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KOHNEN'S FORMULA AND A CONJECTURE OF DARMON AND TORNARÍA

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ABSTRACT. We generalize a result of W. Kohnen (1985) to explicit Waldspurger lifts constructed by E. M. Baruch and Z. Mao (2007). As an application, we prove a conjecture formulated by H. Darmon and G. Tornaría (2008).

1. INTRODUCTION

The aim of this note is to extend a result of Kohnen [13, Thm. 3] to Waldspurger lifts of elliptic modular forms constructed in [1] and use this formula to prove a conjecture of H. Darmon and G. Tornaría in [7].

Let us first explain the generalization of Kohnen's formula in which we are interested. Suppose that $f=\sum_{n\geq 1}a_nq^n$ is a modular form of weight $2k$ for $\Gamma_0(M)$, the usual congruence subgroup of level M in $\mathrm{SL}_2(\mathbb{Z})$, where $M\geq 1$ is a square free odd integer. Thanks to the work of Baruch and Mao [1] one attaches to f and a divisor $M'\mid M$ a form $g=\sum_{n\geq 1}c_nq^n$ of weight $k+1/2$ and with respect to the congruence group $\Gamma_1(4MM')$. Let s_0 be the cardinality of the set S_0 of primes dividing M'. Let $\mathcal{D}(f,S_0)$ be the set of fundamental discriminants D such that $\left(\frac{D}{\ell}\right)=-w_\ell$ if $\ell\mid M'$ and $\left(\frac{D}{\ell}\right)=+w_\ell$ if $\ell\mid M/M'$, where w_ℓ is the sign of the Atkin-Lehner involution acting on f. We are interested in fundamental discriminants satisfying the following condition:

$$(*)\qquad D\in\mathcal{D}(f,S_0)\quad\text{and}\quad(-1)^{s_0+k}=\mathrm{sgn}(D).$$

Suppose D_1 and D_2 are fundamental discriminants satisfying $(*)$. Kohnen's formula relates the product $c_{|D_1|}\cdot\overline{c_{|D_2|}}$ to certain linear combinations of explicit Shintani integrals, namely, integrals of the differential form $f(z)dz$ along geodesic cycles in the upper half plane. In [13] one assumes, apart from a natural parity condition on D_1 and D_2, that $\left(\frac{D_1}{\ell}\right)=\left(\frac{D_2}{\ell}\right)=+w_\ell$ for all $\ell\mid M$; therefore, in our setting, this corresponds to the case $S_0=\emptyset$. Our first task is to generalize Kohnen's result to a discriminant satisfying the more general condition $(*)$ above. However, the proof of this result is not a direct generalization of the proof of loc. cit., which has a more combinatoric flavour. Instead, our proof is based on methods from [1] and [19], working in the context of automorphic forms. Finally, let us point out that the above result is proved in the more general setting of automorphic forms over

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totally real number fields (in Proposition 2.9 below), although for the application to the Darmon-Tornaría conjecture we only need the case of rational numbers. As a general observation, it might be helpful to recall that the interest of [13, Thm. 3] that we generalize arises from the fact that, as a corollary, Kohnen obtains a formula in [13, Cor. 1] relating $|c_D|^2$ with the special value $L(f, \chi_D, k)$, where D is a fundamental discriminant and χ_D is the associated quadratic character. Special values formulas analogue to [13, Cor. 1] have already been generalized in our setting in [1] and by Prasanna in [25] in the context of modular forms on quaternion algebras, using the approach of Waldspurger [35], [34], [33], [36]; however, for the arithmetic applications to the Darmon-Tornaría conjecture we need a more refined formula analogue to [13, Thm. 3].

We now briefly explain the application to elliptic curves, and the content of the Darmon-Tornaría conjecture. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = Mp$, where $M > 1$ is an odd square free positive integer and $p \nmid 2M$ is a prime number. Fix as above a divisor $M' \mid M$ and let g be the generalized Kohnen-Waldspurger lift in [1] of the modular form f attached to E. This is a modular form of weight $3/2$ for $\Gamma_1(4NM')$ with Fourier expansion $\sum_{n \geq 1} c_n q^n$. This expansion is only well-defined up to (non-zero) scalar, and therefore we may form the quotients $\tilde{c}_n := c_n/c_{n_0}$, where $c_{n_0} \neq 0$ (the existence of such an integer n_0 follows from the main result of Baruch-Mao, combined with standard non-vanishing results for L-series [5]). We show that these coefficients can be seen as the value at 1 of rigid analytic functions $\tilde{c}_n(k)$, defined on a neighborhood $\mathcal{U}$ of 1 in a suitable weight space, which incorporate similarly defined quotients of Fourier coefficients of the generalized Kohnen-Waldspurger lifts of classical even weight modular forms in the Hida family passing through f. We fix a fundamental discriminant D satisfying the following particular case of $(*)$ (in the case $k = 1$) in which we place a further condition at p (recall that, as above, w_ℓ for $\ell \mid N$ is the sign of the Atkin-Lehner involution on E and that s_0 is the cardinality of the fixed set S_0):

$$(\dagger\dagger) \qquad \left(\frac{D}{p}\right) = -w_p.$$

It turns out that $(\dagger\dagger)$ implies $\tilde{c}_{|D|}(1) = 0$ ([1, Thm. 1.1]), although the function $\tilde{c}_D(k)$ is not a priori identically zero in $\mathcal{U}$; note that D satisfies $(*)$ with respect to all the newforms of level $\Gamma_0(M)$ appearing in the Hida family (actually in the Hida family the associated p-stabilized forms of level $\Gamma_0(N)$ appear) and therefore these coefficients are not forced to be zero by sign considerations. Our main result (Theorem 4.6 below, a consequence of Theorem 4.5) is the following: There exists a family of points $P_D \in E(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$, one for each D as above, which is non-zero if and only if $L'(E, \chi_D, 1) \neq 0$, and such that

$$(1) \qquad \log_E(P_D) = \left(\frac{d}{dk}\tilde{c}_{|D|}(k)\right)_{|k=1}.$$

Further, if $D < 0$, then we may take $P_D \in E(\sqrt{D}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$. Here $\log_E$ is the formal group logarithm on E (see §4.1 for a precise definition). Further, the point P_D arises from Darmon's theory of his eponymous points, also known as Stark-Heegner points, introduced in [6] and developed by several authors (see for example [3], [4], [10], [21], [27], [16], [17]). With $S = \emptyset$, Theorem 4.5 corresponds to [7, Thm. 1.5].

We finally use this result to prove Theorem 4.7, a conjecture of Darmon and Tornaría ([7, Conj. 5.3]). To discuss this conjecture, let D be a fundamental discriminant satisfying the following particular case of (∗), in which we put the following condition at p:

$$(\dagger) \qquad \left(\frac{D}{p}\right) = +w_p.$$

So (†) is the opposite of (††) above; in the following, those discriminants satisfying (†) (resp. (††)) are called of type I (resp. type II), following a similar terminology in [7]. One considers certain p-adic analogues of Shintani integrals, which are denoted $\vartheta(f, D, D')$ in §4.1, whose definition depends on the choice of an auxiliary fundamental discriminant D' satisfying (††). In the particular setting of [7] in which $S = \emptyset$, one can prove that these p-adic numbers $\vartheta(f, D, D')$ are proportional to the Fourier coefficients of a modular form of weight 3/2, and that the coefficient of proportionality, which of course depends on the choice of D', is non-zero if and only if $L'(E, \chi_{D'}, 1) \neq 0$; we refer the reader to [7, Thm 5.1] for more details. However, p-adic Shintani integrals $\vartheta(f, D, D')$ can be defined in the more general situation in which $S \neq \emptyset$, and therefore the natural conjecture [7, Conj. 5.3] by Darmon-Tornaría predicts the existence of a modular form of weight 3/2 whose Fourier coefficients are proportional to $\vartheta(f, D_1, D_2)$; further, the coefficient of proportionality is required to be non-zero if and only if $L'(E, \chi_{D'}, 1) \neq 0$. We prove this conjecture showing that the modular form of weight 3/2 whose existence is predicted in [7, Conj. 5.3] is the complex conjugate g^* of the generalized Kohnen-Waldspurger lift g of the newform f attached to E.

2. Kohnen's formula

2.1. **Generalized Kohnen-Shintani correspondence.** Let $f = \sum_{n\geq 1} a_n q^n$ be a newform of even integral weight $2k$, square-free odd level M and trivial character. We let S be a set of primes dividing M, whose cardinality we denote $s = |S|$. Fix a subset $S_0 \subset S$, write M' for the product of the primes in S_0, and let $s_0 = |S_0|$ be the cardinality of S_0. Let $\mathcal{D}(f, S_0)$ be the set of fundamental discriminants D defined in the Introduction. We also denote by χ_D the quadratic character $a \mapsto \left(\frac{D}{a}\right)$ attached to the fundamental discriminant D. Fix a Dirichlet character χ' of $(\mathbb{Z}/(4M))^\times$ such that $\chi'_\ell = 1$ if $\ell \mid (M/M')$, $\chi'_\ell(-1) = -1$ if $\ell \mid M'$ and $\chi'(-1) = 1$. We can consider χ' as a character of $(\mathbb{Z}/(4MM'))^\times$. Attached to f and the choice of the auxiliary character χ', we may consider the explicit Waldspurger's lift of Baruch-Mao relative to S_0 ([1], [33], [36], [34]),

$$g = \sum_{n\geq 1} c_n q^n \in S_{k+1/2}(\Gamma_0(4MM', \chi'))$$

which satisfies the following properties (see [1, Thm. 1.1]):

(a) g is a Shimura lift of f.
(b) g belongs to the Kohnen's plus space: $c_n = 0$ if $(-1)^{s_0+k} n \equiv 2, 3 \mod 4$.
(c) $c_{|D|} = 0$ if $(-1)^{s_0+k} D > 0$ and $D \notin \mathcal{D}(f, S_0)$.
(d) If D satisfies (∗) in the Introduction, then

$$(2) \qquad \frac{|c_{|D|}|^2}{\langle g, g\rangle} = \frac{L(f, \chi_D, k)}{\langle f, f\rangle} \cdot \frac{2^{|S|} \cdot |D|^{k-1/2} \cdot (k-1)!}{\pi^k} \cdot \prod_{\ell | S_0} \frac{\ell}{\ell+1}.$$

Otherwise, if $D \in \mathcal{D}(f, S_0)$ and $(-1)^{s_0+k} \neq \mathrm{sgn}(D)$, then $L(f, \chi_D, k) = 0$.

2.2. **Kohnen's formula.** Let $K=\mathbb{Q}(\sqrt{\Delta})$ be a real quadratic field of fundamental discriminant Δ such that all primes dividing M are split in K.

Fix $\tau=\left(\begin{smallmatrix}\tau_1 & \tau_3\\ \tau_2 & -\tau_1\end{smallmatrix}\right)$ with integer entries such that $\tau_1^2+\tau_2\tau_3=\Delta$ and $2M\mid\tau_2$, $2\mid\tau_3$. Let $\mathcal{F}_\Delta$ denote the set of binary integral primitive quadratic forms

$$Q(x,y)=Ax^2+Bxy+Cy^2$$

of discriminant Δ satisfying the following properties: (1) $M\mid A$ and (2) $B\equiv\tau_1$ modulo M. The group $\Gamma_0(M)$ acts on $\mathcal{F}_\Delta$ from the right via

$$(Q|\gamma)(x,y):=Q(ax+by,cx+dy)$$

for $\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$.

Recall that canonical projection induces a bijection between $\mathcal{F}_\Delta/\Gamma_0(M)$ and the group of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral primitive binary quadratic forms of discriminant Δ equipped with Gaussian composition law. This is identified, by class field theory, with the Galois group $G_K^+=\mathrm{Gal}(H_K^+/K)$ of the strict Hilbert class field H_K^+ of K.

Let $r+s\sqrt{\Delta}$ be a fundamental unit of norm 1 in $\mathcal{O}_\Delta=\mathbb{Z}[(\Delta+\sqrt{\Delta})/2]$ normalized with $r>0$ and $s>0$ and define $\gamma_Q:=\left(\begin{smallmatrix}r+sB & 2Cs\\ -2As & r-sB\end{smallmatrix}\right)$, an element in $\Gamma_0(M)$. Define Shintani integrals attached to f and $Q\in\mathcal{F}_\Delta$ as

$$r(f,Q):=\int_{z_0}^{\gamma_Q(z_0)}f(z)Q(z,1)^{k-1}dz \tag{3}$$

(one can check that this is independent on the choice of $z_0\in\mathcal{H}$ and only depends on the $\Gamma_0(M)$-equivalence class of Q). Fix a genus character χ_{D_1,D_2} of K, attached to the pair of Dirichlet characters (χ_{D_1},χ_{D_2}), where $\Delta=D_1\cdot D_2$, and $(D_1,D_2)=1$. Set as in [23]

$$r(f;D_1,D_2;\tau):=\sum_{[Q]\in\mathcal{F}_\Delta/\Gamma_0(M)}\chi_{D_1,D_2}(Q)\cdot r(f,Q).$$

Definition 2.1. Let χ be the character of $(\mathbb{Z}/M')^\times$ such that $\chi_\ell=\chi'_\ell$ when $\ell\mid M'$. The *normalized complex period* attached to f, χ, D_1 and D_2 is the complex number:

$$r_\chi(f;D_1,D_2):=\chi(\tau_1)\cdot r(f;D_1,D_2;\tau).$$

Remark 2.2. The value of $(r(f;D_1,D_2;\tau))^2$ does not depend on the choice of τ. Also, $r_\chi(f;D_1,D_2)$ is independent of the choice of τ.

