STRONG FAILURES OF HIGHER ANALOGS
OF HINDMAN’S THEOREM

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This paper is dedicated to the memory of András Hajnal (1931–2016)

Abstract. We show that various analogs of Hindman’s theorem fail in a
strong sense when one attempts to obtain uncountable monochromatic sets:

Theorem 1. There exists a colouring \( c : \mathbb{R} \to \mathbb{Q} \), such that for every \( X \subseteq \mathbb{R} \)
with \( |X| = |\mathbb{R}| \), and every colour \( \gamma \in \mathbb{Q} \), there are two distinct elements \( x_0, x_1 \)
of \( X \) for which \( c(x_0 + x_1) = \gamma \). This forms a simultaneous generalization
of a theorem of Hindman, Leader and Strauss and a theorem of Galvin and
Shelah.

Theorem 2. For every abelian group \( G \), there exists a colouring \( c : G \to \mathbb{Q} \)
such that for every uncountable \( X \subseteq G \) and every colour \( \gamma \), for some large
enough integer \( n \), there are pairwise distinct elements \( x_0, \ldots, x_n \) of \( X \) such
that \( c(x_0 + \cdots + x_n) = \gamma \). In addition, it is consistent that the preceding
statement remains valid even after enlarging the set of colours from \( \mathbb{Q} \) to \( \mathbb{R} \).

Theorem 3. Let \( \oplus_\kappa \) assert that for every abelian group \( G \) of cardinality \( \kappa \),
there exists a colouring \( c : G \to G \) such that for every positive integer \( n \), every
\( X_0, \ldots, X_n \in [G]^\kappa \), and every \( \gamma \in G \), there are \( x_0 \in X_0, \ldots, x_n \in X_n \) such
that \( c(x_0 + \cdots + x_n) = \gamma \). Then \( \oplus_\kappa \) holds for unboundedly many uncountable
cardinals \( \kappa \), and it is consistent that \( \oplus_\kappa \) holds for all regular uncountable
cardinals \( \kappa \).

1. Introduction

In one of its more general forms, Hindman’s theorem (see [14, Corollary 5.9] for
the general form; the particular case \( G = \mathbb{N} \) was originally proved in [12]) asserts
that whenever a commutative cancellative semigroup \( G \) is partitioned into two cells
(i.e., coloured with two colours), there exists an infinite \( X \subseteq G \) such that the set of its finite sums

\[
\text{FS}(X) := \left\{ \sum_{x \in a} x \mid a \in [X]^{<\omega} \right\}
\]

is completely contained in one of the cells of the partition (i.e., \( \text{FS}(X) \) is monochromatic). The infinite set \( X \subseteq G \) constructed in the proof of this theorem is countable, so it is natural to ask whether it is possible to find, given a colouring of an uncountable commutative cancellative semigroup \( G \), a subset \( X \subseteq G \) of a given

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uncountable cardinality such that \( \text{FS}(X) \) is monochromatic. This question was answered in the negative in [6], where, given a commutative cancellative semigroup \( G \), a colouring with two colours of \( G \) is exhibited such that no uncountable \( X \subseteq G \) can have \( \text{FS}(X) \) monochromatic. A related result for the particular case of the group \( \mathbb{R} \) can be found in [13], where a colouring with two colours is exhibited, satisfying that whenever \( X \subseteq \mathbb{R} \) has the same cardinality as \( \mathbb{R} \), then not only is \( \text{FS}(X) \) not monochromatic, but even \( \text{FS}_2(X) := \{ x + y \mid x, y \in X \text{ distinct} \} \) is not monochromatic. In particular, assuming the Continuum Hypothesis (\( \text{CH} \)), this result implies that for the aforementioned colouring of \( \mathbb{R} \), every uncountable subset \( X \) is such that \( \text{FS}_2(X) \) is not monochromatic.

In this paper we shall consider stronger versions of these results, where colourings of uncountable commutative cancellative semigroups \( G \) are obtained, with more than two colours, satisfying that for every uncountable \( X \) not only is \( \text{FS}(X) \) not monochromatic, but in fact \( \text{FS}(X) \) contains occurrences of every possible colour.

In order to properly state our results, we will introduce some new notation, which is inspired by the standard notation from Ramsey theory. Following [5, p. 56], for cardinals \( \kappa, \lambda, \theta, \mu \), write

\[
\kappa \rightarrow [\lambda]_\mu^{\theta}
\]

to assert that for every colouring of \( [\kappa]^\mu \) in \( \theta \) many colours, it is possible to find an \( X \subseteq \kappa \) with \( |X| = \lambda \) such that \( [X]^\mu \) omits at least one of the colours\(^2\). Note that the negation of the above, denoted

\[
\kappa \not\rightarrow [\lambda]_\mu^{\theta},
\]

asserts the existence of a colouring \( c : [\kappa]^\mu \rightarrow \theta \) such that for every \( \lambda \)-sized subset \( X \) of \( \kappa \), we have \( c^[X]^\mu = \theta \). So Ramsey’s theorem is just the assertion that \( \omega \rightarrow [\omega]^2_2 \) holds. However, when \( \omega \) is replaced by larger cardinals, typically one gets negative relations, sometimes quite strong (i.e., involving a large number of colours). Negative square bracket partition relations have been studied extensively. For instance, in the realm of \( \omega_1 \), we have a sequence of results starting with Sierpiński’s uncountable poset [30] that admits no uncountable chains nor uncountable antichains, thereby witnessing \( \omega_1 \not\rightarrow [\omega_1]^3_2 \). Later, Blass [11] improved this to \( \omega_1 \not\rightarrow [\omega_1]^3_3 \), and Galvin and Shelah [10] improved further to \( \omega_1 \not\rightarrow [\omega_1]^{2^\omega}_2 \). Then, Todorcević [33] managed to gain control on the maximal number of colours, proving that \( \omega_1 \not\rightarrow [\omega_1]^{2^\omega}_2 \) holds. Recently, even more complicated statements were proven by Moore [20] and Peng and Wu [21].

In analogy with the above, we now define a negative partition relation for commutative semigroups, involving finite sums (FS), bounded finite sums (FS\(_n\)), and sumsets (SuS).

**Definition.** For a commutative semigroup \( G \), cardinals \( \lambda, \theta, \) and an integer \( n \):

1. \( G \not\rightarrow [\lambda]_\mu^{\text{FS}} \) asserts the existence of a colouring \( c : G \rightarrow \theta \) such that for every \( \lambda \)-sized subset \( X \) of \( G \), we have \( c[\text{FS}(X)] = \theta \);
(2) $G \to [\lambda]^\text{FS}_n$ asserts the existence of a colouring $c : G \to \theta$ such that for every $\lambda$-sized subset $X$ of $G$, we have $c(\text{FS}_n(X)) = \theta$, where $\text{FS}_n(X) := \{x_1 + \cdots + x_n \mid x_1, \ldots, x_n \in X \text{ are all distinct}\};
\]

(3) $G \to [\lambda]^\text{SuS}_n$ asserts the existence of a colouring $c : G \to \theta$ such that for every integer $m \geq 2$ and all $\lambda$-sized subsets $X_1, \ldots, X_m$ of $G$, we have $c[X_1 + \cdots + X_m] = \theta$, where $X_1 + \cdots + X_m := \{x_1 + \cdots + x_m \mid \forall i (x_i \in X_i)\}.
\]

Note that for all $\lambda \leq \chi$, $G \subseteq G'$, $\theta \leq \theta'$, and $x \in \{\text{FS}_n, \text{FS}, \text{SuS} \mid n < \omega\}$, $G' \to [\lambda]_\theta^\chi$ implies $G \to [\lambda]_\theta^\chi$. Also note that for every $n$, $G \to [\lambda]_\theta^{\text{FS}_n}$ implies $G \to [\lambda]_\theta^{\text{FS}}$.

Finally, note that for every infinite cardinal $\lambda$ and every integer $n \geq 2$, $G \not\to [\lambda]_\theta^{\text{SuS}_n}$ implies $G \not\to [\lambda]_\theta^{\text{FS}_n}$ simply because any infinite set $X$ may be partitioned into $\bigcup_{i=1}^n X_i$ in such a way that $|X_i| = |X|$ for all $i$. And indeed, $G \not\to [\lambda]_\theta^{\text{SuS}_n}$ will be the strongest negative partition relation considered in this paper.

Using the above notation, we can now succinctly state the relevant web of results (in chronological order):

- The generalized Hindman theorem (see [14, Corollary 5.9]) asserts that $G \to [\omega]^2_2$ holds (that is, $G \to [\omega]^2_2$ fails) for every infinite commutative cancellative semigroup $G$.
- Milliken [19] proved that $G \to [\kappa^+]^{\text{FS}_2}_\kappa$ holds whenever $|G| = \kappa^+ = 2^\kappa$ for some infinite cardinal $\kappa$.
- Hindman, Leader and Strauss [13, Theorem 3.2] proved that $R \to [\omega]^2_\omega$ holds for every integer $n \geq 2$;
- The first author [6] proved that $G \not\to [\omega_1]^2_2$ holds for every uncountable commutative cancellative semigroup $G$;
- Komjáth [18], and independently Soukup and Weiss [31], proved that $R \not\to [\omega_1]^2_\omega$ holds for every integer $n \geq 2$. They also pointed out [31, Corollary 2.3] that by a theorem of Shelah [27, Theorem 2.1], it is consistent with $\text{ZFC}$ (modulo a large cardinal hypothesis) that $R \not\to [\omega_1]^2_\omega$ fails for every $n \geq 2$.

The main results of this paper read as follows:

**Theorem A.** $G \not\to [\omega_1]^2_\omega$ holds for every uncountable commutative cancellative semigroup $G$.

We shall show that the superscript $\text{FS}$ in the preceding is optimal by exhibiting an uncountable abelian group $G$ for which $G \not\to [\omega_1]^2_\omega$ fails for all $n$. We shall also address a stronger form of Theorem A, proving that things can go both ways:

**Theorem B1.** It is consistent with $\text{ZFC}$ that $G \not\to [\omega_1]^2_\omega$ holds for every uncountable commutative cancellative semigroup $G$.

**Theorem B2.** Modulo a large cardinal hypothesis, it is consistent with $\text{ZFC}$ that $R \not\to [\omega_1]^2_\omega$ fails.

It turns out that partition relations for $\text{FS}_n$ sets can also go both ways. To exemplify on the real line:

**Theorem C1.** If $\kappa$ is a successor cardinal (e.g., assuming $\text{CH}$), then $R \not\to [\omega]^2_\omega$ holds, and hence, so does $R \not\to [\omega]^n_\omega$ for every integer $n \geq 2$.

\footnote{Consequently, $G \not\to [\omega]^n_\omega$ holds for every $n < \omega$.}
\footnote{However, it is consistent with $\text{ZF}$ that $2^\kappa > \kappa^+$ for every infinite cardinal $\kappa$.}
Theorem C2. Modulo a large cardinal hypothesis, it is consistent with ZFC that $R \nrightarrow [\kappa]^{|\omega_n}|^{FS_n}$ fails for every integer $n \geq 2$.

The preceding raises the question of which negative partition relations for $R$ are consequences of ZFC. For this, we have the following simultaneous generalization of [10, Theorem 1] and [13, Theorem 3.2]:

Theorem C3. $R \nrightarrow [\kappa]^{|\omega_n}|^{FS_n}$ holds for every integer $n \geq 2$.

Finally, we establish that the negative partition relation of the strongest form is quite a prevalent phenomenon:

Theorem D. Denote by $\oplus_\kappa$ the assertion that $G \nrightarrow [\kappa]^{|\omega_\kappa}|^{SuS_\kappa}$ holds for every commutative cancellative semigroup $G$ of cardinality $\kappa$. Then:

- $\oplus_\kappa$ holds for $\kappa = \aleph_1, \aleph_2, \ldots, \aleph_n, \ldots$. In fact, $\oplus_\kappa$ holds for every $\kappa$ which is a successor of a regular cardinal.
- $\oplus_\kappa$ holds whenever $\kappa = \lambda^+ = 2^\lambda$ or whenever $\kappa$ is a regular uncountable cardinal admitting a nonreflecting stationary set. In particular:
  - It is consistent with ZFC that $\oplus_\kappa$ holds for every regular uncountable cardinal $\kappa$.

Organization of this paper. Theorems A, B1 and B2 are proved in Section 2. In Section 3 we establish some new results on the partition calculus of uncountable cardinals. In Section 4, the machinery of Section 3 is invoked in proving, among other things, Theorem D. Section 5 focuses on the real line, and Theorems C1, C2, C3 are derived there as corollaries.

2. Colourings for finite sums

We open this section by stating a structural result that will allow us to pass from elements of commutative cancellative semigroups of cardinality $\kappa$ to finite subsets of $\kappa$, so that we are able to apply some machinery from partition relations on cardinals to our semigroups.

For this, we will need to lay down some terminology.

Definition 2.1. Given a sequence of groups $\langle G_\alpha \mid \alpha < \kappa \rangle$, define its direct sum to be the group

$$\bigoplus_{\alpha < \kappa} G_\alpha := \left\{ x \in \prod_{\alpha < \kappa} G_\alpha \mid x(\alpha) \text{ equals the identity for all but finitely many } \alpha \right\}.$$

Recall that a divisible group is an abelian group $G$ such that for every $x \in G$ and every $n \in \mathbb{N}$, there exist some $z \in G$ such that $nz = x$.

Lemma 2.2. Suppose that $G$ is an infinite commutative cancellative semigroup. Denote $\kappa := |G|$. Then there exists a sequence of countable divisible groups, $\langle G_\alpha \mid \alpha < \kappa \rangle$, such that $G$ embeds in $\bigoplus_{\alpha < \kappa} G_\alpha$.

Proof. It is well-known that every commutative cancellative semigroup $G$ can be embedded in an abelian group $G'$, which can furthermore be assumed to have the same cardinality as $G$.

Next, since $G'$ is an abelian group, it can be embedded in a

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5The noncommutative case will be handled in a forthcoming paper.

6This is done by means of the same process which embeds the additive group $\mathbb{N}$ into $\mathbb{Z}$ or the multiplicative group $\mathbb{Z} \setminus \{0\}$ into $\mathbb{Q} \setminus \{0\}$. This process yields $G'$ as a quotient of the semigroup $G \times G$; therefore $|G'| = |G|$.
divisible group $G''$ by [8, Theorem 24.1]. Finally, by [8, Theorem 23.1], each divisible group is isomorphic to a direct sum of some copies of $Q$ with some quasicyclic groups $\mathbb{Z}(p^\infty)$. Thus, we may assume that $G'' = \bigoplus_{\alpha < \kappa} G_\alpha$, where each $G_\alpha$ is equal to either $Q$ or $\mathbb{Z}(p^\infty)$. By removing spurious summands (i.e., the $\alpha < \lambda$ such that $x(\alpha) = 0$ for every $x \in G$) we can assume that the number of summands is $|G|$, in other words, that $\lambda = \kappa$. Thus, $G$ embeds into $\bigoplus_{\alpha < \kappa} G_\alpha$, and each $G_\alpha$ (being either $Q$ or $\mathbb{Z}(p^\infty)$ for some $p$) is a countable divisible group.

