section C, these results are further generalized and reformulated in terms of the respective flow maps.

As we consider general shear flows and also the setting of a finite periodic channel, Fourier methods are not available anymore. We therefore obtain results by duality in analogy to classical stationary phase arguments and as an extension of [11] and [5, Appendix A.1].

Theorem 3.1 (Generalization of [11, Theorem 3]; [20])**.** *Let Ω be either the infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$, or the finite periodic channel, $\mathbb{T}_L \times [0,1]$. Let ω be a solution to the linearized Euler equations,* (3)*, around a strictly monotone shear flow $U(y)$, on the domain Ω. Suppose further that $\frac{1}{U'} \in W^{2,\infty}(\Omega)$. Then the following statements hold:*

(1) *If $W(t) \in \dot{H}_x^{-1} H_y^1(\Omega)$ for all times, then*

$$\|v(t) - \langle v \rangle_x\|_{L^2(\Omega)} = \mathcal{O}(t^{-1}) \|W(t)\|_{\dot{H}_x^{-1} H_y^1(\Omega)}, \text{ as } t \to \pm\infty.$$

(2) *If $\dot{W}(t) \in H_x^{-1} H_y^2(\Omega)$ for all times, then*

$$\|v_2(t)\|_{L^2(\Omega)} = \mathcal{O}(t^{-2}) \|W(t)\|_{\dot{H}_x^{-1} H_y^2(\Omega)}, \text{ as } t \to \pm\infty.$$

Remark 6. We note that for the linearized Euler equations around a shear flow, $\langle \omega \rangle_x$ is a conserved quantity. By linearity of the equation, we may thus restrict to initial data ω_0 such that $\langle \omega \rangle(t,y) = \langle \omega_0 \rangle(y) \equiv 0$. In this case, $\langle v \rangle_x = (c, 0)$ for a constant $c \in \mathbb{R}$, which by Galilean invariance can be chosen to be zero.

However, when considering the nonlinear Euler equations, $\langle \omega \rangle_x$ and $\langle v \rangle_x$ are no longer conserved by the evolution.

Proof. The results are established by using a dual formulation of the kinetic energy and integrating by parts. More precisely, in the infinite channel case, denoting the stream function by ϕ, v satisfies

$$\begin{aligned} \|v - \langle v \rangle_x\|_{L^2(\mathbb{T}_L \times \mathbb{R})}^2 &= \iint_{\mathbb{T}_L \times \mathbb{R}} |\nabla^\perp \phi - \nabla^\perp \langle \phi \rangle_x|^2 = \iint_{\mathbb{T}_L \times \mathbb{R}} |\nabla \phi - \nabla \langle \phi \rangle_x|^2 \\ &= - \iint_{\mathbb{T}_L \times \mathbb{R}} (\phi - \langle \phi \rangle_x) \Delta (\phi - \langle \phi \rangle_x) \\ &= - \iint_{\mathbb{T}_L \times \mathbb{R}} (\phi - \langle \phi \rangle_x)(\omega - \langle \omega \rangle_x), \end{aligned} \tag{6}$$

where we used that ϕ decays sufficiently rapidly for $|y| \to \infty$ and that $\Delta \phi = \omega$. Hence,

$$\|v - \langle v \rangle_x\|_{L^2(\mathbb{T}_L \times \mathbb{R})} \lesssim \sup_{\psi \in H^1(\mathbb{T}_L \times \mathbb{R}), \|\psi\|_{H^1} \leq 1, \langle \psi \rangle_x = 0} \iint_{\mathbb{T}_L \times \mathbb{R}} \psi \omega. \tag{7}$$

It can be shown (see [10, Lemma 3]) that an estimate of this form also holds in the setting of a finite channel, where the supremum is instead taken over elements of $\hat{H}^1 := \{\psi \in H^1(\mathbb{T}_L \times [0,1]) : \psi = 0 \text{ for } y \in \{0,1\}, \langle \psi \rangle_x = 0\}$, i.e.

$$\|v - \langle v \rangle_x\|_{L^2(\mathbb{T}_L \times [0,1])} \lesssim \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \iint_{\mathbb{T}_L \times [0,1]} \psi \omega. \tag{8}$$

Indeed, let ϕ be the stream function corresponding to v. Then

$$\begin{aligned} \nabla^\perp (\phi - \langle \phi \rangle_x) &= v - \langle v \rangle_x, \\ \phi - \langle \phi \rangle_x |_{y=0,1} &= 0, \end{aligned}$$

where we used that on the boundary $y \in \{0,1\}$,

$$0 = v_2 = \partial_x \phi,$$

and hence $\phi - \langle \phi \rangle_x |_{y=0,1} = 0$. An integration by parts as in (6) thus yields no boundary contributions and hence the same estimate.

For simplicity of notation, in the following we use $\hat{H}^1$ to also denote $H^1(\mathbb{T}_L \times \mathbb{R})$, so that both (7) and (8) read the same.

We further introduce $f_k(t,y) := \mathcal{F}_x W(t,k,y)$. Then,

$$\begin{aligned}\|v - \langle v\rangle_x\|_{L^2(\Omega)} &\lesssim \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \left| \iint_\Omega \psi\omega \right| \\ &= \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \left| \sum_{k \neq 0} \int \psi_{-k} f_k e^{iktU(y)} \right|.\end{aligned}$$

We integrate by parts to obtain

$$\int \psi_{-k} f_k e^{iktU(y)} dy = -\int \frac{e^{iktU(y)}}{ikt} \partial_y \left(\frac{\psi_{-k} f_k}{U'} \right) dy, \tag{9}$$

where, in the case of a finite channel, the boundary terms

$$\frac{e^{iktU(y)}}{iktU'(y)} \psi_{-k} f_k \Big|_{y=0}^{1}$$

vanish as ψ vanishes on the boundary. Using the strict monotonicity of U and Hölder's inequality, we thus bound

$$\|v(t) - \langle v\rangle_x\|_{L^2(\Omega)} \lesssim \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \mathcal{O}(t^{-1}) \|W(t)\|_{H_x^{-1} H_y^1} \|\psi\|_{H^1}, \tag{10}$$

which establishes the first statement.

In order to bound v_2, we proceed slightly differently. Note that v_2 satisfies

$$\Delta v_2 = \partial_x \omega. \tag{11}$$

We thus introduce an additional potential ψ (not to be confused with the stream function ϕ) such that

$$\Delta\psi = v_2.$$

In the case of an infinite channel, we require that $\nabla^\perp \psi \in L^2(\mathbb{T}_L \times \mathbb{R})$. For the finite channel we additionally require zero Dirichlet conditions, i.e.

$$\psi = 0, \text{ for } y \in \{0,1\}. \tag{12}$$

Therefore,

$$\begin{aligned}\iint_{\mathbb{T}_L \times [0,1]} \partial_x \omega \psi &= \iint_{\mathbb{T}_L \times [0,1]} \Delta v_2 \psi \\ &= \int_0^1 \psi \partial_x v_2|_{x=0}^{L} dy + \int_{\mathbb{T}_L} \psi \partial_y v_2|_{y=0}^{1} dx - \iint_{\mathbb{T}_L \times [0,1]} \nabla v_2 \cdot \nabla \psi \\ &= -\int_0^1 v_2 \partial_x \psi|_{x=0}^{L} dy - \int_{\mathbb{T}_L} v_2 \partial_y \psi|_{y=0}^{1} dx + \iint v_2 \Delta\psi \\ &= \|v_2\|^2_{L^2(\mathbb{T}_L \times [0,1])},\end{aligned}$$

where we used periodicity in x and that v_2 and ψ vanish whenever $y \in \{0,1\}$. Hence, for both the infinite and finite channels,

$$\|v\|^2_{L^2(\Omega)} = \iint_\Omega \partial_x \omega \psi. \tag{13}$$

Using (13), we compute

$$\begin{aligned}\|v_2\|_{L^2(\Omega)}^2 &= \iint \partial_x \omega \psi = \sum_k \int ike^{iktU(y)} f_k \psi_{-k} \\ &= \sum_k \int \frac{e^{iktU(y)}}{t} \partial_y \left(\frac{f_k \psi_{-k}}{U'} \right).\end{aligned}$$

Integrating by parts once more, we obtain

$$-\frac{1}{t^2} \sum_k \int \frac{e^{iktU}}{ik} \partial_y \left(\frac{1}{U'} \partial_y \left(\frac{f_k \psi_{-k}}{U'} \right) \right)$$

and an additional boundary term in the setting of a finite channel:

$$\frac{1}{t^2} \sum \frac{e^{iktU(y)}}{ikU'} \partial_y \left(\frac{f_k \psi_{-k}}{U'} \right) \Big|_{y=0}^{1}.$$

Using Hölder's inequality, trace estimates and that $\frac{1}{U'} \in W^{2,\infty}(\Omega)$, we hence obtain

$$\|v_2(t)\|_{L^2(\Omega)}^2 \lesssim \mathcal{O}(t^{-2}) \|W(t)\|_{H_x^{-1} H_y^2(\Omega)} \|\psi\|_{H^2(\Omega)}. \tag{14}$$

By classic elliptic regularity theory for the Laplacian, $\|\psi\|_{H^2(\Omega)} \lesssim \|v_2\|_{L^2(\Omega)}$. Thus, dividing by $\|v\|_{L^2(\Omega)}$ yields the result. ☐

Remark 7.
- Assuming that $\|W(t)\|_{H_x^{-1} H_y^2}$ is bounded uniformly in t, we hence obtain damping with the optimal algebraic rates. Furthermore, slightly slower decay still holds if the growth of the norms of $W(t)$ can be adequately controlled. Consider for example the last inequality (14):

$$\|v_2(t)\|_{L^2} \lesssim \mathcal{O}(t^{-2}) \|W(t)\|_{H_x^{-1} H_y^2}.$$

  If $\|W(t)\|_{H_x^{-1} H_y^2}$ grows with a rate of $\mathcal{O}(t^\alpha), \alpha < 2$, then $\|v(t)\|_{L^2} = \mathcal{O}(t^{\alpha-2})$ still decays.
- Analogously to Theorem 2.1, it is possible to interpolate between the two estimates of Theorem 3.1 and hence obtain

$$\|v(t) - \langle v \rangle_x\|_{L^2(\Omega)} = \mathcal{O}(t^{-s}) \|W(t)\|_{H_x^{-1} H_y^s(\Omega)},$$

  for $1 < s < 2$, provided $W(t) \in H_x^{-1} H_y^s(\Omega)$ for all times.

Consider the linearized Euler equations, (3), in either the finite or infinite channel and introduce

$$V_2(t, x, y) := v_2(t, x - tU(y), y).$$

Then W satisfies

$$\partial_t W = U''(y) V_2. \tag{15}$$

Furthermore, since

$$(x, y) \mapsto (x - tU(y), y)$$

is an L^2 isometry,

$$\|V_2\|_{L^2(\Omega)} = \|v_2\|_{L^2(\Omega)}.$$

Integrating (15), sufficient decay of $\|v_2\|_{L^2}$ hence implies a scattering result.

Theorem 3.2 (Scattering). *Let Ω be either the infinite periodic channel or finite periodic channel and let ω be a solution of the linearized Euler equations,* (3)*, on Ω with initial datum $\omega_0 \in L^2_x H^2_y(\Omega)$. Further let U satisfy the assumptions of Theorem* 3.1*, $U'' \in L^\infty(\Omega)$ and suppose that, for all times t, W satisfies*

$$\|W(t) - \langle W \rangle_x\|_{L^2_x H^2_y(\Omega)} < C < \infty.$$

Then there exist asymptotic profiles $W^{\pm\infty} \in L^2_x H^2_y(\Omega)$ such that

$$W(t) \xrightarrow{L^2} W^{\pm\infty},$$

as $t \to \pm\infty$.

Proof. By Duhamel's formula, which in our scattering formulation is just integrating (15), W satisfies

$$W(t) = \omega_0 + \int_0^t U'' V_2(\tau) d\tau. \tag{16}$$

By Theorem 3.1, we control

$$\left\| \int_0^t U'' V_2(\tau) d\tau \right\|_{L^2(\Omega)} \leq \|U''\|_{L^\infty(\Omega)} \int_0^t \mathcal{O}(\tau^{-2}) d\tau.$$

Therefore, the limits $W^{\pm\infty}$ of (16) as $t \to \pm\infty$ exist in $L^2(\Omega)$ and by weak compactness of the unit ball of $L^2_x H^2_y(\Omega)$ and lower semi-continuity, also $W^{\pm\infty} \in L^2_x H^2_y(\Omega)$. $\square$

In the Appendix C, we further generalize the conditional damping results from shear flows, $(x, y) \mapsto (x - tU(y), y)$, to diffeomorphisms Y, which are structurally similar to shear flows.

4. Asymptotic stability for an infinite channel

The results of section 3 have been *conditional* under the assumption that our scattered solution, W, of

$$\begin{aligned} \partial_t \omega + U(y)\partial_x \omega &= U'' v_2, \text{ on } \mathbb{T}_L \times \mathbb{R} \times \mathbb{R} \ni (x, y, t), \\ v_2 &= \partial_x \Delta^{-1} \omega, \\ W(t, x, y) :&= \omega(t, x - tU(y), y) \end{aligned} \tag{17}$$

stays regular in the sense that the L^2, H^1 and H^2 norms of W remain uniformly bounded or at least grow very slowly. As the main result of this section and the following section, we remove this restriction and establish stability and scattering for W.

In the case of L^2 stability, there are classical stability results due to Rayleigh, [17], Fjørtoft, [7, page 132], and Arnold, [2]. However, these results use fundamentally different mechanisms, namely orthogonality, cancellation or convexity, while we use mixing by shearing. In particular, our flows are in general not covered by any of these classical stability results. Furthermore, we show that the shearing mechanism is more robust in the sense that it can also be used to derive stability results in higher Sobolev norms.

Before stating the main result, we introduce coordinate transformations, notation and perform a Fourier transform in x to simplify the equation.

As $U : \mathbb{R} \mapsto \mathbb{R}$ is strictly monotone, it is also bijective and invertible. We hence introduce a change of variables, $y \mapsto z = U(y)$, as well as functions

$$\begin{aligned} f(z) &:= U''(U^{-1}(z)), \\ g(z) &:= U'(U^{-1}(z)). \end{aligned} \tag{18}$$

Here, it is convenient to assume that U' is not only bounded from below but also from above so that the change of variables is bilipschitz. For simplicity of notation, we often also assume that $g > 0$, i.e. U is strictly monotonically increasing, but all described results remain valid for strictly monotonically decreasing U as well.

In the new coordinates, the linearized Euler equations are given by

$$\begin{aligned} \partial_t \omega + z\partial_x \omega &= f(z)\partial_x \phi, \\ (\partial_x^2 + (g(z)\partial_z)^2)\phi &= \omega. \end{aligned} \tag{19}$$

The underlying transport structure hence turns into Couette flow, which is particularly useful for computing derivatives and applications of a Fourier transform. As a trade off, the equation for the stream function is no longer given by the Laplacian. However, the equation is still elliptic *if and only if* g is bounded away from zero, i.e. iff U is strictly monotone.

Changing to a *scattering formulation*, i.e. introducing

$$\begin{aligned} W(t,x,z) &:= \omega(t, x-tz, z), \\ \Phi(t,x,z) &:= \phi(t, x-tz, z), \end{aligned} \tag{20}$$

the left-hand-side of (19) simplifies and we obtain

$$\begin{aligned} \partial_t W &= f(z)\partial_x \Phi, \\ (\partial_x^2 + (g(z)(\partial_z - t\partial_x))^2)\Phi &= W. \end{aligned}$$

We further note that, like Couette flow, the x average $\langle W\rangle_x = \langle \omega\rangle_x$ satisfies

$$\partial_t \langle W\rangle_x = f(z)\langle \partial_x \Phi\rangle \equiv 0$$

and is thus conserved. We may therefore subtract $\langle \omega_0\rangle_x$ from ω_0 and assume that

$$\langle W\rangle_x(t,y) \equiv 0.$$

As f and g do not depend on x, after a Fourier transform in x the system *decouples* and the frequency k plays the role of a parameter

$$\begin{aligned} \partial_t \hat{W} &= f(z)ik\hat{\Phi}, \\ (-k^2 + (g(z)(\partial_z - ikt))^2)\hat{\Phi} &= \hat{W}, \\ (k,y,t) &\in (\mathbb{Z}\setminus\{0\}) \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Furthermore, we adjust the definition of Φ by dividing by k^2, which is well-defined, as we assumed that

$$\langle W\rangle_x(t,z) = \hat{W}(k=0,t,\eta) \equiv 0.$$

Relabeling z as y, we thus obtain the following *linearized Euler equations in scattering formulation*:

$$\begin{aligned}
&\partial_t \hat{W} = \frac{if}{k}\hat{\Phi}, \\
&(-1 + (g(\frac{\partial_y}{k} - it))^2)\hat{\Phi} = \hat{W}, \qquad (21)\\
&(k, y, t) \in (\mathbb{Z} \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}.
\end{aligned}$$

Our main result of this section is given by the following stability theorem, which is proved in subsection 4.3.

Theorem 4.1 (Sobolev stability for the infinite periodic channel)**.** *Let* $s \in \mathbb{N}_0$ *and* $f, g \in W^{s+1,\infty}(\mathbb{R})$ *and suppose that there exists* $c > 0$ *such that*

$$0 < c < g < c^{-1} < \infty.$$

Suppose further that

$$L\|f\|_{W^{s+1,\infty}}$$

is sufficiently small. Then for all $m \in \mathbb{N}_0$ *and* $\omega_0 \in H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})$*, the solution* W *of the linearized Euler equations in scattering formulation,* (21)*, with initial datum* ω_0 *satisfies*

$$\|W(t)\|_{H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})} \lesssim \|\omega_0\|_{H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})}.$$

Remark 8. As (21) decouples with respect to k, in our stability results we actually prove that, for any given k,

$$\|\hat{W}(t, k, \cdot)\|_{H^s_y(\mathbb{R})} \lesssim \|\hat{\omega}_0\|_{H^s_y(\mathbb{R})}.$$

The results for $H^m_x H^s_y$ are then obtained by summing in k. In particular, any result for $L^2_x H^s_y$ can be easily shown to also hold for $H^m_x H^s_y$.

