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HOW MANY VARIETIES OF CYLINDRIC ALGEBRAS ARE THERE

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Abstract. Cylindric algebras, or concept algebras as another name, form an interface between algebra, geometry and logic; they were invented by Alfred Tarski around 1947. We prove that there are $2^{|\alpha|}$ many varieties of geometric (i.e., representable) α-dimensional cylindric algebras, which means that $2^{|\alpha|}$ properties of definable relations of (possibly infinitary) models of first order theories can be expressed by formula schemes using α variables, where α is infinite. This solves Problem 4.2 in the 1985 Henkin-Monk-Tarski monograph [*Cylindric algebras. Part II*, Studies in Logic and the Foundations of Mathematics, vol. 115, North-Holland, Amsterdam, 1985]; the problem is restated by Németi [*On varieties of cylindric algebras with applications to logic*, Ann. Pure Appl. Logic 36 (1987), no. 3, 235–277] and Andréka, Monk, and Németi [*Algebraic logic*, Colloq. Math. Soc. János Bolyai, Vol. 54, North-Holland, Amsterdam, 1991]. For solving this problem, we devise a new construction, which we then use to solve Problem 2.13 of the 1971 Henkin-Monk-Tarski monograph [*Cylindric algebras. Part I*, Studies in Logic and the Foundations of Mathematics, vol. 64, North-Holland, Amsterdam, 1971] which concerns the structural description of geometric cylindric algebras. There are fewer varieties generated by locally finite-dimensional cylindric algebras, and we get a characterization of these among all the $2^{|\alpha|}$ varieties. As a by-product, we get a simple recursive enumeration of all the equations true of geometric cylindric algebras, and this can serve as a solution to Problem 4.1 of the 1985 Henkin-Monk-Tarski monograph. All of this has logical content and implications concerning ordinary first order logic with a countable number of variables.

1. Introduction

Cylindric algebras (or concept algebras as another name) are an algebraic form of first order logic, analogous to Boolean algebras which are an algebraic form of propositional logic. Cylindric algebras were created by Alfred Tarski around 1947, they are Boolean algebras endowed with complemented closure operators—one for each quantifier $\exists v_i$—and constants $v_i = v_j$ for representing equality. Set cylindric algebras analogous to set Boolean algebras are called representable cylindric algebras, these latter are equationally definable (i.e., form a variety), this was proved by Tarski in 1952. However, unlike the propositional case, not all cylindric algebras are representable, and this cannot be repaired easily since J. Donald Monk proved in 1969 that the representable algebras are not finite schema axiomatizable. This gap between "abstract" and representable cylindric algebras is in the center of algebraic logic studies, it is a source of insights and problems.

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A set cylindric algebra is the algebra of all definable relations of a model. Representable cylindric algebras are the same, just for classes of models, i.e., for theories, in place of single models. We often call them concept algebras, since they are natural algebras of concepts of a theory. (The name "cylindric algebras" on the other hand refers to the geometrical meaning of these algebras.) Properties of a model, or of a theory, can be read off its concept algebra; e.g., whether dense linear order can be defined in the model can be expressed by an equation. Equations of concept algebras talk about properties of all definable relations in a model, as opposed to talking about the primitive relations only. An equation in the algebraic language corresponds to a formula schema of first order logic where the formula variables range over formulas with free variables of the language, this way they "talk about" definable relations of models. In this context, the problem asking about the number of varieties of representable cylindric algebras asks how many properties of definable relations we can express by first order logic schemata. In this connection, the present work is not unrelated to Shelah's classification theory. (For some connections, see [7, 40].)

Let α denote the number of variables we have in the first order language, the algebraic versions of which we investigate. In this paper we deal with infinite α only, let RCA_α denote the class of representable cylindric algebras of this language. Then we have α many closure operations called cylindrifications, and $\alpha \times \alpha$ many constants called diagonal constants apart from the Boolean operations, so our equational language has size α which means that there are at most $2^{|\alpha|}$ many different equational theories in the algebraic language. It was known that RCA_α has at least continuum many subvarieties ([19, Thm. 4.1.24]) and [19, Problem 4.2] asks whether there are $2^{|\alpha|}$ many for uncountable α. There are theorems pointing to the answer being continuum (i.e., 2^ω, where ω is the least infinite ordinal) and there are theorems pointing to the answer being $2^{|\alpha|}$. For example, it is proved in [34] that in concept algebras of finite models we can only express the size of models and nothing else; in the algebraic language the theorem says that the concept algebras of models of size n generate a variety which is an atom of the lattice of varieties of RCA_α. On the other hand, we can say a lot of things about definable relations in infinite models. For example, the equational theory of the concept algebra of a single infinite model in which we have no primitive relations (i.e., in which all the definable relations are the ones we can define by using the equality only) is not recursively enumerable; see [34, Thm. 1(i)]. We prove in this paper that there are $2^{|\alpha|}$ many subvarieties of RCA_α, but we also show that this is only part of the answer since there is a large subclass of the distinguished representable cylindric algebras which has only continuum many subvarieties.

From the logical point of view, at first, the subject of the above discussed problem might look "esoteric", since it concerns the difference between using countably or uncountably infinite number of variables in the logical languages having only formulas built up from finitely many variables. However, our solution has impact on the countable case, too. We mentioned already that the gap between abstract and representable cylindric algebras is a source of insights and problems. One of the insights has been that this gap is concerned with the number of "free" extra variables v_i we have for a formula or proof in the logical language. The crucial thing is not that each first order formula uses only finitely many variables, but instead that for each finite set of formulas there is always at least one variable that they do not use. Let us call a cylindric algebra dimension-complemented if to each finite subset there is a cylindrification of which all elements of this finite subset are fixed-points; one can prove then that all dimension-complemented cylindric algebras are representable. The strongest and most beautiful form of this insight is Leon Henkin's theorem saying that an abstract cylindric algebra is representable if and only if it can be embedded to one having infinitely many more cylindric operations so that each of the elements of the original algebra are fixed-points of all the new operations. Efforts were made to find structural properties of abstract cylindric algebras, similar to being dimension-complemented as mentioned above, which refer to only the original cylindric algebra without comparing it to others. Theorem 2.6.50 in [18] summarizes how far they got in this respect, and [18, Problem 2.13] asks if their last description (which we call here endo-dimension-complemented) captures being representable or not.

Our full answer to [19, Problem 4.2] shows that when proving that there are many varieties of RCA_α, we used unusual features of representable cylindric algebras, so we had to devise a new kind of construction. This contributes to understanding the structural properties of representable cylindric algebras. Namely, our construction shows that "endo-dimension-complemented" is not the final answer in [18, Thm. 2.6.50], since our new constructions are representable yet not endo-dimension-complemented, solving Problem 2.13 in the negative. However, here, too, something positive can be added as a result of this solution. A new property extending "endo-dimension-complemented" toward representability emerges by an analysis of our construction. We call an abstract cylindric algebra inductive iff from the fact that an equation holds for all elements which are fixed-points of a given cylindric operation, we can infer that this equation holds for the whole algebra, provided that the given cylindrification does not occur in the equation. In a way, this property also talks about extra free variables. We prove that each endo-dimension-complemented algebra is inductive and each inductive algebra is representable, but none of the reverse implications holds. In this way we get a new representability theorem extending the chain of classes in [18, Thm.2.6.50], and we get a new insight about representable algebras.

A cylindric algebra is called locally finite-dimensional if each of its elements is the fixed-point of all but finitely many cylindrifications, Lf_α denotes their class. These algebras correspond to theories of first order logic in which all primitive relations have finite rank. The rest of the representable algebras correspond to theories of first order logic when the primitive relations may have arbitrary ranks, let us call this logic infinitary first order logic to distinguish it from the former which we call finitary or ordinary (following [19]). It was known that the infinitary and finitary first order logics have the same valid formulas, in algebraic form this theorem says that Lf_α generates the variety RCA_α. However, it was not known whether the two logics have the same theories, i.e., whether all subvarieties of RCA_α are generated as varieties by their locally finite-dimensional members. In this paper we give a negative answer to this: there are subvarieties of RCA_α not generated by their locally finite-dimensional members, even in the $\alpha = \omega$ case, so the theories of finitary and infinitary first order logics are different. The notion of inductivity is the key notion to finding this answer. The property of a cylindric algebra being inductive can easily be converted to a new logical rule, which we call inductive rule. We prove that this rule is admissible for ordinary first order logic but it is not admissible for infinitary first order logic. In particular, we prove that a theory is one of ordinary first order logic iff it is closed under the inductive rule.

The above theorem readily yields a simple recursive enumeration for all equations valid in RCA_α. This gives a possible solution to [19, Problem 4.1], this problem is restated in numerous other places, e.g., in [4, Problem 19, p. 735], or in logical form as [34, Problem 2.9].

The structure of the paper is the following. In the rest of the introduction we recall the definitions, notation, and background needed in the paper. Section 2 contains the proof that there are $2^{|\alpha|}$ varieties of RCA_α, section 3 contains the proof of an analogous statement for abstract cylindric algebras. Section 4 is concerned with subclasses where there are only continuum many varieties and with the results we can get from the proof contained in section 2. We introduce the notions of symmetric, endo-dimension-complemented (endo-dc in short), and inductive algebras. Section 4.1 investigates the relationship between symmetric and endo-dc algebras leading to the solution of [18, Problem 2.13]. Section 4.2 reveals connections with the polyadic substitution operations. Section 4.3 concerns inductive algebras. It places this notion between endo-dc and symmetric and shows representability of inductive algebras by proving that they are exactly the algebras equationally indistinguishable from a member of Lf_α. As a corollary we get a Δ_2-formula separating Lf_α and RCA_α. Section 4.4 contains a simple recursive enumeration of the equational theory of RCA_α and the characterization of varieties generated by subclasses of Lf_α. Finally, section 4.5 contains a proof that locally finite-dimensional set algebras with infinite bases have indeed continuum many subvarieties.

1.1. **Background.** Where not specified otherwise, we use the notation of [18, 19], but we try to be self-contained.

Let α be any set. An algebra $\langle A, +, -, c_i, d_{ij} : i, j \in \alpha\rangle$ is a *cylindric algebra* of dimension α if $\langle A, +, -\rangle$ is a Boolean algebra and for all distinct $i, j, k \in \alpha$ the following hold. The operations c_i are unary, they are commuting complemented closure operators, i.e., for all $x, y \in A$ we have $c_i c_j x = c_j c_i x$, $x \le c_i x = c_i c_i x$, $c_i(x + y) = c_i x + c_i y$, $c_i - c_i x = -c_i x$. The operations d_{ij} are nullary, i.e., they are constants and they satisfy the following equations: $d_{ii} = 1$, $d_{ij} = d_{ji}$, $c_j(d_{ij} \cdot d_{jk}) = d_{ik}$, $c_i d_{ij} = 1$, $d_{ij} \cdot c_i(d_{ij} \cdot x) = d_{ij} \cdot x$. In the above, $\cdot$ is the Boolean intersection defined from $+, -$ in the usual way. The extra-Boolean operations c_i and d_{ij} are called *cyindrifications* and *diagonal constants*, respectively. The schemata of equations (C_0)–(C_7) in [18, p. 161] express the above. CA_α denotes the class of all cylindric algebras.

The cylindric *set algebras* of dimension α are Boolean set algebras of α-dimensional spaces, where the extra-Boolean operations have natural geometric interpretations. Let U be any set, the points of the α-dimensional space ${}^\alpha U$ over U are the U-termed α-sequences, i.e., the set of all functions from α to U. We use the terms sequence and function to mean the same thing. If s is any sequence, then $\mathsf{Dom}(s), \mathsf{Rg}(s)$ denote its domain and range, s_i denotes $s(i)$, and $s(i/u)$ denotes the sequence we get from s by changing its value at i to be u if $i \in \mathsf{Dom}(s)$ (and $s(i/u)$ denotes s if $i \notin \mathsf{Dom}(s)$). For $V \subseteq {}^\alpha U$ the *full cylindric set algebra with unit* V is

$$\langle \mathcal{P}(V), \cup, -, C_i^V, D_{ij}^V : i, j \in \alpha\rangle, \quad \text{where } \mathcal{P}(V) \text{ is the powerset of } V, \text{ and}$$
$$C_i^V X := \{s(i/u) \in V : s \in X, u \in U\}, \quad D_{ij}^V := \{s \in V : s_i = s_j\}.$$

Algebras isomorphic to subalgebras of full cylindric set algebras with units as disjoint unions of α-dimensional spaces are called *geometric* or *representable*, RCA_α denotes their class. (For technical reasons, when α is a one-element set, RCA_α is defined as the class of subdirect products of these.) Cylindric set algebras with units of form ${}^\alpha U$ are called simply *cylindric set algebras*, U is called their *base set*, and their class is denoted as Cs_α.

Geometric cylindric algebras have natural *logical meaning*, too. Let us have a first order equality language $\mathcal{L}$ with variables as α (or v_i with $i \in \alpha$), and with the logical connectives $\vee, \neg, \exists v_i, v_i = v_j$ for $i \in \alpha$, and let $\mathfrak{M}$ be a relational structure with universe U and with at most α-place primitive relations. The points of the α-dimensional space over U are also evaluations of variables. For any formula φ in this language let $\varphi^{\mathfrak{M}}$ denote the set of evaluations of variables under which φ is true in $\mathfrak{M}$. Then $(\varphi \vee \psi)^{\mathfrak{M}} = \varphi^{\mathfrak{M}} \cup \psi^{\mathfrak{M}}$, $(\neg\varphi)^{\mathfrak{M}} = {}^\alpha U - \varphi^{\mathfrak{M}}$, $(\exists v_i \varphi^{\mathfrak{M}}) = C_i \varphi^{\mathfrak{M}}$ and $(v_i = v_j)^{\mathfrak{M}} = D_{ij}$ (with $V = {}^\alpha U$). Thus

$$\mathrm{Ca}^{\mathfrak{M}} := \{\varphi^{\mathfrak{M}} : \varphi \in \mathcal{L}\}$$

is a subuniverse of the full cylindric set algebra with unit ${}^\alpha U$. We call the subalgebra with this universe the *concept algebra* of $\mathfrak{M}$, its universe is the set of all definable relations over $\mathfrak{M}$. Concept algebras for theories can be defined analogously, the unit of such a concept algebra is a disjoint union of sets of the form ${}^\alpha U$, so it is in RCA_α. Two theories are definitionally equivalent iff their concept algebras are isomorphic, and homomorphisms from one concept algebra to another correspond exactly to the interpretations between the two theories. Thus, the category RCA_α with homomorphisms corresponds to the category of all theories and interpretations between them (see [19, sec. 4.3]). This feature of concept algebras comes in handy in applications of logic, e.g., in physics. For this kind of applications see, e.g., [5, 8, 26–28, 46].

Equations holding for concept algebras have several logical interpretations; see, e.g., [19, sec. 4.3]. The most direct of these is where we interpret the algebraic variables occurring in the equation as schemes for formulas in the corresponding logical form. In this way we get the logic of formula schemata; for definitions see e.g., [34], [36, sec. 3.7]. This schema logic talks directly about the definable relations in a model. Alternatively, we can interpret the algebraic variables as primitive α-place relation symbols (full restricted first order logic in [19]), or as primitive relation symbols of unspecified but finite arity (type-free logic in [19]). Via these logical interpretations, results about varieties of RCA_α imply corresponding results about these logics. We deal with the algebraic aspects in this paper, the logical consequences are dealt with in a separate paper.

A class of algebras that can be axiomatized/defined by a set of equations is called a *variety*. Let K be a class of algebras of the same similarity class. Then $\mathsf{Eq}(\mathsf{K})$ denotes the set of all equations (using a countable set of prespecified algebraic variables) valid in all algebras in K, and K is a variety iff K consists of all algebras in which $\mathsf{Eq}(\mathsf{K})$ is true. The variety generated by K is the least variety containing K, this is the class of all algebras in which $\mathsf{Eq}(\mathsf{K})$ holds. Varieties are one of the main subjects of universal algebra, Birkhoff's theorem says, e.g., that all members of the variety generated by K can be obtained from members of K as homomorphic images of subalgebras of products of members of K. The class of algebras isomorphic to members of K is denoted by $\mathbf{I}\mathsf{K}$, the class of subalgebras of members of K is denoted by $\mathbf{S}\mathsf{K}$.

The following are the main facts, in connection with the present paper, known about varieties of cylindric algebras, these are contained in [18–20] if not specified otherwise. $\mathsf{RCA}_\alpha \subseteq \mathsf{CA}_\alpha$ is a variety, it is not finitely axiomatizable iff $|\alpha| > 2$, where if α is any set, $|\alpha|$ denotes its cardinality. J. D. Monk [30, 31] characterized the lattice of all subvarieties of RCA_α for $|\alpha| = 1$, and a similar characterization for $|\alpha| = 2$ is contained in [9]. Let α be infinite. Some of the distinguished subvarieties of RCA_α are $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ and ${}_n\mathsf{RCA}_\alpha$ for finite n. $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ is the class of all algebras isomorphic to a cylindric set algebra with unit of form ${}^\alpha U$ for infinite U, and ${}_n\mathsf{RCA}_\alpha$ is the class of all algebras isomorphic to cylindric set algebras with unit as disjoint union of sets of form ${}^\alpha U$ where $|U| = n$. It is proved in [34] that ${}_n\mathsf{RCA}_\alpha$ are atoms in the lattice of subvarieties of RCA_α, but $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ is not an atom. The structure of subvarieties is interesting and is investigated for other kinds of algebras related to logic as well; see, e.g., [3, 10, 11, 16, 24, 25].

Monographs and books on these algebras and their logical applications include [2, 4, 12, 14, 15, 17–20, 23, 29, 32, 36].

2. Number of subvarieties of RCA_α

Let α be any infinite ordinal, throughout the rest of the paper. (We assume that α is an ordinal, and not just any set, for convenience, this way α is ordered by the elementhood relation.) It is proved in [19, 4.1.24] that there are at least 2^ω many subvarieties of RCA_α. Since in the language of CA_α there are $|\alpha|$ many equations, there can be at most $2^{|\alpha|}$ many subvarieties of any $\mathsf{K} \subseteq \mathsf{CA}_\alpha$. Problem 4.2 in [19] asks if there are $2^{|\alpha|}$ many subvarieties of RCA_α and of $\mathbf{I}\mathsf{Cs}_\alpha \subseteq \mathsf{RCA}_\alpha$, if α is uncountable. The problem is restated in [4, Problem 41, p. 738]. In this section, we prove that indeed there are a maximum number of subvarieties of RCA_α as well as of $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha \subseteq \mathbf{I}\mathsf{Cs}_\alpha$. Note that both RCA_α and $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ are varieties but $\mathbf{I}\mathsf{Cs}_\alpha$ is not.

