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RADIAL POSITIVE DEFINITE FUNCTIONS AND SCHOENBERG MATRICES WITH NEGATIVE EIGENVALUES

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ABSTRACT. The main object under consideration is a class $\Phi_n\backslash\Phi_{n+1}$ of radial positive definite functions on $\mathbb{R}^n$ which do not admit *radial positive definite continuation* on $\mathbb{R}^{n+1}$. We find certain necessary and sufficient conditions on the Schoenberg representation measure ν_n of $f \in \Phi_n$ for $f \in \Phi_{n+k}$, $k \in \mathbb{N}$. We show that the class $\Phi_n\backslash\Phi_{n+k}$ is rich enough by giving a number of examples. In particular, we give a direct proof of $\Omega_n \in \Phi_n\backslash\Phi_{n+1}$, which avoids Schoenberg's theorem; Ω_n is the Schoenberg kernel. We show that $\Omega_n(a\cdot)\Omega_n(b\cdot) \in \Phi_n\backslash\Phi_{n+1}$ for $a \neq b$. Moreover, for the square of this function we prove the surprisingly much stronger result $\Omega_n^2(a\cdot) \in \Phi_{2n-1}\backslash\Phi_{2n}$. We also show that any $f \in \Phi_n\backslash\Phi_{n+1}$, $n \geq 2$, has infinitely many negative squares. The latter means that for an arbitrary positive integer N there is a finite Schoenberg matrix $\mathcal{S}_X(f) := \|f(|x_i - x_j|_{n+1})\|_{i,j=1}^m$, $X := \{x_j\}_{j=1}^m \subset \mathbb{R}^{n+1}$, which has at least N negative eigenvalues.

1. INTRODUCTION

Positive definite functions have a long history, being an important chapter in various areas of harmonic analysis. They can be traced back to papers of Carathéodory, Herglotz, Bernstein, culminating in Bochner's celebrated theorem from 1932–1933.

In this paper we will be dealing primarily with radial positive definite functions. Such functions have significant applications in probability theory, statistics, and approximation theory, where they occur as the characteristic functions or Fourier transforms of spherically symmetric probability distributions. Denote the class of radial positive definite functions on $\mathbb{R}^n$ by Φ_n.

We follow the standard notation for the inner product $(u,v)_n = (u,v) = u_1v_1 + \ldots + u_nv_n$ of two vectors $u = (u_1,\ldots,u_n)$ and $v = (v_1,\ldots,v_n)$ in $\mathbb{R}^n$, and $|u|_n = |u| = \sqrt{(u,u)}$ for the Euclidean norm of u.

Definition 1.1. Let $n \in \mathbb{N}$. A real-valued and continuous function f on $\mathbb{R}_+ = [0,\infty)$, $f \in C(\mathbb{R}_+)$, is called a *radial positive definite (RPD) function on* $\mathbb{R}^n$ if for an arbitrary finite set $\{x_1,\ldots,x_m\}$, $x_k \in \mathbb{R}^n$, and $\{\xi_1,\ldots,\xi_m\} \in \mathbb{C}^m$,

$$\sum_{k,j=1}^{m} f(|x_k - x_j|_n)\xi_j\overline{\xi}_k \geq 0. \tag{1.1}$$

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In other words, RPD functions f are exactly those for which $f(|\cdot|_n)$ are positive definite functions on $\mathbb{R}^n$ or, equivalently, the Schoenberg matrices $\mathcal{S}_X(f) := \|f(|x_i - x_j|_n)\|_{i,j=1}^m$ are positive definite for any $X := \{x_j\}_{j=1}^m \subset \mathbb{R}^n$.

The characterization of radial positive definite functions is a fundamental result of I. Schoenberg [13, 14] (see, e.g., [2, Theorem 5.4.2]).

Theorem 1.2. *A function $f \in \Phi_n$, $f(0) = 1$, if and only if there exists a probability measure ν on $\mathbb{R}_+$ such that*

$$f(r) = \int_0^\infty \Omega_n(rt)\,\nu(dt), \qquad r \in \mathbb{R}_+, \tag{1.2}$$

where the Schoenberg kernel

$$\Omega_n(s) := \Gamma(q+1)\left(\frac{2}{s}\right)^q J_q(s) = \sum_{j=0}^\infty \frac{\Gamma(q+1)}{j!\,\Gamma(j+q+1)}\left(-\frac{s^2}{4}\right)^j, \quad q := \frac{n}{2} - 1, \tag{1.3}$$

J_q is the Bessel function of the first kind and order q. Moreover,

$$\Omega_n(|x|) = \int_{S^{n-1}} e^{i(u,x)}\sigma_n(du), \qquad x \in \mathbb{R}^n, \tag{1.4}$$

where σ_n is the normalized surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

So it is not surprising that various properties of the Bessel functions (recurrence and differential relations, bounds and asymptotics, integrals) come up repeatedly throughout the paper. The first three functions Ω_n, $n = 1, 2, 3$, can be computed as

$$\Omega_1(s) = \cos s, \quad \Omega_2(s) = J_0(s), \quad \Omega_3(s) = \frac{\sin s}{s}\,. \tag{1.5}$$

It is well known that the classes Φ_n are nested, and inclusion $\Phi_{n+1} \subset \Phi_n$ is proper for any $n \in \mathbb{N}$. The result is mentioned in the pioneering paper of Schoenberg [13] (without proof) and then duplicated in Akhiezer's book [2] (and in a number of later papers and books). The main goal of Section 2 is to study the classes $\Phi_n \backslash \Phi_{n+1}$. We start out with two proofs of the known fact that $\Omega_n \in \Phi_n \backslash \Phi_{n+1}$; the first one is based on the Schoenberg theorem and some rudiments of the Stieltjes moment problem. The second one is direct and has nothing to do with Schoenberg's theorem. We just manufacture a set $X = \{x_j\}_{j=1}^{n+3} \subset \mathbb{R}^{n+1}$ such that the corresponding matrix $\mathcal{S}_X(\Omega_n) = \|\Omega_n(|x_i - x_j|)\|_{i,j=1}^{n+3}$, called the *Schoenberg matrix* (see [5] for a detailed account of this object), has at least one negative eigenvalue (Proposition 2.2).

Let us emphasize that the inclusion $f \in \Phi_n \backslash \Phi_{n+1}$ means that f is an RPD function on $\mathbb{R}^n$ but does not admit *radial* positive definite continuation on $\mathbb{R}^{n+1}$, while positive definite continuations obviously exist.

As it turns out, the class $\Phi_n \backslash \Phi_{n+1}$ is rich enough. We give some sufficient conditions in terms of Schoenberg's measure $\nu_n = \nu_n(f)$ for f to belong to $\Phi_n \backslash \Phi_{n+1}$. A key ingredient here is the following relation called a *transition formula*.

Let $f \in \Phi_n$, $n \geq 2$. Then $f \in \Phi_m$ for $m = 1, 2, \ldots, n-1$, and according to Schoenbeerg's theorem f admits representation (1.2) with some measure $\nu_m(f)$. We show (see Theorem 3.1) that the relation between the measures ν_n and ν_m is given by

$$\nu_m(dx) = p_m(x)\,dx, \quad p_m(x) = \frac{2x^{m-1}}{B\left(\frac{m}{2}, \frac{n-m}{2}\right)} \int_x^\infty \left(1 - \frac{x^2}{u^2}\right)^{\frac{n-m}{2}-1} \frac{\nu_n(du)}{u^m}\,, \tag{1.6}$$

where $B(a,b)$ is the Euler beta function. So, each ν_m is *absolutely continuous* and $\nu_m\{(0,\varepsilon)\} > 0$ for any $\varepsilon > 0$. In other words, if $\nu_n(f)$ is either not purely absolutely continuous (contains a singular component) or $\nu_n\{(0,\varepsilon)\} = 0$ for some $\varepsilon > 0$, then $f \notin \Phi_{n+1}$.

Besides, we investigate the smoothness and decaying properties of the distribution function generated by Schoenberg's measure ν_m in (1.6). For instance, it is shown in Theorem 3.9 that $\nu_m \in AC^{[k/2]}_{\rm loc}(\mathbb{R}_+)$ and moreover, $x^j p_m^{(j)}(x) \in L^p(\mathbb{R}_+)$, $j = 0, 1, \dots, [k/2]-1$, $k = n-m$, whenever ν_n is absolutely continuous, $\nu_n = p_n\,dx$, and $p_n \in L^p(\mathbb{R}_+)$ for some $1 \le p < \infty$.

We put a function $f \in \Phi_m$ to the subclass Φ_m^{ac} if its Schoenberg representing measure $\nu_m = \nu_m(f)$ is absolutely continuous $\nu_m(dx) = p_m(x)\,dx$, $p_m(\cdot) \in L^1(\mathbb{R}_+)$. We equip Φ_m^{ac} with the L^1–norm by setting $\|\nu_m\|_{\Phi_m^{ac}} := \|p_m\|_{L^1(\mathbb{R}_+)}$. Denoting by $M_{m,m+k}$ the set of (necessarily absolutely continuous) representing measures of functions $f \in \Phi_{m+k}$ considered as functions from Φ_m we show that the set $M_{m,m+k}$ *forms a closed nowhere dense subset in* Φ_m^{ac}.

Clearly, $\Omega_n(a\cdot) \in \Phi_n$ for each $a > 0$. Since the Schur (entrywise) product of two nonnegative matrices is again a nonnegative matrix, the product of two Schoenberg kernels $\Omega_n(a\cdot)\Omega_n(b\cdot) \in \Phi_n$, $a, b > 0$. Moreover, the following result is proved in Section 4.

Theorem 1.3. *For* $n \in \mathbb{N}$*,*

(i) $\Omega_n(at)\Omega_n(bt) \in \Phi_n \backslash \Phi_{n+1}$, $\quad a \ne b$;
(ii) $\Omega_n^2(t) \in \Phi_{2n-1} \backslash \Phi_{2n}$.

The problem we deal with in Section 5 concerns the number of negative squares of a function $f \in \Phi_n \backslash \Phi_{n+1}$. Namely, since such f does not admit RPD continuation on $\mathbb{R}^{n+1}$, the quadratic forms (1.1) associated with $f(|\cdot|_{n+1})$ in place of $f(|\cdot|_n)$ might have negative squares. We are interested in the maximal number of negative squares of such forms. One reformulates this concept in terms of the maximal number of negative eigenvalues of the corresponding Schoenberg matrices $\mathcal{S}_X(f) := \|f(|x_i - x_j|_{n+1})\|_{i,j=1}^m$ with $X = \{x_j\}_{j=1}^m \subset \mathbb{R}^{n+1}$. To be precise, given a real-valued and continuous function g on $\mathbb{R}_+$ and a finite set $X \subset \mathbb{R}^n$, denote by $\kappa^-(g, X)$ a number of negative eigenvalues of the finite Schoenberg matrix $\mathcal{S}_X(g)$, and by

$$\kappa_n^-(g) := \sup\{\kappa^-(g,X) : X \text{ runs through all finite subsets of } \mathbb{R}^n\}.$$

Certainly, $\kappa_n^-(g) = 0$ for $g \in \Phi_n$. The question is whether $\kappa_n^-(g)$ can be finite for $g \notin \Phi_n$.