In what follows, given an infinite commutative cancellative semigroup $G$ of infinite cardinality $\kappa$, we will always implicitly fix an embedding of $G$ into $\bigoplus_{\alpha < \kappa} G_\alpha$, where each $G_\alpha$ is countable, as per Lemma [2.4]. This allows us to define the support of an element $x \in G$ to be the finite set

$$\text{supp}(x) := \{\alpha < \kappa \mid x(\alpha) \neq 0\}.$$  

**Definition 2.3.** A family of sets $\mathcal{X}$ is said to be a $\Delta$-system with root $r$ if for every two distinct $x, x' \in \mathcal{X}$, we have $x \cap x' = r$.

A $\Delta$-system $\mathcal{X}$ is said to be of the head-tail-tail form if:

- $\text{supp}(r) < \min(x \setminus r)$ for all $x \in \mathcal{X}$;
- for any two distinct $x, x' \in \mathcal{X}$, either $\text{supp}(x) < \min(x' \setminus r)$ or $\text{supp}(x') < \min(x \setminus r)$.

A standard fact from set theory states that for every regular uncountable cardinal $\kappa$ and every family $\mathcal{X}$ consisting of $\kappa$ many finite sets, there exists $\mathcal{X}' \subseteq \mathcal{X}$ with $|\mathcal{X}'| = \kappa$ such that $\mathcal{X}'$ forms a $\Delta$-system. In the special case that $\mathcal{X} \subseteq [\kappa]^\omega$, a $\Delta$-subsystem $\mathcal{X}' \subseteq \mathcal{X}$ may be found which is moreover of the head-tail-tail form.

**Proposition 2.4.** Suppose that $x_1, \ldots, x_n$ are elements of a direct sum $\bigoplus_{\alpha < \kappa} G_\alpha$, and there exist a fixed $r \in [\kappa]^\omega$ and pairwise disjoint $s_1, \ldots, s_n \in [\kappa]^\omega$ satisfying $\text{supp}(x_i) = r \cup s_i$ for all $1 \leq i \leq n$ (that is, the set of corresponding supports forms a $\Delta$-system with root $r$). Then

$$s_1 \cup \cdots \cup s_n \subseteq \text{supp}(x_1 + \cdots + x_n) \subseteq r \cup s_1 \cup \cdots \cup s_n.$$  

**Proof.** Let $\alpha < \kappa$ be arbitrary. If $\alpha \notin r \cup s_1 \cup \cdots \cup s_n$, then $\alpha \notin \text{supp}(x_i)$ for any $i$. Thus

$$(x_1 + \cdots + x_n)(\alpha) = x_1(\alpha) + \cdots + x_n(\alpha) = 0 + \cdots + 0 = 0;$$

therefore $\text{supp}(x_1 + \cdots + x_n) \subseteq r \cup s_1 \cup \cdots \cup s_n$. Now, if $\alpha \in s_1 \cup \cdots \cup s_n$, then there exists a unique $1 \leq i \leq n$ with $\alpha \in s_i$. This means that $\alpha \in \text{supp}(x_i)$, but $\alpha \notin \text{supp}(x_j)$ for $j \neq i$. In other words, $x_i(\alpha) \neq 0$ but $x_j(\alpha) = 0$ for $j \neq i$. Therefore

$$(x_1 + \cdots + x_n)(\alpha) = x_1(\alpha) + \cdots + x_{i-1}(\alpha) + x_i(\alpha) + x_{i+1}(\alpha) + x_n(\alpha)$$

$$= 0 + \cdots + 0 + x_i(\alpha) + 0 + \cdots + 0 = x_i(\alpha) \neq 0;$$

thus $s_1 \cup \cdots \cup s_n \subseteq \text{supp}(x_1 + \cdots + x_n)$. \hfill $\square$

The main offshoot of Proposition [2.4] is that whenever we want to determine $\text{supp}(x_1 + \cdots + x_n)$ for $x_1, \ldots, x_n$ satisfying the corresponding hypothesis, the only

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7Recall that for a prime number $p$, the $p$-quasicyclic group (also known as the Prüfer $p$-group) is a countable divisible subgroup of $\mathbb{R}/\mathbb{Z}$, defined by $\mathbb{Z}(p^\infty) := \{\frac{a}{p^n} + \mathbb{Z} \mid a \in \mathbb{Z} \& n < \omega\}$. 

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coordinates \( \alpha \) that require careful inspection are the \( \alpha \in r \), where one must determine whether \( x_1(\alpha) + \cdots + x_n(\alpha) \) is equal to 0. In the particular case where \( n = 2 \), we obtain the fact, which will be useful later, that

\[
\text{supp}(x_1) \triangle \text{supp}(x_2) \subseteq \text{supp}(x_1 + x_2) \subseteq \text{supp}(x_1) \cup \text{supp}(x_2).
\]

In [6] Theorem 5], it was established that for every uncountable commutative cancellative semigroup \( G \), \( G \rightarrow [\omega_1]^{FS}_\omega \). This was done by colouring an element \( x \in G \) according to the parity of \( |\log_2 \text{supp}(x)|| \). Similarly, for every uncountable commutative cancellative semigroup \( G \) and every positive integer \( m \), it is possible to define a colouring \( c : G \rightarrow m \) by declaring \( c(x) \) to be the class of \( |\log_2 |\text{supp}(x)|| \) modulo \( m \). Arguing in the same way as in the proof of [6] Theorem 5], one can show that every uncountable \( X \subseteq G \) satisfies \( c[\text{FS}(X)] = m \), so that every uncountable commutative cancellative semigroup \( G \) satisfies \( G \rightarrow [\omega_1]^{FS}_\omega \) for every \( m < \omega \). In this section, among other things, we will show that every uncountable commutative cancellative semigroup \( G \) moreover satisfies \( G \rightarrow [\omega_1]^{FS}_\omega \).

In order to simplify certain steps of the proof of the main theorem, let us introduce the following notion.

**Definition 2.5.** Let \( G \) be a commutative semigroup and \( X \subseteq G \). We say that \( Y \) is a **condensation** of \( X \) if there exists a family \( A \subseteq [X]^{<\omega} \) consisting of pairwise disjoint sets, such that

\[
Y = \left\{ \sum_{a \in A} x \mid a \in A \right\}.
\]

The importance of this notion stems from the evident fact that if \( Y \) is a condensation of \( X \), then \( \text{FS}(Y) \subseteq \text{FS}(X) \). Thus the notion of “passing to a condensation” constitutes another, purely algebraic, form of thinning-out a family of elements of \( G \) which can be useful if we are dealing with finite sums without restrictions on the number of summands.

The following argument was first used in [6] (in the proof of the Claim within the proof of Theorem 5 therein) and is encapsulated here as a lemma both for the convenience of the reader and for future reference.

**Lemma 2.6.** Suppose that \( G \) is a commutative cancellative semigroup and \( X \subseteq G \) is a subset of regular uncountable cardinality. Fix an embedding of \( G \) into a direct sum \( \bigoplus_{\alpha < \kappa} G_\alpha \), as per Lemma 2.2. Then there exists a condensation \( Y \) of \( X \) such that:

- \( \text{supp}[Y] := \{ \text{supp}(y) \mid y \in Y \} \) forms a \( \Delta \)-system of cardinality \( |X| \);
- for all \( y \in Y \) and all \( \alpha \) in the root of \( \text{supp}[Y] \), \( y(\alpha) \) has an infinite order in \( G_\alpha \);
- for every positive integer \( n \) and every \( y_1, \ldots, y_n \in Y \),

\[
\text{supp}(y_1 + \cdots + y_n) = \text{supp}(y_1) \cup \cdots \cup \text{supp}(y_n).
\]

**Proof.** Note that as \( x \mapsto \{ (\alpha, x(\alpha)) \mid \alpha \in \text{supp}(x) \} \) is an injection and each \( G_\alpha \) is countable, we have \( |\text{supp}[Y]| = |Y| \) for every uncountable \( Y \subseteq X \). In particular, \( |\text{supp}[X]| \) is regular and uncountable, and by passing to an equipotent subset of \( X \), we may assume that \( \text{supp}[X] \) forms a \( \Delta \)-system.

Let \( r \) denote the root of this system. If \( r \) is empty, then we are done, as a consequence of Proposition 2.4. Thus, suppose that \( r \) is nonempty. Now, a finite
number of applications of the pigeonhole principle allows us to thin out $X$, without changing its cardinality, in such a way that for every $\alpha \in r$, there exists a fixed $g_\alpha \in G_\alpha$ such that $x(\alpha) = g_\alpha$ for all $x \in X$. Let $r_\infty := \{\alpha \in r \mid g_\alpha$ is of infinite order$\}$, and $M := \{\text{ord}(g_\alpha) \mid \alpha \in r \setminus r_\infty\}$. Since $r$ is finite, so is $M$. Thus we can take $m := \prod_{\alpha \in M} n$ (in the understanding that the empty product equals 1), thereby ensuring that $mg_\alpha = 0$ for all $\alpha \in r \setminus r_\infty$. Now we obtain a condensation $Y$ of $X$ by fixing some family $A \subseteq [X]^m$ consisting of exactly $|X|$ many pairwise disjoint $m$-sized sets and then letting

$$Y := \left\{ \sum_{x \in a} x \mid a \in A \right\}.$$ 

Claim 2.6.1. $\text{supp}[Y]$ forms a $\Delta$-system of cardinality $|X|$ with root $r_\infty$.

Proof. For each $x \in X$, denote $s_x := \text{supp}(x) \setminus r$, so that $\{s_x \mid x \in X\}$ is a family of pairwise disjoint sets. Now, for any $y \in Y$, there is an $a \in A$ (with $|a| = m$) such that $y = \sum_{x \in a} x$, and we claim that

$$\text{supp}(y) = r_\infty \uplus \left( \bigcup_{x \in a} s_x \right).$$

This will show that $\text{supp}[Y]$ forms a $\Delta$-system with root $r_\infty$. This will also show that the map $a \mapsto \text{supp}(\sum_{x \in a} x)$ is an injection from $A$ to $\text{supp}[Y]$, so that $|\text{supp}[Y]| = |A| = |X|$.

Here goes. By Proposition 2.4, we know that $\bigcup_{x \in a} s_x \subseteq \text{supp}(y) \subseteq r \cup \left( \bigcup_{x \in a} s_x \right)$. Thus it suffices to prove that for every $\alpha \in r$: $\alpha \in \text{supp}(y) \iff \alpha \in r_\infty$.

- For each $\alpha \in r_\infty$ and $x \in a$, we have $x(\alpha) = g_\alpha$, which is an element of infinite order. Hence

$$y(\alpha) = \sum_{x \in a} x(\alpha) = mg_\alpha \neq 0.$$ 

- For each $\alpha \in r \setminus r_\infty$ and $x \in a$, we have $x(\alpha) = g_\alpha$, which is an element whose order is a divisor of $m$. Hence

$$y(\alpha) = \sum_{x \in a} x(\alpha) = mg_\alpha = 0.$$ 

Altogether, we have $\text{supp}(y) = r_\infty \uplus \left( \bigcup_{x \in a} s_x \right)$.

Finally, given any two distinct $y, y' \in Y$, pick $a, a' \in A$ such that $y = \sum_{x \in a} x$ and $y' = \sum_{x \in a'} x$. Then by $a \cap a' = \emptyset$ and since the $s_x$ are pairwise disjoint, we have that

$$\text{supp}(y) \cap \text{supp}(z) = \left( r_\infty \cup \bigcup_{x \in a} s_x \right) \cap \left( r_\infty \cup \bigcup_{x \in b} s_x \right) = r_\infty,$$

which shows that $\text{supp}[Y]$ forms a $\Delta$-system, with root $r_\infty$. \hfill \square

Now let $n \in \mathbb{N}$, and let $y_1, \ldots, y_n \in Y$ be $n$ many distinct elements. It remains to prove that $\text{supp}(y_1 + \cdots + y_n) = \text{supp}(y_1) \cup \cdots \cup \text{supp}(y_n)$. Denote $s_i := \text{supp}(y_i) \setminus r_\infty$. By Proposition 2.4,

$$s_1 \cup \cdots \cup s_n \subseteq \text{supp}(y_1 + \cdots + y_n) \subseteq r_\infty \cup s_1 \cup \cdots \cup s_n = \text{supp}(y_1) \cup \cdots \cup \text{supp}(y_n),$$
and therefore it suffices to prove that \( r_\infty \subseteq \text{supp}(y_1 + \cdots + y_n) \). So let \( \alpha \in r_\infty \) be arbitrary. We have already noticed that \( y_i(\alpha) = mg_\alpha \) for each \( 1 \leq i \leq n \), where \( g_\alpha \) is an element of infinite order. Consequently

\[(y_1 + \cdots + y_n)(\alpha) = y_1(\alpha) + \cdots + y_n(\alpha) = mg_\alpha + \cdots + mg_\alpha = (nm)g_\alpha \neq 0,
\]

which shows that \( \alpha \in \text{supp}(y_1 + \cdots + y_n) \), and we are done. \( \square \)

We now arrive at the main technical result of this section.

**Theorem 2.7.** Suppose that \( G \) is a commutative cancellative semigroup of uncountable cardinality \( \kappa \). Then there exists a transformation \( d : G \to [\kappa]^{<\omega} \) with the property that for every uncountable \( X \subseteq G \), there exists \( A \subseteq \kappa \) such that \( |A| = |X| \) and \( d^{-}\text{FS}(X) \supseteq [A]^{<\omega} \).

**Proof.** We commence with an easy observation.

**Claim 2.7.1.** There exists a surjection \( f : \omega \to [\omega]^{<\omega} \) satisfying both of the following:

- for all \( k < \omega \), \( f(k) \subseteq k \);
- for all \( m, n < \omega \) and \( \Omega \in [\omega]^{<\omega} \), there are infinitely many \( k < \omega \) such that \( f(m + nk) = \Omega \).

**Proof.** For all \( m, n, a, b < \omega \), the set

\[ D(m, n, a, b) := \{ \varphi \in \omega^\omega \mid \exists k < \omega[k \geq a \& \varphi(m + nk) = b]\} \]

is dense open in the Baire space \( \omega^\omega \). So, by the Baire category theorem,

\[ \bigcap \{D(m, n, a, b) \mid m, n, a, b < \omega\} \neq \emptyset. \]

Pick \( \varphi \) from that intersection, along with an arbitrary surjection \( \psi : \omega \to [\omega]^{<\omega} \). Then, define \( f : \omega \to [\omega]^{<\omega} \) by stipulating

\[ f(k) := \begin{cases} (\psi \circ \varphi)(k), & \text{if } (\psi \circ \varphi)(k) \subseteq k, \\ 0, & \text{otherwise.} \end{cases} \]

Clearly, \( f \) is as sought. \( \square \)

Fix \( f \) as in the preceding. For every finite set of ordinals \( z \), let us denote by \( \sigma_z : |z| \leftrightarrow z \) the order-preserving bijection, so that \( \sigma_z(i) \) stands for the \( i \)th-element of \( z \).