Theorem 2.1 (Solution of [19, Problem 4.2]). *Let α be infinite. There are $2^{|\alpha|}$ many subvarieties of RCA_α as well as of $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$.*

The proof of Theorem 2.1 is contained in subsections 2.1–2.4. The idea of the proof is the following. We exhibit a set of equations E of cardinality $|\alpha|$ which is independent in the sense that no element of E follows from the rest of the equations in E. All these equations will be variants of a single equation e such that we rename the indices of the operations occurring in e. We will show independence of E by constructing one algebra $\mathfrak{A} \in \mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ in which e fails but the rest of the equations in E hold. Then the algebras in which we rename the operations c_i and d_{ij} according to appropriate permutations of α will show independence of the whole set E. This will show that all the subvarieties of $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ specified by the $2^{|\alpha|}$ many subsets of E are distinct. We begin with constructing the "witness" algebra $\mathfrak{A}$, because it will give intuition for writing up the "master" equation e.

2.1. Construction of the witness algebra $\mathfrak{A}$.

Let $\langle V_i : i \in \alpha\rangle$ be a system of sets such that

$$V_0 = V_1 = V_2 \text{ is the set of rational numbers,}$$

and all the other V_i's are pairwise disjoint two-element sets disjoint also from V_0 (i.e., $V_i \cap V_j = \emptyset$ for $2 \le i < j < \alpha$). Let U be the union of these sets, i.e.,

$$U := \bigcup \{V_i : i \in \alpha\},$$

let p be an α-sequence such that $p_i \in V_i$ for all $i \in \alpha$ and let V be the set of U-termed α-sequences that deviate from p only at finitely many places, i.e.,

$$V := {}^{\alpha}U^{(p)} = \{s \in {}^{\alpha}U : |\{i \in \alpha : s_i \ne p_i\}| < \omega\}.$$

This V will be the unit of our algebra. Since V_0 is the set of rational numbers, we will use the usual operations and ordering $<$ between rational numbers. Our algebra is generated by a single element, namely by

$$\begin{aligned} g := \{s \in V : s_0 < s_1 < s_2 \text{ and } s_i \in V_i \text{ for all } i \in \alpha \text{ and} \\ s_1 = (s_0 + s_2)/2 \quad \text{if } |\{s_i : s_i \ne p_i, i > 2\}| \text{ is even}, \\ s_1 \ne (s_0 + s_2)/2 \quad \text{if } |\{s_i : s_i \ne p_i, i > 2\}| \text{ is odd}\}. \end{aligned}$$

Let $\mathfrak{A}$ denote the subalgebra of the full set algebra with unit V that is generated by g.

Sets of form ${}^{\alpha}U^{(p)}$ for some set U and $p \in {}^{\alpha}U$ are called *weak spaces* and algebras with unit a weak space are called *weak set algebras*, their class is denoted by Ws_α. It is proved in [19, 3.1.102] that $\mathsf{Ws}_\alpha \subseteq \mathbf{I}\mathsf{Cs}_\alpha$, thus our above constructed algebra $\mathfrak{A}$ is in $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ since the fact that U is infinite is reflected by an equation holding in $\mathfrak{A}$ (see, [18, 2.4.61]). (We note that we could have used for our witness algebra the subalgebra of the full cylindric set algebra with unit ${}^{\alpha}U$ generated by g, the proofs would be only slightly more complicated.) The set U is called the *base set* of $\mathfrak{A}$.

2.2. **Describing the elements of $\mathfrak{A}$.** The idea behind the construction of the witness algebra $\mathfrak{A}$, defined in the previous section, is that we put some information in g at the indices $0, 1, 2$; this information cannot be transferred by the cylindric operations to higher indices $i, j, k \in \alpha$. Then we express lack of this information by an equation e. If we succeed with realizing this idea, then e would fail in $\mathfrak{A}$ at $0, 1, 2$ while e would be valid in $\mathfrak{A}$ at higher indices. We now turn to elaborating this idea.

Let $R \subseteq {}^{n}U$ be an n-place relation on U, where n is any ordinal. We say that X is a *sensitive cut* of R if $c_i X = c_i(R - X)$ for all $i < n$. (Here, c_i denotes cylindrification of the full Cs_n with base set U, i.e., $c_i X = \{s(i/u) \in {}^{n}U : s \in X\}$.) Thus, as soon as we apply a cylindrification to X, the information on how X cuts R into two parts is lost. This technique is widely applicable; cf., e.g., [1, 39, 41, 42]. We are going to show that our generator g is a sensitive cut of

$$T := \{s \in V : s_0 < s_1 < s_2 \text{ and } s_i \in V_i \text{ for all } i \in \alpha\}.$$

Intuitively, the proof will be a kind of "flip-flop" play between the two independent conditions $s_1 = (s_0 + s_2)/2$ and $|\{s_i : s_i \ne p_i, i > 2\}|$ being even in the definition of g. Indeed, let $i \in \alpha$ and $s \in T$. We show that $s(i/u) \in g$ while $s(i/v) \in T - g$ for some $u, v \in U$. This will show that $c_i g = c_i(T - g)$. By $s \in T$ we have $s_0 < s_1 < s_2$. Let $\Sigma = |\{s_i : s_i \ne p_i, i > 2\}|$, $u = (s_0 + s_2)/2$, and let v be such that $v \ne u$, $s_0 < v < s_2$. Assume first that $i = 1$. Now, if Σ is even, then $s(1/u) \in g$, $s(1/v) \in T - g$, and if Σ is odd, then $s(1/u) \in T - g$, $s(1/v) \in g$. For $i = 0$ choose $u = 2s_1 - s_2$ and $v < u$, for $i = 2$ choose $u = 2s_1 - s_0$ and $v > u$, for these choices the same is true as in the case of $i = 1$. Assume $i > 2$ and let v be the element of V_i distinct from s_i. Assume $s_1 = (s_0 + s_2)/2$. Then $s \in g$, $s(i/v) \in T - g$ if Σ is even, and otherwise $s \in T - g$, $s(i/v) \in g$. Assume $s_1 \neq (s_0 + s_2)/2$. Then just the other way round, namely $s \in T - g$, $s(i/v) \in g$ if Σ is even, and otherwise $s \in g$, $s(i/v) \in T - g$. We have shown that

$$(1) \qquad c_i g = c_i(T - g) = c_i T \quad \text{for all } i \in \alpha.$$

Next we show that this last property implies that the elements of A are those that are generated by T and perhaps one of g, $T - g$ added.

Lemma 2.1. *Let $\mathfrak{B}$ be the weak set algebra of dimension α with unit V and generated by T. Then*

$$A = \{x + h : x \in B \text{ and } h \in \{0, g, T - g\}\}.$$

Proof. Each element of the form $x+h$ is generated by g, since $T = c_0 g \cdot c_2 g$. To finish the proof, we are going to show that A is closed under the cylindric operations c_i and d_{ij} as well as under the Boolean operations $+, -$. Let $i, j \in \alpha$. Now, A is closed under c_i by (1) since $c_i(x + h) = c_i x + c_i h \in B$ by $x \in B$ and $c_i h \in \{0, c_i T\} \subseteq B$. Also, $d_{ij} \in A$ by $d_{ij} \in B$. The set A is closed under Boolean addition $+$ by its definition and by $g + (T - g) = T \in B$.

To see that A is closed under Boolean complementation, first we show that T is an atom in B. We will use the following property of $<$ later on, as well:

(*) Assume that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. There is an automorphism π of $\langle V_0, < \rangle$ mapping $a_1, \ldots, a_n$ to $b_1, \ldots, b_n$, respectively. If $a_1 = b_1$ and $a_n = b_n$, then π can be chosen such that it is the identity on elements smaller than a_1 or bigger than a_n.

To prove (*), for $a < b$ let $[a, b] = \{x \in V_0 : a \leq x \leq b\}$ denote the closed interval between a and b. Let $u, v \in V_0$ be such that $u < a_0$, $u < b_0$ and $v > a_n$, $v > b_n$ and define $a_0 := b_0 := u$ and $a_{n+1} := b_{n+1} := v$. For $k \leq n$ and $x \in [a_k, a_{k+1}]$ let $\pi_k(x) := (x - a_k) \cdot (b_{k+1} - b_k)/(a_{k+1} - a_k) + b_k$. Then π_k is an isomorphism between $\langle [a_k, a_{k+1}], < \rangle$ and $\langle [b_k, b_{k+1}], < \rangle$, thus their union $\sigma := \pi_0 \cup \cdots \cup \pi_n$ is an automorphism of $\langle [a_0, a_{n+1}], < \rangle$ taking $a_0, \ldots, a_{n+1}$ to $b_0, \ldots, b_{n+1}$, respectively. We now can choose π to be the identity outside $[a_0, a_{n+1}]$ and σ on the interval. This proves (*).

Returning to showing that T is an atom, let $s, z \in T$ be arbitrary. Then $s_0 < s_1 < s_2$ and $z_0 < z_1 < z_2$ and $s_i, z_i \in V_i$ for all $i \in \alpha$ by the definition of T. Let π be a permutation of U which takes s_i to z_i for all $i \in \alpha$, is a permutation of V_i for all $i \in \alpha$, and is an automorphism of $\langle V_0, < \rangle$. By (*), there is such a π. Then clearly, π takes s to z in the sense that $z = \pi(s) := \langle \pi(s_k) : k \in \alpha \rangle$ while leaving T as well as V fixed, i.e., $T = \pi(T) := \{\pi(q) : q \in T\}$ and $V = \pi(V) := \{\pi(q) : q \in V\}$. This implies that $\pi(x) = x$ for all $x \in B$ (see, e.g., [19, 3.1.36]), so $s \in x$ implies $z \in x$ for all $x \in B$. Since $s, z \in T$ were chosen arbitrarily, this shows that T is an atom in B. We are ready to show that A is closed under complementation. Let $x \in B$ and $h \in \{0, g, T - g\}$. If T is disjoint from x, then $V - (x + h) = (V - x) + (T - h) \in A$, and if $T \leq x$, then $x = x + h$ and we are done with proving Lemma 2.1. □

2.3. **The set E of independent equations.** The "master equation" e will express about an element that it is not similar to our generator g. Namely, for an element x, it will say that either x is not a sensitive cut of its closure $c_0 x \cdot c_2 x$ (this is T

in the case of g), or else this closure is not like T in the sense that the first two coordinates of c_2x form a strict linear order $<_x$ and the ternary beginning of the closure is $\{\langle u, v, w\rangle : u <_x v <_x w\}$. An equation can talk about finitely many indices only, our equation will concern the first three indices $0, 1, 2$.

We begin writing the equation e. First we write a term we will use in checking that x is not a sensitive cut of its closure $z := c_0x \cdot c_2x$ (precise statements about the meanings of the terms below can be found in the proof of Lemma 2.2 that is to be stated soon).

$$\beta(x) := c_0x \oplus c_0(z - x) + c_1x \oplus c_1(z - x) + c_2x \oplus c_2(z - x),$$

where $\oplus$ denotes the Boolean symmetric difference, i.e., $x \oplus y := (x \cdot y) + (-x \cdot -y)$. In writing the rest of the terms, it will be convenient to use the following notation. It concerns rearranging sequences in set algebras (see (4)–(7) somewhat later):

$$\begin{aligned} \mathsf{s}^i_j x &:= c_i(d_{ij} \cdot x) \quad \text{for } i \neq j,\\ \mathsf{s}^{01}_{12} x &:= \mathsf{s}^0_1 \mathsf{s}^1_2 x,\\ \mathsf{s}^{01}_{10} x &:= {}_2s(0,1)c_2x = \mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^1_2 c_2x,\\ \mathsf{s}^{12}_{01} x &:= \mathsf{s}^2_1\mathsf{s}^1_0 x. \end{aligned}$$

Next we write the terms we use in expressing that it is not the case that the binary relations at places 01 and 12 of x coincide:

$$\gamma(x) := c_2x \oplus s^{12}_{01}c_0x.$$

Finally, we write the terms for expressing that c_2x is not a strict linear order:

$$\begin{aligned} \iota(x) &:= c_2x \cdot d_{01} && \text{not irreflexive,}\\ \sigma(x) &:= c_2x \cdot s^{01}_{10}c_2x && \text{not antisymmetric,}\\ \tau(x) &:= c_2x \cdot s^{01}_{12}c_2x - s^1_2c_2x && \text{not transitive,}\\ \lambda(x) &:= c_1c_2x \cdot c_0c_2x \cdot -c_2x \cdot -s^{01}_{10}c_2x && \text{not linear,}\\ \o(x) &:= \iota(x) + \sigma(x) + \tau(x) + \lambda(x). \end{aligned}$$

Let

$$e(x) \quad := \quad x \leq c_{(3)}(\beta(x) + \gamma(x) + \o(x)), \tag{2}$$

where $c_{(3)}y := c_0c_1c_2y$.

We now turn to stating precisely what the equation $e(x)$ expresses in a set-algebra about an element x. We will use the following notation extensively. Let U be a set, $s \in {}^\alpha U$, let $n \in \omega$, let $H \in {}^n\alpha$ be an injection (i.e., repetition-free as a sequence), and let $q \in {}^nU$. Then $s(H/q)$ denotes the sequence we get from s by changing $s(H_k)$ to q_k, simultaneously, for all $k < n$. We will write finite sequences $\langle i_0, i_1, \ldots, i_n\rangle$ in the simplified form $i_0i_1 \ldots i_n$ when this is not likely to lead to confusion; e.g., $s(12/uv) = s(1/u)(2/v)$. Further, if $x \subseteq {}^\alpha U$, then $x[s, H]$ denotes the n-place relation defined as

$$x[s, H] := \{q \in {}^nU : s(H/q) \in x\}.$$

For example, $x[s, 01] = \{uv : s(01/uv) \in x\}$ and $x[s, 0] = \{u : s(0/u) \in x\}$. Finally, we say that a binary relation R is *linear on* W iff W is both the domain and range of R and $\langle u, v\rangle \in R$ or $\langle v, u\rangle \in R$ for all $u, v \in W$.

Lemma 2.2. *Let $\mathfrak{C}$ be any α-dimensional set algebra with unit a disjoint union of weak spaces. Then the equation e as defined in (2) is true in $\mathfrak{C}$ at $x \in C$ iff for all $s \in x$ at least one of the following is true:*

(i) *$x[s, 012]$ is not a sensitive cut of $Z := (c_0 x \cdot c_2 x)[s, 012]$;*
(ii) *$<_x := c_2 x[s, 01]$ is not a strict linear order on $W := c_1 c_2 x[s, 0]$;*
(iii) *$Z \neq \{uvw \in {}^3W : u <_x v <_x w\}$.*

Proof. Recall that the equation e is written up by the use of the terms $\beta, \gamma, \emptyset$. We begin with expounding the meanings of these three terms in the set algebra $\mathfrak{C}$. Let $x \in C$ and $s \in x$.

To deal with the term β, let $z := c_0 x \cdot c_2 x$, then $Z = z[s, 012]$. We are going to prove

(3) $$s \notin c_{(3)}\beta(x) \quad \text{iff} \quad x[s, 012] \text{ is a sensitive cut of } z[s, 012].$$

Indeed,

$$\begin{aligned}
&s \in -c_{(3)}(c_0 x \oplus c_0(z - x)) && \text{iff}\\
&s(012/uvw) \notin c_0 x \oplus c_0(z - x) \text{ for all } u, v, w && \text{iff}\\
&s(012/uvw) \in (c_0 x \cdot c_0(z - x) + (-c_0 x \cdot -c_0(z - x)) \text{ for all } u, v, w && \text{iff}\\
&(c_0 x)[s, 012] = (c_0(z - x))[s, 012] && \text{iff}\\
&c_0(x[s, 012]) = c_0(z[s, 012] - x[s, 012]).
\end{aligned}$$

In the last step we used $(c_0 x)[s, 012] = c_0(x[s, 012])$ and $(z - x)[s, 012] = z[s, 012] - x[s, 012]$; these statements are easy to verify by using the definitions. All of the above hold also for c_1 and c_2 in place of c_0, so we get (3).

For dealing with the other two terms, we make some preparations. Namely we expound the set theoretical meanings of s^{ij}_{kl} occurring in them. Indeed, assume $i \neq j$. We are going to check the following:

(4) $$s \in \mathsf{s}^i_j x \quad \text{iff} \quad s(i/s_j) \in x,$$

(5) $$s \in \mathsf{s}^{12}_{01} x \quad \text{iff} \quad s(12/s_0 s_1) \in x,$$

(6) $$s \in \mathsf{s}^{01}_{10} c_2 x \quad \text{iff} \quad s(01/s_1 s_0) \in c_2 x,$$

(7) $$s \in \mathsf{s}^{01}_{12} x \quad \text{iff} \quad s(01/s_1 s_2) \in x.$$

Indeed, (4) is true because

$$\begin{aligned}
&s \in \mathsf{s}^i_j x = c_i(d_{ij} \cdot x) && \text{iff}\\
&s(i/u) \in d_{ij} \cdot x \quad \text{for some } u && \text{iff}\\
&u = s_j \text{ and } s(i/u) \in x \quad \text{for some } u && \text{iff}\\
&s(i/s_j) \in x.
\end{aligned}$$

(5) is true because

$$\begin{aligned}
&s \in \mathsf{s}^{12}_{01} x = \mathsf{s}^2_1 \mathsf{s}^1_0 x && \text{iff, by (4)}\\
&s(2/s_1) \in \mathsf{s}^1_0 x && \text{iff, by (4)}\\
&(s(2/s_1))(1/s_0) \in x && \text{iff}\\
&s(12/s_0 s_1) \in x.
\end{aligned}$$

Checking (6):

$$\begin{aligned}
&s \in \mathsf{s}^{01}_{10} c_2 x = \mathsf{s}^2_0 \mathsf{s}^0_1 \mathsf{s}^1_2 c_2 x && \text{iff, by (4)}\\
&s(2/s_0) \in \mathsf{s}^0_1 \mathsf{s}^1_2 c_2 x && \text{iff, by (4)}\\
&(s(2/s_0))(0/s_1) \in \mathsf{s}^1_2 c_2 x && \text{iff, by (4)}\\
&((s(2/s_0))(0/s_1))(1/s_0) \in c_2 x && \text{iff}\\
&s(012/s_1 s_0 s_0) \in c_2 x && \text{iff}\\
&s(01/s_1 s_0) \in c_2 x.
\end{aligned}$$

Checking (7):

$$\begin{aligned}
&s \in \mathsf{s}^{01}_{12} x = \mathsf{s}^0_1 \mathsf{s}^1_2 x && \text{iff, by (4)}\\
&s(0/s_1) \in \mathsf{s}^1_2 x && \text{iff, by (4)}\\
&(s(0/s_1))(1/s_2) \in x && \text{iff}\\
&s(01/s_1 s_2) \in x.
\end{aligned}$$

We are ready to continue with the terms occurring in e. We expound the set theoretical meaning of γ in (11), we proceed in steps.