Theorem 1.4. *Let g be a continuous function on $\mathbb{R}_+$ such that the limit*

$$\lim_{t\to\infty} g(t) = g(\infty) \ge 0 \tag{1.7}$$

exists. If $\kappa^-(g, Y) \ge 1$ for some set $Y \in \mathbb{R}^m$, then $\kappa_m^-(g) = +\infty$. In particular,

$$\kappa_{n+1}^-(g) = +\infty \quad \text{for each} \quad g \in \Phi_n \backslash \Phi_{n+1}, \quad n \ge 2. \tag{1.8}$$

The case $n = 1$ is more subtle, since $\Omega_1(t) = \cos t$ has no limit at infinity. Nonetheless we believe that the conclusion in (1.8) holds in this case as well.

Conjecture. *For each function $f \in \Phi_1 \setminus \Phi_2$ the relation $\kappa_2^-(f) = +\infty$ holds.*

We confirm this conjecture for $f \in \Phi_1 \backslash \Phi_2$ under certain additional assumptions on the Schoenberg measure $\nu_1(f)$ (see (5.13)).

In connection with relation (1.8) we note that functions with a *finite* number of negative squares (indefinite analogs of positive definite functions) appear naturally in various extension problems. According to Theorem 1.4 this is not the case for functions $f \in \Phi_n \backslash \Phi_{n+1}$.

We note also that indefinite analogs of positive definite and more general classes of functions have thoroughly been investigated by M. Krein and H. Langer (see [9] and references therein).

2. Functions from $\Phi_n \backslash \Phi_{n+1}$: Algebraic approach

As we mentioned above the classes Φ_n are nested, and inclusion $\Phi_{n+1} \subset \Phi_n$ is proper for any $n \in \mathbb{N}$. For some examples of functions $f \in \Phi_n \backslash \Phi_{n+1}$ see, e.g., [6, Remark 3.5], [19].

There is a simple way to show that $\Omega_n \notin \Phi_{n+1}$, which relies on some basics from the Stieltjes moment problem. We sketch the proof without getting into much detail. Assume on the contrary that $\Omega_n \in \Phi_{n+1}$, and so

$$\Omega_n(r) = \int_0^\infty \Omega_{n+1}(rt)\, \sigma(dt), \qquad \sigma(\mathbb{R}_+) = 1.$$

As the function on the left hand side is an even entire function, it is easy to see that σ has all moments finite. By using the Taylor series expansions (1.3) for both Ω_n and Ω_{n+1} we come thereby to the following moment problem:

$$s_0 = 1, \quad s_{2k} := \int_0^\infty t^{2k} \sigma(dt) = \frac{(n+1)(n+3)\dots(n+2k-1)}{n(n+2)\dots(n+2k-2)}, \qquad k = 1, 2, \dots,$$

in particular,

$$s_2 = \frac{n+1}{n}, \qquad s_4 = \frac{(n+1)(n+3)}{n(n+2)}.$$

But such a moment problem has no solution, since, e.g.,

$$\det \begin{bmatrix} s_0 & s_2 \\ s_2 & s_4 \end{bmatrix} = \frac{(n+1)(n+3)}{n(n+2)} - \left(\frac{n+1}{n}\right)^2 < 0.$$

Our goal here is to suggest a direct proof of the relation $\Omega_n \notin \Phi_{n+1}$, which avoids Schoenberg's theorem. In other words, we construct an explicit finite set $X \subset \mathbb{R}^{n+1}$ so that the Schoenberg matrix $\mathcal{S}_X(\Omega_n)$ has at least one negative eigenvalue, whereas $\mathcal{S}_Y(\Omega_n) \ge 0$ for each finite set $Y \subset \mathbb{R}^n$. Moreover, we show that there is no upper bound for the number of negative eigenvalues of $\mathcal{S}_X(\Omega_n)$ for an appropriate choice of the set X.

Let $E_{l,m}$ be an $l \times m$ matrix composed of 1's, that is, $(E_{l,m})_{ij} = 1$, $1 \le i \le l$, $1 \le j \le m$, $E_m := E_{m,m}$. It is clear that $rank\, E_m = 1$ and its spectrum $\sigma(E_m) = \{0^{(m-1)}, m\}$. Let I_p be the unit matrix of order p.

Lemma 2.1. *Let*

$$\mathcal{S} := \begin{bmatrix} A & B \\ B^* & I_k \end{bmatrix}, \tag{2.1}$$

where a $p \times p$ matrix A and a $q \times k$ matrix B are defined as

$$A = \begin{bmatrix} 1 & a & a & \dots & a \\ a & 1 & a & \dots & a \\ \vdots & & & & \vdots \\ a & a & a & \dots & 1 \end{bmatrix} = (1-a)I_p + aE_p, \qquad B = \begin{bmatrix} b & b & \dots & b \\ \vdots & & & \vdots \\ b & b & \dots & b \end{bmatrix} = bE_{p,k} \tag{2.2}$$

with real entries a, b. $\mathcal{S}$ has at least one negative eigenvalue if and only if either of two inequalities holds:

$$a > 1 \quad or \quad \lambda := 1 + (p-1)a - kpb^2 < 0. \tag{2.3}$$

Proof. A general linear algebraic identity for the block matrix $\mathcal{S}$

$$\mathcal{S} = \begin{bmatrix} I_p & B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A - BB^* & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} I_p & 0 \\ B^* & I_k \end{bmatrix} \tag{2.4}$$

shows that $\mathcal{S}$ has the same number of negative eigenvalues as the block diagonal matrix in the right hand side of (2.4) or the same number of negative eigenvalues as the matrix $A - BB^*$.

In our particular case the spectrum of the latter matrix can be computed explicitly. Indeed, as

$$E^*_{l,m} = E_{m,l}, \qquad E_{m,l}E_{l,m} = lE_m,$$

then by (2.2),

$$D = A - BB^* = (1-a)I_p + (a-kb^2)E_p, \qquad \sigma(D) = \{(1-a)^{(p-1)}, 1 + (p-1)a - kpb^2\},$$

and the result follows. □

The matrix $\mathcal{S}$ (2.1)–(2.2) (with $k = 1$) will arise as the Schoenberg matrix $\mathcal{S}_X(f)$ for a certain configuration X in $\mathbb{R}^{n+1}$.

Proposition 2.2. *Let $f \in C(\mathbb{R}_+)$ be a real-valued function, $f \le 1$, analytic at the origin with the Taylor series expansion*

$$f(z) = \sum_{j=0}^{\infty} (-1)^j a_j z^{2j}, \qquad a_0 = 1, \quad a_j > 0, \quad j \in \mathbb{N}. \tag{2.5}$$

Then $f \notin \Phi_{m+1}$ provided that

$$\frac{2m+6}{m+1}\, a_2 < a_1^2. \tag{2.6}$$

In particular, $\Omega_n \in \Phi_n \backslash \Phi_{n+1}$.

Proof. Let $X = \{x_j\}_{j=1}^{m+3}$ be a configuration in $\mathbb{R}^{m+1}$ such that the first $m+2$ points are the vertices of a regular simplex in R^{m+1} with the edge length t and let x_{m+3} be the center of this simplex. Clearly,

$$|x_i - x_j| = t, \quad i,j = 1,2,\dots,m+2, \quad |x_{m+3} - x_i| = \rho_n t, \quad \rho_m := \sqrt{\frac{m+1}{2(m+2)}}, \tag{2.7}$$

where ρ_m is the radius of the circumscribed sphere. It is easily seen that $\mathcal{S}_X(f) = \mathcal{S}$, where $\mathcal{S}$ is given by (2.1) with

$$p = m+2, \quad k = 1, \quad a = a(t) = f(t), \quad b = b(t) = f(\rho_m t).$$

From (2.5) and (2.6) we see that for small enough t,

$$\lambda_m(t) = 1 + (m+1)f(t) - (m+2)f^2(\rho_m t) = \sum_{j=1}^{\infty} (-1)^j \lambda_{m,j} t^{2j}$$

with

$$\lambda_{m,1} = 0, \quad \lambda_{m,2} = \frac{m+1}{4(m+2)}\big((2m+6)a_2 - (m+1)a_1^2\big) < 0.$$

So

$$\lambda_m(t) = \lambda_{m,2} t^4 + O(t^6) < 0$$

for small enough t, and the result follows from Lemma 2.1.

The series expansion for Ω_n is

$$\Omega_n(t) = 1 - \frac{t^2}{2n} + \frac{t^4}{8n(n+2)} + O(t^6), \qquad t \to 0,$$

so

$$a_1^2 - \frac{2n+6}{n+1}\, a_2 = \frac{1}{4n}\Big(\frac{1}{n} - \frac{n+3}{(n+1)(n+2)}\Big) = \frac{1}{2n^2(n+1)(n+2)} > 0.$$

The proof is complete. □

Remark 2.3. Proposition 2.2 applies to a wide class of functions, e.g., to the powers of the Schoenberg kernels Ω_n^p, $p \in \mathbb{N}$, but the result obtained this way is far from being optimal (at least for $p = 2$). For instance, for the squares and cubes we find

$$\frac{a_1^2(\Omega_n^2)}{a_2(\Omega_n^2)} = \frac{2n+4}{n+1}\,, \qquad \frac{a_1^2(\Omega_n^3)}{a_2(\Omega_n^3)} = \frac{6n+12}{3n+4}\,,$$

and (2.6) holds with $m \ge 2n+2$ and $m \ge 3n+4$, respectively. Hence, $\Omega_n^2 \notin \Phi_{2n+3}$ and $\Omega_n^3 \notin \Phi_{3n+5}$. As a matter of fact we show later in Theorem 4.3 that $\Omega_n^2 \notin \Phi_{2n}$.

Remark 2.4. If $X = \{x_k\}_{k=1}^{n+2}$ is the set of vertices of a regular simplex (without its center), then the corresponding Schoenberg matrix $\mathcal{S}_X(\Omega_n)$ is positive definite. Nonetheless, we conjecture that a negative eigenvalue can already be seen on a certain configuration $Y = \{y_k\}_{k=1}^{n+2}$ of $n+2$ points in $\mathbb{R}^{n+1}$.

3. When a function from the class Φ_n belongs to Φ_{n+k}

3.1. **First approach.** As a matter of fact, the class $\Phi_n \backslash \Phi_{n+1}$ is rather rich. We give certain necessary and sufficient conditions in terms of Schoenberg's measure $\nu_n = \nu_n(f)$ for f to belong to $\Phi_n \backslash \Phi_{n+1}$.

Theorem 3.1. *Let (const $\neq$)$f \in \Phi_m$, and let ν_m be its Schoenberg measure. Then $f \in \Phi_{m+k}$, $k \in \mathbb{N}$, if and only if there is a finite positive Borel measure ν on $\mathbb{R}_+$ of the total mass 1 such that*

$$\nu_m(dx) = p_m(x)\,dx, \qquad p_m(x) = \frac{2x^{m-1}}{B\big(\frac{m}{2}, \frac{k}{2}\big)} \int_x^{\infty} \Big(1 - \frac{x^2}{u^2}\Big)^{\frac{k}{2}-1} \frac{\nu(du)}{u^m}. \tag{3.1}$$

In this case $\nu = \nu_{m+k}$ with ν_{m+k} being the Schoenberg representing measure of f as a function from Φ_{m+k}. In particular, ν_m is absolutely continuous and $\nu_m\{(0,\varepsilon)\} > 0$ for any $\varepsilon > 0$.