Remark 2.3. When $D_1=D_2$ and $\tau=\left(\begin{smallmatrix}D_1 & \\ & -D_1\end{smallmatrix}\right)$, we denote $r(f;D_1,D_2;\tau)$ by $r(f;D_1,D_1)$.

Theorem 2.4. *Suppose $\Delta=D_1\cdot D_2$ is a fundamental discriminant with D_1, D_2 satisfying $(*)$ above and D_1 odd. Then*

$$\frac{c_{|D_2|}\overline{c_{|D_1|}}}{\langle g,g\rangle}=(-2i)^k\cdot 2^{|S|}\cdot\chi(|D_1|)^{-1}\cdot\prod_{\ell\in S_0}\frac{\ell}{1+\ell}\cdot\frac{r_\chi(f;D_2,D_1)}{\langle f,f\rangle}.$$

Moreover $r_\chi(f;D_2,D_1)=r_\chi(f;D_1,D_2)$.

Remark 2.5. The difference in constant (a factor of $2^{|S|}$) between the above theorem and [13, Theorem 3] lies in the difference of $r_\chi(f;D_2,D_1)$ (which is defined through a sum of *oriented* optimal embeddings in [23]) and $r_{k,N}(f;D_1,D_2)$ in [13] (which is a sum over non-oriented optimal embeddings).[1]

[1] A similar factor should also appear in [7, Theorem 2.1].

Remark 2.6. A combination of (2), [3, Eq. (28)] and [3, Eq. (29)] already shows, at least in the cases of weight $2k > 2$ which will be relevant for the following sections, that the square norm of the above formula is true.

The proof of the theorem is based on results in [1] and [19]. Before proving the theorem, we first give a generalization of Kohnen's formula in the setting of automorphic forms over a totally real number field.

2.3. **Theta correspondence.** Let F be a totally real number field, $\mathbb{A}$ its adele ring. Fix an additive character ψ on $\mathbb{A}/F$ which is non-trivial. We will recall now the theta correspondence (Shimura correspondence) studied by Waldspurger [33].

Let M be the space of 2×2 matrices, and M^0 be the subspace consisting of matrices with trace 0. Let Φ be in $\mathcal{S}(\mathrm{M}^0(\mathbb{A}))$ the space of Schwartz functions on $\mathrm{M}^0(\mathbb{A})$. Let ω_ψ be the Weil representation of $\mathrm{PGL}_2 \times \widetilde{\mathrm{SL}}_2$ associated to ψ, (see for example [1] for a definition). We can construct a theta function Θ_Φ^ψ on $\mathrm{PGL}_2 \times \widetilde{\mathrm{SL}}_2$:

$$\Theta_\Phi^\psi(g,h) = \sum_{x \in \mathrm{M}^0(F)} \omega_\psi(g,h)\Phi(x), \quad g \in \mathrm{PGL}_2(\mathbb{A}),\ h \in \widetilde{\mathrm{SL}}_2(\mathbb{A}).$$

Then for any cusp form φ on PGL_2 and $\Phi \in \mathcal{S}(\mathrm{M}^0(\mathbb{A}))$ define

$$\theta_\Phi^\psi(\varphi)(h) = \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})} \Theta_\Phi^\psi(g,h)\varphi(g)\ dg.$$

For irreducible cuspidal representations π of PGL_2, the space

$$\{\theta_\Phi^\psi(\varphi) : \varphi \in \pi, \Phi \in \mathcal{S}(\mathrm{M}^0(\mathbb{A}))\}$$

is an irreducible cuspidal representation $\tilde{\pi}$ (which could be trivial). We denote $\tilde{\pi} = \theta^\psi(\pi)$ and call it the theta correspondence of π.

2.4. **Transition of periods.** Let $\tau \in \mathrm{GL}_2(F)$ such that $\tau^2 = \left(\begin{smallmatrix} D & \\ & D \end{smallmatrix}\right)$, $D \in F^\times$. Let T_τ be the centralizer of τ in PGL_2.

Define

$$P_\tau(\varphi) = \int_{T_\tau(F)\backslash T_\tau(\mathbb{A})} \varphi(t)\ dt,\ \ \varphi \in \pi,$$

and

$$\tilde{W}^D(\tilde{\varphi}) = \mathrm{vol}(F\backslash\mathbb{A})^{-1} \int_{F\backslash\mathbb{A}} \tilde{\varphi}(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right))\psi(-Dx)\ dx,\ \ \tilde{\varphi} \in \tilde{\pi}.$$

Here and in the following we use a convention that $g \in \mathrm{SL}_2$ denotes also the element $(g,1)$ in $\widetilde{\mathrm{SL}}_2$. This will not cause confusion as the few multiplications between elements in $\widetilde{\mathrm{SL}}_2$ in this section always satisfy $(g_1,1)(g_2,1) = (g_1g_2,1)$.

The following lemma is the analogue of [19, Proposition 2.1], we will skip the proof as it is identical to the proof given in ibid.

Lemma 2.7. *If* $\tilde{\varphi} = \theta_\Phi^\psi(\varphi)$, *then*

$$\tilde{W}^D(\tilde{\varphi}) = P_\tau(f_{\Phi,\tau} * \varphi) \tag{4}$$

where $f_{\Phi,\tau}$ *is a function on* PGL_2 *satisfying*

$$\int_{T_\tau(\mathbb{A})} f_{\Phi,\tau}(tg)\ dt = \Phi(g^{-1}\tau g) \tag{5}$$

and

$$f_{\Phi,\tau} * \varphi = \int_{\mathrm{PGL}_2(\mathbb{A})} f_{\Phi,\tau}(g)\varphi(\cdot g)\ dg.$$

Of course $f_{\Phi,\tau}$ is not uniquely determined; the discussion below applies to any choice of $f_{\Phi,\tau}$. For the special case $\tau = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$, we write $f_\Phi := f_{\Phi,\tau}$, $T := T_\tau$ and $P := P_\tau$. We will also use the notation $f_{\Phi,\tau} * \varphi$ in the local setting for the corresponding local integral.

2.5. **A generalization of Kohnen's formula.** Let π be such that $L(\pi, \frac{1}{2}) \neq 0$. Then it is well known that $\tilde{\pi} = \theta^\psi(\pi) \neq 0$.

For $\varphi \in \pi$, define the corresponding Whittaker function

$$W_\varphi(g) := \mathrm{vol}(F\backslash\mathbb{A})^{-1} \int_{F\backslash\mathbb{A}} \varphi(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g)\psi(-x)\ dx.$$

Let $W(\varphi) = W_\varphi(e)$. By uniqueness of the Whittaker model, if

$$\varphi = \bigotimes_{v\in\Sigma} \varphi_v \bigotimes_{v\notin\Sigma} \varphi_{v,0},$$

where $\varphi_{v,0}$ is a fixed unramified vector in π_v, we can write for $g \in F_S = \bigotimes_{v\in\Sigma} F_v$, $W_\varphi(g) = \prod_{v\in\Sigma} W_v(g)$ for a compatible choice of functions W_v in the local Whittaker spaces of π_v.

On the other hand, from [36] we get locally the space of $T_{\tau,v}$ invariant forms on π_v is of dimension at most one. Any such invariant form (on the Whittaker space of π_v) is a scalar multiple of

$$P_\tau(W'_v) := \int_{T_{\tau,v}} W'_v(t)\ dt.$$

(The above integral converges as π_v is a unitary representation.) Thus there is a constant $c_{\pi,\Sigma}$ depending on π and Σ (and not on φ) such that

$$P_\tau(\varphi) = c_{\pi,\Sigma} \prod_{v\in\Sigma} P_\tau(W_v). \tag{6}$$

Remark 2.8. The following discussion holds if we let the set of bad places Σ to be large enough, and fix the local measures on $T_{\tau,v}$ and T_v for $v \notin \Sigma$ so that $\mathrm{vol}(T_{\tau,v}(R_v)) = \mathrm{vol}(T_v(R_v)) = 1$. The global measures are taken to be the products of local measures.

By [19, Lemma 3.1], if Φ_v and τ_v are unramified, in the sense that Φ_v is the characteristic function of $\mathrm{M}^0(R_v)$ where R_v is the ring of integers in F_v and $\tau_v \in \mathrm{GL}_2(R_v)$, we can let $f_{\Phi,\tau,v} = f_{0,v}$ the characteristic function of $\mathrm{PGL}_2(R_v)$. Thus when W_v is also unramified,

$$f_{\Phi,\tau,v} * W_v = f_{0,v} * W_v = \mathrm{vol}(\mathrm{PGL}_2(R_v))W_v.$$

We have for Σ large enough

$$\tilde{W}^D(\theta^\psi_\Phi(\varphi)) = c_{\pi,\Sigma} \prod_{v\notin\Sigma} \mathrm{vol}(\mathrm{PGL}_2(R_v)) \prod_{v\in\Sigma} P_\tau(f_{\Phi,\tau,v} * W_v). \tag{7}$$

We can apply the discussion to the special case $\tau = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Using $c^1_{\pi,\Sigma}$ to denote the constant in (6), we have

$$\tilde{W}(\theta^{\psi}_{\Phi}(\varphi)) = c^1_{\pi,\Sigma} \prod_{v \notin \Sigma} \mathrm{vol}(\mathrm{PGL}_2(R_v)) \prod_{v \in \Sigma} P(f_{\Phi,v} * W_v). \tag{8}$$

Let $\tilde{\varphi} = \theta^{\psi}_{\Phi}(\varphi)$ be such that $\tilde{W}(\tilde{\varphi}) \neq 0$, and $\tilde{\varphi}' = \theta^{\psi}_{\Phi'}(\varphi)$, we get from (7) and (8) that (for Σ large enough)

$$\frac{\tilde{W}^D(\tilde{\varphi}')}{\tilde{W}(\tilde{\varphi})} = \frac{c_{\pi,\Sigma} \prod_{v\in\Sigma} P_\tau(f_{\Phi',\tau,v} * W_v)}{c^1_{\pi,\Sigma} \prod_{v\in\Sigma} P(f_{\Phi,v} * W_v)}. \tag{9}$$

Recall the (special case of) main formula in [1]:

$$\frac{|\tilde{W}(\tilde{\varphi})|^2}{\|\tilde{\varphi}\|^2} = \frac{|W(\varphi)|^2}{\|\varphi\|^2} L(\pi, \frac{1}{2}) \prod_{v\in\Sigma} E_v(\varphi_v, \tilde{\varphi}_v, \psi), \tag{10}$$

where $E_v(\varphi_v, \tilde{\varphi}_v, \psi)$ are local constants defined in [1]; $\|\tilde{\varphi}\|^2$ and $\|\varphi\|^2$ are Peterson norm with respect to a pair of compatible measures on $\widetilde{\mathrm{SL}}_2$ and PGL_2, fixed in [1].

It follows from the well-known Hecke theory for GL_2 (see [8, Theorem 11]) that for $v \notin \Sigma$ with W_v unramified,

$$P(W_v) = L(\pi_v, \frac{1}{2}) W_v(e) \tag{11}$$

and thus $c^1_{\pi,\Sigma} = L^{\Sigma}(\pi, \frac{1}{2})$ the value at $\frac{1}{2}$ of the partial L-function $L^{\Sigma}(\pi, s) = \prod_{v\notin\Sigma} L(\pi_v, s)$. Multiplying equations (9) and (10) and using (6), we get:

Proposition 2.9. *Let $\Phi = \bigotimes \Phi_v$ and $\Phi' = \bigotimes \Phi'_v$ in $\mathcal{S}(\mathrm{M}^0(\mathbb{A}))$, $\varphi = \bigotimes \varphi_v \in \pi$ and $\tilde{\varphi} = \theta^{\psi}_{\Phi}(\varphi)$ be such that $\tilde{W}(\tilde{\varphi}) \neq 0$. Let $\tilde{\varphi}' = \theta^{\psi}_{\Phi'}(\varphi)$. Let Σ be large enough so that for $v \notin \Sigma$, $\psi_v, \varphi_v, \tilde{\varphi}_v, \tilde{\varphi}'_v, \Phi_v, \Phi'_v$ are unramified and $\tau_v \in \mathrm{GL}_2(R_v)$. Assume $W_\varphi = \prod_{v\in\Sigma} W_v$ (over F_Σ) be such that $P_\tau(W_v) \neq 0$. Then:*

$$\frac{\tilde{W}^D(\tilde{\varphi}')\overline{\tilde{W}(\tilde{\varphi})}}{\|\tilde{\varphi}\|^2} = P_\tau(\varphi) \frac{|W(\varphi)|^2}{\|\varphi\|^2} \prod_{v\in\Sigma} \left(\frac{P_\tau(f_{\Phi',\tau,v} * W_v)}{P_\tau(W_v) P(f_{\Phi,v} * W_v)} L(\pi_v, \frac{1}{2}) E_v(\varphi_v, \tilde{\varphi}_v, \psi) \right). \tag{12}$$

In the special case where $\Phi = \Phi'$, the above equation gives a relation between the product of distinct Whittaker functionals of $\tilde{\varphi}$ and the period P_τ of φ, up to some local factors. We can consider it as a generalization of Kohnen's formula in [13, Theorem 3], which is a formula for product of distinct Fourier coefficients of a half integral weight form.