(8) $$s \notin c_{(3)}\gamma(x) \quad \text{iff} \quad c_2 x[s, 01] = c_0 x[s, 12].$$

Indeed,

$$\begin{aligned}
&s \in -c_{(3)}\gamma(x) = -c_{(3)}(c_2 x \oplus \mathsf{s}^{12}_{01} c_0 x) && \text{iff}\\
&s(012/uvw) \notin c_2 x \oplus \mathsf{s}^{12}_{01} c_0 x \ \text{ for all } u, v, w && \text{iff}\\
&s(012/uvw) \in (c_2 x \cdot \mathsf{s}^{12}_{01} c_0 x) + (-c_2 x \cdot -\mathsf{s}^{12}_{01} c_0 x) \ \text{ for all } u, v, w && \text{iff}\\
&s(012/uvw) \in c_2 x \text{ iff } s(012/uvw) \in \mathsf{s}^{12}_{01} c_0 x \ \text{ for all } u, v, w && \text{iff, by (5)}\\
&uv \in c_2 x[s, 01] \text{ iff } s(012/wuv) \in c_0 x \ \text{ for all } u, v, w && \text{iff}\\
&uv \in c_2 x[s, 01] \text{ iff } uv \in c_0 x[s, 12], \ \text{ for all } u, v && \text{iff}\\
&c_2 x[s, 01] = c_0 x[s, 12].
\end{aligned}$$

By this, (8) has been proved. Recall that $Z = (c_0 x \cdot c_2 x)[s, 012]$.

(9) $$uvw \in Z \quad \text{iff} \quad uv \in c_2 x[s, 01] \text{ and } vw \in c_0 x[s, 12].$$

Indeed,

$$\begin{aligned}
&uvw \in Z && \text{iff}\\
&s(012/uvw) \in c_0 x \cdot c_2 x && \text{iff}\\
&s(012/uvw) \in c_0 x \text{ and } s(012/uvw) \in c_2 x && \text{iff}\\
&s(12/vw) \in c_0 x \text{ and } s(01/uv) \in c_2 x && \text{iff}\\
&vw \in c_0 x[s, 12] \text{ and } uv \in c_2 x[s, 01].
\end{aligned}$$

By this, (9) has been proved. Recall that $<_x = c_2 x[s, 01]$ and $W = c_1 c_2 x[s, 0] = \mathsf{Dom}(<_x)$.

(10) $$c_2 x[s, 01] = c_0 x[s, 12] \text{ implies } \mathsf{Rg}(<_x) = \mathsf{Dom}(<_x)\ .$$

Indeed,

$u \in \mathsf{Rg}(c_2x[s,01])$		iff
$vu \in c_2x[s,01]$	for some v	iff
$s(012/vuw) \in x$	for some v, w	iff
$s(012/vuw) \in c_0x$	for some v, w	iff
$uw \in c_0x[s,12]$	for some w	iff
$u \in \mathsf{Dom}(c_0x[s,12])$		iff, by $c_0x[s,12] = c_2x[s,01]$
$u \in \mathsf{Dom}(c_2x[s,01])$,		

and

(11) $s \notin c_{(3)}\gamma(x)$ iff $Z = \{uvw \in {}^3W : u <_x v <_x w\}$ and $W = \mathsf{Rg}(<_x)$.

Indeed, assume $s \notin c_{(3)}\gamma(x)$. Then $W = \mathsf{Rg}(<_x)$ by (10). Also, by (9) we have that $uvw \in Z$ holds iff we have $vw \in c_0x[s,12]$ and $uv \in c_2x[s,01]$, and this, by (8), holds iff $vw \in c_2x[s,01]$ and $uv \in c_2x[s,01]$, which by the definition of $<_x$ holds iff $u <_x v <_x w$. Note that $\{uvw : u <_x v <_x w\} = \{uvw \in {}^3W : u <_x v <_x w\}$ when $W = \mathsf{Rg}(<_x)$. To show the other direction, by (8), it is enough to show $c_2x[s,01] = c_0x[s,12]$. Indeed, $uv \in c_2x[s,01]$ iff $u <_x v$, and by $W = \mathsf{Rg}(<_x)$ this holds iff $w <_x u <_x v$ for some w. By our assumption on Z, this last statement holds iff $wuv \in Z$ for some w, which by (9) holds iff $w <_x u$ and $uv \in c_0x[s,12]$ for some w, which in turn by $W = \mathsf{Rg}(<_x)$ holds iff $uv \in c_0x[s,12]$. By this, (11) has been proved.

We turn to the last term ø. Let us assume for (12) that the domain and range of $<_x$ coincide. We may do this because we will use (12) only when $s \notin c_{(3)}\gamma(x)$.

(12) $s \notin c_{(3)}ø(x)$ iff $c_2x[s,01]$ is a strict linear order on $c_1c_2x[s,0]$.

Indeed,

$s \in -c_{(3)}\iota(x) = -c_{(3)}(c_2x \cdot d_{01})$	iff
$s(012/uvw) \notin (c_2x \cdot d_{01})$ for all u, v, w	iff
$s(012/uvw) \in c_2x$ implies $u \neq v$ for all u, v, w	iff
$u \neq v$ for all $uvw \in c_2x[s,012]$	iff
$u \neq v$ for all $uv \in c_2x[s,01]$	iff
$c_2x[s,01]$ is irreflexive.	

$s \in -c_{(3)}\sigma(x) = -c_{(3)}(c_2x \cdot \mathsf{s}^{01}_{10}c_2x)$	iff
$s(012/uvw) \notin (c_2x \cdot \mathsf{s}^{01}_{10}c_2x)$ for all u, v, w	iff
$s(012/uvw) \in c_2x$ implies $s(012/uvw) \notin \mathsf{s}^{01}_{10}c_2x$	iff, by (6)
$uv \in c_2x[s,01]$ implies $vu \notin c_2x[s,01]$	iff
$c_2x[s,01]$ is antisymmetric.	

$$\begin{array}{ll}
s \in -c_{(3)}\tau(x) = -c_{(3)}(c_2x \cdot \mathsf{s}^{01}_{12}c_2x - \mathsf{s}^1_2c_2x) & \text{iff}\\
s(012/uvw) \notin (c_2x \cdot s^{01}_{12}c_2x - s^1_2c_2x) \ \text{ for all } u,v,w & \text{iff}\\
s(012/uvw) \in c_2x \cdot \mathsf{s}^{01}_{12}c_2x \text{ implies } s(012/uvw) \in \mathsf{s}^1_2c_2x & \text{iff, by (7),(4)}\\
uv, vw \in c_2x[s,01] \text{ implies } uw \in c_2x[s,01] & \text{iff}\\
c_2x[s,01] \text{ is transitive.} &
\end{array}$$

$$\begin{array}{ll}
s \in -c_{(3)}\lambda(x) = -c_{(3)}(c_1c_2x \cdot c_0c_2x \cdot -c_2x \cdot -\mathsf{s}^{01}_{10}c_2x) & \text{iff}\\
s(012/uvw) \notin (c_1c_2x \cdot c_0c_2x \cdot -c_2x \cdot -\mathsf{s}^{01}_{10}c_2x) \text{ for all } u,v,w & \text{iff}\\
s(012/uvw) \in c_1c_2x \cdot c_0c_2x \Rightarrow s(012/uvw) \in c_2x + \mathsf{s}^{01}_{10}c_2x & \text{iff, by (6)}\\
u \in \mathsf{Dom}(c_2x[s,01]), v \in \mathsf{Rg}(c_2x[s,01]) & \\
\qquad\qquad \Rightarrow uv \in c_2x[s,01] \text{ or } vu \in c_2x[s,01] & \text{iff}\\
c_2x[s,01] \text{ is linear on its field if } \mathsf{Dom}(c_2x[s,01]) = \mathsf{Rg}(c_2x[s,01]). &
\end{array}$$

By the above, (12) has been proved and we can round up the proof of Lemma 2.2. Let $x \in C$. By (2), then $\mathfrak{C} \models e(x)$ iff for all $s \in x$ we have $s \in c_{(3)}(\beta(x)+\gamma(x)+\o(x))$. Now, $s \in c_{(3)}(\beta(x)+\gamma(x)+\o(x))$ iff $s \in c_{(3)}\beta(x)$ or $s \in c_{(3)}\gamma(x)$ or we have $s \in c_{(3)}\o(x)$. We have to show that this holds exactly when one of (i)–(iii) in the statement of the lemma holds. Assume that $W \neq \mathsf{Rg}(<_x)$. Then (ii) holds, and also $s \in c_{(3)}\gamma$ by (11), and we are done. Assume that $W = \mathsf{Rg}(<_x)$. Then we are done by (3), (11) and (12). □

2.4. **Checking the equations in the witness algebra.**

Lemma 2.3. *The equation e fails in $\mathfrak{A}$.*

Proof. Recall that g is the generator element of our master-algebra $\mathfrak{A}$ defined in section 2.1. We show that e fails in $\mathfrak{A}$ at g. Let $s \in g$ be arbitrary such that s agrees with p on all indices $i > 2$. There is such a sequence. Then, $<_g = c_2g[s,01] = c_2T[s,01] = \{\langle u,v\rangle : u,v \in V_0, u < v\}$ is a strict linear order on $c_1c_2g[s,0] = c_1c_2T[s,0] = V_0$. Also, $Z = (c_0g \cdot c_2g)[s,012] = T[s,012] = \{\langle u,v,w\rangle \in {}^3V_0 : u < v < w\}$. Finally, $g[s,012] = \{\langle u,(u+v)/2,v\rangle : u < v, u,v \in V_0\}$ is a sensitive cut of Z. Then Lemma 2.2 implies that e fails in $\mathfrak{A}$ at g. □

Let $i,j,k \in \alpha - \{0,1,2\}$ be distinct and let e_{ijk} denote the equation we get from e by replacing the indices $0,1,2$ everywhere with i,j,k, respectively. We are going to show that e_{ijk} holds in $\mathfrak{A}$. Lemma 2.2 is true by systematically replacing the indices $0,1,2$ with i,j,k. Thus, to show that e_{ijk} holds in $\mathfrak{A}$, we have to show that $x[s,ijk]$ is not a sensitive cut of a ternary relation built up from a linear order $<_x$ in the way T is built up from $<$, for all $x \in A$ and $s \in x$. In proving this, we will use the following lemma, which says that certain permutations of U leave the relations $x[s,ijk]$, that determine the validity of e_{ijk}, fixed. We agree on some terminology first.

Let π be a permutation of U, and let n be an ordinal. Then $\pi(s) := \langle \pi(s_i) : i < n\rangle$ if $s \in {}^nU$, and $\pi(R) := \{\pi(s) : s \in R\}$ for $R \subseteq {}^nU$. We say that π leaves R fixed iff $\pi(R) = R$. In the present section we shall often use a certain property of permutations of U, so we give it a temporary name.

Definition 2.1. We say that π is *good* iff it satisfies (i)–(iii) below.

(i) π leaves $<$ fixed, i.e., $u < v$ iff $\pi(u) < \pi(v)$ for all $u, v \in V_0$,
(ii) π leaves all the V_m's fixed, i.e., $\pi(V_m) = V_m$ for all $m \in \omega$,
(iii) π is the identity on all but a finite number of V_m's, i.e., $\{m \in \alpha : \forall u \in V_m(\pi(u) = u)\}$ is a co-finite subset of α.

Lemma 2.4. *For any $x \in A$ and $s \in V$ there is a finite $S \subseteq U$ such that any good permutation of U which is the identity on S leaves $x[s, ijk]$ fixed.*

Proof. Let $x \in A$. Then $x = y + h$ for some y generated by T and for some $h \in \{0, g, T - g\}$, by Lemma 2.1. Assume that $y = \xi(T)$ for a term ξ. Let $\Delta \subseteq \alpha$ be finite such that it contains all the indices occurring in ξ as well as $0, 1, 2$. We show that for all s, s'

$$s \in T \leftrightarrow s' \in T, \quad s \restriction \Delta = s' \restriction \Delta \qquad \text{imply} \qquad s \in y \leftrightarrow s' \in y. \tag{13}$$

We prove (13) by induction on elements z generated from T by the use of indices from Δ. Clearly, (13) holds for T and d_{mn} for $m, n \in \Delta$. Assume that (13) holds for z, z'. Then clearly it holds for $-z$ and $z \cdot z'$. Let $m \in \Delta$ and assume that s, s' satisfy the conditions. Now,

$s \in c_m z$	iff, by the definition of c_m
$s(m/u) \in z$ for some u	iff, by the induction hyp. (see details below)
$s'(m/u) \in z$ for the same u	iff, by the definition of c_m
$s' \in c_m z$.	

Above, in the step from the second to third line we used that $s'(m/u)$ agrees with $s(m/u)$ on Δ and $s(m/u) \in T$ iff $s'(m/u) \in T$ (by $s \in T$ iff $s' \in T$, the definition of T, and $0, 1, 2 \in \Delta$). By this, (13) has been proved.

Now let $s \in V$ and

$$S := \{s_m : m \in \Delta\} \cup V_i \cup V_j \cup V_k.$$

Then $S \subseteq U$ is finite. Let π be a good permutation of U which is the identity on S, we want to show that π leaves $x[s, ijk]$ fixed. Recall that $x = y + h$ where $h \in \{0, g, T - g\}$. Then $x[s, ijk] = y[s, ijk] + h[s, ijk]$. Thus $\pi(x[s, ijk]) = \pi(y[s, ijk]) + \pi(h[s, ijk])$, so it is enough to show that π leaves both $y[s, ijk]$ and $h[s, ijk]$ fixed. We begin with the second. Indeed, $h[s, ijk] \subseteq V_i \times V_j \times V_k$ when $h \in \{0, g, T - g\}$, thus $\pi(uvw) = uvw$ for all $uvw \in h[s, ijk]$ by $V_i \cup V_j \cup V_k \subseteq S$ and π being the identity on S.

We turn to $y[s, ijk]$. First we note that π being good implies that $\pi(V) = V$ by Definition 2.1(iii), and then $\pi(T) = T$ by Definition 2.1(i),(ii). Since y is generated by T we then have (by, e.g., [19, 3.1.36])

$$\pi(y) = y. \tag{14}$$

We want to show that

$$uvw \in y[s, ijk] \quad \text{iff} \quad \pi(uvw) \in y[s, ijk]. \tag{15}$$

Indeed,

$$\begin{array}{ll} uvw \in y[s,ijk] & \text{iff, by the definition of } y[s,ijk] \\ s(ijk/uvw) \in y & \text{iff, by (14)} \\ \pi(s(ijk/uvw)) \in y & \text{iff, by (13) and see below} \\ s(ijk/\pi(uvw)) \in y & \text{iff, by the definition of } y[s,ijk] \\ \pi(uvw) \in y[s,ijk]. & \end{array}$$

In the argument from the third to fourth line we used that $\pi(s(ijk/uvw))$ and $s(ijk/\pi(uvw))$ agree on i,j,k by their definitions, they agree on $\Delta - \{i,j,k\}$ by π being the identity on $S \supseteq \{s_m : m \in \Delta\}$; further, one of them is in T iff the other is:

$$\begin{array}{ll} \pi(s(ijk/uvw)) \in T & \text{iff, by } \pi(T) = T \\ s(ijk/uvw) \in T & \text{iff, by Definition 2.1(ii)} \\ s(ijk/\pi(uvw)) \in T. & \end{array}$$

This proves (15), and Lemma 2.4 has been proved. □

Lemma 2.5. *The equation e_{ijk} is valid in $\mathfrak{A}$ when $i,j,k \in \alpha - \{0,1,2\}$ are distinct.*

Proof. To check the validity of e_{ijk}, we will use Lemma 2.2 (with $0,1,2$ systematically replaced by i,j,k). Let $x \in A$, $s \in x$ and $R := x[s,ijk]$, $W := c_j c_k x[s,i]$, $Z := (c_i x \cdot c_k x)[s,ijk]$. Assume that $<_x := c_k x[s,ij]$ is a strict linear order on W and $Z = \{\langle u,v,w\rangle \in {}^3W : u <_x v <_x w\}$. We have to show that R is not a sensitive cut of Z. Let $S \subseteq U$ be such that

(16) $\pi(R) = R$ for all good permutations π of U that are identity on S.

There is such an S by Lemma 2.4. We note that

(17) $\pi(R) = R$ implies $\pi(<_x) = <_x$ and $\pi(W) = W$.

This is true because $<_x = c_2 R$ by their definitions: $<_x = c_k x[s,ij] = \{uv : s(ij/uv) \in c_k x\} = c_2\{uvw : s(ijk/uvw) \in x\} = c_2 R$. Below, we will use (*) from the proof of Lemma 2.1 several times.

