THE GYSIN TRIANGLE VIA LOCALIZATION AND A^1-HOMOTOPY INVARIANCE

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ABSTRACT. Let X be a smooth scheme, Z a smooth closed subscheme, and U the open complement. Given any localizing and $\mathbb{A}^1$-homotopy invariant of dg categories E, we construct an associated Gysin triangle relating the value of E at the dg categories of perfect complexes of X, Z, and U. In the particular case where E is homotopy K-theory, this Gysin triangle yields a new proof of Quillen's localization theorem, which avoids the use of devissage. As a first application, we prove that the value of E at a smooth scheme belongs to the smallest (thick) triangulated subcategory generated by the values of E at the smooth projective schemes. As a second application, we compute the additive invariants of relative cellular spaces in terms of the bases of the corresponding cells. Finally, as a third application, we construct explicit bridges relating motivic homotopy theory and mixed motives on the one side with noncommutative mixed motives on the other side. This leads to a comparison between different motivic Gysin triangles as well as to an étale descent result concerning noncommutative mixed motives with rational coefficients.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *differential graded (=dg) category* $\mathcal{A}$, over a base field k, is a category enriched over complexes of k-vector spaces; see §4.1. Every (dg) k-algebra A naturally gives rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X admits a canonical dg enhancement[1] $\mathrm{perf}_{\mathrm{dg}}(X)$; see Keller [24, §4.6]. Let us denote by $\mathrm{dgcat}(k)$ the category of (essentially small) dg categories, and by $\mathrm{Hmo}(k)$ its localization at the class of Morita equivalences.

A functor $E\colon \mathrm{dgcat}(k) \to \mathcal{T}$, with values in a triangulated category, is called:

(C1) a *localizing invariant* if it inverts the Morita equivalences (or equivalently if it factors through the category $\mathrm{Hmo}(k)$) and sends short exact sequences of dg categories (see §4.2) to distinguished triangles

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0 \quad \mapsto \quad E(\mathcal{A}) \longrightarrow E(\mathcal{B}) \longrightarrow E(\mathcal{C}) \xrightarrow{\partial} \Sigma E(\mathcal{A})$$

in a way which is functorial for strict morphisms of exact sequences;

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[1]When X is quasi-projective this dg enhancement is unique; see [34, Thm. 2.12], [9, Thm. B].

(C2) an $\mathbb{A}^1$-*homotopy invariant* if it inverts the canonical dg functors $\mathcal{A} \to \mathcal{A}[t]$, where $\mathcal{A}[t]$ stands for the tensor product of $\mathcal{A}$ and $k[t]$.

Example 1.1 (Homotopy K-theory)**.** Let $\mathrm{Ho}(\mathrm{Spt})$ be the homotopy category of spectra. Weibel's homotopy K-theory gives rise to a functor $KH\colon \mathrm{dgcat}(k) \to \mathrm{Ho}(\mathrm{Spt})$ which satisfies conditions (C1)-(C2); see [44, §2][47, §5.3]. When applied to A, resp. to $\mathrm{perf}_{\mathrm{dg}}(X)$, this functor computes the homotopy K-theory of A, resp. of X.

Example 1.2 (Nonconnective algebraic K-theory with coefficients)**.** Let l be a prime. When $l \nmid \mathrm{char}(k)$, mod-l^ν nonconnective algebraic K-theory gives rise to a functor $I\!K(-; \mathbb{Z}/l^\nu)\colon \mathrm{dgcat}(k) \to \mathrm{Ho}(\mathrm{Spt})$ which satisfies conditions (C1)-(C2); see [48, §1]. When $l \mid \mathrm{char}(k)$, we can consider the functor $I\!K(-) \otimes \mathbb{Z}[1/l]$. When applied to A, resp. to $\mathrm{perf}_{\mathrm{dg}}(X)$, these functors compute the nonconnective algebraic K-theory with coefficients of A, resp. of X.

Example 1.3 (Étale K-theory)**.** Let l be an odd prime. Dwyer-Friedlander's étale K-theory gives rise to a functor $K^{\mathrm{et}}(-; \mathbb{Z}/l^\nu)\colon \mathrm{dgcat}(k) \to \mathrm{Ho}(\mathrm{Spt})$ which satisfies conditions (C1)-(C2); see [47, §5.4]. When $l \nmid \mathrm{char}(k)$ and X is moreover regular and of finite type over $\mathbb{Z}[1/l]$, $K^{\mathrm{et}}(\mathrm{perf}_{\mathrm{dg}}(X); \mathbb{Z}/l^\nu)$ agrees with the étale K-theory of X.

Notation 1.4. Let $E\colon \mathrm{dgcat}(k) \to \mathcal{C}$ be a functor, with values in an arbitrary category, and X a quasi-compact quasi-separated k-scheme. In order to simplify the exposition, we will write $E(X)$ instead of $E(\mathrm{perf}_{\mathrm{dg}}(X))$.

Example 1.5 (Periodic cyclic homology)**.** Let k be a field of characteristic zero and $\mathcal{D}^\pm(k)$ the derived category of $\mathbb{Z}/2$-graded k-vector spaces. Periodic cyclic homology gives rise to a functor $HP\colon \mathrm{dgcat}(k) \to \mathcal{D}^\pm(k)$ which satisfies conditions (C1)-(C2); see [25, §1.5][44, §3]. When applied to A, resp. to $\mathrm{perf}_{\mathrm{dg}}(X)$, this functor computes the periodic cyclic homology of A, resp. of X. When X is moreover smooth, the classical Hochschild-Kostant-Rosenberg theorem yields the following identifications with de Rham cohomology:

$$(1.6) \qquad HP^+(X) \simeq \bigoplus_{n \text{ even}} H_{dR}^n(X), \qquad HP^-(X) \simeq \bigoplus_{n \text{ odd}} H_{dR}^n(X).$$

Example 1.7 (Noncommutative motives)**.** Let $\mathrm{Mot}(k)$ be the (closed) symmetric monoidal triangulated category of noncommutative motives constructed in [47, §2]; denoted by $\mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)$ in loc. cit. By construction, this category comes equipped with a symmetric monoidal functor $U\colon \mathrm{dgcat}(k) \to \mathrm{Mot}(k)$ which satisfies conditions (C1)-(C2). Roughly speaking,[2] U is the initial functor satisfying these conditions and preserving filtered homotopy colimits; for further information on noncommutative motives we invite the reader to consult the recent book [42].

Remark 1.8. As explained in [42, §8-9], there exist four different (closed) symmetric monoidal triangulated categories of noncommutative motives, that are related by

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[2]The precise formulation of this universal property makes use of the language of derivators.

symmetric monoidal triangulated functors:

$$
\begin{array}{ccc}
\mathrm{Mot}_{\mathrm{add}}(k) & \longrightarrow & \\
\downarrow & & \mathrm{Mot}_{\mathrm{add}}^{\mathbb{A}^1}(k) \\
\mathrm{Mot}_{\mathrm{loc}}(k) & \searrow & \downarrow \\
& & \mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)\,.
\end{array}
$$

Roughly speaking, the underscript add, resp. loc, stands for additivity, resp. localization, and the upperscript $\mathbb{A}^1$ for $\mathbb{A}^1$-homotopy invariance. Since the main results of this article make use of localization and $\mathbb{A}^1$-homotopy invariance, they apply solely to the category $\mathrm{Mot}(k) := \mathrm{Mot}_{\mathrm{loc}}^{\mathbb{A}^1}(k)$.

Our main result is the following:

Theorem 1.9 (Gysin triangle)**.** *Let X be a smooth k-scheme, $i\colon Z \hookrightarrow X$ a smooth closed subscheme, and $j\colon U \hookrightarrow X$ the open complement of Z. For every functor $E\colon \mathrm{dgcat}(k) \to \mathcal{T}$ which satisfies conditions* (C1)-(C2), *we have an induced triangle*

$$
(1.10) \qquad E(Z) \xrightarrow{E(i_*)} E(X) \xrightarrow{E(j^*)} E(U) \xrightarrow{\partial} \Sigma E(Z)\,,
$$

where i_*, resp. j^*, stands for the push-forward, resp. pull-back, dg functor.

Remark 1.11 (Generalizations)*.* Theorem 1.9 admits the following generalizations:
  (G1) We may replace the schemes X, Z, and U by algebraic spaces; consult §7.
  (G2) Given a dg category $\mathcal{A}$, we may replace the dg categories $\mathrm{perf}_{\mathrm{dg}}(Z)$, $\mathrm{perf}_{\mathrm{dg}}(X)$, $\mathrm{perf}_{\mathrm{dg}}(U)$, by their tensor product with $\mathcal{A}$. In the case where $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(Y)$, with Y a quasi-compact quasi-separated k-scheme, this corresponds to replacing the schemes X, Z, U by their product with Y over k; consult Lemma 4.26.

Let $\mathrm{perf}_{\mathrm{dg}}(X)_Z \subset \mathrm{perf}_{\mathrm{dg}}(X)$ be the full dg subcategory of those perfect complexes of $\mathcal{O}_X$-modules that are supported on Z. The bulk of the proof of Theorem 1.9 consists of showing that the morphism $E(i_*)\colon E(Z) \to E(\mathrm{perf}_{\mathrm{dg}}(X)_Z)$ is invertible; see Theorem 6.3. This result, which is of independent interest, should be considered as a new "dévissage" theorem. Its proof is based on the description of the dg category $\mathrm{perf}_{\mathrm{dg}}(X)_Z$ in terms of a formal dg k-algebra (when X is affine) and on a Zariski[3] descent argument.