In the following sections, we hence consider k as a fixed parameter in (21) and study the stability of $\hat{W}(t, k, \cdot) \in H^s(\mathbb{R})$.

Remark 9. A main difficulty in establishing stability results such as Theorem 4.1 is that the operator

$$W \mapsto \Phi,$$

interpreted as an operator from L^2 to L^2, does not improve in time, as multiplication by e^{ikty} is a unitary operation. More precisely, for any given k, the operator norm of the solution operator to

$$e^{-ikty}(-k^2 + (g\partial_y)^2)e^{ikty} \qquad (22)$$

is independent of time. As a consequence, the uniform damping results of section 3 necessarily sacrifice regularity in order to obtain uniform decay. In the proof of Theorem 4.1, we therefore have to use the more subtle mode-wise decay, where for each fixed frequency, (k, η), the solution operator of (22) decays with rate $\mathcal{O}(|\eta - kt|^{-2})$.

In the following, we first introduce the mechanism of our proof in a simplified setting of a constant coefficient model, for which we can also compute the solution explicitly. Using a perturbation argument, we establish L^2 stability for the general setting in section 4.2 and subsequently extend the result to higher Sobolev norms in section 4.3.

4.1. **A constant coefficient model.** In order to obtain a better understanding of the dynamics of the linearized Euler equations, in the following we consider a simplified model. Here, we formally replace $f(y)$ and $g(y)$ in (21) by constants to recover the decoupling:

$$\begin{aligned} \partial_t \Lambda &= c\Psi, \\ (-1+(\frac{\partial_y}{k}-it)^2)\Psi &= \Lambda, \\ \Lambda|_{t=0} &= \hat{\omega}_0(k,\cdot), \\ (k,y,t) &\in L(\mathbb{Z}\setminus\{0\})\times\mathbb{R}\times\mathbb{R}. \end{aligned} \tag{CC}$$

Here, $c \in \mathbb{C}$ should be thought of as small and not necessarily imaginary. For simplicity of notation, we choose the constant in front of $(\frac{\partial_y}{k}-it)^2$ to be 1. In general, $\min(g^2)>0$ is the natural choice.

Like the linearized Euler equations in scattering formulation, (21), the model problem, (CC), decouples with respect to k (cf. Remark 8). In the following, we hence write $\Lambda(t) \in H^s = H^s(\mathbb{R})$ to denote that, for given k,

$$\Lambda(t,k,\cdot) \in H^s(\mathbb{R}).$$

Estimates in the Sobolev spaces $H^m_x H^s_y(\mathbb{T}_L\times\mathbb{R})$ can then be obtained by summing in k.

By our choice of constant coefficients in (CC), the model problem further decouples after a Fourier transform in y and is explicitly solvable:

Theorem 4.2. *Let $\omega_0 \in L^2$. Then the solution of the constant coefficient problem,* (CC)*, is given by*

$$\Lambda = \mathcal{F}^{-1}\exp\left(c\left(\arctan(\frac{\eta}{k}-t)-\arctan(\frac{\eta}{k})\right)\right)\mathcal{F}\omega_0. \tag{23}$$

In particular, for any $s\in\mathbb{N}$ such that $\omega_0 \in H^s$, also $\Lambda(t)\in H^s$ and

$$\|\Lambda(t)\|_{H^s} \leq e^{|c|\pi}\|\hat{\omega}_0(k,\cdot)\|_{H^s}$$

uniformly in time.

Remark 10. An estimate by $\pi|\Re(c)|$ would of course also be possible in this case. However, dropping the imaginary part of c corresponds to using anti-symmetry and orthogonality, which is more difficult to employ in the variable coefficient setting. As we seek to obtain a robust strategy, we therefore limit ourselves to using the shearing mechanism only.

While the constant coefficient case allows for an explicit solution, in the general case a more indirect proof is required, which we introduce in the following.

The underlying method of our proof is to introduce a weight that decreases at the right places at a large enough rate to counter potential growth. This method of proof is reminiscent of integrating factors in ODE theory and is sometimes called *ghost energy*, [1]. Recent applications of similar methods can, for example, be found in a more sophisticated form in the work of [4].

For simplicity of notation, in the following we assume that $c>0$ in order to avoid writing absolute values.

Theorem 4.3. *Let $c > 0$ and let Λ be the solution of the constant coefficient problem,* (CC), *with initial data ω_0. Let $C > 0$ and define*

$$E(t) := \langle \Lambda, \mathcal{F}_\eta^{-1} \exp\left(C \arctan(\frac{\eta}{k} - t)\right) \mathcal{F}_y \Lambda\rangle =: \langle \Lambda, A(t)\Lambda\rangle. \tag{24}$$

Then for $|c| \ll C$ sufficiently small, $E(t)$ is nonincreasing and uniformly comparable to $\|W(t)\|_{L^2}^2$. In particular,

$$e^{-C\pi}E(t) \leq \|\Lambda(t)\|_{L^2}^2 \leq e^{C\pi}E(t) \leq e^{C\pi}E(0) \leq e^{2C\pi}\|\omega_0\|_{L^2}^2. \tag{25}$$

Remark 11. As can be seen from the explicit solution, the assumptions of Theorem 4.3 and the factors in (25) are not optimal for our decoupling model. For example, even for large c, choosing $C \geq c$ would work. However, in the general case, we additionally have to control the commutator of A and multiplication by $\frac{f}{ik}$. Hence, at least for finite times, we cannot avoid incurring an operator norm, $e^{C\pi}$, and thus a condition of the form

$$c < Ce^{-C\pi},$$

which does not improve for large C. This is discussed in more detail in section 4.2. Therefore, we think of C as approximately 1 and require c to be small.

Proof of Theorem 4.3. We compute the time-derivative of $E(t)$:

$$\partial_t E(t) = \langle \Lambda, \dot{A}\Lambda\rangle + 2\Re\langle A(t)\Lambda, c\Psi\rangle. \tag{26}$$

By our choice of A, $\dot{A}$ is a negative semidefinite symmetric operator. For the proof of our theorem it hence suffices to show that

$$\langle \Lambda, \dot{A}\Lambda\rangle \leq 0$$

is negative enough to absorb the possible growth of

$$|2\Re\langle A(t)\Lambda, c\Psi\rangle|.$$

This therefore ensures that $\partial_t E(t) \leq 0$.

Using Plancherel, it suffices to show that

$$\int_{\mathbb{R}} \frac{-Ce^{C\arctan(\frac{\eta}{k}-t)}}{1+(\frac{\eta}{k}-t)^2}|\tilde{\Lambda}(t,k,\eta)|^2 d\eta + 2\int_{\mathbb{R}} \Re(c)\frac{e^{C\arctan(\frac{\eta}{k}-t)}}{1+(\frac{\eta}{k}-t)^2}|\tilde{\Lambda}(t,k,\eta)|^2 d\eta \leq 0, \tag{27}$$

for arbitrary functions $|\tilde{\Lambda}(t,k,\eta)|$, which in this case holds if

$$2|c| \leq C.$$

□

4.2. **L^2 stability for monotone shear flows.** In the following, we adapt the L^2 stability result, Theorem 4.3 of section 4.1, to the linearized Euler equations in scattering formulation, (21),

$$\begin{aligned} \partial_t W &= \frac{if}{k}\Phi, \\ (-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi &= W, \\ (k,y,t) &\in L(\mathbb{Z}\setminus\{0\}) \times \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{28}$$

where for simplicity we dropped the hats, $\hat{\cdot}$, from our notation. As noted in Remark 8, (28) decouples with respect to k. For the remainder of this article, we thus follow the same convention as in section 4.1 and use $W \in H^s(\mathbb{R})$ to denote that, for given k,

$$W(t,k,\cdot) \in H^s(\mathbb{R}).$$

In analogy to the constant coefficient model, (CC), for a given solution W of (28), we introduce the *constant coefficient stream function* Ψ:

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi = W. \tag{29}$$

We stress that, starting from this section, Ψ does not correspond to a solution of the constant coefficient problem, (CC), but only to a given right-hand-side W in (29).

More generally, we introduce the following notation:

Definition 4.1 (Constant coefficient stream function)**.** Let $k \in L(\mathbb{Z} \setminus \{0\})$ and let $R(t) \in L^2(\mathbb{R})$ be a given function. Then the *constant coefficient stream function*, $\Psi[R](t)$, is defined as the solution of

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[R](t,y) = R(t,y). \tag{30}$$

Further let W be a solution of (28); then for any k, t,

$$\Psi(t,k,y) := \Psi[W(t,k,\cdot)](t,y). \tag{31}$$

Since Φ and $\Psi = \Psi[W]$ satisfy very similar (shifted elliptic) equations, (28) and (29), with the same right-hand-side, we can estimate Sobolev norms of Φ in terms of Ψ, as is shown in Lemma 4.1. We note that, for this purpose, W need not solve (28), but can be any given L^2 function.

Lemma 4.1. *Let $\frac{1}{g} \in W^{1,\infty}$ and assume there exists $c > 0$ such that*

$$0 < c < g < c^{-1} < \infty.$$

Then for any $W(t) \in L^2(\mathbb{R})$, the solutions Φ, Ψ of

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi = W, \tag{32}$$

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi = W \tag{33}$$

satisfy

$$\|\Phi\|_{\tilde{H}^1}^2 := \|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2 \lesssim \|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2.$$

In the following, we establish L^2 stability of (28) using Lemma 4.1 and subsequently give a proof of Lemma 4.1.

Theorem 4.4 (L^2 stability for the infinite periodic channel)**.** *Let W be a solution to the linearized Euler equations,* (28)*, and assume that g satisfies the assumptions of Lemma* 4.1*. Further let A be defined as in Theorem* 4.3*, i.e.*

$$I(t) := \langle W, A(t)W \rangle_{L^2(\mathbb{R})} := \int |\tilde{W}(t,k,\eta)|^2 \exp\Big(C \arctan(\frac{\eta}{k} - t)\Big)\, d\eta, \tag{34}$$

and suppose that

$$\|f\|_{W^{1,\infty}} L$$

is sufficiently small. Then, for any initial datum $\omega_0 \in L^2(\mathbb{R})$, $I(t)$ *is nonincreasing and satisfies*

$$\|W(t)\|_{L^2}^2 \lesssim I(t) \leq I(0) \lesssim \|\omega_0\|_{L^2}^2.$$

Proof of Theorem 4.4. Let $\Psi[AW]$ be as in Definition 4.1; i.e. $\Psi[Aw] \in L^2$ is the solution of

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[AW] = AW.$$

Then, by integration by parts, the time-derivative of $I(t)$ satisfies

$$\begin{aligned} \partial_t I(t) &= \langle W, \dot{A}W\rangle + 2\Re\langle AW, \frac{if}{k}\Phi\rangle \\ &\leq \langle W, \dot{A}W\rangle + 2\|\frac{f}{k}\|_{W^{1,\infty}} \|\Psi[AW]\|_{\tilde{H}^1} \|\Phi\|_{\tilde{H}^1}. \end{aligned} \tag{35}$$

By Lemma 4.1, the last term is further controlled by

$$C_1 \|\frac{f}{k}\|_{W^{1,\infty}} \|\Psi[AW]\|_{\tilde{H}^1} \|\Psi\|_{\tilde{H}^1}. \tag{36}$$

As A is a bounded Fourier multiplier and commutes with the Fourier multiplier $u \mapsto \Psi[u]$, we control

$$\|\Psi[AW]\|_{\tilde{H}^1} \leq \|A\| \|\Psi\|_{\tilde{H}^1} \leq \|A\| \sqrt{|\langle W, \Psi\rangle|}, \tag{37}$$

where we used that

$$\|\Psi\|_{\tilde{H}^1}^2 := \|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2 = -\langle W, \Psi\rangle. \tag{38}$$

Furthermore,

$$\begin{aligned} -\langle W, A\Psi\rangle &= -\langle(-k^2 + (\frac{\partial_y}{ik} - t)^2)\Psi, A\Psi\rangle \\ &= \int (k^2 + (\frac{\eta}{k} - t)^2) \exp\left(C \arctan(\frac{\eta}{k} - t)\right) |\tilde{\Psi}(t, k, \eta)|^2 d\eta. \end{aligned} \tag{39}$$

Therefore,

$$\|\Psi\|_{\tilde{H}^1}^2 \leq \|A\|(-\langle W, A\Psi\rangle) \leq \|A\|^2 \|\Psi\|_{\tilde{H}^1}^2,$$

where we used that A^{-1} has the same operator norm as A. Thus, (36) is further controlled by

$$C_1 \|\frac{f}{k}\|_{W^{1,\infty}} \|A\|^2 |\langle W, A\Psi\rangle|. \tag{40}$$

Hence, combining (35) and (40), $I(t)$ satisfies

$$\partial_t I(t) \leq \langle W, \dot{A}W\rangle + C_2 \|A\|^2 \|\frac{f}{k}\|_{W^{1,\infty}} |\langle W, A\Psi[W]\rangle|.$$

Using the explicit characterization of A and Ψ in Fourier space, we conclude as in the proof of Theorem 4.3, provided

$$c := C_2 \|f\|_{W^{1,\infty}} \|A\|^2 \sup_{k \neq 0} \frac{1}{|k|} \lesssim e^{2C\pi} \|f\|_{W^{1,\infty}} L \tag{41}$$

is sufficiently small. □

Proof of Lemma 4.1. Testing (32) with $\frac{1}{g}\Phi$ and integrating by parts, we obtain

$$\int \frac{1}{g}|\Phi|^2 + g|(\frac{\partial_y}{k} - it)\Phi|^2 = \langle W, \frac{1}{g}\Phi\rangle. \tag{42}$$

As by our assumption, $c < g < c^{-1}$, the left-hand-side is bounded from below by

$$c(\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2) \gtrsim \|\Phi\|_{\tilde{H}^1}^2.$$

Hence, it remains to estimate $\langle W, \frac{1}{g}\Phi\rangle$ from above.

Using (33) and integrating by parts, we obtain

$$\begin{aligned}\langle W, \frac{1}{g}\Phi\rangle =& \left\langle (-1 + (\frac{\partial_y}{k} - it)^2)\Psi, \frac{1}{g}\Phi \right\rangle \\ \leq& \sqrt{\|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2}\sqrt{\|\frac{1}{g}\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\frac{1}{g}\Phi\|_{L^2}^2} \\ \lesssim& \|\frac{1}{g}\|_{W^{1,\infty}}\sqrt{\|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2}\sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2}.\end{aligned}$$

Dividing by $\|\Phi\|_{\tilde{H}^1} = \sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2}$, we thus obtain the result. $\square$

Remark 12. Testing (32) with Φ instead of $\frac{1}{g}\Phi$ has the small drawback of introducing commutators involving gg' on the left-hand-side, which one can control either by a smallness or sign condition. The right-hand-side however is simplified.

Testing (33) with Ψ and integrating

$$\langle W, \Psi\rangle = \langle (-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi, \Psi\rangle \tag{43}$$

by parts, we analogously obtain that

$$\|\Psi\|_{\tilde{H}^1} \lesssim \|\Phi\|_{\tilde{H}^1}.$$

One can more generally show that, up to a factor, both Φ and Ψ attain

$$\|W\|_{\tilde{H}^{-1}} := \sup\{\langle W, \mu\rangle_{L^2} : \|\mu\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\mu\|_{L^2}^2 \leq 1\}.$$

Remark 13. It is possible to reduce the requirements of Theorem 4.4 for large $\|A\|$ slightly by noting that

$$\Psi[AW] = A\Psi[W],$$

as Fourier multipliers commute and that, as a positive multiplier, we can split $A = A^{1/2}A^{1/2}$ for the purpose of our L^2 bound. Hence, in (35), instead of estimating

$$2\Re\langle AW, \frac{if}{k}\Phi\rangle \lesssim C_1\|\frac{f}{k}\|_{W^{1,\infty}}\|A\|^2|\langle W, A\Psi\rangle|,$$

it suffices to obtain an estimate of the form

$$\|A^{1/2}\frac{if}{k}\Phi\|_{\tilde{H}^1} \lesssim \|A^{1/2}\Psi\|_{\tilde{H}^1}.$$

However, we note that for nonconstant f, even for $\Phi = \Psi$, such an estimate would have to control

$$A^{1/2}fA^{-1/2} \tag{44}$$

as an operator from $\tilde{H}^1$ to $\tilde{H}^1$. Asymptotically, i.e. for $t \to \pm\infty$, $\arctan(\eta - t) \to \pm\frac{\pi}{2}$ and thus $A^{\pm 1} \rightharpoonup e^{\pm C\frac{\pi}{2}} Id$. Therefore, for all C,

$$A^{1/2} f A^{-1/2} \rightharpoonup f,$$

as $t \to \pm\infty$. However, for each finite time we obtain commutators involving

$$C(\arctan(\eta_1 - t) - \arctan(\eta_2 - t)),$$

which are not bounded uniformly in C. Hence, at least for finite times, the operator norm corresponding to (44) is not better than

$$e^{c_1 C} \|f\|_{W^{1,\infty}}$$

for some $c_1 > 0$, and thus only provides a small improvement over (41).

4.3. **Iteration to arbitrary Sobolev norms.** Thus far we have only shown L^2 stability. In order to derive damping, it remains to extend the result to ensure stability in higher Sobolev norms.

In the constant coefficient model, this generalization is trivial as our equation is invariant under taking derivatives. Hence, after relabeling, we may apply the L^2 result to $\partial_y^s \Lambda$.