Proof. Assume that $f \in \Phi_{m+k}$. Then f admits two representations:

$$f(r) = \int_0^\infty \Omega_m(ru)\nu_m(du) = \int_0^\infty \Omega_{m+k}(ru)\nu_{m+k}(du). \tag{3.2}$$

It is not hard to obtain a relation between the measures ν_m and ν_{m+k}. Indeed, recall Sonine's integral [1, formula (4.11.11), p. 218]:

$$J_\mu(t) = \frac{2}{\Gamma(\mu-\lambda)}\left(\frac{t}{2}\right)^{\mu-\lambda}\int_0^1 J_\lambda(ts)s^{\lambda+1}(1-s^2)^{\mu-\lambda-1}\,ds, \qquad \mu > \lambda \ge -\frac{1}{2}.$$

For the values

$$\lambda = \frac{m}{2}-1, \qquad \mu = \frac{m+k}{2}-1 = \lambda + \frac{k}{2}$$

one has in terms of Ω's

$$\Omega_{m+k}(t) = \frac{2}{B\left(\frac{m}{2},\frac{k}{2}\right)}\int_0^1 \Omega_m(ts)s^{m-1}(1-s^2)^{\frac{k}{2}-1}\,ds. \tag{3.3}$$

We plug the latter equality into (3.2) to obtain

$$\begin{aligned} f(r) &= \frac{2}{B\left(\frac{m}{2},\frac{k}{2}\right)}\int_0^\infty \nu_{m+k}(du)\int_0^1 \Omega_m(rus)s^{m-1}(1-s^2)^{\frac{k}{2}-1}\,ds \\ &= \frac{2}{B\left(\frac{m}{2},\frac{k}{2}\right)}\int_0^\infty \frac{\nu_{m+k}(du)}{u^m}\int_0^u \Omega_m(rx)x^{m-1}\left(1-\frac{x^2}{u^2}\right)^{\frac{k}{2}-1}dx \\ &= \frac{2x^{m-1}}{B\left(\frac{m}{2},\frac{k}{2}\right)}\int_0^\infty \Omega_m(rx)dv\int_x^\infty \left(1-\frac{x^2}{u^2}\right)^{\frac{k}{2}-1}\frac{\nu_{m+k}(du)}{u^m}. \end{aligned}$$

Due to the uniqueness of Schoenberg's representation we arrive at (3.1).

Conversely, starting from (3.1) and reversing the argument we come to (3.2) with $\nu_{m+k} = \nu$.

Since $f \ne \text{const}$, $\nu_{m+k} \ne \delta_0$, we see that ν_m is absolutely continuous and $\nu_m\{(0,\varepsilon)\} > 0$ for any $\varepsilon > 0$, as claimed. □

Remark 3.2. A closely related result is obtained in [16, Theorem 6.3.5], where a certain condition for $f \in \Phi_n$ to belong to Φ_m with $m > n$ is given in terms of f itself.

We call (3.1) the *k-step transition formula.*

We can paraphrase the statement of Theorem 3.1 as follows: $f \in \Phi_m \backslash \Phi_{m+1}$ as long as ν_m is either not purely absolutely continuous or $\nu_m\{(0,\varepsilon)\} = 0$ for some $\varepsilon > 0$.

Corollary 3.3. *Let $f \in \Phi_m$, and let its Schoenberg measure ν_m be a pure point one. Then $f \in \Phi_m \setminus \Phi_{m+1}$. In particular,*

$$f(t) = \sum_{k=1}^\infty \alpha_k \Omega_n(r_k t) \in \Phi_n \backslash \Phi_{n+1} \tag{3.4}$$

for any sequence of nonnegative numbers $\{r_k\}_{k\ge1}$ and $\{\alpha_k\}_{k\ge1} \in l^1(\mathbb{N})$.

Example 3.4. Let

$$f_1(r) = e^{-r}, \qquad f_2(r) = e^{-r^2}.$$

Both functions belong to Φ_n for all $n \in \mathbb{N}$, and their Schoenberg measures are known explicitly (see, e.g., [15, Chapter 1]):
(3.5)

$$\nu_m(f_1) = \frac{2}{B\big(\frac{m}{2}, \frac{1}{2}\big)}\, \frac{u^{m-1}}{(1+u^2)^{\frac{m+1}{2}}}\, du, \qquad \nu_m(f_2) = \frac{1}{\Gamma(\frac{m}{2})} \Big(\frac{u}{2}\Big)^{m-1} \exp\Big\{-\frac{u^2}{4}\Big\}\, du.$$

It is a matter of simple (though lengthy) computations to verify formula (3.1) for each of these sequences of measures for all $m, k \in \mathbb{N}$.

Let us single out the simplest case $k = 2$.

Corollary 3.5. *$f \in \Phi_m$ belongs to Φ_{m+2} if and only if there is a finite positive Borel measure ν on $\mathbb{R}_+$ of the total mass 1 such that*

$$\nu_m(dx) = p_m(x)\, dx, \qquad p_m(x) = m x^{m-1} \int_x^\infty \frac{\nu(du)}{u^m}, \quad \nu = \nu_{m+2}. \tag{3.6}$$

The problem we address now concerns the smoothness properties and the rate of decay of measures ν_m in (3.1) for the case $k = 2j$, $j \in \mathbb{N}$.

We start with the following auxiliary statement. Let $C^k(\mathbb{R}_+)$ denote the Freshet space of k-smooth continuous functions defined on an open set $\mathbb{R}_+ = (0, \infty)$, and let $AC[0, a]$ be the space of absolutely continuous functions on $[0, a]$. We also put

$$AC_{\rm loc}(\mathbb{R}_+) := \{f \in AC[0,a]\ \forall a > 0\},$$
$$AC^k_{\rm loc}(\mathbb{R}_+) := \{f \in C^{k-1}(\mathbb{R}_+) :\ f^{(k-1)} \in AC_{\rm loc}(\mathbb{R}_+)\}.$$

Lemma 3.6. *Let σ be a function of bounded variation on $\mathbb{R}_+$, $\varphi(x,\cdot) \in AC_{\rm loc}(\mathbb{R}_+)$ for each $x \in \mathbb{R}_+$, $D_t\varphi(\cdot, t) \in C^k(0,\infty)$ for each $t \in \mathbb{R}_+$, and let*

$$g(x) := \int_x^\infty \varphi(x,t) d\sigma(t). \tag{3.7}$$

Assume also that there exist functions $\zeta_0 \in L^1([\varepsilon,\infty); d\sigma)$ and $\psi_j \in L^1[\varepsilon,\infty)$ for each $\varepsilon > 0$, and such that

$$|\varphi(x,t)| \le \zeta_0(t), \quad |D_x^j D_t \varphi(x,t)| \le \psi_j(t), \quad x \in \mathbb{R}_+, \quad j = 0, 1, \dots, k-1. \tag{3.8}$$

Moreover, assume that

$$\varphi(x,x) = 0 \quad \text{and} \quad D_x^j D_t\varphi(x,t)|_{t=x} = 0, \quad j0, 1, \dots, k-2, \tag{3.9}$$

and $\lim_{t\to\infty} \varphi(x,t)\sigma(t) = 0$ for each $x \in \mathbb{R}_+$. Then $g \in C^k(\mathbb{R}_+)$. If in addition

$$\lim_{t\to\infty} D_x^k \varphi(x,t) = 0 \quad \text{for each } x \in \mathbb{R}_+, \tag{3.10}$$

then

$$g^{(k)}(x) = \sigma(x)\big([D_x^{k-1}(D_x + D_t)\varphi(x,t)]|_{t=x}\big) + \int_x^\infty D_x^k \varphi(x,t) d\sigma(t). \tag{3.11}$$

Proof. Since $\varphi(x,\cdot) \in AC_{\rm loc}(\mathbb{R}_+)$ and $\varphi(x,\cdot) \in L^1([x,\infty); d\sigma)$ for each $x \in \mathbb{R}_+$, one gets after integrating by parts with account of the first relation in (3.9)

$$g(x) = -\varphi(x,x)\sigma(x) - \int_x^\infty \varphi'_t(x,t)\sigma(t) dt = -\int_x^\infty \varphi'_t(x,t)\sigma(t)dt. \tag{3.12}$$

On the other hand, since σ is bounded, $\psi_j\sigma \in L^1[\varepsilon,\infty)$ for each $\varepsilon > 0$ and $j = 0, 1, \dots, k-1$, and according to (3.8)

$$(D_x^j D_t \varphi(x,t))\sigma(t) \le \psi_j(t)\sigma(t), \quad x \in \mathbb{R}_+.$$

Therefore one can differentiate (3.12) subsequently with account of (3.9) to obtain by induction that $g \in C^j(0,\infty)$ and

$$g^{(j)}(x) = -\int_x^\infty D_x^j D_t \varphi(x,t)\sigma(t)dt, \quad j \in \{0,1,\dots,k-1\}, \tag{3.13}$$

and

$$g^{(k)}(x) = \sigma(x)\left(D_x^{k-1} D_t \varphi(x,t)|_{t=x}\right) - \int_x^\infty D_x^k D_t \varphi(x,t)\sigma(t)dt. \tag{3.14}$$

Integrating identity (3.14) by parts and taking (3.10) into account we arrive at (3.11). ☐

Remark 3.7. Note that we do not assume the majorant $\zeta_0 \in L^1[\varepsilon,\infty)$ for each $\varepsilon > 0$. The existence of the Lebesgue integral in (3.12) is implied by estimate (3.8) with $j = 0$.

Corollary 3.8. *Assume that* $\varphi_0 \in AC^k[0,1]$ *and*

$$\varphi(x,t) := \frac{1}{t}\varphi_0\left(\frac{x}{t}\right), \ t \ge x, \quad \textit{and} \quad \varphi_0(1) = \varphi_0'(1) = \ldots = \varphi_0^{(k-1)}(1) = 0. \tag{3.15}$$

Let also $\sigma(\cdot)$ *be of bounded variation and let* $g(\cdot)$ *be given by* (3.7)*. Then*

$$g^{(k)}(x) = \int_x^\infty D_x^k \varphi(x,t) d\sigma(t). \tag{3.16}$$

Proof. Clearly, conditions (3.9) are implied by conditions (3.15). Moreover, estimates (3.8) hold with $\psi_j(t) = C_j t^{-(j+1)}$ and $C_j = \|\varphi_0^{(j)}\|_{C[0,1]}$, $j \in \{1,\dots,k\}$. Besides, $|\varphi(x,t)| \le C_0 t^{-1}$, where $C_0 = \|\varphi_0\|_{C[0,1]}$, hence $\varphi(x,t) \in L^1([x,\infty); d\sigma)$ for each $x \in \mathbb{R}_+$.