Let us now illustrate the general Theorem 1.9 in some particular cases.

Example 1.12 (Fundamental theorem)**.** When X is the affine line $\mathrm{Spec}(k[t])$, Z is the closed point $t = 0$, and U is the punctured affine line $\mathrm{Spec}(k[t, t^{-1}])$, the general Gysin triangle (1.10) reduces to the following distinguished triangle:

$$
(1.13) \qquad E(k) \xrightarrow{E(i_*)} E(k) \xrightarrow{E(j^*)} E(k[t, t^{-1}]) \xrightarrow{\partial} \Sigma E(k)\,.
$$

In this case we have $E(i_*) = 0$; see [44, Lem. 4.2]. Consequently, (1.13) gives rise to an isomorphism $E(k[t, t^{-1}]) \simeq E(k) \oplus \Sigma E(k)$. The generalization (G2) yields an isomorphism $E(\mathcal{A}[t, t^{-1}]) \simeq E(\mathcal{A}) \oplus \Sigma E(\mathcal{A})$ for every dg category $\mathcal{A}$. By taking $E = KH$, resp. $E = HP$, we hence recover the fundamental theorems in homotopy

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[3]When X is an algebraic space $\mathcal{X}$ we use instead a Nisnevich descent argument.

K-theory, resp. in periodic cyclic homology, established by Weibel in [52, Thms. 1.2 and 6.11], resp. by Kassel in [23, Cor. 3.12]; consult [44] for further details.

Example 1.14 (Quillen's localization theorem). Homotopy K-theory agrees with Quillen's algebraic K-theory on smooth schemes. Therefore, when $E = KH$ the general Gysin triangle (1.10) reduces to the localization theorem

$$(1.15) \qquad K(Z) \xrightarrow{K(i_*)} K(X) \xrightarrow{K(j^*)} K(U) \xrightarrow{\partial} \Sigma K(Z)$$

established by Quillen in [39, Chap. 7 §3]. Quillen's proof is based on the dévissage theorem for abelian categories and on the equivalence between K-theory and G-theory for smooth schemes. As mentioned above, our proof is different! Moreover, following the generalization (G1), it applies also to algebraic spaces.

Example 1.16 (Six-term exact sequence). The maps $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$ give rise to homomorphisms on de Rham cohomology $H_{dR}^n(i_*)\colon H_{dR}^n(Z) \to H_{dR}^{n+2c}(X)$ and $H_{dR}^n(j^*)\colon H_{dR}^n(X) \to H_{dR}^n(U)$ where $c := \operatorname{codim}(i)$. Therefore, when $E = HP$ the long exact sequence associated to the general Gysin triangle (1.10) reduces, via the identification (1.6), to the following six-term exact sequence:

$$
\begin{array}{ccccc}
\bigoplus_{n\,\text{even}} H_{dR}^n(Z) & \xrightarrow{\oplus_n H_{dR}^n(i_*)} & \bigoplus_{n\,\text{even}} H_{dR}^n(X) & \xrightarrow{\oplus_n H_{dR}^n(j^*)} & \bigoplus_{n\,\text{even}} H_{dR}^n(U) \\
\Big\uparrow{\scriptstyle\partial} & & & & \Big\downarrow{\scriptstyle\partial} \\
\bigoplus_{n\,\text{odd}} H_{dR}^n(U) & \xleftarrow[\oplus_n H_{dR}^n(j^*)]{} & \bigoplus_{n\,\text{odd}} H_{dR}^n(X) & \xleftarrow[\oplus_n H_{dR}^n(i_*)]{} & \bigoplus_{n\,\text{odd}} H_{dR}^n(Z)\,.
\end{array}
$$

One may check that this sequence is the "2-periodization" of the Gysin long exact sequence on de Rham cohomology constructed by Hartshorne in [14, Chap. II §3].

Example 1.17 (Noncommutative motivic Gysin triangle). When $E = U$ the general Gysin triangle (1.10) reduces to the *noncommutative motivic Gysin triangle*:

$$(1.18) \qquad U(Z) \xrightarrow{U(i_*)} U(X) \xrightarrow{U(j^*)} U(U) \xrightarrow{\partial} \Sigma U(Z)\,.$$

Consult Remarks 3.4 and 3.9 for the relation between (1.18) and the motivic Gysin triangle(s) constructed by (Morel-)Voevodsky.

We conclude this section with the following remark:

Remark 1.19. Theorem 1.9 is *false* if we assume (C1) but *not* (C2). For example, cyclic homology gives rise to a localizing invariant $HC\colon \mathrm{dgcat}(k) \to \mathcal{D}(k)$ which is *not* $\mathbb{A}^1$-homotopy invariant; see [24, §5.3], [25, §1.5]. Following Kassel [23, §3.4], we have the following computation:

$$HC_n(k[t,t^{-1}]) \simeq \left\{ \begin{array}{ll} HC_n(k) \oplus HC_{n-1}(k) \oplus k \oplus I, & n = 0, \\ HC_n(k) \oplus HC_{n-1}(k) \oplus I, & n \neq 0\,, \end{array} \right.$$

where I stands for the augmentation ideal of $k[t,t^{-1}]$. Therefore, we conclude from Example 1.12 that Theorem 1.9 is false when $E = HC$.

2. Applications

2.1. Reduction to smooth projective schemes.

Theorem 2.1. *Let k be a perfect field of characteristic $p \geq 0$ and $E \colon \mathrm{dgcat}(k) \to \mathcal{T}$ a functor which satisfies conditions* (C1)-(C2). *Let us write $\mathcal{T}^{sp}$ for the smallest triangulated subcategory of $\mathcal{T}$ containing the objects $E(Y)$, with Y a smooth projective k-scheme, and $\overline{\mathcal{T}^{sp}}$ for the thick closure of $\mathcal{T}^{sp}$ inside $\mathcal{T}$. Given a smooth k-scheme X, the following hold:*

  (i) *When $p = 0$, the object $E(X)$ belongs to $\mathcal{T}^{sp}$.*
  (ii) *When $p > 0$ and $\mathcal{T}$ is $\mathbb{Z}[1/p]$-linear, the object $E(X)$ belongs to $\overline{\mathcal{T}^{sp}}$.*

Remark 2.2. The proof of item (ii) makes use of three ingredients:[4] (a) the Gysin triangles provided by Theorem 1.9; (b) Gabber's refined version of de Jong's theory of alterations; (c) a "globalization" argument which allows us to pass from $\mathbb{Z}_{(l)}$-linearity for all $l \neq p$ to $\mathbb{Z}[1/p]$-linearity. Making use of ingredients (a)-(b), and of different "globalization" arguments, Bondarko [7, Thm. 2.2.1] and Kelly [27, Prop. 5.5.3] established an analogue of item (ii) in the particular case where $\mathcal{T}$ is the Voevodsky's triangulated category of (effective) geometric motives. Their argument relies on a certain compact generation statement, which does not hold[5] in the case of the triangulated category of noncommutative motives $\mathrm{Mot}(k)$.

Corollary 2.3. *Let $E \colon \mathrm{dgcat}(k) \to \mathcal{T}$ be a functor as in Theorem* 2.1. *Assume furthermore that $\mathcal{T}$ is well generated (see* [36, Def. 1.15]), *symmetric monoidal,[6] and that the tensor product $- \otimes -$ preserves arbitrary direct sums in both variables. Under these assumptions, if the functor E is moreover symmetric monoidal, then the objects $E(X)$, with X a smooth k-scheme, are strongly dualizable.*

Proof. Given an object $b \in \mathcal{T}$, the functor $- \otimes b \colon \mathcal{T} \to \mathcal{T}$ is exact and preserves arbitrary direct sums. Therefore, thanks to [36, Thm. 8.4.4], it admits a right adjoint $\underline{\mathrm{Hom}}(b, -)$ which by definition is the internal-Hom functor. This shows that the symmetric monoidal structure of $\mathcal{T}$ is closed.

As proved in [42, Thm. 1.43], the strongly dualizable objects of the category $\mathrm{Hmo}(k)$ are the smooth proper dg categories; see §4.1. Since by assumption the functor E is symmetric monoidal, we conclude that the objects $E(X)$, with X a smooth projective k-scheme, are strongly dualizable. The result follows now from Theorem 2.1 and from the well-known fact that the strongly dualizable objects of a closed symmetric monoidal triangulated category are stable under distinguished triangles and direct summands. □

2.2. Additive invariants of relative cellular spaces.

2.2.1. *Additive invariants.* Our next application is for so-called *additive invariants* which are a weaker type of invariant than the kind we have considered up to now. Every localizing invariant is an additive invariant but the converse is not true.[7]

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[4]The proof of item (i) makes use of ingredient (a) and of resolution of singularities.
[5]The fact that the noncommutative motive $\mathrm{U}(X)$ of a smooth k-scheme X is a compact object of $\mathrm{Mot}(k)$ is now a consequence of Theorem 2.1.
[6]We assume throughout the article that a symmetric monoidal structure on a triangulated category is exact in both variables.
[7]Quillen's algebraic K-theory as well as Karoubi-Villamayor's K-theory are examples of additive invariants which are not localizing; consult [42] for details.

Examples of additive invariants include algebraic K-theory and all its variants, cyclic homology and all its variants, topological Hochschild homology, etc.