Corollary 4.1. *Let $s \in \mathbb{N}$, $\omega_0 \in H^s(\mathbb{R})$ and let Λ be the solution of the constant coefficient problem,* (CC)*, with initial data ω_0. Then $\partial_y^s \Lambda$ solves the constant coefficient problem,* (CC)*, with initial data $\partial_y^s \omega_0$ and for $c < Ce^{-C\pi}$,*

$$\|\partial_y^s \Lambda\|_{L^2} \lesssim \|\partial_y^s \omega_0\|_{L^2}.$$

When taking derivatives of the linearized Euler equations, we obtain additional lower order corrections due to commutators. More precisely, for given $j \in \mathbb{N}$, $\partial_y^j W$ satisfies

$$\begin{aligned} \partial_t \partial_y^j W &= \frac{i}{k} \partial_y^j (f\Phi) =: \frac{i}{k} \sum_{j' \leq j} c_{jj'} (\partial_y^{j-j'} f) \partial_y^{j'} \Phi, \\ (-1 + (g(\frac{\partial_y}{k} - it))^2) \partial_y^{j'} \Phi &= \partial_y^{j'} W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^{j'}] \Phi. \end{aligned} \tag{45}$$

In order to control these corrections, we introduce a family of energies

$$I_j(t) = \langle \partial_y^j W, A \partial_y^j W \rangle \tag{46}$$

and a combined energy

$$E_j(t) = \sum_{j' \leq j} I_{j'}(t). \tag{47}$$

With this notation our main theorem is:

Theorem 4.5 (Sobolev stability for the infinite periodic channel)**.** *Let $j \in \mathbb{N}$ and assume f, g satisfy the assumptions of Theorem* 4.4*, $f, g \in W^{j+1,\infty}(\mathbb{R})$ and that*

$$\|f\|_{W^{j+1,\infty}} L$$

is sufficiently small. Then for any initial datum $\omega_0 \in H^j(\mathbb{R})$, $E_j(t)$ is nonincreasing and satisfies

$$\|W(t)\|_{H^j}^2 \lesssim E_j(t) \leq E_j(0) \lesssim \|\omega_0\|_{H^j}^2.$$

As in the previous proof, we compare with constant coefficient potentials Ψ:

Lemma 4.2. *Let $j \in \mathbb{N}$ and let g satisfy the assumptions of Theorem* 4.5. *Then,*

$$\|\partial_y^j \Phi\|_{\tilde{H}^1} \lesssim \sum_{j' \leq j} \|\partial_y^{j'} \Psi\|_{\tilde{H}^1}.$$

Proof of Theorem 4.5. For any $j' \leq j$, $I_{j'}$ satisfies

$$\begin{aligned} \partial_t I_{j'}(t) &= \langle \partial_y^{j'} W, \dot{A} \partial_y^{j'} W \rangle + \langle A \partial_y^{j'} W, \partial_y^{j'} \frac{if}{k} \Phi \rangle \\ &\leq \langle \partial_y^{j'} W, \dot{A} \partial_y^{j'} W \rangle + \|\Psi[A \partial_y^{j'} W]\|_{\tilde{H}^1} \left\| \frac{f}{k} \right\|_{W^{j'+1,\infty}} \sum_{j'' \leq j'} \|\partial_y^{j''} \Phi\|_{\tilde{H}^1}. \end{aligned} \tag{48}$$

Summing over all $j' \leq j$ and using Lemma 4.2 and Young's inequality, we hence obtain

$$\begin{aligned} \partial_t E_j(t) &\leq \sum_{j' \leq j} \langle \partial_y^{j'} W, \dot{A} \partial_y^{j'} W \rangle + \left\| \frac{f}{k} \right\|_{W^{j+1,\infty}} \left(\sum_{j' \leq j} \|\Psi[A \partial_y^{j'} W]\|_{\tilde{H}^1}^2 + \|\partial_y^{j'} \Phi\|_{\tilde{H}^1}^2 \right) \\ &\lesssim \sum_{j' \leq j} \langle \partial_y^{j'} W, \dot{A} \partial_y^{j'} W \rangle + \left\| \frac{f}{k} \right\|_{W^{j+1,\infty}} \left(\sum_{j' \leq j} \|\Psi[A \partial_y^{j'} W]\|_{\tilde{H}^1}^2 + \|\partial_y^{j'} \Psi\|_{\tilde{H}^1}^2 \right). \end{aligned} \tag{49}$$

We further note that $\partial_y^{j'} \Psi = \Psi[\partial_y^{j'} W]$. Hence, relabeling and applying the constant coefficient L^2 result, Theorem 4.3, we obtain that for any j' and for c sufficiently small

$$\langle \partial_y^{j'} W, \dot{A} \partial_y^{j'} W \rangle + c(\|\Psi[\partial_y^{j'} W]\|_{\tilde{H}^1}^2 + \|\Psi[A \partial_y^{j'} W]\|_{\tilde{H}^1}^2) \leq 0. \tag{50}$$

Supposing that

$$\sup_{k \neq 0} \|\frac{f}{k}\|_{W^{j+1,\infty}} = \|f\|_{W^{j+1,\infty}} L \ll c,$$

summing (50) with respect to j and (49) hence implies that

$$\partial_t E_j(t) \leq 0, \tag{51}$$

which concludes our proof. $\square$

Proof of Lemma 4.2. We prove the result by induction in j. The case $j = 0$ has been proven as Lemma 4.1 in section 4.2. Hence, it suffices to show the induction step $j - 1 \mapsto j$:

$$\|\partial_y^j \Phi\|_{\tilde{H}^1} \lesssim \|\partial_y^j \Psi\|_{\tilde{H}^1} + \sum_{j' \leq j-1} \|\partial_y^{j'} \Phi\|_{\tilde{H}^1}, \tag{52}$$

for $j \geq 1$.

Recall that $\partial_y^j \Phi$ satisfies (45):

$$(-1 + (g(\frac{\partial_y}{k} - it))^2) \partial_y^j \Phi = \partial_y^j W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^j] \Phi.$$

Proceeding as in the proof of Lemma 4.1, we thus test (45) with $\frac{1}{g} \partial_y^j \Phi$ to obtain an estimate by

$$\|\partial_y^j \Phi\|_{\tilde{H}^1}^2 \lesssim \|\partial_y^j \Psi\|_{\tilde{H}^1} \|\partial_y^j \Phi\|_{\tilde{H}^1} + \langle \partial_y^j \Phi, [(g(\frac{\partial_y}{k} - it))^2, \partial_y^j] \Phi \rangle, \tag{53}$$

where we used that

$$|\langle \partial_y^j \Phi, \partial_y^j W \rangle| = |\langle \partial_y^j \Phi, (-1 + (\frac{\partial_y}{k} - it)^2)\partial_y^j \Psi \rangle| \leq \|\partial_y^j \Psi\|_{\tilde{H}^1} \|\partial_y^j \Phi\|_{\tilde{H}^1}. \tag{54}$$

In order to estimate the contribution of the commutator,

$$[(g(\frac{\partial_y}{k} - it))^2, \partial_y^j]\Phi, \tag{55}$$

we note that at least one of the derivatives ∂_y^j has to fall on the coefficient function g. Hence, (55) can be expressed in terms of

$$\partial_y^{j'} \Phi, \ (g(\frac{\partial_y}{k} - it))\partial_y^{j'} \Phi$$

and

$$(g(\frac{\partial_y}{k} - it))^2 \partial_y^{j'} \Phi, \tag{56}$$

with $j' \leq j - 1$. Integrating $(\frac{\partial_y}{k} - it)$ by parts in the case (56), (53) is thus further estimated by

$$\|\partial_y^j \Phi\|_{\tilde{H}^1}^2 \lesssim \|\partial_y^j \Psi\|_{\tilde{H}^1} \|\partial_y^j \Phi\|_{\tilde{H}^1} + C(g)\|\partial_y^j \Phi\|_{\tilde{H}^1} \sum_{j' \leq j-1} \|\partial_y^{j'} \Phi\|_{\tilde{H}^1}, \tag{57}$$

where $C(g)$ depends on all derivatives of g up to order j.

Dividing (57) by $\|\partial_y^j \Phi\|_{\tilde{H}^1}$ hence proves the induction step, (52), and concludes our proof. □

As we discuss in section 6, Theorem 4.5 in particular provides a uniform control of

$$\|W\|_{L_x^2 H_y^2(\mathbb{T}_L \times \mathbb{R})},$$

and hence allows us to close our strategy and thus prove linear inviscid damping with the optimal decay rates for a large class of monotone shear flows in an infinite periodic channel. Furthermore, as discussed in section 3, as a consequence of sufficiently fast damping, we obtain a scattering result via Duhamel's formula.

Prior to this, however, in section 5 we prove a similar stability result in the case of a finite channel $\mathbb{T}_L \times [0, 1]$ with impermeable walls. There, boundary effects are shown to have a nonnegligible effect on the dynamics.

5. Asymptotic stability for a finite channel

Inspired by the Fourier proof in the whole space case, in the following we establish stability in the setting of a finite periodic channel $\mathbb{T}_L \times [a, b]$ (see Figure 5). The physically natural boundary conditions in this setting are that the boundary in y is impermeable:

$$v_2 = 0, \quad \text{for } y \in \{a, b\}. \tag{58}$$

As the stream function ϕ satisfies

$$v_2 = \partial_x \phi,$$

this, in particular, implies that ϕ restricted to the boundary only depends on time.

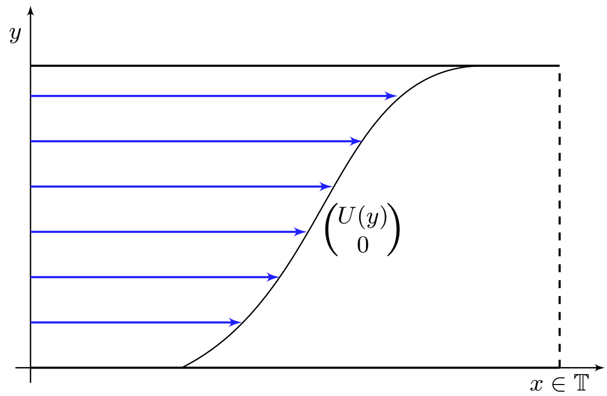

FIGURE 5. An example of a shear flow in a finite periodic channel.

Following the same reduction steps as in section 4.1, in particular removing the mean $\langle W \rangle_x$, ϕ and thus Φ vanishes identically on the boundary. The linearized Euler equations in scattering formulation are hence given by

$$
\begin{aligned}
\partial_t W &= \frac{if(y)}{k}\Phi, \\
(-1 + (g(y)(\frac{\partial_y}{k} - it))^2)\Phi &= W, \\
\Phi|_{y=U(a),U(b)} &= 0, \\
(t,k,y) &\in \mathbb{R} \times L(\mathbb{Z} \setminus \{0\}) \times [U(a), U(b)].
\end{aligned}
\tag{59}
$$

In order to simplify notation, we translate in y and rescale L by a factor (using Galilean symmetry and the scaling symmetry of the (linearized) Euler equations) to reduce to $[U(a), U(b)] = [0, 1]$.

As in section 4 (cf. Remark 8), the equations (59) decouple with respect to k. Hence, in the following we again consider k as a given parameter and write $W(t) \in H^s$ to denote that

$$W(t, k, \cdot) \in H^s([0, 1]).$$

Our main result is given by the following theorem and proved in section 5.3.

Theorem 5.1. *Let W be a solution of* (59), $f, g \in W^{3,\infty}([0, 1])$, *and suppose that there exists $c > 0$ such that*

$$0 < c < g < c^{-1} < \infty.$$

Suppose further that

$$\|f\|_{W^{1,\infty}} L$$

is sufficiently small.

Then, for any $\omega_0 \in H^2([0, 1])$ with $\omega_0|_{y=0,1} = 0$ and for any time t,

$$\|W(t)\|_{H^2} \lesssim \|\omega_0\|_{H^2}.$$

As we show in the following, the case of a finite channel is not only technically more involved, due to the lack of Fourier methods as well as the loss of the multiplier structure for Φ (even for Couette flow), but the qualitative behavior also changes due to boundary effects.

When differentiating the equation, $\partial_y^n \Phi$ satisfies nonzero Dirichlet boundary conditions. Computing the boundary conditions explicitly, we show that asymptotic H^2 stability holds if and only if ω_0 satisfy zero Dirichlet conditions, $\omega_0|_{y=0,1} = 0$. Higher Sobolev norms in turn would require even stronger conditions, as we discuss in Appendix B.

As the damping results provide the sharp algebraic decay rates already for H^2 regularity, we restrict ourselves to considering only L^2, H^1 and H^2 stability.

5.1. L^2 **stability via shearing.** As in section 4.2, we consider the linearized Euler equations, (59), this time in the finite periodic channel, $\mathbb{T}_L \times [0,1]$,

$$
\begin{aligned}
\partial_t W &= \frac{if(y)}{k}\Phi, \\
\left(-1 + \left(g(y)\left(\frac{\partial_y}{k} - it\right)\right)^2\right)\Phi &= W, \\
\Phi|_{y=0,1} &= 0, \\
(t,k,y) &\in \mathbb{R} \times L(\mathbb{Z}\setminus\{0\}) \times [0,1],
\end{aligned}
\tag{60}
$$

and additionally introduce the constant coefficient stream function Ψ

$$
\begin{aligned}
\left(-1 + \left(\frac{\partial_y}{k} - it\right)^2\right)\Psi &= W, \\
\Psi_{y=0,1} &= 0.
\end{aligned}
\tag{61}
$$

As in Definition 4.1 of section 4.2, we introduce constant coefficient stream functions for a given right-hand-side, where we additionally prescribe boundary conditions:

Definition 5.1 (Constant coefficient stream function for a finite periodic channel)**.** Let $k \in L(\mathbb{Z}\setminus\{0\})$ and let $R(t) \in L^2([0,1])$ be a given function. Then the *constant coefficient stream function*, $\Psi[R](t)$, is defined as the solution of

$$
\begin{aligned}
(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[R](t,y) &= R(t,y), \\
\Psi[R](t,y)|_{y=0,1} &= 0.
\end{aligned}
\tag{62}
$$

Further let W be a solution of (60); then for any k, t, we define

$$
\Psi(t,k,y) := \Psi[W(t,k,\cdot)](t,y).
\tag{63}
$$

If we considered periodic boundary conditions in a Fourier expansion, $\Psi[\cdot]$ would again be given by a multiplier and could be estimated explicitly in the same way as in the setting of an infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$. As we however have zero Dirichlet conditions, we can no longer solve the evolution of a constant coefficient model explicitly, but rather have to establish control of boundary effects and growth of norms, using more indirect methods. Thus, stability results are already nontrivial even for a constant coefficient model.

Emulating the proof of the L^2 stability with a decreasing weight $A(t)$ as in section 4.2, a natural replacement for the Fourier transform is given by the expansion in an L^2-basis (e_n).

In view of our zero Dirichlet conditions a natural choice of such a basis is

$$\sin(ny), n \in \mathbb{N}.$$

For the current purpose of L^2 stability, however, it is advantageous to instead consider an expansion in the Fourier basis

$$e^{iny}, n \in 2\mathbb{Z},$$

for which calculations greatly simplify, at the cost of worse mapping properties in higher Sobolev spaces. This trade-off and the role of the choice of basis are discussed in more detail in Appendix A.

In the following we introduce several lemmata, which allow us to prove L^2 stability in Theorem 5.2:

- Lemma 5.3 provides a definition of a decreasing weight A, as in Theorem 4.3, and proves that the constant coefficient stream function Ψ can be controlled in terms of this weight. In the case of an infinite channel as in section 4, this result immediately followed from the explicit Fourier characterization. In the setting of a finite channel, however, additional boundary effects have to be controlled, which is accomplished by the basis computations in Lemmata 5.1 and 5.2.
- Lemma 5.4 provides an estimate of Φ in terms of Ψ and hence a reduction similar to Lemma 4.1 of section 4.2.

Lemma 5.1. *Let $n \in 2\pi\mathbb{Z}$ and let $\Psi[e^{iny}]$ be given by Definition* 5.1*; i.e. let $\Psi[e^{iny}]$ be the solution of*

$$\begin{aligned}(-k^2 + (\partial_y - ikt)^2)\Psi[e^{iny}] &= e^{iny},\\ \Psi[e^{iny}]|_{y=0,1} &= 0.\end{aligned}$$

Then, for any $m \in 2\pi\mathbb{Z}$,

$$\langle \Psi[e^{iny}], e^{imy}\rangle = \frac{\delta_{nm}}{k^2 + (n-kt)^2} + \frac{k}{(k^2 + (m-kt)^2)(k^2+(n-kt)^2)}(a-b),$$

where a, b solve

$$\begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = -\begin{pmatrix} 1 \\ 1\end{pmatrix}.$$

Lemma 5.2. *Let $\Psi, W \in L^2$ solve*

$$\begin{aligned}(-k^2 + (\partial_y - ikt)^2)\Psi &= W,\\ \Psi|_{y=0,1} &= 0.\end{aligned}$$

Denote the basis expansion of W with respect to e^{iny}, $n \in 2\pi\mathbb{Z}$, by

$$W(y) = \sum_n W_n e^{iny}.$$

Then W satisfies

$$|\langle W, \Psi\rangle| \lesssim k^{-2}\sum_n \langle \frac{n}{k} - t\rangle^{-2} |W_n|^2.$$

Lemma 5.3. *Define the operator* $A(t)$ *by*

$$A(t) : e^{iny} \mapsto \exp\left(-\int^t \langle \frac{n}{k} - \tau\rangle^{-2} d\tau\right) e^{iny} = \exp\left(\arctan\left(\frac{n}{k} - t\right)\right) e^{iny}.$$

Then $A : L^2 \to L^2$ *is a uniformly bounded, symmetric, positive operator and satisfies*

$$\|W\|_{L^2}^2 \lesssim \langle W, AW\rangle \lesssim \|W\|_{L^2}^2,$$

where the estimates are uniform in t*. Furthermore, the time derivative* $\dot{A}$ *is symmetric and nonpositive and there exists a constant* $C > 0$ *such that, for* Ψ *as in Definition* 5.1*,*

$$|\langle W, A\Psi\rangle| C + \langle W, \dot{A}W\rangle \leq 0.$$

Lemma 5.4. *Let* $W \in L^2$*,* $0 < c < g < c^{-1} < \infty$*,* $\frac{1}{g(y)}, f(y) \in W^{1,\infty}$ *and* A *as in Lemma* 5.3*. Let* W, Φ, Ψ *solve*

$$\begin{aligned} (k^2 + (g(\partial_y - ikt))^2)\Phi &= W, \\ \Phi_{y=0,1} &= 0, \end{aligned} \tag{64}$$

$$\begin{aligned} (k^2 + (\partial_y - ikt)^2)\Psi &= W, \\ \Psi_{y=0,1} &= 0. \end{aligned} \tag{65}$$

Then there exists a constant C *such that*

$$|\langle AW, \frac{if}{k}\Phi\rangle| \leq \frac{C}{k} |\langle AW, \Psi\rangle|.$$

With these lemmata we can now prove L^2 stability.