We turn to showing that R is not a sensitive cut of Z. By $s \in x$ we have that $s_i s_j \in c_k x[s,ij]$ so $<_x$ is nonzero. Since by our assumption $<_x$ is a strict linear order on W, it does not have a maximal element (by $W = \mathsf{Dom}(<_x) = \mathsf{Rg}(<_x)$), so W is infinite by $W \neq \emptyset$. Assume $m \geq 3$ is such that S is disjoint from V_m. We show that V_m is disjoint from W. Assume $W \cap V_m \neq \emptyset$. Let π be the permutation of U that interchanges the elements of V_m and it leaves all the other elements of U fixed. (Recall that the V_m's for $m \geq 3$ have two elements.) Then π is good and it is identity on S, so it leaves W as well as $<_x$ fixed, by (16),(17). This implies that $V_m \subseteq W$ by $V_m \cap W \neq \emptyset$, so by $<_x$ being linear on W, we have $a <_x b$ for some distinct $a,b \in V_m$. By π leaving $<_x$ fixed, then we have $b <_x a$ (by $\pi(a) = b$, $\pi(b) = a$). This contradicts $<_x$ being antisymmetric.

Thus W intersects only finitely many of the V_ms. Then $W \cap V_0$ is infinite because all V_m disjoint from V_0 are finite. Let

$$K := W \cap V_0 \cap S \qquad \text{and} \qquad W' := (W \cap V_0) - S.$$

Thus K is finite and W' is infinite. Therefore, there are distinct $u,v \in W'$ such that no element of K lies in between u,v according to $<$. (Indeed, let $K = \{k_1,\dots,k_n\}$

with $k_1 < \cdots < k_n$. Then there are at least two elements of W' that lie in the same interval determined by the k_ms, and they will do.) We may assume $u < v$. Since $<_x$ is linear on W and $u, v \in W$, we have either $u <_x v$ or $v <_x u$. We assume $u <_x v$, the case $v <_x u$ will be completely analogous; see (19). So assume

$$[u,v] \cap K = \emptyset, \quad u < v, \quad \text{and} \quad u <_x v,$$

and we are going to show that for all $w \in U$ we have

$$u <_x w <_x v \quad \text{iff} \quad u < w < v. \tag{18}$$

Indeed, to prove (18), first let $u < w < v$ be arbitrary, we want to show $u <_x w <_x v$. Let $u' < u$ be such that there is no element of K between u' and u, there is such a u' because $<$ is dense and K is finite. Then there is no element of K between u' and v. Take π as in (*) for $u' < u < v$ and $u' < w < v$ and extend it to U by being the identity on $U - V_0$. Then this π is identity on S and it is good. So it leaves $<_x$ fixed, by (16),(17). By $u <_x v$ then we have $w = \pi(u) <_x \pi(v) = v$. By a similar argument we get $u <_x w$. (Indeed, choose $v' > v$ such that there is no element of K between v and v' and apply (*) with $u < v < v'$ and $u < w < v'$.) We have seen $u <_x w <_x v$. To prove the other direction, assume that $w \in U$ and it is not the case that $u < w < v$. Thus either $w < u < v$ or $u < v < w$. In either case, there is a good permutation π of U which is identity on S, leaves w fixed and takes u to v, and there is also a good permutation π of U which is identity on S, leaves w fixed and takes v to u. (Indeed, take $u' < u$ and $v < v'$ such that no element of $K \cup \{w\}$ lies between u' and v', and then apply (*).) Hence $w <_x u$ iff $w <_x v$ and $u <_x w$ iff $v <_x w$. Hence, it is not the case that $u <_x w <_x v$, as it was desired. The equation (18) has been proved.

Now assume the other case, i.e., that

$$[u,v] \cap K = \emptyset, \quad u < v, \quad \text{and} \quad v <_x u.$$

We are going to show that for all $w \in U$ we have

$$v <_x w <_x u \quad \text{iff} \quad u < w < v. \tag{19}$$

To prove (19), first let $u < w < v$ be arbitrary, we want to show $u <_x w <_x v$. Let $u' < u$ be such that there is no element of K between u' and u. Take π as in (*) for $u' < u < v$ and $u' < w < v$ and extend it to U by being the identity on $U - V_0$. Then this π is identity on S and it is good. So it leaves $<_x$ fixed, by (16),(17). By $v <_x u$, we then have $v = \pi(v) <_x \pi(u) = w$. We get $w <_x u$ by choosing $v' > v$ such that there is no element of K between v and v' and applying (*) with $u < v < v'$ and $u < w < v'$. We have seen $v <_x w <_x u$. The proof of the other direction is the same as in the proof for (18). The equation (19) has been proved.

We are ready to prove that R is not a sensitive cut of Z. Assume that $Z \subseteq c_1 R$, we will show that $Z \not\subseteq c_1(Z - R)$. By (18),(19) and $<$ being dense we have that $uwv \in Z$ for some w (and the u, v chosen as before), so $uw'v \in R$ for some w' by $Z \subseteq c_1 R$. Let $u < w'' < v$ be arbitrary and take a good permutation π of U that takes w' to w'' and leaves everything outside the open interval (u, v) fixed. There is such a π by (*). This π leaves S fixed since no element of S lies between u and v (according to $<$). Then it leaves R fixed by (16). So $uw''v \in R$ by $uw'v \in R$ and $\pi(uv'v) = uv''v$. By (18),(19) this means that $R(uw'v)$ for all w' such that $uw'v \in Z$. Hence $uwv \in Z$ is such that $uwv \notin c_1(Z - R)$, and we are done with proving Lemma 2.5. $\square$

We now round up the proof of Theorem 2.1. Let $I \subseteq \alpha \times \alpha \times \alpha$ be such that $|I| = |\alpha|$ and for all distinct $ijk, lmn \in I$ we have $\{i,j,k\} \cap \{l,m,n\} = \emptyset$, $|\{i,j,k\}| = 3$ and $012 \in I$. There is such an I since α is infinite. Let

$$E := \{e_{ijk} : ijk \in I\}.$$

Then $|E| = |\alpha|$. For all $ijk \in I$ let $\mathfrak{A}_{ijk}$ denote the algebra we get from $\mathfrak{A}$ by interchanging (renaming) the operations c_m, d_{mn} for $m,n \in \{0,1,2\}$ with those for $m,n \in \{i,j,k\}$, respectively. This algebra is denoted by $\mathfrak{Rd}^\rho\mathfrak{A}$ where ρ is the permutation of α which interchanges i,j,k with $0,1,2$, respectively; see [18, Def. 2.6.1]. By Lemmas 2.3 and 2.5 we then have

$$(20) \qquad \mathfrak{A}_{ijk} \not\models e_{ijk} \quad \text{while} \quad \mathfrak{A}_{ijk} \models e_{klm} \quad \text{for all } klm \in I, klm \neq ijk.$$

Also, $\mathfrak{A}_{ijk} \in \mathbf{I}\mathsf{Ws}_\alpha$ by [19, 3.1.119] and, so $\mathfrak{A}_{ijk} \in \mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$ by [19, 3.1.102] and because the fact that $\mathfrak{A}$ has an infinite base is reflected on its equational theory. (We note that $\mathfrak{A}_{ijk}$ is isomorphic to the algebra we get from $\mathfrak{A}$ by replacing in its construction $0,1,2$ with i,j,k systematically, and changing nothing else. An isomorphism showing this takes $x \in \mathfrak{A}_{ijk}$ to $\{\rho(s) : s \in x\}$.)

For $G \subseteq E$ define

$$\mathsf{V}_G := \{\mathfrak{B} \in \mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha : \mathfrak{B} \models G\}.$$

Then V_G is a subvariety of $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$. Assume $G, H \subseteq E$ are distinct. Then there is $ijk \in I$ distinguishing them, we may assume $ijk \in G$ and $ijk \notin H$. By (20) we have that $\mathfrak{A}_{ijk} \notin \mathsf{V}_G$ but $\mathfrak{A}_{ijk} \in \mathsf{V}_H$, so $\mathsf{V}_G \neq \mathsf{V}_H$. Therefore, there are $2^{|\alpha|}$ distinct subvarieties of $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha$. The same is true for RCA_α by $\mathbf{I}\,{}_\infty\mathsf{Cs}_\alpha \subseteq \mathsf{RCA}_\alpha$. Theorem 2.1 has been proved.

We close this section by discussing some properties necessary for our construction $\mathfrak{A}$ to work.

Remark 2.1. (i) It is necessary that the base of $\mathfrak{A}$ (i.e., the set U) be at least of cardinality $|\alpha|$. This is true because algebras of smaller base are diagonal (roughly: each of their elements intersects many diagonal elements, for a precise definition see [18, p. 416]), and we will prove that all diagonal algebras are symmetric; see Theorem 4.1. Clearly, $\mathfrak{A}$ has to be nonsymmetric to play its role in the proof.

(ii) In the equation e it was necessary to code a property that can occur on an infinite set only, this is the role of using the ordering on rational numbers $V_0 = V_1 = V_2$ in the definition of the generator element g. In more detail: an equation $e(x)$ using indices from $\{i,j,k\}$ can talk in a set algebra about the ternary relation $x[s, ijk]$ only. However, all ternary relations on a finite set occur as $x[s, ijk]$ in a set algebra when the base set is infinite. (This is the main idea used in [34].) Since we want $e_{012}(x)$ to hold and $e_{ijk}(x)$ to fail in our witness algebra, $e(x)$ has to code a property of $x[s, ijk]$ which can be realized only on infinite sets.

(iii) The equation e fails in $\mathfrak{A}$ at g, but e is true in $\mathfrak{A}$ for all elements that are closed to at least one cylindrification c_i. Indeed, we can see that e holds for $c_i a \in A$ as follows. Lemma 2.1 and (1) together with $T \in B$ imply that $c_i a \in B$. The proof of Lemma 2.4 works for $x \in B$ and $ijk = 012$, and then the proof of Lemma 2.5 works to show that e holds for $c_i a$. Thus, in $\mathfrak{A}$ cylindrification-closed elements satisfy more equations than all the elements. This behavior of our witness algebra $\mathfrak{A}$ is necessary, because each algebra in which no such behavior occurs is symmetric (see Theorem 4.5(i) in subsection 4.3).

3. Subvarieties of CA_α containing RCA_α

This section contains an unpublished theorem from [33]. The proof is analogous to the proof of Theorem 2.1.

Theorem 3.1. *There are* $2^{|\alpha|}$ *distinct subvarieties of* CA_α *all containing* RCA_α*.*

Proof. We are going to exhibit an equation e valid in RCA_α and an algebra $\mathfrak{A} \in \mathsf{CA}_\alpha$ such that $\mathfrak{A} \not\models e$ while $\mathfrak{A} \models e_{i1}$ for appropriate versions e_{i1} of e. This e is Henkin's equation $e_{ij}(x,y)$ with ij taken as 01:

$$e_{ij}(x,y) \quad := \quad c_j(x \cdot y \cdot c_i(x-y)) \leq c_i(c_j x - d_{ij}); \tag{21}$$

see [19, 3.2.65]. For a simplified version of this equation see [44, chap. 3.5], and for a drawing see [35, p. 551]. Henkin's equation expresses that if the domains of R and S coincide and this common domain is a singleton, then R and S are disjoint iff their ranges are disjoint. Now, $\mathsf{RCA}_\alpha \models e_{ij}(x,y)$, by, e.g., [19, 3.2.65].

We now turn to constructing our "witness" algebra $\mathfrak{A}$. It is obtained from a representable algebra $\mathfrak{B}$ in which we split an atom whose domain is a singleton into two parts both having the same domain and range as the original atom. Henkin's equation then will fail for the split elements. In some sense this will be a "nonrepresentable counterpart" of the construction we used in the proof of Theorem 2.1.

Let $\langle V_i : i \in \alpha\rangle$ be a system of disjoint sets such that V_0 is a singleton, and V_i for $i \geq 1$ have more than one element. Let U be the union of these sets, let $V := {}^\alpha U$ and let g be the direct product of the V_i, i.e.,

$$g := \prod\langle V_i : i \in \alpha\rangle := \{s \in V : s_i \in V_i \text{ for all } i \in \alpha\}.$$

Let $\mathfrak{B}$ denote the cylindric set algebra with base set U and generated by g. In $\mathfrak{B}$, the element g is an atom, this can be seen by using permutations of U exactly as in the proof of Lemma 2.1. Now, g is below all the diversity elements $-d_{ij}$, $i < j < \alpha$, so we can split it into two disjoint parts g', g'' obtaining the algebra $\mathfrak{A} \in \mathsf{CA}_\alpha$ defined as follows. Let $\langle A, +, -\rangle$ be the Boolean algebra which contains $\langle B, +, -\rangle$ as a subalgebra, in which g' and g'' are disjoint nonzero elements such that $g = g' + g''$ and which is generated by $B \cup \{g'\}$. Then the elements of A are

$$A = \{b + h : b \in B \text{ and } h \in \{0, g', g''\}\}.$$

The cylindric operations are defined in $\mathfrak{A}$ so that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and

$$c_i(b + g') := c_i(b + g'') := c_i(b + g) \qquad \text{for all } b \in B.$$

Now, $\mathfrak{A} \in \mathsf{CA}_\alpha$ can be checked directly by checking that the cylindric equations (C_0)–(C_7) of [18, 1.1.1] hold in $\mathfrak{A}$, or by checking that $\mathfrak{A}$ is the algebra we get from $\mathfrak{B}$ by splitting g in it by the method in [18, 2.6.12].

We show that $\mathfrak{A} \not\models e_{01}(g, g')$. Indeed, $c_1(g \cdot g' \cdot c_0(g - g')) = c_1(g' \cdot c_0(g'')) = c_1(g' \cdot c_0 g) = c_1 g$ while $c_0(c_1 g - d_{01}) = c_0(V_0 \times (U - V_0) \times V_2 \times \dots)$ which does not contain $c_1 g = V_0 \times U \times V_2 \times \dots$.

Next we show that $\mathfrak{A} \models e_{ij}(x,y)$ when $i \neq 0$, i.e., we show

$$\mathfrak{A} \models c_i(x \cdot y \cdot c_j(x-y)) \leq c_i(c_j x - d_{ij}).$$

Let $x, y \in B$ be arbitrary. Then $x \cdot y, x - y$ are of form $a + h$, $b + k$ with $a, b \in B$, $h, k \in \{0, g', g''\}$ and a, b, g pairwise disjoint as well as h, k disjoint, by our construction of $\mathfrak{A}$. Since negation $-$ occurs in the equation only in the form of $-d_{ij}$, the terms at the two sides of the equation are additive, and since $a, b \in B$

and $\mathfrak{B} \subseteq \mathfrak{A}$ is representable, we have that the equation is true for a, b. So, if both h and k are 0, then we are done. Assume therefore that $h + k \neq 0$. Then we get a bigger term on the lhs of the inequality if we replace h, k with g, g, respectively. We then get

$$c_i(x \cdot y \cdot c_j(x - y)) = c_i((a + g) \cdot c_j(b + g)) = c_i(a \cdot c_j(b + g)) + c_i(g \cdot c_j(b + g)).$$

On the other side of the inequality we have

$$c_i(c_j x - d_{ij}) = c_i(c_j(a + b + h + k) - d_{ij}) = c_i(c_j(a + b + g) - d_{ij}).$$

(We used $c_j(h + k) = c_j g$ in the last step.) This is now an equation concerning the representable algebra $\mathfrak{B}$ since all the elements occurring are in $\mathfrak{B}$. Now, $c_i(a \cdot c_j(b + g)) \leq c_i(c_j(a + b + g) - d_{ij})$ since this is an instance of Henkin's equation by a and $b + g$ being disjoint. We only have to show $c_i(g \cdot c_j(b + g)) \leq c_i(c_j(a + b + g) - d_{ij})$. The inequality

$$c_i(g) \leq c_i(c_j g - d_{ij}) \tag{22}$$

holds because V_i has at least two elements: $c_i(c_j g - d_{ij}) = c_i\{s \in V : (\forall k \neq j) s_k \in V_k \text{ and } s_j \neq s_i\} = c_i g$. Then

$c_i(g \cdot c_j(b + g))$	$=$	by g being an atom
$c_i(g)$	$\leq$	by (22)
$c_i(c_j g - d_{ij})$	$\leq$	by monotony of the terms involved
$c_i(c_j(a + b + g) - d_{ij})$.		

To finish the proof of Theorem 3.1, let $E := \{e_{i1} : i \in \alpha, i \neq 1\}$ and for all $H \subseteq E$ let V_H be the subvariety of CA_α axiomatized by H. Then $\mathsf{V}_H \supseteq \mathsf{RCA}_\alpha$ for all $H \subseteq E$, by $\mathsf{RCA}_\alpha \models E$. Also, $\mathsf{V}_H \neq \mathsf{V}_G$ for distinct $H, G \subseteq E$ since if, say, $e_{i1} \in G - H$, then $\mathfrak{Rd}^\rho \mathfrak{A} \in (\mathsf{V}_H - \mathsf{V}_G)$ whenever ρ is a permutation of α with $\rho(i) = 0$. □

4. Counterpoint: Classes with only continuum many varieties

Let us call a cylindric algebra $\mathfrak{A}$ *symmetric* iff $\mathfrak{A} \models e$ implies $\mathfrak{A} \models \rho(e)$ for all permutations ρ of α, where $\rho(e)$ denotes the equation we get from e by systematically replacing each index $i \in \alpha$ in it with $\rho(i) \in \alpha$. The proofs of the previous theorems were based on the existence of nonsymmetric algebras. We will show that, surprisingly, many CA_αs, almost all in some sense, are symmetric. In particular, all dimension-complemented, all diagonal cylindric algebras, and more generally, all the algebras occurring in [18, Thm. 2.6.50(i)-(iii)] are symmetric. Clearly, symmetric algebras can generate at most continuum many varieties since their equational theories are determined by equations written in the first ω indices. In section 4.5 we show that this maximal possible number 2^ω is indeed achieved using only a small subclass of symmetric CA_αs: locally finite-dimensional regular cylindric set algebras with infinite bases indeed generate continuum many varieties, for all infinite α.

Thus, RCA_α has $2^{|\alpha|}$ subvarieties, but locally finite-dimensional ones generate only 2^ω many. What is the property that the Lf_α-generated varieties have but not all of the subvarieties have? Clearly, being symmetric is such a distinguishing property. (We call a variety symmetric iff it is generated by symmetric algebras.) However, being symmetric does not characterize the Lf_α-generated subvarieties: we will show that there is a symmetric subvariety of RCA_α that is not generated by a subclass of Lf_α. In section 4.3 we introduce the notion of *inductive* algebras and inductive varieties and we prove that this property characterizes the subvarieties generated by Lf_α, the property of being inductive singles out the 2^ω many Lf_α-generated subvarieties among all the $2^{|\alpha|}$ many subvarieties of RCA_α. By this we also get a simple characterization, and recursive enumeration, of the equational theory of RCA_α, simpler than either one of the three enumerations presented in [19, pp. 112-119]. These results contribute to solving [19, Problem 4.1] which is asking for a simple equational basis for RCA_α.