Further, it is easily seen that

$$(D_x + D_t)D_x^{k-1}\varphi(x,t) = (D_x + D_t)\left(\frac{1}{t^k}\varphi_0^{(k-1)}\left(\frac{x}{t}\right)\right)$$

$$= (-1)^{k-1}\left[\frac{1}{t^{k+1}}\varphi_0^{(k)}\left(\frac{x}{t}\right)\left[1 - \frac{x}{t}\right] - \frac{k}{t^{k+1}}\varphi_0^{(k-1)}\left(\frac{x}{t}\right)\right]. \tag{3.17}$$

It follows by taking (3.15) into account that

$$[D_x^{k-1}(D_x + D_t)\varphi(x,t)]|_{t=x} = (-1)^k \frac{k}{x^{k+1}}\varphi_0^{(k-1)}(1) = 0. \tag{3.18}$$

Thus conditions of Lemma 3.6 are met and (3.16) follows by combining (3.11) with (3.18). ☐

Recall a classical result of Hardy, Littlewood and Pólya [7, Theorem 319].

Theorem HLP. *Let* T *be a measurable function on* $\mathbb{R}_+^2$ *such that it is homogeneous of the degree* -1*, that is,* $T(\lambda x, \lambda t) = \lambda^{-1}T(x,t)$ *for* $\lambda > 0$*, and for some* p*,* $1 \le p < \infty$*,*

$$\tau_p := \int_0^\infty |T(1,t)| t^{-1/p}\, dt < \infty. \tag{3.19}$$

Then the integral operator generated by T*,*

$$(Th)(x) := \int_0^\infty T(x,t)h(t)\, dt,$$

is bounded in $L^p(\mathbb{R}_+)$*, and its norm is* $\|T\| \le \tau_p$*.*

A typical example which will be of particular interest for us is

$$T(x,t) = \begin{cases} \frac{1}{t}\,T\big(\frac{x}{t}\big), & t \ge x; \\ 0, & t < x, \end{cases} \tag{3.20}$$

where T is a polynomial. Now

$$\tau_p = \int_0^1 |T(u)|u^{1/p-1}\,du < \infty, \quad p \ge 1.$$

Theorem 3.9. *Let* (const $\ne$)$f \in \Phi_m$ *with Schoenberg's measure ν_m, $m \in \mathbb{N}$. Assume that $f \in \Phi_{m+2j}$, $j \in \mathbb{N}$, with Schoenberg's measure $\nu = \nu_{m+2j}$. Then the following relations hold:*

(i) $\nu_m \in AC^j_{\rm loc}(\mathbb{R}_+)$.

(ii) *If ν is absolutely continuous, $\nu = p_{m+2j}\,dx$, and $p_{m+2j} \in L^p(\mathbb{R}_+)$ for some $1 \le p < \infty$, then*

$$x^k p_m^{(k)}(x) \in L^p(\mathbb{R}_+), \qquad k = 0,1,\dots,j-1. \tag{3.21}$$

Moreover,

$$p_m^{(k)}(x) = o\left(\frac{1}{x^{k+1}}\right), \quad x \to \infty, \qquad k = 0,1,\dots,j-1. \tag{3.22}$$

(iii) *Let ν satisfy*

$$I := \int_0^\infty \frac{\nu(du)}{u} < \infty. \tag{3.23}$$

Then

$$p_m(0) = \lim_{x\to 0+} p_m(x) = 0, \quad m \ge 2, \qquad p_1(0) = I. \tag{3.24}$$

Furthermore,

$$x^k p_m^{(k)}(x) \in L^\infty(\mathbb{R}_+), \qquad k = 0,1,\dots j-1. \tag{3.25}$$

Proof. (i) For $m, j \in \mathbb{N}$ we put

$$\begin{aligned} P_{m,j}(u) &:= \frac{u^{m-1}(1-u^2)^{j-1}}{B\big(\frac{m}{2},j\big)}, \\ Q_{m,j}(x,t) &:= \frac{1}{t}\,P_{m,j}\left(\frac{x}{t}\right) = \frac{2x^{m-1}}{B\big(\frac{m}{2},j\big)t^m}\left(1-\frac{x^2}{t^2}\right)^{j-1}, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &Q_{m,j,k}(x,t) := x^k D_x^k Q_{m,j}(x,t) = x^k t^{-k-1} P_{m,j}^{(k)}\left(\frac{x}{t}\right), \quad k \in \{1,2,\dots,j-1\}, \\ &Q_{m,j,0} = Q_{m,j}. \end{aligned} \tag{3.27}$$

It is easily seen that the kernel $\varphi = Q_{m,j}$ is homogeneous of degree -1 and meets the hypothesis of Corollary 3.8. So, Corollary 3.8 applied to relation (3.1) provides $p_m \in AC^{j-1}_{\rm loc}(\mathbb{R}_+)$ and

$$x^k p_m^{(k)}(x) = \int_x^\infty Q_{m,j,k}(x,t)\,\nu(dt), \qquad k = 0,1,\dots,j-1. \tag{3.28}$$

(ii) It is clear that the kernels $Q_{m,j,k}$, $k \in \{0,\dots,j-1\}$, are homogeneous of degree -1, and (3.19) holds for all $1 \le p < \infty$. By the HLP theorem, the integral operators

$$(Q_{m,j,k}h)(x) = \int_x^\infty Q_{m,j,k}(x,t)h(t)\,dt, \qquad k = 0,\dots,j-1,$$

are bounded in $L^p(\mathbb{R}_+)$ for all $1 \le p < \infty$. The latter relation with $h = p_{m+2j}$ leads directly to (3.21) in view of (3.28). Note that $p_{m+2j} \in L^1(\mathbb{R}_+)$ (so (3.21) holds automatically for $p = 1$) since ν is a finite measure.

Next, it easily follows from (3.26) and (3.27) that

$$\sum_{k=0}^{j-1} |Q_{r,m,k}(x,t)| \le \frac{C_{m,j}}{t}, \quad t \ge x, \tag{3.29}$$

and since the measure ν is finite, we have from (3.28)

$$|x^k p_m^{(k)}(x)| \le C_{m,j} \int_x^\infty \frac{\nu(dt)}{t} \le \frac{C_{m,j}}{x} \int_x^\infty \nu(dt) = o\left(\frac{1}{x}\right), \quad x \to \infty, \tag{3.30}$$

for $1 \le k \le j-2$. For $k = j-1$ the estimate is a consequence of (3.29), (3.26) and (3.28), and so (3.22) follows.

(iii) Limit relations (3.24) follow easily from (3.1), the bound

$$0 < \left(1 - \frac{x^2}{t^2}\right)^{\frac{k}{2}-1} \left(\frac{x}{t}\right)^{m-1} < 1, \qquad t \ge x,$$

and the Dominated Convergence Theorem in view of assumption (3.23).

Relations (3.25) arise directly from (3.24) and (3.22). The proof is complete. □

We turn now to the case $k = \infty$ in Theorem 3.1. Recall that

$$\Phi_\infty := \bigcap_{m=1}^\infty \Phi_m.$$

A theorem of Schoenberg states that $f \in \Phi_\infty$ if and only if there is a positive measure σ on $\mathbb{R}_+$ with the total mass 1 such that

$$f(r) = \int_0^\infty e^{-sr^2}\,\sigma(ds). \tag{3.31}$$

Proposition 3.10. *Let* (const $\neq$)$f \in \Phi_m$*, and let* ν_m *be its Schoenberg measure. Then* $f \in \Phi_\infty$ *if and only if there is a finite positive Borel measure* σ *on* $\mathbb{R}_+$ *of the total mass* 1 *such that*

(3.32)

$$\nu_m(dx) = p_{m,\sigma}(x)\,dx, \qquad p_{m,\sigma}(x) := \frac{x^{m-1}}{2^{\frac{m}{2}-1}\Gamma(\frac{m}{2})} \int_0^\infty (2s)^{-m/2} \exp\left(-\frac{x^2}{4s}\right)\sigma(ds).$$

The density $p_{m,\sigma}$ *can be extended as an analytic function to the sector* $\{|\arg x| < \frac{\pi}{4}\}$.

Proof. We argue in the same manner as in Theorem 3.1. The only difference is that we use (3.31) instead of Schoenberg's representation in Φ_{m+k} and the equality

$$e^{-sr^2} = \frac{1}{2^q\Gamma(q+1)} \int_0^\infty \Omega_n(ru) \frac{u^{n-1}}{(2s)^{n/2}} \exp\left(-\frac{u^2}{4s}\right) du$$

in place of Sonine's integral. □

Remark 3.11. Let $f \in \Phi_m$ with Schoenberg's measure ν_m (3.32). It is clear that (3.31) holds then for f. The following problem arises naturally. Given $f \in \Phi_\infty$, is it possible to obtain (3.32) as a limit case of (3.1) as $k \to \infty$, proving thereby the result of Schoenberg (3.31)?

3.2. **Second approach.**

Proposition 3.12. *Let $f \in \Phi_m$ with $m \ge 3$; i.e., f admits representation* (1.2) *with the measure σ_m. Then it admits the representation*

$$f(r) = \int_0^\infty \Omega_{m-2}(rx)\,\sigma_{m-2}(dx) \tag{3.33}$$

with the measure σ_{m-2} given by

$$\sigma_{m-2}(x) = \frac{1}{2}\int_0^x u^{m-3}du \int_u^\infty \frac{\sigma_m(dt)}{t^{m-2}}\,dx. \tag{3.34}$$

Conversely, if f admits representation (3.33) *with σ_{m-2} of the form* (3.34)*, then $f \in \Phi_m$.*

Proof. Our considerations rely on the following identity [18, Section III.3.2]:

$$\frac{d}{dx}\Big((rx)^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)\Big) = r(rx)^{\frac{m-2}{2}} J_{\frac{m-4}{2}}(rx). \tag{3.35}$$

We let

$$\widehat{\sigma}_m(x) = -\int_x^\infty \frac{\sigma_m(dt)}{t^{m-2}} \le 0. \tag{3.36}$$

Recall that

$$J_\nu(x) = O(x^\nu), \quad x \to 0 \quad \text{and} \quad J_\nu(x) = O\Big(\frac{1}{\sqrt{x}}\Big), \quad x \to \infty. \tag{3.37}$$

Using the first of these relations and applying the Dominated Convergence Theorem to the function $(x/t)^{m-2}$ as $x \to 0$ with the majorant $1 \in L^1(\mathbb{R}_+, d\sigma_m)$, we get from (3.36)

$$\begin{aligned} &\lim_{x\to 0} x^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)\widehat{\sigma}_m(x) = \lim_{x\to 0} x^{m-2}\widehat{\sigma}_m(x) \\ &= -\lim_{x\to 0}\int_x^\infty \Big(\frac{x}{t}\Big)^{m-2}\sigma_m(dt) = 0, \qquad m \ge 2. \end{aligned} \tag{3.38}$$

Further, $\widehat{\sigma}_m(x) = o(\frac{1}{x^{m-2}})$. Therefore,

$$\lim_{x\to\infty} x^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)\widehat{\sigma}_m(x) = \lim_{x\to\infty} x^{\frac{m-3}{2}}\widehat{\sigma}_m(x) = 0. \tag{3.39}$$

Transformation of (1.2) in view of (3.36) and integration by parts taking (3.35), (3.38), and (3.39) into account give