Next, embed \( G \) into \( \bigoplus_{\alpha < \kappa} G_\alpha \), with each \( G_\alpha \) a countable abelian group, as per Lemma 2.2. Then, define a colouring \( d : G \to [\kappa]^{<\omega} \) by stipulating

\[ d(x) := \sigma_{\text{supp}(x)}(\alpha | \text{supp}(x)) \]

To see that \( d \) works, let \( X \) be some uncountable subset of \( G \). The proof now splits into two cases, depending on \( \lambda := |X| \).

**Case 1.** Suppose that \( \lambda \) is regular.

Let \( Y \) be given by Lemma 2.0 with respect to \( X \). In particular, \( \text{supp}(Y) \) forms a \( \Delta \)-system of cardinality \( \lambda \), with root, say, \( r \). Denote \( m := |r| \). Clearly, by passing to an equipotent subset of \( Y \), we may assume the existence of some positive integer \( n \) and a strictly increasing function \( h : m \to m + n \) such that for every \( y \in Y \):

- \( |\text{supp}(y)| = m + n \), and
- \( \sigma_r = \sigma_{\text{supp}(y)} \circ h. \)
There exist Claim 2.7.2.

Proof. By each of $z$ we have a sequence $\lambda$ such that (see Figure 1):

- $\{\sigma_{\text{supp}(y_i)}(a + b) \mid i < \lambda\}$ forms a head-tail-tail $\Delta$-system with root $\sigma_r[a]$;
- $i \mapsto \sigma_{\text{supp}(y_i)}(a)$ is strictly increasing over $\lambda$;
- $\sigma_{\text{supp}(y_i)}(a + b) = \text{supp}(y_i) \cap \delta$ for all $i < \lambda$.

Next, for all $j \leq m + n$, let $\Lambda_j := \{i < \lambda \mid \text{supp}(y_i) \cap \delta = \sigma_{\text{supp}(y_i)}(\delta)\}$. This defines a partition of $\lambda$ into finitely many sets, and we may pick some positive integer $b$ such that $|\Lambda_{a+b}| = \lambda$.

Finally, recursively construct a (strictly-increasing) function $g : \lambda \to \Lambda_{a+b}$ as follows:

- Let $g(0) := \min(\Lambda_{a+b})$.
- If $i < \lambda$ is nonzero and $g \upharpoonright i$ has already been defined, let

$$\beta := \sup\{\text{supp}(y_{g(j)}) \cap \delta \mid j < i\}.$$ 

By $i < \lambda = \text{cf}(\delta)$, we have $\beta < \delta$, so we may let $g(i) := \min\{j \in \Lambda_{a+b} \mid \sigma_{\text{supp}(y_j)}(a) > \beta\}$.

Clearly, $b, \delta$ and the sequence $\langle y_{g(i)} \mid i < \lambda \rangle$ are as sought. \hfill \Box

Let $b, \delta$ and $\langle y_i \mid i < \lambda \rangle$ be as in the statement of the preceding claim. Denote $z_i := \text{supp}(y_i) \setminus (r \cap \delta)$. Notice that $\min(z_i) = \min(\text{supp}(y_i) \setminus r) = \sigma_{\text{supp}(y_i)}(a)$.

We claim that $d^{\text{FS}(X)} \supseteq [A]^{<\omega}$ for the $\lambda$-sized set $A := \{\min(z_i) \mid i < \lambda\}$.

To see this, fix an arbitrary $p \in [A]^{<\omega}$. Let $\{\alpha_j \mid j < c\}$ be the increasing enumeration of $p$, so that $c = |p|$. Set $\Omega := \{a + bj \mid j < c\}$. By the choice of the function $f$, let us fix some integer $k > c$ such that $f(m + nk) = \Omega$.

Let $\langle i_j \mid j < k \rangle$ be a strictly increasing sequence of ordinals in $\lambda$ such that for all $j < c$, $i_j$ is the unique ordinal to satisfy $\alpha_j = \min(z_{i_j})$. Put $x := \sum_{j < k} x_j$, where $x_j := y_{i_j}$. By $\langle y_{i_j} \mid j < k \rangle \subseteq Y$, we have:

- $x \in \text{FS}(X)$;
- $\text{supp}(x) = \bigcup_{j < k} \text{supp}(x_j) = r \uplus (\text{supp}(x_0) \setminus r) \uplus \cdots \uplus (\text{supp}(x_{k-1}) \setminus r)$;
- $\text{supp}(x) \cap \delta = \sigma_r[a] \sqcup \sigma_{\text{supp}(x_0)}^\omega[a, a + b] \sqcup \cdots \sqcup \sigma_{\text{supp}(x_{k-1})}^\omega[a, a + b]$.

\footnote{Here, $w = z \sqcup z'$ asserts that $w = z \cup z'$ and $\sup(z) < \min(z')$.}
In particular, $|\text{supp}(x)| = m + nk > a + bc$, and
\[
d(x) = \sigma_{\text{supp}(x)} f(m + nk) = \sigma_{\text{supp}(x)} \Omega
\]
\[
= \{\sigma_{\text{supp}(x)}(a + bj) \mid j < c\}
\]
\[
= \{\sigma_{\text{supp}(x_j)}(a) \mid j < c\} = \{\sigma_{\text{supp}(y_j)}(a) \mid j < c\}
\]
\[
= \{\min(z_i j) \mid j < c\} = \{\alpha_j \mid j < c\} = p,
\]
as sought.

**Case 2.** Suppose that $\lambda$ is singular.

Let $\langle \lambda_\gamma : \gamma < \text{cf}(\lambda) \rangle$ be a strictly increasing sequence of regular cardinals converging to $\lambda$, with $\lambda_0 > \text{cf}(\lambda)$. Let $\langle X_\gamma : \gamma < \text{cf}(\lambda) \rangle$ be a partition of $X$ with $|X_\gamma| = \lambda_\gamma$ for all $\gamma < \text{cf}(\lambda)$.

For each $\gamma < \text{cf}(\lambda)$, appeal to Lemma 2.6 with $X_\gamma$, to obtain a set $Y_\gamma$. In particular, $\text{supp}(Y_\gamma)$ forms a $\Delta$-system of cardinality $\lambda_\gamma$, with root, say, $r_\gamma$. Let
\[
\delta_\gamma := \min\{\delta \leq \kappa \mid |\{\text{supp}(y) \cap (\delta \setminus r_\gamma) \mid y \in Y_\gamma\}| = \lambda_\gamma\}.
\]
Clearly, $\text{cf}(\delta_\gamma) = \lambda_\gamma$. In particular, $\gamma \mapsto \delta_\gamma$ is injective over $\text{cf}(\lambda)$, and we may find some cofinal subset $\Gamma \subseteq \text{cf}(\lambda)$ over which $\gamma \mapsto \delta_\gamma$ is strictly increasing. Put $\delta := \sup_{\gamma \in \Gamma} \delta_\gamma$ and $r := \bigcup_{\gamma \in \Gamma} r_\gamma$.

For all $\gamma \in \Gamma$, by $\lambda_\gamma > \text{cf}(\lambda) = \text{cf}(\delta)$, let us fix a large enough $\beta_\gamma < \delta$ for which
\[
Y_\gamma^0 := \{y \in Y_\gamma \mid \text{min}(\text{supp}(y) \setminus r_\gamma) < \delta_\gamma \land \text{supp}(y) \cap \delta \subseteq \beta_\gamma\}
\]
has cardinality $\lambda_\gamma$. Let $\bar{\Gamma}$ be some sparse enough cofinal subset of $\Gamma$ such that
\[
\text{supp}\{\beta_\gamma' \mid \gamma' \in \bar{\Gamma} \cap \gamma\} < \delta_\gamma
\]
for all $\gamma' < \gamma$ both from $\bar{\Gamma}$.

For all $\gamma \in \Gamma$, by minimality of $\delta_\gamma$ and by $\lambda_\gamma > |[r]^{<\omega}|$, we infer that the following set has size $\lambda_\gamma$:
\[
Y_\gamma^1 := \{y \in Y_\gamma^0 \mid (\text{sup} \{\beta_\gamma' \mid \gamma' \in \bar{\Gamma} \cap \gamma\} < \text{min}(\text{supp}(y) \setminus r_\gamma)) \land (y \cap r = r_\gamma)\}.
\]
Put $Y := \bigcup_{\gamma \in \Gamma} Y_\gamma^1$. For all $y \in Y$, let $\gamma(y)$ denote the unique ordinal $\gamma \in \bar{\Gamma}$ such that $y \in Y_\gamma^1$.

Consider the following subset of $\delta$:
\[
A := \{\text{min}(\text{supp}(y) \setminus r_{\gamma(y)}) \mid y \in Y\} \setminus r.
\]

**Claim 2.7.3.** For each $\alpha \in A$, there exists a unique $y \in Y$ such that $\alpha \in \text{supp}(y)$. In particular, $|A| = \lambda$.

**Proof.** Let $\alpha \in A$ be arbitrary. Fix some $y_\alpha \in Y$ such that $\alpha = \text{min}(\text{supp}(y_\alpha) \setminus r_{\gamma(y_\alpha)})$. Towards a contradiction, suppose that there exists $y \in Y \setminus \{y_\alpha\}$ with $\alpha \in \text{supp}(y)$. There are three cases to consider, each of which leads to a contradiction:

- **Suppose that $\gamma(y_\alpha) = \gamma(y)$.**

  Then $\alpha \in \text{supp}(y_\alpha) \cap \text{supp}(y) \subseteq r_{\gamma(y)} \subseteq r$, contradicting the fact that $A \cap r = \emptyset$.

- **Suppose that $\gamma(y_\alpha) \in \bar{\Gamma} \cap \gamma(y)$.**

  Then $\alpha \in \text{supp}(y_\alpha) \cap \delta \subseteq \beta_{\gamma(y_\alpha)} < \text{min}(\text{supp}(y) \setminus r_{\gamma(y)})$.

  So, by $\alpha \in \text{supp}(y)$, it must be the case that $\alpha \in r_{\gamma(y)} \subseteq r$, contradicting the fact that $A \cap r = \emptyset$. 
Suppose that \( \gamma(y) \in \hat{\Gamma} \cap \gamma(y_\alpha) \).

Then \( \alpha \in \text{supp}(y) \cap \delta \leq \beta_\gamma(y) < \min(\text{supp}(y_\alpha) \setminus r_\gamma(y_\alpha)) = \alpha \). This is a contradiction. \( \square \)

To see that \( d^+\text{FS}(X) \supseteq [A]^{<\omega} \), let \( p \) be an arbitrary element of \([A]^{<\omega}\). For each \( \alpha \in p \), pick \( y_\alpha \in Y \) such that \( \min(\text{supp}(y_\alpha) \setminus r_\gamma(y_\alpha)) = \alpha \). Write \( z := \sum_{\alpha \in p} y_\alpha \).

By Claim 2.7.3, we have \( p \subseteq \text{supp}(z) \).

Fix a large enough \( \gamma \in \hat{\Gamma} \) such that \( \gamma > \gamma(y_\alpha) \) for all \( \alpha \in p \). Put \( \epsilon := \min \left( \left\{ \text{supp}(y) \setminus r_\gamma \mid y \in Y_\gamma^1 \right\} \right) \). By the choice of \( \gamma \) and the definition of \( Y_\gamma^1 \), we have \( p \subseteq \epsilon \).

Next, by \( |\prod_{\beta \in r_\gamma} G_\beta| \leq \omega \) and the pigeonhole principle, let us fix an uncountable \( Y_\gamma^2 \subseteq Y_\gamma^1 \), a sequence \( \langle g_\beta \mid \beta \in r_\gamma \rangle \), and a positive integer \( n \), such that for all \( y \in Y_\gamma^2 \):

- \( y(\beta) = g_\beta \) for all \( \beta \in r_\gamma \);
- \( |\text{supp}(y) \setminus r_\gamma| = n \).

Let \( \beta \in r_\gamma \) be arbitrary. As \( Y_\gamma \) was provided by Lemma 2.6 and \( \beta \) belongs to the root of \( \text{supp}(Y_\gamma) \), we get that for all \( y \in Y_\gamma^2 \), \( g_\beta = y(\beta) \) has an infinite order. In particular, \( |\{ k < \omega \mid z(\beta) + kg_\beta = 0_{G_\beta} \}| \leq 1 \).

As \( r_\gamma \) is finite, let us pick a large enough \( K < \omega \) such that \( \{ k < \omega \mid \exists \beta \in r_\gamma [z(\beta) + kg_\beta = 0_{G_\beta}] \} \subseteq K \), so that \( z(\beta) + kg_\beta \neq 0_{G_\beta} \) for all \( \beta \in r_\gamma \) and \( k \geq K \).

Pick an injective sequence \( \langle y_i \mid i < K \rangle \) of elements of \( Y_\gamma^1 \). Write \( z' := z + \sum_{i < K} y_i \), and \( m := | \text{supp}(z') | \).

Put \( Y_\gamma^2 := Y_\gamma^1 \setminus \{ y_i \mid i < K \} \). Recalling Claim 2.7.3, we infer that for every finite \( \mathcal{Y} \subseteq Y_\gamma^2 \):

- \( p \cup r_\gamma \subseteq \text{supp}(z' + \sum_{y \in \mathcal{Y}} y) \);
- \( |\text{supp}(z' + \sum_{y \in \mathcal{Y}} y)| = m + n|\mathcal{Y}| \);
- \( \text{supp}(z') \cap \epsilon \) is an initial segment of \( \text{supp}(z' + \sum_{y \in \mathcal{Y}} y) \).

By \( p \subseteq \text{supp}(z') \cap \epsilon \), let us fix \( \Omega \in [\text{supp}(z') \cap \epsilon]^{<\omega} \) such that \( \sigma_{\text{supp}(z') \cap \epsilon}^{\text{supp}(z')} \Omega = p \).

By the choice of the function \( f \), let us fix some \( k < \omega \) such that \( f(m + nk) = \Omega \).

Pick \( \mathcal{Y} \in \{ Y_\gamma^2 \}^k \), and set \( z' := z' + \sum_{y \in \mathcal{Y}} y \).