Being inductive is a nice property: inductive algebras are all representable, they are symmetric, and their equational theories coincide with the one of an Lf_α. We show that there are more inductive algebras than the widest class dealt with in [18, 2.6.50]. This provides us with a new representation theorem for CA_αs. All this strengthens, extends and improves [18, 2.6.50], whose significance was discussed in the introduction to the present paper. The notion of being inductive can be described by a set of Δ_2 first order logic formulas. Since inductive algebras are all symmetric and we have constructed a nonsymmetric representable algebra in section 2.1 here, we get a Δ_2-formula distinguishing Lf_α and RCA_α.

4.1. **Endo-dimension-complemented algebras are symmetric.** Let Lf_α, Dc_α, and Di_α denote the classes of all locally finite-dimensional, dimension-complemented, and diagonal CAs, respectively. Let us call the elements of the wider class introduced in (iii) of [18, 2.6.50] *endo-dimension-complemented* (*endo-dc* in short): an algebra $\mathfrak{A} \in \mathsf{CA}_\alpha$ is called endo-dc if for each finite $\Gamma \subseteq \alpha$ and each nonzero $x \in A$ there are a $\kappa \in \alpha - \Gamma$ and an endomorphism h of the Γ-*reduct* $\mathfrak{Rd}_\Gamma\mathfrak{A} := \langle A, +, -, c_i, d_{ij}\rangle_{i,j\in\Gamma}$ of $\mathfrak{A}$ such that $h(x) \neq 0$ and each element of the range of h is κ-closed, i.e., $c_\kappa h(a) = h(a)$ for all $a \in A$. Let Edc_α denote the class of all endo-dimension-complemented CA_αs. It is proved in [18, 2.6.50] that $\mathsf{Lf}_\alpha \subset \mathsf{Dc}_\alpha \subset \mathsf{Di}_\alpha \subset \mathsf{Edc}_\alpha \subseteq \mathsf{RCA}_\alpha$ and it is asked as [18, Problem 2.13] whether the last inclusion is proper or not. We are going to show that this inclusion is proper: the algebra constructed in section 2.1 here is representable but not endo-dc. More specifically, we will show that each endo-dc algebra is symmetric, which implies $\mathsf{Edc}_\alpha \neq \mathsf{RCA}_\alpha$ since our witness algebra $\mathfrak{A}$ in the proof of Theorem 2.1 was designed to be non-symmetric but it is representable. We also show that RCA_α is close to Edc_α in the sense that RCA_α is the closure of Edc_α under taking subalgebras. On the other hand, to indicate the distance between Edc_α and RCA_α we show that the class $\mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha$ of symmetric representable algebras lies strictly in between Edc_α and RCA_a, i.e., $\mathsf{Edc}_\alpha \subset \mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha \subset \mathsf{RCA}_\alpha$.

Theorem 4.1. *Each endo-dc algebra is symmetric.*

Proof. Because the notion of a symmetric algebra involves renaming indices of operations, in this and the coming proofs we will often deal with renaming operations in equations and algebras. Therefore we begin the proof by introducing notation for these. We will use these notation, except for $\mathsf{ind}(e)$, only in proofs.

If τ is a term in the language of CA_α and $\rho : \alpha \to \alpha$, then $\rho(\tau)$, the term we get from τ by renaming the indices occurring in it according to ρ, is defined by induction as $\rho(d_{ij}) := d_{\rho(i)\rho(j)}$, $\rho(c_i\sigma) := c_{\rho(i)}\rho(\sigma)$ and $\rho(x) := x$ if x is a variable, $\rho(\sigma + \delta) := \rho(\sigma) + \rho(\delta)$, $\rho(-\sigma) := -\rho(\sigma)$. If e is an equation of form $\tau = \sigma$, then $\rho(e)$ is $\rho(\tau) = \rho(\sigma)$.

$\mathsf{ind}(\tau)$ denotes the set of indices occurring in τ, this is defined by induction as follows. $\mathsf{ind}(d_{ij}) := \{i,j\}$, $\mathsf{ind}(c_i\tau) := \{i\} \cup \mathsf{ind}(\tau)$, and $\mathsf{ind}(x) := \emptyset$, $\mathsf{ind}(\tau+\sigma) := \mathsf{ind}(\tau) \cup \mathsf{ind}(\sigma)$, $\mathsf{ind}(-\tau) := \mathsf{ind}(\tau)$. If e is an equation of form $\tau = \sigma$, then $\mathsf{ind}(e) := \mathsf{ind}(\tau) \cup \mathsf{ind}(\sigma)$.

Assume $\mathfrak{A} \in \mathsf{CA}_\alpha$, Γ is any set and $\rho : \Gamma \to \alpha$ is a one-to-one function. Then $\mathfrak{Rd}^\rho\mathfrak{A}$ denotes an algebra whose signature is that of CA_Γ, whose Boolean reduct $\langle A, +, -\rangle$ is the same as that of $\mathfrak{A}$, whose operation denoted by c_i for $i \in \Gamma$ is the operation of $\mathfrak{A}$ denoted by $c_{\rho(i)}$, and similarly for the diagonals, d_{ij} of $\mathfrak{Rd}^\rho\mathfrak{A}$ is the same as $d_{\rho(i)\rho(j)}$ of $\mathfrak{A}$. In symbols,

$$\mathfrak{Rd}^\rho\mathfrak{A} := \langle A, +, -, c_{\rho(i)}, d_{\rho(i)\rho(j)} : i, j \in \Gamma\rangle.$$

It is not difficult to check that $\mathfrak{Rd}^\rho\mathfrak{A} \in \mathsf{CA}_\Gamma$ and $\mathfrak{Rd}^\rho\mathfrak{A} \models e$ iff $\mathfrak{A} \models \rho(e)$, for any equation e. This algebra is called a generalized reduct of $\mathfrak{A}$ and it is introduced in [18, 2.6.1].

We begin the proof of Theorem 4.1. Assume $\mathfrak{A} \in \mathsf{CA}_\alpha$ is endo-dc, we want to show that it is symmetric. This means showing that $\mathfrak{A} \models e$ implies $\mathfrak{A} \models \rho(e)$ for all equations e and permutations ρ of α. For this, it is enough to prove

$$\mathfrak{A} \not\models e \quad \text{implies} \quad \mathfrak{A} \not\models \rho(e), \qquad \text{for all } e \text{ and } \rho, \tag{23}$$

since each equation e is of form $\rho(e')$ and $\rho^{-1}\rho(e') = e'$. Assume $\mathfrak{A} \not\models e$. We may assume that e is of form $\tau = 0$ for some τ. Let $\Gamma := \mathsf{ind}(\tau)$ and let $\rho : \Gamma \to \Delta$ be a bijection. We have $\tau(a) \neq 0$ for some $a \in A$ by $\mathfrak{A} \not\models e$, and we want to show that $\rho(\tau)(b) \neq 0$ for some $b \in A$. (In fact, τ may have more than one variable, so we should use a sequence $\overline{a}$ in place of $a \in A$. For simplicity, we write out the present proof for the case when τ contains one variable.)

We aim for getting a homomorphism $\mathfrak{Rd}_\Gamma\mathfrak{A} \to \mathfrak{Rd}^\rho\mathfrak{A}$ which takes $\tau(a)$ to a nonzero element. The idea of the proof is as follows. Assume $\Delta = \{k_1, \dots, k_n\}$ is disjoint from $\Gamma = \{i_1, \dots, i_n\}$. Then the substitution operation $x \mapsto \mathsf{s}^{i_1}_{k_1} \dots \mathsf{s}^{i_n}_{k_n}(x)$ is such a homomorphism, but only on the Δ-closed elements x, i.e., when $x = c_{(\Delta)}x := c_{k_1} \dots c_{k_n}x$. There are two obstacles to deal with: Δ may not be disjoint from Γ, and $\tau(a)$ may not be Δ-closed. We deal with the first obstacle by finding J which is disjoint both from Γ and Δ, and finding desired homomorphisms from Γ to J and then from J to Δ. We deal with the second obstacle by using the condition $\mathfrak{A} \in \mathsf{Edc}_\alpha$ for finding a homomorphism from Γ to Γ which takes $\tau(a)$ to a J-closed nonzero element. We begin now to elaborate the just outlined idea.

By [18, (2), p. 416], $\mathfrak{A} \in \mathsf{Edc}_\alpha$ implies that there is $J \subseteq \alpha - (\Gamma \cup \Delta)$ with $|J| = |\Gamma|$ and there is a homomorphism $h : \mathfrak{Rd}_\Gamma\mathfrak{A} \to \mathfrak{Rd}_\Gamma\mathfrak{A}$ such that $h(\tau(a)) \neq 0$ and $h(x) = c_{(J)}h(x)$ for all $x \in A$. By h being a homomorphism on $\mathfrak{Rd}_\Gamma\mathfrak{A}$ and $\mathsf{ind}(\tau) \subseteq \Gamma$ we have $\tau(h(a)) = h(\tau(a)) \neq 0$.

Now that h provided us with J-closed elements, we can use the usual substitution operations s^i_j to get the homomorphism we seek, as follows. Let $c^*_j\mathfrak{A}$ denote the algebra whose elements are the c_j-closed elements of $\mathfrak{A}$ and whose operations are those of $\mathfrak{A}$ except c_j, d_{jk}, d_{kj} for $k \in \alpha$. This is indeed an algebra, it is $\mathfrak{Nr}_{(\alpha-\{j\})}\mathfrak{A}$ in the terminology of [18], but we will use the shorter notation $c^*_j\mathfrak{A}$ in the present proof. We will use $c^*_{(J)}\mathfrak{A}$ for the analogous algebra (where $J \subset \alpha$). Let $[i/j]$ denote the function that takes i to j and takes k to k for all $k \in (\alpha - \{i,j\})$. Then $\mathfrak{Rd}^{[i/j]}c^*_i\mathfrak{A}$ is the algebra $c^*_i\mathfrak{A}$ except that we rename the operations c_j, d_{jk}, d_{kj} (of $c^*_i\mathfrak{A}$) as c_i, d_{ik}, d_{ki}, respectively. Thus the similarity types of $c^*_j\mathfrak{A}$ and $\mathfrak{Rd}^{[i/j]}c^*_i\mathfrak{A}$

are equal. We are going to show, by using [18, sec. 1.5], that

$$\mathsf{s}^i_j : c^*_j\mathfrak{A} \to \mathfrak{Rd}^{[i/j]}c^*_i\mathfrak{A} \quad \text{is an isomorphism.} \tag{24}$$

Indeed, s^i_j is a Boolean homomorphism by 1.5.3, it is a homomorphism for c_k, d_{km} for $k, m \in \alpha - \{i, j\}$ by 1.5.8(ii), 1.5.4(ii), and it takes d_{ik}, d_{ki} to d_{jk}, d_{kj} by 1.5.4(i). For the next two steps we need to use that we are mapping c_j-closed elements. s^j_i is the inverse of s^i_j on c_j-closed elements because $\mathsf{s}^j_i\mathsf{s}^i_j c_j x = c_j x$ by 1.5.10(i), 1.5.8(i). s^i_j takes the operation c_i on c_j-closed elements to c_j because $c_j\mathsf{s}^i_j a = c_i\mathsf{s}^j_i a = c_i\mathsf{s}^j_i c_j a = c_i c_j a = c_i a = \mathsf{s}^i_j c_i a$, by 1.5.8(i), 1.5.9(i). We are done with proving (24).

Recall that $J \subseteq \alpha - (\Gamma \cup \Delta)$ and $|J| = |\Gamma|$. Let $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ be injective enumerations of Γ and J, respectively. Let $\eta : \Gamma \to J$ be such that $\eta(i_1) = j_1, \ldots, \eta(i_n) = j_n$. Define

$$\mathsf{s}(\eta) := \mathsf{s}^{i_1}_{j_1} \ldots \mathsf{s}^{i_n}_{j_n}.$$

By using (24) successively, we get

$$\mathsf{s}(\eta) : \mathfrak{Rd}_\Gamma c^*_{(J)}\mathfrak{A} \to \mathfrak{Rd}^\eta c^*_{(\Gamma)}\mathfrak{A} \quad \text{is an isomorphism.} \tag{25}$$

By letting $k_\ell := \rho(i_\ell)$ and $\xi(j_\ell) := k_\ell$ for $1 \leq \ell \leq n$ we get that $k_1, \ldots, k_n$ is an injective enumeration of Δ, and $\rho = \xi \circ \eta$. By repeating the process leading to (25) we get

$$\mathsf{s}(\xi) : \mathfrak{Rd}^\eta c^*_{(\Gamma)}\mathfrak{A} \to \mathfrak{Rd}^\rho c^*_{(\Delta)}\mathfrak{A} \quad \text{is an isomorphism.} \tag{26}$$

Putting these two isomorphisms together we get

$$\mathsf{s}(\rho) : \mathfrak{Rd}_\Gamma c^*_{(J)}\mathfrak{A} \to \mathfrak{Rd}^\rho c^*_{(\Delta)}\mathfrak{A} \quad \text{is an isomorphism.} \tag{27}$$

Let $g := \mathsf{s}(\rho) \circ h$, then $g(\tau(a)) = \mathsf{s}(\rho)h(a) \neq 0$ by $h(a) \neq 0$, so

$$g : \mathfrak{Rd}_\Gamma\mathfrak{A} \to \mathfrak{Rd}^\rho c^*_{(\Delta)}\mathfrak{A} \quad \text{is a homomorphism with } \ g(\tau(a)) \neq 0. \tag{28}$$

Now, $\rho(\tau)$ in $\mathfrak{A}$ is the same as τ in $\mathfrak{Rd}^\rho\mathfrak{A}$, by definition. Therefore, $\rho(\tau)(ga)$ in $\mathfrak{A}$ is the same as $\tau(ga)$ in $\mathfrak{Rd}^\rho\mathfrak{A}$, which is the same as $g(\tau(a))$ which is nonzero by (28) and $\tau(a) \neq 0$. We are done with showing that $\mathfrak{A}$ is symmetric. □

Lemma 4.1. *Each full cylindric set algebra with unit a disjoint union of weak spaces is endo-dc.*

Proof. The proof in [18, 2.6.51, p. 417] for showing "(iii) does not imply (ii)" in fact proves the present Lemma 4.1. □

Theorem 4.2. (Solution of [18, Problem 2.13]) *There is an* RCA_α *which is not endo-dc, but each* RCA_α *can be embedded into an endo-dc algebra. In symbols:* $\mathsf{Edc}_\alpha \subset \mathbf{S}\mathsf{Edc}_\alpha = \mathsf{RCA}_\alpha$.

Proof. The algebra we based the proof of Theorem 2.1 on is not symmetric, hence not endo-dc by Theorem 4.1. Clearly, it is representable. This shows $\mathsf{Edc}_\alpha \neq \mathsf{RCA}_\alpha$. $\mathsf{Edc}_\alpha \subseteq \mathsf{RCA}_\alpha$ is proved as (iii)⇒(iv) in [18, Thm. 2.6.50]. $\mathsf{RCA}_\alpha = \mathbf{S}\mathsf{Edc}_\alpha$ follows from Lemma 4.1 immediately, since each representable algebra is embeddable into a full one. □

Remark 4.1. In the proof above, we used Theorem 4.1 to show that the algebra $\mathfrak{A}$ we used in the proof of Theorem 2.1 is not endo-dc. A concrete $\Gamma \subseteq \alpha$ and nonzero $a \in A$ for which there are no $\kappa \in \alpha$ and endomorphism h with the required properties are $\{0,1,2\}$ and g. Indeed, let $\tau := x - c_{(3)}(\beta + \gamma + \emptyset)$; see (2). Then $e(x)$ fails iff $\tau(x) \neq 0$, by (2). Hence, $\tau(g) \neq 0$ but $\tau(c_\kappa x) = 0$ for all κ by Remark 2.1(iii), and this implies that there is no endomorphism h of $\mathfrak{Rd}_{\{0,1,2\}}\mathfrak{A}$ with range inside $c^*_\kappa A$ and $h(g) \neq 0$.

Theorem 4.3. *Not all symmetric algebras are representable, and not all representable algebras are symmetric. In symbols,*

$$\mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha \subset \mathsf{Sy}_\alpha \quad \textit{and} \quad \mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha \subset \mathsf{RCA}_\alpha.$$

Proof. To exhibit a symmetric algebra that is nonrepresentable, take any nonrepresentable $\mathfrak{A} \in \mathsf{CA}_\alpha$, we "turn" it symmetric. Indeed, let

$$\mathfrak{B} := \prod \langle \mathfrak{Rd}^\rho \mathfrak{A} : \rho \text{ is a permutation of } \alpha \rangle.$$

That $\mathfrak{B}$ is symmetric can be seen by

$\mathfrak{B} \models e$	iff	by the definition of $\mathfrak{B}$
$\mathfrak{Rd}^\rho\mathfrak{A} \models e$ for all ρ	iff	by the definition of $\mathfrak{Rd}^\rho$
$\mathfrak{A} \models \rho(e)$ for all ρ	iff	by the nature of permutations
$\mathfrak{A} \models \rho(\eta(e))$ for all ρ, η	iff	by previous step
$\mathfrak{Rd}^\rho\mathfrak{A} \models \eta(e)$ for all ρ	iff	by first step
$\mathfrak{B} \models \eta(e)$.		

That $\mathfrak{B}$ is not representable follows from the facts that $\mathfrak{A} \notin \mathsf{RCA}_\alpha$ is a homomorphic image of $\mathfrak{B}$ (as $\mathfrak{A} = \mathfrak{Rd}^\rho\mathfrak{A}$ with ρ being the identity permutation of α) and RCA_α, being a variety ([19, 3.1.103]), is closed under homomorphic images.