Let I be the dg category with objects $\{1, 2\}$ and complexes of morphisms $I(1,1) = I(2,2) = I(1,2) = k$ and $I(2,1) = 0$, with k supported in degree zero. Given a dg category $\mathcal{A}$, let $T(\mathcal{A}) := \mathcal{A} \otimes I$. We have two inclusion dg functors $\iota_1, \iota_2 \colon \mathcal{A} \to T(\mathcal{A})$. A functor $F \colon \mathrm{dgcat}(k) \to \mathrm{A}$, with values in an additive category, is called an *additive invariant* if it inverts the Morita equivalences and sends the dg categories $T(\mathcal{A})$ to direct sums

$$[F(\iota_1) \; F(\iota_2)] \colon F(\mathcal{A}) \oplus F(\mathcal{A}) \xrightarrow{\simeq} F(T(\mathcal{A})) \,.$$

As explained in [45, §13], the notion of additive invariant can be equivalently formulated in terms of *split* short exact sequences of dg categories. Therefore, every localizing invariant is in particular an additive invariant.

Remark 2.4. As proved in [46] (consult also §4.3), there exists a *universal additive invariant* $\mathrm{U}_{\mathrm{add}} \colon \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ with values in a suitable additive category[8] $\mathrm{Hmo}_0(k)$. This implies that an additive invariant can be alternatively characterized as a functor $F \colon \mathrm{dgcat}(k) \to \mathrm{A}$ which factors through $\mathrm{U}_{\mathrm{add}}$.

2.2.2. *Relative cellular spaces.* A flat map of k-schemes $f \colon X \to Y$ is called an *affine fibration* of relative dimension d if for every point $y \in Y$ there exists a Zariski open neighborhood $y \in V$ such that $X_V := f^{-1}(V) \simeq Y \times \mathbb{A}^d$ with $f_V \colon X_V \to Y$ isomorphic to the projection onto the first factor. Following Karpenko [22, Def. 6.1], a smooth projective k-scheme X is called a *relative cellular space* if there exists a filtration by closed subschemes

$$(2.5) \qquad \varnothing = X_{-1} \hookrightarrow X_0 \hookrightarrow \cdots \hookrightarrow X_i \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$$

and affine fibrations $p_i \colon X_i \backslash X_{i-1} \to Y_i, 0 \leq i \leq n$, of relative dimension d_i with Y_i a smooth projective k-scheme. The smooth schemes $X_i \backslash X_{i-1}$ are called the *cells* and the smooth projective schemes Y_i the *bases of the cells*.

Example 2.6 ($\mathbb{G}_m$-schemes)**.** The celebrated Bialynicki-Birula decomposition [3] provides a relative cellular space structure on smooth projective k-schemes equipped with a $\mathbb{G}_m$-action in which the bases of the cells are given by the connected components of the fixed point locus; consult also [8, Thm. 3.1], [15, 18]. This class of relative cellular spaces includes the isotropic flag varieties considered originally by Karpenko [22] as well as the isotropic homogeneous spaces considered later by Chernousov-Gille-Merkurjev [10].

Our main result concerning relative cellular spaces is the following:

Theorem 2.7. *Let X be a relative cellular space. For every additive invariant F, we have an induced isomorphism $F(X) \simeq \bigoplus_{i=0}^{n} F(Y_i)$.*

Remark 2.8 (Strategy of the proof). In order to prove Theorem 2.7 we consider first the special case $F = \mathrm{U}$, i.e., we establish first an induced isomorphism

$$(2.9) \qquad \mathrm{U}(X) \simeq \bigoplus_{i=0}^{n} \mathrm{U}(Y_i)$$

in the category of noncommutative motives $\mathrm{Mot}(k)$. This decomposition is analogous to a similar result for Chow motives proved by Karpenko [22] using refined

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[8]The category $\mathrm{Hmo}_0(k)$ is different from the category $\mathrm{Mot}_{\mathrm{add}}(k)$ of Remark 1.8. The latter category is triangulated while the former is only additive.

properties of Chow and K-cohomology groups. The proof of (2.9) uses in an essential way the fact that U satisfies conditions (C1)-(C2) (note that we do *not* require this for F in Theorem 2.7!). It is based on the invariance of noncommutative motives under affine fibrations and on the observation that the Gysin triangles associated to the filtration (2.5) are actually split! We cannot immediately obtain Theorem 2.7 from (2.9) since F will in general not factor through U. However, using the fact that all the schemes in the motivic decomposition (2.9) are smooth projective, we prove that (2.9) implies a similar decomposition

$$\mathrm{U_{add}}(X) \simeq \bigoplus_{i=0}^{n} \mathrm{U_{add}}(Y_i)$$

in the additive category $\mathrm{Hmo}_0(k)$. To finish the proof we use the fact that F, being additive, factors (uniquely) through $\mathrm{U_{add}}$.

Example 2.10 (Knörrer periodicity). The following application of the Bialynicki-Birula decomposition was inspired by the work of Brosnan [8]. Let $q = fg + q'$ where f, g, and q', are forms of degree $a > 0$, $b > 0$, and $a + b$, in disjoint sets of variables $(x_i)_{i=1,\dots,m}$, $(y_j)_{j=1,\dots,n}$, and $(z_k)_{k=1,\dots,p}$, respectively. Let us write Q and Q' for the projective hypersurfaces defined by q and q', respectively. Assume that Q is smooth. Under this notation and assumptions, there is a $\mathbb{G}_m$-action on Q given by $\lambda \cdot (\underline{x}, \underline{y}, \underline{z}) := (\lambda^b \underline{x}, \lambda^{-a} \underline{y}, \underline{z})$ with fixed point locus $\mathbb{P}^{m-1} \coprod \mathbb{P}^{n-1} \coprod Q'$; note that this implies that Q' is also smooth. By combining Theorem 2.7 and Example 2.6 with the fact that $\mathrm{U_{add}}(\mathbb{P}^n) \simeq \mathrm{U_{add}}(k)^{\oplus(n+1)}$ (see [42, §2.4.2]), we hence obtain an induced isomorphism

$$(2.11) \qquad\qquad F(Q) \simeq F(k)^{\oplus(m+n)} \oplus F(Q')$$

for every additive invariant F. Intuitively speaking, isomorphism (2.11) shows that the "nontrivial parts" of $F(Q)$ and $F(Q')$ are the same. Finally, recall that the preceding computation holds for all isotropic quadratic forms since it is well known that these can be written as $xy + q'(\underline{z})$.

3. Motives versus noncommutative motives

3.1. Motivic homotopy theory versus noncommutative mixed motives.
The reduction Theorem 2.1 (and Corollary 2.3) allows us to improve the bridge between Morel-Voevodsky's motivic homotopy theory and Kontsevich's noncommutative mixed motives originally constructed in [43].

Kontsevich introduced in [30] a (rigid) symmetric monoidal triangulated category of noncommutative mixed motives $\mathrm{KMM}(k)$. As explained in [43, §4], this category can be described as the smallest thick triangulated subcategory of $\mathrm{Mot}(k)$ containing the objects $\mathrm{U}(\mathcal{A})$ with $\mathcal{A}$ a smooth proper dg category. In the same vein, let us write $\mathrm{KMM}(k)^{\oplus}$ for the smallest triangulated subcategory of $\mathrm{Mot}(k)$ which contains $\mathrm{KMM}(k)$ and is stable under arbitrary direct sums.

Morel and Voevodsky introduced in [35, 51] the stable $\mathbb{A}^1$-homotopy category of $(\mathbb{P}^1, \infty)$-spectra $\mathrm{SH}(k)$. By construction, this category comes equipped with a symmetric monoidal functor $\Sigma^\infty(-_+) \colon \mathrm{Sm}(k) \to \mathrm{SH}(k)$ defined on smooth k-schemes. Let $\mathrm{KGL} \in \mathrm{SH}(k)$ be the ring $(\mathbb{P}^1, \infty)$-spectrum representing homotopy K-theory (see [12, 41]) and $\mathrm{Mod}(\mathrm{KGL})$ the homotopy category of KGL-modules.

Theorem 3.1.   (i) *Let k be a field of characteristic zero. Then there exists a fully-faithful, symmetric monoidal, triangulated functor Φ making the following diagram commute:*

(3.2)

$$
\begin{array}{ccc}
\mathrm{Sm}(k) & \xrightarrow{\ X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)\ } & \mathrm{dgcat}(k) \\
\downarrow{\scriptstyle \Sigma^\infty(-_+)} & & \downarrow{\scriptstyle \mathrm{U}} \\
\mathrm{SH}(k) & \mathrm{KMM}(k) \longrightarrow & \mathrm{Mot}(k) \\
\downarrow{\scriptstyle -\wedge\mathrm{KGL}} & \downarrow{\scriptstyle (-)^\vee} & \downarrow{\scriptstyle \underline{\mathrm{Hom}}(-,\mathrm{U}(k))} \\
\mathrm{Mod}(\mathrm{KGL}) & \xrightarrow{\ \Phi\ } \mathrm{KMM}(k)^{\oplus} \longrightarrow & \mathrm{Mot}(k)\,,
\end{array}
$$

where $\underline{\mathrm{Hom}}(-,-)$ stands for the internal-Hom of the category $\mathrm{Mot}(k)$.