2.6. **Specification of the formula** (12)**.** Now consider the case $F = \mathbb{Q}$. Let M be a square free odd number. Let π' be the irreducible cuspidal representation of PGL_2 associated to a newform f of weight $2k$ and level M. Let τ be as in §2.2. In particular $(\Delta, M) = 1$ and ℓ splits in $\mathbb{Q}(\sqrt{\Delta})$ for all primes $\ell \mid M$. Write the positive fundamental discriminant Δ as $D_1 D_2$ where D_1 is odd and satisfies the condition $(*)$; then D_2 is coprime to D_1 and also satisfies $(*)$.

Let $\pi = \pi' \otimes \chi_{D_1}$. (We use as above χ_a to denote the quadratic character of $\mathbb{A}^\times$ attached to $a \in \mathbb{Q}^\times$.) Let $\psi(x) = \psi_0(|D_1|x)$ where $\psi_0(x)$ is fixed as follows: $\psi_{0,v}(x) = e^{2\pi i x}$ if v is finite and $e^{-2\pi i x}$ if $v = \infty$. We assume $L(f, \chi_{D_1}, 1) \neq 0$; then $\tilde{\pi} = \theta^{\psi}(\pi) \neq 0$. Moreover with our restrictions on D_1 (that it satisfies the condition $(*)$), the representation $\tilde{\pi}$ is independent of our choice of parameter D_1, (see [36], summarized in [1, Theorem 3.2]).

In principle we can derive Theorem 2.4 from (12) by explicitly computing the local factors appearing in (12). However in [1] the local constants $E_v(\varphi_v, \tilde{\varphi}_v, \psi)$ are already computed, and the equation (10) is explicated into the form (2). Thus our strategy is to explicate the equation (9), and then multiply it with (2) to get Theorem 2.4. This amounts to computing over local places $v \in \Sigma$:

$$C(W_v, \Phi_v, \Phi'_v) := \frac{P_\tau(f_{\Phi',\tau,v} * W_v)}{P(f_{\Phi,v} * W_v)}. \tag{13}$$

We take $\varphi' = \bigotimes \varphi'_v$ to be the vector in π' such that φ'_v is the new vector at all finite places v and φ'_∞ is the lowest weight vector. Let $\varphi(g) = \varphi'(g) \cdot \chi_{D_1}(\det g)$ be a vector in π. Then we can take the set of places Σ to be $\Sigma_0 := \{\infty, 2\} \cup \{l : l | DM\}$, provided that $\Phi_v = \Phi'_v$ are characteristic functions of $\mathrm{M}^0(\mathbb{Z}_v)$ when $v \notin \Sigma_0$, (see Proposition 2.9 for the condition on Σ). Note that the place 2 is never an unramified place when considering representations of $\widetilde{\mathrm{SL}}_2$.

We will compute $C(W_v, \Phi_v, \Phi'_v)$ with $v \in \Sigma_0$ for some specific choice of Φ_v, Φ'_v, (note that with our φ being fixed, W_v is determined up to a scalar multiple). We choose Φ_v so that $\tilde{\varphi}$ would be a vector corresponding to $g(z)$ in Theorem 2.4. This means $\tilde{\varphi}_l$ is unramified for l not a divisor of M or 2 and ∞; when $l \mid M$, $\tilde{\varphi}_l$ is the vector of lowest level; $\tilde{\varphi}_2$ is the Kohnen vector ([1, (9.4)]); $\tilde{\varphi}_\infty$ is the lowest weight vector. These conditions determine $\tilde{\varphi}$ uniquely up to scalar multiple. For Φ'_v, we choose it for the convenience of computation, and in many cases $\Phi'_v = \Phi_v$.

With our specific choices of Φ'_v, we will check in most cases

$$f_{\Phi',\tau,v} * W_v = \alpha'_v W_v \tag{14}$$

for some non-zero constant α'_v; thus

$$C(W_v, \Phi_v, \Phi'_v) := \frac{\alpha'_v P_\tau(W_v)}{P(f_{\Phi,v} * W_v)}.$$

If we also have

$$f_{\Phi,v} * W_v = \alpha_v W_v, \tag{15}$$

then the above local factor is just

$$C(W_v, \Phi_v, \Phi'_v) = \frac{\alpha'_v P_\tau(W_v)}{\alpha_v P(W_v)}. \tag{16}$$

It is convenient to use the following notation: for $\alpha \in \mathrm{PGL}_2$, $\alpha * W(g) := W(g\alpha)$; for $\tilde{\alpha} \in \widetilde{\mathrm{SL}}_2$, $\tilde{\alpha} * \tilde{W}(g) = \tilde{W}(g\tilde{\alpha})$ and $\tilde{\alpha} * \Phi = \omega_\psi(\tilde{\alpha})\Phi$; $\underline{a} = \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right)$. Recall the local theta correspondence defines a map from $\pi_v \otimes \mathcal{S}(\mathrm{M}^0(\mathbb{Q}_v))$ to $\tilde{\pi}_v$, we denote it by $\tilde{\varphi}_v = \theta(\varphi_v, \Phi_v, \psi_v) \in \tilde{\pi}_v$, (this is only defined up to a scalar multiple). Sometimes we also denote the absolute value $|D_1| \in \mathbb{Q}^+$ by $D_1^\sharp$, to distinguish from $|D_1|_v$. We need the following formula for Weil representation action (see for example [19, (2.3)]):

$$\underline{a} * \Phi(X) = \Phi(aX)|a|^{\frac{3}{2}} \gamma_\psi(a) \tag{17}$$

where $\gamma_\psi(a)$ is a root of unity.

There are six distinct cases to consider for the places in Σ_0: the infinite place, the place 2, the odd place l which is a divisor of D_1, of D_2, the odd place $l \in S_0$, or $l \in S \backslash S_0$. Below we discuss the calculation of the local constant for each case in a separate subsection. In each case, we first describe the choice of Φ_l, Φ'_l, then we check that $\tilde{\varphi}_l$ is the type of vector we specified above so that $\tilde{\varphi}$ is corresponding to $g(z)$. Finally we calculate $C(W_v, \Phi_v, \Phi'_v)$.

2.7. **Specification of measures.** The local factor $C(W_v, \Phi_v, \Phi'_v)$ is not dependent on any choice of local measure. Indeed both numerator and denominator in (13) are only dependent on measures on PGL_2 and the quotient is independent of measure. However for convenience we fix measure on $T_{\tau,v}$ over a local place $\mathbb{Q}_v$ (with $v \in \Sigma_0$) as follows. Let $K_v = \mathbb{Q}_v(\tau)$ be a quadratic algebra over $\mathbb{Q}_v$, take the measure on F_v to be self dual with respect to $\psi_{0,v}$, the measure on K_v to be self dual with respect to $\psi_{0,v} \circ tr_{K_v/\mathbb{Q}_v}$, and on $K_v^\times$ to be $\zeta_{K_v}(1)\frac{dx}{|N_{K_v/F_v}x|_v}$, on $\mathbb{Q}_v^\times$ to be $\zeta_{\mathbb{Q}_v}(1)\frac{dx}{|x|_v}$. (Here $\zeta_{\mathbb{Q}_v}$ and ζ_{K_v} are the L-functions associated to $\mathbb{Q}_v$ and K_v when v is finite place, and set to 1 when v is infinite place). The measure on $T_{\tau,v}$ is just the quotient measure on $\mathbb{Q}_v^\times \backslash K_v^\times$. The choice of local measure on PGL_2 is not important for the discussion below.

We note that if we fix the measures over $v \notin \Sigma_0$ in the same way, then the condition on local measure set in Remark 2.8 is satisfied.

2.8. $l \mid D_2$ **is odd.** In this case π_l is unramified and W_l is the unramified vector. We take $\Phi_l = \Phi'_l$ to be the characteristic function of $\mathrm{M}^0(\mathbb{Z}_l)$. Then $\tilde{\varphi}_l$ is an unramified vector in $\tilde{\pi}$.

As $\tau = k^{-1}\left(\begin{smallmatrix} & \Delta \\ 1 & \end{smallmatrix}\right)k$ for an element $k \in \mathrm{PGL}_2(\mathbb{Z}_l)$, by [19, Lemma 3.1], we can take $f_{\Phi,l} = 1_{\mathrm{PGL}_2(\mathbb{Z}_l)}$ and $f_{\Phi',\tau,l} = |\Delta|_l^{-\frac{1}{2}} 1_{\mathrm{PGL}_2(\mathbb{Z}_l)}$. Thus (14) holds with $\alpha'_l = |\Delta|_l^{-\frac{1}{2}} \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))$ and (15) holds with $\alpha_l = \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))$. The local constant is

$$C(W_l, \Phi_l, \Phi'_l) = \frac{|\Delta|_l^{-\frac{1}{2}} P_\tau(W_l)}{P(W_l)}. \tag{18}$$

2.9. $l \mid D_1$**.** Since D_1 is odd this implies l is odd. In this case π' is unramified and W'_l is the unramified vector, however π and $W_l = W'_l \chi_{D_1}(\det \cdot)$ is no longer unramified.

We take Φ'_l to be as chosen in [19, Lemma 3.2]. Namely $\Phi'_l(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right))$ is 0 if one of a, b, c is not integral, or both b and c are prime in $\mathbb{Z}_l$, or $a^2 + bc$ is in $\mathbb{Z}_l^\times$. Otherwise, we set $\Phi'_l(\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)) = \chi_{D_1}(c)$ if $c \in \mathbb{Z}_l^\times$ or $\chi_{D_1}(-b)$ if $b \in \mathbb{Z}_l^\times$. In [24, Proposition 3.4] it is shown that $\theta(\varphi_l, \Phi'_l, \psi'_l)$ where $\psi'_l(x) := e^{2\pi i x/D_1^\sharp}$ is unramified. This easily implies that $\theta(\varphi_l, \underline{D_1^\sharp} * \Phi'_l, \psi_l)$ is unramified. We let $\Phi_l = \underline{D_1^\sharp} * \Phi'_l$. Thus the corresponding $\tilde{\varphi}_l$ is unramified.

By [19, Lemma 3.2], $f_{\Phi',\tau,l} = 1_{\mathrm{PGL}_2(\mathbb{Z}_l)} |D_1|_l^{-\frac{1}{2}} \chi_{D_1}(\det \cdot)$. (Note that $L(\chi_D, 1) = 1$ in [19, Lemma 3.2].) Thus

$$f_{\Phi',\tau,l} * W_l(g) = \int_{\mathrm{PGL}_2(\mathbb{Z}_l)} W'_l(gk)\chi_{D_1}(\det gk)|D_1|_l^{-\frac{1}{2}} \chi_{D_1}(\det k)\, dk.$$

Since χ_{D_1} is quadratic and $W'_l(gk) = W'_l(g)$, we get (14) holds with

$$\alpha'_l = \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))|D_1|_l^{-\frac{1}{2}}.$$

On the other hand by (17) we have $\Phi_l(X) = \Phi'_l(D_1^\sharp X)|D_1|_l^{\frac{3}{2}} \mathfrak{g}$ where $\mathfrak{g}$ is a root of unity; explicitly $\mathfrak{g}$ is equal to $l^{-\frac{1}{2}}$ times the Gaussian sum associated to the quadratic character on the finite field F_l.