The algebra used in the proof of Theorem 2.1 is representable and nonsymmetric, this proves the second part of the theorem, i.e., $\mathsf{RCA}_\alpha \cap \mathsf{Sy}_\alpha \subset \mathsf{RCA}_\alpha$. ☐

4.2. **Polyadic algebras are symmetric.** We have seen in the proof of Theorem 4.1 that substitution operations are useful in proving that an algebra is symmetric. In fact, the proof of Theorem 2.1 hinges over the fact that the polyadic substitution operations p_{ij} are not expressible in the witness algebra $\mathfrak{A}$. In this section we very briefly talk about Halmos' polyadic algebras. We show that α-dimensional quasi-polyadic equality algebras indeed have only 2^ω many subvarieties, since all their members are symmetric (in an appropriate sense). We then state some of the corollaries of our construction that concern polyadic algebras.

Polyadic equality algebras (PEA_αs) were introduced by Paul Halmos [17], they are basically cylindric algebras endowed with unary substitution operations s_ρ for $\rho : \alpha \to \alpha$. In the set algebras with unit ${}^\alpha U$ these are interpreted as

$$S_\rho(X) := \{s \in {}^\alpha U : \rho \circ s \in X\}.$$

Quasi-polyadic equality algebras were also defined by Halmos in [17], they retain only those substitutions where ρ is finite. Let QPEA_α denote their class; for precise definition see, e.g., [19, p. 266, item 9] or [39].

Theorem 4.4. QPEA_α *has exactly* 2^ω *many subvarieties.*

Proof. The idea of the proof is to show that each QPEA_α is symmetric in the sense analogous to the notion used in CA_α. However, the indices of the QPEA_α-operations have some structure, it is not so clear how we are to change the indices in an equation systematically/uniformly. (For more on this see the introduction of [39].) Therefore, we will use the more index-friendly version FPEA_α of QPEA_α defined in [39]. Since the two varieties are term-definitionally equivalent, proved as in [39, Thm. 1(ii)], it is enough to show that FPEA_α has only 2^ω subvarieties.

We are going to show that each element of FPEA_α is symmetric in the very analogous sense to CA_α, this will prove that QPEA_α has at most continuum many subvarieties (since each equation is equivalent to one which uses indices from ω only). That QPEA_α has indeed continuum many varieties can be seen by repeating the proof of [19, Thm. 4.1.24] for QPEA_α.

The extra-cylindric operations in an FPEA_α are denoted as p_{ij} for $i, j \in \alpha$. The operation p_{ij} stands for s_ρ where ρ is $[i, j]$, the latter being the permutation of α that interchanges i and j and leaves all the other elements fixed. Now, the definitions of $\rho(\tau)$ and $\mathsf{ind}(\tau)$ for FPEA_α-terms τ can easily be extended from CA_α. We will show that each $\mathfrak{A} \in \mathsf{FPEA}_\alpha$ is symmetric in the sense that

$$\mathfrak{A} \models e \quad \text{iff} \quad \mathfrak{A} \models \rho(e), \qquad \text{for all permutations } \rho \text{ of } \alpha.$$

Indeed, let $\mathfrak{A} \in \mathsf{FPEA}_\alpha$ and let τ be a term in the language of FPEA_α, let ρ be a permutation of α. Then $\mathsf{ind}(\tau)$ is finite, so we may assume that ρ is finite, too. Each finite permutation is a composition of transpositions $[i, j]$, so we may assume that ρ is indeed a transposition $[i, j]$ with $i \neq j$. In what follows we will write $\tau(\bar{x})$ and $\tau(\mathsf{p}_{ij}\bar{x})$ for $\tau(x_1, \ldots, x_n)$ and $\tau(\mathsf{p}_{ij}x_1, \ldots, \mathsf{p}_{ij}x_n)$. The following can be proved by induction on τ:

$$\mathsf{FPEA}_\alpha \models \quad \mathsf{p}_{ij}\tau(\bar{x}) = \rho(\tau(\mathsf{p}_{ij}\bar{x})) \tag{29}$$

with the use of the following equations that can be proved to hold in FPEA_α:

$$\mathsf{p}_{ij}(x + y) = \mathsf{p}_{ij}x + \mathsf{p}_{ij}y, \quad \mathsf{p}_{ij}(-x) = -\mathsf{p}_{ij}x,$$
$$\mathsf{p}_{ij}\mathsf{p}_{ij}x = x, \quad \mathsf{p}_{ij}x = \mathsf{p}_{ji}x,$$
$$\mathsf{p}_{ij}c_k x = c_{k'}\mathsf{p}_{ij}x, \quad \mathsf{p}_{ij}d_{kl} = d_{k'l'}, \quad \mathsf{p}_{ij}\mathsf{p}_{kl}x = \mathsf{p}_{k'l'}\mathsf{p}_{ij}x,$$

where $k' = \rho(k)$ and $l' = \rho(l)$. Now, let e be any equation, we may assume that it is of form $\tau(\bar{x}) = 1$:

$\mathfrak{A} \models e$	iff	by e being $\tau = 1$
$\mathfrak{A} \models \tau = 1$	implies	by $\mathsf{p}_{ij}1 = 1$
$\mathfrak{A} \models \mathsf{p}_{ij}\tau = 1$	iff	by (29)
$\mathfrak{A} \models \rho(\tau(\mathsf{p}_{ij}\bar{x})) = 1$	implies	by $\mathsf{p}_{ij}\mathsf{p}_{ij}x = x$
$\mathfrak{A} \models \rho(\tau) = 1$	iff	by e being $\tau = 1$
$\mathfrak{A} \models \rho(e)$.		

□

It is proved in [19, 5.4.18] that the cylindric reducts of PEA_αs are all representable, in symbols $\mathsf{Rd}_{ca}\mathsf{PEA}_\alpha \subseteq \mathsf{RCA}_\alpha$. Our results imply that this inclusion is a strict one. Further, the cylindric reducts of (quasi-)polyadic (equality)-algebras are not closed under subalgebras.

Corollary 4.1. *Not every representable cylindric algebra is the cylindric reduct of a polyadic equality algebra, hence the class of the latter is not closed under subalgebras. Formally:*

$$\mathsf{Rd}_{ca}\mathsf{PEA}_\alpha \subset \mathsf{RCA}_\alpha = \mathbf{S}\mathsf{Rd}_{ca}\mathsf{PEA}_\alpha.$$

Further, $\mathsf{Rd}_{ca}\mathsf{QPEA}_\alpha \subset \mathbf{S}\mathsf{Rd}_{ca}\mathsf{QPEA}_\alpha$.

Proof. It follows from the proof of Theorem 4.4 that the cylindric reduct of any quasi-polyadic equality algebra is symmetric. We have seen in Theorem 4.3 that not all representable algebras are symmetric. Take a nonsymmetric RCA_α, it is not in $\mathsf{Rd}_{ca}\mathsf{QPEA}_\alpha$, hence it is not in $\mathsf{Rd}_{ca}\mathsf{PEA}_\alpha$, either. Since all full cylindric set algebras are reducts of PEA_α, our nonsymmetric RCA_α is in $\mathbf{S}\mathsf{Rd}_{ca}\mathsf{PEA}_\alpha$. □

4.3. **Inductive algebras.** Let us call a cylindric algebra $\mathfrak{A}$ *inductive* iff $\mathfrak{A} \models e(c_ix_1,\dots,c_ix_n)$ implies $\mathfrak{A} \models e(x_1,\dots,x_n)$ whenever e is an equation and i does not occur as an index of an operation in e. Let Ind_α denote the class of all inductive CA_αs. While $\mathfrak{A} \models e$ implies $\mathfrak{A} \models e(c_ix)$ always holds, the converse of this would be thought to hold only in rather special cases, if at all. We are going to show that, on the contrary, there is a great variety of inductive algebras: each endo-dc algebra is inductive and we have already seen that there is a great variety of endo-dc algebras. There are even more inductive algebras than endo-dc algebras: $\mathsf{Edc}_\alpha \subset \mathsf{Ind}_\alpha$. We then prove that each inductive algebra is representable and symmetric (but the converse does not hold). Thus, we refine the chain $\mathsf{Lf}_\alpha \subset \mathsf{Dc}_\alpha \subset \mathsf{Di}_\alpha \subset \mathsf{Edc}_\alpha \subset \mathsf{Sy} \cap \mathsf{RCA}_\alpha \subset \mathsf{RCA}_\alpha$ by inserting a new class into it: $\mathsf{Edc}_\alpha \subset \mathsf{Ind}_\alpha \subset \mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha$. This is also a new representation theorem, a sharpening of [18, 2.6.50], since in the chain presented in [18, 2.6.50] the widest representable class was Edc_α. The new class Ind_α has an additional significance, namely an algebra is inductive iff it is equationally indistinguishable from an Lf_α. So, inductive algebras are in intimate connection with Lf_α. This will give us a specific Δ_2 formula distinguishing Lf_α and RCA_α.

Theorem 4.5.

(i) *Each endo-dc algebra is inductive, and each inductive algebra is symmetric and representable but the converses of these statements do not hold, i.e.,*

$$\mathsf{Edc}_\alpha \subset \mathsf{Ind}_\alpha \subset \mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha.$$

(ii) *An algebra is inductive iff there is an* Lf_α *with the same equational theory, i.e.,*

$$\mathfrak{A} \textit{ is inductive} \quad \textit{iff} \quad \mathsf{Eq}(\mathfrak{A}) = \mathsf{Eq}(\mathfrak{B}) \textit{ for some } \mathfrak{B} \in \mathsf{Lf}_\alpha.$$

Proof. First we prove part of (i), namely we prove $\mathsf{Edc}_\alpha \subseteq \mathsf{Ind}_\alpha$. This follows almost directly from the definitions and from Theorem 4.1. Let $\mathfrak{A} \in \mathsf{Edc}_\alpha$ and let $e(x_1,...,x_n)$ be an equation, $i \in \alpha$ such that i does not occur in e. In the sequel we will write $e(\bar{x})$ and $e(c_i\bar{x})$ in place of $e(x_1,\dots,x_n)$ and $e(c_ix_1,\dots,c_ix_n)$, respectively. We want to show $\mathfrak{A} \models e(c_i\bar{x})$ implies $\mathfrak{A} \models e(\bar{x})$. To this end, we assume $\mathfrak{A} \not\models e(\bar{x})$ and we show that $\mathfrak{A} \not\models e(c_i\bar{x})$. Let $a_1,\dots,a_n \in A$ be such that $\mathfrak{A} \not\models e(\bar{a})$. We may assume that e is of form $\tau = 0$, so we have $\tau(\bar{a}) \neq 0$ in $\mathfrak{A}$. Let $\Gamma := \mathsf{ind}(\tau)$. By $\mathfrak{A}$ being endo-dc, there are a homomorphism $h : \mathfrak{Rd}_\Gamma\mathfrak{A} \to \mathfrak{Rd}_\Gamma\mathfrak{A}$ and a $\kappa \in \alpha - \Gamma$ such that $h(\tau(\bar{a})) \neq 0$ and $h(b) = c_\kappa h(b)$ for all $b \in A$. Now, $h(\tau(\bar{a})) = \tau(h(\bar{a})) = \tau(h(a_1),\dots,h(a_n))$ by h being a homomorphism w.r.t. the operations occurring in τ. By $h(a_1) = c_\kappa h(a_1),\dots,h(a_n) = c_\kappa h(a_n)$ we then have

$\tau(c_\kappa h(\bar{a})) \neq 0$ in $\mathfrak{A}$. This means that $\mathfrak{A} \not\models e(c_\kappa \bar{x})$. Since $\mathfrak{A}$ is symmetric by Theorem 4.1 and $\kappa, i \notin \mathsf{ind}(e)$, we get that $\mathfrak{A} \not\models e(c_i \bar{x})$ as was desired.

Next we prove (ii). For proving the "only-if" part, let $\mathfrak{A}$ be inductive, we will show that it is equationally indistinguishable from an Lf_α. Let $\mathfrak{C}$ be an elementary α-saturated extension of $\mathfrak{A}$, and let $\mathfrak{B}$ be the greatest locally finite-dimensional subalgebra of $\mathfrak{C}$. (This exists by [18, 2.1.5(ii)].) We are going to show that $\mathfrak{A}$ and $\mathfrak{B}$ are equationally indistinguishable. Let e be an equation. If $\mathfrak{A} \models e$, then $\mathfrak{C} \models e$ because $\mathfrak{C}$ is an elementary extension of $\mathfrak{A}$, and thus $\mathfrak{B} \models e$ because $\mathfrak{B}$ is a subalgebra of $\mathfrak{C}$. Assume now $\mathfrak{A} \not\models e$. Let $\Delta := \{i_1, \ldots, i_n\}$ be disjoint from $\mathsf{ind}(e)$ with $i_1, \ldots, i_n$ being all distinct. Then $\mathfrak{A} \not\models e(c_{i_1}\bar{x})$ since $\mathfrak{A}$ is inductive and $\mathfrak{A} \not\models e$. But then $\mathfrak{A} \not\models e(c_{i_1} c_{i_2} \bar{x})$ because $i_2 \notin \mathsf{ind}(e(c_{i_1}\bar{x}))$, and so on, showing that $\mathfrak{A} \not\models e(c_{(\Delta)}\bar{x})$. Let

$$\Sigma(\bar{x}) := \{\neg e(\bar{x}), c_i \bar{x} = \bar{x} : i \in \alpha - \mathsf{ind}(e)\}.$$

Each finite subset of Σ is satisfiable in $\mathfrak{C}$ by $\mathfrak{A} \not\models e(c_{(\Delta)}\bar{x})$ for all finite $\Delta \subseteq \alpha - \mathsf{ind}(e)$. By $\mathfrak{C}$ being α-saturated, this implies that there are $b_1, \ldots, b_n \in C$ for which $\Sigma(\bar{b})$ holds in $\mathfrak{C}$. These b_j's are finite dimensional (by $c_i(b_j) = b_j$ for all $i \in \alpha - \mathsf{ind}(e)$), and $\mathfrak{C} \not\models e(\bar{b})$ (by $\neg e(\bar{x}) \in \Sigma(\bar{x})$). Hence $b_1, \ldots, b_n \in B$ and $\mathfrak{C} \not\models e(\bar{b})$, hence $\mathfrak{B} \not\models e(\bar{b})$, i.e., $\mathfrak{B} \not\models e$. This finishes the "only-if" part of the proof of (ii). For the "if" part, we have to show that each $\mathfrak{B} \in \mathsf{Lf}_\alpha$ is inductive. Indeed, $\mathsf{Lf}_\alpha \subseteq \mathsf{Edc}_\alpha$ by [18, 2.6.50], and $\mathsf{Edc}_\alpha \subseteq \mathsf{Ind}_\alpha$ by that part of (i) that we have already proved.

It remains to prove the rest of (i). We have already shown $\mathsf{Edc}_\alpha \subseteq \mathsf{Ind}_\alpha$. To show $\mathsf{Ind}_\alpha \subseteq \mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha$ we use (ii), [18, 2.6.50] and Theorem 4.1, as follows. Let $\mathfrak{A} \in \mathsf{Ind}_\alpha$. Then $\mathsf{Eq}(\mathfrak{A}) = \mathsf{Eq}(\mathfrak{B})$ for some $\mathfrak{B} \in \mathsf{Lf}_\alpha$, by (ii). Now, $\mathsf{Lf}_\alpha \subseteq \mathsf{Edc}_\alpha \subseteq \mathsf{Sy}_\alpha$ by [18, 2.6.50] and Theorem 4.1, $\mathsf{Lf}_\alpha \subseteq \mathsf{RCA}_\alpha$ by [19]. Thus, $\mathfrak{A} \in \mathsf{Sy}_\alpha \cap \mathsf{RCA}_\alpha$. We turn to proving that the stated inclusions are proper.

First we want to exhibit an inductive algebra that is not endo-dc. The difference between the two notions, and this will be reflected in the algebra $\mathfrak{D}$ we exhibit, is that the notion of being inductive talks about the equational theory of the algebra, while the notion endo-dc talks about the inner structure of the algebra. The algebra $\mathfrak{D}$ is a direct product of the ω-generated free RCA_α-algebra $\mathfrak{F}$ and another representable algebra $\mathfrak{A}$. By this, it is already ensured that $\mathfrak{D}$ is inductive, as follows:

$\mathfrak{D} \models e(c_i\bar{x})$	implies	by $\mathfrak{F}$ being a homomorphic image of $\mathfrak{D}$
$\mathfrak{F} \models e(c_i\bar{x})$	implies	by $\mathfrak{F}$ being a free algebra of RCA_α
$\mathsf{RCA}_\alpha \models e(c_i\bar{x})$	implies	by $\mathsf{Lf}_\alpha \subseteq \mathsf{RCA}_\alpha$
$\mathsf{Lf}_\alpha \models e(c_i\bar{x})$	implies	by $\mathsf{Lf}_\alpha \subseteq \mathsf{Ind}_\alpha$
$\mathsf{Lf}_\alpha \models e(\bar{x})$	implies	by $\mathsf{RCA}_\alpha = \mathbf{Var}(\mathsf{Lf}_\alpha)$
$\mathsf{RCA}_\alpha \models e(\bar{x})$	implies	by $\mathfrak{D} \in \mathsf{RCA}_\alpha$
$\mathfrak{D} \models e(\bar{x})$.		

The role of the algebra $\mathfrak{A}$ in the direct product is to destroy the property endo-dc. The idea is that we split an α-dimensional atom in $\mathfrak{A}$ into more parts than there are i-closed elements (for some $i \in \alpha$) in $\mathfrak{D}$, so each required endomorphism will have to collapse all of the split parts to 0. We begin to elaborate this idea. Let W be a set of cardinality bigger than $|\alpha|$ and let $\langle W, +, z\rangle$ be any commutative group on W where z is the zero-element of $+$ (i.e., $w + z = w$ for all $w \in W$).