Then \( x \in \text{FS}(X) \), \( |\text{supp}(x)| = m + nk \), and

\[
d(x) = \sigma_{\text{supp}(x)} f(m + nk) = \sigma_{\text{supp}(x)} \Omega
= (\sigma_{\text{supp}(x)} \upharpoonright |\text{supp}(z') \cap \epsilon|) \Omega
= \sigma_{\text{supp}(z') \cap \epsilon} \Omega = p,
\]

as sought. \( \square \)

**Corollary 2.8.** For every uncountable cardinal \( \lambda \leq \kappa \) and a cardinal \( \theta \), the following are equivalent:

1. \( \kappa \rightarrow [\lambda]^{<\omega}_\theta \) holds;
2. \( G \rightarrow [\lambda]^{\text{FS}}_\theta \) holds for every commutative cancellative semigroup \( G \) of cardinality \( \kappa \);
3. \( G \rightarrow [\lambda]^{\text{FS}}_\theta \) holds for some commutative cancellative semigroup \( G \) of cardinality \( \kappa \).

---

10If \( p = \emptyset \), then \( z \) stands for \( 0_G \) (the identity element of \( G \)).
Proof. (1) $\implies$ (2) Suppose that $\lambda, \kappa, \theta$ are as above and that $c : [\kappa]^{<\omega} \to \theta$ witnesses $\kappa \hookrightarrow [\lambda]^{<\omega}_\theta$. That is, for every $A \subseteq [\kappa]^\lambda$, we have $c^{-1}[A]^{<\omega} = \emptyset$. Now, given a commutative cancellative semigroup $G$ of cardinality $\kappa$, let $d : G \to [\kappa]^{<\omega}$ be given by Theorem 2.7. Clearly, $c \circ d$ witnesses $G \hookrightarrow [\lambda]^{\text{FS}}_\theta$.

(2) $\implies$ (3) This is trivial.

(3) $\implies$ (1) Suppose that $G$ is a commutative cancellative semigroup of cardinality $\kappa$ and that $d : G \to \theta$ is a colouring witnessing $G \hookrightarrow [\lambda]^{\text{FS}}_\theta$. Fix an injective enumeration $\{x_\alpha \mid \alpha < \kappa\}$ of the elements of $G$, and define $c : [\kappa]^{<\omega} \to \theta$ by stipulating $c(\{\alpha_0, \ldots, \alpha_n\}) := d(x_{\alpha_0} + \cdots + x_{\alpha_n})$. Clearly, $c$ witnesses $\kappa \hookrightarrow [\lambda]^{<\omega}_\theta$. \hfill \qed

Recall that $(\kappa, \mu) \hookrightarrow (\lambda, \theta)$ asserts that for every structure $(A, R, \ldots)$ for a countable first-order language with a distinguished unary predicate, if $(|A|, |R|) = (\kappa, \mu)$, then there exists an elementary substructure $(B, S, \ldots) \subseteq (A, R, \ldots)$ with $(|B|, |S|) = (\lambda, \theta)$.

To exemplify, let us point out that if $\theta$ is an infinite cardinal and there exists a $\theta^+$-Kurepa tree, then $(\theta^+, \theta^+) \hookrightarrow (\theta^+ + \theta^+) \hookrightarrow (\theta^+, \theta)$ fails. Also note that the instance $(\omega_2, \omega_1) \hookrightarrow (\omega_2, \omega)$ is known as Chang’s conjecture.

**Corollary 2.9.** For every infinite regular cardinal $\theta$ and every cardinal $\kappa > \theta$, the following are equivalent:

- $(\kappa, \theta^+) \hookrightarrow (\theta^+, \theta)$ fails;
- $G \hookrightarrow [\theta^+]^{\text{FS}}_{\theta^+}$ holds for every commutative cancellative semigroup $G$ of cardinality $\kappa$.

Proof. By a standard coding argument (using Skolem functions), the failure of $(\kappa, \theta^+) \hookrightarrow (\theta^+, \theta)$ is equivalent to $\kappa \hookrightarrow [\theta^+]^{<\omega}_{\theta^+ \theta}$. Thus, recalling Corollary 2.8, it suffices to prove that $\kappa \hookrightarrow [\theta^+]^{<\omega}_{\theta^+ \theta}$ is equivalent to $\kappa \hookrightarrow [\theta^+]^{<\omega}_{\theta^+}$. Of course, only the forward implication requires an argument. Now, as $\theta$ is regular, we get from [33] that $\theta^+ \hookrightarrow [\theta^+]^{2}_{\theta^+}$ holds. Then, as shown in [29], the conjunction of $\kappa \hookrightarrow [\theta^+]^{<\omega}_{\theta^+}$ with $\theta^+ \hookrightarrow [\theta^+]^{2}_{\theta^+}$ entails $\kappa \hookrightarrow [\theta^+]^{<\omega}_{\theta^+}$. \hfill \qed

**Corollary 2.10.** For every uncountable commutative cancellative semigroup $G$, $G \hookrightarrow [\omega_1]^{\text{FS}}_{\omega}$ holds.

Proof. The map $z \mapsto |z|$ witnesses that $\kappa \hookrightarrow [\omega]^{<\omega}_{\omega}$ holds for every infinite cardinal $\kappa$. In particular, $\kappa \hookrightarrow [\omega_1]^{<\omega}_{\omega}$ holds for every uncountable cardinal $\kappa$. Now, appeal to Corollary 2.8 with $\lambda = \omega_1$ and $\theta = \omega$. \hfill \qed

Modulo a large cardinal hypothesis, the preceding is optimal:

**Proposition 2.11.** If there exists an $\omega_1$-Erdős cardinal, then in some forcing extension, $\mathbb{R} \hookrightarrow [\omega_1]^{\text{FS}}_{\omega_1}$ fails. Furthermore, in this forcing extension, for every semigroup $(G, \ast)$ of size continuum and every colouring $c : G \to \omega_1$, there exists an uncountable subset $X \subseteq G$ for which $\{c(x_0 \ast \cdots \ast x_n) \mid n < \omega, x_0, \ldots, x_n \in X\}$ is countable.

Proof. Let $\kappa$ denote the $\omega_1$-Erdős cardinal. So $\kappa$ is strongly inaccessible and satisfies that for every $\theta < \kappa$ and every colouring $d : [\kappa]^{<\omega} \to \theta$, there exists some $H \in [\kappa]^{\omega_1}$ such that $d \restriction [H]^n$ is constant for all $n < \omega$.

Let $\mathbb{P}$ denote the notion of forcing for adding $\kappa$ many Cohen reals. We claim that the forcing extension $V^\mathbb{P}$ is as sought.

\[\textit{For a proof, see Theorem 8.1 of [17].}\]
Suppose that \( \hat{c} \) is a \( \mathbb{P} \)-name for a colouring \( c : G \to \omega_1 \) of a given semigroup \( (G, \ast) \) of size continuum. As \( V^P \models c = \kappa \), let us simplify the matter and just assume that the underlying set \( G \) is in fact \( \kappa \).

Working in \( V \), define a colouring \( d : [\kappa]^{<\omega} \to [\omega_1]^{<\omega} \) by letting for all \( n < \omega \) and all \( \alpha_0 < \cdots < \alpha_n < \kappa \):

\[
d(\alpha_0, \ldots, \alpha_n) := \{ \delta < \omega_1 \mid \exists \sigma : \{0, \ldots, n\} \to \{0, \ldots, n\} \exists p \in \mathbb{P} [p \Vdash \langle \alpha_0, \ldots, \alpha_n \rangle = \delta^n] \}.
\]

As \( \mathbb{P} \) is ccc, the range of \( d \) indeed consists of countable subsets of \( \omega_1 \). As \( \kappa \) is an \( \omega_1 \)-Erdős cardinal, \( |[\omega_1]^{<\omega}| < \kappa \) and we may pick some \( H \in [\kappa]^{\omega_1} \) such that \( d[H]^n \) is a singleton, say \( \{A_n\} \), for every positive integer \( n \). Then \( H \) is an uncountable subset of \( G \), \( A := \bigcup_{n=1}^{\infty} A_n \) is a countable subset of \( \omega_1 \), and

\[
\Vdash \langle \forall n \in \omega \forall x_0, \ldots, x_n \in H[\hat{c}(x_0 \ast \cdots \ast x_n) \in A] \rangle.
\]

It is also consistent that the number of colours in Corollary 2.10 may be increased to the maximal possible value. To see this, simply take \( \theta \) to be \( \omega_1 \) in the next statement:

**Corollary 2.12.** It is consistent with \( \text{ZFC + GCH} \) that for every commutative cancellative semigroup \( G \), \( G \twoheadrightarrow [\theta]^\text{FS} \) holds for every uncountable cardinal \( \theta \).

**Proof.** If there exists an inaccessible cardinal in Gödel’s constructible universe \( L \), then let \( \mu \) denote the least such one and work in \( L_\mu \). Otherwise, work in \( L \).

In both cases, we end up with a model of \( \text{ZFC + GCH} \), satisfying \( \kappa \twoheadrightarrow [\theta]^{\omega_1} \) for every uncountable cardinal \( \theta \leq \kappa \) \[29\]. Now, appeal to Corollary 2.10. \( \square \)

The preceding is quite surprising (think of the instance \( |G| = \beth_{\theta+\omega} \)), but of course we are standing on the shoulders of Rowbottom \[26\].

### 3. A set-theoretic interlude

This section is dedicated to the study of the following new set-theoretic principles.

**Definition 3.1.** For infinite cardinals \( \chi \leq \kappa \) and an arbitrary cardinal \( \theta \leq \kappa \):

- \( S(\kappa, \theta) \) asserts the existence of a colouring \( d : [\kappa]^{<\omega} \to \theta \) satisfying the following. For every \( \kappa \)-sized family \( \mathcal{X} \subseteq [\kappa]^{<\omega} \) and every \( \delta < \theta \), there exist two distinct \( x, y \in \mathcal{X} \) such that \( d(z) = \delta \) whenever \( (x \triangle y) \subseteq z \subseteq (x \cup y) \).
- \( S^*(\kappa, \theta, \chi) \) asserts the existence of a colouring \( d : [\kappa]^{<\chi} \to \theta \) satisfying the following. For every integer \( n \geq 2 \), every sequence \( \langle \mathcal{X}_i \mid i < n \rangle \in \prod_{i<n} [\kappa]^{<\chi} \), and every \( \delta < \theta \), there exists \( \langle x_i \mid i < n \rangle \in \prod_{i<n} \mathcal{X}_i \) such that \( \langle \sup(x_i) \mid i < n \rangle \) is strictly increasing, and such that \( d(z) = \delta \) whenever

\[
\bigcup_{i<n} \left( x_i \setminus \bigcup_{j \in n \setminus \{i\}} x_j \right) \subseteq z \subseteq \bigcup_{i<n} x_i.
\]

Hindman’s theorem is well-known to be equivalent to a Ramsey-theoretic statement concerning block sequences. Thus, the reader may want to observe that whenever \( d : [\kappa]^{<\omega} \to \theta \) witnesses \( S(\kappa, \theta) \), then for every block sequence \( \vec{x} = \langle x_\alpha \mid \alpha < \kappa \rangle \) of finite subsets of \( \kappa \) (i.e., satisfying \( \max(x_\alpha) < \min(x_\beta) \) for all \( \alpha < \beta < \kappa \)), we
have \( d^{\ast}FU(\bar{x}) = \emptyset \) (indeed, for every \( \delta < \theta \), there exist \( \alpha < \beta < \kappa \) such that \( d(x_\alpha \cup x_\beta) = \delta \)). Observe that if \( d \) furthermore witnesses \( S^*(\kappa, \theta, \omega) \), then for every positive \( n < \omega \) and every \( \delta < \theta \), there exist \( \alpha_0 < \cdots < \alpha_n < \kappa \) such that \( d(x_{\alpha_0} \cup \cdots \cup x_{\alpha_n}) = \delta \).

**Proposition 3.2.** For all infinite \( \chi \leq \chi' \leq \kappa \) and all \( \theta \leq \theta' \leq \kappa \):

1. \( S^*(\kappa, \theta', \chi') \) entails \( S^*(\kappa, \theta, \chi) \);
2. \( S^*(\kappa, \theta, \chi) \) entails \( \kappa \rightarrow [\kappa; \kappa]_\theta^2 \);
3. \( S^*(\kappa, \theta, \chi) \) entails \( S(\kappa, \theta) \);
4. \( S(\kappa, \theta) \) entails \( \kappa \rightarrow [\kappa]_\theta^2 \).

**Proof.** (1) This is obvious.

(2) Let \( d : [\kappa]^{<\chi} \rightarrow \theta \) be a witness to \( S^*(\kappa, \theta, \chi) \). Define \( c : [\kappa]^2 \rightarrow \theta \) by letting \( c(\alpha, \beta) := d(\{\alpha, \beta\}) \). Now, suppose that we are given \( X, Y \subseteq [\kappa]^\kappa \). We need to verify that for all \( \delta < \theta \), there exist \( \alpha \in X \) and \( \beta \in Y \) such that \( \alpha < \beta \) and \( c(\alpha, \beta) = \delta \). Fix \( x \in X \) and \( y \in Y \) such that \( \sup(x) < \sup(y) \) and \( d(\delta) = \delta \) whenever \( (x \triangle y) \subseteq z \subseteq (x \cup y) \). Let \( \alpha := \sup(x) \) and \( \beta := \sup(y) \). Then \( \alpha \in X, \beta \in Y, \alpha < \beta \) and \( c(\alpha, \beta) = \delta \), as sought.

(3) Let \( d : [\kappa]^{<\chi} \rightarrow \theta \) be a witness to \( S^*(\kappa, \theta, \chi) \). We claim that \( d \upharpoonright [\kappa]^{<\omega} \) witnesses \( S(\kappa, \theta) \). To avoid trivialities, suppose that \( \theta > 1 \). In particular, by clause (2) and Ramsey’s theorem, \( \kappa \) is uncountable.

Given a \( \kappa \)-sized family \( X \subseteq [\kappa]^{<\omega} \), pick a partition \( \chi = X_0 \cup X_1 \) with \( |X_0| = |X_1| = \kappa \). Then, by the choice of \( d \), for every \( \delta < \theta \), there exist \( x \in X_0 \) and \( y \in X_1 \) such that \( d(\delta) = \delta \) whenever \( (x \triangle y) \subseteq z \subseteq (x \cup y) \). Clearly, \( x, y \) are distinct elements of \( \chi \).

(4) Similar to the proof of clause (2). \( \square \)

**Proposition 3.3.** Suppose that \( \lambda > \text{cf}(\lambda) = \kappa \) are infinite cardinals. Then for all cardinals \( \theta, \chi \):

1. \( S(\kappa, \theta) \) entails \( S(\lambda, \theta) \);
2. \( S^*(\kappa, \theta, \chi) \) entails \( S^*(\lambda, \theta, \chi) \), provided that \( \mu^{<\chi} < \lambda \) for every cardinal \( \mu < \lambda \).

**Proof.** Pick a club \( \Lambda \) in \( \lambda \) with \( \text{otp}(\Lambda) = \kappa \), and derive a mapping \( h : \lambda \rightarrow \kappa \) by stipulating \( h(\alpha) := \text{otp}(\Lambda \cap \alpha) \).