Let $\eta = \begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix}$. It satisfies $\eta \begin{pmatrix} & D_1^\sharp \\ (D_1^\sharp)^{-1} & \end{pmatrix} \eta^{-1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. We can take

$$f_{\Phi,l} = \mathfrak{g}|D_1|^{\frac{1}{2}}(1-l^{-1})1_{\mathrm{PGL}_2(\mathbb{Z}_l)}\left(\eta^{-1}\cdot\right)\chi_{D_1}(\det \eta \cdot).$$

To prove this fact we just need to show (5) (local version) in the setting where τ is replaced by $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. By the support condition of $f_{\Phi,l}$, the integral $\int_{T_l} f_{\Phi,l}(tg)\, dt$ is non-zero only when

$$g^{-1}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g = h^{-1}\begin{pmatrix} & D_1^\sharp \\ (D_1^\sharp)^{-1} & \end{pmatrix} h$$

for some $h = h_g \in \mathrm{PGL}_2(\mathbb{Z}_l)$. From the comments before [19, Lemma 3.2], we see $\Phi_l(g^{-1}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g) \neq 0$ only when g satisfies the same condition. Moreover assuming the condition is satisfied for some $h_g \in \mathrm{PGL}_2(\mathbb{Z}_l)$, then

$$\Phi_l(g^{-1}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g) = \chi_{D_1}(\det h_g)|D_1|_l^{\frac{3}{2}}\mathfrak{g}.$$

Meanwhile under the same assumption

$$\int_{T_l} f_{\Phi,l}(tg)\, dt = \mathfrak{g}|D_1|^{\frac{1}{2}}(1-l^{-1})\chi_{D_1}(\det h_g)\,\mathrm{vol}(T_l \cap \eta\,\mathrm{PGL}_2(\mathbb{Z}_l)\eta^{-1}).$$

The volume in the above expression is $(l-1)^{-1}$, thus we verified the identity (local version of) (5).

From the description of $f_{\Phi,l}$ we get

$$f_{\Phi,l} * W_l(\cdot) = \mathfrak{g}|D_1|^{\frac{1}{2}}(1-l^{-1})\,\mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_l))W_l(\cdot\eta).$$

Using the Iwasawa decomposition $\eta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D_1^\sharp & \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & -D_1^\sharp \end{pmatrix}$, we get

$$P(W_l(\cdot\begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix})) = \int_{\mathbb{Q}_l^*} \psi_l(t) W_l\left(\begin{pmatrix} tD_1^\sharp & \\ & 1 \end{pmatrix}\right)\, d^*t.$$

It is clear that $W_l(\begin{pmatrix} a & \\ & 1 \end{pmatrix}) = 0$ if $a \notin l^{-1}\mathbb{Z}_l$. Thus we can restrict the above integration to $|t|_l \leq l^2$. On the other hand over the domain $|t|_l = c$ where $c \leq l$ is fixed, we have $\psi_l(t) = 1$ and the integrand is a constant multiple of $\chi_{D_1}(t)$. Thus the integration over $|t|_l = c \leq l$ is 0. We are left with

$$\begin{aligned} P(W_l(\cdot\begin{pmatrix} 1 & D_1^\sharp \\ 1 & -D_1^\sharp \end{pmatrix})) &= \int_{|t|_l = l^2} \psi_l(t) W_l(\begin{pmatrix} tD_1^\sharp & \\ & 1 \end{pmatrix})\, d^*t \\ &= (1-l^{-1})^{-1} W_l(\begin{pmatrix} (D_1^\sharp)^{-1} & \\ & 1 \end{pmatrix})|D_1|^{\frac{1}{2}}\mathfrak{g}^{-1}. \end{aligned}$$

In conclusion the local factor is

$$C(\varphi_l, \Phi_l, \Phi_l') = |D_1|_l^{-\frac{3}{2}} \frac{P_\tau(W_l)}{W_l(\begin{pmatrix} (D_1^\sharp)^{-1} & \\ & 1 \end{pmatrix})}. \tag{19}$$

2.10. $l \in S \backslash S_0$**.** Here $l \mid M$ and $l \notin S_0$. In this case with our assumption on D_1, $\epsilon(\pi_l, \frac{1}{2}) = 1$. Take $\Phi_l = \Phi_l'$ to be the characteristic function of $\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\}$ where $c \in l\mathbb{Z}_l$ and $a, b \in \mathbb{Z}_l$. Then $\tilde{\varphi} = \tilde{\varphi}'$ is the vector in $\tilde{\pi}_l$ of the lowest level, i.e., a multiple of the vector $\tilde{\varphi}$ appearing in [1, Lemma 8.3]; the representation $\tilde{\pi}_l$ is a special representation.

Let $K_{0,l} := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathbb{Z}_l^\times, b \in \mathbb{Z}_l, c \in l\mathbb{Z}_l\}$, $w_l := \begin{pmatrix} & 1 \\ l & \end{pmatrix}$. We claim that we can take $f_{\Phi',\tau,l} = f_{\Phi,l} = 1_{K_{0,l} \cup w_l K_{0,l}}$. This amounts to check (5) (local version).

We first check the identity for $f_{\Phi,1}$. When $g \in T_l(K_{0,l} \cup w_l K_{0,l})$, it is clear $\Phi_l(g^{-1}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g) = 1$, and $\int_{T_l} f_{\Phi,l}(tg)\, dt = \mathrm{vol}(T_l \cap \mathrm{PGL}_2(\mathbb{Z}_l)) = 1$. On the other hand if g is not in $T_l(K_{0,l} \cup w_l K_{0,l})$ (as an element of $\mathrm{PGL}_2(\mathbb{Q}_l)$), then it is easy to see both the integral $\int_{T_l} f_{\Phi,l}(tg)\, dt$ and $\Phi_l(g^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g)$ equal 0. Thus the equation (5) holds when we consider $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ in place of τ.

To check the identity (5) for $f_{\Phi',\tau,l}$, we recall with our assumption in §2.2, $\tau \in K_{0,l}$. Thus there is an element $\eta \in K_{0,l}$ such that $\tau_0 \eta \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \eta^{-1} = \tau$. Here $\tau_0 \in \mathbb{Z}_l^\times$ with $\tau_0^2 = \Delta$. Thus

$$\int_{T_{\tau,l}} f_{\Phi',\tau,l}(tg)\, dt = \int_{T_l} f_{\Phi',\tau,l}(\eta t \eta^{-1} g)\, dt = \int_{T_l} f_{\Phi,l}(tg')\, dt$$

where $g' = \eta^{-1} g \eta$; here we use the fact $f_{\Phi',\tau,l} = f_{\Phi,l}$ is both left and right $K_{0,l}$ invariant. Similarly $\Phi_l'(g^{-1}\tau g) = \Phi_l((g')^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g')$ since Φ_l is invariant both under conjugation of $K_{0,l}$ and under multiplication by $\mathbb{Z}_l^\times$. Thus the identity (5) in this case follows from the case of $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$.

Now we verify the properties (14) and (15). The vector W_l being of lowest level is fixed by $K_{0,l}$. Since $\epsilon(\pi_l, \frac{1}{2}) = 1$, we have also $\pi(w_l)W_l = W_l$. Thus $f_{\Phi',\tau,l} * W_l = f_{\Phi,l} * W_l = 2\,\mathrm{vol}(K_{0,l})W_l$. From (16)

$$C(\varphi_l, \Phi_l, \Phi_l') = \frac{P_\tau(W_l)}{P(W_l)}. \tag{20}$$

2.11. $l \in S_0$**.** In this case with our assumption on D_1, $\epsilon(\pi_l, \frac{1}{2}) = -1$. Take $\Phi_l = \Phi_l'$ whose value at $\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\}$ is zero unless $a \in \mathbb{Z}_l^\times$, $b \in \mathbb{Z}_l$ and $c \in l\mathbb{Z}_l$. Otherwise it is $\chi_l(a)$ where χ_l is an odd character on $\mathbb{Z}_l^\times/(1 + l\mathbb{Z}_l)$. Then $\tilde\varphi = \tilde\varphi'$ is the vector described by [1, Proposition 8.5]. It is still the vector of lowest level in the space of $\tilde\pi_l$, this time however $\tilde\pi_l$ is a supercuspidal representation.

We can let $f_{\Phi',\tau,l} = (1_{K_{0,l}} - 1_{w_l K_{0,l}})\chi_l(\tau_1)$ and $f_{\Phi,l} = 1_{K_{0,l}} - 1_{w_l K_{0,l}}$. As in the previous case the identity (5) for case τ follows from the case $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. The only difference is that now Φ_l' is no longer invariant under multiplication by an element in $\mathbb{Z}_l^\times$, but rather equivariant with the character χ_l. We have

$$\Phi_l'(g^{-1}\tau g) = \Phi_l(\tau_0 (g')^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g') = \chi_l(\tau_0)\Phi_l((g')^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g').$$

Since $\tau_0^2 = \Delta$, we have $\chi_l(\tau_0) = \pm\chi_l(\tau_1)$; we can check from the definition of τ_0 that indeed $\chi_l(\tau_0) = \chi_l(\tau_1)$, thus we adjust $f_{\Phi',\tau,l}$ to be $f_{\Phi,l}\chi_l(\tau_1)$.

We now check (5) in case $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. As in the case of $l \in S \backslash S_0$, by support condition if g is not in $T_l(K_{0,l} \cup w_l K_{0,l})$ (as an element of $\mathrm{PGL}_2(\mathbb{Q}_l)$), both the integral $\int_{T_l} f_{\Phi,l}(tg)\, dt$ and $\Phi_l(g^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g)$ equal 0. Consider the case $g \in T_l(K_{0,l} \cup w_l K_{0,l})$; when $g \in T_1 K_{0,l}$, $\Phi_l(g^{-1} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g) = 1$ and $\int_{T_l} f_{\Phi,l}(tg)\, dt = 1$; when $g \in T_1 w_l K_{0,l}$, both expressions equal -1. Thus (5) holds.

Next we verify the properties (14) and (15). Again the vector W_l being of lowest level is fixed by $K_{0,l}$. Since $\epsilon(\pi_l, \frac{1}{2}) = -1$ now, we have in this case $\pi(w_l)W_l = -W_l$. Thus $f_{\Phi,l} * W_l = 2\,\mathrm{vol}(K_{0,l})W_l$ and

$$f_{\Phi',\tau,l} * W_l = 2\,\mathrm{vol}(K_{0,l})\chi_l(\tau_1)W_l.$$

We get

$$C(\varphi_l, \Phi_l, \Phi_l') = \frac{\chi_l(\tau_1)P_\tau(W_l)}{P(W_l)}. \tag{21}$$

2.12. $l = 2$. Take $\Phi_l = \Phi_l'$ whose value at $\{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\}$ is zero unless $a \in \mathbb{Z}_l$ and $b, c \in 2\mathbb{Z}_l$. Then $\tilde{\varphi} = \tilde{\varphi}'$ is the Kohnen vector described by [1, (9.4)]. Note that we assumed M and D_1 are odd, so π_2 is unramified and W_2 is an unramified vector. However Δ can be even for our discussion.

By [19, Lemma 3.4], $f_{\Phi',\tau,2} = |\Delta|_2^{-\frac{1}{2}} 1_{\mathrm{PGL}_2(\mathbb{Z}_2)}$ and $f_{\Phi,2} = 1_{\mathrm{PGL}_2(\mathbb{Z}_2)}$. We get the relations (14) and (15) hold with $\alpha' = |\Delta|_2^{-\frac{1}{2}} \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_2))$ and $\alpha = \mathrm{vol}(\mathrm{PGL}_2(\mathbb{Z}_2))$. It follows from (16) that

$$C(\varphi_2, \Phi_2, \Phi_2') = \frac{|\Delta|_2^{-\frac{1}{2}} P_\tau(W_2)}{P(W_2)}. \tag{22}$$

2.13. $v = \infty$. Take Φ_∞ to be the function in [24, p. 544]. It follows from [31, Remark 2.1] that $\tilde{\varphi}$ is the lowest weight vector in $\tilde{\pi}_\infty$.