(3.40)

$$\begin{aligned} f(r) &= \int_0^\infty \left(\frac{2}{rx}\right)^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)\sigma_m(dx) = \frac{2^{\frac{m-2}{2}}}{r^{m-2}} \int_0^\infty \frac{(rx)^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)}{x^{m-2}} \sigma_m(dx) \\ &= \frac{2^{\frac{m-2}{2}}}{r^{m-2}} \int_0^\infty (rx)^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)\widehat{\sigma}_m(dx) \\ &= \frac{2^{\frac{m-2}{2}}}{r^{m-2}} (rx)^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)\widehat{\sigma}_m(x)\Big|_0^\infty \\ &\quad - \frac{2^{\frac{m-2}{2}}}{r^{m-2}} \int_0^\infty \widehat{\sigma}_m(x)\frac{d}{dx}\big((rx)^{\frac{m-2}{2}} J_{\frac{m-2}{2}}(rx)\big)dx \\ &= -\frac{2^{\frac{m-2}{2}}}{r^{m-3}} \int_0^\infty r^{m-3}\left[(rx)^{\frac{4-m}{2}} J_{\frac{m-4}{2}}(rx)\right] x^{m-3}\widehat{\sigma}_m(x)\,dx. \end{aligned}$$

Setting

$$\sigma_{m-2}(x) := -\frac{1}{2}\int_0^x t^{m-3}\widehat{\sigma}_m(t)dt \ (\geq 0), \tag{3.41}$$

we note that σ_{m-2} is a nonnegative increasing bounded function. Indeed,

$$\begin{aligned} \sigma_{m-2}(\infty) &= \frac{1}{2}\int_0^\infty x^{m-3}dx \int_x^\infty \frac{\sigma_m(dt)}{t^{m-2}} \\ &= \frac{1}{2}\int_0^\infty \frac{\sigma_m(dt)}{t^{m-2}} \int_0^t x^{m-3}dx = \frac{1}{2(m-2)}\int_0^\infty \sigma_m(dt) < \infty. \end{aligned} \tag{3.42}$$

Therefore we can rewrite representation (3.40) in the form (3.33). □

Corollary 3.13. *Let $f \in \Phi_1$, i.e.,*

$$f(r) = \int_0^\infty \cos(rx)d\sigma_1(x). \tag{3.43}$$

Then $f \in \Phi_3$ if and only if σ_1 is upper convex. In the latter case it admits a representation

$$f(r) = \int_0^\infty \Omega_3(rx)d\sigma_3(x) = \int_0^\infty \frac{\sin(rx)}{rx}d\sigma_3(x) \tag{3.44}$$

where

$$\sigma_3(x) = \sigma_3(0) - \int_0^x td\sigma_1'(t)d. \tag{3.45}$$

In particular, σ_3 is a monotonically decreasing, bounded function, which is absolutely continuous with respect to σ_1'. In fact, the measures $d\sigma_3$ and $d\sigma_1'$ are equivalent: $d\sigma_1'(x) = -\frac{1}{2x}d\sigma_3(x)$.

Proof. Necessity. It follows from (3.41) with $m = 3$ that

$$\sigma_1(x) = \frac{1}{2}\int_0^x \big(-\widehat{\sigma}_3(t)\big)dt. \tag{3.46}$$

On the other hand, according to (3.36) $\widehat{\sigma}_3(\geq 0)$ decreases. Hence $\sigma_1(\cdot)$ is upper convex (see [10]).

Sufficiency. Let σ_1 in representation (3.43) be upper convex. Then it is locally absolutely continuous, hence admits a representation $\sigma_1(x) = \sigma_1(0) + \int_0^x d\sigma_1'(t)dt$ with an increasing function $\sigma_1'(\cdot)$ (see [10]). Since σ_1 monotonically increases and is bounded, the Fatou theorem applies and gives $\int_0^\infty \sigma_1'(x)dx \le \sigma_1(+\infty) - \sigma_1(0) < \infty$. Using decaying of σ_1' we get

$$0 \le \lim_{x\to\infty} x\sigma_1'(x) \le \lim_{x\to\infty} 2\int_{x/2}^x \sigma_1'(t)dt = 0. \tag{3.47}$$

Integration by parts in (3.43) in view of (3.47), definition (3.45), and the inclusion $\sigma_1 \in AC_{\rm loc}(\mathbb{R}_+)$ gives

$$\begin{aligned} f(r) &= \int_0^\infty \cos(rx)\sigma_1'(x)dx = \frac{1}{r}\int_0^\infty \sigma_1'(x)d\sin(rx) \\ &= \frac{\sin rx}{r}\sigma_1'(x)\Big|_0^\infty - \int_0^\infty \frac{\sin(rx)}{rx} x d\sigma_1'(x) = \int_0^\infty \frac{\sin(rx)}{rx} d\sigma_3(x). \end{aligned} \tag{3.48}$$

It remains to show that $\sigma_3(\infty) < \infty$. Integrating by parts in (3.45) and tending $x \to \infty$, by taking (3.47) into account, we derive

$$\sigma_3(\infty) - \sigma_3(0) = -\int_0^\infty t d\sigma_1'(t) = -t\sigma_1'(t)\Big|_0^\infty + \int_0^\infty \sigma_1'(t)dt \le \sigma_1(\infty) - \sigma_1(0) < \infty.$$

Thus $\sigma_3(\cdot)$ is bounded and the inclusion $f \in \Phi_3$ is proved. □

Remark 3.14. In the case $k = 2$ equation (3.34) is equivalent to the differential equation

$$\frac{d}{d\sigma_m}\left(\frac{1}{x^{m-3}}\frac{d}{dx}\right)\sigma_{m-2}(x) = -\frac{1}{2x^{m-2}}, \quad m \in \mathbb{N}, \tag{3.49}$$

subject to certain boundary conditions. For $m = 3$ this equation is just Krein's string equation with respect to the unknown monotone function σ_{m-2} and the mass distribution function σ_m.

3.3. **How reachable is the set of transition measures?** We put a function $f \in \Phi_m$ into the subclass Φ_m^{ac} if its Schoenberg representing measure $\nu_m = \nu_m(f)$ is absolutely continuous $\nu_m(dx) = p_m(x)\,dx$, $p_m(\cdot) \in L^1(\mathbb{R}_+)$. We equip Φ_m^{ac} with the L^1–norm by setting

$$\|\nu_m\|_{\Phi_m^{ac}} := \|p_m\|_{L^1(\mathbb{R}_+)}. \tag{3.50}$$

Clearly, the set Φ_m^{ac} constitutes a "positive" conus in $L^1(\mathbb{R}_+)$. In accordance with Theorem 3.1 for each $k \in \mathbb{N}$ the embedding $\Phi_{m+k} \hookrightarrow \Phi_m^{ac}$ holds.

In what follows we identify the representing measure $\nu_m = \nu_m(f)$ of $f \in \Phi_{m+k}$ as a function from Φ_m, with its density p_m.

Denote by $M_{m,m+k}$ the set of (necessarily absolutely continuous) representing measures of functions $f \in \Phi_{m+k}$ considered as functions from Φ_m.

Proposition 3.15. *The set $M_{m,m+k}$ forms a closed nowhere dense subset in Φ_m^{ac}.*

Proof. (i) Let us prove that $M_{m,m+k}$ is a closed subset of Φ_m^{ac} assuming for simplicity that $k = 2$. Let $f_j \in \Phi_{m+k} \subset \Phi_m^{ac}$, $j \in \mathbb{N}$, and let $\sigma_{m,j}(dx) = p_{m,j}(x)dx$ be a sequence of the corresponding representing Schoenberg measures (see (1.2)).

Assume that the sequence $\{p_{m,j}(\cdot)\}_{j\in\mathbb{N}}$ converges in $L^1(\mathbb{R}_+)$; i.e., there exists $\widetilde{p}_m(\cdot) \in L^1(\mathbb{R}_+)$ such that

$$\lim_{j\to\infty} \|p_{m,j} - \widetilde{p}_m\|_{L^1(\mathbb{R}_+)} = 0. \tag{3.51}$$

This relation ensures existence of a subsequence p_{m,j_k} such that

$$\lim_{k\to\infty} p_{m,j_k}(x) = \widetilde{p}_m(x) \quad \text{for a.e.} \quad x \in \mathbb{R}_+, \tag{3.52}$$

hence $\widetilde{p}(\cdot) \in L^1_+(\mathbb{R}_+)$. It follows from (3.51) that

$$\sigma_{m,j}(x) = \int_0^x p_{m,j}(t)dt \to \int_0^x \widetilde{p}_m(t)dt =: \widetilde{\sigma}_m(x) \quad \text{as } j \to \infty, \quad x \in \mathbb{R}_+. \tag{3.53}$$

Clearly, $\widetilde{\sigma}_m(\cdot)$ is a bounded monotonically increasing function.

Further, according to representation (3.34),

$$\begin{aligned} \sigma_{m,j}(x) &= \frac{1}{2}\int_0^x u^{m-1}du \int_u^\infty \frac{\sigma_{m+2,j}(dt)}{t^m} \\ &= \frac{1}{2(m-2)}\left[\int_0^x \sigma_{m+2,j}(dt) + \int_x^\infty \left(\frac{x}{t}\right)^m \sigma_{m+2,j}(dt)\right]. \end{aligned} \tag{3.54}$$

Moreover, (3.51) ensures the existence of $N \in \mathbb{N}$ such that

$$\|p_{m,j}\|_{L^1(\mathbb{R}_+)} \le \|\widetilde{p}_m\|_{L^1(\mathbb{R}_+)} + 1 \quad \text{for } j \ge N.$$

Combining this estimate with (3.54) and assuming for definiteness that $\sigma_{m+2,j}(0) = 0, j \in \mathbb{N}$, we derive

$$\sigma_{m+2,j}(x) \le 2(m-2)\sigma_{m,j}(x) \le \|\widetilde{p}_m\|_{L^1(\mathbb{R}_+)} + 1, \quad x \in \mathbb{R}_+, \quad \text{for} \quad j \ge N. \tag{3.55}$$

So, the system $\{\sigma_{m+2,j}(\cdot)\}_{j=1}^\infty$ of monotone functions is uniformly bounded, and according to the second Helly theorem, there exist a subsequence $\{\sigma_{m+2,j_k}(\cdot)\}_{k=1}^\infty$ and a (necessarily bounded) monotone function $\widetilde{\sigma}_{m+2}(\cdot)$ such that

$$\lim_{k\to\infty} \sigma_{m+2,j_k}(x) = \widetilde{\sigma}_{m+2}(x) \quad \text{for each} \quad x \in \mathbb{R}_+. \tag{3.56}$$

Note that for each fixed $x \in \mathbb{R}_+$ one has $\lim_{t\infty}(\frac{x}{t})^m = 0$; hence the function $(x/t)^m$ is continuous at infinity. Therefore, by the second Helly theorem, one can pass to the limit in (3.54) as $j_k \to \infty$ to obtain

$$\begin{aligned} \widetilde{\sigma}_m(x) &= \frac{1}{2(m-2)}\left[\int_0^x \widetilde{\sigma}_{m+2}(dt) + \int_x^\infty \left(\frac{x}{t}\right)^m \widetilde{\sigma}_{m+2}(dt)\right] \\ &= \frac{1}{2}\int_0^x u^{m-1}du \int_u^\infty \frac{1}{t^m}\widetilde{\sigma}_{m+2}(dt). \end{aligned} \tag{3.57}$$

The latter means that $\widetilde{\sigma}_m(\cdot) \in M_{m,m+2}$ and the set $M_{m,m+2}$ is closed in $L^1_+(\mathbb{R}_+)$.