Theorem 5.2 (L^2 stability for the finite periodic channel)**.** *Let* $f, g \in W^{1,\infty}([0, 1])$ *and suppose that there exists* $c > 0$ *such that*

$$0 < c < g < c^{-1} < \infty.$$

Suppose further that

$$L\|f\|_{W^{1,\infty}}$$

is sufficiently small. Then for all $\omega_0 \in L^2$*, the solution* W *of the linearized Euler equations,* (59)*, with initial datum* ω_0*, for any time* t*, satisfies*

$$\|W(t)\|_{L^2} \lesssim \|\omega_0\|_{L^2}.$$

Proof of Theorem 5.2. The time derivative of $I(t) := \langle W, AW\rangle$ is controlled by

$$2|\langle AW, \frac{if}{k}\Phi\rangle| + \langle W, \dot{A}W\rangle.$$

By Lemma 5.4 there exists a constant C_1 such that

$$\dot{I}(t) \leq \frac{C_1}{|k|} |\langle AW, \Psi\rangle| + \langle W, \dot{A}W\rangle.$$

Requiring $|k|$ to be sufficiently large, $\frac{C_1}{|k|} \leq C$. Thus, Lemma 5.3 yields

$$\dot{I}(t) \leq |\langle W, A\Psi[W]\rangle| C + \langle W, \dot{A}W\rangle \leq 0.$$

In particular,

$$\|W(t)\|_{L^2}^2 \lesssim I(t) \leq I(0) \lesssim \|\omega_0\|_{L^2}^2.$$

$\square$

It remains to prove the previously stated Lemmata 5.1-5.4.

Proof of Lemma 5.1. The constant coefficient stream function for e^{iny} is given by

$$\Psi[e^{iny}] = \frac{1}{k^2+(n-kt)^2}(e^{iny} + ae^{ky+ikty} + be^{-ky+ikty}),$$

where a, b solve

$$\begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Integrating against another basis function e^{imy}, we obtain

$$\begin{aligned}
\langle \Psi[e^{iny}], e^{imy} \rangle &= \frac{1}{k^2+(n-kt)^2}\left(\delta_{nm} + \frac{e^{ky+ikty}\big|_{y=0}^1}{k+i(kt-m)}a + \frac{e^{ky+ikty}\big|_{y=0}^1}{-k+i(kt-m)}\right) \\
&= \frac{1}{k^2+(n-kt)^2}\left(\delta_{nm} + \frac{k\ e^{ky+ikty}\big|_{y=0}^1}{k^2+(kt-m)^2}a - \frac{k\ e^{ky+ikty}\big|_{y=0}^1}{k^2+(kt-m)^2}b\right. \\
&\quad \left. - \frac{i(kt-m)}{k^2+(kt-m)^2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right) \\
&= \frac{\delta_{nm}}{k^2+(n-kt)^2} + \frac{k}{(k^2+(m-kt)^2)(k^2+(n-kt)^2)}(a-b).
\end{aligned}$$

□

Proof of Lemma 5.2. Using Lemma 5.1, we expand $\langle W, \Psi \rangle$ in our basis and explicitly compute

$$\begin{aligned}
\langle W, \Psi \rangle &= \sum_{n,m} \overline{W}_m \langle e^{imy}, \Psi[e^{iny}] \rangle W_n \\
&= \sum_{n,m} \overline{W}_m \left(\frac{\delta_{nm}}{k^2+(n-kt)^2} + \frac{k}{(k^2+(m-kt)^2)(k^2+(n-kt)^2)}(a-b) \right) W_n \\
&= \sum_n \frac{1}{k^2+(n-kt)^2}|W_n|^2 + k(a-b)\left(\sum_n \frac{1}{k^2+(n-kt)^2}W_n\right) \\
&\quad \times \left(\sum_m \frac{1}{k^2+(m-kt)^2}\overline{W}_m\right) \\
&\leq \left(\sum_n \frac{|W_n|^2}{k^2+(n-kt)^2}\right)\left(1 + |k(a-b)| \left\| \frac{1}{\sqrt{k^2+(m-kt)^2}} \right\|_{l^2_m}^2\right) \\
&\lesssim \sum_n \frac{|W_n|^2}{k^2+(n-kt)^2}.
\end{aligned}$$

□

Proof of Lemma 5.3. Expressed in the Fourier basis, e^{iny}, $A(t)$ is a diagonal operator with positive, monotonically decreasing coefficients that are uniformly bounded from above and below by $\exp(\pm \|\langle t \rangle^{-2}\|_{L^1_t}) = e^{\pm\pi}$. It remains to show

$$|\langle W, A\Psi[W] \rangle| C + \langle W, \dot{A}W \rangle \leq 0.$$

Modifying the proof of Lemma 5.2 slightly, we obtain that

$$\begin{aligned}
|\langle W, A\Psi[W]\rangle| &\lesssim \sum \langle \frac{n}{k} - t\rangle^{-2} |W_n|^2 \\
&\lesssim \sum \langle \frac{n}{k} - t\rangle^{-2} \exp\left(\int^t \langle \frac{n}{k} - \tau\rangle^{-2} d\tau\right) |W_n|^2 \\
&= -\langle W, \dot{A} W\rangle.
\end{aligned}$$

□

Proof of Lemma 5.4. Let $\Psi[AW]$ solve

$$\begin{aligned}
(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[AW] &= AW, \\
\Psi[AW]_{y=0,1} &= 0.
\end{aligned}$$

By integration by parts, we then obtain

$$\begin{aligned}
|\langle AW, \frac{if}{k}\Phi\rangle| &\leq \sqrt{\|\Psi[AW]\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi[AW]\|_{L^2}^2} \\
&\quad \times \|f\|_{W^{1,\infty}} \frac{1}{|k|} \sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2} \\
&=: \|\Psi[AW]\|_{\tilde{H}^1} \|f\|_{W^{1,\infty}} \frac{1}{|k|} \|\Phi\|_{\tilde{H}^1}.
\end{aligned}$$

By our basis characterization and as A is a bounded, positive multiplier on our basis,

$$\|\Psi[AW]\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi[AW]\|_{L^2}^2 = |\langle AW, \Psi[AW]\rangle| \lesssim \|A\| |\langle AW, \Psi\rangle|,$$

so it only remains to control the factors involving Φ. Testing (64) with $-\frac{1}{g}\Phi$ and using (65), we obtain

$$\begin{aligned}
\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2 &\lesssim -\Re\langle W, \frac{1}{g}\Phi\rangle = \Re\langle (1 - (g(\frac{\partial_y}{k} - it))^2)\Phi, \frac{1}{g}\Phi\rangle \\
&\lesssim \|\frac{1}{g}\|_{W^{1,\infty}} \|\Psi\|_{\tilde{H}^1} \|\Phi\|_{\tilde{H}^1}.
\end{aligned}$$

Dividing by $\|\Phi\|_{\tilde{H}^1} := \sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - ikt)\Phi\|_{L^2}^2}$ then provides the desired estimate. □

This concludes our proof of Theorem 5.2 and thus establishes L^2 stability for a large class of strictly monotone shear flows in a finite periodic channel. Unlike the setting of an infinite periodic channel, where in section 4.3 the L^2 stability results could be extended to arbitrarily high Sobolev norms, in the following subsections we show that boundary effects introduce additional correction terms (even in the constant coefficient model), which qualitatively change the stability behavior of the equations.

5.2. H^1 **stability.** In order to extend the stability results to H^1, we proceed as in section 4.3 and differentiate the linearized Euler equations for a finite periodic channel, (60). We note that $\partial_y \Psi$ and $\partial_y \Phi$ no longer satisfy zero Dirichlet boundary conditions and thus split $\partial_y \Phi = \Phi^{(1)} + H^{(1)}$:

$$
\begin{aligned}
\partial_t \partial_y W &= \frac{if}{k}(\Phi^{(1)} + H^{(1)}) + \frac{if'}{k}\Phi, \\
(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(1)} &= \partial_y W + [(g(\partial_y - it))^2, \partial_y]\Phi, \\
\Phi^{(1)}_{y=0,1} &= 0, \\
H^{(1)} &= \partial_y \Phi - \Phi^{(1)}.
\end{aligned} \tag{66}
$$

The *homogeneous correction*, $H^{(1)}$, hence satisfies

$$
\begin{aligned}
(-1 + (g(\frac{\partial_y}{k} - it))^2)H^{(1)} &= 0, \\
H^{(1)}|_{y=0,1} &= \partial_y \Phi_{y=0,1}.
\end{aligned} \tag{67}
$$

The control of the contributions by Φ and $\Phi^{(1)}$ is obtained as in section 5.1, while the control of the boundary corrections due to $H^{(1)}$ is given by the following lemmata.

Lemma 5.5 (H^1 boundary contributions)**.** *Let $A(t)$ be a diagonal operator comparable to the identity, i.e.*

$$
\begin{aligned}
A(t) &: e^{iny} \mapsto A_n(t) e^{iny}, \\
1 &\lesssim A_n(t) \lesssim 1,
\end{aligned}
$$

$\omega_0 \in H^1, f, g \in W^{2,\infty}$*, and suppose that* $0 < c < g < c^{-1} < \infty$*. Further let W be the solution of* (66)*.*

Then, for any $0 < \gamma, \beta < \frac{1}{2}$ there exists a constant

$$C = C(\gamma, \beta, \|f\|_{W^{2,\infty}}, c, \|g\|_{W^{2,\infty}}),$$

such that

$$|\langle A\partial_y W, f H^{(1)}\rangle| \leq C\langle t\rangle^{-2(1-\gamma)}\|\omega_0\|^2_{H^1} + C\sum_n \langle t\rangle^{-2\gamma}\langle \frac{n}{k} - t\rangle^{-2\beta}|(\partial_y W)_n|^2.$$

If additionally $\omega_0|_{y=0,1} \equiv 0$, then for any $0 < \beta < \frac{1}{2}$ there exists a constant $C = C(\gamma, \beta, \|f\|_{W^{2,\infty}}, c, \|g\|_{W^{2,\infty}})$ such that

$$|\langle A\partial_y W, f H^{(1)}\rangle| \leq C\sum_n \langle t\rangle^{-1}\langle \frac{n}{k} - t\rangle^{-2\beta}|(\partial_y W)_n|^2.$$

Lemma 5.6 (H^1 stream function estimate)**.** *Let A, W, f, g satisfy the assumptions of Lemma* 5.5*. Then*

$$
\begin{aligned}
&|\langle A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi\rangle| \\
&\quad \lesssim |k|^{-1}\|f\|_{W^{2,\infty}}(\|\Psi[A\partial_y W]\|^2_{\tilde{H}^1} + \|\Psi[\partial_y W]\|^2_{\tilde{H}^1} + \|\Psi[W]\|^2_{\tilde{H}^1}).
\end{aligned}
$$

Using Lemmata 5.5, 5.6 and the lemmata of section 5.1, we prove H^1 stability.

Theorem 5.3 (H^1 stability for the finite periodic channel)**.** *Let W be a solution of the linearized Euler equations,* (66)*, and suppose that $f, g \in W^{2,\infty}$ and that there exists $c > 0$ such that*

$$0 < c < g < c^{-1} < \infty.$$

Further define a diagonal weight $A(t)$:

$$(68)\qquad \begin{aligned} &A(t) : e^{iny} \mapsto A_n(t) e^{iny}, \\ &A_n(t) = \exp\left(- \int_0^t \langle \frac{n}{k} - \tau \rangle^{-2} + \langle \tau \rangle^{-2\gamma} \langle \frac{n}{k} - \tau \rangle^{-2\beta} d\tau \right), \end{aligned}$$

where $0 < \beta, \gamma < \frac{1}{2}$ and $2\gamma + 2\beta > 1$. Also suppose that

$$\|f\|_{W^{2,\infty}} L$$

is sufficiently small. Then, for any $\omega_0 \in H^1([0,1])$, the solution W of (60) *(and hence* (66)*) with initial datum ω_0 satisfies*

$$\|W(t)\|_{H^1}^2 \lesssim I(t) := \langle A(t) W, W \rangle + \langle A(t) \partial_y W, \partial_y W \lesssim I(0) \lesssim \|\omega_0\|_{H^1}.$$

If additionally $\omega_0|_{y=0,1} \equiv 0$, then $I(t)$ is nonincreasing.

Proof of Theorem 5.3. Let W be a solution of (66); then we compute

$$\begin{aligned} \frac{d}{dt} \langle \partial_y W, A \partial_y W \rangle = \langle \dot{A} \partial_y W, \partial_y W \rangle + 2\Re \langle A \partial_y W, \frac{if}{k} \Phi^{(1)} + \frac{if'}{k} \Phi \rangle \\ + 2\Re \langle A \partial_y W, \frac{if}{k} H^{(1)} \rangle. \end{aligned}$$

Using Lemma 5.6 in combination with Lemma 5.2, we estimate the second term by

$$\begin{aligned} &2\Re \langle A \partial_y W, \frac{if}{k} \Phi^{(1)} + \frac{if'}{k} \Phi \rangle \\ &\lesssim \frac{\|f\|_{W^{2,\infty}}}{|k|} (\|\Psi[A \partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2) \\ &\lesssim \frac{\|f\|_{W^{2,\infty}}}{|k|} (|\langle W, \dot{A} W \rangle| + |\langle \partial_y W, \dot{A} \partial_y W \rangle|). \end{aligned}$$

Using Lemma 5.5, the last term is controlled by

$$\begin{aligned} &2\Re \langle A \partial_y W, \frac{if}{k} H^{(1)} \rangle \\ &\lesssim \frac{\|f\|_{W^{2,\infty}}}{|k|} \langle t \rangle^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2 + \frac{\|f\|_{W^{2,\infty}}}{|k|} \sum_n \langle t \rangle^{-2\gamma} \langle \frac{n}{k} - t \rangle^{-2\beta} |(\partial_y W)_n|^2 \end{aligned}$$

or by

$$C_1 \frac{\|f\|_{W^{2,\infty}}}{|k|} \sum_n \langle t \rangle^{-1} \langle \frac{n}{k} - t \rangle^{-2\beta} |(\partial_y W)_n|^2$$

if $\omega_0|_{y=0,1} \equiv 0$.

Hence, for

$$\sup_{k \neq 0} \frac{\|f\|_{W^{2,\infty}}}{|k|}$$

sufficiently small,

$$2\Re\langle A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi\rangle + 2\Re\langle A\partial_y W, \frac{if}{k}H^{(1)}\rangle$$

can be absorbed by

$$\langle \dot{A}\partial_y W, \partial_y W\rangle = -\sum_n A_n(t)\left(\langle \frac{n}{k}-t\rangle^{-2} + \langle t\rangle^{-2\gamma}\langle \frac{n}{k}-t\rangle^{-2\beta}\right)|(\partial_y W)_n|^2 \leq 0.$$

Thus, $I(t)$ satisfies

$$\frac{d}{dt}I(t) \lesssim \langle t\rangle^{-2(1-\gamma)}\|\omega_0\|_{H^1}^2$$

or, in the case of vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$,

$$\frac{d}{dt}I(t) \leq 0.$$

Integrating these inequalities in time concludes the proof. □

Proof of Lemma 5.5. Similar to the construction of Lemma 5.1, let u_j, $j = 1, 2$, be solutions of

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)u_j = 0$$

with boundary values

$$\begin{aligned} u_1(0) = u_2(1) = 1, \\ u_1(1) = u_2(0) = 0. \end{aligned} \tag{69}$$

Recalling the sequence of transformations turning ϕ into Φ, the functions u_j are given by linear combinations of the homogeneous solutions

$$e^{\pm kG(y)+ikty},$$

where $G(y) = U^{-1}(y)$ satisfies $G(y)' = g(y)$.