Let $U_i := W \times \{i\}$, let $p := \langle (z, i) : i \in \alpha\rangle$, let $U := \bigcup\{U_i : i \in \alpha\}$ and $T := \{s \in {}^{\alpha}U : s_i \in U_i$ for all $i \in \alpha$ and $|\{i \in \alpha : s_i \neq p_i\}| < \omega\}$. Let $\mathfrak{B}$ be the weak cylindric set algebra of dimension α with unit element ${}^{\alpha}U^{(p)}$ and generated by T. Then $|B| = |\alpha|$ and T is an atom in $\mathfrak{B}$. We now split T in $\mathfrak{B}$ into $|W|$ many parts. For all $g \in W$ let

$$T_g := \{s \in T : \sum\{w : s_i = (w, i) \text{ for some } i \in \alpha\} = g\}.$$

Then the T_g's ($g \in W$) form a disjoint union of T such that

$$c_i T_g = c_i T \quad \text{for all } i \in \alpha \text{ and } g \in W. \tag{30}$$

Let $\mathfrak{A}$ be the weak set algebra with unit element ${}^{\alpha}U^{(p)}$ and generated by T together with T_g, $g \in W$. It is not hard to check, by using (30), that each element of A is of form

$$b + \sum\{T_g : g \in X\},$$

where $b \in B$ and X is a finite or co-finite subset of W. Thus, all elements of $A - B$ are α-dimensional. We now show that $\mathfrak{D} = \mathfrak{F} \times \mathfrak{A}$ is not endo-dc. Let us fix a $g \in W$, let $a := \langle 0, T_g\rangle \in D$, let $\Gamma := \{0\}$, we want to show that there are no endomorphism h of $\mathfrak{Rd}_{\Gamma}\mathfrak{D}$ and $\kappa \in \alpha$ such that h takes a to a nonzero element and each element of the range of h is κ-closed. For, assume the contrary, that h and κ are as described above, we will derive a contradiction. Since there are only α many κ-closed elements of $\mathfrak{A}$, hence of $\mathfrak{D}$ by $|F| = |\alpha|$, and there are more than α many split parts of T, the endomorphism h has to take two of the elements $\langle 0, T_w\rangle \in D$ to the same element. But these are all disjoint from each other, so $h(\langle 0, T_w\rangle) = 0$ for some $w \in W$. But then $h(c_0\langle 0, T_w\rangle) = c_0 h(\langle 0, T_w\rangle = 0$ since h is a homomorphism w.r.t. c_0. However, $c_0\langle 0, T_w\rangle = \langle 0, c_0 T_w\rangle = \langle 0, c_0 T\rangle \geq \langle 0, T\rangle \geq \langle 0, T_g\rangle$, showing that $h(\langle 0, T_g\rangle) = h(a) = 0$, and this contradicts our assumption $h(a) \neq 0$. Thus the algebra $\mathfrak{D}$ is inductive but not endo-dc.

Finally, we exhibit a symmetric representable algebra which is not inductive. Here, both notions refer to the equational theory of the algebra, but they make different restrictions on it. Symmetry requires that if an equation holds, then its versions where we rename the indices hold also, and inductivity requires that the same equations hold for the some-cylindrification-closed elements than for the whole algebra. Our algebra $\mathfrak{A}$ that we used in the proof of Theorem 2.1 is not symmetric, hence it is not inductive, either, by the already proved part of (i) of the present theorem. We will modify the algebra $\mathfrak{A}$ so that it becomes symmetric, but the above mentioned difference between the some-cylindrification-closed and α-dimensional elements remains intact. Let R denote the set of all permutations of α. Define $\mathfrak{B}$ as the direct product of all the ρ-reducts of $\mathfrak{A}$ for $\rho \in R$, i.e.,

$$\mathfrak{B} := \prod\langle \mathfrak{Rd}^{\rho}\mathfrak{A} : \rho \in R\rangle.$$

Clearly, $\mathfrak{B}$ is symmetric and representable. We show that it is not inductive. Take the equation e used in the proof of Theorem 2.1. We have seen in Remark 2.1 that $\rho(e(c_0 x))$ is valid in $\mathfrak{A}$ for all $\rho \in R$. Hence, $e(c_0 x)$ is valid in all $\mathfrak{Rd}^{\rho}\mathfrak{A}$, hence in $\mathfrak{B}$ by its construction. However, $e(x)$ is not valid in $\mathfrak{B}$ since it is not valid in $\mathfrak{A}$. This shows that $\mathfrak{B}$ is not inductive. □

It is known that the same universal formulas are valid in Lf_{α} as in RCA_{α}, see [19, 4.1.29]. There is no existential formula distinguishing Lf_{α} and RCA_{α}, either because each RCA_α has a subalgebra in Lf_α. The next complexity class is Δ_2-formulas, and our theorems so far imply that Lf_α indeed can be distinguished from RCA_α by a Δ_2-formula. We note that it was known that there is a Π_2-formula distinguishing Lf_α and RCA_α (see [18, 2.6.53]).

Corollary 4.2. *There is a Δ_2-formula which is valid in Lf_α but is not valid in RCA_α.*

Proof. The property of being inductive is defined by a set D of formulas of form $\forall\bar{x}e_1(\bar{x}) \to \forall\bar{x}e_2(\bar{x})$ where e_1, e_2 are equations using variables occurring in $\bar{x}$. All such formulas are known to be Δ_2. Indeed, let φ denote the previous formula. Then φ is equivalent both to the Π_2-formula $\forall\bar{x}\exists\bar{y}(\neg e_2(\bar{x}) \to \neg e_1(\bar{y}))$, and to the Σ_2 formula $\exists\bar{x}\forall\bar{y}(\neg e_1(\bar{x}) \vee e_2(\bar{y}))$. There is a representable algebra $\mathfrak{A}$ which is not inductive, by Theorem 4.5(i). Since $\mathfrak{A}$ is not inductive, there is a Δ_2-formula $\varphi \in D$ which is not valid in $\mathfrak{A}$. Then $\mathsf{RCA}_\alpha \not\models \varphi$ by $\mathfrak{A} \in \mathsf{RCA}_\alpha$. However, $\mathsf{Lf}_\alpha \models \varphi$ since $\mathsf{Lf}_\alpha \subseteq \mathsf{Ind}_\alpha$ by Theorem 4.5(ii). ☐

Remark 4.2. (i) We can get a concrete Δ_2 formula separating Lf_α and RCA_α by using Remark 2.1(ii).

(ii) From the fact that there are more subvarieties of RCA_α than generated by $\mathsf{Lf}_\alpha \subseteq \mathsf{RCA}_\alpha$ we can immediately get that there is a subvariety V of RCA_α which is not generated by its Lf_α-members, i.e., V is not generated by $\mathsf{V} \cap \mathsf{Lf}_\alpha$.

(iii) Using (ii) above, from the fact that there are more subvarieties of RCA_α than generated by $\mathsf{Lf}_\alpha \subseteq \mathsf{RCA}_\alpha$ we can immediately get that there is a Δ_2-formula distinguishing Lf_α and RCA_α, because the structure of subvarieties of a variety V is determined by its Δ_2-theory. Indeed, assume that K and L have the same Δ_2-theories. Then the same varieties are generated by subclasses of K and L, since all of the formulas of form $\forall\bar{x}e_1(\bar{x}) \wedge \cdots \wedge \forall\bar{x}e_n(\bar{x}) \to \forall\bar{x}e_0(\bar{x})$ are Δ_2. Indeed, let $\mathsf{K}_0 \subseteq \mathsf{K}$, let $E_0 = \mathsf{Eq}(\mathsf{K}_0)$ and let $\mathsf{L}_0 = \{\mathfrak{A} \in \mathsf{L} : \mathfrak{A} \models E_0\}$, then $\mathsf{Eq}(\mathsf{L}_0) = E_0$, since $E_0 \subseteq \mathsf{Eq}(\mathsf{L}_0)$ by the definition of L_0 and for all $e \notin E_0$ we have $\mathsf{K} \not\models \Sigma \to e$ for all finite $\Sigma \subseteq E$, so the same is true for L.

(iv) From what we said so far, it follows that for any $\mathsf{Lf}_\alpha \subseteq \mathsf{K} \subseteq \mathsf{Sy} \cap \mathsf{RCA}_\alpha$ we have that the Δ_2-theories of K and RCA_α are different but the corresponding universal and existential theories coincide.

4.4. **Characterization of the equational theory of RCA_α.** In this section we concentrate on sets of equations, rather than on algebras. Assume E is a set of equations in the language of CA_α, it contains the cylindric axioms (C_0)–(C_7) axiomatizing CA_α and it is semantically closed (i.e., $e \in E$ iff $E \models e$). We call E *inductive* iff $e(c_ix_1, \ldots, c_ix_n) \in E$ implies $e(x_1, \ldots, x_n) \in E$ whenever i does not occur as an index of an operation in e. Thus, an algebra is inductive iff its equational theory is such. However, we will see that not all models of an inductive set of equations are inductive. In the next theorem we characterize the inductive sets of equations. We obtain that they coincide with the equational theories of subclasses of Lf_α. Equational theories of subclasses of Lf_α are important, because Lf_αs correspond to ordinary first order logic theories ([19, 4.3.28(iii)]).

Theorem 4.6.

$$E \text{ is inductive} \quad \text{iff} \quad E = \mathsf{Eq}(\mathsf{K}) \text{ for some } \mathsf{K} \subseteq \mathsf{Lf}_\alpha.$$

Proof. Assume that E is inductive. Let $\mathfrak{F}$ be the E-free ω-generated algebra. Then $E = \mathsf{Eq}(\mathfrak{F})$ and $\mathfrak{F}$ is inductive, by E being inductive. So, there is $\mathfrak{B} \in \mathsf{Lf}_\alpha$ with

$\mathsf{Eq}(\mathfrak{F}) = \mathsf{Eq}(\mathfrak{B})$, by Theorem 4.5(ii). This shows that $E = \mathsf{Eq}(\mathsf{K})$ for $\mathsf{K} = \{\mathfrak{B}\} \subseteq \mathsf{Lf}_\alpha$. Assume now $\mathsf{K} \subseteq \mathsf{Lf}_\alpha$ and let $E = \mathsf{Eq}(\mathsf{K})$. Then E contains the cylindric axioms (C_0)–(C_7) and is semantically closed. Also, E is inductive by Theorem 4.5(ii). □

Let us call *inductive rule* the rule according to which from $e(c_i x_1, \dots, c_i x_n)$ we can infer $e(x_1, \dots, x_n)$ provided that $i \notin \mathsf{ind}(e)$. Note that this is a decidable rule, because given any equation we can decide whether it is of form $e(c_i x_1, \dots, c_i x_n)$ for an equation e such that $i \notin \mathsf{ind}(e)$.

Corollary 4.3. *The equational theory of* RCA_α *is the least set of equations which contains the equations* (C_0)–(C_7) *which define* CA_α*, is closed under the five rules of equational logic, and is closed under the inductive rule defined above.*

Proof. By definition, a set E of equations contains (C_0)–(C_7), is closed under the five rules of equational logic, and is closed under the inductive rule iff E is inductive. This is so because equational logic is complete for its five rules. By Theorem 4.6, the least such set axiomatizes the variety generated by the largest subclass of Lf_α, which subclass is Lf_α itself. Now, the variety generated by Lf_α is RCA_α, e.g., by [19, 4.1.29]. □

Corollary 4.3 above gives a simple enumeration for the equational theory of RCA_α. It can be considered as a solution to [19, Problem 4.1] which asks for a simple equational base for $\mathsf{Eq}(\mathsf{RCA}_\alpha)$. Certainly, the enumeration based on the above Corollary 4.3 is simpler than any of the three such enumerations given in [19, sec. 4.1]. It has some resemblance to the second and third enumerations given in [19]. An advantage of the present enumeration is that it stays strictly in the equational language of CA_α while the second method given in [19] uses all first order logic formulas in the language of CA_α, and the third method even uses symbols outside the language of CA_α. A drawback of the present enumeration is that it works only for infinite α, while the three methods given in [19] work for finite α as well. We note that possible solutions for Problems 4.1 and the related Problem 4.16 were also given in Simon [43] and Venema [45]. The root of [19, Problem 4.1] is Monk's theorem saying that RCA_α is not finite schema axiomatizable, exposing a gap between abstract and representable cylindric algebras. As we mentioned in the Introduction, this gap is addressed many ways in algebraic logic, some works in this direction are [6, 13, 21, 22, 37, 38].

Remark 4.3. (i) Not all models of an inductive set of equations are inductive. An example is $\mathsf{Eq}(\mathsf{RCA}_\alpha)$. It is inductive because RCA_α is generated by Lf_α and it has a noninductive algebra by Theorem 4.5(i). Exceptions are the equational theories of the minimal cylindric algebras in the sense that all members of these varieties are inductive. We wonder whether these are the only such exceptions or not.

(ii) Any variety of cylindric algebras generated by a class of locally finite-dimensional algebras is also generated by a single Lf_α. This was known, but this also follows from Theorems 4.5, 4.6 as follows. Let V be generated by $\mathsf{K} \subseteq \mathsf{Lf}_\alpha$. Then $\mathsf{Eq}(\mathsf{V})$ is inductive by Theorem 4.6, so the free algebra $\mathfrak{F}$ of V is inductive, then it is equationally indistinguishable from a $\mathfrak{B} \in \mathsf{Lf}_\alpha$ by Theorem 4.5, and then $\mathsf{Eq}(V) = \mathsf{Eq}(\mathfrak{B})$.

(iii) A set E is inductive iff there is an ordinary first order logic theory Th such that E is the equational theory of all the concept algebras of models of Th. We briefly sketch a proof for this, we deal with the logical connections in detail in another paper. Let E be any inductive set. Then, by (ii) above, it is the equational theory of a single $\mathfrak{B} \in \mathsf{Lf}_\alpha$. Then $\mathfrak{B}$ is the Lindenbaum-Tarski algebra of an ordinary theory Th, by [19, 4.3.28(ii)]. It is not difficult to see that the Lindenbaum-Tarski algebra is a subdirect product of $\{\mathsf{Ca}^{\mathfrak{M}} : \mathfrak{M} \models \mathsf{Th}\}$, which finishes the proof.

4.5. **Continuum many inductive varieties.** We close the paper by showing that subclasses of concept algebras of ordinary first order logic with infinite universes generate continuum many subvarieties. The proof of this theorem will be analogous to, but simpler than, the proof of Theorem 2.1. Concept algebras of ordinary first order logic with finite universes also generate continuum many varieties; a slightly modified version of the proof of [19, 4.1.24] shows this. This is why we deal with concept algebras of models with infinite universes below.

Let $a_m := c_{(m)} \prod\{-d_{ij} : i < j < m\}$, for $m \in \omega$; cf. [18, 2.4.61]. We call a cylindric algebra *of infinite base* iff $\{e_m : m \in \omega\}$ is valid in it, and ${}_\infty\mathsf{Lf}_\alpha$ denotes the class of Lf_αs of infinite bases. An *inductive variety of infinite base* is a variety whose equational theory is inductive and which contains the equations $\{a_m = 1 : m \in \omega\}$. The inductive varieties of infinite base are exactly the varieties generated by subclasses of ${}_\infty\mathsf{Lf}_\alpha$, by Theorem 4.5. Also, they are exactly the varieties generated by concept algebras of ordinary first order logic with infinite bases, by Remark 4.3(iii).

The following is a counterpoint to Theorem 2.1. We know that there can be only continuum many inductive varieties for all α because inductive varieties are also symmetric. The following theorem says that there are indeed continuum many inductive varieties for all α, even if we require the bases to be infinite.

Theorem 4.7. *The number of varieties generated by subclasses of ${}_\infty\mathsf{Lf}_a$ is 2^ω, for all infinite α. In other words, there are exactly continuum many inductive varieties of infinite base.*

Proof. As in the proof of Theorem 2.1, we will use a set of independent equations, in this case we will use a countable set of independent equations. The n-th equation e_n will express that there is no partition of the universe (in the form of an equivalence relation as element of the algebra) all of whose blocks have size n. Then, for each $n \in \omega$ we will exhibit an algebra $\mathfrak{A}_n \in {}_\infty\mathsf{Lf}_\alpha$ in which e_n fails, but e_k holds for all $k \in \omega - 2$, $k \neq n$.

We begin to write the term expressing that "x is not an equivalence relation on the whole base set with each equivalence block having size n". The following terms express the parts of this statement (in the final equation we will replace x with $c_2 \dots c_n x$). Let $n \geq 2$.

The domain of x is not the base set:

$$\delta(x) := c_0 - c_1 x.$$

x is not symmetric:

$$\sigma(x) := c_0 c_1 ({}_2\mathsf{s}(0,1)x \oplus x).$$

x is not transitive:

$$\tau(x) := c_0 c_1 c_2 (x \cdot \mathsf{s}^{01}_{12} x - \mathsf{s}^{01}_{02} x).$$

x is not reflexive:

$$\rho(x) := c_0 c_1 (d_{01} - x).$$

There is a block in x with size $< n$:

$$\mu_<(x) := c_0 - c_1 \cdots - c_{n-1}(\prod\{-d_{ij} : i < j < n\} \cdot \prod\{\mathsf{s}^{01}_{ij}x : i < j < n\}).$$

There is a block in x with size $> n$:

$$\mu_>(x) := c_{(n+1)}(\prod\{-d_{ij} : i < j \le n\} \cdot \prod\{\mathsf{s}^{01}_{ij}x : i < j \le n\}).$$

The sum of all of these is

$$\eta(x) := \delta(x) + \sigma(x) + \tau(x) + \rho(x) + \mu_<(x) + \mu_>(x).$$

The equation e_n is defined as

$$e_n(x) \quad := \quad \eta(c_2 \dots c_n x) = 1.$$

Lemma 4.2. *Let $\mathfrak{A} \in \mathsf{Cs}_\alpha$, let $n \ge 2$ and let $a = c_2 \dots c_n a \in A$. Then $\mathfrak{A} \models e_n(a)$ iff for all $s \in a$ it is true that $a[s, 01]$ is not an equivalence relation on the base set with each block of size n.*

Proof of Lemma 4.2. Assume the conditions of the lemma, then $\mathfrak{A} \models e_n(a)$ iff for all $s \in {}^\alpha U$, where U is the base set of $\mathfrak{A}$, we have $s \in \eta(a) = \delta(a) + \cdots + \mu_>(a)$. Let $R := a[s, 01] = \{(u, v) : s(01/uv) \in a\} \subseteq U \times U$. We have

$$\text{(31)} \qquad s \notin \delta(a) \quad \text{iff} \quad \text{the domain of } R \text{ is } U.$$

Indeed, $s \notin \delta(a)$ iff $s \in -\delta(a) = -c_0 - c_1 a$ iff for all $u \in U$ there is $v \in U$ with $s(01/uv) \in a$, which means $(u, v) \in a[s, 01]$.

$$\text{(32)} \qquad s \notin \sigma(a) \quad \text{iff} \quad R \text{ is symmetric.}$$

Indeed, $s \notin \sigma(a) = c_0 c_1({}_2\mathsf{s}(0, 1)a \oplus a)$ iff for all $u, v \in U$ we have $s(01/uv) \notin ({}_2\mathsf{s}(0, 1)a \oplus a)$, this last thing holds iff $s(01/vu) \in a \Leftrightarrow s(01/uv) \in a$, which means that R is symmetric:

$$\text{(33)} \qquad s \notin \tau(a) \quad \text{iff} \quad R \text{ is transitive.}$$

Indeed, $s \notin \tau(a) = c_0 c_1 c_2(a \cdot \mathsf{s}^{01}_{12}a - \mathsf{s}^{01}_{02}a)$ iff for all $u, v, w \in U$ whenever $s(012/uvw) \in a \cdot \mathsf{s}^{01}_{12}a$ we have $s(012/uvw) \in \mathsf{s}^{01}_{02}a$. Now, $s(012/uvw) \in a \cdot \mathsf{s}^{01}_{12}a$ means that $(u, v) \in R$ and $(v, w) \in R$ (we used $c_2 a = a$). Similarly, $s(012/uvw) \in \mathsf{s}^{01}_{02}a$ means that $(u, w) \in R$. Putting these together, we get that R is transitive.

$$\text{(34)} \qquad s \notin \rho(a) \quad \text{iff} \quad R \text{ is reflexive.}$$

Indeed, $s \notin \rho(a) = c_0 c_1(d_{01} - a)$ iff for all $u, v \in U$ we have $u = v$ implies $(u, v) \in R$, i.e., R is reflexive.