(1) Let \( d : [\kappa]^{<\omega} \rightarrow \theta \) be a witness to \( S(\kappa, \theta) \). Define \( d_h : [\lambda]^{<\omega} \rightarrow \theta \) by stipulating \( d_h(z) := d(h[z]) \).

To see that \( d_h \) witnesses \( S(\lambda, \theta) \), suppose that we are given a \( \lambda \)-sized family \( \chi \subseteq [\lambda]^{<\omega} \) and a prescribed colour \( \delta < \theta \). Put \( X_h := \{h[x] \mid x \in X\} \). As \( |\mu|^{<\omega} < \lambda = |X| \) for all \( \mu \in \Lambda \), we infer that \( X_h \) is a \( \kappa \)-sized subfamily of \( [\kappa]^{<\omega} \).

Thus, by the choice of \( d \), we may pick two distinct \( x', y' \in X_h \) such that \( d(z') = \delta \) whenever \( (x' \triangle y') \subseteq z' \subseteq (x' \cup y') \). Now, find \( x, y \in X \) such that \( h[x] = x' \) and \( h[y] = y' \). Clearly, \( x \) and \( y \) are distinct. Finally, suppose that \( z \) is some set satisfying \( (x \triangle y) \subseteq z \subseteq (x \cup y) \). Then \( (x' \triangle y') \subseteq h[z] \subseteq (x' \cup y') \), and hence \( d_h(z) = d(h[z]) = \delta \), as sought.

(2) Let \( d : [\kappa]^{<\chi} \rightarrow \theta \) be a witness to \( S^*(\kappa, \theta, \chi) \). Define \( d_h : [\lambda]^{<\chi} \rightarrow \theta \) by stipulating \( d_h(z) := d(h[z]) \). Note that for every \( \lambda \)-sized family \( \chi \subseteq [\lambda]^{<\chi} \), \( X_h := \{h[x] \mid x \in X\} \) is a \( \kappa \)-sized subfamily of \( [\kappa]^{<\chi} \), because \( |\mu|^{<\chi} < \lambda = |\chi| \) for all \( \mu \in \Lambda \). The rest of the verification is similar to that of clause (1). \( \square \)
Recall that $\Pr_1(\kappa, \kappa, \theta, \chi)$ asserts the existence of a colouring $c : [\kappa]^2 \to \theta$ satisfying that for every $\gamma < \theta$ and every $A \subseteq [\kappa]^{<\chi}$ of size $\kappa$, consisting of pairwise disjoint sets, there exist $x, y \in A$ with $\sup(x) < \min(y)$ for which $c[x \times y] = \{\gamma\}$.

**Lemma 3.4.** Suppose that $\Pr_1(\kappa, \kappa, \theta, \chi)$ holds for given infinite cardinals $\chi \leq \theta \leq \kappa = \text{cf}(\kappa)$. If $\kappa$ is uncountable and $\mu^{<\chi} < \kappa$ for every cardinal $\mu < \kappa$, then $S^*(\kappa, \theta, \chi)$ holds.

**Proof.** Let $c : [\kappa]^2 \to \theta$ be a witness to $\Pr_1(\kappa, \kappa, \theta, \chi)$. Fix a bijection $\pi : \theta \leftrightarrow \theta \times \chi$. Define $c_0 : [\kappa]^2 \to \theta$ and $c_1 : [\kappa]^2 \to \chi$ in such a way that if $c(\alpha, \beta) = \gamma$ and $\pi(\gamma) = (\delta, \epsilon)$, then $c_0(\alpha, \beta) = \delta$ and $c_1(\alpha, \beta) = \epsilon$.

Now, define $d : [\kappa]^{<\chi} \to \theta$ as follows. Let $z \in [\kappa]^{<\chi}$ be arbitrary. If $M_z := \{\langle \alpha, \beta \rangle \in [z]^2 | c_1(\alpha, \beta) = \sup(c_i(\bar{z})^2)\}$ is nonempty, then let $d(z) := c_0(\alpha, \beta)$ for an arbitrary choice of $\langle \alpha, \beta \rangle$ from $M_z$. Otherwise, let $d(z) := 0$.

To see that $d$ works, suppose we are given a sequence $\langle X_i | i < n \rangle \in \prod_{i<n}[[\kappa]^{<\chi}]^\kappa$, for some integer $n \geq 2$, along with some prescribed colour $\delta \in \theta$.

By thinning out, we may assume that for all $i < n$, $X_i$ forms a $\Delta$-system with root, say, $r_i$. In particular, for all $i < n$, $\{x \mid r_i \cap X_i \neq \emptyset \}$ consists of $\kappa$ many pairwise disjoint bounded subsets of $\kappa$. Consequently, we can construct (e.g., by recursion on $\gamma < \kappa$) a matrix $\langle x_i^\gamma | \gamma < \kappa, i < n \rangle$ in such a way that for all $\gamma < \gamma' < \kappa$ and $i < j < n$:

- $r_i \uplus x_i^\gamma \in X_i$;
- $\sup(r_0 \cup \cdots \cup r_{n-1}) < \min(x_i^\gamma) \leq \sup(x_i^\gamma) < \min(x_j^\gamma) \leq \sup(x_j^\gamma) < \min(x_0^\gamma)$.

By the pigeonhole principle, let us fix $\Gamma \in [\kappa]^{<\kappa}$ and $\epsilon < \chi$ such that for all $\gamma \in \Gamma$:

- $\sup(c_i(\bar{z})^2) = \epsilon$.

Denote $a_\gamma := x_0^\gamma \uplus \cdots \uplus x_{n-1}^\gamma$. As $A := \{a_\gamma | \gamma \in \Gamma\}$ is a $\kappa$-sized subfamily of $[\kappa]^{<\chi}$ consisting of pairwise disjoint sets, we may now pick $\gamma < \gamma'$ both from $\Gamma$ for which $c[a_\gamma \times a_{\gamma'}] = \{\pi^{-1}(\delta, \epsilon + 1)\}$.

Write $\bar{x}_0 := r_0 \uplus x_0^\gamma$, and for all nonzero $i < n$, write $\bar{x}_i := r_i \uplus x_i^\gamma$. Clearly, $\langle \bar{x}_i | i < n \rangle \in \prod_{i<n}X_i$, and $\langle \sup(\bar{x}_i) | i < n \rangle$ is strictly increasing. Next, suppose that we are given $z$ satisfying

$$
\bigcup_{i<n} \left( \bar{x}_i \setminus \bigcup_{j \in \mathbb{N} \setminus \{i\}} \bar{x}_j \right) \subseteq z \subseteq \bigcup_{i<n} \bar{x}_i.
$$

**Claim 3.4.1.** $M_z$ is a nonempty subset of $a_\gamma \times a_{\gamma'}$.

**Proof.** Let $\langle \alpha, \beta \rangle \in [z]^2$ be arbitrary. As $\bigcup_{i<n} \bar{x}_i = (\bigcup_{i<n} r_i) \uplus x_0^\gamma \uplus \left( \bigcup_{0<i<n} x_i^\gamma \right)$, we consider the following cases:

1. Suppose that $\alpha \in (\bigcup_{i<n} r_i)$.
   - By $\beta \in (r_0 \cup \cdots \cup r_{n-1} \cup x_0^\gamma \uplus x_1^\gamma \uplus x_{n-1}^\gamma)$, we have $c_1(\alpha, \beta) \leq \epsilon$.
2. Suppose that $\alpha \in x_0^\gamma$.
   - (a) If $\beta \in (r_0 \cup \cdots \cup r_{n-1} \cup x_0^\gamma)$, then $c_1(\alpha, \beta) \leq \epsilon$.
   - (b) If $\beta \in \bigcup_{0<i<n} x_i^\gamma$, then $\langle \alpha, \beta \rangle \in a_\gamma \times a_{\gamma'}$ and hence $c(\alpha, \beta) = \pi^{-1}(\delta, \epsilon + 1)$, so that $c_1(\alpha, \beta) = \epsilon + 1$.

\[\text{This is where we use the hypothesis that } \mu^{<\chi} < \kappa \text{ for every cardinal } \mu < \kappa.\]
(3) Suppose that $\alpha \in \bigcup_{0<i<n} x_i^\gamma$.

(a) If $\beta \in x_0^\gamma$, then $\alpha > \beta$, which gives a contradiction to $\langle \alpha, \beta \rangle \in [z]^2$.

(b) If $\beta \in r_0 \cup \cdots \cup r_n \cup x_1^\gamma \cup \cdots \cup x_n^\gamma$, then $c_1(\alpha, \beta) \leq \epsilon$.

Finally, by

$$z \supseteq \bigcup_{i<n} \left( \bar{x}_i \setminus \bigcup_{j \in n \setminus \{i\}} \bar{x}_j \right) \supseteq \left( \bigcup_{i<n} \bar{x}_i \setminus \bigcup_{i<n} r_i \right),$$

we have $x_0^\gamma \times x_1^\gamma \subseteq [z]^2$, so that case (2)(b) is indeed feasible. Consequently, $M_z$ is a nonempty subset of $a_i \times a_j$.

Let $\langle \alpha, \beta \rangle \in M_z$ be such that $d(z) = c_0(\alpha, \beta)$. By the preceding claim, $c(\alpha, \beta) = \pi^{-1}(\delta, \epsilon + 1)$, so that $d(z) = c_0(\alpha, \beta) = \delta$, as sought.

The colouring principle $Pr_1(\cdots)$ was studied extensively by many authors, including Eisworth, Galvin, Rinot, Shelah, and Todorcevic. To mention a few results:

**Fact 3.5.** Suppose that $\kappa$ is a regular uncountable cardinal. Then $Pr_1(\kappa, \kappa, \theta, \chi)$ holds in all of the following cases:

1. $\kappa = \theta = \beta = \omega_1$ and $\chi = \omega$;
2. $\kappa = \theta > \chi^+$, and $\square(\kappa)$ holds
3. $\kappa = \theta > \chi^+$, and $E_{\geq \chi}^\kappa$ admits a nonreflecting stationary set;
4. $\kappa = \theta = \lambda^+ > \chi^+$, and $\lambda$ is regular;
5. $\kappa = \theta = \lambda^+$, $\lambda$ is singular, $\chi = cf(\lambda)$, and $pp(\lambda) = \lambda^+$ (e.g., $\lambda^{cf(\lambda)} = \lambda^+$);
6. $\kappa = \theta = \lambda^+$, $\lambda$ is singular, $\chi = cf(\lambda)$, and there exists a collection of $< cf(\lambda)$ many stationary subsets of $\kappa$ that do not reflect simultaneously;
7. $\kappa = \lambda^+$, $\lambda$ is singular, and $\theta = \chi = cf(\lambda)$.

**Proof.**

(1) By Lemma 1.0 of [34, Section 1].

(2) By Theorem B of [24].

(3) By Corollary 3.2 of [25].

(4) By clause (3) above.

(5) By Corollary 6.2 of [4].

(6) By Corollary 3.3 of [23].

(7) By Conclusion 4.1 of [28].

It follows that $S^*(\lambda^+, \lambda^+, \omega)$ holds for every regular cardinal $\lambda \geq \omega_1$. Now, what about $S^*(\omega_1, \omega_1, \omega)$?

- Galvin proved [9] that the failure of $Pr_1(\omega_1, \omega_1, \omega, \omega)$ is consistent with ZFC, and hence Lemma 3.3 is inapplicable here.

- Getting just $S(\omega_1, \omega_1)$ turns out to be ready-made. It follows from Theorem 2.6 of [25] that $S(\mu^+, \mu^+)$ holds for every infinite cardinal $\mu$ satisfying $\mu^{< \mu} = \mu$.

Altogether, there is a need for a dedicated proof of $S^*(\omega_1, \omega_1, \omega)$. This is our next task.

**Theorem 3.6.** $S^*(\omega_1, \omega_1, \omega)$ holds.

**Proof.** As the product of $c$ many separable topological spaces is again separable, let us pick a countable dense subset $\{f_\iota \mid \iota < \omega\}$ of the product space $\omega^{\omega_1}$. Notice
that this means that for every finite subset \( a \subseteq \omega_1 \) and every function \( f : a \rightarrow \omega \), there exists some \( \iota < \omega \) such that \( f_\iota \upharpoonright a = f \).

Next, by Theorem 1.5 of [20], let us pick a function \( \operatorname{osc} : [\omega_1]^2 \rightarrow \omega \) satisfying that for all positive integers \( k, l \), all uncountable families \( A \subseteq [\omega_1]^k \) and \( B \subseteq [\omega_1]^l \), each consisting of pairwise disjoint sets, and every \( s < \omega \), there exist \( a \in A \) and a sequence \( (b_m \mid m < s) \) of elements in \( B \) such that for all \( m < s \):

\[
\begin{align*}
&\bullet \max(a) < \min(b_m), \\
&\bullet \operatorname{osc}(a(i), b_m(j)) = \operatorname{osc}(a(i), b_0(j)) + m \text{ for all } i < k \text{ and } j < l.
\end{align*}
\]

For every nonzero \( \alpha < \omega_1 \), fix a surjection \( \psi_\alpha : \omega \rightarrow \alpha \). Finally, define the function \( d : [\omega_1]^{<\omega} \rightarrow \omega_1 \) as follows. Let \( z \in [\omega_1]^{<\omega} \) be arbitrary. If \( M_z := \{ (\alpha, \beta) \in [z]^2 \mid \operatorname{osc}(\alpha, \beta) = \sup(\operatorname{osc} "[z]^2") \} \) is empty, then let \( d(z) := \emptyset \). Otherwise, pick an arbitrary \( (\alpha, \beta) \) from \( M_z \), let \( \iota \) be the maximal natural number to satisfy that \( 2^\iota \) divides \( \operatorname{osc}(\alpha, \beta) \), and then put \( d(z) := \psi_\alpha(f_\iota(\alpha)) \).