Further recalling the boundary conditions in (67), $H^{(1)}$ is hence given by

$$H^{(1)} = \partial_y\Phi(0)u_1 + \partial_y\Phi(1)u_2.$$

In order to compute $\partial_y\Phi|_{y=0,1}$, we test the equation for Φ in (60), i.e.

$$\begin{aligned} (-1 + (g(y)(\frac{\partial_y}{k} - it))^2)\Phi = W, \\ \Phi|_{y=0,1} = 0, \end{aligned} \tag{70}$$

with u_j:

$$\begin{aligned} \langle W, u_j\rangle &= \langle(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi, u_j\rangle \\ &= u_j g(\frac{\partial_y}{k} - it)(g\Phi)\Big|_{y=0}^{1} - g\Phi(\frac{\partial_y}{k} - it)(gu_j)\Big|_{y=0}^{1} \\ &\quad + \langle\Phi, (-1 + (g(\frac{\partial_y}{k} - it))^2)u_j\rangle \\ &= u_j\frac{g^2}{k}\partial_y\Phi\Big|_{y=0}^{1}, \end{aligned}$$

where we used that $\Phi|_{y=0,1} = 0$. Using the boundary values of u_j, (69),

$$u_1 \frac{g^2}{k} \partial_y \Phi \Big|_{y=0}^{1} = -\frac{g^2(0)}{k} \partial_y \Phi|_{y=0},$$
$$u_2 \frac{g^2}{k} \partial_y \Phi \Big|_{y=0}^{1} = \frac{g^2(1)}{k} \partial_y \Phi|_{y=1}.$$

As $k \neq 0$ and $g^2 > c > 0$, we may solve for $\partial_y \Phi|_{y=0,1}$:

$$H^{(1)} = \frac{k}{g^2(0)} \langle W, u_1 \rangle u_1 - \frac{k}{g^2(1)} \langle W, u_2 \rangle u_2. \tag{71}$$

The boundary contribution can thus be explicitly computed in terms of u_1, u_2:

$$\langle A\partial_y W, fH^{(1)} \rangle = \frac{k}{g^2(0)} \langle W, u_1 \rangle \langle A\partial_y W, fu_1 \rangle - \frac{k}{g^2(1)} \langle W, u_2 \rangle \langle A\partial_y W, fu_2 \rangle.$$

As the homogeneous solutions $e^{\pm kG(y)+ikty}$ and thus u_1, u_2 are highly oscillatory, we integrate $k\langle W, u_j \rangle$ by parts and use that the evolution of (59) preserves boundary values, i.e. $W|_{y=0,1} = \omega_0|_{y=0,1}$. Denoting primitive functions of u_j by U_j and using that

$$e^{\pm kG(y)+ikty} = \frac{1}{\pm kg + ikt} \partial_y e^{\pm kG(y)+ikty},$$

we therefore obtain

$$\begin{aligned} k\langle W, u_j \rangle &= kU_j \omega_0|_{y=0}^{1} - \langle \partial_y W, kU_j \rangle \\ &\leq \mathcal{O}(t^{-1})(\|\omega_0\|_{H^1} + |\langle \partial_y W, u_1 \rangle| + |\langle \partial_y W, u_2 \rangle|). \end{aligned} \tag{72}$$

Using Young's inequality, this yields a bound by

$$\begin{aligned} \left| \langle A\partial_y W, fH^{(1)} \rangle \right| &\lesssim \langle t \rangle^{-1} (|\langle \partial_y W, u_1 \rangle|^2 + |\langle A\partial_y W, fu_j \rangle|^2) \\ &\quad + \langle t \rangle^{-1} |\langle A\partial_y W, fu_j \rangle| \|\omega_0\|_{H^1} \\ &\lesssim \langle t \rangle^{-2\gamma} (|\langle \partial_y W, u_1 \rangle|^2 + |\langle A\partial_y W, fu_j \rangle|^2) \\ &\quad + \langle t \rangle^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2, \end{aligned} \tag{73}$$

where $0 < \gamma < \frac{1}{2}$ is chosen close to $\frac{1}{2}$.

Expanding $\partial_y W$ in our basis and choosing $0 < \beta < \frac{1}{2}$ close to $\frac{1}{2}$, we further estimate

$$\begin{aligned} |\langle \partial_y W, u_j \rangle| &\lesssim \sum_n |(\partial_y W)_n| |\langle e^{iny}, u_j \rangle| \lesssim \sum_n |(\partial_y W)_n| \frac{1}{|k + i(n-kt)|} \\ &\leq \frac{1}{k} \|(\partial_y W)_n \langle \frac{n}{k} - t \rangle^{-\beta} \|_{l^2_n} \| \langle \frac{n}{k} - t \rangle^{-1+\beta} \|_{l^2_n} \\ &\lesssim_\beta \|(\partial_y W)_n \langle \frac{n}{k} - t \rangle^{-\beta} \|_{l^2}. \end{aligned}$$

A similar bound also holds for $\langle A\partial_y W, fu_j \rangle$, where the constant further includes a factor $\|f\|_{W^{1,\infty}}$.

Thus, (73) can further be controlled by

$$\left| \langle A\partial_y W, fH^{(1)} \rangle \right| \lesssim \langle t \rangle^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2 + \sum_n \langle t \rangle^{-2\gamma} \langle \frac{n}{k} - t \rangle^{-2\beta} |(\partial_y W)_n|^2.$$

The improved result for $\omega_0|_{y=0,1} \equiv 0$ similarly follows from (73), as in that case the term $\langle t\rangle^{-1}|\langle A\partial_y W, fu_j\rangle|$ is not present. □

Proof of Lemma 5.6. Using the vanishing boundary values of Φ and $\Phi^{(1)}$ and introducing

$$\begin{aligned}(-1+(\frac{\partial_y}{k}-it)^2)\Psi[A\partial_y W] &= A\partial_y W,\\ \Psi[A\partial_y W]|_{y=0,1} &= 0,\end{aligned}$$

we integrate by parts to bound by

$$\begin{aligned}&\left|\left\langle(-1+(\frac{\partial_y}{k}-it)^2)\Psi[A\partial_y W], \frac{if}{k}\Phi^{(1)}+\frac{if'}{k}\Phi\right\rangle\right|\\ &\quad\leq \left(\|\Psi\|_{L^2}+\|(\frac{\partial_y}{k}-it)\Psi\|_{L^2}\right)\\ &\qquad\times\frac{\|f\|_{W^{2,\infty}}}{k}\left(\|\Phi\|_{L^2}+\|(\frac{\partial_y}{k}-it)\Phi\|_{L^2}+\|\Phi^{(1)}\|_{L^2}+\|(\frac{\partial_y}{k}-it)\Phi^{(1)}\|_{L^2}\right)\\ &\quad\leq\frac{\|f\|_{W^{2,\infty}}}{k}(\|\Psi\|_{\tilde H^1}^2+\|\Phi\|_{\tilde H^1}^2+\|\Phi^{(1)}\|_{\tilde H^1}^2).\end{aligned}$$

In order to further estimate $\|\Phi^{(1)}\|_{\tilde H^1}$, we again use the vanishing boundary values of $\Phi^{(1)}$ and test

$$\begin{aligned}(-1+(g(\frac{\partial_y}{k}-it))^2)\Phi^{(1)} &= \partial_y W+[(g(\partial_y-it))^2,\partial_y]\Phi,\\ \Phi^{(1)}_{y=0,1} &= 0,\end{aligned}$$

with $-\frac{1}{g}\Phi^{(1)}$, to obtain that

$$\begin{aligned}\|\Phi^{(1)}\|_{\tilde H^1}^2 &\lesssim -\langle(-1+(g(\frac{\partial_y}{k}-it))^2)\Phi^{(1)},\frac{1}{g}\Phi^{(1)}\rangle\\ &\leq -\langle(-1+(\frac{\partial_y}{k}-it)^2)\Psi[\partial_y W],\frac{1}{g}\Phi^{(1)}\rangle+\langle[(g(\frac{\partial_y}{k}-it))^2,\partial_y]\Phi,\Phi^{(1)}\rangle\\ &\lesssim\|\Psi[\partial_y W]\|_{\tilde H^1}\|\Phi^{(1)}\|_{\tilde H^1}+\|\Phi\|_{\tilde H^1}\|\Phi^{(1)}\|_{\tilde H^1}.\end{aligned}$$

Using this inequality and Lemma 5.4 to estimate $\|\Phi\|_{\tilde H^1}\lesssim\|\Psi\|_{\tilde H^1}$ then concludes the proof. □

As a consequence of the H^1 stability result, Theorem 5.3, Theorem 3.1 of section 3 yields damping with rate t^{-1}, i.e.

$$\begin{aligned}\|v-\langle v\rangle_x\|_{L^2(\mathbb{T}_L\times[0,1])}\leq\mathcal{O}(t^{-1})\|W(t)\|_{L^2(\mathbb{T}_L\times[0,1])}&\leq\mathcal{O}(t^{-1})\|\omega_0\|_{L^2(\mathbb{T}_L\times[0,1])},\\ \|v_2\|_{L^2(\mathbb{T}_L\times[0,1])}&\leq\mathcal{O}(t^{-1})\|\omega_0\|_{L^2(\mathbb{T}_L\times[0,1])}.\end{aligned}$$

As discussed in section 3, the first estimate thus already attains the optimal damping rate and regularity requirements. The estimate for v_2, however, does not yet provide an integrable decay rate, $\mathcal{O}(t^{-1-\epsilon})$, and thus, in particular, is not sufficient to prove scattering.

In the following section, we thus prove H^2 stability and hence linear inviscid damping with the optimal rates as well as scattering. There, we additionally require our perturbations to satisfy zero Dirichlet boundary conditions, $\omega_0|_{y=0,1}=0$.

As we discuss in Appendix B, this is not only a technical restriction: We show that otherwise $\partial_y W$ asymptotically develops a logarithmic singularity at the boundary, which by the trace theorem in particular forbids stability in any Sobolev space more regular than $H_y^{\frac{3}{2}}$.

5.3. H^2 **stability.** Following a similar approach as in subsection 5.2, we obtain H^2 stability and hence linear inviscid damping with the optimal rates and scattering for a large class of monotone shear flows in a finite periodic channel. As we discuss in Appendix B, for this stability result it is necessary to restrict to perturbations with zero Dirichlet data, $\omega_0|_{y=0,1} = 0$.

We again differentiate our equation and introduce homogeneous correction terms $H^{(1)}, H^{(2)}$. Thus let W be a solution of (60); then $\partial_y^2 W$ satisfies

$$\begin{aligned}
\partial_t \partial_y^2 W &= \frac{if}{k}(\Phi^{(2)} + H^{(2)}) + \frac{2f'}{ik}(\Phi^{(1)} + H^{(1)}) + \frac{f''}{ik}\Phi, \\
(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(2)} &= \partial_y^2 W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi, \\
\Phi^{(2)}_{y=0,1} &= 0.
\end{aligned} \tag{74}$$

Here the *homogeneous correction* $H^{(2)}$ satisfies

$$\begin{aligned}
(-1 + (g(\frac{\partial_y}{k} - it))^2)H^{(2)} &= 0, \\
H^{(2)}|_{y=0,1} &= \partial_y^2 \Phi|_{y=0,1}.
\end{aligned}$$

We recall that the equations satisfied by $\partial_y W, \Phi^{(1)}, H^{(1)}$ are given by (66) and (67), respectively.

As in section 5.2, we introduce several lemmata to control boundary corrections. Using these lemmata, we then prove the main stability result, Theorem 5.4.

Lemma 5.7 (H^2 boundary contribution I). *Let $A(t)$ be a diagonal operator comparable to the identity, i.e.*

$$\begin{aligned}
A : e^{iny} &\mapsto A_n e^{iny}, \\
1 \lesssim A_n &\lesssim 1,
\end{aligned}$$

and let W be a solution of (74) *with initial datum $\omega_0 \in H^2([0,1])$ with $\omega_0|_{y=0,1} = 0$. Suppose further that $f, g \in W^{3,\infty}$ and k (or L respectively) satisfy the assumptions of the H^1 stability result, Theorem* 5.3. *Then $H^{(1)}$ satisfies*

$$\|H^{(1)}\|^2_{\tilde{H}^1} \lesssim \langle t\rangle^{-2}\|W\|^2_{H^1} \lesssim \langle t\rangle^{-2}\|\omega_0\|^2_{H^1}$$

and for any $0 < \beta, \gamma < \frac{1}{2}$,

$$\begin{aligned}
|\langle A\partial_y^2 W, \frac{if'}{k}H^{(1)}\rangle| \lesssim_{\beta,\gamma} \|f\|_{W^{2,\infty}} k^{-1}\Big(&\log^2(t)\langle t\rangle^{-2(1-\gamma)}\|\omega_0\|^2_{H^2} \\
&+ \sum_n \langle t\rangle^{-2\gamma}\langle \frac{n}{k} - t\rangle^{-2\beta}|(A\partial_y^2 W)_n|^2\Big).
\end{aligned}$$

Lemma 5.8 (H^2 boundary contribution II)**.** *Let A, f, g, W, k be as in Lemma 5.7. Then for $0 < \gamma, \beta < \frac{1}{2}$ there exists a constant $C = C(f, g, k, \beta, \gamma)$ such that*

$$\begin{aligned}|\langle A\partial_y^2 W, \frac{if}{k} H^{(2)}\rangle| &\leq C \log^2(t) \langle t\rangle^{-2(1-\gamma)} \|\omega_0\|_{H^2}^2 \\ &\quad + C\sum_n \langle t\rangle^{-2\gamma} \langle \frac{n}{k} - t\rangle^{-2\beta} |(\partial_y^2 W)_n|^2.\end{aligned}$$

Lemma 5.9 (H^2 stream function estimate I)**.** *Let A, f, g, W, k be as in Lemma 5.7. Then,*

$$|\langle A\partial_y W, \frac{if}{k}\Phi^{(2)}\rangle| \lesssim k^{-1}\|f\|_{W^{1,\infty}} (\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2).$$

Lemma 5.10 (H^2 stream function estimate II)**.** *Let A, f, g, k, W be as in Lemma 5.7. Then,*

$$\begin{aligned}&|\langle A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi\rangle| \\ &\quad \lesssim \frac{1}{|k|}\|f\|_{W^{2,\infty}} (\|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2).\end{aligned}$$

Theorem 5.4 (H^2 stability for the finite periodic channel)**.** *Let f, g, W, k be as in Lemma 5.7 and let $A(t)$ be defined as in Theorem 5.3; i.e. let $A(t)$ be a diagonal weight*

$$\begin{aligned}&A(t) : e^{iny} \mapsto A_n(t)e^{iny}, \\ &A_n(t) = \exp\left(-\int_0^t \langle \frac{n}{k} - \tau\rangle^{-2} + \langle\tau\rangle^{-2\gamma}\langle\frac{n}{k} - \tau\rangle^{-2\beta} d\tau\right),\end{aligned} \tag{75}$$

where $\beta, \gamma < \frac{1}{2}$ and $2\gamma + 2\beta > 1$. Further suppose that

$$\|f\|_{W^{3,\infty}} L$$

is sufficiently small. Then, for any $\omega_0 \in H^2([0, 1])$ with vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$,

$$E_2(t) := \langle A(t)W, W\rangle + \langle A(t)\partial_y W, \partial_y W\rangle + \langle A(t)\partial_y^2 W, \partial_y^2 W\rangle$$

satisfies

$$\|W(t)\|_{H^2} \lesssim E_2(t) \lesssim E_2(0) \lesssim \|\omega_0\|_{H^2}.$$

Proof of Theorem 5.4*.* The control of

$$\langle A(t)W, W\rangle + \langle A(t)\partial_y W, \partial_y W\rangle$$

has been established in Theorem 5.3.

Differentiating $\langle A(t)\partial_y^2 W, \partial_y^2 W\rangle$ in time, we have to control

$$\langle A\partial_y^2 W, \frac{if}{k}\Phi^{(2)} + \frac{2f'}{ik}\Phi^{(1)} + \frac{f''}{ik}\Phi\rangle + \langle A\partial_y^2 W, \frac{if}{k}H^{(2)} + \frac{2f'}{ik}H^{(1)}\rangle.$$

As $\Phi^{(2)}, \Phi^{(1)}$ and Φ have zero boundary values, we integrate

$$\begin{aligned}(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[A\partial_y^2 W] &= A\partial_y^2 W, \\ \Psi[A\partial_y^2 W]_{y=0,1} &= 0,\end{aligned}$$

by parts and bound by the $\tilde{H}^1$ norm:

$$\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1}\frac{\|f\|_{W^{3,\infty}}}{k}\left(\|\Phi^{(2)}\|_{\tilde{H}^1}+\|\Phi^{(1)}\|_{\tilde{H}^1}+\|\Phi\|_{\tilde{H}^1}\right).$$

Lemmata 5.9 and 5.10 provide control by

$$\frac{1}{|k|}\|f\|_{W^{3,\infty}}(\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1}^2+\|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2+\|\Psi[\partial_y W]\|_{\tilde{H}^1}^2+\|\Psi[W]\|_{\tilde{H}^1}^2). \tag{76}$$

Supposing that

$$\sup\frac{1}{|k|}\|f\|_{W^{3,\infty}} \tag{77}$$

is sufficiently small and using Lemma 5.3, we find that (76) can be absorbed by

$$\langle W,\dot{A}W\rangle+\langle\partial_y W,\dot{A}\partial_y W\rangle+\langle\partial_y^2 W,\dot{A}\partial_y^2 W\rangle. \tag{78}$$

Using Lemmata 5.7 and 5.8 and supposing again that (77) is sufficiently small, the boundary contributions

$$\langle A\partial_y^2 W,\frac{if}{k}H^{(2)}+\frac{2f'}{ik}H^{(1)}\rangle$$

can be partially absorbed in (78), with the remaining terms estimated by

$$\langle t\rangle^{-2}\|\omega_0\|_{H^1}^2+\|f\|_{W^{2,\infty}}|\frac{1}{k}|\langle t\rangle^{-2(1-\gamma)}\|\omega_0\|_{H^2}^2+\log^2(t)\langle t\rangle^{-2(1-\gamma)}\|\omega_0\|_{H^2}^2. \tag{79}$$

We thus obtain that $E_2(t)$ satisfies

$$\begin{aligned}\partial_t E_2(t)&\leq\frac{d}{dt}\left(\langle A(t)W,W\rangle+\langle A(t)\partial_y W,\partial_y W\rangle+\langle A(t)\partial_y^2 W,\partial_y^2 W\rangle\right)\\&\lesssim(\langle t\rangle^{-2}+\log^2(t)\langle t\rangle^{-2(1-\gamma)})\|\omega_0\|_{H^2}^2.\end{aligned}$$

As $0<\gamma<\frac{1}{2}$, this is integrable and thus yields the result. □

It remains to prove Lemmata 5.7-5.10.