Assume now that $s \notin (\delta(a) + \sigma(a) + \tau(a) + \rho(a))$. Then, by the above, we have that R is an equivalence relation on U.

$$\text{(35)} \qquad s \in \mu_<(a) \quad \text{iff} \quad \text{there is a block in } R \text{ with size } < n.$$

Indeed, $s \in \mu_<(a) = c_0 - c_1 \cdots - c_{n-1}(\prod\{-d_{ij} : i < j < n\} \cdot \prod\{\mathsf{s}^{01}_{ij}a : i < j < n\})$ iff there is $u_0 \in U$ such that there are no $u_1, \dots, u_{n-1} \in U$ such that $u_0, \dots, u_{n-1}$ are all distinct and $(u_i, u_j) \in R$ for all $i < j < n$. This means that there is a block in R with size $< n$.

$$\text{(36)} \qquad s \in \mu_>(a) \quad \text{iff} \quad \text{there is a block in } R \text{ with size } > n.$$

Indeed, $s \in \mu_>(a) = c_{(n+1)}(\prod\{-d_{ij} : i < j \le n\} \cdot \prod\{\mathsf{s}^{01}_{ij}a : i < j \le n\})$ iff there are $u_0, \dots, u_n \in U$ such that they are all distinct and $(u_i, u_j) \in R$ for all $i < j \le n$, and this means that there is a block in R with size $> n$.

By the above we have that $s \in \eta(a)$ iff whenever $R = s[a, 01]$ is an equivalence relation on U, there is either a block with size $< n$ or else there is a block with size $> n$. This proves Lemma 4.2. $\square$

Let $n \in \omega, n \geq 2$, let U be an infinite set, and let R be an equivalence relation on U with each block of size n. Let $\mathfrak{A}_n$ be the Cs_α with base U and generated by $g := \{s \in U^\alpha : (s_0, s_1) \in R\}$.

Lemma 4.3. *$\mathfrak{A}_n \not\models e_n$ but $\mathfrak{A}_n \models e_k$ for all $k \neq n$, $k \in \omega - 2$.*

Proof of Lemma 4.3. $\mathfrak{A}_n \not\models e_n$ by Lemma 4.2 and $g \in A_n$, since clearly $s[g, 01] = R$ for all $s \in g$ and R is an equivalence relation of the kind e_n prohibits. Let $k \in \omega - 2$, $k \neq n$, we want to show that $\mathfrak{A}_n \models e_k$. By Lemma 4.2, it is enough to show that $a[s, 01]$ is not an equivalence relation on U with all blocks of size k, whenever $s \in a = c_2 \dots c_n a$. We begin doing this.

We call an $X \subseteq {}^\alpha U$ *regular* if, intuitively, X is determined by its restriction to its dimension set $\Delta(X)$, formally

$$s \in X \quad \text{iff} \quad z \in X, \qquad \text{whenever } s, z \in {}^\alpha U \text{ and } s, z \text{ agree on } \Delta(X),$$

where $\Delta(X) := \{i \in \alpha : c_i X \neq X\}$. Since $\mathfrak{A}_n$ is generated by g which is a locally finite regular element, we have that a is also a regular locally finite element, by [19, 3.1.64]. Let $S' := \mathsf{Rg}(s \restriction \Delta(a) \cup \{0\})$ and let $S := \{u \in U : (\exists v \in S')(u, v) \in R\}$. Then S is finite since $\Delta(a)$ is finite and each block of R is finite. Assume that $E := a[s, 01]$ is an equivalence relation on U with all blocks finite, and ≥ 2. If E is not such, then we are done by Lemma 4.2 and $k \geq 2$.

Let $u, v \in U - S$ such that $(u, v) \in E - R$, we will derive a contradiction. Let $w \in U - S - u/R$ be arbitrary. There are infinitely many such w. We want to show that $w \in u/E$, contradicting our assumption that u/E is finite. Let $\pi : U \to U$ be a permutation of U which leaves R fixed, is identity on $S \cup \{u\}$ and takes v to w. There is such a permutation since $v/R \cup w/R$ is disjoint from $S \cup \{u\}$ by our assumptions. Since π leaves R fixed and $\mathfrak{A}_n$ is generated by g, we have that a is closed under π, i.e., $z \in a$ iff $\pi \circ z \in a$ for all z. Now, $(u, v) \in E = a[s, 01]$ means that $s(01/uv) \in a$. Therefore $z := \pi \circ (s(01/uv)) \in a$. Now, $z_0 = \pi(u) = u$, $z_1 = \pi(v) = w$, and z agrees with $s(01/uw)$ on $\Delta(a)$ by π being the identity on S. Hence $s(01/uw) \in a$ by $z \in a$ and a being regular. This means $(u, w) \in E$, i.e., $w \in u/R$ as was to be shown.

Assume now that $u, v \in U - S$ such that $(u, v) \in R - E$, we will derive a contradiction. Let $(w, v) \in E$, $w \neq v$. There is such by our assumption that each block of E has at least two elements. Let $\pi : U \to U$ be a permutation of U which is identity on $U - \{u, v\}$ and interchanges u and v. This π is identity on S by $u, v \notin S$ and it leaves R fixed by $(u, v) \in R$. Thus, a is closed under this π, too. As before, $(w, v) \in E = a[s, 01]$ means that $s(01/wv) \in a$, therefore $z := \pi \circ (s(01/wv)) \in a$. Then $z_0 = \pi(w) = w$, $z_1 = \pi(v) = u$, and z agrees with $s(01/wu)$ on $\Delta(a)$ by π being the identity on S. Hence $s(01/wu) \in a$ by $z \in a$ and a being regular. This means $(w, u) \in E$, contradicting $(u, v) \notin E$ and $(v, w) \in E$.

We have seen that R and E agree on the infinite set $U - S$. Since each block of R has n elements, this means that E has at least one block with exactly n elements. So, $e_k(a)$ holds in $\mathfrak{A}$ by $k \neq n$ and Lemma 4.2. By this, Lemma 4.3 has been proved. $\square$

We are ready for completing the proof of Theorem 4.7. For each $H \subseteq \omega - 2$ let V_H be the variety generated by $\mathsf{K}_H := \{\mathfrak{A}_n : n \in H\} \subseteq {}_\infty\mathsf{Lf}_\alpha$. Assume $G, H \subseteq \omega - 2$ are distinct, say $n \in H - G$. Then $\mathfrak{A}_n \in \mathsf{V}_G - \mathsf{V}_H$ by Lemma 4.3, so V_H and V_G are distinct. This shows that there are at least continuum many varieties generated by subclasses of ${}_\infty\mathsf{Lf}_\alpha$. $\square$

References

[1] H. Andréka, S. D. Comer, J. X. Madarász, I. Németi, and T. Sayed Ahmed, *Epimorphisms in cylindric algebras and definability in finite variable logic*, Algebra Universalis **61** (2009), no. 3-4, 261–282, DOI 10.1007/s00012-009-0022-2. MR2565854

[2] Hajnal Andréka, Miklós Ferenczi, and István Németi (eds.), *Cylindric-like algebras and algebraic logic*, Bolyai Society Mathematical Studies, vol. 22, János Bolyai Mathematical Society, Budapest; Springer, Berlin, 2012. MR3137681

[3] Hajnal Andréka, Steven Givant, and István Németi, *The lattice of varieties of representable relation algebras*, J. Symbolic Logic **59** (1994), no. 2, 631–661, DOI 10.2307/2275414. MR1276639

[4] H. Andréka, J. D. Monk, and I. Németi (eds.), *Algebraic logic*, Colloquia Mathematica Societatis János Bolyai, vol. 54, North-Holland Publishing Co., Amsterdam, 1991. Papers from the colloquium held in Budapest, August 8–14, 1988. MR1153415

[5] Hajnal Andréka and István Németi, *Comparing theories: the dynamics of changing vocabulary*, Johan van Benthem on logic and information dynamics, Outst. Contrib. Log., vol. 5, Springer, Cham, 2014, pp. 143–172, DOI 10.1007/978-3-319-06025-5_6. MR3329289

[6] H. Andréka and R. J. Thompson, *A Stone-type representation theorem for algebras of relations of higher rank*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 671–682, DOI 10.2307/2000932. MR961607

[7] M. Assem, T. Sayed-Ahmed, G. Sági, and D. Sziráki, *The number of countable models via agebraic logic*, Manuscript 2013. http://real.mtak.hu/id/eprint/16985

[8] Thomas William Barrett and Hans Halvorson, *Morita equivalence*, Rev. Symb. Log. **9** (2016), no. 3, 556–582, DOI 10.1017/S1755020316000186. MR3569170

[9] Nick Bezhanishvili, *Varieties of two-dimensional cylindric algebras*, Cylindric-like algebras and algebraic logic, Bolyai Soc. Math. Stud., vol. 22, János Bolyai Math. Soc., Budapest, 2012, pp. 37–59, DOI 10.1007/978-3-642-35025-2_3. MR3156394

[10] W. J. Blok, *Varieties of interior algebras*, PhD Dissertation, University of Amsterdam, 1976.

[11] W. J. Blok, *The lattice of modal logics: an algebraic investigation*, J. Symbolic Logic **45** (1980), no. 2, 221–236, DOI 10.2307/2273184. MR569394

[12] William Craig, *Logic in algebraic form: Three languages and theories*, Studies in Logic and the Foundations of Mathematics, Vol. 72, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1974. MR0411962

[13] Miklós Ferenczi, *The polyadic generalization of the Boolean axiomatization of fields of sets*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 867–886, DOI 10.1090/S0002-9947-2011-05332-8. MR2846356

[14] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev, *Many-dimensional modal logics: theory and applications*, Studies in Logic and the Foundations of Mathematics, vol. 148, North-Holland Publishing Co., Amsterdam, 2003. MR2011128

[15] S. R. Givant, *Relation algebras, vol. I: Arithmetic and algebra; vol. II: Complete extensions, representations, varieties, and atom structures*, Springer-Verlag, to appear.

[16] Robert Goldblatt, *Varieties of complex algebras*, Ann. Pure Appl. Logic **44** (1989), no. 3, 173–242, DOI 10.1016/0168-0072(89)90032-8. MR1020344

[17] Paul R. Halmos, *Algebraic logic*, Chelsea Publishing Co., New York, 1962. MR0131961

[18] Leon Henkin, J. Donald Monk, and Alfred Tarski, *Cylindric algebras. Part I*, Studies in Logic and the Foundations of Mathematics, vol. 64, North-Holland Publishing Co., Amsterdam, 1985. Reprint of the 1971 original. MR781929

[19] Leon Henkin, J. Donald Monk, and Alfred Tarski, *Cylindric algebras. Part II*, Studies in Logic and the Foundations of Mathematics, vol. 115, North-Holland Publishing Co., Amsterdam, 1985. MR781930

[20] Leon Henkin, J. Donald Monk, Alfred Tarski, Hajnal Andréka, and István Németi, *Cylindric set algebras*, Lecture Notes in Mathematics, vol. 883, Springer-Verlag, Berlin-New York, 1981. MR639151

[21] Robin Hirsch and Ian Hodkinson, *Step by step—building representations in algebraic logic*, J. Symbolic Logic **62** (1997), no. 1, 225–279, DOI 10.2307/2275740. MR1450522

[22] Robin Hirsch and Ian Hodkinson, *Axiomatising various classes of relation and cylindric algebras*, Log. J. IGPL **5** (1997), no. 2, 209–229, DOI 10.1093/jigpal/5.2.209. MR1433257

[23] Robin Hirsch and Ian Hodkinson, *Relation algebras by games*, Studies in Logic and the Foundations of Mathematics, vol. 147, North-Holland Publishing Co., Amsterdam, 2002. MR1935083

[24] Peter Jipsen and Henry Rose, *Varieties of lattices*, Lecture Notes in Mathematics, vol. 1533, Springer-Verlag, Berlin, 1992. MR1223545

[25] Bjarni Jónsson, *Varieties of relation algebras*, Algebra Universalis **15** (1982), no. 3, 273–298, DOI 10.1007/BF02483728. MR689767

[26] K. Lefever and G. Székely, *Interpretation of special relativity in the language of Newtonian kinematics*, Logic, Relativity and Beyond, 2nd International Conference, August 9-13, 2015, Budapest, talk in the Symposium of Equivalences of Theories part. http://www.renyi.hu/conferences/lrb15/slides/LRB15-Lefever–Szekely.pdf

[27] J. X. Madarász, *Logic and relativity (in the light of definability theory)*, PhD Dissertation, ELTE Budapest, 2002, xviii+367pp.

[28] J. X. Madarász and G. Székely, *Comparing relativistic and Newtonian dynamics in first-order logic*, in The Vienna Circle in Hungary, Veröffentlichungen des Instituts Wiener Kreis, Vol. 16, A. Máté, M. Rédei, and F. Stadler, eds., Springer, Vienna, 2011, pp. 155–179.

[29] Maarten Marx and Yde Venema, *Multi-dimensional modal logic*, Applied Logic Series, vol. 4, Kluwer Academic Publishers, Dordrecht, 1997. MR1427056

[30] J. D. Monk, *On the lattice of equational classes of one- and two-dimensional polyadic algebras*, Notices Amer. Math. Soc. **16** (1969), 183.

[31] Donald Monk, *On equational classes of algebraic versions of logic. I*, Math. Scand. **27** (1970), 53–71, DOI 10.7146/math.scand.a-10987. MR0280345

[32] Roger D. Maddux, *Relation algebras*, Studies in Logic and the Foundations of Mathematics, vol. 150, Elsevier B. V., Amsterdam, 2006. MR2269199

[33] I. Németi, *Varieties of cylindric algebras*, Preprint, Budapest, 1985.

[34] I. Németi, *On varieties of cylindric algebras with applications to logic*, Ann. Pure Appl. Logic **36** (1987), no. 3, 235–277, DOI 10.1016/0168-0072(87)90019-4. MR915900

[35] István Németi, *Algebraization of quantifier logics, an introductory overview*, Studia Logica **50** (1991), no. 3-4, 485–569, DOI 10.1007/BF00370684. MR1170186

[36] Vladimir V. Rybakov, *Admissibility of logical inference rules*, Studies in Logic and the Foundations of Mathematics, vol. 136, North-Holland Publishing Co., Amsterdam, 1997. MR1454360

[37] Ildikó Sain, *On the search for a finitizable algebraization of first order logic*, Log. J. IGPL **8** (2000), no. 4, 497–591, DOI 10.1093/jigpal/8.4.497. MR1776151

[38] Ildikó Sain and Viktor Gyuris, *Finite schematizable algebraic logic*, Log. J. IGPL **5** (1997), no. 5, 699–751, DOI 10.1093/jigpal/5.5.699. MR1465620

[39] Ildikó Sain and Richard J. Thompson, *Strictly finite schema axiomatization of quasipolyadic algebras*, Algebraic logic (Budapest, 1988), Colloq. Math. Soc. János Bolyai, vol. 54, North-Holland, Amsterdam, 1991, pp. 539–571. MR1153440

[40] Gábor Sági and Dorottya Sziráki, *Some variants of Vaught's conjecture from the perspective of algebraic logic*, Log. J. IGPL **20** (2012), no. 6, 1064–1082, DOI 10.1093/jigpal/jzr049. MR2999224

[41] T. Sayed-Ahmed, *Splitting methods in algebraic logic: proving results on non-atom-canonicity, non-finite axiomatizability and non-first-order definability for cylindric and relation algebras*, Preprint arXiv:1503.02189, 2015.

[42] György Serény, *Isomorphisms of finite cylindric set algebras of characteristic zero*, Notre Dame J. Formal Logic **34** (1993), no. 2, 284–294, DOI 10.1305/ndjfl/1093634658. MR1231290

[43] András Simon, *Finite schema completeness for typeless logic and representable cylindric algebras*, Algebraic logic (Budapest, 1988), Colloq. Math. Soc. János Bolyai, vol. 54, North-Holland, Amsterdam, 1991, pp. 665–670, DOI 10.1007/BF02631111. MR1153445

[44] Venema, Y., Many-dimensional modal logic. PhD Dissertation, Amsterdam, 1992. 177pp.

[45] Yde Venema, *Cylindric modal logic*, J. Symbolic Logic **60** (1995), no. 2, 591–623, DOI 10.2307/2275853. MR1335139

[46] James Owen Weatherall, *Are Newtonian gravitation and geometrized Newtonian gravitation theoretically equivalent?*, Erkenntnis **81** (2016), no. 5, 1073–1091, DOI 10.1007/s10670-015-9783-5. MR3547774

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