(ii) *Let k be a perfect field of positive characteristic $p > 0$. Then there exists a $\mathbb{Z}[1/p]$-linear, fully-faithful, symmetric monoidal, triangulated functor $\Phi_{1/p}$ making the following diagram commute (the shorthand $1/p$ stands for $\mathbb{Z}[1/p]$):*

(3.3)

$$
\begin{array}{ccc}
\mathrm{Sm}(k) & \xrightarrow{\ X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)\ } & \mathrm{dgcat}(k) \\
\downarrow{\scriptstyle \Sigma^\infty(-_+)_{1/p}} & & \downarrow{\scriptstyle \mathrm{U}(-)_{1/p}} \\
\mathrm{SH}(k)_{1/p} & \mathrm{KMM}(k)_{1/p} \longrightarrow & \mathrm{Mot}(k)_{1/p} \\
\downarrow{\scriptstyle -\wedge\mathrm{KGL}_{1/p}} & \downarrow{\scriptstyle (-)^\vee} & \downarrow{\scriptstyle \underline{\mathrm{Hom}}(-,\mathrm{U}(k)_{1/p})} \\
\mathrm{Mod}(\mathrm{KGL}_{1/p}) & \xrightarrow{\ \Phi_{1/p}\ } \mathrm{KMM}(k)^{\oplus}_{1/p} \longrightarrow & \mathrm{Mot}(k)_{1/p}\,.
\end{array}
$$

Proof. The outer commutative square of diagram (3.2) was constructed in [43, Cor. 2.5(i)]. The inner commutative squares follow from Theorem 2.1(i) and Corollary 2.3 applied to the functor $E = \mathrm{U}$. Similarly to the proof of Theorem 2.1(ii) (see §8), one can refine the proof of Ayoub [2, Prop. 2.2.27-2] using Gabber's refined theory of alterations. Using [43, Thm. 2.1(iii)], we hence obtain the outer commutative square of diagram (3.3). The inner commutative squares follow from Theorem 2.1(ii) and Corollary 2.3 applied to the functor $E = \mathrm{U}$.   □

Intuitively speaking, Theorem 3.1 formalizes the conceptual idea that the difference between the motivic homotopy theory and the theory of noncommutative mixed motives is measured solely by the existence of a KGL-module structure.

Remark 3.4 (Morel-Voevodsky's motivic Gysin triangle). Let X be a smooth scheme, $i\colon Z \hookrightarrow X$ a smooth closed subscheme with normal vector bundle N, and $j\colon U \hookrightarrow X$ the open complement of Z. Making use of the Nisnevich topology and of homotopy purity, Morel-Voevodsky constructed in [35, §3.2][51, §4] a motivic Gysin triangle

(3.5) $\Sigma^\infty(U_+) \xrightarrow{\Sigma^\infty(j_+)} \Sigma^\infty(X_+) \longrightarrow \Sigma^\infty(\mathrm{Th}(N)) \xrightarrow{\partial} \Sigma(\Sigma^\infty(U_+))$

in $\mathrm{SH}(k)$, where $\mathrm{Th}(N)$ stands for the Thom space of N. Since homotopy K-theory is an orientable and periodic cohomology theory, $\Sigma^\infty(\mathrm{Th}(N)) \wedge \mathrm{KGL}$ is isomorphic to $\Sigma^\infty(Z_+) \wedge \mathrm{KGL}$. Using the commutative diagram (3.2), we hence conclude that

the image of (3.5) under the composed functor $\Phi \circ (- \wedge \mathrm{KGL}) \colon \mathrm{SH}(k) \to \mathrm{KMM}(k)^{\oplus}$ agrees with the dual of the noncommutative motivic Gysin triangle (1.18). Roughly speaking, (1.18) is the dual of the KGL-linearization of (3.5).

3.2. Mixed motives versus noncommutative mixed motives.
The reduction Theorem 2.1 (and Corollary 2.3) allows us also to improve the bridge between Voevodsky's mixed motives and noncommutative mixed motives constructed in [43].

Voevodsky introduced in [50, §2] the triangulated category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k)$ (over a perfect field k). By construction, this category comes equipped with a symmetric monoidal functor $M \colon \mathrm{Sm}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$.

Let $\mathrm{HZ} \in \mathrm{SH}(k)$ be the ring $(\mathbb{P}^1, \infty)$-spectrum representing motivic cohomology; see [51, §6.1]. Thanks to Bloch's work [4], we have $\mathrm{KGL}_{\mathbb{Q}} \simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{HZ}_{\mathbb{Q}}(i)[2i]$. Moreover, $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ identifies with the full triangulated subcategory of compact objects of $\mathrm{DM}(k)_{\mathbb{Q}} := \mathrm{Mod}(\mathrm{HZ}_{\mathbb{Q}})$; see [40]. As a consequence, base-change along $\mathrm{HZ}_{\mathbb{Q}} \to \mathrm{KGL}_{\mathbb{Q}}$ gives rise to a functor $\mathrm{DM}(k)_{\mathbb{Q}} \to \mathrm{Mod}(\mathrm{KGL}_{\mathbb{Q}})$. By composing it with $\Phi_{\mathbb{Q}}$, we hence obtain a $\mathbb{Q}$-linear, symmetric monoidal, triangulated functor

$$(3.6) \qquad \mathrm{R} \colon \mathrm{DM}(k)_{\mathbb{Q}} \longrightarrow \mathrm{Mod}(\mathrm{KGL}_{\mathbb{Q}}) \xrightarrow{\Phi_{\mathbb{Q}}} \mathrm{KMM}(k)_{\mathbb{Q}}^{\oplus}.$$

Theorem 3.7. *Let k be a perfect field. The functor (3.6) gives rise to a $\mathbb{Q}$-linear, fully-faithful, symmetric monoidal functor $\overline{\mathrm{R}}$ making the following diagram commute:*

$$(3.8)$$

where $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}/_{-\otimes \mathbb{Q}(1)[2]}$ stands for the orbit category of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ with respect to the Tate motive $\mathbb{Q}(1)[2]$ (consult [43, §3.5] for the notion of orbit category).

Proof. The outer commutative square of diagram (3.8) was constructed in [43, Thm. 2.8]. The inner commutative squares follow from Theorem 2.1 and Corollary 2.3 applied to the functor $E = \mathrm{U}$. $\qquad \square$

Intuitively speaking, Theorem 3.7 formalizes the conceptual idea that the commutative world embeds fully-faithfully into the noncommutative world as soon as we "⊗-trivialize" the Tate motive $\mathbb{Q}(1)[2]$.

Remark 3.9 (Voevodsky's motivic Gysin triangle). Let X be a smooth scheme, $i \colon Z \hookrightarrow X$ a smooth closed subscheme of codimension c, and $j \colon U \hookrightarrow X$ the open complement of Z. Making use of algebraic geometric arguments such as the projective bundle theorem and the deformation to the normal cone, Voevodsky constructed in [50, §2] a motivic Gysin triangle

$$(3.10) \qquad M(U)_{\mathbb{Q}} \xrightarrow{M(j)_{\mathbb{Q}}} M(X)_{\mathbb{Q}} \longrightarrow M(Z)_{\mathbb{Q}}(c)[2c] \xrightarrow{\partial} \Sigma M(U)_{\mathbb{Q}}$$

in $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$. Using the commutative diagram (3.8), we hence conclude that the image of (3.10) under the composed functor $\overline{\mathrm{R}} \circ \pi \colon \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}} \to \mathrm{KMM}(k)_{\mathbb{Q}}$ agrees

with the dual of the rationalized noncommutative motivic Gysin triangle (1.18).
Roughly speaking, $(1.18)_{\mathbb{Q}}$ is the dual of the Tate $\otimes$-trivialization of (3.10).

Remark 3.11 (Levine's mixed motives). Levine introduced in [32, Part I] a triangu-
lated category of mixed motives $\mathcal{DM}(k)$ and a *contravariant* symmetric monoidal
functor $h\colon \mathrm{Sm}(k) \to \mathcal{DM}(k)$. As proved by Ivorra in [19, Thm. 4.2], when k is
a perfect field, the assignment $h(X)_{\mathbb{Q}}(n) \mapsto \underline{\mathrm{Hom}}(M(X)_{\mathbb{Q}}, \mathbb{Q}(n))$ gives rise to an
equivalence of categories $\mathcal{DM}(k)_{\mathbb{Q}} \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ whose precomposition with $h(-)_{\mathbb{Q}}$
identifies with $X \mapsto M(X)_{\mathbb{Q}}^{\vee}$. Thanks to Theorem 3.7, there exists then a $\mathbb{Q}$-linear,
fully-faithful, symmetric monoidal functor $\overline{\mathrm{R}}$ making the following diagram com-
mute:

(3.12)

$$
\begin{array}{ccc}
\mathrm{Sm}(k) & \xrightarrow{\ X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)\ } & \mathrm{dgcat}(k) \\
\ \downarrow{\scriptstyle h(-)_{\mathbb{Q}}} & & \ \downarrow{\scriptstyle \mathrm{U}(-)_{\mathbb{Q}}} \\
\mathcal{DM}(k)_{\mathbb{Q}} & & \\
\ \downarrow{\scriptstyle \pi} & & \\
\mathcal{DM}(k)_{\mathbb{Q}}/_{-\otimes \mathbb{Q}(1)[2]} & \xrightarrow[\ \ \overline{\mathrm{R}}\ \ ]{} & \mathrm{KMM}(k)_{\mathbb{Q}} \subset \mathrm{Mot}(k)_{\mathbb{Q}}\,.
\end{array}
$$