(ii) Let us show that $M_{m,m+2}$ has no interior points. Indeed, any fixed $p_m(\cdot) \in M_{m,m+2}$ admits representation (3.6); hence the function $p_m(x)/x^{m-1}$ *necessarily* monotonically decreases. For any $\varepsilon > 0$ and $x_0 \in \mathbb{R}_+$ we put

$$\widetilde{p}_m(x) := \begin{cases} p_m(x), & x \in \mathbb{R}_+ \setminus (x_0, x_0+1), \\ \varphi_\varepsilon(x), & x \in (x_0, x_0+1). \end{cases}$$

Moreover, we choose $\varphi_\varepsilon(\cdot)$ in such a way that $\widetilde{p}_m(x)/x^{m-1}$ is not monotonically decreasing and

$$\|\widetilde{p}_m - p_m\|_{L^1(\mathbb{R}_+)} \le \varepsilon.$$

Clearly, $\widetilde{p}(\cdot) \notin M_{m,m+2}$. In other words, in any small neighborhood of $p_m(\cdot) \in M_{m,m+2}$ there exist points not belonging to $M_{m,m+2}$; i.e., $p_m(\cdot)$ is not an interior point. Since $p_m(\cdot) \in M_{m,m+2}$ is arbitrary, $M_{m,m+2}$ has no interior points. ☐

4. Products of Schoenberg's kernels

We demonstrate here a power and capability of the transition formula from Theorem 3.1 by applying it to products and squares of the Schoenberg kernels Ω_n.

Theorem 4.1 (= Theorem 1.3(i))**.** *For any pair of points $\{a, b\} \subset \mathbb{R}_+$, $a \neq b$, we have the function*

$$\Omega_n(a, b; t) := \Omega_n(at) \cdot \Omega_n(bt) \in \Phi_n \backslash \Phi_{n+1}.$$

Proof. The case $ab = 0$ is trivial, so we let $a, b > 0$. Clearly, $\Omega_n(a, b) \in \Phi_n$ since the Schur (entrywise) product of two nonnegative matrices is again a nonnegative matrix. Next, by (1.4) $\Omega_n(a, b; |\cdot|)$ is the Fourier transform of the convolution of the Lebesgue measures on the spheres S_a^{n-1} and S_b^{n-1}. As is well known (see, e.g., [11, Theorem 6.2.3]) the support of a convolution equals a closure of the algebraic sum of the supports for the components, so the support of the convolution in question is the spherical annulus $\{x \in \mathbb{R}^n : |a - b| \leq |x| \leq a + b\}$. The Schoenberg measure of $\Omega_n(a, b; |\cdot|)$ comes up as the spherical projection of the above convolution of two Lebesgue measures on the spheres, so its support is the interval $[|a-b|, a+b]$, disjoint from the origin (for an explicit expression of this measure see [12]). An application of Theorem 3.1 completes the proof. ☐

Remark 4.2. Given two functions $f_1, f_2 \in \Phi_n \backslash \Phi_{n+1}$, let their Schoenberg measures $\nu(f_1)$ and $\nu(f_2)$ have disjoint supports

$$\operatorname{supp} \nu(f_j) \subset [a_j, b_j], \quad j = 1, 2, \qquad a_1 < b_1 < a_2 < b_2.$$

The same argument shows that $f_1 f_2 \in \Phi_n \backslash \Phi_{n+1}$.

The case $a = b$ is much more delicate.

Theorem 4.3 (= Theorem 1.3(ii))**.** $\Omega_n^2 \in \Phi_{2n-1} \backslash \Phi_{2n}$, $n \in \mathbb{N}$.

Proof. Assume first that $n \geq 2$. We begin with the known formula from the Hankel transforms theory (see, e.g., [3, (22), p. 24]):

$$\int_0^2 J_\nu(tx)\sqrt{tx}\, \frac{dx}{\sqrt{x(4 - x^2)}} = \frac{\pi}{2}\sqrt{t}\, J_{\nu/2}^2(t), \qquad \nu > -1. \tag{4.1}$$

Now put $\nu = n - 2$, so in terms of Ω's,

$$\Omega_n(y) = \Gamma\Big(\frac{n}{2}\Big) \left(\frac{2}{y}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(y), \qquad \Omega_{2n-2}(y) = \Gamma(n-1) \left(\frac{2}{y}\right)^{n-2} J_{n-2}(y),$$

and hence

$$\Omega_n^2(t) = \int_0^2 \Omega_{2n-2}(tx)\, p_{2n-2}(x) dx, \quad p_{2n-2}(x) = C_n \, \frac{x^{n-2}}{\sqrt{4 - x^2}}, \quad C_n = \frac{2\Gamma^2\left(\frac{n}{2}\right)}{\pi\Gamma(n-1)}. \tag{4.2}$$

We see thereby that $f = \Omega_n^2 \in \Phi_{2n-2}$.

The algorithm we apply now may be called "one step backward and two steps forward". First we find the Schoenberg measure $\nu_{2n-3}(f)$ from the 1-step transition formula (3.1) with $m = 2n-3$:

$$p_{2n-3}(x) = x^{2n-4}\int_x^2 \frac{p_{2n-2}(u)du}{u^{2n-4}\sqrt{u^2-x^2}} = C_n x^{2n-4}\int_x^2 \frac{u^{2-n}du}{\sqrt{(u^2-x^2)(4-u^2)}}\,.$$

The latter integral can be computed by means of the change of variables $u^2 = 4-(4-x^2)v$ and so

$$p_{2n-3}(x) = 2^{-n}C_n x^{2n-4}\int_0^1 \left(1-\frac{4-x^2}{4}v\right)^{\frac{1-n}{2}} \frac{du}{\sqrt{v(1-v)}}\,.$$

Recall the Euler formula for the hypergeometric function [1, Theorem 2.2.1]

$$F(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 \frac{v^{b-1}(1-v)^{c-b-1}}{(1-vz)^a}\,dv\,, \quad c>b>0, \tag{4.3}$$

which gives

$$p_{2n-3}(x) = C'_n x^{2n-4}\,F\Big(\frac{n-1}{2},\frac{1}{2};1;\frac{4-x^2}{4}\Big). \tag{4.4}$$

The first part of the algorithm is accomplished. We go now two steps forward by using 2-step transition formula (3.6) again with $m = 2n-3$. Having in mind a formula for the derivative

$$F'(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z) \tag{4.5}$$

we put

$$\varphi(x) := \frac{n-1}{8}\,x^{2n-2}F\Big(\frac{n+1}{2},\frac{3}{2};2;\frac{4-x^2}{4}\Big).$$

It is clear that $\varphi(x) = O(1)$ as $x \to 2-0$ and (see, e.g., [1, Theorem 2.1.3])

$$\varphi(x) = x^{2n-2}O(x^{-n}) = O(x^{n-2}), \qquad x \to 0+0, \tag{4.6}$$

so φ is bounded and positive on $[0,2]$. By (4.5)

$$\frac{\varphi(x)}{x^{2n-3}} = -\frac{d}{dx}\,F\Big(\frac{n-1}{2},\frac{1}{2};1;\frac{4-x^2}{4}\Big),$$
$$x^{2n-4}\int_x^2 \frac{\varphi(u)}{u^{2n-3}}\,du = C''_n p_{2n-3}(x) - x^{2n-4}.$$

Formula (3.6) now implies

$$p_{2n-3}(x) = \frac{2x^{2n-4}}{B\big(\frac{2n-3}{2},1\big)}\int_x^2 \frac{\nu_{2n-1}(du)}{u^{2n-3}}\,, \quad \nu_{2n-1}(du) = A_n\varphi(x)dx + B_n\delta\{2\}, \tag{4.7}$$

where A_n, B_n are positive constants. So we see that $\Omega_n^2 \in \Phi_{2n-1}$, and the corresponding Schoenberg measure ν_{2n-1} has a singular component. But Theorem 3.1 states that in this case $\Omega_n^2 \notin \Phi_{2n}$, as claimed.

Finally, let $n = 1$. Then

$$\Omega_1^2(x) = \cos^2 x = \frac{1+\cos 2x}{2}\,, \qquad \nu_1(\Omega_1^2) = \frac{\delta\{0\}+\delta\{2\}}{2},$$

and again $\Omega_1^2 \notin \Phi_2$. The proof is complete. □

Remark 4.4. We are unaware of the analog of formula (4.2) for Ω_n^k with $k \geq 3$. Our conjecture with regard to the powers of Ω_n reads $\Omega_n^k \in \Phi_{kn-k+1} \backslash \Phi_{kn-k+2}$ (cf. Remark 2.3).

By Theorem 4.3, each function

$$f(r) := \int_0^\infty \Omega_n^2(rt)\sigma(dt) \tag{4.8}$$

belongs to the class Φ_{2n-2} whenever σ is a probability measure on $\mathbb{R}_+$. We put f in the subclass $\Phi_{2n-2}^{(2)} \subset \Phi_{2n-2}$ if it admits representation (4.8).

The following result describes the class $\Phi_{2n-2}^{(2)}$ in terms of the corresponding Schoenberg measures.

Corollary 4.5. *Let $f \in \Phi_{2n-2}$ and let ν_{2n-2} be its Schoenberg measure. Then $f \in \Phi_{2n-2}^{(2)}$ if and only if*

$$\nu_{2n-2}(du) = C_n \int_{u/2}^\infty \left(\frac{u}{t}\right)^{n-2} \frac{\sigma(dt)}{\sqrt{4t^2-u^2}}\, du, \tag{4.9}$$

where σ is a probability measure on $\mathbb{R}_+$. In particular, ν_{2n-2} is absolutely continuous with respect to σ, and $\nu_{2n-2}\{(0,\varepsilon)\} > 0$ for any $\varepsilon > 0$.

Proof. Let $f \in \Phi_{2n-2}^{(2)}$. Then according to (4.2),

$$\begin{aligned} f(r) &= \int_0^\infty \sigma(dt) \int_0^2 \Omega_{2n-2}(rtx) p_{2n-2}(x)\, dx \\ &= C_n \int_0^2 \frac{x^{n-2}\, dx}{\sqrt{4-x^2}} \int_0^\infty \Omega_{2n-2}(rtx)\, \sigma(dt) \\ &= C_n \int_0^\infty \sigma(dt) \int_0^{2t} \Omega_{2n-2}(ru) \left(\frac{u}{t}\right)^{n-2} \frac{du}{\sqrt{4t^2-u^2}} \\ &= C_n \int_0^\infty \Omega_{2n-2}(ru)\, du \int_{u/2}^\infty \left(\frac{u}{t}\right)^{n-2} \frac{\sigma(dt)}{\sqrt{4t^2-u^2}}\,. \end{aligned}$$

This proves representation (4.9).

The converse statement is proved by reversing the reasoning. □

5. Schoenberg matrices with infinitely many negative eigenvalues

In view of the above results it seems reasonable to introduce the following notation.