Recall $\Delta > 0$. Then there exists an element $\gamma \in \mathrm{PGL}_2(\mathbb{R})$ such that

$$\gamma^{-1}\tau\gamma = \sqrt{\Delta}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R}). \tag{23}$$

Let $t_{\Delta,\infty} = \left(\begin{smallmatrix} \sqrt{\Delta} & \\ & \sqrt{\Delta}^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

Lemma 2.10. *Let $\Phi_\infty' = t_{\Delta,\infty}^{-1} * \Phi_\infty$. Then*

$$P_\tau(f_{\Phi',\tau,\infty} * W_\infty) = |\Delta|_\infty^{-\frac{3}{4}} P_\tau(\gamma * (f_{\Phi,\infty} * W_\infty)).$$

Proof. First observe

$$\begin{aligned} P_\tau(f_{\Phi',\tau,\infty} * W_\infty) &= \int_{T_{\tau,\infty}} \int_{\mathrm{PGL}_2(\mathbb{R})} f_{\Phi',\tau,\infty}(g) W_\infty(tg)\, dg\, dt \\ &= \int_{T_{\tau,\infty}} \int_{\mathrm{PGL}_2(\mathbb{R})} f_{\Phi',\tau,\infty}(tg) W_\infty(g)\, dg\, dt. \end{aligned}$$

Using the local version of (5) we get the above is:

$$\int_{\mathrm{PGL}_2(\mathbb{R})} \Phi_\infty'(g^{-1}\tau g) W_\infty(g)\, dg.$$

Now from (23) this is:

$$\begin{aligned} &\int_{\mathrm{PGL}_2(\mathbb{R})} \Phi_\infty'(g^{-1}\gamma\sqrt{\Delta}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)\gamma^{-1}g) W_\infty(g)\, dg \\ &\qquad = \int_{\mathrm{PGL}_2(\mathbb{R})} \Phi_\infty'(g^{-1}\sqrt{\Delta}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) g) W_\infty(\gamma g)\, dg. \end{aligned}$$

From (17), $t_{\Delta,\infty} * \Phi_\infty'(X) = |\Delta|_\infty^{\frac{3}{4}} \Phi_\infty'(\sqrt{\Delta}X)$ (we use the fact that over $\mathbb{R}$ the constant $\gamma_\psi(a) = 1$ when $a > 0$). The above becomes

$$\begin{aligned} &|\Delta|_\infty^{-\frac{3}{4}} \int_{\mathrm{PGL}_2(\mathbb{R})} \Phi_\infty(g^{-1}\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) g) W_\infty(\gamma g)\, dg \\ &\qquad = |\Delta|_\infty^{-\frac{3}{4}} \int_{\mathrm{PGL}_2(\mathbb{R})} \int_{T_\infty} f_{\Phi,\infty}(tg) W_\infty(\gamma g)\, dt\, dg. \end{aligned}$$

Making a change of variable $g \mapsto t^{-1}g$ we get that the above is:

$$|\Delta|_\infty^{-\frac{3}{4}} \int_{T_\infty} f_{\Phi,\infty} * W_\infty(\gamma t^{-1})\, dt = |\Delta|_\infty^{-\frac{3}{4}} \int_{T_{\tau,\infty}} f_{\Phi,\infty} * W_\infty(t\gamma)\, dt$$

where the last identity comes from the fact that the Jacobian for the change of variable $t \mapsto \gamma^{-1}t^{-1}\gamma$ is 1. □

It follows from the proof of [24, Proposition 3.5] that $f_{\Phi,\infty} * W_\infty = \alpha_\infty W_\infty$ for a non-zero scalar α_∞. Then from Lemma 2.10

$$C(W_\infty, \Phi_\infty, \Phi'_\infty) = \frac{P_\tau(f_{\Phi',\tau,\infty} * W_\infty)}{\alpha_\infty P(W_\infty)} = |\Delta|_\infty^{-\frac{3}{4}} \frac{P_\tau(\gamma * W_\infty)}{P(W_\infty)}. \tag{24}$$

Note (for some non-zero β) $W_\infty(\left(\begin{smallmatrix} a & \\ & 1\end{smallmatrix}\right)) = \beta a^k e^{-2\pi D_1^\sharp a}$ when $a > 0$ and 0 otherwise. Thus

$$\begin{aligned} P(W_\infty) &= \beta \int_0^\infty a^k e^{-2\pi D_1^\sharp a} \frac{da}{a} = \beta(2\pi D_1^\sharp)^{-k}\Gamma(k) \\ &= \frac{(k-1)!}{\pi^k} 2^{-k} e^{2\pi} W_l\left(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right)\right). \end{aligned} \tag{25}$$

2.14. **Summary.** We can apply (9) with $\Sigma = \Sigma_0$ taken to be the union of $\infty, 2$ and the prime divisors of $M\Delta$. From (18), (19), (20), (21), (22), (24) and (9), we get:

$$\frac{\tilde{W}^\Delta(\tilde{\varphi}')}{\tilde{W}(\tilde{\varphi})} = \Delta^{-\frac{1}{4}} D_1^\sharp \chi(\tau_1) \frac{c_{\pi,\Sigma_0}}{c^1_{\pi,\Sigma_0}} \frac{P_\tau(\gamma * W_\infty) \prod_{l \in \Sigma_0 \setminus \infty} P_\tau(W_l)}{\prod_{l|D_1} W_l(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right)) \prod_{l \in \Sigma_0, l \nmid D_1} P(W_l)}. \tag{26}$$

Here χ is an odd character on $(\mathbb{Z}/\prod_{l \in S_0} l)^\times$ associated to $\{\chi_l : l \in S_0\}$. The identity holds for $\tilde{\varphi} = \theta^\psi_\Phi(\varphi)$, and $\tilde{\varphi}' = \theta^\psi_{\Phi'}(\varphi)$ with Φ and Φ' described as above. We note that with our choices, $\tilde{\varphi}$ is the vector corresponding to the half integral weight form in $S_{k+\frac{1}{2}}(4M\prod_{l \in S_0} l, \chi)$ defined in [1, Theorem 10.1]. In particular $\tilde{W}(\tilde{\varphi}) \neq 0$ since we assumed $L(f, \chi_{D_1}, k) \neq 0$, (note $\tilde{W}(\tilde{\varphi})$ corresponds to the $D_1^\sharp$-th Fourier coefficient of the half integral weight form).

Use $\hat{\gamma}$ to denote the element in $\mathrm{GL}_2(\mathbb{A})$ whose infinite component is γ and all other local components are identity. Then from (6) we have

$$c_{\pi,\Sigma_0} P_\tau(\gamma * W_\infty) \prod_{l \in \Sigma_0 \setminus \infty} P_\tau(W_l) = P_\tau(\hat{\gamma} * \varphi). \tag{27}$$

Next note that for any finite place $l \nmid D_1$:

$$P(W_l) = L(\pi_l, \frac{1}{2}) W_l(e) = L(\pi_l, \frac{1}{2}) W_l\left(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right)\right).$$

The first equation is well known (see [8]–in the unramified setting it is just (11)), the second follows from the fact that W_l is $K_{0,l}$ invariant. When $l|D_1$, π is not unramified and $L(\pi, s) \equiv 1$ (again see [8]). Thus

$$\begin{aligned} & c^1_{\pi,\Sigma_0} \prod_{l|D_1} W_l(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right)) \prod_{l \in \Sigma_0, l \nmid D_1} P(W_l) \\ &= L^\infty(\pi, \frac{1}{2}) W_\varphi(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right)) / W_\infty(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right)). \end{aligned} \tag{28}$$

Here $L^\infty(\pi, s)$ is the partial L–function $\prod_{v \neq \infty} L(\pi_v, s)$. From (26), (27), (28) and (25) we get

$$\frac{\tilde{W}^\Delta(\tilde{\varphi}')}{\tilde{W}(\tilde{\varphi})} = \Delta^{-\frac{1}{4}} D_1^\sharp \chi(\tau_1) \frac{\pi^k}{(k-1)!} 2^k e^{-2\pi} \frac{P_\tau(\hat{\gamma} * \varphi)}{L^\infty(\pi, \frac{1}{2}) W_\varphi(\left(\begin{smallmatrix} (D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right))}. \tag{29}$$

We also need to interpret the Whittaker functional $\tilde{W}^{\Delta}(\tilde{\varphi}')$ in terms of the Whittaker functional of $\tilde{\varphi}$. Let $\hat{\delta}$ be an element in $\mathbb{A}^\times$ whose infinite component is $\sqrt{\frac{D_1}{D_2}}$ and all other components are 1.

Lemma 2.11.

$$\tilde{W}^{\Delta}(\tilde{\varphi}') = \chi(D_1^\sharp)\tilde{W}_{\tilde{\varphi}}^{\Delta/D_1^2}(\underline{\hat{\delta}}).$$

Proof. First recall the relation between our choices of Φ_l and Φ_l': if $l \mid D_1$, $\Phi_l = \underline{D_1^\sharp} * \Phi_l'$; at infinite place, $\Phi_\infty = t_{\Delta,\infty} * \Phi_\infty'$; at all other places $\Phi_l = \Phi_l'$. Denote by $\delta' \in \mathbb{A}^\times$ such that its infinite component is $\sqrt{\Delta}$, component at a place $l \mid D_1$ is $D_1^\sharp$ and at all other places its component is 1. Then $\Phi = \underline{\delta'} * \Phi'$, which implies $\tilde{\varphi} = \tilde{\varphi}'(\cdot\underline{\delta'})$.

From [34, p. 386], given the fact $\tilde{\varphi}$ is the vector corresponding to the half integral weight form in $S_{k+\frac{1}{2}}(4M\prod_{l\in S_0} l, \chi)$, we have

$$\tilde{\varphi} = \tilde{\varphi}(\cdot\underline{\delta''})\chi(D_1^\sharp)^{-1}$$

where $\delta'' \in \mathbb{A}^\times$ is such that its infinite component and component at a place $l \mid D_1$ is 1 and at all other places its component is $D_1^\sharp$.

Since $\delta'\delta'' = D_1^\sharp(\hat{\delta})^{-1}$, we get $\tilde{\varphi}' = \chi(D_1^\sharp)\tilde{\varphi}(\cdot\underline{D_1^\sharp}^{-1}\underline{\hat{\delta}})$. Thus

$$\begin{aligned}\tilde{W}^{\Delta}(\tilde{\varphi}') &= \chi(D_1^\sharp)\int_{\mathbb{Q}\backslash\mathbb{A}} \tilde{\varphi}(\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)\underline{D_1^\sharp}^{-1}\underline{\hat{\delta}})\psi(-\Delta x)\,dx\\ &= \chi(D_1^\sharp)\int_{\mathbb{Q}\backslash\mathbb{A}} \tilde{\varphi}(\underline{D_1^\sharp}^{-1}\left(\begin{smallmatrix}1 & (D_1^\sharp)^2 x\\ & 1\end{smallmatrix}\right)\underline{\hat{\delta}})\psi(-\Delta x)\,dx.\end{aligned}$$

Now use the fact $\underline{D_1^\sharp}^{-1}$ is a rational element thus $\tilde{\varphi}(\underline{D_1^\sharp}^{-1}\cdot) = \tilde{\varphi}(\cdot)$. Making a change of variable $x \mapsto x(D_1^\sharp)^{-2}$ we get the lemma. □

2.15. **Proof of Theorem 2.4.** We connect the notions in (29) to concepts in modular forms language.

We associate to f the automorphic form φ' of PGL_2 using the dictionary of [1, Section 9], and let $\varphi = \varphi'\chi_{D_1}(\det\cdot)$. Then $P_\tau(\hat{\gamma} * \varphi)$ is the integral $\ell(\varphi)$ defined in [23, (6.1.2)]. (It is easy to check that $\chi_{D_1}\circ\det$ when restricted to T_τ is a genus character on T_τ.) From [23, (6.1.7)]:

$$P_\tau(\hat{\gamma} * \varphi) = \Delta^{-\frac{k}{2}} i^{-k} r(f; D_2, D_1; \tau). \tag{30}$$

(Note the measure chosen by [23] differs from our choice in §2.7 by a factor of $\Delta^{\frac{1}{2}}$; see [23, p. 830].) We observe here $\chi_{D_1}\circ\det = \chi_{D_2}\circ\det$ on T_τ, thus the relation:

$$r(f; D_2, D_1; \tau) = r(f; D_1, D_2; \tau).$$

Moreover we have

$$W_\varphi\left(\left(\begin{smallmatrix}(D_1^\sharp)^{-1} & \\ & 1\end{smallmatrix}\right)\right) = W_\varphi^{1/D_1^\sharp}(e) = e^{-2\pi}, \tag{31}$$

here the first equation follows from the similar calculation done in Lemma 2.11, the second is in [1, §9.1] (note $\psi(x) = \psi_0(D_1^\sharp x)$ and recall $D_1^\sharp = |D_1|$).

Next let $g(z) = \sum_{n\geq 1} c_n e^{2\pi i n z}$ be the half integral weight form corresponding to $\tilde{\varphi}$ by the recipe of [34] (recalled in [1, §9.2]). By [34, Lemme 3], we have

$$\tilde{W}_{\tilde{\varphi}}^{\xi}(\underline{\widehat{\sqrt{a}}}) = a^{\frac{k}{2}+\frac{1}{4}} e^{-2\pi a\xi|D_1|} c(\xi|D_1|).$$

Here $\hat{x}$ is an element in $\mathrm{GL}_2(\mathbb{A})$ whose infinity component is x and all other components are identity. Thus (with $a = \xi = 1$)

$$\tilde{W}(\tilde{\varphi}) = e^{-2\pi|D_1|}c(|D_1|), \tag{32}$$

and from Lemma 2.11

$$\tilde{W}^{\Delta}(\tilde{\varphi}') = \chi(D_1^{\sharp})e^{-2\pi|D_1|}(\frac{D_1}{D_2})^{\frac{k}{2}+\frac{1}{4}}c(|D_2|). \tag{33}$$

Finally note that $L^{\infty}(\pi, \frac{1}{2}) = L(f, \chi_{D_1}, k)$. Using (30), (31), (32) and (33), the equation (29) becomes:

$$(\frac{D_1}{D_2})^{\frac{k}{2}+\frac{1}{4}}\frac{c(|D_2|)}{c(|D_1|)} = \Delta^{-\frac{1}{4}-\frac{k}{2}}|D_1|i^{-k}\chi(\frac{\tau_1}{|D_1|})\frac{\pi^k}{(k-1)!}2^k\frac{r(f; D_2, D_1; \tau)}{L(f, \chi_{D_1}, k)}.$$

Now we apply (2), and get:

$$\frac{|c_{|D_1|}|^2}{\langle g, g\rangle} = \frac{L(f, \chi_{D_1}, k)}{\langle f, f\rangle}|D_1|^{k-\frac{1}{2}}\frac{(k-1)!}{\pi^k}2^{|S|}\prod_{\ell\in S_0}\frac{\ell}{1+\ell}.$$

Our theorem (with the assumption $L(f, \chi_{D_1}, k) \neq 0$) follows from multiplying the above two equations. When $L(f, \chi_{D_1}, k) = 0$, by the above equation $c_{|D_1|} = 0$. Meanwhile it is known (see [23]) the norm square of $r(f; D_2, D_1; \tau)$ is a multiple of $L(f, \chi_{D_1}, k)$ thus equals 0. So the theorem holds also in this case.