3.3. **Étale descent of noncommutative mixed motives.** Let $\mathrm{DM}^{\mathrm{et}}(k)$ be the
étale variant of $\mathrm{DM}(k)$ introduced by Voevodsky in [50, §3.3]. As proved in loc. cit.,
we have an equivalence of categories $\mathrm{DM}(k)_{\mathbb{Q}} \simeq \mathrm{DM}^{\mathrm{et}}(k)_{\mathbb{Q}}$; consult also Ayoub's
ICM survey [1]. Theorem 3.7 leads then to the following étale descent result:

Theorem 3.13. *The presheaf of noncommutative mixed motives*

$$
\mathrm{Sm}(k)^{\mathrm{op}} \longrightarrow \mathrm{KMM}(k)_{\mathbb{Q}}^{\oplus}, \qquad X \mapsto \mathrm{U}(X)_{\mathbb{Q}}
$$

satisfies étale descent, *i.e. for every* $X \in \mathrm{Sm}(k)$ *and étale cover* $\mathcal{U} = \{U_i \to X\}_{i \in I}$
of X, *we have an induced isomorphism* $\mathrm{U}(X)_{\mathbb{Q}} \simeq \mathrm{holim}_{n \geq 0}\mathrm{U}(\check{C}_n \mathcal{U})_{\mathbb{Q}}$, *where* $\check{C}_{\bullet}\mathcal{U}$
stands for the Čech simplicial scheme associated to the cover $\mathcal{U}$.

Proof. Thanks to the equivalence of categories $\mathrm{DM}(k)_{\mathbb{Q}} \simeq \mathrm{DM}^{\mathrm{et}}(k)_{\mathbb{Q}}$, we have an
induced isomorphism $M(X)_{\mathbb{Q}} \simeq \mathrm{hocolim}_{n \geq 0}M(\check{C}_n \mathcal{U})_{\mathbb{Q}}$ in $\mathrm{DM}(k)_{\mathbb{Q}}$. Since by con-
struction the functor (3.6) preserves homotopy colimits, we hence conclude from
Theorem 3.7 that $\mathrm{U}(X)_{\mathbb{Q}}^{\vee} \simeq \mathrm{hocolim}_{n \geq 0}\mathrm{U}(\check{C}_n \mathcal{U})_{\mathbb{Q}}^{\vee}$. The proof follows now from the
fact that the functor $\underline{\mathrm{Hom}}(-, \mathrm{U}(k)_{\mathbb{Q}})\colon \mathrm{Mot}(k)_{\mathbb{Q}} \to \mathrm{Mot}(k)_{\mathbb{Q}}$ interchanges homo-
topy colimits with homotopy limits and restricts to a (contravariant) equivalence
of categories $(-)^{\vee}\colon \mathrm{KMM}(k)_{\mathbb{Q}} \to \mathrm{KMM}(k)_{\mathbb{Q}}$. $\square$

4. Preliminaries

4.1. **Dg categories.** Let $(\mathcal{C}(k), \otimes, k)$ be the category of (cochain) complexes of k-
vector spaces; we use cohomological notation. A *differential graded (=dg) category*
$\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a *dg functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor enriched
over $\mathcal{C}(k)$; for further details consult Keller's ICM survey [24].

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$
and $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right dg $\mathcal{A}$-module* is a dg functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with
values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k-vector spaces. Let us write $\mathcal{C}(\mathcal{A})$
for the category of right dg $\mathcal{A}$-modules. Following [24, §3.2], the derived category

$\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. Let $\mathcal{D}_c(\mathcal{A})$ be the triangulated subcategory of compact objects.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence of categories $\mathcal{D}(\mathcal{B}) \overset{\cong}{\Rightarrow} \mathcal{D}(\mathcal{A})$; see [24, §4.6]. As proved in [46, Thm. 5.3], dgcat(k) admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by Hmo(k) the associated homotopy category.

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A} \otimes \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$. As explained in [24, §2.3], this construction gives rise to a symmetric monoidal structure on dgcat(k), which descends to the homotopy category Hmo(k).

An $\mathcal{A}$-$\mathcal{B}$-*bimodule* B is a dg functor B$: \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ or equivalently a right dg $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module. A standard example is the $\mathcal{A}$-$\mathcal{B}$-bimodule

(4.1) $_F\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \qquad (x, z) \mapsto \mathcal{B}(z, F(x)),$

associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$.

Recall from Kontsevich [28–31] that a dg category $\mathcal{A}$ is called *smooth* if the $\mathcal{A}$-$\mathcal{A}$-bimodule $_{\mathrm{id}}\mathcal{B}$ belongs to the triangulated category $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$ and *proper* if $\sum_i \dim H^i \mathcal{A}(x, y) < \infty$ for any ordered pair of objects (x, y). Examples include the finite dimensional k-algebras of finite global dimension (when k is perfect) as well as the dg categories $\mathrm{perf}_{\mathrm{dg}}(Y)$ associated to smooth proper k-schemes Y.

4.2. **Localizing invariants.** Let $E: \mathrm{dgcat}(k) \to \mathcal{T}$ be a functor, with values in a triangulated category, which inverts Morita equivalences. Thanks to the universal property of the homotopy category Hmo(k), we have an induced functor $E: \mathrm{Hmo}(k) \to \mathcal{T}$. Recall from [24, Thm. 4.11] that the homotopy category Hmo(k) is pointed[9] and that a short exact sequence of dg categories

(4.2) $0 \longrightarrow \mathcal{A} \overset{I}{\longrightarrow} \mathcal{B} \overset{P}{\longrightarrow} \mathcal{C} \longrightarrow 0$

consists of morphisms I and P in Hmo(k) such that $P \circ I = 0$, I is the kernel of P, and P is the cokernel of I. A "generic" example is given by the Drinfeld's dg quotient $\mathcal{A} \subset \mathcal{B} \to \mathcal{B}/\mathcal{A}$ of an inclusion of dg categories; consult [11] for details.

Definition 4.3. A functor $E: \mathrm{dgcat}(k) \to \mathcal{T}$ as above is called a *localizing invariant* if the induced functor $E: \mathrm{Hmo}(k) \to \mathcal{T}$ sends short exact sequences of dg categories (4.2) to distinguished triangles

$$E(\mathcal{A}) \overset{E(I)}{\longrightarrow} E(\mathcal{B}) \overset{E(P)}{\longrightarrow} E(\mathcal{C}) \overset{\partial}{\longrightarrow} \Sigma E(\mathcal{A})$$

in a way which is functorial for strict morphisms of exact sequences.

Remark 4.4. Using the methods in [25], one may show that the functoriality of E on strict morphisms of exact sequences of dg categories implies that E is functorial on morphisms between exact sequences of dg categories in Hmo(k).

Example 4.5 (Mixed complex)**.** Following Kassel [23], a *mixed complex* is a (right) dg module over the algebra of dual numbers $\Lambda := k[\epsilon]/\epsilon^2$ with $\deg(\epsilon) = -1$ and $d(\epsilon) = 0$. As proved by Keller in [25, §1.5], the mixed complex construction gives rise to a localizing invariant $C: \mathrm{dgcat}(k) \to \mathcal{D}(\Lambda)$. Since Hochschild homology, cyclic homology, negative cyclic homology, and periodic cyclic homology factor through C, they are also examples of localizing invariants; consult [26, §2.2] for details.

---

[9]The dg category with one object and one morphism is the initial=terminal object of Hmo(k).

Remark 4.6 (Quillen model). Let $\mathcal{M}$ be a stable Quillen model category and $E\colon \mathrm{dgcat}(k) \to \mathcal{M}$ a functor which sends the Morita equivalences to weak equivalences and the Drinfeld's dg quotients $\mathcal{A} \subset \mathcal{B} \to \mathcal{B}/\mathcal{A}$ to homotopy (co)fiber sequences $E(\mathcal{A}) \to E(\mathcal{B}) \to E(\mathcal{B}/\mathcal{A})$. Consider the associated composition

$$(4.7) \qquad \mathrm{dgcat}(k) \xrightarrow{E} \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M})$$

with values in the homotopy triangulated category. Clearly, the functor (4.7) inverts Morita equivalences. Moreover, it sends in a functorial way the Drinfeld's dg quotients to distinguished triangles. As proved by Keller in [25, §4], every short exact sequence of dg categories (4.2) can be "strictified", in a functorial way, into the Drinfeld's dg quotient $\mathcal{A} \subset \mathcal{B} \to \mathcal{B}/\mathcal{A}$ of an inclusion of dg categories. This hence implies that (4.7) is a localizing invariant.

Example 4.8. Examples 1.1-1.3 and 1.7 fit into the framework of Remark 4.6. Further examples include nonconnective algebraic K-theory, topological Hochschild homology, topological cyclic homology, etc.; consult [42] for details.