Given a finite set $Y \subset \mathbb{R}^n$, denote by $\kappa^-(g, Y)$ a number of negative eigenvalues of the finite Schoenberg matrix $\mathcal{S}_Y(g)$ counting multiplicity, and denote

$$\kappa_n^-(g) := \sup\{\kappa^-(g, Y) : Y \text{ runs through all finite subsets of } \mathbb{R}^n\}.$$

Certainly, $\kappa_n^-(g) = 0$ for $g \in \Phi_n$, and κ_n is a nondecreasing function of n.

We turn to the case when the Schoenberg matrix $\mathcal{S}_X(g)$ can have arbitrarily many negative eigenvalues. As we will see shortly, the cases $n \geq 2$ and $n = 1$ should be discerned.

5.1. **The case $n \ge 2$.**

Theorem 5.1 (= Theorem 1.4). *Let g be a continuous function on $\mathbb{R}_+$ such that the limit*

$$\lim_{t\to\infty} g(t) = g(\infty) \ge 0 \tag{5.1}$$

exists. If $\kappa^-(g, Y) \ge 1$ for some set $Y \in \mathbb{R}^m$, then $\kappa_m^-(g) = +\infty$. In particular, $\kappa_{n+1}^-(g) = +\infty$ for each $g \in \Phi_n \backslash \Phi_{n+1}$ with $n \ge 2$.

Proof. By the assumption there is a finite set $Y = \{y_k\}_{k=1}^p \subset \mathbb{R}^m$ so that $\mathcal{S}_Y(g)$ has at least one negative eigenvalue. Given an arbitrary positive integer N, consider a collection of the shifts of Y of the form $Y_j = Y + w_j$, $j = 1, 2, \ldots, N$, with

$$w_j = (u_j, 0, \ldots, 0), \qquad 0 = u_1 < u_2 < \ldots < u_N;$$

u_j are chosen later. Take $X = \bigcup_j Y_j$. The Schoenberg matrix $\mathcal{S}_X(g)$ is now a block matrix with $p \times p$ blocks

$$\begin{aligned} \mathcal{S}_X(g) &= \begin{bmatrix} \mathcal{S}_Y(g) & B_{12} & B_{13} & \ldots & B_{1N} \\ B_{21} & \mathcal{S}_Y(g) & B_{23} & \ldots & B_{2N} \\ \vdots & \vdots & & & \vdots \\ B_{N1} & B_{N2} & B_{N3} & \ldots & \mathcal{S}_Y(g) \end{bmatrix} \\ &= \operatorname{diag}(\mathcal{S}_Y(g), \mathcal{S}_Y(g), \ldots, \mathcal{S}_Y(g)) + \Delta. \end{aligned} \tag{5.2}$$

The block diagonal matrix in the right hand side of (5.2) has at least N negative eigenvalues.

Assume first that $g(\infty) = 0$. Since the block entries of Δ are

$$\Delta_{rs} = \|g(|y_i - y_j + w_r - w_s|)\|_{i,j=1}^p, \qquad r \ne s,$$

the appropriate choice of $\{u_j\}$ with large enough differences $|u_{j+1} - u_j|$ provides $\|\Delta_{rs}\| \le \varepsilon_1$. So

$$\|\mathcal{S}_X(g) - \operatorname{diag}(\mathcal{S}_Y(g), \mathcal{S}_Y(g), \ldots, \mathcal{S}_Y(g))\| \le \varepsilon,$$

and $\mathcal{S}_X(g)$ has at least N negative eigenvalues, as claimed.

Now let $g(\infty) > 0$. Then for $h = g - g(\infty)$ one has

$$\mathcal{S}_Y(h) = \mathcal{S}_Y(g) - g(\infty)\, E_p \le \mathcal{S}_Y(g).$$

So $\mathcal{S}_Y(h)$ has at least one negative eigenvalue, and $\lim_{t\to\infty} h(t) = 0$. By the above argument $\kappa_m^-(h) = +\infty$, so for any large N there is a finite set $Z = \{z_j\}_{j=1}^l$ such that $\mathcal{S}_Z(h)$ has at least N negative eigenvalues. But

$$\mathcal{S}_Z(g) = \mathcal{S}_Z(h) + g(\infty)\, E_l.$$

The matrix E_l is nonnegative and has rank one, so $\mathcal{S}_Z(g)$ has at least $N-1$ negative eigenvalues. Hence $\kappa_m^-(g) = +\infty$, as claimed.

To prove the second statement note that for $n \ge 2$,

$$\Omega_n(x) = O(x^{\frac{1-n}{2}}), \qquad x \to \infty,$$

in view of (1.3) and the well-known asymptotic behavior of the Bessel function $J_q(x) = O(x^{-1/2})$, $x \to \infty$. As $\Omega_n(0) = 1$, by the Dominated Convergence Theorem

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \int_0^\infty \Omega_n(xt)\nu(dt) = \nu(\{0\}) \ge 0.$$

The result follows from the first statement. □

Remark 5.2. If $\kappa^-(g,Y) \geq 2$ for some set $Y \in \mathbb{R}^m$, then $\kappa_m^-(g) = +\infty$ under assumption (5.1) with an arbitrary value $g(\infty)$.

5.2. **The case $n = 1$.** In the case $n = 1$ the Schoenberg representation of $g \in \Phi_1$ reads (see (1.2) and (1.5))

$$g(t) = \int_0^{+\infty} \cos(tu)\,\nu(du), \qquad t \in \mathbb{R}_+, \tag{5.3}$$

with a finite positive Borel measure $\nu = \nu_1$. Since $\Omega_1(s) = \cos s$ has no limit as $s \to \infty$, condition (5.1) is in general false, and Theorem 5.1 does not apply directly. By using an ad hoc argument we are able to prove $\kappa_2^-(g) = +\infty$ under certain additional assumptions on ν.

Let us begin with a simple (and perhaps well-known) result about spectra of certain Toeplitz matrices.

Lemma 5.3. *Let $a = \{a_j\}_{j\in\mathbb{Z}}$ be an m-periodic sequence of complex numbers, $a_{j+m} = a_j$, $T_m(a) := \|a_{k-j}\|_{k,j=0}^{m-1}$. Then the spectrum of $T_m(a)$ is*

$$\sigma(T_m(a)) = \{\lambda_{km}\}_{k=1}^m, \qquad \lambda_{km} = \sum_{j=1}^m a_j e^{-\frac{2\pi i k}{m} j}. \tag{5.4}$$

If in addition a_j are real numbers and $a_{-j} = a_j$, then

$$\lambda_{km} = \sum_{j=1}^m a_j \cos\frac{2\pi k}{m} j\,. \tag{5.5}$$

Proof. It is easy to see that

$$T_m(a) = a_0 + a_{-1}Z + a_{-2}Z^2 + \ldots + a_{-(m-1)}Z^{m-1},$$

where Z is the permutation matrix

$$Z = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & \ldots & \ldots & \ldots & 0 & 0 \end{bmatrix}.$$

The spectrum of Z is well known, namely,

$$\sigma(Z) = \{e^{\frac{2\pi i}{m} j}\}_{j=0}^{m-1},$$

so by the Spectral Mapping Theorem the spectrum $\sigma(T_m(a)) = \{\lambda_{km}\}_{k=1}^m$ is

$$\lambda_{km} = \sum_{l=0}^{m-1} a_{-l} e^{\frac{2\pi i k}{m} l} = \sum_{l=0}^{m-1} a_{m-l} e^{\frac{2\pi i k}{m} l} = \sum_{j=1}^{m} a_j e^{-\frac{2\pi i k}{m} j},$$

as claimed. The second statement is obvious. □

It turns out that the spectrum of the Schoenberg matrix $\mathcal{S}_Y(g)$ can be computed explicitly when Y is a set of vertices of the regular m-gon on the complex plane.

Proposition 5.4. *Let $Y_m(r)$ be the set of vertices of the regular m-gon of radius r, that is,*

$$Y_m(r) = \{y_k(r)\}_{k=1}^m, \qquad y_k = y_k(r) := re^{\frac{2\pi ik}{m}}, \quad k = 1, 2, \dots, m. \tag{5.6}$$

Given an even continuous function g on $\mathbb{R}$, let $\mathcal{S}_{Y_m(r)}(g) = \|g(|y_j - y_k|)\|_{j,k=1}^m$ be the corresponding Schoenberg matrix. Then for its spectrum one has

$$\sigma\big(\mathcal{S}_{Y_m(r)}(g)\big) = \{\lambda_{km}(r)\}_{k=1}^m, \qquad \lambda_{km}(r) = \sum_{j=1}^m g\left(2r\sin\frac{\pi j}{m}\right)\cos\frac{2\pi k}{m}j\,. \tag{5.7}$$

In particular,

$$\lim_{m\to\infty}\frac{\lambda_{km}(r)}{m} = \widehat{g}(k, r) := \frac{1}{2\pi}\int_0^{2\pi} g\left(2r\sin\frac{t}{2}\right)\cos kt\,dt. \tag{5.8}$$

Proof. It is clear that the Schoenberg matrix $\mathcal{S}_{Y_m(r)}(g)$ is of the type considered in Lemma 5.3:

$$\mathcal{S}_{Y_m(r)}(g) = \|g(|y_j(r) - y_k(r)|)\|_{j,k=1}^m = \left\|g\left(2r\sin\frac{\pi|k-j|}{m}\right)\right\|_{j,k=1}^m.$$

Hence the result is immediate from (5.5). Equality (5.7) divided by m gives an integral sum for the integral in (5.8). The proof is complete. □

Given a sequence of (not necessarily different) numbers $\alpha = \{\alpha_k\}_{k=1}^l \subset \mathbb{R}$, $l \in \mathbb{N}\cup\{\infty\}$, we denote by $\kappa_-(\alpha)$ a number of negative entries in α counting multiplicity. Similarly, given a symmetric matrix A we denote by $\kappa_-(A)$ a number of its negative eigenvalues counting multiplicity.

Corollary 5.5. *Let $\mathcal{K}(r) := \{\widehat{g}(k, r)\}_{k\in\mathbb{N}}$. If $\sup_{r>0}\kappa_-(\mathcal{K}(r)) = +\infty$, then $\kappa_2^-(g) = +\infty$. In particular, if $\kappa_-(\mathcal{K}(r_0)) = +\infty$ for some $r_0 > 0$, then $\kappa_2^-(g) = +\infty$.*

Proof. By the assumption, for an arbitrary $N \in \mathbb{N}$ there is $r > 0$ so that $\mathcal{K}(r)$ contains at least N negative numbers $\{\widehat{g}(k_j, r)\}_{j=1}^N$. By Proposition 5.4 for large enough m

$$\lambda_{k_j,m} < 0, \qquad j = 1, 2, \dots, N,$$

so there are at least N negative eigenvalues counting multiplicity of the Schoenberg matrix $\mathcal{S}_{Y_m(r)}(g) = \|g(|y_j - y_k|)\|_{j,k=1}^m$. Hence $\kappa_2^-(g) = +\infty$, as claimed. □

Clearly, if the measure ν in (5.3) is absolutely continuous, $\nu(du) = p(u)du$, then by the Riemann–Lebesgue Lemma $g(t) \to 0$ as $t \to \infty$. So, Theorem 5.1 applies and $\kappa_2^-(g) = +\infty$ as long as $g \notin \Phi_2$. In what follows we will focus upon the opposite case when ν contains a singular component.