Proof of Lemma 5.7. We recall from the proof of Lemma 5.5 that $H^{(1)}$ is explicitly given by

$$H^{(1)}=\partial_y\Phi(0)u_1+\partial_y\Phi(1)u_2.$$

By the triangle inequality, we thus estimate by

$$\|H^{(1)}\|_{\tilde{H}^1}\lesssim|\partial_y\Phi(t,0)|\|u_1(t)\|_{\tilde{H}^1}+|\partial_y\Phi(t,1)|\|u_1(t)\|_{\tilde{H}^1}.$$

We further recall that the homogeneous solutions u_1,u_2 are of the form

$$a(t)e^{kG(y)+ikty}+b(t)e^{kG(y)+ikty},$$

where $a(t),b(t)$ are chosen to satisfy the boundary conditions, (69). Hence, for any time t

$$\begin{aligned}\|u_j(t)\|_{\tilde{H}^1}&\leq|a(t)|\|e^{kG(y)+ikty}\|_{\tilde{H}^1}+|b(t)|\|e^{-kG(y)+ikty}\|_{\tilde{H}^1}\\&=|a(t)|\|e^{kG(y)}\|_{H^1}+|b(t)|\|e^{-kG(y)}\|_{H^1}.\end{aligned}$$

Therefore, by direct computation of the coefficients a,b (analogously to Lemma 5.1),

$$\|u_j(t)\|_{\tilde{H}^1}<C<\infty,$$

uniformly in time.

Thus, $H^{(1)}$ satisfies

$$\|H^{(1)}(t)\|_{\tilde{H}^1} \lesssim |\partial_y \Phi(t,0)| + |\partial_y \Phi(t,1)|.$$

Recalling the explicit characterization of $\partial_y \Phi|_{y=0,1}$, (71),

$$\begin{aligned} \partial_y \Phi|_{y=0} &= \frac{k}{g^2(0)} \langle W, u_1 \rangle, \\ \partial_y \Phi|_{y=1} &= -\frac{k}{g^2(1)} \langle W, u_2 \rangle, \end{aligned}$$

and the subsequent estimate, (72),

$$|k \langle W, u_j \rangle| \lesssim \langle t \rangle^{-1} (\|\omega_0\|_{H^1} + |\langle \partial_y W, u_1 \rangle| + |\langle \partial_y W, u_2 \rangle|),$$

we further control

$$\|H^{(1)}\|_{\tilde{H}^1} \lesssim \langle t \rangle^{-1} \|W\|_{H^1}, \tag{80}$$

which yields the first result.

In order to estimate

$$\begin{aligned} \langle A\partial_y^2 W, \frac{if'}{k} H^{(1)} \rangle &= \partial_y \Psi(0) \langle A\partial_y^2 W, u_1 \rangle + \partial_y \Psi(1) \langle A\partial_y^2 W, u_2 \rangle \\ &\lesssim \langle t \rangle^{-1} |\langle A\partial_y^2 W, u_j \rangle|, \end{aligned}$$

we proceed as in Lemma 5.5 and expand $\langle A\partial_y^2 W, u_j \rangle$ in our basis. Thus, we obtain

$$|\langle A\partial_y^2 W, u_j \rangle| \lesssim_\beta \|(A\partial_y^2 W)_n \langle \frac{n}{k} - t \rangle^{-\beta}\|_{l^2},$$

for $0 < \beta < \frac{1}{2}$. Hence, by Young's inequality:

$$\begin{aligned} \langle A\partial_y^2 W, \frac{if'}{k} H^{(1)} \rangle &\lesssim \langle t \rangle^{-1} \|W\|_{H^1} \|(A\partial_y^2 W)_n \langle \frac{n}{k} - t \rangle^{-\beta}\|_{l^2} \\ &\lesssim \langle t \rangle^{-2(1-\gamma)} \|W\|_{H^1}^2 + \sum_n \langle t \rangle^{-2\gamma} \langle \frac{n}{k} - t \rangle^{-2\beta} |(A\partial_y^2 W)_n|^2. \end{aligned}$$

□

Proof of Lemma 5.8. We follow the same strategy as in the proof of Lemma 5.5 and Lemma 5.7 and explicitly compute

$$H^{(2)} = \partial_y^2 \Phi(0) u_1 + \partial_y^2 \Phi(1) u_2,$$

$$\begin{aligned} \langle A\partial_y^2 W, \frac{if}{k} H^{(2)} \rangle &= \partial_y^2 \Phi(0) \langle A\partial_y^2 W, \frac{if}{k} u_1 \rangle + \partial_y^2 \Phi(1) \langle A\partial_y^2 W, \frac{if}{k} u_2 \rangle \\ &\lesssim |\partial_y^2 \Phi|_{y=0,1}| \left\| \langle \frac{n}{k} - t \rangle^{-\beta} (A\partial_y^2 W)_n \right\|_{l_n^2}. \end{aligned} \tag{81}$$

It hence remains to estimate $\partial_y^2 \Phi|_{y=0,1}$. We thus expand the equation for the stream function Φ in the linearized Euler equations, (60),

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi = W,$$

and obtain

$$-\Phi + g^2 k^{-2} \partial_y^2 \Phi + k^{-1} g g' (\frac{\partial_y}{k} - it)\Phi - g^2 it \frac{\partial_y}{k} \Phi + g^2 t^2 \Phi = W.$$

Thus, using that Φ and W vanish at the boundary, $\partial_y^2\Phi|_{y=0,1}$ satisfies

$$g^2\partial_y^2\Phi|_{y=0,1} = (-gg' + iktg^2)\partial_y\Phi|_{y=0,1}.$$

Dividing by g^2 (which we required to be bounded away from zero), we may thus solve for $\partial_y^2\Phi_{y=0,1}$:

$$\partial_y^2\Phi|_{y=0,1} = \frac{-gg' + iktg^2}{g^2}\partial_y\Phi|_{y=0,\psi} = \mathcal{O}(kt)\partial_y\Phi|_{y=0,1}.$$

Again recalling the explicit characterization of $\partial_y\Phi|_{y=0,1}$, (71),

$$\begin{aligned}\partial_y\Phi|_{y=0} &= \frac{k}{g^2(0)}\langle W, u_1\rangle,\\ \partial_y\Phi|_{y=1} &= -\frac{k}{g^2(1)}\langle W, u_2\rangle,\end{aligned}$$

from the proof of Lemma 5.5, we further compute

$$\begin{aligned}\mathcal{O}(kt)\partial_y\Phi|_{y=0,1} &\lesssim k^2t\langle W, u_j\rangle \lesssim k\langle \partial_y W, u_j\rangle + ku_jW|_{y=0}^1\\ &\lesssim \langle t\rangle^{-1}\langle\partial_y^2 W, u_j\rangle + \langle t\rangle^{-1}\partial_y W u_j|_{y=0}^1 + ku_jW|_{y=0}^1\\ &= \langle t\rangle^{-1}\langle\partial_y^2 W, u_j\rangle + \langle t\rangle^{-1}\partial_y W u_j|_{y=0}^1,\end{aligned}$$

where we again used that $W|_{y=0,1} = \omega_0|_{y=0,1} \equiv 0$. The first term can again be estimated by

$$\langle t\rangle^{-1}\|\langle\frac{n}{k} - t\rangle^{-\beta}(A\partial_y^2 W)_n\|_{l_n^2}$$

and thus yields a contribution of the desired form.

To estimate the second term,

$$\langle t\rangle^{-1}\partial_y W u_j|_{y=0}^1, \tag{82}$$

we restrict the evolution equation for $\partial_y W$, (66), to the boundary and obtain

$$\partial_t\partial_y W|_{y=0,1} = \frac{f'}{ik}\Phi|_{y=0,1} + \frac{f}{ik}\partial_y\Phi_{y=0,1} = \frac{f}{ik}\partial_y\Phi \lesssim |\langle W, u_j\rangle|.$$

Controlling the right-hand-side by $\mathcal{O}(t^{-1})\|W\|_{H^1}$ and using the H^1 stability result, Theorem 5.3, we thus obtain a logarithmic control

$$|\partial_y W|_{y=0,1}| = \mathcal{O}(\log(t))\|\omega_0\|_{H^1}.$$

Hence, (82) can be bounded by

$$\langle t\rangle^{-1}\partial_y W u_j|_{y=0}^1 \lesssim \log(t)\langle t\rangle^{-1}\|\omega_0\|_{H^1}.$$

Using these estimates, we may further estimate equation (81) by

$$\begin{aligned}\langle A\partial_y^2 W, \frac{if}{k}H^{(2)}\rangle \lesssim &(\langle t\rangle^{-1}\|\langle\frac{n}{k} - t\rangle^{-\beta}(A\partial_y^2 W)_n\|_{l_n^2} + \langle t\rangle^{-1}\log(t)\|\omega_0\|_{H^1})\\ &\cdot\|\langle\frac{n}{k} - t\rangle^{-\beta}(A\partial_y^2 W)_n\|_{l_n^2}\\ \lesssim &\log^2(t)\langle t\rangle^{-2(1-\gamma)}\|\omega_0\|_{H^1}^2\\ &+ \sum_n\langle t\rangle^{-2\gamma}\langle\frac{n}{k} - t\rangle^{-2\beta}|(A\partial_y^2 W)_n|^2\\ &+ \sum_n\langle t\rangle^{-1}\langle\frac{n}{k} - t\rangle^{-2\beta}|(A\partial_y^2 W)_n|^2.\end{aligned}$$

□

Proof of Lemma 5.9. We introduce

$$(-1+(\frac{\partial_y}{k}-it)^2)\Psi[A\partial_y^2 W]=A\partial_y^2 W,$$
$$\Psi[A\partial_y^2 W]_{y=0,1}=0$$

and use the vanishing boundary values of $\Phi^{(2)}$ to integrate by parts and obtain

$$|\langle A\partial_y W,\frac{if}{k}\Phi^{(2)}\rangle|\lesssim k^{-1}\|f\|_{W^{1,\infty}}(\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1}^2+\|\Phi^{(2)}\|_{\tilde{H}^1}^2).$$

It thus remains to control $\|\Phi^{(2)}\|_{\tilde{H}^1}^2$. Testing

$$(-1+(g(\frac{\partial_y}{k}-it))^2)\Phi^{(2)}=\partial_y^2 W+[(g(\frac{\partial_y}{k}-it))^2,\partial_y^2]\Phi,$$
$$\Phi^{(2)}_{y=0,1}=0,$$

with $-\frac{1}{g}\Phi^{(2)}$, we estimate

$$\begin{aligned}\|\Phi^{(2)}\|_{\tilde{H}^1}^2\lesssim&-\langle(-1+(g(\frac{\partial_y}{k}-it))^2)\Phi^{(2)},\frac{1}{g}\Phi^{(2)}\rangle\\ \lesssim&\|\Psi[\partial_y^2 W]\|_{\tilde{H}^1}\|\Phi^{(2)}\|_{\tilde{H}^1}+\langle[(g(\frac{\partial_y}{k}-it))^2,\partial_y^2]\Phi,\frac{1}{g}\Phi^{(2)}\rangle\\ \lesssim&\|\Phi^{(2)}\|_{\tilde{H}^1}(\|\Psi[\partial_y^2 W]\|_{\tilde{H}^1}+\|\partial_y\Phi\|_{\tilde{H}^1}+\|\Phi\|_{\tilde{H}^1}).\end{aligned}$$

Using the triangle inequality

$$\|\partial_y\Phi\|_{\tilde{H}^1}\lesssim\|\Phi^{(1)}\|_{\tilde{H}^1}+\|H^{(1)}\|_{\tilde{H}^1},$$

Lemma 5.7 and Lemma 5.6 then provide the desired control. □

Proof of Lemma 5.10. We introduce

$$(-1+(\frac{\partial_y}{k}-it)^2)\Psi[A\partial_y^2 W]=\partial_y^2 W,$$
$$\Psi[A\partial_y^2 W]_{y=0,1}=0$$

and use the vanishing boundary values of $\Phi^{(1)}$ and Φ to integrate by parts to obtain

$$\begin{aligned}&|\langle A\partial_y W,\frac{if}{k}\Phi^{(1)}+\frac{if'}{k}\Phi\rangle|\\ &\quad\lesssim|k^{-1}|\|f\|_{W^{2,\infty}}(\|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2+\|\Phi^{(1)}\|_{\tilde{H}^1}^2+\|\Phi[W]\|_{\tilde{H}^1}^2).\end{aligned}$$

Lemma 5.6 then provides the desired control. □

With these stability results, we now have the desired control on $\|W\|_{H^2}$ and hence, as we discuss in the following section, can prove linear inviscid damping with the optimal algebraic decay rates for a large class of strictly monotone shear flows in a finite periodic channel.

6. Linear inviscid damping, scattering and consistency

In this section, we combine the results of the previous sections and thus close our strategy to prove linear inviscid damping for monotone shear flows (see Figure 6).

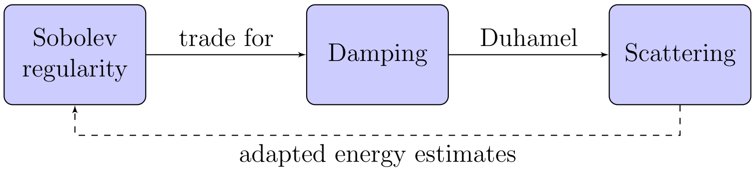

FIGURE 6. The stability results of sections 4 and 5 allow us prove linear inviscid damping and scattering.

Theorem 6.1 (Linear inviscid damping for the infinite periodic channel and finite periodic channel)**.** *Let* $\omega_0 \in L^2_x H^2_y$ *with* $\langle \omega_0 \rangle_x \equiv 0$ *and let* W *solve*

$$\begin{aligned} \partial_t W &= f \partial_x \Phi, \\ (\partial_x^2 + (g(\partial_y - t\partial_x))^2)\Phi &= W, \end{aligned} \tag{83}$$

either on the infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$*, or finite periodic channel,* $\mathbb{T}_L \times [0,1]$*. Suppose that there exists* $c > 0$ *such that*

$$\begin{aligned} &c < g < c^{-1}, \\ &\frac{1}{g}, f \in W^{3,\infty} \end{aligned}$$

and that

$$L \left\| \frac{f}{k} \right\|_{W^{3,\infty}}$$

is sufficiently small. In the case of a finite periodic channel, additionally assume that

$$\omega_0(x,0) \equiv 0 \equiv \omega_0(x,1).$$

Then there exists a function $W_\infty \in L^2_x H^2_y$ *such that*

$$\begin{aligned} &\text{(Stability)} && \|W\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y}, \\ &\text{(Damping)} && \|v - \langle v \rangle_x\|_{L^2} = \mathcal{O}(t^{-1}), \\ & && \|v_2\|_{L^2} = \mathcal{O}(t^{-2}), \\ &\text{(Scattering)} && W(t) \to_{L^2} W_\infty, \end{aligned}$$

as $t \to \infty$*.*

Proof. Let $\omega_0 \in L^2_x H^2_y$ and f, g be given. Then by the stability results for the infinite channel, Theorem 4.5, and for the finite channel, Theorem 5.4, W satisfies

$$\|W\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y}.$$

As the mean in x is conserved, i.e.

$$\langle W(t) \rangle_x \equiv \langle \omega_0 \rangle_x \equiv 0,$$

we may apply Poincaré's theorem to deduce that

$$\|W\|_{H^{-1}_x H^2_y} \lesssim \|W\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y}.$$

The damping result, Theorem 3.1, of section 3 then implies decay of the velocity field with the optimal algebraic rates.

Duhamel's formula in our scattering formulation is just integrating (83) in time and leads to

$$W(t,x,y) = \omega_0(x, U^{-1}(y)) + \int_0^t f(y) V_2(\tau, x, y) d\tau, \tag{84}$$

where

$$V_2(t,x,y) = \partial_x \Phi(t,x,z) = v_2(t, x - ty, U^{-1}(y)).$$

Hence, as the change of variables $(x,y) \mapsto (x - tU(y), y)$ is an isometry and $y \mapsto U(y)$ is bilipschitz,

$$\|fV_2\|_{L^2} \leq \|f\|_{L^\infty} \|V_2\|_{L^2} \lesssim \|f\|_{L^\infty} \|v_2\|_{L^2} = \mathcal{O}(t^{-2}).$$

Thus, the integral in (84) is uniformly bounded in L^2 for all t, and the improper integral for $t \to \pm\infty$ exists as a limit in L^2. Therefore,

$$W \xrightarrow{L^2} W_{\pm\infty} := \omega_0 + \lim_{t \to \pm\infty} \int_0^t f V_2(\tau) d\tau.$$

As $\|W\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y}$ uniformly in time, weak compactness and lower semi-continuity imply $W_{\pm\infty} \in L^2_x H^2_y$ and

$$\|W_{\pm\infty}\|_{H^2_y L^2_x} \lesssim \|w_0\|_{L^2_x H^2_y}.$$

□

Corollary 6.1 (L^2 scattering)**.** *Let f, g, L be as in Theorem 6.1 and let $\omega_0 \in L^2$. Then there exists $W_\infty \in L^2$ such that*

$$W(t) \xrightarrow{L^2} W_\infty, \text{ as } t \to \infty.$$

Proof. Let $H^2 \ni \omega_0^j \xrightarrow{L^2} \omega_0$ as $j \to \infty$. Then by the previous theorem there exist W_∞^j such that

$$W^j(t) \xrightarrow{L^2} W_\infty^j, \text{ as } t \to \infty.$$

By the L^2 stability results, Theorem 4.4 of section 4 and Theorem 5.2 of section 5, and letting t tend to infinity, W_∞^j is a Cauchy sequence in L^2. Denoting the limit by W_∞, a diagonal sequence argument yields $W(t) \xrightarrow{L^2} W_\infty$ as $t \to \infty$. □

A natural question following these linear inviscid damping and scattering results is, of course, whether such behavior also persists under the nonlinear evolution. Bedrossian and Masmoudi, [4], answer this question positively in the case of Couette flow in an infinite periodic channel, where the perturbations are required to be small in Gevrey regularity to control nonlinear effects.