4.3. **Universal additive invariant and its relation with NC motives.** We start by recalling from [46] the construction of the universal additive invariant. As proved in [46, Cor. 5.10], there is a natural bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B})$ and the set of isomorphism classes of the full triangulated subcategory $\mathrm{rep}(\mathcal{A}, \mathcal{B}) \subset \mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ of those $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every $x \in \mathcal{A}$ the right dg $\mathcal{B}$-module B$(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Under this bijection, the composition law of $\mathrm{Hmo}(k)$ corresponds to the tensor product of bimodules. Since the bimodules (4.1) belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, we have the tautological functor

$$(4.9) \qquad \mathrm{dgcat}(k) \to \mathrm{Hmo}(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad F \mapsto {}_F\mathrm{B} \,.$$

The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects and with abelian groups of morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B})$ given by the Grothendieck group $K_0\mathrm{rep}(\mathcal{A}, \mathcal{B})$ of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. By construction, we have the following functor:

$$(4.10) \qquad \mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad \mathrm{B} \mapsto [\mathrm{B}] \,.$$

Let us denote by $\mathrm{U}_{\mathrm{add}}$ the composition (4.10)∘(4.9). As proved in [46, Thms. 5.3 and 6.3], the functor $\mathrm{U}_{\mathrm{add}}\colon \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ is the *universal additive invariant*, i.e., given any additive category A we have an induced equivalence of categories

$$(4.11) \qquad \mathrm{U}_{\mathrm{add}}\colon \mathrm{Fun}_{\mathrm{additive}}(\mathrm{Hmo}_0(k), \mathrm{A}) \xrightarrow{\cong} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{A}) \,,$$

where the left-hand side denotes the category of additive functors and the right-hand side the category of additive invariants (see §2.2.1).

As mentioned in §2.2.1, every localizing invariant is in particular an additive invariant. Therefore, since the functor $\mathrm{U}\colon \mathrm{dgcat}(k) \to \mathrm{Mot}(k)$ (see Example 1.7) is a localizing invariant, and hence an additive invariant, it factors as follows:

$$(4.12) \qquad \begin{array}{ccc} \mathrm{dgcat}(k) & \xrightarrow{\ \mathrm{U}\ } & \mathrm{Mot}(k) \\ {\scriptstyle \mathrm{U}_{\mathrm{add}}}\downarrow & \nearrow_{\overline{\mathrm{U}}} & \\ \mathrm{Hmo}_0(k) & & \end{array} \,.$$

The following Proposition 4.13 and Lemma 4.15, concerning the functors U and $\mathrm{U}_{\mathrm{add}}$, will play a key role in the proof of Theorem 2.7.

Proposition 4.13. *Given a smooth proper k-scheme X and a smooth k-scheme Y, there are isomorphisms of abelian groups*

$$(4.14) \qquad \mathrm{Hom}_{\mathrm{Mot}(k)}(\mathrm{U}(X), \Sigma^n \mathrm{U}(Y)) \simeq K_{-n}(X \times Y), \qquad n \in \mathbb{Z}.$$

In particular, the abelian group (4.14) is zero whenever $n > 0$.

Proof. As proved in [47, Cor. 2.7], the left-hand side of (4.14) is isomorphic to $KH_{-n}(X \times Y)$. The proof follows then from the fact that homotopy K-theory agrees with Quillen's algebraic K-theory on smooth schemes. $\square$

Lemma 4.15. *Given smooth k-schemes X and Y, with X proper, the morphism*

$$(4.16) \qquad \mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathrm{U}_{\mathrm{add}}(X), \mathrm{U}_{\mathrm{add}}(Y)) \longrightarrow \mathrm{Hom}_{\mathrm{Mot}(k)}(\mathrm{U}(X), \mathrm{U}(Y)),$$

induced by the additive functor $\overline{\mathrm{U}}$, is invertible.

Proof. Thanks to Proposition 4.13, the right-hand side of (4.16) is naturally isomorphic to $K_0(X \times Y)$. For the left-hand side we have the identifications

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathrm{U}_{\mathrm{add}}(X), \mathrm{U}_{\mathrm{add}}(Y)) \quad &\simeq \quad K_0(\mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{op}}, \mathrm{perf}_{\mathrm{dg}}(Y))) \\
(4.17) \qquad\qquad\qquad\qquad\qquad\qquad &\simeq \quad K_0(\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{op}} \otimes \mathrm{perf}_{\mathrm{dg}}(Y)) \\
(4.18) \qquad\qquad\qquad\qquad\qquad\qquad &\simeq \quad K_0(\mathrm{perf}_{\mathrm{dg}}(X) \otimes \mathrm{perf}_{\mathrm{dg}}(Y)) \\
(4.19) \qquad\qquad\qquad\qquad\qquad\qquad &\simeq \quad K_0(\mathrm{perf}_{\mathrm{dg}}(X \times Y)) \\
&\simeq \quad K_0(X \times Y),
\end{aligned}$$

where (4.17) follows from the fact that $\mathrm{rep}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ for any two dg categories with $\mathcal{A}$ smooth and proper, (4.18) follows from the Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{op}} \to \mathrm{perf}_{\mathrm{dg}}(X)$ given by $\mathcal{F} \mapsto \underline{\mathrm{Hom}}_X(\mathcal{F}, \mathcal{O}_X)$, and (4.19) from Lemma 4.26. The proof follows now from the construction of the diagram (4.12). $\square$

4.4. Quasi-coherent sheaves and their derived categories.

Given a quasi-compact quasi-separated k-scheme X, let us write $\mathrm{Mod}(X)$ for the Grothendieck category of $\mathcal{O}_X$-modules, $\mathrm{Qcoh}(X)$ for the full subcategory of quasi-coherent $\mathcal{O}_X$-modules, $\mathcal{D}(X) := \mathcal{D}(\mathrm{Mod}(X))$ for the derived category of X, and $\mathcal{D}_{\mathrm{Qcoh}}(X) \subset \mathcal{D}(X)$ for the full triangulated subcategory of those complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology. In the same vein, given a closed subscheme $Z \hookrightarrow X$, let us write $\mathcal{D}(X)_Z \subset \mathcal{D}(X)$ and $\mathcal{D}_{\mathrm{Qcoh}}(X)_Z \subset \mathcal{D}_{\mathrm{Qcoh}}(X)$ for the full triangulated subcategories of those complexes of $\mathcal{O}_X$-modules that are supported on Z.

Theorem 4.20 (Compact generation). *Assume that the open complement of Z is also quasi-compact quasi-separated. Under this assumption, the triangulated category $\mathcal{D}_{\mathrm{Qcoh}}(X)_Z$ is compactly generated. Moreover, its full triangulated subcategory of compact objects identifies with $\mathrm{perf}(X)_Z$.*

Proof. Simply imitate the proof of [6, Thm. 3.1.1]; consult also [37], [21, Tag 0AEC, Lem. 62.14.5]. $\square$

The following result will play a key role in the proof of Theorem 5.3.

Proposition 4.21. *Let X be a quasi-compact quasi-separated scheme, $p\colon V \hookrightarrow X$ a quasi-compact open subscheme, and $W \hookrightarrow X$ the closed complement of V. For every closed subscheme $i\colon Z \hookrightarrow X$ with quasi-compact complement, we have an induced short exact sequence of dg categories*

$$(4.22) \qquad 0 \longrightarrow \mathrm{perf}_{\mathrm{dg}}(X)_{Z \cap W} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(X)_Z \xrightarrow{p^*} \mathrm{perf}_{\mathrm{dg}}(V)_{Z \cap V} \longrightarrow 0.$$

Proof. As explained by Keller in [24, Thm. 4.11], (4.22) is a short exact sequence of dg categories if and only if the associated sequence of triangulated categories

$$(4.23) \qquad \operatorname{perf}(X)_{Z \cap W} \longrightarrow \operatorname{perf}(X)_Z \xrightarrow{p^*} \operatorname{perf}(V)_{Z \cap V}$$

is exact in the sense of Verdier. We claim that there is a short exact sequence:

$$(4.24) \qquad 0 \longrightarrow \mathcal{D}_{\mathrm{Qcoh}}(X)_{Z \cap W} \longrightarrow \mathcal{D}_{\mathrm{Qcoh}}(X)_Z \xrightarrow{p^*} \mathcal{D}_{\mathrm{Qcoh}}(V)_{Z \cap V} \longrightarrow 0 \,.$$

As usual, this follows from the following facts: (i) if $\mathcal{F} \in \mathcal{D}_{\mathrm{Qcoh}}(V)_{Z \cap V}$, then $p_*(\mathcal{F})$ belongs to $\mathcal{D}_{\mathrm{Qcoh}}(X)_Z$; (ii) if $\mathcal{G} \in \mathcal{D}_{\mathrm{Qcoh}}(X)_Z$, then $\operatorname{cone}(\mathcal{G} \to p_*p^*(\mathcal{G}))$ belongs to $\mathcal{D}_{\mathrm{Qcoh}}(X)_{Z \cap W}$. In what concerns (i), note that by base-change the restriction of $p_*(\mathcal{F})$ to the complement of Z is zero. In what concerns (ii), restrict to V. Thanks to Theorem 4.20, the category $\mathcal{D}_{\mathrm{Qcoh}}(X)_{Z \cap W}$ is generated by perfect complexes. Therefore, by applying Neeman's celebrated result [38, Thm. 2.1] to (4.24), we conclude that (4.23) is also a short exact sequence of triangulated categories. $\square$

The following "excision" result will be used in the proof of Theorems 5.3 and 6.8.