To see that \( d \) works, suppose that we are given a sequence \( \langle X_i \mid i < n \rangle \) for all \( \gamma < \omega \) and \( i < j < n \):

\[
\begin{align*}
&\bullet r_i \uplus x_i^\gamma \in X_i; \\
&\bullet \sup(r_0 \cup \cdots \cup r_{n-1}) < \min(x_i^\gamma) \leq \max(x_i^\gamma) < \min(x_j^\gamma) < \min(x_0^\gamma).
\end{align*}
\]

For all \( \gamma < \omega_1 \), denote \( a^\gamma := x_0^\gamma \) and \( b^\gamma := x_1^\gamma \uplus \cdots \uplus x_n^\gamma \). By the pigeonhole principle, let us fix an uncountable \( \Gamma \subseteq \omega_1 \) along with \( k, l, \iota, \epsilon < \omega \) such that for all \( \gamma \in \Gamma \):

\[
\begin{align*}
&\bullet |a^\gamma| = k \text{ and } |b^\gamma| = l; \\
&\bullet \psi_\alpha(f_\iota(\alpha)) = \delta \text{ for all } \alpha < a^\gamma; \\
&\bullet \max([\epsilon, \uplus (r_0 \cup \cdots \cup r_{n-1} \cup a^\gamma \uplus b^\gamma)]^2) = \epsilon.
\end{align*}
\]

Put \( s := \epsilon + 1 + 2^\iota + 1 \). Consider \( A := \{ a^\gamma \mid \gamma \in \Gamma \} \) and \( B := \{ b^\gamma \mid \gamma \in \Gamma \} \). As \( A \subseteq [\omega_1]^k \) and \( B \subseteq [\omega_1]^l \) are uncountable families, each consisting of pairwise disjoint sets, we may pick \( a \in A \) and a sequence \( \langle b_m \mid m < s \rangle \) of elements in \( B \) such that for all \( m < s \):

\[
\begin{align*}
&\bullet \max(a) < \min(b_m), \\
&\bullet \operatorname{osc}(a(i), b_m(j)) = \operatorname{osc}(a(i), b_0(j)) + m \text{ for all } i < k \text{ and } j < l.
\end{align*}
\]

Write \( m := \max(\operatorname{osc}(a \times b_0)) \). Let \( t \) be the unique natural number to satisfy \( 0 \leq t < 2^\iota \) and \( m + \epsilon + 1 \equiv t \pmod{2^\iota} \). Put \( m := \epsilon + 1 + 2^\iota - t \). Then \( m < s \) and \( t \) is the maximal natural number to satisfy that \( 2^\iota \) divides \( m + m \).

Fix \( \gamma < \gamma' < \omega_1 \) such that \( a = a^\gamma \) and \( b_m = b_m^{\gamma'} \). Write \( x_0 := r_0 \uplus x_0^\gamma \), and for all nonzero \( i < n \), write \( x_i := r_i \uplus x_i^{\gamma'} \). Clearly, \( \langle x_i \mid i < n \rangle \in \prod_{i<n} X_i \), and \( \langle \sup(x_i) \mid i < n \rangle \) is strictly increasing. Next, suppose that we are given \( z \) satisfying

\[
\bigcup_{i<n} \left( \bar{x}_i \setminus \bigcup_{j \in n \setminus \{i\}} \bar{x}_j \right) \subseteq z \subseteq \bigcup_{i<n} \bar{x}_i.
\]

\[15\text{Here, } a(i) \text{ stands for the unique } \alpha \in a \text{ to satisfy } |a \cap \alpha| = i. \text{ The interpretation of } b_m(j) \text{ is similar.} \]
Claim 3.6.1. \( M_z \) is a nonempty subset of \( a \times b_m \), and \( \text{osc}(\alpha, \beta) = m + m \) for all \( \langle \alpha, \beta \rangle \in M_z \).

Proof. Let \( \langle \alpha, \beta \rangle \in [z]^2 \) be arbitrary. As \( \bigcup_{i<n} \bar{x}_i = (\bigcup_{i<n} r_i) \uplus a^\gamma \uplus b^{\gamma'} \), we consider the following cases:

1. Suppose that \( \alpha \in (\bigcup_{i<n} r_i) \).
   By \( \beta \in (r_0 \cup \cdots \cup r_{n-1} \cup a^\gamma \cup b^{\gamma'}) \), we have \( \text{osc}(\alpha, \beta) \leq \epsilon \).

2. Suppose that \( \alpha \in a^\gamma \).
   a. If \( \beta \in (r_0 \cup \cdots \cup r_{n-1} \cup a^\gamma) \), then \( \text{osc}(\alpha, \beta) \leq \epsilon \).
   b. If \( \beta \in b^{\gamma'} \), then \( \langle \alpha, \beta \rangle \in a \times b_m \), so that writing \( i := a \cap \alpha \) and \( j := b_m \cap \beta \), we have \( \text{osc}(\alpha, \beta) = \text{osc}(a(i), b_m(j)) = \text{osc}(a(i), b_0(j)) + m \geq m > \epsilon \).

3. Suppose that \( \alpha \in b^{\gamma'} \).
   a. If \( \beta \in a^\gamma \), then \( \alpha > \beta \), which gives a contradiction to \( \langle \alpha, \beta \rangle \in [z]^2 \).
   b. If \( \beta \in r_0 \cup \cdots \cup r_{n-1} \cup b^{\gamma'} \), then \( \text{osc}(\alpha, \beta) \leq \epsilon \).

Finally, by

\[
\begin{align*}
 z &\supseteq \bigcup_{i<n} \left( \bar{x}_i \setminus \bigcup_{j \in \{i\} \setminus \{i\}} \bar{x}_j \right) \\
 &\supseteq \left( \bigcup_{i<n} \bar{x}_i \setminus \bigcup_{i<n} r_i \right),
\end{align*}
\]

we have \( a^\gamma \times b^{\gamma'} \subseteq [z]^2 \), so that case (2)(b) is indeed feasible, and \( M_z \) is a nonempty subset of \( a^\gamma \times b^{\gamma'} = a \times b_m \). It follows that \( M_z = \{ \langle \alpha, \beta \rangle \in a \times b_m \mid \text{osc}(\alpha, \beta) = m + m \} \). \( \square \)

Let \( \langle \alpha, \beta \rangle \in M_z \) be arbitrary. By the choice of \( m \), we know that \( \iota \) is the maximal natural number to satisfy that \( 2^\iota \) divides \( m + m \), and hence \( d(z) = \psi_\alpha(f_\iota) = \delta \), as sought. \( \square \)

Corollary 3.7. Suppose that \( \kappa \) is a regular uncountable cardinal that admits a nonreflecting stationary set (e.g., \( \kappa \) is the successor of an infinite regular cardinal). Then:

- \( \text{S}'(\kappa, \kappa, \omega) \) holds. In particular:
- There exists a colouring \( d : [\kappa]^<\omega \rightarrow \kappa \) such that for every \( X \subseteq [\kappa]^<\omega \) of size \( \kappa \) and every colour \( \delta < \kappa \), there exist two distinct \( x, y \in X \) satisfying \( d(x \cup y) = \delta \).

Proof. For \( \kappa = \aleph_1 \), use Theorem 3.6. For \( \kappa > \aleph_1 \), use Lemma 3.4 together with Fact 3.5(3). \( \square \)

Note that the second bullet of the preceding generalizes the celebrated result from [33, p. 285] asserting that \( \kappa \rightarrow [\kappa]^2 \) holds for every regular uncountable cardinal \( \kappa \) that admits a nonreflecting stationary set.

As colouring of the real line is of special interest, and as the results so far only shed a limited amount of light on cardinals of the form \( 2^\lambda \), our next task is proving the following.

Theorem 3.8. Suppose that \( \lambda \) is an infinite cardinal satisfying \( 2^{<\lambda} = \lambda \). Then \( S(\text{cf}(2^\lambda), \omega) \) holds.
Proof. By $2^{<\lambda} = \lambda$ and a classic theorem of Sierpiński, there exists a linear ordering of size $2^\lambda$ with a dense subset of size $\lambda$. Then, by Theorem 3 of [32] (independently, also by the main result of [3]), we may fix a linear order $(L, <)$ of size $\kappa := cf(2^\lambda)$ which is $\kappa$-entangled. The latter means that for every $\mu < \omega$, every $\Omega \subseteq \mu$, and every injective sequence $\langle f_\alpha : \mu \to L \mid \alpha < \kappa \rangle$ of order-preserving maps from $(\mu, \in)$ to $(L, <)$, with pairwise disjoint images, there exist $\alpha < \beta < \kappa$ such that for all $\iota < \mu$: $f_\alpha(\iota) < f_\beta(\iota)$ iff $\iota \in \Omega$.\footnote{Note that the definition of a $(\kappa, \mu)$-entangled ordering in [32] only guarantees “$\alpha \neq \beta$”. However, “$\alpha < \beta$” can be ensured by appealing to the $(\kappa, 2\mu)$-entangledness of the ordering.}

Fix a sequence of injections $\langle l_\alpha : \omega \to L \mid \alpha < \kappa \rangle$ such that $\text{Im}(l_\alpha) \cap \text{Im}(l_\beta) = \emptyset$ for all $\alpha < \beta < \kappa$. Define a colouring $c : [\kappa]^2 \to \omega$ as follows. For all $\alpha < \beta < \kappa$, if there exists some $\tau < \omega$ such that $l_\alpha(\tau) < l_\beta(\tau)$, let $c(\alpha, \beta)$ be the least such $\tau$. Otherwise, let $c(\alpha, \beta) := 0$.

By $\kappa \leq 2^\lambda$, let $\langle g_\alpha : \lambda \to 2 \mid \alpha < \kappa \rangle$ be a sequence of pairwise distinct functions. For all $\alpha < \beta < \kappa$, let $\Delta(\alpha, \beta) := \min\{\varepsilon < \lambda \mid g_\alpha(\varepsilon) \neq g_\beta(\varepsilon)\}$ . Now, to define $d : [\kappa]^2 \to \omega$, let $z \in [\kappa]^2$ be arbitrary. If $M_z := \{\langle \alpha, \beta \rangle \in [z]^2 \mid \Delta(\alpha, \beta) = \sup(\Delta^\varepsilon[\alpha, \beta])\}$ is nonempty, then let $d(z) := c(\alpha, \beta)$ for an arbitrary choice of $\langle \alpha, \beta \rangle$ from $M_z$. Otherwise, let $d(z) := 0$.

To see that $d$ witnesses $S(\kappa, \omega)$, suppose that we are given a $\kappa$-sized family $\mathcal{X} \subseteq [\kappa]^{<\omega}$ and a prescribed colour $\delta < \omega$. By the $\Delta$-system lemma, we may find a sequence $\langle a_\gamma \mid \gamma < \kappa \rangle$ along with $r \in [\kappa]^{<\omega}$ and $m < \omega$ such that for all $\gamma < \gamma' < \kappa$:

- $|a_\gamma| = m$;
- $r \not\in a_\gamma \in \mathcal{X}$;
- $sup(r) < min(a_\gamma) \leq max(a_\gamma) < min(a_{\gamma'})$.

For each $\gamma < \kappa$, let $f_\gamma : m(\delta + 1) \to L$ denote the unique order-preserving map from $(m(\delta + 1), \in)$ to $(L, <)$ such that $\text{Im}(f_\gamma) = \{l_\alpha(\tau) \mid \alpha \in a_\gamma, \tau \leq \delta\}$. Also, fix some enumeration $\{a_{\gamma}(j) \mid j < m\}$ of $a_\gamma$.

Next, by an iterated application of the pigeonhole principle, let us fix $\Gamma \in [\kappa]^{\kappa}$ together with $\epsilon < \lambda$, $t : m \to (\epsilon + 1)2$ and $h : m \times (\delta + 1) \leftrightarrow m(\delta + 1)$ such that for all $\gamma \in \Gamma$:

- $max(\Delta^\varepsilon[r \not\in a_\gamma]) = \epsilon$;
- $\langle a_\gamma(j) \mid (\epsilon + 1) \mid j < m\rangle = t$;\footnote{Note that $\left| m(\epsilon + 1)2 \right| \leq 2^{<\lambda} = \lambda < \kappa$}
- $f_\gamma(h(j, \tau)) = a_{\gamma}(j)(\tau)$ for all $j < m$ and $\tau \leq \delta$.

Put $\Omega := h[m \times \{\delta\}]$. As $(L, <)$ is $\kappa$-entangled, let us pick $\gamma < \gamma'$ both from $\Gamma$ such that for all $\iota < m(\delta + 1)$: $f_\gamma(\iota) < f_{\gamma'}(\iota)$ iff $\iota \in \Omega$. Write $x := r \not\in a_\gamma$ and $y := r \not\in a_{\gamma'}$. Clearly, $x, y$ are two distinct elements of $\mathcal{X}$. Next, suppose that we are given $z$ satisfying $x \Delta y \subseteq z \subseteq x \cup y$.

Claim 3.8.1. $M_z$ is a nonempty subset of $\{\langle a_{\gamma}(j), a_{\gamma'}(j) \rangle \mid j < m\}$.

Proof. Let $\langle \alpha, \beta \rangle \in [z]^2$ be arbitrary. As $x \cup y = r \not\in a_\gamma \not\in a_{\gamma'}$, we consider the following cases:

1. Suppose that $\alpha \in r$.

2. By $\beta \in r \not\in a_\gamma \not\in a_{\gamma'}$, we have $\Delta(\alpha, \beta) \leq \epsilon$. 

(2) Suppose that $\alpha \in a_\gamma$.
   (a) If $\beta \in (r \cup a_\gamma)$, then $\Delta(\alpha, \beta) \leq \epsilon$.
   (b) If $\beta \in a_\gamma$, then let $j_\alpha, j_\beta < m$ be such that $\alpha = a_\gamma(j_\alpha)$ and $\beta = a_\gamma(j_\beta)$. There are two cases to consider:
      (i) If $j_\alpha = j_\beta$, then $g_\alpha \upharpoonright (\epsilon + 1) = g_{a_\gamma(j_\alpha)} \upharpoonright (\epsilon + 1) = t(j_\alpha) = t(j_\beta) = g_{a_\gamma(j_\beta)} \upharpoonright (\epsilon + 1) = g_\beta \upharpoonright (\epsilon + 1)$, so that $\Delta(\alpha, \beta) > \epsilon$.
      (ii) If $j_\alpha \neq j_\beta$, then $g_\alpha \upharpoonright (\epsilon + 1) = t(j_\alpha) = g_{a_\gamma(j_\alpha)} \upharpoonright (\epsilon + 1)$, and hence
          \[
          \Delta(\alpha, a_\gamma(j_\alpha)) > \epsilon = \max(\Delta^\omega[a_\gamma]^2) > \Delta(a_\gamma(j_\alpha), \beta),
          \]
          so that $\Delta(\alpha, \beta) \leq \epsilon$.

(3) Suppose that $\alpha \in a_\gamma$.
   (a) If $\beta \in a_\gamma$, then $\alpha > \beta$, which gives a contradiction to $\langle \alpha, \beta \rangle \in [z]^2$.
   (b) If $\beta \in r \cup a_\gamma$, then $\Delta(\alpha, \beta) \leq \epsilon$.