As a small step in the direction of similar results for monotone shear flows, we follow Bouchet and Morita, [5], and answer the simpler question of consistency. In the derivation of the linearized Euler equations

$$\begin{aligned} \partial_t \omega + U(y) \partial_x \omega &= U'' \partial_x \phi, \\ \Delta \phi &= \omega, \end{aligned} \tag{85}$$

one neglects the nonlinearity:

$$v \cdot \nabla \omega = v_1 \partial_x \omega + v_2 \partial_y \omega. \tag{86}$$

For a consistency result, we show that the nonlinearity, *when evolved with the linear dynamics*, is an integrable perturbation in the sense that

$$\sup_{T>0} \left\| \int_0^T v \cdot \nabla \omega dt \right\|_{L^2} < C < \infty.$$

In view of Theorem 6.1, at first sight we would expect decay of (86) with a rate of only $\mathcal{O}(t^{-1})$, as

$$\begin{aligned} \|v_1\|_{L^2} &= \mathcal{O}(t^{-1}), \\ \|v_2\|_{L^2} &= \mathcal{O}(t^{-2}), \\ \|\partial_y \omega\|_{L^2} &= \|(\partial_y - tU'\partial_x)W\|_{L^2} = \mathcal{O}(t). \end{aligned}$$

However, there is some additional cancellation, which can be used. In scattering coordinates $v \cdot \nabla \omega$ is given by

$$-(\partial_y - tU'\partial_x)\Phi \partial_x W + \partial_x \Phi (\partial_y - tU'\partial_x)W = \nabla^\perp \Phi \cdot \nabla W.$$

Combining the stability results on ∇W and the damping results on $\nabla^\perp \Phi$ of sections 4.3 and 5.2, we obtain quadratic decay

$$\|\nabla^\perp \Phi\|_{L^2} = \mathcal{O}(t^{-2})$$

and thus consistency.

Lemma 6.1 (Consistency)**.** *Let W be a solution to the linearized 2D Euler equation,* (83)*, on $\mathbb{T} \times \mathbb{R}$ with initial datum $\omega_0 \in H^3_{x,y}(\mathbb{T}_L \times \mathbb{R})$. Suppose further that the assumptions of the Sobolev regularity result, Theorem* 4.5*, for $j = 3$, as well as of the damping result, Theorem* 3.1*, are satisfied. Then*

$$\|\nabla^\perp \Phi \cdot \nabla W\|_{L^2} = \mathcal{O}(t^{-2}).$$

In particular,

$$W(t) + \int^t \nabla^\perp \Phi(\tau) \nabla W(\tau) d\tau$$

is close to $W(t)$ in L^2 uniformly in time and there exist asymptotic profiles $W^{\pm\infty}_{con}$ such that

$$W(t) + \int^t \nabla^\perp \Phi(\tau) \nabla W(\tau) d\tau \xrightarrow{L^2} W^{\pm\infty}_{con}, \text{ as } t \to \pm\infty.$$

Proof. By Theorem 4.5, W satisfies

$$\|W(t)\|_{H^3_{x,y}} \lesssim \|\omega_0\|_{H^3_{x,y}}.$$

Hence, by Theorem 3.1,

$$\|\nabla^\perp \Phi\|_{L^2} = \mathcal{O}(t^{-2}) \|W(t)\|_{H^3_{x,y}} = \mathcal{O}(t^{-2}) \|\omega_0\|_{H^3_{x,y}}.$$

Using the Sobolev embedding,

$$\|\nabla W\|_{L^\infty_{x,y}} \lesssim \|W(t)\|_{H^3_{x,y}} \lesssim \|\omega_0\|_{H^3_{x,y}}.$$

An application of Hölder's inequality then gives the desired bound:

$$\|\nabla^\perp \Phi \nabla W\|_{L^2} \leq \|\nabla^\perp \Phi\|_{L^2} \|\nabla W\|_{L^\infty} = \mathcal{O}(t^{-2}) \|\omega_0\|^2_{H^3_{x,y}}.$$

□

We remark that the regularity assumptions on ω_0 are not sharp. As we are, however, only interested in the qualitative property of consistency, we assume sufficiently much regularity to use a Sobolev embedding.

In the setting of a finite periodic channel $\mathbb{T} \times [0,1]$, we thus far have only established stability in H^2, which is not sufficient for integrable decay of $\|\nabla^\perp \Phi\|_{L^2}$. Furthermore, in two dimensions $H^2_{x,y}$ regularity is critical for the Sobolev embedding. Hence, control of $\|W\|_{H^2(\mathbb{T}_L\times[0,1])}$ only yields $\nabla_{x,y}W \in \mathrm{BMO}(\mathbb{T}_L \times [0,1])$ instead of $L^\infty(\mathbb{T}_L \times [0,1])$.

A natural question is thus whether the stability result in a finite periodic channel can be improved to higher Sobolev spaces. As we sketch in Appendix B, stability in H^3 is in general not possible.

Appendix A. Bases and mapping properties

In this section, we elaborate on the role of boundary conditions, the choice of basis and the mapping properties of

$$\begin{aligned} W \mapsto &\langle W, \Psi\rangle, \\ (k^2 - (\partial_y - ikt)^2)\Psi &= W, \\ \Psi|_{y=0,1} &= 0. \end{aligned}$$

In analogy to the whole space setting, a first natural approach is via a Fourier basis, which we used in section 5.1. There, the coefficients of Ψ have been computed in Lemma 5.1.

Lemma A.1. *Let Ψ be as above, $n, m \in 2\pi\mathbb{Z}$. Then*

$$\langle \Psi[e^{iny}], e^{imy}\rangle = \frac{\delta_{nm}}{k^2 + (n-kt)^2} + \frac{k}{(k^2 + (m-kt)^2)(k^2+(n-kt)^2)}(a-b),$$

where a, b solve

$$\begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This choice of basis has a distinct advantage in its simplicity and good decoupling multiplier structure. In particular, we may easily prove Lemma 5.2 using Cauchy-Schwarz. We, however, see that we cannot obtain a bounded map in $H^s, s \geq \frac{1}{2}$, in this way, as the decay is not fast enough and thus

$$\frac{n^s}{k^2 + (n-kt)^2} \not\in l^2.$$

When trying to use Schur's test instead, one encounters the problem of slow decay as $n, m \to \infty$ at an even earlier stage of our proof:

$$\begin{aligned} &\sup_m \sum_n \langle \frac{n}{k} - t\rangle^\alpha \langle \frac{m}{k} - t\rangle^\alpha |\langle \Psi[e^{iny}], e^{imy}\rangle| \\ &\leq 1 + k^{-3} \sum_n \langle \frac{n}{k} - t\rangle^{\alpha-2} \lesssim_\alpha 1 + k^{-2}. \end{aligned} \tag{87}$$

Therefore, this approach does not even provide an l^2 estimate with optimal decay, but only a weaker variant of Lemma 5.2 with $\alpha < 1$.

Furthermore, testing against homogeneous solutions, we only obtain slow decay:

$$\langle e^{iny}, e^{\pm y+ity}\rangle = \frac{1}{\pm 1 + i(t-n)} e^{\pm y + i(t-n)y}|_{y=0}^{1} = \mathcal{O}(\langle n-t\rangle^{-1}),$$
$$\langle \sin(ny), e^{\pm y+ity}\rangle = -n\frac{1}{\pm 1+it}\langle \cos(ny), e^{\pm y+ity}\rangle = \mathcal{O}(n\langle t\rangle^{-1}\langle n-t\rangle^{-1}).$$

Considering a sin basis instead, we may make use of vanishing boundary terms to obtain additional cancellations and better coefficients:

Lemma A.2. *Let $n \in \pi\mathbb{N}$ and let $\Psi[\sin(ny)]$ be the solution of*

$$\begin{aligned}(k^2 - (\partial_y - ikt)^2)\Psi[\sin(ny)] &= \sin(ny),\\ \Psi[\sin(ny)]|_{y=0,1} &= 0.\end{aligned}$$

Then, for any $m \in \pi\mathbb{N}$,

$$\begin{aligned}&\langle \Psi[\sin(ny)], \sin(my)\rangle = \delta_{nm}\left(\frac{1}{k^2+(n-kt)^2} + \frac{1}{k^2+(n+kt)^2}\right)\\ &+ dk\left(\frac{1}{k^2+(kt+n)^2} - \frac{1}{k^2+(kt-n)^2}\right)\left(\frac{1}{k^2+(kt+m)^2} - \frac{1}{k^2+(kt-m)^2}\right)\\ &+ i((-1)^{n+m}-1)nmkt\\ &\cdot \frac{k^4t^4 + 2k^4t^2 + 2k^4 - 2k^2t^2(m^2+n^2) + 2k^2(m^2+n^2) + 2m^2n^2}{(k^2+(kt+m)^2)(k^2+(kt-m)^2)(k^2+(kt+n)^2)(k^2+(kt-n)^2)(n^2-m^2)},\end{aligned}$$

where

$$d = -((-1)^{n+m}-1) + 2\frac{(-1)^{n+m}e^{-k} + e^k}{e^k - e^{-k}} + 2\frac{(-1)^m e^{ikt} - (-1)^n e^{-ikt}}{e^k - e^{-k}}.$$

Before proving this result, let us comment on some of the implications and the relation to the results of section 5.

- While these coefficients are much less simple than for a Fourier basis, they asymptotically decay with rates $n^{-3}m^{-3}$. Hence, an argument via Schur's test as in (87) does not have to require $\alpha < 1$. Furthermore, the rapid decay suggests that the mapping

(88)
$$\begin{aligned}W &\mapsto \Psi,\\ L^2 &\to L^2\end{aligned}$$

  can be extended to a bounded mapping on the fractional Sobolev spaces:

$$\sum_n n^{2s}|W_n|^2,$$

  for $s > 0$ not too large.
- Using that $n, m, kt \geq 0$, one may roughly bound

$$\frac{k^2t^2}{\sqrt{k^2+(n+kt)^2}\sqrt{k^2+(m+kt)^2}} \leq 1,$$

  and thus trade the additional decay for the convenience of a uniform bound. While this is far from optimal, it reduces estimates to the ones for the Fourier basis.

- In section 5 we use a different approach and consider boundary terms separately. That is, we decompose $\partial_y \Phi$ into a function, $\Phi^{(1)}$, with zero Dirichlet conditions

$$
\begin{aligned}
(k^2 - (g(\partial_y - ikt))^2)\Phi^{(1)} &= \partial_y W + [(g(\partial_y - ikt))^2, \partial_y]\Phi, \\
\Psi^{(1)}|_{y=0,1} &= 0,
\end{aligned}
$$

  and a *homogeneous correction*

$$
\begin{aligned}
(k^2 - (\partial_y - ikt)^2)H^{(1)} &= 0, \\
H^{(1)}|_{y=0,1} &= \partial_y \Phi|_{y=0,1}.
\end{aligned}
$$

  The estimate of

$$
\partial_y W + [(g(\partial_y - ikt))^2, \partial_y]\Phi \mapsto \Phi^{(1)}
$$

  is then similar to the estimate of Φ in terms of W. In order to control $H^{(1)}$, we make additional use of the dynamics and study the evolution of

$$
\partial_y \Phi|_{y=0,1}.
$$

- We further note that by our choice of basis, for $\frac{1}{2} < s < 1$, $\partial_y W \in H^s$ would also imply that $\partial_y W|_{y=0,1}$ vanishes for all times. However, $\partial_y W|_{y=0,1}$ is not conserved by the linearized Euler equations.

Proof of Lemma A.2. The streamfunction $\Psi[\sin(ny)]$ is given by

$$
\begin{aligned}
\Psi[\sin(ny)] = &\left(\frac{1}{k^2 + (n - kt)^2} + \frac{1}{k^2 + (n + kt)^2}\right)\sin(ny) \\
&+ i\left(\frac{1}{k^2 + (n - kt)^2} - \frac{1}{k^2 + (n + kt)^2}\right) \\
&\times \left(\cos(ny) + ae^{ky + ikty} + be^{-ky + ikty}\right),
\end{aligned}
$$

where a, b solve

$$
\begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} (-1)^n \\ 1 \end{pmatrix}.
$$

Integrating against another basis function, $\sin(my)$, we obtain

$$
\begin{aligned}
&\langle \Psi[\sin(ny)], \sin(my) \rangle \\
&= \delta_{nm}\left(\frac{1}{k^2 + (n - kt)^2} + \frac{1}{k^2 + (n + kt)^2}\right) + i\left(\frac{1}{k^2 + (n - kt)^2} - \frac{1}{k^2 + (n + kt)^2}\right) \\
&\cdot \left(\frac{m((-1)^{n+m} - 1)}{n^2 - m^2} + \frac{1}{2i}\left(\frac{1}{k + i(kt + m)} - \frac{1}{k + i(kt - m)}\right)((-1)^m e^{k+ikt} - 1)a\right. \\
&\left.+ \frac{1}{2i}\left(\frac{1}{-k + i(kt + m)} - \frac{1}{-k + i(kt - m)}\right)((-1)^m e^{-k+ikt} - 1)b\right).
\end{aligned}
$$

As the δ_{nm} term is already of the desired form, in the following we consider only the remaining terms. Using the equation for a, b, we obtain

(89)
$$\begin{aligned}&\left(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2}\right)\left(\frac{im((-1)^{n+m}-1)}{n^2-m^2}-\frac{1}{2}\left(\frac{1}{k+i(kt+m)}\right.\right.\\&-\left.\frac{1}{k+i(kt-m)}+\frac{1}{-k+i(kt+m)}-\frac{1}{-k+i(kt-m)}\right)((-1)^{n+m}-1)\\&+\left.\frac{1}{2}\left(\frac{1}{k+i(kt+m)}-\frac{1}{k+i(kt-m)}-\frac{1}{-k+i(kt+m)}+\frac{1}{-k+i(kt-m)}\right)d\right),\end{aligned}$$

where

$$\begin{aligned}d&=((-1)^m e^{k+ikt}-1)a-((-1)^m e^{-k+ikt}-1)b\\&=((-1)^{n+m}-1)-2((-1)^m e^{-k+ikt}-1)b\\&=-((-1)^{n+m}-1)+2\frac{((-1)^m e^{-k+ikt}-1)((-1)^n-e^{k+ikt})}{e^{k+ikt}-e^{-k+ikt}}\\&=-((-1)^{n+m}-1)+2\frac{(-1)^{n+m}e^{-k}+e^k}{e^k-e^{-k}}+2\frac{(-1)^m e^{ikt}-(-1)^n e^{-ikt}}{e^k-e^{-k}}.\end{aligned}$$

We, in particular, note that

$$d(t,k,n,m)=\overline{d(t,k,m,n)}=d(-t,k,m,n)$$

and that d is uniformly bounded if k is bounded away from zero. Furthermore, consider k large and $n+m$ even. Then in (89) only the contribution involving d is present and

$$\begin{aligned}(-1)^{n+m}-1&=0,\\d=2\frac{e^k}{e^k-e^{-k}}+\mathcal{O}(e^{-k})&=2+\mathcal{O}(e^{-k})>1.\end{aligned}$$

The factor in front of d and $((-1)^{n+m}-1)$ in (89) are given by the real and imaginary parts of

$$\begin{aligned}&\frac{1}{k+i(kt+m)}-\frac{1}{k+i(kt-m)}=\frac{k-i(kt+m)}{k^2+(kt+m)^2}-\frac{k-i(kt-m)}{k^2+(kt-m)^2}\\&=(k-ikt)\left(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2}\right)\\&\quad-im\left(\frac{1}{k^2+(kt+m)^2}+\frac{1}{k^2+(kt-m)^2}\right)\\&=k\left(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2}\right)+i\frac{kt(4ktm)-m(2k^2+2k^2t^2+2m^2)}{(k^2+(kt-m)^2)(k^2+(kt-m)^2)}\\&=k\left(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2}\right)-i\frac{m(2k^2-2k^2t^2+2m^2)}{(k^2+(kt-m)^2)(k^2+(kt-m)^2)}.\end{aligned}$$

The coefficients $c_{nm}(t,k)$ are hence explicitly given by

$$\begin{aligned}
&\delta_{nm}(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2}) \\
&\quad + dk\left(\frac{1}{k^2+(kt+n)^2}-\frac{1}{k^2+(kt-n)^2}\right)\left(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2}\right) \\
&\quad + i((-1)^{n+m}-1)\left(\frac{1}{k^2+(kt+n)^2}-\frac{1}{k^2+(kt-n)^2}\right) \\
&\quad \cdot\left(-\frac{m(2k^2-2k^2t^2+2m^2)}{(k^2+(kt+m)^2)(k^2+(kt-m)^2)}+\frac{m}{n^2-m^2}\right) \\
&= \delta_{nm}(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2}) \\
&\quad + dk\left(\frac{1}{k^2+(kt+n)^2}-\frac{1}{k^2+(kt-n)^2}\right)\left(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2}\right) \\
&\quad + i((-1)^{n+m}-1) \\
&\quad \cdot nmkt\frac{k^4t^4+2k^4t^2+2k^4-2k^2t^2(m^2+n^2)+2k^2(m^2+n^2)+2m^2n^2}{(k^2+(kt+m)^2)(k^2+(kt-m)^2)(k^2+(kt+n)^2)(k^2+(kt-n)^2)(n^2-m^2)}.
\end{aligned}$$

□

Appendix B. Stability and boundary conditions

In section 5.3, we required ω_0 to satisfy zero Dirichlet conditions to establish decay of $\partial_y^2\Phi$ and $\partial_y^2\Psi$. In this section, we show that this condition is necessary, both for the explicit example

$$\omega_0(x,y)=2\cos(x), \quad (x,y)\in\mathbb{T}_\pi\times[0,1],$$

as well as for general functions with $\hat{\omega}_0(k)\in H^2([0,1])$. For simplicity, we first consider linearized Couette flow.