Theorem 4.25 (See [49, Thm 2.6.3]). *Let $f : X' \to X$ be a flat morphism of quasi-compact quasi-separated k-schemes and let $Z \hookrightarrow X$ be a closed subscheme with quasi-compact complement such that $Z' := X' \times_X Z \to Z$ is an isomorphism of k-schemes. Then the functors (f_*, f^*) define inverse equivalences of categories between $\mathcal{D}(X)_Z$ and $\mathcal{D}(X')_{Z'}$ and between $\operatorname{perf}(X)_Z$ and $\operatorname{perf}(X')_{Z'}$.*

Proof. As proved in loc. cit., the functors (f_*, f^*) define inverse equivalences between $\mathcal{D}^-(X)_Z$ and $\mathcal{D}^-(X')_{Z'}$. However, since $\mathcal{D}(X)_Z$ and $\mathcal{D}(X')_{Z'}$ admit arbitrary direct sums and are generated by $\mathcal{D}^-(X)_Z$ and $\mathcal{D}^-(X')_{Z'}$, respectively, we conclude that the functors (f_*, f^*) also define inverse equivalences between $\mathcal{D}(X)_Z$ and $\mathcal{D}(X')_{Z'}$. By restriction to compact objects, we hence obtain inverse equivalences between $\operatorname{perf}(X)_Z$ and $\operatorname{perf}(X')_{Z'}$; see Theorem 4.20. $\square$

The following result will be used in the proof of Generalization (G2).

Lemma 4.26. *Let X and Y be two quasi-compact quasi-separated k-schemes. Then there is a Morita equivalence:*

$$\operatorname{perf}_{\mathrm{dg}}(X) \otimes \operatorname{perf}_{\mathrm{dg}}(Y) \longrightarrow \operatorname{perf}_{\mathrm{dg}}(X \times Y), \qquad (\mathcal{F}, \mathcal{F}') \mapsto \mathcal{F} \boxtimes \mathcal{F}' \,.$$

Proof. Let $\mathcal{G}$ and $\mathcal{G}'$ be compact generators of the triangulated categories $\mathcal{D}_{\mathrm{Qcoh}}(X)$ and $\mathcal{D}_{\mathrm{Qcoh}}(Y)$, respectively; see Theorem 4.20. According to [6, Lem. 3.4.1], $\mathcal{G} \boxtimes \mathcal{G}'$ is a compact generator of $\mathcal{D}_{\mathrm{Qcoh}}(X \times Y)$. Hence, it is sufficient to show that the dg k-algebras $\mathbf{R}\mathrm{End}_{X \times Y}(\mathcal{G} \boxtimes \mathcal{G}')$ and $\mathbf{R}\mathrm{End}_X(\mathcal{G}) \otimes \mathbf{R}\mathrm{End}_Y(\mathcal{G}')$ are quasi-isomorphic. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the projection maps. Since $p^*(\mathcal{G})$ and $q^*(\mathcal{G}')$ are perfect complexes, we have

$$\mathbf{R}\mathrm{End}_{X \times Y}(\mathcal{G} \boxtimes \mathcal{G}') = \mathbf{R}\Gamma(X \times Y, \mathbf{R}\underline{\mathrm{End}}_{X \times Y}(p^*(\mathcal{G}) \otimes^{\mathbf{L}}_{X \times Y} q^*(\mathcal{G}'))),$$

$$\begin{aligned} \mathbf{R}\underline{\mathrm{End}}_{X \times Y}(p^*(\mathcal{G}) \otimes^{\mathbf{L}}_{X \times Y} q^*(\mathcal{G}')) &\simeq p^*(\mathbf{R}\underline{\mathrm{End}}_X(\mathcal{G})) \otimes^{\mathbf{L}}_{X \times Y} q^*(\mathbf{R}\underline{\mathrm{End}}_Y(G')) \\ &\simeq \mathbf{R}\underline{\mathrm{End}}_X(\mathcal{G}) \boxtimes \mathbf{R}\underline{\mathrm{End}}_Y(\mathcal{G}') \,. \end{aligned}$$

Therefore, it suffices to show that $\mathbf{R}\Gamma(X \times Y, \mathcal{F} \boxtimes \mathcal{F}') \simeq \mathbf{R}\Gamma(X, \mathcal{F}) \otimes \mathbf{R}\Gamma(Y, \mathcal{F}')$ for any two complexes with quasi-coherent cohomology $\mathcal{F}$ and $\mathcal{F}'$. We have

$$
\begin{aligned}
\mathbf{R}\Gamma(X \times Y, \mathcal{F} \boxtimes \mathcal{F}') &\simeq \mathbf{R}\Gamma(X \times Y, p^*(\mathcal{F}) \otimes_{X \times Y}^{\mathbf{L}} q^*(\mathcal{F}')) \\
(4.27) \qquad &\simeq \mathbf{R}\Gamma(Y, q_* p^*(\mathcal{F}) \otimes_Y^{\mathbf{L}} (\mathcal{F}')) \\
(4.28) \qquad &\simeq \mathbf{R}\Gamma(Y, (\mathbf{R}\Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_Y) \otimes_Y^{\mathbf{L}} \mathcal{F}') \\
&\simeq \mathbf{R}\Gamma(Y, \mathbf{R}\Gamma(X, \mathcal{F}) \otimes_k \mathcal{F}') \\
&\simeq \mathbf{R}\Gamma(X, \mathcal{F}) \otimes \mathbf{R}\Gamma(Y, \mathcal{F}') \, ,
\end{aligned}
$$

where (4.27) follows from the projection formula for q and (4.28) from flat base change for $\mathbf{R}\Gamma(Y, -)$. This concludes the proof. $\qquad\square$

4.5. **Notation.** Let X be a k-scheme, $Z \hookrightarrow X$ a closed subscheme, $\mathcal{A}$ a dg category, and $E \colon \mathrm{dgcat}(k) \to \mathcal{C}$ a functor with values in an arbitrary category. In order to simplify the exposition, we will write

$$
E(X; Z; \mathcal{A}) := E(\mathrm{perf}(X)_Z \otimes \mathcal{A}) \, .
$$

If $Z = X$ or $\mathcal{A} = k$, then we will omit the corresponding symbols from the notation.

5. Nisnevich descent in the supported setting

Consider the cartesian square of quasi-compact quasi-separated k-schemes

$$
(5.1) \qquad
\begin{array}{ccc}
V_{12} := V_1 \times_X V_2 & \longrightarrow & V_2 \\
\downarrow & & \downarrow p_2 \\
V_1 & \xrightarrow{\ p_1\ } & X \, ,
\end{array}
$$

where p_1 is an open immersion and p_2 is an étale map inducing an isomorphism of reduced schemes $p_2^{-1}(X \backslash V_1)_{\mathrm{red}} \simeq (X \backslash V_1)_{\mathrm{red}}$. As proved by Morel and Voevodsky in [35, §3.1 Prop. 1.4], the Nisnevich topology is generated by the distinguished squares (5.1). The Zariski topology is generated by those distinguished squares (5.1) in which p_2 is also an open immersion.

Notation 5.2. A sequence of maps $a \to a' \to a'' \to \Sigma a$ in a triangulated category $\mathcal{T}$ is called an *LES-triangle* if it becomes a Long Exact Sequence after applying $\mathrm{Hom}_{\mathcal{T}}(b, -)$, for every object b of $\mathcal{T}$. Distinguished triangles are, of course, LES-triangles but the converse is false.

Let $i \colon Z \hookrightarrow X$ be a closed subscheme with quasi-compact complement, $Z_1 := Z \cap V_1$, $Z_2 := p_2^{-1}(Z)$, and $Z_{12} := Z_1 \times_Z Z_2$. Under this notation, and those of §4.5, we have the following Nisnevich descent result:

Theorem 5.3. *For every localizing invariant $E \colon \mathrm{dgcat}(k) \to \mathcal{T}$ we have a "Mayer-Vietoris" LES-triangle*[10]

$$
(5.4) \qquad E(X; Z) \xrightarrow{\ \pm\ } E(V_1; Z_1) \oplus E(V_2; Z_2) \longrightarrow E(V_{12}; Z_{12}) \xrightarrow{\ \delta\ } \Sigma E(X; Z) \, .
$$

---

[10]If the functor E is suitably enhanced, then (5.4) can be made into an actual distinguished triangle. However, we will not need this extra assumption/result.

Proof. Let us write W for the (reduced) closed complement $(X \backslash V_1)_{\mathrm{red}}$ of V_1. Under this notation, we have the commutative diagram

$$(5.5) \qquad \begin{array}{ccccc} \mathrm{perf}_{\mathrm{dg}}(X)_{Z \cap W} & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(X)_Z & \xrightarrow{p_1^*} & \mathrm{perf}_{\mathrm{dg}}(V_1)_{Z_1} \\ \downarrow & & \downarrow{\scriptstyle p_2^*} & & \downarrow \\ \mathrm{perf}_{\mathrm{dg}}(V_2)_{Z_2 \cap p_2^{-1}(W)} & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(V_2)_{Z_2} & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(V_{12})_{Z_{12}} \end{array}$$

in the homotopy category $\mathrm{Hmo}(k)$. Thanks to Proposition 4.21, both rows are short exact sequences of dg categories. Moreover, since $p_2^{-1}(Z \cap W) = Z_2 \cap p_2^{-1}(W)$ and $p_2 \colon V_2 \to X$ is étale, the left-hand side vertical morphism is a Morita equivalence by Theorem 4.25. By applying E to (5.5), we obtain an induced morphism between distinguished triangles with invertible outer vertical morphisms

$$(5.6) \qquad \begin{array}{ccccccc} E(X; Z \cap W) & \longrightarrow & E(X; Z) & \xrightarrow{E(p_1^*)} & E(V_1; Z_1) & \xrightarrow{\partial} & \Sigma E(X; Z \cap W) \\ {\scriptstyle \simeq}\downarrow & & {\scriptstyle E(p_2^*)}\downarrow & & \downarrow & & {\scriptstyle \simeq}\downarrow \\ E(V_2; Z_2 \cap p_2^{-1}(W)) & \rightarrow & E(V_2; Z_2) & \rightarrow & E(V_{12}; Z_{12}) & \xrightarrow{\partial} & \Sigma E(V_2; Z_2 \cap p_2^{-1}(W)) \,. \end{array}$$