We begin with the simplest case $\nu = \delta\{s\}$.

Proposition 5.6. *Let $g_s(r) = \cos sr$, $s > 0$. Then $\kappa_2^-(g_s) = +\infty$.*

Proof. To apply Corollary 5.5 we compute the cosine Fourier coefficients of the function $g_s\big(2r\sin\frac{t}{2}\big)$. Fortunately, this can be done explicitly. Precisely, we have (see [18, p. 21])

$$\begin{aligned}\widehat{g}_s(k, r) &:= \frac{1}{2\pi}\int_0^{2\pi}\cos\left(2sr\sin\frac{t}{2}\right)\cos kt\,dt \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(2sr\sin\theta)\cos 2k\theta\,d\theta = J_{2k}(2sr).\end{aligned} \tag{5.9}$$

According to Corollary 5.5 the problem is reduced to the study of a number of negative terms in the sequence $\{J_{2k}(x)\}_{k=1}^{\infty}$ for $x > 0$ large enough. Our argument relies heavily on the well-known identity [18, p. 17]

$$\frac{J_{p-1}(x) + J_{p+1}(x)}{p} = \frac{2J_p(x)}{x}. \tag{5.10}$$

Write (5.10) with $p = 2k - 1$, $p = 2k + 1$:

$$\frac{J_{2k-2}(x)}{2k-1} + \frac{J_{2k}(x)}{2k-1} = \frac{2J_{2k-1}(x)}{x},$$
$$\frac{J_{2k}(x)}{2k+1} + \frac{J_{2k+2}(x)}{2k+1} = \frac{2J_{2k+1}(x)}{x},$$

take their sum, and apply again (5.10) with $p = 2k$ to obtain

(5.11)
$$\frac{J_{2k-2}(x)}{2k-1} + \frac{4k}{4k^2-1}\, J_{2k}(x) + \frac{J_{2k+2}(x)}{2k+1} = \frac{2}{x}\big(J_{2k-1}(x) + J_{2k+1}(x)\big),$$
$$\frac{J_{2k-2}(x)}{2k-1} + 4k\left(\frac{1}{4k^2-1} - \frac{2}{x^2}\right) J_{2k}(x) + \frac{J_{2k+2}(x)}{2k+1} \equiv 0.$$

Denote by Z a set of all positive roots of at least one function J_{2k}, $k \in \mathbb{N}$. We assume later on that $x \notin Z$, so $J_{2k}(x) \neq 0$ for all k (note that Z is a countable subset of $\mathbb{R}_+$).

Given $N \in \mathbb{N}$ put $x = 9N + \varepsilon_N$, $0 < \varepsilon_N < 1$, so that $x \notin Z$. It is clear that

$$\frac{1}{4k^2-1} - \frac{2}{x^2} > 0, \qquad k = 1, 2, \dots, 3N,$$

so by (5.11) at least one of the numbers $J_{2k-2}(x)$, $J_{2k}(x)$, $J_{2k+2}(x)$ is negative. There are exactly N such triples in the set $\{J_{2p}(x)\}_{p=1}^{3N}$, so

$$\kappa_-\big(\{J_{2p}(x)\}_{p=1}^{3N}\big) \geq N. \tag{5.12}$$

Hence, by Corollary 5.5, $\kappa_2^-(g_s) = +\infty$, as claimed. □

Remark 5.7. It is easy to see that the same conclusion holds for the function

$$g_s^2(r) = \cos^2 sr = \frac{1 + \cos 2sr}{2}.$$

Indeed, the Schoenberg matrix $\mathcal{S}_X(g_s^2)$ is the rank one perturbation of the Schoenberg matrix $\mathcal{S}_X(g_s)$, and the latter can have arbitrarily many negative eigenvalues for an appropriate choice of the set $X \subset \mathbb{R}^2$.

We show that the same conclusion remains valid within a certain class of Schoenberg's measures.

Proposition 5.8. *Suppose that for a function g (5.3) the support of ν is separated from the origin, i.e., $\operatorname{supp}\nu \subset [a, \infty)$ for some $a > 0$. Next, assume that*

$$l(\nu) := \limsup_{r\to\infty} |h(r)| > 0, \qquad h(r) := \frac{1}{2\pi^{3/2}} \int_0^{\infty} \frac{\cos\left(2rs - \frac{\pi}{4}\right)}{\sqrt{s}}\, \nu(ds). \tag{5.13}$$

Then $\kappa_2^-(g) = +\infty$.

Proof. It follows from (5.3) and (5.9) that

$$\widehat{g}(k,r) = \frac{1}{2\pi}\int_0^\infty J_{2k}(2rs)\nu(ds). \tag{5.14}$$

We wish to show that the number of negative terms in the sequence $\{\widehat{g}(k,r)\}_{k\in\mathbb{N}}$ grows unboundedly as $r\to\infty$.

Write the asymptotic expansion [1, formula (4.8.5)]

$$J_{2k}(x) = \sqrt{\frac{2}{\pi x}}\cos\Big(x - k\pi - \frac{\pi}{4}\Big) + \varepsilon_{2k}(x) = (-1)^k\sqrt{\frac{2}{\pi x}}\cos\Big(x - \frac{\pi}{4}\Big) + \varepsilon_{2k}(x),$$

and so

$$\widehat{g}(k,r) = (-1)^k\frac{h(r)}{\sqrt{r}} + \frac{1}{2\pi}\int_0^\infty \varepsilon_{2k}(2rs)\nu(ds). \tag{5.15}$$

The following uniform bound for the reminder ε_{2k} is known [8, Theorem 10]:

$$\varepsilon_{2k}(x) \le c_1\frac{k^2}{x^{3/2}}, \qquad x>0,$$

where c_p below stands for some absolute constants. Hence

$$|\sqrt{r}\widehat{g}(k,r) - (-1)^k h(r)| \le c_2\,\frac{k^2}{r}\int_a^\infty \frac{\nu(ds)}{s^{3/2}}\,. \tag{5.16}$$

By assumption (5.13) there is a sequence of positive numbers $\{r_j\}_{j\ge1}$ so that

$$\lim_{j\to\infty} r_j = +\infty, \qquad |h(r_j)| \ge \frac{l(\nu)}{2} > 0.$$

Take large enough $j_0 = j_0(\nu)$ so that

$$c_2\int_a^\infty \frac{\nu(ds)}{s^{3/2}} < \frac{l(\nu)}{4}\,r_j^{1/2}, \quad j\ge j_0.$$

It follows now from (5.16) that

$$|\sqrt{r}\widehat{g}(k,r) - (-1)^k h(r)| < \frac{l(\nu)}{4}, \qquad k = 1,2,\dots,[r_j^{1/4}], \quad j\ge j_0,$$

and the number of negative terms in $\{\widehat{g}(k,r)\}_{k\in\mathbb{N}}$ is at least $c_3 r_j^{1/4}$, so it grows to infinity as $j\to\infty$. An application of Corollary 5.5 completes the proof. □

Remark 5.9. As a matter of fact, the assumption $\operatorname{supp}\nu\subset[a,\infty)$ can be relaxed to

$$\int_0^\infty \frac{\nu(ds)}{\sqrt{s}} < \infty, \qquad \int_x^\infty \frac{\nu(ds)}{s^{3/2}} = o(x^{-1}), \quad x\to 0. \tag{5.17}$$

Example 5.10. Let

$$\nu(ds) = \sum_{k=1}^\infty a_k\delta\{s_k\}, \qquad \inf_k s_k > 0,$$

with $a_k\ge 0$, $\{a_k\}_{k\ge1}\in\ell^1(\mathbb{N})$. Then (5.13) is true since h is almost periodic on $\mathbb{R}_+$.

Remark 5.11. A similar problem for a general Bochner's class P_n of positive-definite functions on $\mathbb{R}^n$ can be easily resolved. Recall that a continuous function f on $\mathbb{R}^n$ belongs to P_n if for an arbitrary finite set $\{x_1, \dots, x_m\}$, $x_k \in \mathbb{R}^n$, and $\{\xi_1, \dots, \xi_m\} \in \mathbb{C}^m$

$$\sum_{k,j=1}^{m} f(x_k - x_j)\xi_j\overline{\xi}_k \ge 0.$$

Each function from the Bochner class admits the integral representation

$$f(x) = \int_{\mathbb{R}^n} e^{i(x,t)_n}\, \sigma(dt), \tag{5.18}$$

where σ is a finite positive Borel measure on $\mathbb{R}^n$.

Although inclusion of the classes P_n has no sense now, let us write $f_{n+1} \in P_{n-1}\backslash P_n$ for a function f_n on $\mathbb{R}^n$ if $f_n \notin P_n$ but its restriction on $\mathbb{R}^{n-1}$ is

$$f_{n-1}(x_1, x_2, \dots, x_{n-1}) := f_n(x_1, x_2, \dots, x_{n-1}, 0) \in P_{n-1}.$$

Let μ be a finite positive Borel measure on $\mathbb{R}^n$ with the only atom at the origin $\mu\{0\} > 0$ and the Fourier transform f_n (5.18). Put

$$\mu_1 = \mu - \varepsilon\, \delta_a, \qquad 0 < \varepsilon < \mu\{0\}, \quad a = (0, \dots, 0, 1) \in \mathbb{R}^n, \tag{5.19}$$

a finite charge (sign measure) on $\mathbb{R}^n$. By the construction, μ is not a measure, so the Fourier transform

$$g_n(x) = \int_{\mathbb{R}^n} e^{i(x,t)_n}\, \mu_1(dt) = f_n(x) - \varepsilon\, e^{ix_n} \notin P_n. \tag{5.20}$$

So

$$g_{n-1}(x) = f_{n-1}(x) - \varepsilon = \int_{\mathbb{R}^{n-1}} e^{i(x,t)_{n-1}}\, \tilde{\mu}(dt) - \varepsilon,$$

where $\tilde{\mu}$ is a projection of μ on $\mathbb{R}^{n-1}$; that is, $\tilde{\mu}(E) = \mu(E \times \mathbb{R})$ for each Borel set $E \subset \mathbb{R}^{n-1}$. By the choice of ε (5.19) and $\tilde{\mu}\{0\} \ge \mu\{0\}$ we have $g_n \in P_{n-1}$, so $g_n \in P_{n-1}\backslash P_n$.

Given a finite set $Y = \{y_j\}_{j=1}^N \subset \mathbb{R}^n$ and a function g on $\mathbb{R}^n$, we define a finite Schoenberg matrix by $\mathcal{S}_Y(g) = \|f(y_i - y_j)\|_{i,j=1}^N$ and define the values $\kappa^-(g, Y)$ and $\kappa_n^-(g)$ as in Section 1. It follows from (5.20) that $\mathcal{S}_Y(g_n) = T_1 - T_2$, where $T_1 \ge 0$ and $\operatorname{rk} T_2 = 1$. So $\kappa(g_n, Y) \ge 1$, and for some particular choice of Y $\kappa(g_n, Y) = 1$. Hence, $\kappa_n^-(g_n) = 1$, as needed.

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