3. Families of modular forms

We keep from now on the following notation: f is a weight 2 newform of level N, square free and odd, trivial character and rational Fourier coefficients, corresponding to an elliptic curve E. Fix a prime number $p \mid N$ and put $M := N/p$.

Choose an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. We will then identify algebraic numbers with p-adic numbers by means of this embedding without making this explicit in the following.

3.1. **Hida families.**

Let f_∞ be the Hida family passing through f, the weight 2 modular form attached to E by Taylor-Wiles's modularity theorem. More precisely, and to fix notation, there exists a compact open neighborhood $\mathcal{U}$ of 1 in $\mathbb{Z}_p$, contained in the residue class of 1 modulo $p-1$, and a formal series expansion

$$f_\infty(k) = \sum_{n\geq 1} a_n(k)q^n$$

where $a_n(k)$ are $\mathbb{C}_p$-valued rigid analytic functions on $\mathcal{U}$ (and $\mathbb{C}_p$ is the completion of a fixed algebraic closure of $\mathbb{Q}_p$), such that:

(1) For any integer $k \geq 1$ in $\mathcal{U}$, $f_k := f_\infty(k)$ is the q-expansion of a p-ordinary cusp form of weight $2k$, level $\Gamma_0(N)$ and trivial character, which is an eigenform for all Hecke operators;
(2) $f_1 = f$.

For integers $k > 1$ in $\mathcal{U}$, f_k is not p-new, and we let $f_k^{\sharp} = \sum_{n\geq 1} a_n^{\sharp}(k)q^n$ be the weight $2k$ cusp form of level $\Gamma_0(M)$ and trivial character whose p-stabilization is f_k.

3.2. **Analytic continuation of generalized Kohnen's lift.** Fix a set of divisors S_0 of M and let M' be the product of the prime numbers in S_0. Let $g = \sum_{n\geq 1} c_n q^n$ and $g_k^\sharp = \sum_{n\geq 1} c_n^\sharp(k) q^n$ be the lifts of f and $f_k^\sharp$, respectively, relative to this choice of S_0 and the choice of an auxiliary character χ' as in §2.1. Recall that s_0 is the cardinality of S_0 and χ is a character of $(\mathbb{Z}/M')^*$ determined by χ'.

Definition 3.1. Let D be a fundamental discriminant of a quadratic field such that $p \nmid D$ and the following conditions are satisfied:

(1) $\left(\frac{D}{\ell}\right) = w_\ell$ if $\ell \mid (M/M')$;
(2) $\left(\frac{D}{\ell}\right) = -w_\ell$ if $\ell \mid M'$;
(3) $(-1)^{s_0+1} = \mathrm{sgn}(D)$.

We say that D is of *type* I or *type* II if

(I) $\left(\frac{D}{p}\right) = w_p$;
(II) $\left(\frac{D}{p}\right) = -w_p$.

Remark 3.2. Conditions (1), (2) and (3) above correspond to $(*)$ of the Introduction; discriminants of type I (resp. type II) are those satisfying (†) (resp. (††)) of the Introduction.

Note that $L(f, \chi_D, 1) = 0$ and $c_{|D|} = 0$ for all D of type II, while non-vanishing results for L-functions show that there are infinitely many fundamental discriminant D_0 of type I such that $L(f, \chi_{D_0}, 1) \neq 0$ (cf. [20, Cor. 2], for example), and consequently we also have $c_{|D_0|}(1) \neq 0$. We fix such a choice of D_0 from now on.

Lemma 3.3. *There exists a neighborhood $\mathcal{U}$ of 1 in $\mathbb{Z}_p$ such that the coefficients $c^\sharp_{|D_0|}(k)$ do not vanish for all $k \in \mathcal{U}$.*

Proof. By (2), this is equivalent to showing that the same is true for the values $L(f_k^\sharp, \chi_{D_0}, k)$.

We begin by fixing for each integer $k > 1$ in $\mathcal{U}$, Shimura periods $\Omega^\pm_{f_k^\sharp}$ satisfying the additional property that

$$I^\pm(f_k^\sharp, P, r, s) := \frac{\int_r^s f_k^\sharp(z)P(z)dz \pm \overline{\int_r^s f_k^\sharp(z)P(z)dz}}{2\Omega^\pm_{f_k^\sharp}}$$

belongs to the field $K_{f_k^\sharp}$ generated over $\mathbb{Q}$ by the Fourier coefficients of $f_k^\sharp$, for all polynomials P of degree at most $k-2$ and all $r, s \in \mathbb{P}^1(\mathbb{Q})$, where $\xi \mapsto \overline{\xi}$ is complex conjugation. Define the algebraic part of the special values of the relevant L-functions to be

$$L^*(f_k^\sharp, \chi_{D_0}, k) := \frac{(k-1)! \cdot \tau(\chi_{D_0})}{(-2\pi i)^{k-1} \cdot \Omega^{w_\infty}_{f_k^\sharp}} \cdot L(f_k^\sharp, \chi_{D_0}, k),$$

and, in weight 2,

$$L^*(f, \chi_{D_0}, 1) := \frac{\tau(\chi_{D_0})}{\Omega_f^{w_\infty}} \cdot L(f, \chi_{D_0}, 1),$$

where $\tau(\chi_{D_0})$ is the Gauss sum ([3, §3.1], for example). These are algebraic numbers, which we can see as p-adic numbers by the fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Then, one can equivalently show that the values $L^*(f_k^\sharp, \chi_{D_0}, k)$ do not vanish in a neighborhood of 1.

Recall the Mazur-Kitagawa p-adic L-function $L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k, s)$, in two variables k and s for which we use the notation in [3, Sec. 3] (except that here the weight variable is $2k$ instead of k in loc. cit.; to avoid confusion, we require that, for a fixed $k = k_0 \in \mathbb{Z} \cap \mathcal{U}$, $k_0 \geq 1$, the function $L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k_0, s)$ is the cyclotomic p-adic L-function of f_{2k_0} instead of f_{k_0} as in [3]). Its definition requires the choice of a sign at infinity w_∞ corresponding to the choice of the w_∞ eigencomponent for the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on modular symbols, and the choice of the corresponding Shimura period $\Omega_{f_k^\sharp}^{w_\infty}$ and $\Omega_f^{w_\infty}$ (cf. [3, §1.1]). We make the choice of w_∞ so that (note k is odd)

$$\chi_{D_0}(-1) = (-1)^{k-1} w_\infty = w_\infty.$$

Then, by [3, Theorem 3.1] we have

$$L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, 1, 1) = (1 - \chi_{D_0}(p) a_p^{-1}) \cdot L^*(f, \chi_{D_0}, 1) \tag{34}$$

and, by [3, Corollary 2.3], we have

$$L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k, k) = \lambda(k) \cdot (1 - \chi_{D_0}(p) a_p^{-1}(k) p^{k-1})^2 \cdot L^*(f_k^\sharp, \chi_{D_0}, k).$$

The $\bar{\mathbb{Q}}_p$-valued function $\lambda \mapsto \lambda(k)$ is non-zero in a neighborhood of 1, by [2, Proposition 1.7]. The choice of D_0 implies that $\chi_{D_0}(p) = w_p$, and since $a_p = -w_p$, we see that $1 - \chi_{D_0}(p) a_p^{-1} \neq 0$. Thus, since also $L(f, \chi_{D_0}, 1) \neq 0$, the Mazur-Kitagawa p-adic L-function does not vanish at $(1, 1)$; since it is p-adic analytic in a neighborhood of $(1, 1)$, it follows then that there exists a neighborhood of 1 where it is non-zero, which proves the non-vanishing of the algebraic parts of the L-functions of $f_k^\sharp$ in a neighborhood of 1. □

Fix $\mathcal{U}$ as in Lemma 3.3. Define for each $k \in \mathcal{U} \cap \mathbb{Z}$, $k > 1$, and D a fundamental discriminant, the normalized Fourier coefficients

$$\tilde{c}_{|D|}(k) := \frac{1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)}{1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)} \cdot \frac{c_{|D|}^\sharp(k)}{c_{|D_0|}^\sharp(k)}. \tag{35}$$

Let $\mathbb{Q}(\chi)$ be the subfield of $\bar{\mathbb{Q}}$ generated by the values of χ. Via our fixed embedding, we will also view $\mathbb{Q}(\chi)$ as a subfield of $\bar{\mathbb{Q}}_p$.

Proposition 3.4. *Let D be a fundamental discriminant of type* I *or* II. *After replacing $\mathcal{U}$ by a smaller neighborhood of* 1 *in $\mathbb{Z}_p$, we can ensure that the normalized coefficients $\tilde{c}_{|D|}(k)$ extend to a p-adic analytic function on $\mathcal{U}$, whose value at* 1 *is*

$$\tilde{c}_{|D|}(1) = \frac{c_{|D|}}{c_{|D_0|}}.$$

Finally, $\tilde{c}_{|D|}(1)$ belongs to $\mathbb{Q}(\chi)$.

Proof. Fix a neighborhood as in Lemma 3.3 to start with. Note that

$$\frac{c_{|D|}^\sharp(k)}{c_{|D_0|}^\sharp(k)} = \frac{c_{|D|(k)}^\sharp \cdot \overline{c_{|D_0|}^\sharp(k)}}{|c_{|D_0|}^\sharp(k)|^2}.$$

Assume first that $(D, D_0) = 1$. Combining (2) and Theorem 2.4 we find:

$$\tilde{c}_{|D|}(k) = \frac{1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)}{1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)} \cdot \frac{(-2\pi i)^k \cdot \chi(|D|)^{-1}}{|D_0|^{k-1/2} \cdot (k-1)!} \cdot \frac{r_\chi(f_k^\sharp, D, D_0)}{L(f_k^\sharp, \chi_{D_0}, k)}.$$

Using the expression in the proof of the lemma above for $L(f_k^\sharp, \chi_{D_0}, k)$ in terms of the Mazur-Kitagawa p-adic L-function, we find

$$\tilde{c}_{|D|}(k) = \frac{-\tau(\chi_{D_0}) \cdot \chi(|D|)^{-1}}{|D_0|^{k-1/2}} \cdot \frac{1}{L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, k, k)} \cdot \big(1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\big) \cdot \big(1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\big) \cdot \frac{\lambda(k) \cdot (2\pi i) \cdot r_\chi(f_k^\sharp, D, D_0)}{\Omega_{f_k^\sharp}^{\omega_\infty}}.$$

Here, as in the proof of the above lemma, we make the choice of $w_\infty = \chi_{D_0}(-1)$.