Lemma B.1. *Consider the linearized Couette flow in scattering formulation*

$$\begin{aligned}
\partial_t W &= 0, \\
(-k^2+(\partial_y-ikt)^2)\Psi &= W,
\end{aligned}$$

with initial datum $\hat{\omega}_0(k,y)=\delta_1(k)+\delta_{-1}(k)$. Then there exists a sequence $t_n\to\infty$ such that $e^{-kt_ny}\partial_y^2\Psi(t_n,k,y)$ converges to a nontrivial limit in L^2_y.

Proof. By symmetry it suffices to consider $k=1$. The stream function Ψ is then given by

$$\frac{1}{1+t^2}(-1+a(t)e^{y+ity}+b(t)e^{-y+ity}),$$

where a,b solve

$$\begin{pmatrix} e^{1+it} & e^{-1+it} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Differentiating twice, we obtain

$$\Xi := e^{-ity}\partial_y^2\Psi=\frac{(1+it)^2}{1+t^2}a(t)e^y+\frac{(-1+it)^2}{1+t^2}b(t)e^{-y}.$$

As $a(t), b(t)$ depend on t only via e^{it}, for any $c \in \mathbb{R}, m \in 2\pi\mathbb{Z}$,

$$a(c) = a(c+m),$$
$$b(c) = b(c+m).$$

We may thus, for example, consider sequences $t_{1,n} \in 2\pi\mathbb{Z}$ and $t_{2,n} \in 2\pi\mathbb{Z} + \pi$ tending to $\pm\infty$. Along these sequences a, b are constant and nontrivial, while

$$\frac{(\pm 1 + it)^2}{1+t^2} \to -1.$$

Therefore,

$$\Xi(t_n) \to -ae^y - be^{-y} \neq 0,$$

which yields the desired result.

□

A similar result also holds for generic ω_0:

Lemma B.2. *Consider the linearized Couette flow in scattering formulation*

$$\partial_t W = 0,$$
$$(-k^2 + (\partial_y - ikt)^2)\Psi = W.$$

Further let $\hat{\omega}_0(k,\cdot) \in H^2([0,1])$ and suppose that for some $k \neq 0$, $\hat{\omega}_0(k,\cdot)|_{y=0,1}$ is nontrivial. Then $e^{-ity}\partial_y^2\Psi$ does not converge to zero in L^2 as $t \to \pm\infty$.

Proof. Splitting $\partial_y^2\Psi = \Psi^{(2)} + H^{(2)}$ as in section 5.3, we obtain

$$\|\Psi^{(2)}\|_{L^2}^2 \leq \|\Psi^{(2)}\|_{\tilde{H}^1}^2 = \langle \Psi^{(2)}, \partial_y^2 W\rangle \leq \sum_n \langle \frac{n}{k} - t\rangle^{-2} |(\partial_y^2 W)_n|^2.$$

Using a similar argument as in the proof of Theorem 3.1, one can show that $\|\Psi^{(2)}\|_{L^2} \to 0$. Here we use that, for Couette flow, W is preserved in time and hence an L^2 estimate suffices. In the more general case, for this argument one would either need some additional control of the L^2 integrability, e.g.

$$\lim_{N\to\infty} \sup_{t>0} \sum_{|n|\geq N} |(\partial_y^2 W)_n|^2 = 0,$$

or control in a fractional Sobolev space.

It thus remains to consider

$$e^{-ikty} H^{(2)} = \partial_y^2\Psi(0) e^{-ikty} u_1 + \partial_y^2 \Psi(1) e^{-ikty} u_2.$$

For convenience of notation, we again set $k = 1$.

Restricting to sequences $t_n \in 2\pi\mathbb{N}$, $e^{-ity}u_1$ and $e^{-ity}u_2$ do not depend on t and are linearly independent. It thus suffices to show that $\partial_y^2\Psi(0)$ and $\partial_y^2\Psi(1)$ cannot both converge to zero unless ω_0 satisfies zero Dirichlet conditions.

Solving

$$(-1 + (\partial_y - it)^2)\Psi = \hat{\omega}_0$$

for $\partial_y^2\Psi$, we obtain

$$\partial_y^2 \Psi|_{y=0,1} = \hat{\omega}_0|_{y=0,1} + 2it\partial_y\Psi|_{y=0,1}.$$

Testing the above equation with u_j yields

$$\begin{aligned}\partial_y \Psi|_{y=0,1} &= \langle \hat{\omega}_0, u_j \rangle = \langle \hat{\omega}_0, e^{ity}(ae^y + be^{-y}) \rangle \\ &= \frac{1}{it}\hat{\omega}_0|_{y=0,1} - \frac{1}{it}\int e^{ity}\partial_y \left(\hat{\omega}_0(ae^y + be^{-y})\right) \\ &= \frac{1}{it}\hat{\omega}_0|_{y=0,1} + \|\hat{\omega}_0\|_{H^2}\mathcal{O}(t^{-2}).\end{aligned}$$

Here we used that $e^{it_n y}|_{y=0,1} = 1$ for our sequence of t_n. Therefore,

$$\partial_y^2 \Psi|_{y=0,1} = 3\hat{\omega}_0|_{y=0,1} + \mathcal{O}(t_n^{-1}) \not\to 0,$$

which concludes the proof. □

The above method of proof also allows us to derive an instability result for flows other than Couette flow:

Theorem B.1. *Let $f, g \in W^{2,\infty}([0,1])$ such that*

$$0 < c < g < c^{-1} < \infty,$$

and suppose that $f|_{y=0,1} \neq 0$. Then for any $\omega_0 \in H^2$ with $\omega_0|_{y=0,1} \neq 0$, the solution to the linearized Euler equations, (60)*,*

$$\begin{aligned}\partial_t W &= \frac{if(y)}{k}\Phi, \\ \left(-1 + \left(g(y)\left(\frac{\partial_y}{k} - it\right)\right)^2\right)\Phi &= W, \\ \Phi|_{y=0,1} &= 0, \\ (t,k,y) &\in \mathbb{R} \times L(\mathbb{Z} \setminus \{0\}) \times [0,1],\end{aligned}$$

satisfies

$$\sup_{t \geq 0} \|W(t)\|_{H^2} = \infty.$$

Proof of Theorem B.1. Assume for the sake of contradiction that

$$\sup_{t \geq 0} \|W(t)\|_{H^2} = C < \infty. \tag{90}$$

We then claim that $\partial_y \Phi|_{y=0,1}$ satisfies

$$\partial_y \Phi|_{y=0,1} = \frac{1}{ikt}\frac{k}{g^2}\omega_0|_{y=0,1} + C\mathcal{O}(t^{-2}). \tag{91}$$

Considering the evolution of $\partial_y W$ restricted to the boundary,

$$\partial_t \partial_y W|_{y=0,1} = \frac{if}{k}\partial_y \Phi|_{y=0,1},$$

and integrating in time, we thus obtain that, as $T \to \infty$,

$$|\partial_y W(T)|_{y=0,1}| \gtrsim \left| \int_1^T \frac{if}{k}\frac{1}{ikt}\frac{k}{g^2}\omega_0|_{y=0,1} dt \right| + \mathcal{O}(1) \gtrsim \log(T).$$

Hence,

$$\sup_{t \geq 0} \|\partial_y W\|_{C^0} = \infty,$$

which by the Sobolev embedding contradicts our assumption (90) and concludes the proof.

It remains to show the claim, (91). In equation (71) of section 5.2, we have shown that $\partial_y \Phi|_{y=0,1}$ can be computed as

$$\begin{aligned}\partial_y \Phi|_{y=0} &= \frac{k}{g^2(0)}\langle W, u_1\rangle,\\ \partial_y \Phi|_{y=1} &= -\frac{k}{g^2(1)}\langle W, u_2\rangle,\end{aligned}$$

where u_j are solutions of

$$(-1 + (g(y)(\frac{\partial_y}{k} - it))^2)u_j = 0,$$

with boundary conditions, (69),

$$\begin{aligned}u_1(0) = u_2(1) = 1,\\ u_1(1) = u_2(0) = 0.\end{aligned}$$

It hence suffices to show that

$$\begin{aligned}\langle W, u_1\rangle &= -\frac{1}{ikt}\omega_0|_{y=0} + C\mathcal{O}(t^{-2}),\\ \langle W, u_2\rangle &= \frac{1}{ikt}\omega_0|_{y=1} + C\mathcal{O}(t^{-2}).\end{aligned}$$

For this purpose we note that $u_j(t, y)$ satisfy

$$\begin{aligned}u_1(t, y) &= e^{ikty}u_1(0, y),\\ u_2(t, y) &= e^{ikt(y-1)}u_2(0, y).\end{aligned}$$

Integrating e^{ikty} by parts twice, we hence obtain

$$\begin{aligned}\langle W, u_1(t, y)\rangle &= \frac{1}{ikt}Wu_1(t, y)|_{y=0}^1 + \frac{1}{k^2t^2}e^{ikty}\partial_y(Wu_1(0, \cdot))|_{y=0}^1\\ &\quad - \frac{1}{k^2t^2}\langle e^{ikty}, \partial_y^2(Wu_1(0, \cdot))\rangle\\ &= \frac{1}{ikt}Wu_1(t, y)|_{y=0}^1 + C\mathcal{O}(t^{-2}),\end{aligned}$$

where we used a trace estimate to control the second boundary term. □

Using the same approach, one can obtain similar results for higher Sobolev norms involving boundary values of higher derivatives. However, for non-Couette flow the boundary values of higher derivatives are not conserved by the evolution, and therefore conditions of the form

$$\partial_y^n W|_{y=0,1} \equiv 0$$

are in general never satisfied for $n \geq 1$. Instead, one would have to derive necessary and sufficient conditions under which $\partial_y^n W|_{y=0,1} \to 0$ as $t \to \pm\infty$.

Appendix C. Diffeomorphisms with shearing structure

Consider the full 2D Euler equations in either the infinite periodic channel, $\mathbb{T}\times\mathbb{R}$, or the finite periodic channel, $\mathbb{T}\times[0,1]$:

$$\begin{aligned} \partial_t\omega + v\cdot\nabla\omega &= 0,\\ \nabla\times v &= \omega,\\ \nabla\cdot v &= 0,\\ \omega|_{t=0} &= \omega_0, \end{aligned} \tag{92}$$

where, in the case of a finite periodic channel, we consider impermeable walls, i.e.

$$v_2 = 0, \text{ for } y\in\{0,1\}. \tag{93}$$

Restricting to sufficiently regular solutions, we may equivalently consider the evolution of the flow maps X_t (cf. [12, Chapter 2.5]):

$$\begin{aligned} \partial_t X_t &= v(t,X_t),\\ X_0 &= Id,\\ \omega(t,X_t) &= \omega_0. \end{aligned} \tag{94}$$

We further recall that, as v is divergence-free, DX satisfies

$$\det(DX)\equiv 1$$

and is thus measure preserving and invertible. Hence, if $\omega_0\in L^p(\Omega)$, then for any time t also $\omega(t)\in L^p(\Omega)$ and

$$\|\omega(t)\|_{L^p(\Omega)} = \|\omega_0\|_{L^p(\Omega)}.$$

However, we note that in an infinite periodic channel, for solutions close to a monotone shear flow, $U(y)$, in general $\omega_0\notin L^p(\mathbb{T}_L\times\mathbb{R})$, since $\nabla\times(U(y),0) = -U'(y)\notin L^p(\mathbb{T}_L\times\mathbb{R})$. Furthermore, if X_t is not a shear, then

$$\langle\omega\rangle_x = \langle\omega_0\circ X\rangle_x \neq \langle\omega_0\rangle_x\circ X \neq \langle\omega_0\rangle_x.$$

Thus, unlike in the linear setting, the “underlying shear”

$$\langle v\rangle_x = \begin{pmatrix}\langle v_1\rangle_x(t,y)\\ 0\end{pmatrix} \tag{95}$$

corresponding to

$$\begin{aligned} \nabla\times\langle v\rangle_x &= \langle\omega\rangle_x,\\ \nabla\cdot\langle v\rangle_x &= 0, \end{aligned}$$

is no longer time-independent.

In the following, we thus instead consider $\langle\omega\rangle_x(t,y)$ and $\langle v\rangle_x(t,y)$ as given functions and let Y_t denote the flow by $\langle v\rangle_x$, i.e. the solution map of

$$\partial_t f + \langle v\rangle_x\cdot\nabla f = 0.$$

The flow, Y_t, is then of the form

$$Y_t : (x,y)\mapsto (x-u(t,y),y), \tag{96}$$

where

$$u(t,y) = \int_0^t \langle v_1\rangle_x(\tau,y)d\tau. \tag{97}$$

In particular, denoting

$$W(t) := (\omega - \langle \omega \rangle_x) \circ Y_t^{-1},$$

we observe that, unlike (95),

$$\langle W(t) \rangle_x = \langle W(t) \rangle_x \circ Y_t = 0. \tag{98}$$

Similar to Theorem 3.1, in the following theorem we *assume* that Y_t is a good approximation to X in the sense that $W(t) \in H^2_{x,y}(\Omega)$, uniformly in time.

We then study under which assumptions on Y_t the perturbation to the velocity field $v - \langle v \rangle_x$,

$$\begin{aligned} \nabla \times (v - \langle v \rangle_x) &= \omega - \langle \omega \rangle_x = W \circ Y_t, \\ \nabla \cdot (v - \langle v \rangle_x) &= 0, \end{aligned} \tag{99}$$

decays with algebraic rates.

Theorem C.1 (Damping in terms of the flow map Y and W)**.** *Let $W(t) \in L^2_x H^1_y(\Omega)$ be such that for all times*

$$\langle W(t) \rangle_x = 0, \tag{100}$$

$$\|W(t)\|_{L^2_x H^1_y(\Omega)} < C < \infty. \tag{101}$$

Further let Y_t be given by

$$Y_t : (x, y) \mapsto (x - u(t, y), y), \tag{102}$$

and suppose $\partial_y u(t, y) \in W^{2,\infty}$ satisfies

$$\inf_{t,y} \frac{1}{t} \partial_y u(t, y) > c > 0. \tag{103}$$

Then, for any test function $\psi \in H^1(\Omega)$ with compact support in y:

$$\begin{aligned} \iint \psi W \circ Y &= \iint \psi \frac{d}{dx} \left(\frac{d}{dx} \right)^{-1} W \circ Y \\ &= \iint \psi \frac{1}{\partial_y u(t, y)} \frac{d}{dy} \left(\frac{d}{dx} \right)^{-1} W \circ Y + \left(\frac{d}{dx} \right)^{-1} ((\partial_y W) \circ Y) \\ &= \iint \frac{1}{\partial_y u(t, y)} \psi \left(\frac{d}{dx} \right)^{-1} ((\partial_y W) \circ Y) \\ &\quad - \frac{d}{dy} \left(\frac{1}{\partial_y u(t, y)} \psi \right) \left(\frac{d}{dx} \right)^{-1} W \circ Y. \end{aligned} \tag{104}$$

In particular, taking the supremum over all test functions ψ such that $\|\psi\|_{H^1(\Omega)} \leq 1$, we obtain

$$\|v(t) - \langle v \rangle_x\|_{L^2} \lesssim \frac{1}{ct} \|W(t)\|_{H^{-1}_x H^1_y} \lesssim \frac{1}{ct} \|W(t)\|_{L^2_x H^1_y} = \mathcal{O}(t^{-1}).$$

Proof of Theorem C.1. As W satisfies $\langle W \rangle_x = 0$ and as this property is preserved under composition with Y, $\left(\frac{d}{dx}\right)^{-1} W \circ Y$ is well-defined and

$$W \circ Y = \frac{d}{dx} \left(\frac{d}{dx} \right)^{-1} W \circ Y = \frac{d}{dx} \left(\left(\frac{d}{dx} \right)^{-1} W \right) \circ Y.$$

We further note that, by the chain rule

$$
\begin{aligned}
\frac{d}{dx} W \circ Y &= \partial_x Y_1 (\partial_x W) \circ Y + \partial_x Y_2 (\partial_y W) \circ Y, \\
\frac{d}{dy} W \circ Y &= \partial_y Y_1 (\partial_x W) \circ Y + \partial_y Y_2 (\partial_y W) \circ Y,
\end{aligned} \tag{105}
$$

and that

$$
\det \begin{pmatrix} \partial_x Y_1 & \partial_y Y_1 \\ \partial_x Y_2 & \partial_y Y_2 \end{pmatrix} \equiv 1. \tag{106}
$$

Thus,

$$
\frac{d}{dx} W \circ Y = \frac{\partial_x Y_2}{\partial_y Y_1} \frac{d}{dy} W \circ Y + \frac{1}{\partial_y Y_1} (\partial_y W) \circ Y.
$$

The equation (104) hence follows using integration by parts.

In order to prove the desired damping result, we recall from the proof of Theorem 3.1 that

$$
\|v - \langle v \rangle_x\|_{L^2} \lesssim \sup_{\psi : \|\psi\|_{H^1(\Omega)} \leq 1} \iint \psi (\omega - \langle \omega \rangle_x).
$$

Using (104), the proof hence concludes by an application of Hölder's inequality and using that

$$
\frac{1}{\partial_y u(t,y)} < \frac{1}{ct}.
$$

□

As seen in the proof, the theorem can be formulated for flows not of the form (96), and we can also allow $\det(DY)$ to be nonconstant. In this case, (104) is replaced by

$$
\begin{aligned}
\iint \psi W \circ Y = \iint & \frac{\det(DY)}{\partial_y Y_1} \psi \left(\frac{d}{dx}\right)^{-1} ((\partial_y W) \circ Y) \\
& - \frac{d}{dy} \left(\frac{\partial_x Y_2}{\partial_y Y_1} \psi \right) \left(\frac{d}{dx} \right)^{-1} W \circ Y.
\end{aligned}
$$

However, in order to use $\left(\frac{d}{dx}\right)^{-1} W \circ Y$, we have to require that

$$
\langle W \circ Y \rangle_x = 0,
$$

which heavily restricts the possible choices for Y and W. In particular, in general one cannot choose $W = \omega_0 - \langle \omega_0 \rangle_x$ and $Y = X$.

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