   Finally, by $z \supseteq (x \Delta y) = a_\gamma \cup a_\gamma$, we have $a_\gamma \times a_\gamma \subseteq [z]^2$, so that case (2)(b)(i) is indeed feasible. \(\square\)

As $\Omega = h[m \times \{\delta\}]$, we have $f_\gamma(\iota) \triangleleft f_\gamma(\iota)$ if $\iota \in h[m \times \{\delta\}]$. Let $j < m$ be such that $d(z) = c(a_\gamma(j), a_\gamma(j))$. Then, for all $\tau \leq \delta$: $f_\gamma(h(j, \tau)) \triangleleft f_\gamma(h(j, \tau))$ if $h(j, \tau) \in h[m \times \{\delta\}]$ if $\tau = \delta$. That is, for all $\tau \leq \delta$: $l_{a_\gamma(j)}(\tau) \triangleleft l_{a_\gamma(j)}(\tau)$ if $\tau = \delta$. Recalling the definition of $c$, we altogether infer that $d(z) = c(a_\gamma(j), a_\gamma(j)) = \delta$, as sought. \(\square\)

Corollary 3.9. For every successor ordinal $\alpha$:
   (1) $S(\underline{\alpha}, \omega)$ and $S^*(\aleph_\alpha, \text{cf}(\aleph_{\alpha-1}), \omega)$ hold.\(^{18}\)
   (2) if $\underline{\alpha} = \aleph_\alpha$, then $S^*(\aleph_\alpha, \aleph_\alpha, \omega)$ holds.\(^{19}\)

   For every limit ordinal $\alpha$:
   (3) if $\text{cf}(\alpha)$ is uncountable and admitting a nonreflecting stationary set, then $S^*(\aleph_\alpha, \text{cf}(\alpha), \omega)$ holds;
   (4) if $\text{cf}(\alpha)$ is a successor of an infinite cardinal of cofinality $\theta$, then $S^*(\aleph_\alpha, \theta, \omega)$ holds.

Proof. (1) Suppose that $\alpha = \beta + 1$.

   Then $\lambda := \underline{\beta}$ is a strong limit cardinal, and hence $2^{<\lambda} = \lambda$. So, by Theorem 3.8 $S(\text{cf}(2^\lambda), \omega)$ holds. But then, by Proposition 3.3 $S(2^\lambda, \omega)$ holds. That is, $S(\underline{\alpha}, \omega)$ holds.

   As for the second part of clause (1):
   - If $\aleph_\beta$ is a regular cardinal, then by Corollary 3.7 $S^*(\aleph_\alpha, \aleph_\alpha, \omega)$ holds.
   - If $\aleph_\beta$ is a singular cardinal, then by Fact 3.3, $\text{Pr}_1(\aleph_\alpha, \aleph_\alpha, \text{cf}(\aleph_\beta), \text{cf}(\aleph_\beta))$ holds. Then, by Lemma 3.4 $S^*(\aleph_\alpha, \text{cf}(\aleph_\beta), \omega)$ holds.

   (2) Suppose that $\alpha = \beta + 1$ and $\underline{\alpha} = \aleph_\alpha$. Write $\lambda := \aleph_\beta$.

   - If $\lambda$ is a regular cardinal, then by Corollary 3.7 $S^*(\aleph_\alpha, \aleph_\alpha, \omega)$ holds.
   - If $\lambda$ is a singular cardinal, then $\text{pp}(\lambda) \leq 2^\lambda = 2^{\aleph_\beta} \leq 2^{\underline{\alpha}} = \aleph_\alpha = \aleph_\alpha^+ \leq \aleph_\alpha^+$, and then by Fact 3.3, $\text{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \text{cf}(\lambda))$ holds. So, by Lemma 3.4 $S^*(\aleph_\alpha, \aleph_\alpha, \omega)$ holds.

\(^{18}\)Note that if $\aleph_{\alpha-1}$ is regular, then $S^*(\aleph_\alpha, \text{cf}(\aleph_{\alpha-1}), \omega)$ is equivalent to $S^*(\aleph_\alpha, \aleph_\alpha, \omega)$.

\(^{19}\)Note that GCH is equivalent to the assertion that $\underline{\alpha} = \aleph_\alpha$ for all ordinals $\alpha$, and that if ZFC is consistent, then so is $\text{ZFC + GCH}$ + every regular uncountable cardinal is of the form $\aleph_\alpha$ for some successor ordinal $\alpha$. 
(3) Suppose that $\text{cf}(\alpha) = \kappa$, where $\kappa$ is an uncountable cardinal admitting a nonreflecting stationary set. By Fact 3.5.3, $\Pr_1(\kappa, \kappa, \kappa, \omega)$ holds. Then, by Lemma 3.4, $\text{S}^*(\kappa, \kappa, \omega)$ holds. As $\text{cf}(\kappa) = \text{cf}(\omega)$, we infer from Proposition 3.3 that $\text{S}^*(\kappa, \text{cf}(\kappa), \omega)$ holds.

(4) Suppose that $\text{cf}(\alpha) = \mu^+$ for some infinite cardinal $\mu$ of cofinality, say, $\theta$. Given clause (3), we may assume that $\mu$ is singular. By Fact 3.5.7, $\Pr_1(\mu^+, \mu^+, \theta, \theta)$ holds. Then, by Lemma 3.4, $\text{S}^*(\mu^+, \theta, \omega)$ holds. As $\text{cf}(\omega) = \text{cf}(\kappa) = \mu^+$, we infer from Proposition 3.3 that $\text{S}^*(\kappa, \theta, \omega)$ holds. □

**Corollary 3.10.** For every ordinal $\alpha$ such that $\text{cf}(\alpha)$ is a successor cardinal, $\text{S}(\alpha, \omega)$ holds.

**Proof.** By Corollary 3.9 and Proposition 3.2 □

**Remarks.**

(i) The restriction to cofinality of a successor cardinal is necessary, as it follows from Proposition 3.2(4) that $\text{S}(\alpha, \omega)$ fails for any ordinal $\alpha$ satisfying $\alpha \rightarrow [\alpha]_2^2$.

(ii) It is unknown whether the conclusion $\text{S}(\alpha, \omega)$ may be replaced by the stronger conclusion $\text{S}^*(\alpha, \omega)$. In fact, whether ZFC implies $\exists \alpha \rightarrow [\alpha; \alpha]_2^2$ is already a longstanding open problem.

4. Colourings for sumsets and bounded finite sums

The main goal of this section is to show that for unboundedly many regular uncountable cardinals $\kappa$, if $|G| = \kappa$, then $G \not\rightarrow [\kappa]_{FS}^2$ and even $G \not\rightarrow [\kappa]_{FS}^\Sigma$. A minor goal is to prove some no-go theorems.

4.1. Sumsets. We commence with a lemma that will simplify some reasoning concerning sumsets.

**Lemma 4.1.** Let $G$ be a commutative cancellative semigroup of cardinality $\kappa > \omega$, let $\theta \leq \kappa$ be an arbitrary cardinal, and let $c : G \rightarrow \theta$ be some colouring. Then for each $\delta < \theta$, the following two conditions are equivalent:

1. For every $X, Y \subseteq G$ with $|X| = |Y| = \kappa$, we have $\delta \in c[X + Y]$ (that is, there are $x \in X$ and $y \in Y$ such that $c(x + y) = \delta$).

2. For every integer $n \geq 2$ and $\kappa$-sized sets $X_1, \ldots, X_n \subseteq G$, we have $\delta \in c[X_1 + \cdots + X_n]$ (that is, there are $x_1 \in X_1, \ldots, x_n \in X_n$ such that $c(x_1 + \cdots + x_n) = \delta$).

**Proof.** We focus on the nontrivial implication $1 \implies 2$.

Proceed by induction on $n \in \{2, 3, \ldots\}$. Suppose the statement holds true for a given $n$, and suppose we are given $\kappa$-sized sets $X_1, \ldots, X_n, X_{n+1}$. First, notice that $X_1 + \cdots + X_n$ has cardinality $\kappa$ (since the elements $x_1 + \cdots + x_{n-1} + y$, where the $x_i \in X_i$ are fixed and $y$ ranges over $X_n$, are all distinct by cancellativity). Thus, by our assumption we can find $x_1 + \cdots + x_n \in X_1 + \cdots + X_n$ and $x_{n+1} \in X_{n+1}$ such that $c(x_1 + \cdots + x_n + x_{n+1}) = c((x_1 + \cdots + x_n) + x_{n+1}) = \delta$. □

**Theorem 4.2.** Suppose that $G$ is a commutative cancellative semigroup of cardinality, say, $\kappa$. If $\text{S}^*(\kappa, \theta, \omega)$ holds, then so does $G \not\rightarrow [\kappa]_{FS}^\Sigma$. 

Proof. Let \(d : [\kappa]^{<\omega} \to \theta\) be a witness to \(S^*(\kappa, \theta, \omega)\). Using Lemma 2.2, embed \(G\) into a direct sum \(\bigoplus_{\alpha < \kappa} G_\alpha\), with each \(G_\alpha\) a countable abelian group. Then, define \(c : G \to \theta\) by stipulating
\[
c(x) := d(\text{supp}(x)).
\]

While the axiom \(S^*(\kappa, \theta, \omega)\) allows us to handle any finite number of \(\kappa\)-sized families, we shall take advantage of Lemma 4.1, which reduces the algebraic problem into looking at sumsets of two sets.

Thus, let \(X\) and \(Y\) be two \(\kappa\)-sized subsets of \(G\), and let \(\delta < \theta\) be arbitrary. Since each \(x \in G\) has a finite support, and each of the \(G_\alpha\) are countable, there are only countably many elements of \(G\) with a given support. Therefore, both \(X := \{\text{supp}(x) \mid x \in X\}\) and \(Y := \{\text{supp}(y) \mid y \in Y\}\) are \(\kappa\)-sized subfamilies of \([\kappa]^{<\omega}\). Now, as \(d\) witnesses \(S^*(\kappa, \theta, \omega)\), we may pick \(\langle x, y \rangle \in X \times Y\) such that \(d(z) = \delta\) whenever \(\text{supp}(x) \triangle \text{supp}(y) \subseteq z \subseteq \text{supp}(x) \cup \text{supp}(y)\); in particular \(z := \text{supp}(x + y)\) satisfies the above equation by Proposition 2.4, and hence \(c(x + y) = d(z) = \delta\).

Corollary 4.3. Let \(G\) be any commutative cancellative semigroup of cardinality, say, \(\lambda\). If \(\kappa := \text{cf}(\lambda)\) is an uncountable cardinal satisfying at least one of the following conditions:

1. \(\square(\kappa)\) holds;
2. \(\kappa\) admits a nonreflecting stationary set (e.g., \(\kappa = \mu^+\) for \(\mu\) regular);
3. \(\kappa = \mu^+, \mu\) is singular, and \(\text{pp}(\mu) = \mu^+\) (e.g., \(\mu^{\text{cf}(\mu)} = \mu^+\)),

then \(G \rightarrow [\lambda]^{\text{SuS}}\) holds.

Proof. By Lemma 3.4, Fact 3.5 and Theorem 3.6 any of the above hypotheses imply that \(S^*(\kappa, \kappa, \omega)\) holds. Then by Proposition 3.3 \(S^*(\lambda, \kappa, \omega)\) holds, as well. Now, appeal to Theorem 4.2.

Corollary 4.4. It is consistent with ZFC that for every infinite commutative cancellative semigroup \(G\), letting \(\kappa := |G|\), \(G \rightarrow [\kappa]^{\text{SuS}}\) holds iff \(\text{cf}(\kappa) > \omega\).

Proof. If there exists a weakly compact cardinal in Gödel’s constructible universe \(L\), then let \(\mu\) denote the least such one and work in \(L_\mu\). Otherwise, work in \(L\). In both cases, we end up with a model of ZFC in which \(\square(\kappa)\) holds for every regular uncountable cardinal \(\kappa\) [16, Theorem 6.1] and in which every singular cardinal is a strong limit [11]. Now, there are three cases to consider:

- If \(\text{cf}(\kappa) > \omega\), then \(\square(\text{cf}(\kappa))\) holds and then \(G \rightarrow [\kappa]^{\text{SuS}}_{\text{cf}(\kappa)}\) holds as a consequence of Corollary 4.3.
- If \(\kappa > \text{cf}(\kappa) = \omega\) and \(G \rightarrow [\kappa]^{\text{SuS}}_{\text{cf}(\kappa)}\) holds, then so does \(G \rightarrow [\kappa]^{\text{FS}_2}_{\omega}\). But then by Proposition 4.10 below, \(\kappa \rightarrow [\kappa]^{2}_{\omega}\) holds, contradicting Theorem 54.1 of [5] and the fact that \(\kappa\) is a strong limit.
- If \(\kappa = \omega\) and \(G \rightarrow [\kappa]^{\text{SuS}}_{\text{cf}(\kappa)}\) holds, then so does \(G \rightarrow [\omega]^{2}_{\omega}\), contradicting Hindman’s theorem.

We conclude this subsection with an analog of Corollary 2.8 in the context of sumsets.
Corollary 4.5. For every infinite cardinal $\lambda$ and every cardinal $\theta$, the following are equivalent:

1. $\lambda^+ \rightarrow [\lambda^+]^2_\theta$ holds;
2. $G \rightarrow [\lambda^+]^{SS}_{\theta}$ holds for every commutative cancellative semigroup $G$ of cardinality $\lambda^+$;
3. $G \rightarrow [\lambda^+]^{SS}_{\theta}$ holds for some commutative cancellative semigroup $G$ of cardinality $\lambda^+$.

Proof. Let $\lambda$ and $\theta$ be as above. If $\lambda$ is regular, then all of the three clauses hold as a consequence of Corollary 3.7 and Theorem 4.2.

Next, as the implication $(2) \implies (3)$ is trivial and the implication $(3) \implies (1)$ follows immediately from Proposition 4.10 below, let us suppose that $\lambda$ is a singular cardinal for which $\lambda^+ \rightarrow [\lambda^+]^2_\theta$ holds.

Then, by [23, Theorem 1], $\Pr_1(\lambda^+, \lambda^+, \theta, \cf(\lambda))$ holds. Then, by Lemma 3.4 and Theorem 4.2, $G \rightarrow [\lambda^+]^{SS}_{\theta}$ holds for every commutative cancellative semigroup $G$ of cardinality $\lambda^+$. □

4.2. Finite sums. An immediate corollary to Theorem 4.2 reads as follows.

Corollary 4.6. Suppose that $G$ is a commutative cancellative semigroup of cardinality, say, $\kappa$. If $S^*(\kappa, \theta, \omega)$ holds, then so does $G \rightarrow [\kappa]^{FS_\omega}_{\theta}$ for all integers $n \geq 2$.

Our next goal is to derive statements about $FS_n$ from the weaker principle $S(\kappa, \theta)$. We first deal with the case where $n = 2$.

Theorem 4.7. Suppose that $G$ is a commutative cancellative semigroup of cardinality, say, $\kappa$. If $S(\kappa, \theta)$ holds, then so does $G \rightarrow [\kappa]^{FS_2}_{\theta}$.

Proof. Let $d : [\kappa]^{<\omega} \rightarrow \theta$ be a witness to $S(\kappa, \theta)$. Using Lemma 2.2, embed $G$ into a direct sum $\bigoplus_{\alpha < \kappa} G_\alpha$, with each $G_\alpha$ a countable abelian group. Then, define $c : G \rightarrow \theta$ by stipulating $c(x) := d(\text{supp}(x))$.