Suppose that D is of type II and $(D, D_0) = 1$. Since D_0 and D are of different types, we have

$$\big(1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\big) \cdot \big(1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\big) = 1 - a_p^{-2}(k)p^{2k-2}.$$

One observes now that

$$\lambda(k) \cdot \big(1 - a_p^{-2}(k)p^{2k-2}\big) \cdot \frac{(2\pi i) \cdot r_\chi(f_k^\sharp, D, D_0)}{\Omega_{f_k^\sharp}^{\omega_\infty}} = \mathcal{L}_p^{\mathrm{BD}}(f_\infty/\mathbb{Q}(\sqrt{D \cdot D_0}), \chi_{D,D_0}, k)$$

where the RHS is (up to a constant multiple) the p-adic L-function defined by Bertolini and Darmon in [3, Definition 3.4, (1)] (note that the prime p is inert in $\mathbb{Q}(\sqrt{D \cdot D_0})$, and all primes dividing M are split; also note the usual shift of notation in the weight, so actually, and more precisely, by $\mathcal{L}_p^{\mathrm{BD}}(f_\infty/\mathbb{Q}(\sqrt{D \cdot D_0}), \chi_{D,D_0}, k)$ we mean the function $\mathcal{L}_p(f_\infty/\mathbb{Q}(\sqrt{D \cdot D_0}), \chi_{D,D_0}, 2k)$ in loc. cit.). Therefore, we can express $\tilde{c}_{|D|}(k)$ as a product of factors, each of them extending to a p-adic analytic function in a neighborhood of 1, and therefore the extension of normalized coefficients follows. Further, $L_p^{\mathrm{BD}}(f_\infty/\mathbb{Q}(\sqrt{D \cdot D_0}), \chi_{D,D_0}, 1) = 0$ (cf. [3, Sec. 4]) and, since $c_{|D|} = 0$, we have the claimed equality

$$\tilde{c}_{|D|}(1) = c_{|D|} = \frac{c_{|D|}}{c_{|D_0|}}.$$

Suppose now that D is of type I and $(D, D_0) = 1$. Then all primes dividing $N = Mp$ are split in $\mathbb{Q}(\sqrt{D \cdot D_0})$. Since D_0 and D are of the same type, we have

$$\big(1 - \chi_D(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\big) \cdot \big(1 - \chi_{D_0}(p) \cdot p^{k-1} \cdot a_p^{-1}(k)\big) = \big(1 - \chi_{D_0}(p)a_p^{-1}(k)p^{k-1}\big)^2.$$

In this case, we have

$$\lambda(k) \cdot \big(1 - \chi_{D_0}(p)a_p^{-1}(k)p^{k-1}\big)^2 \cdot \frac{(2\pi i) \cdot r_\chi(f_k^\sharp, D, D_0)}{\Omega_{f_k^\sharp}^{\omega_\infty}} = \mathcal{L}_p^{\mathrm{Sh}}(f_\infty/\mathbb{Q}(\sqrt{D \cdot D_0}), \chi_{D,D_0}, k)$$

where the RHS is the p-adic analytic function (up to a constant multiple) defined in S. Shahabi's thesis [30, §. 3.2], and the above formula is [30, Prop. 3.3.1], except for the usual shift of weight. The extension of the normalized coefficients follows. Its value at 1 is given by

$$\frac{-\tau(\chi_{D_0}) \cdot \chi(|D|)^{-1}}{|D_0|^{1/2}} \cdot \frac{\mathcal{L}_p^{\mathrm{Sh}}(f_\infty/\mathbb{Q}(\sqrt{D \cdot D_0}), \chi_{D,D_0}, 1)}{L_p^{\mathrm{MK}}(f_\infty, \chi_{D_0}, 1, 1)}.$$

By [9, Prop. 5.5], or [18, Prop. 4.24], we have

$$\mathcal{L}_p^{\mathrm{Sh}}(f_\infty/\mathbb{Q}(\sqrt{D\cdot D_0}),\chi_{D,D_0},1)=2\cdot\frac{(2\pi i)\cdot r_\chi(f,D,D_0)}{\Omega_f^{w_\infty}}$$

and, by (34),

$$L_p^{\mathrm{MK}}(f_\infty,\chi_{D_0},1,1)=2\cdot\frac{\tau(\chi_{D_0})}{\Omega_f^{w_\infty}}\cdot L(f,\chi_{D_0},1)$$

and therefore the value $\tilde{c}_{|D|}(1)$ is given by

$$\frac{-2\pi i\cdot\chi(|D|)^{-1}}{|D_0|^{1/2}}\cdot\frac{r_\chi(f,D,D_0)}{L(f,\chi_{D_0},1)}.$$

This is equal to $c_{|D|}/c_{|D_0|}$ again by a combination of (2) and Theorem 2.4, and the statement follows.

The rationality of $c_{|D|}/c_{|D_0|}$ follows from results on the rationality of the factors appearing in the above factorizations. More precisely, one may choose Shimura periods $\Omega_f^{w_\infty}$ satisfying $\frac{2\pi\cdot r_\chi(f,D,D_0)}{\Omega_f^{w_\infty}}\in\mathbb{Q}(\chi)$ because f has integer Fourier coefficients (see [32], for example), and then, with this choice of Shimura periods, $\frac{\tau(\chi_{D_0})\cdot L(f,\chi_{D_0},1)}{\Omega_f^{w_\infty}\cdot|D_0|^{1/2}}$ belongs to $\mathbb{Q}(\chi)$ (see [20], for example).

We finally deal with the remaining case of $(D,D_0)\neq 1$. One chooses an auxiliary discriminant D_0', prime to both D and D_0, satisfying the same conditions of D_0 (this is possible by [20]). Then, express $\tilde{c}_{|D|}(k)$ as the product

$$\left(\frac{1-\chi_D(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}{1-\chi_{D_0'}(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}\cdot\frac{c^\sharp_{|D|}(k)}{c^\sharp_{|D_0'|}(k)}\right)\cdot\left(\frac{1-\chi_{D_0'}(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}{1-\chi_{D_0}(p)\cdot p^{k-1}\cdot a_p^{-1}(k)}\cdot\frac{c^\sharp_{|D_0'|}(k)}{c^\sharp_{|D_0|}(k)}\right)$$

and repeat the above argument to each of the two factors in parenthesis appearing above, using the previously proved cases. This concludes the proof. □

Thus, for D_1 of type I and D_2 of type II, we have $\tilde{c}_{D_2}(1)=0$ even if the function $k\mapsto\tilde{c}_{D_2}(k)$ is not a priori identically zero on the neighborhood $\mathcal{U}$ of 1 where it is defined (this follows because $\mathcal{L}_p^{\mathrm{BD}}(f_\infty/\mathbb{Q}(\sqrt{D_1\cdot D_2}),\chi_{D_1,D_2},k)$ is not necessarily zero). It is naturally of interest to investigate then the value at 1 of its p-adic derivative, $\left(\frac{d}{dk}\tilde{c}_{D_2}(k)\right)_{|k=1}$.

Remark 3.5. It might be interesting to prove Proposition 3.4 directly, in a way similar to [7, Prop. 1.3], using arguments borrowed from the proof of [32, Thm. 5.5] (and its sequels [22], [14], [15]). Formally, our proof makes a systematic recourse to p-adic L-functions instead; however, note that the principle of our proof (i.e., the construction of p-adic L-functions) and the proof of [32, Thm. 5.5] share the same fundamental tool, namely, the p-adic interpolation of complex integrals (which are Shintani integrals for p-adic L-functions over real quadratic extensions), and originated from the seminal paper [11].

We finally need to understand the action of complex conjugation on these normalized coefficients. Let

$$i:\mathbb{Q}(\chi)\longrightarrow\bar{\mathbb{Q}}\longrightarrow\bar{\mathbb{Q}}_p$$

be obtained by composition with the fixed embedding $\bar{\mathbb{Q}}\hookrightarrow\bar{\mathbb{Q}}_p$. Let $c\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be a fixed complex conjugation. The composition of c on $\mathbb{Q}(\chi)$, viewed as a subfield of $\bar{\mathbb{Q}}$, with i gives rise to a second embedding

$$i^* := i \circ c : \mathbb{Q}(\chi) \hookrightarrow \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$$

(use that $\mathbb{Q}(\chi)$ is Galois over $\mathbb{Q}$). For each integer $k \in \mathcal{U}$, and each fundamental discriminant D prime to D_0, we thus have p-adic numbers

$$\overline{\tilde{c}_{|D|}(k)} := i^* \left(\tilde{c}_{|D|}(k) \right).$$

The map $i(\mathbb{Q}(\chi)) \to i^*(\mathbb{Q}(\chi))$ defined by $i(a) \mapsto i^*(a)$ clearly extends on the p-adic completions, and we denote the resulting map with the symbol $x \mapsto \bar{x}$. Thus, from Proposition 3.4, the function $k \mapsto \overline{\tilde{c}_{|D|}(k)}$ extends to a p-adic analytic function on $\mathcal{U}$, denoted by the same symbol, whose value at 1 is $\overline{\tilde{c}_{|D|}(1)} = \frac{\overline{c_{|D|}}}{c_{|D_0|}}$, which belongs to $i^*(\mathbb{Q}(\chi)) \subseteq \bar{\mathbb{Q}}_p$.

4. Darmon-Tornaría conjecture

We keep the notation of §3: f is a weight 2 newform of level N, square free and odd, trivial character and rational Fourier coefficients, corresponding to an elliptic curve E; $p \mid N$ is a prime and put $M := N/p$.

4.1. **Rational points on elliptic curves.** Let $K = \mathbb{Q}(\sqrt{\Delta})$ be a quadratic imaginary field where all primes dividing M are split and the prime p is inert. Fix Shimura's periods $\Omega^{\pm}_{f_k^\sharp}$ as in Lemma 3.3 for each integer $k > 1$ in $\mathcal{U}$. Recall the definition of Shintani integrals $r(f_k^\sharp, Q)$ in (3) attached to $f_k^\sharp$ and the quadratic form $Q \in \mathcal{F}_\Delta$, and put

$$\tilde{r}^{\pm}(k, Q) := \left(1 - \frac{p^{2k-2}}{a_p(k)^2}\right) \frac{(r(f_k^\sharp, Q) \pm \overline{r(f_k^\sharp, Q)})}{2\Omega^{\pm}_{f_k^\sharp}}.$$

Then these values belong to $K_{f_k^\sharp}$. Similarly, let $\tau_Q := \frac{-B+\sqrt{\Delta}}{2A}$, where recall that $Q(x,y) = Ax^2 + Bxy + Cy^2$, and define

$$I^{\pm}(k, Q) := \left(1 - \frac{p^{2k-2}}{a_p(k)^2}\right) I^{\pm}(f_k^\sharp, (z - \tau_Q)^{k-2}, z_0, \gamma_Q z_0)$$

which belong again to $K_{f_k^\sharp}$ (this is independent on the choice of z_0 and only depends on the $\Gamma_0(M)$-equivalence class of Q).

Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Then τ_Q, as well as elements in $K_{f_k^\sharp}$, can be alternatively viewed in $\bar{\mathbb{Q}}$ or $\bar{\mathbb{Q}}_p$. We collect here the relevant facts about these integrals (see [7, Sec. 3] for proofs):

(a) The functions $k \mapsto r^{\pm}(k, Q)$ and $k \mapsto I^{\pm}(k, Q)$, defined for integers $k > 1$ in $\mathcal{U}$, extend to p-adic analytic functions on $\mathcal{U}$, taking values in $\bar{\mathbb{Q}}_p$, which we denote by the same symbols $r^{\pm}(k, Q)$ and $I^{\pm}(k, Q)$.

(b) We have $I^{\pm}(1, Q) = 0$.

(c) If we denote by $\xi \mapsto \varsigma(\xi)$ the non-trivial automorphism of the quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$, and we define

$$\vartheta^{\pm}(f, Q) := \left(\frac{d}{dk} I^{\pm}(k, Q)\right)_{|k=1},$$

then $\vartheta^{\pm}(f,Q)$ belongs to $\mathbb{Q}_{p^2}$ and

$$\vartheta^{\pm}(f,Q)+\varsigma(\vartheta^{\pm}(f,Q))=2\left(\frac{d}{dk}\tilde{r}^{\pm}(k,Q)\right)_{|k=1}. \tag{36}$$

Let χ_{D_1,D_2} be the genus character associated to a factorization $\Delta = D_1 \cdot D_2$ of the discriminant of K, as in §2. Recall that χ_{D_1,D_2} is associated with the pair of Dirichlet characters (χ_{D_1},χ_{D_2}), with associated quadratic fields $K_i = \mathbb{Q}(\sqrt{D_i})$, $i = 1,2$. Let $\epsilon = +1$ if the K_i's are both real and $\epsilon = -1$ if the K_i's are both imaginary. Following the terminology in [3, Def. 3.4], the character χ_{D_1,D_2} is said to be *even* in the first case, and *odd* in the second. Since $\chi_{D_1}\cdot\chi_{D_2} = \chi_K$, which is the quadratic character associated with K, then $\chi_{D_1}(\ell) = \chi_{D_2}(\ell)$ for all $\ell \mid M$, while $\chi_{D_1}(p) = -\chi_{D_2}(p)$.

Define the following p-adic number (in $\mathbb{Q}_{p^2}$):

$$\vartheta(f,D_1,D_2;\tau) := \sum_{[Q]\in\mathcal{F}_\Delta/\Gamma_0(M)} \chi_{D_1,D_2}(Q)\vartheta(f,Q)$$

and let H_{D_1,D_2} be the quadratic extension of K determined by χ_{D_1,D_2}. Also, denote by

$$\log_E : E(\bar{\mathbb{Q}}_p) \longrightarrow \bar{\mathbb{Q}}_p$$

the p-adic formal logarithm defined by $\log_E(P) := \log_q(\Phi_{\mathrm{Tate}}^{-1}(P))$, where q is Tate's period of E at p, $\log_q$ is the branch of the p-adic logarithm satisfying $\log_q(q) = 0$ and Φ_{Tate} is Tate's uniformization of the elliptic curve. The following result is [3, Thm. 4.3].

Theorem 4.1 (Bertolini-Darmon). *Suppose that* (χ_{D_1},χ_{D_2}) *satisfies*

$$\chi_{D_1}(-M) = \chi_{D_2}(-M) = -w_M, \qquad \chi_{D_2}(p) = -\chi_{D_1}(p) = -w_p.$$

There exists a point

$$P_{D_1,D_2} \in (E(H_{D_1,D_2})\otimes_{\mathbb{Z}}\mathbb{Q})^{\chi_{D_1,D_2}},$$

in the subspace of $E(H_{D_1,D_2})\otimes_{\mathbb{Z}}\mathbb{Q}$ *where the Galois group* $\mathrm{Gal}(H_{D_1,D_2}/K)$ *acts via the character* χ_{D_1,D_2}*, such that:*

1. $\log_E(P_{D_1,D_2}) = \vartheta(f,D_1,D_2;\tau)$*;*
2. P_{D_1,D_2} *is non-zero if and only if* $L'(E,\chi_{D_2},1)\neq 0$.

4.2. **Darmon points and generalized Kohnen lifts.** We fix the sign ϵ as in §4.1 taking $\epsilon = 1$ for χ_{D_1,D_2} even and $\epsilon = -1$ for χ_{D_1,D_2} odd.

Recall the choice of the periods $\Omega^{\pm}_{f_k^\sharp}$ made in §4 and the fundamental discriminant D_0 of type I chosen in §2. By [7, Lemma 3.3], these periods can be chosen so that, after replacing $\mathcal{U}$ by a smaller neighborhood, the following equality holds:

$$\Omega^{\epsilon}_{f_k^\sharp} = \left(1 - w_p\frac{p^{k-1}}{a_p(k)}\right)^2 r(f_k^\sharp, D_0, D_0). \tag{37}$$

We will assume to have done this choice from now on.

Recall that g is the generalized Kohnen-Waldspurger lift in [1] and $\tilde{c}_{|D|}(k)$ are the normalized coefficients introduced in (35).

Definition 4.2. The *normalized p-adic period* attached to f, χ, D_1 and D_2 is the p-adic number:

$$\vartheta_\chi(f,D_1,D_2) := \chi(\tau_1)\cdot\vartheta(f,D_1,D_2;\tau).$$

Remark 4.3. As in Remark 2.2, note that $\vartheta_\chi(f, D_1, D_2)$ is independent of the choice of τ, justifying the notation.

Proposition 4.4. *Let* D_1 *(resp.* D_2*) be of type* I *(resp. type* II*). Then*

$$\chi(|D_1|)\cdot\overline{\tilde{c}_{|D_1|}(1)}\cdot\left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1} \quad=\quad \vartheta_\chi(f, D_1, D_2).$$

Proof. Notice that, for all integers $k>1$ in $\mathcal{U}$, we have

$$\overline{\tilde{c}_{|D_1|}(k)}\cdot\tilde{c}_{|D_2|}(k)=\frac{\left(1-\frac{p^{2k-2}}{a_p(k)^2}\right)\overline{c^\sharp_{|D_1|}(k)}c^\sharp_{|D_2|}(k)}{\left(1-w_p\frac{p^{k-1}}{a_p(k)}\right)^2|c^\sharp_{|D_0|}(k)|^2}.$$

By Theorem 2.4, we have then

$$\overline{\tilde{c}_{|D_1|}(k)}\cdot\tilde{c}_{|D_2|}(k)=\chi(|D_1|)^{-1}\cdot\frac{\left(1-\frac{p^{2k-2}}{a_p(k)^2}\right)r_\chi(f^\sharp_k, D_1, D_2)}{\left(1-w_p\frac{p^{k-1}}{a_p(k)}\right)^2 r(f^\sharp_k, D_0, D_0)}.$$

Therefore, using (37),

$$\overline{\tilde{c}_{|D_1|}(k)}\cdot\tilde{c}_{|D_2|}(k)=\chi(|D_1|)^{-1}\cdot\left(1-\frac{p^{2k-2}}{a_p(k)^2}\right)\frac{r_\chi(f^\sharp_k, D_1, D_2)}{\Omega^\epsilon_{f^\sharp_k}}. \tag{38}$$

With the above choice of ϵ, we have $r(f^\sharp_k, Q)=\tilde{r}^\epsilon(k, Q)$ (cf. [3, eq. (27)], or [9, Lemma 3.4], [18, §4.3]). Combining (36) and [7, eq. (16)] we get

$$\vartheta_\chi(f, D_1, D_2)=\chi(|D_1|)\cdot\frac{d}{dk}\left(\overline{\tilde{c}_{|D_1|}(k)}\cdot\tilde{c}_{|D_2|}(k)\right)_{|k=1}. \tag{39}$$

Differentiating (38), using that $\tilde{c}_{D_2}(1)=0$ because D_2 is of type II, and substituting (39) we get the result. □

We now apply Theorem 4.1 in this situation. Before doing this, we observe that, for fundamental discriminants D_1 and D_2 of type I and II respectively, the condition

$$\chi_{D_2}(p)=-\chi_{D_1}(p)=-w_p$$

is (I) and (II) in Definition 3.1, respectively, while the condition

$$\chi_{D_1}(-M)=\chi_{D_2}(-M)=-w_M$$

is equivalent to

$$\chi_{D_1}(-1)=\chi_{D_2}(-1)=(-1)^{s_0+1},$$

where recall that s_0 is the cardinality of the set S_0 and M' is the product of the primes in S_0: this is because $\chi_{D_1}(\ell)=\chi_{D_2}(\ell)=w_\ell$ for all primes $\ell\mid(M/M')$ (by (1) in Definition 3.1) and $\chi_{D_1}(\ell)=\chi_{D_2}(\ell)=-w_\ell$ for all primes dividing M' (by (2) in Definition 3.1). Thus, $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$ are both real or imaginary, accordingly with the parity of s_0: odd in the first case, even in the second, and this is precisely condition (3) in Definition 3.1 (which, of course, agrees with $(*)$ required in the Introduction of the paper).

Theorem 4.5. *Let* D_1 *be of type* I *and* D_2 *of type* II*. Also assume that* $c_{|D_1|} \neq 0$*. There exists a point*

$$\mathbf{P} \in (E(H_{D_1,D_2}) \otimes_{\mathbb{Z}} \mathbb{Q}(\chi))^{\chi_{D_1,D_2}}$$

such that:

(1) $\log_E(\mathbf{P}) = \left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1}$;
(2) $\mathbf{P}$ *is non-zero if and only if* $L'(E, \chi_{D_2}, 1) \neq 0$.

Proof. Combining Proposition 4.4 and Theorem 4.1, we see that, for $P = P_{D_1,D_2}$ as in Theorem 4.1,

$$\log_E(P) = \chi\left(\frac{|D_1|}{\tau_1}\right) \cdot \overline{\tilde{c}_{|D_1|}(1)} \cdot \left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1}.$$

Since $\tilde{c}_{|D_1|}(1)$ belongs to $\mathbb{Q}(\chi)$, assertion (1) follows with

$$\mathbf{P} = P \otimes \left(\chi\left(\frac{|D_1|}{\tau_1}\right) \cdot \overline{\tilde{c}_{|D_1|}(1)}\right).$$

Finally, Proposition 3.4 shows that $\tilde{c}_{|D_1|}(1) \neq 0$ if and only if $c_{|D_1|} \neq 0$, so the second part of Theorem 4.1 shows that this point is non-zero if and only if $L'(E, \chi_{D_2}, 1) \neq 0$, thus showing assertion (2) and finishing the proof. □

We close this section with another application, which establishes equation (1) of the Introduction.

Theorem 4.6. *Let* D *be a fundamental discriminant of type* II*. There exists a point* $P_D \in E(\bar{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q}(\chi)$*, which is non-zero if and only if* $L'(E, \chi_D, 1) \neq 0$*, such that*

$$\log_E(P_D) = \left(\frac{d}{dk}\tilde{c}_{|D|}(k)\right)_{|k=1}.$$

Further, if $D < 0$*, then we may take* $P_D \in E(\sqrt{D}) \otimes_{\mathbb{Z}} \mathbb{Q}(\chi)$*.*

Proof. Put $D_2 := D$. Fix a discriminant D_1 of type I such that $(D_1, D_2) = 1$ and $c_{|D_1|}$. Let $\Delta := D_1 \cdot D_2$ be the discriminant of the totally real field $K = \mathbb{Q}(\sqrt{\Delta})$. Then one can apply Theorem 4.7 and obtain the first part of the statement for $P_D = \mathbf{P}$. The second part follows from the proof of [3, Theorem 4.3] because if $D < 0$, then the point P_D actually belongs to the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. □

4.3. **Generating series and a conjecture of Darmon-Tornaría.** Based on the work accomplished thus far, we are in position to address the conjecture of Darmon and Tornaría in [7, Conj. 5.3, Case 1]. Let D_1 (resp. D_2) be of type I (resp. type II), coprime to each other with D_1 and D_2 both positive (resp. negative) if s_0 is odd (resp. even). Put

$$\eta_\chi(f, D_1, D_2) := \chi^{-1}(|D_1|) \cdot \vartheta_\chi(f, D_1, D_2)$$

for D_1 of type I and D_2 of type II. Let g be given as in §2.1 and $g^* := \sum_n \overline{c_n} q^n$ be the form obtained from $g = \sum_n c_n q^n$ by applying the complex conjugation.

Theorem 4.7. *The coefficients* $\eta_\chi(f, D_1, D_2)$ *for* D_1 *of type* I *are proportional to the* $|D_1|$*-th coefficient of* g^**, and they do not vanish identically if and only if* $L'(E, \chi_{D_2}, 1) \neq 0$*.*

Proof. Combining Proposition 4.4 and Proposition 3.4, and using that $c_{D_0} \neq 0$, we have

$$\vartheta_\chi(f, D_1, D_2) = \frac{\chi(|D_1|) \cdot \left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1}}{\overline{c_{|D_0|}}} \overline{c_{|D_1|}}.$$

Theorem 4.5 shows that the coefficient of proportionality $\frac{\left(\frac{d}{dk}\tilde{c}_{|D_2|}(k)\right)_{|k=1}}{c_{|D_0|}}$ is non-zero if and only if $L'(E, \chi_{D_2}, 1) \neq 0$. □

Remark 4.8. We can remove the dependence on the character χ in the above theorem. Fix a congruence class m in $\mathbb{Z}/M'\mathbb{Z}$ and let $g_m := \sum_n c_n^{(m)} q^n$ where $c_n^{(m)} = c_n$ if $n = m \in \mathbb{Z}/M'\mathbb{Z}$ and $c_n^{(m)} = 0$ if $n \neq m \in \mathbb{Z}/M'\mathbb{Z}$. Then g_m is a half integral weight form with respect to the congruence group $\Gamma_1(4MM')$. We get from the above theorem that $\vartheta(f, D_1, D_2, \tau)$ is proportional to the $|D_1|$-th coefficient of g_m^* whenever D_1 is of type I and $|D_1| = m \in \mathbb{Z}/M'\mathbb{Z}$. They do not vanish identically if and only if $L'(E, \chi_{D_2}, 1) \neq 0$.

The above theorem gives a positive answer to [7, Conj. 5.3]. This conjecture says that the coefficients $b(D) := J(f, D, D_2)$ (notation as in [7]; here D_2 is of type II and D_1 of type I as above) are proportional to the D-th coefficient of a modular form of level $\Gamma_0(4N^2)$ and weight 3/2; in our notation, $J(f, D, D_2)$ is $\vartheta(f, D_1, D_2; \tau)$. Note that Darmon and Tornaría also observe that these coefficients are really only defined up to sign, since they depend on the choice of a square root (as our notation makes explicit) and that, therefore, [7, Conj. 5.3] only makes sense up to sign ([7, §6.2]). Seen in this perspective, our result suggests a way to treat this sign with ambiguity, by introducing the above coefficients $\vartheta_\chi(f, D_1, D_2)$ and $\eta_\chi(f, D_1, D_2)$, which do not depend on the choices made, and proves that $\eta_\chi(f, D_1, D_2)$ are the Fourier coefficients of a modular form, in the spirit of [7, Conj. 5.3].

Remark 4.9. The generalization of the main results of Darmon-Tornaría in [7] to higher weight modular forms has been obtained in the Ph.D. thesis of G. Harikumar [12]. In this work one starts with a modular form f of level $\Gamma_0(N)$ and even weight $2k \geq 2$ and considers the Coleman family passing though f. As in [7] and in this paper, one may define p-adic families of Fourier coefficients of Shimura-Shintani lifts of classical forms in the Coleman family. Replacing Darmon points in [7] and in this paper with Darmon cycles introduced in [29], [28], [26], [9], and using results of [9] in combination with Kohnen's formula [13], one proves a relation between the Bloch-Kato logarithm of Darmon cycles and p-adic derivatives of these families of Fourier coefficients of half integral weight modular forms. We refer to [12] for more details. It is clear that all results of this paper can be generalized to higher weight modular forms following the approach of [12], since the automorphic calculations in §2 and the p-adic techniques of [9] hold without any change. The interested reader will easily state and prove the analogue of the main results of [12] in the more general setting of this paper.

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