Let $X \in [G]^{\kappa}$ be arbitrary. Since each $x \in G$ has a finite support and there are only countably many elements of $G$ with a given support, we have that $X := \{\text{supp}(x) \mid x \in X\}$ is a subfamily of $[\kappa]^{<\omega}$ of size $\kappa$. Let $\delta < \theta$ be arbitrary. As $d$ witnesses $S(\kappa, \theta)$, we may now pick two distinct $x, y \in X$ such that $d(z) = \delta$ whenever $\text{supp}(x) \triangle \text{supp}(y) \subseteq z \subseteq \text{supp}(x) \cup \text{supp}(y)$. In particular, by Proposition 2.4, we have that $\text{supp}(x + y)$ is such a $z$, and therefore $c(x + y) = d(\text{supp}(x + y)) = \delta$. □

We would next like to obtain the corresponding result for $FS_n$, with $n > 2$. This will, however, not be very hard under the right circumstances, as the following lemma shows. The idea for the proof of this lemma is adapted from [18]. Recall that, given an $n \in \mathbb{N}$, an abelian group $G$ is said to be $n$-divisible if for every $x \in G$ there exists a $z \in G$ such that $nz = x$ (thus, being divisible is the same as being $n$-divisible for every $n \in \mathbb{N}$).

Lemma 4.8. Let $n \in \mathbb{N}$, and let $G$ be an $n$-divisible abelian group. For every $\lambda, \theta$, if $c$ is a colouring witnessing $G \rightarrow [\lambda]^{FS_\omega}_{\theta}$, then $c$ witnesses $G \rightarrow [\lambda]^{FS_{n+1}}_{\theta}$ as well.

Note that when $G$ is the abelian group $([\kappa]^{<\omega}, \triangle)$, then a colouring witnessing $G \rightarrow [\kappa]^{FS_2}_{\theta}$ is almost a witness to $S(\kappa, \theta)$. 

Proposition 4.10. Suppose that $G$ is an $n$-divisible abelian group such that $G \to [\lambda]^{\text{FS}_n}_\theta$ holds, as witnessed by a colouring $c : G \to \theta$. To see that it is also the case that for all $X \subseteq G$ with $|X| = \lambda$, $c[\text{FS}_{n+1}(X)] = \theta$, grab an arbitrary $X \subseteq G$ with $|X| = \lambda$. Pick an element $x \in X$, and use $n$-divisibility to obtain an element $z \in G$ such that $nz = x$. Now, let

$$Y := \{y + z \mid y \in X \setminus \{x\}\}.$$ 

Then $Y$ is a subset of $G$ of cardinality $\lambda$, so that for each colour $\delta < \theta$ we can find $n$ distinct elements $y_1, \ldots, y_n, z \in Y$ such that the sum

$$(y_1 + z) + \cdots + (y_n + z) = y_1 + \cdots + y_n + nz = y_1 + \cdots + y_n + x$$

is an element of $\text{FS}_{n+1}(X)$ that receives colour $\delta$.


The preceding lemma yields a fairly general result concerning $\text{FS}_n$, where the main piece of information seems to be the cardinality of the commutative cancellative semigroup $G$ rather than the semigroup itself.

Corollary 4.9. Suppose that $\theta \leq \kappa$ are cardinals such that $G \to [\lambda]^{\text{FS}_2}_\theta$ holds for all commutative cancellative semigroups of cardinality $\kappa$. Then, for every commutative cancellative semigroup $G$ of cardinality $\kappa$, there exists a colouring $c$ that simultaneously witnesses $G \to [\lambda]^{\text{FS}_n}_\theta$ for all integers $n \geq 2$.

Proof. Let $G$ be any commutative cancellative semigroup $G$ with $|G| = \kappa$. By Lemma 2.2, we may embed $G$ into a direct sum $G' = \bigoplus_{\alpha < \kappa} G_\alpha$, where each $G_\alpha$ is countable and divisible. This implies that $G'$ is divisible and $|G'| = \kappa$. Thus, by our hypothesis, we may take a colouring $d : G' \to \theta$ witnessing $G' \to [\lambda]^{\text{FS}_2}_\theta$. Since $G'$ is divisible, we can use Lemma 4.8 to inductively prove, for every integer $n \geq 2$, that $d$ witnesses $G' \to [\lambda]^{\text{FS}_n}_\theta$. Therefore $c := d \restriction G$ witnesses the statement $G \to [\lambda]^{\text{FS}_n}_\theta$ for all integers $n \geq 2$.

We now move to proving no-go propositions. These will be obtained using the following simple proxy:

Proposition 4.10. If $G$ is a commutative semigroup satisfying $G \to [\lambda]^{\text{FS}_n}_\theta$, then $\kappa \to [\lambda]^{\theta}_n$ holds, for $\kappa := |G|$.

Proof. Fix an injective enumeration $\{x_\alpha \mid \alpha < \kappa\}$ of a commutative semigroup $G$, along with a colouring $c : G \to \theta$ witnessing $G \to [\lambda]^{\text{FS}_n}_\theta$. Define a colouring $d : [\kappa]^n \to \theta$ by stipulating

$$d(\alpha_1, \ldots, \alpha_n) = c(x_{\alpha_1} + \cdots + x_{\alpha_n}).$$

Now, given $Y \in [\kappa]^\lambda$, we have that $\{x_\alpha \mid \alpha \in Y\}$ is a $\lambda$-sized subset of $G$, and hence for every colour $\delta < \theta$, we may find $\alpha_1, \ldots, \alpha_n$ in $Y$ such that $c(x_{\alpha_1} + \cdots + x_{\alpha_n}) = \delta$, so that $d(\alpha_1, \ldots, \alpha_n) = \delta$.

Corollary 4.11. There exists an uncountable cardinal $\kappa$ such that for every commutative semigroup $G$ of cardinality $\kappa$, $G \to [\kappa]^{\text{FS}_n}_\omega$ fails for all $n < \omega$. In particular, there exists an uncountable abelian group $G$ for which $G \to [\omega_1]^{\text{FS}_n}_\omega$ fails for all $n < \omega$.

Proof. The statement holds true for $\kappa := \beth_\omega$, as otherwise, by Proposition 4.10, there exists some $n < \omega$ for which $\beth_\omega \not\to [\beth_\omega]^{\theta}_n$ holds, contradicting Theorem 54.1 of [5]. In particular, by picking any abelian group $G$ of cardinality $\beth_\omega$, we infer that $G \not\to [\omega_1]^{\text{FS}_n}_\omega$ fails for all $n < \omega$. \hfill \square
Corollary 4.12. Suppose that $\kappa$ is a weakly compact cardinal. Then in the generic extension for adding $\kappa$ many Cohen reals, for every commutative semigroup $G$ of size continuum, $G \to [\mathfrak{c}]^{\text{FS}_n}_{\omega_1}$ fails for every integer $n \geq 2$.

Proof. Let us remind the reader that as $\kappa$ is weakly compact, $\kappa$ is strongly inaccessible and satisfies that for every $\theta < \kappa$, every positive integer $n$, and every colouring $d : [\kappa]^n \to \theta$, there exists some $H \in [\kappa]^{\kappa}_\theta$ such that $d \upharpoonright [H]^n$ is constant.

Let $\mathbb{P}$ be the notion of forcing for adding $\kappa$ many Cohen reals. By Proposition 4.10, it suffices to show that in $V^\mathbb{P}$, we have $\kappa \to [\mathfrak{c}]^{\kappa}_{\omega_1}$ for every integer $n \geq 2$.

Let $\dot{c}$ be a $\mathbb{P}$-name for an arbitrary colouring $c : [\kappa]^n \to \omega_1$ in $V^\mathbb{P}$. Working in $V$, define a colouring $d : [\kappa]^n \to [\omega_1]^{\leq \omega}$ by stipulating

$$d(\alpha_1, \ldots, \alpha_n) := \{ \delta < \omega_1 \mid \exists p \in \mathbb{P} \models \text{“} \dot{c}(\alpha_1, \ldots, \alpha_n) = \delta \text{”} \}.$$ 

Since $\mathbb{P}$ is ccc, the range of $d$ indeed consists of countable subsets of $\omega_1$. As $\kappa$ is weakly compact, $|[\omega_1]^\omega| < \kappa$ and we may pick some $H \in [\kappa]^{\kappa}_\omega$ such that $d\upharpoonright [H]^n$ is a singleton, say, $\{A\}$. Evidently,

$$\models \text{“} \dot{c}([H]^n) \subseteq A \text{”}.$$ 

As $A$ is countable and $\mathbb{P}$ does not collapse cardinals, $\mathbb{P}$ forces that $\dot{c}([H]^n)$ omits at least one colour. \hfill $\Box$

5. Some corollaries concerning the real line

Hindman, Leader and Strauss [13, Theorem 3.2] proved that $\mathbb{R} \to [\mathfrak{c}]^{\text{FS}_n}_{\omega}$ holds for every integer $n \geq 2$. It turns out that it is possible to increase the number of colours from 2 to $\omega$:

Corollary 5.1. $\mathbb{R} \to [\mathfrak{c}]^{\text{FS}_n}_{\omega}$ holds for every integer $n \geq 2$.

Proof. As $|\mathbb{R}| = \mathfrak{c} = 2^{|\omega_0|} = 2^{\aleph_0} = \beth_1$, we infer from Corollary 3.10 that $S(\mathfrak{c}, \omega)$ holds. Now appeal to Theorem 4.2 and Corollary 4.9. \hfill $\Box$

On the grounds of ZFC alone, it is impossible to increase the number of colours to $\omega_1$:

Corollary 5.2. If there exists a weakly compact cardinal, then there exists a model of ZFC in which $\mathbb{R} \to [\mathfrak{c}]^{\text{FS}_n}_{\omega}$ fails for every integer $n \geq 2$. Furthermore, in this model, $\mathfrak{c}$ is an inaccessible cardinal which is weakly compact in Gödel’s constructible universe.

Proof. Suppose that $\kappa$ is a weakly compact cardinal. Work in Gödel’s constructible universe, $L$. Then $\kappa$ is still a weakly compact cardinal (see [15, Theorem 17.22]), and if $\mathbb{P}$ denotes the notion of forcing for adding $\kappa$ many Cohen reals, then by Corollary 4.12, the forcing extension $L^\mathbb{P}$ is a model in which $\mathfrak{c}$ is an inaccessible cardinal that is weakly compact in Gödel’s constructible universe, and $\mathbb{R} \to [\mathfrak{c}]^{\text{FS}_n}_{\omega}$ fails for every integer $n \geq 2$. \hfill $\Box$

However, assuming an anti-large cardinal hypothesis, the number of colours may be increased:

Corollary 5.3. Each of the following implies that $\mathbb{R} \to [\mathfrak{c}]^{\text{SuS}}_{\omega}$ holds:

(1) $\mathfrak{c} = b$ (e.g., Martin’s Axiom holds);
(2) $\mathfrak{c}$ is a successor of a regular cardinal (e.g., CH holds);
(3) $\mathfrak{c}$ is a successor of a singular cardinal of countable cofinality;
(4) \( c \) is a regular cardinal that is not weakly compact in Gödel’s constructible universe.

Proof.  
(1) By Lemma 3.4, Fact 3.5(1), and Theorem 4.2.  
(2) By Corollary 4.3(2).  
(3) If \( c = \lambda^+ \) and cf(\( \lambda \)) = \( \omega \), then \( \lambda^{\text{cf}(\lambda)} = c = \lambda^+ \). Now, appeal to Corollary 4.3(3).  
(4) By [33], if \( \kappa \) is a regular uncountable cardinal which is not a weakly compact cardinal in Gödel’s constructible universe, then \( \boxdot(c) \) holds. Now, appeal to Corollary 4.3(1).  
□

By a theorem of Milliken [19, Theorem 9], CH entails \( R \not\rightarrow [c]^{\text{FS}}_{\omega_1} \). We now derive the same conclusion (and even with superscript SuS) from an (again, optimal) anti-large cardinal hypothesis:

Corollary 5.4. Each of the following implies that \( R \not\rightarrow [c]^{\text{SuS}}_{\omega_1} \) holds:

(1) \( c \) is a successor cardinal (e.g., CH holds);  
(2) cf(\( c \)) is a successor of a cardinal of uncountable cofinality;  
(3) cf(\( c \)) is not weakly compact in Gödel’s constructible universe.

Proof.  
(1) By Corollary 5.3 we may assume that \( c \) is a successor of a singular cardinal of uncountable cofinality \( \theta \). Then, by Fact 3.5(7), Lemma 3.4, and Theorem 4.2, \( R \not\rightarrow [c]^{\text{SuS}}_{\theta} \) holds. In particular, \( R \not\rightarrow [c]^{\text{SuS}}_{\omega_1} \) holds.  
(2) If cf(\( c \)) is a successor of a regular cardinal, then by Corollary 3.7 \( S^+(\text{cf}(\( c \)), \text{cf}(\( c \)), \omega) \) holds. If cf(\( c \)) is a successor of a singular cardinal of cofinality \( \theta \), then, by Fact 3.5(7) and Lemma 3.4, \( S^+(\text{cf}(\( c \)), \theta, \omega) \) holds. Altogether, if cf(\( c \)) is a successor of a cardinal of uncountable cofinality, \( S^+(\text{cf}(\( c \)), \omega_1, \omega) \) holds. So, by Proposition 3.3 \( S^+(c, \omega_1, \omega) \) holds. By Theorem 4.2 then, \( R \not\rightarrow [c]^{\text{SuS}}_{\omega_1} \) holds.  
(3) Let \( \kappa := \text{cf}(\( c \)). \) By König’s lemma, \( \kappa \) is uncountable. By [33], if \( \kappa \) is a regular uncountable cardinal which is not a weakly compact cardinal in Gödel’s constructible universe, then \( \boxdot(\kappa) \) holds. Thus, by Corollary 4.3(1), \( R \not\rightarrow [c]^{\text{SuS}}_{\kappa} \) holds. In particular, \( R \not\rightarrow [c]^{\text{SuS}}_{\omega_1} \) holds.  
□

As for FS sets, we have the following optimal results:

Corollary 5.5. The following are equivalent:

- \( R \not\rightarrow [c]^{\text{FS}}_c \) holds;  
- \( c \) is not a Jónsson cardinal.

Proof. Appeal to Corollary 2.8 with \( \lambda = \kappa = \theta = c \).  
□

Corollary 5.6. The following are equivalent:

- \( R \not\rightarrow [\omega_1]^{\text{FS}}_{\omega_1} \) holds;  
- \( (c, \omega_1) \not\rightarrow (\omega_1, \omega) \) fails.

Proof. Appeal to Corollary 2.9 with \( \kappa = \kappa = \theta = \omega \).  
□

In particular, if there exists a Kurepa tree with \( c \) many branches, then \( R \not\rightarrow [\omega_1]^{\text{FS}}_{\omega_1} \) holds.

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STRONG FAILURES OF HINDMAN’S THEOREM

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