By applying the homological functor $\mathrm{Hom}_{\mathcal{T}}(b, -)$ to the commutative diagram (5.6), for every object b of $\mathcal{T}$, we observe that the middle square forms a "Mayer-Vietoris" LES-triangle (5.4) with boundary morphism δ induced by the composition $E(V_{12}; Z_{12}) \xrightarrow{\partial} \Sigma E(V_2; Z_2 \cap p_2^{-1}(W)) \simeq \Sigma E(X; Z \cap W)$. $\qquad\qquad\square$

Remark 5.7 (Generalization). Given a dg category $\mathcal{A}$, Drinfeld proved in [11, Prop. 1.6.3] that the functor $- \otimes \mathcal{A}$ preserves short exact sequences of dg categories. By tensoring (5.5) with $\mathcal{A}$, we obtain an induced LES-triangle

$$E(X; Z; \mathcal{A}) \xrightarrow{\pm} E(V_1; Z_1; \mathcal{A}) \oplus E(V_2; Z_2; \mathcal{A}) \to E(V_{12}; Z_{12}; \mathcal{A}) \xrightarrow{\delta} \Sigma E(X; Z; \mathcal{A}) \,.$$

6. PROOF OF THEOREM 1.9

Thanks to the work of Thomason-Trobaugh [49, §5] (see also Proposition 4.21), we have the following short exact sequence of dg categories:

$$(6.1) \qquad 0 \longrightarrow \mathrm{perf}_{\mathrm{dg}}(X)_Z \longrightarrow \mathrm{perf}_{\mathrm{dg}}(X) \xrightarrow{j^*} \mathrm{perf}_{\mathrm{dg}}(U) \longrightarrow 0 \,.$$

Consequently, we obtain an induced distinguished triangle

$$E(X; Z) \longrightarrow E(X) \xrightarrow{E(j^*)} E(U) \xrightarrow{\partial} \Sigma E(X; Z) \,.$$

Since the dg functor $i_* \colon \mathrm{perf}_{\mathrm{dg}}(Z) \to \mathrm{perf}_{\mathrm{dg}}(X)$ factors through the inclusion $\mathrm{perf}_{\mathrm{dg}}(X)_Z \subset \mathrm{perf}_{\mathrm{dg}}(X)$, we have also an induced morphism

$$(6.2) \qquad\qquad\qquad E(i_*) \colon E(Z) \longrightarrow E(X; Z) \,.$$

The proof of Theorem 1.9 follows now from the following result:

Theorem 6.3 ("Dévissage"). *The morphism* (6.2) *is invertible.*

The proof of Theorem 6.3 is divided into three main steps:

(i) first, we describe the behavior of an $\mathbb{A}^1$-homotopy invariant with respect to $\mathbb{N}_0$-graded dg categories;

(ii) second, by combining the first step with a formality result of independent interest (see Theorem 6.8), we prove Theorem 6.3 in the affine case;

(iii) third, using the Nisnevich descent results established in §5, we bootstrap the proof of Theorem 6.3 from the affine case to the general case.

Step I: Gradings. A dg category $\mathcal{A}$ is called $\mathbb{N}_0$-*graded* if the (cochain) complexes of k-vector spaces $\mathcal{A}(x, y)$ are equipped with a direct sum decomposition $\bigoplus_{n \geq 0} \mathcal{A}(x, y)_n$ of (cochain) complexes of k-vector spaces, which is preserved by the composition law. Note that, by definition, the $\mathbb{N}_0$-grading of $\mathcal{A}(x, y)$ is respected by the differential. The elements of $\mathcal{A}(x, y)_n$ are called of *pure degree* n. Let $\mathcal{A}_0$ be the dg category with the same objects as $\mathcal{A}$ and $\mathcal{A}_0(x, y) := \mathcal{A}(x, y)_0$. Note that we have an "inclusion" dg functor $\iota_0 \colon \mathcal{A}_0 \to \mathcal{A}$ and a "projection" dg functor $\pi \colon \mathcal{A} \to \mathcal{A}_0$ such that $\pi \circ \iota_0 = \mathrm{id}$.

Remark 6.4. Let $\mathcal{A}$ be a dg category whose (cochain) complexes of k-vector spaces $\mathcal{A}(x, y)$ have zero differential and are supported in nonpositive degrees. In this case, the dg category $\mathcal{A}$ becomes $\mathbb{N}_0$-graded: an element of $\mathcal{A}(x, y)$ is of pure degree n if it is of cohomological degree $-n$.

Remark 6.5. The tensor product of an $\mathbb{N}_0$-graded dg category $\mathcal{A}$ with a dg category $\mathcal{B}$ is again an $\mathbb{N}_0$-graded dg category with $(\mathcal{A} \otimes \mathcal{B})((x, w), (y, z))_n := \mathcal{A}(x, y)_n \otimes \mathcal{B}(w, z)$.

The following result is classical.

Lemma 6.6 (See [16]). *For every* $\mathbb{A}^1$*-homotopy invariant* $E \colon \mathrm{dgcat}(k) \to \mathcal{T}$ *and* $\mathbb{N}_0$*-graded dg category* $\mathcal{A}$*, we have an associated isomorphism* $E(\iota_0) \colon E(\mathcal{A}_0) \to E(\mathcal{A})$.

Proof. Since $\pi \circ \iota_0 = \mathrm{id}$, it suffices to show that $E(\iota_0 \circ \pi) = \mathrm{id}$. Note that we have canonical dg functors $\iota \colon \mathcal{A} \to \mathcal{A}[t]$ and $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathcal{A}[t] \to \mathcal{A}$ satisfying the equalities $\mathrm{ev}_0 \circ \iota = \mathrm{ev}_1 \circ \iota = \mathrm{id}$. Consider the commutative diagram

(6.7)

where H is the dg functor defined by $x \mapsto x$ and $\mathcal{A}(x, y)_n \to \mathcal{A}(x, y)[t], f \mapsto f \otimes t^n$. Since the functor E inverts the morphism ι, it also inverts the morphisms ev_0 and ev_1. Moreover, $E(\mathrm{ev}_0) = E(\mathrm{ev}_1)$. By applying the functor E to (6.7), we hence conclude that $E(\iota_0 \circ \pi) = \mathrm{id}$. $\square$

Step II: Affine case. Our main result in the affine case is the following.

Theorem 6.8 (Formality). *Let* $Z \hookrightarrow X$ *be a closed immersion of smooth affine* k*-schemes and* $\mathcal{I} \subset \mathcal{O}_X$ *the defining ideal of* Z *in* X*. Then the following hold:*

(i) *The sheaf* $\mathcal{O}_X / \mathcal{I} \in \mathrm{perf}(X)_Z$ *is a compact generator of* $\mathcal{D}_{\mathrm{Qcoh}}(X)_Z$*. Consequently, the dg category* $\mathrm{perf}_{\mathrm{dg}}(X)_Z$ *is Morita equivalent to* $\mathrm{perf}_{\mathrm{dg}}(A)$*, where* A *stands for the derived dg algebra of endomorphisms* $\mathbf{R}\mathrm{End}_X(\mathcal{O}_X / \mathcal{I})$.

(ii) *The dg algebra A is formal.*

(iii) *We have an isomorphism $H^*(A) \simeq \Gamma(Z, \bigwedge_Z^*((\mathcal{I}/\mathcal{I}^2)^\vee))$, where $\mathcal{I}/\mathcal{I}^2$ is considered as a vector bundle on Z.*

Proof. (i) Given an object $\mathcal{F} \in \mathcal{D}_{\mathrm{Qcoh}}(X)_Z$, we need to show the implication:

$$(6.9) \qquad \mathrm{Hom}_{\mathcal{D}_{\mathrm{Qcoh}}(X)_Z}(\mathcal{O}_X/\mathcal{I}, \Sigma^n \mathcal{F}) = 0 \quad \forall n \in \mathbb{Z} \quad \Rightarrow \quad \mathcal{F} = 0\,.$$

Since X is affine, the left-hand side of (6.9) is equivalent to $\underline{\mathrm{Hom}}_X(\mathcal{O}_X/\mathcal{I}, \mathcal{F}) = 0$. As proved by Grothendieck in [13, Prop. (19.1.1)], the ideal $\mathcal{I}$ is locally generated by a regular sequence. Given an (affine) open subscheme $U \hookrightarrow X$ such that $Z \cap U = V(f_1, \ldots, f_d)$ for a regular sequence $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_U)$, the object $(\mathcal{O}_X/\mathcal{I})_{|U} := \bigotimes_i \mathrm{cone}(f_i \colon \mathcal{O}_U \to \mathcal{O}_U)$ is well known to be a compact generator of $\mathcal{D}_{\mathrm{Qcoh}}(U)_{Z \cap U}$; see [5, Prop. 6.1]. Using the natural identification between $\underline{\mathrm{Hom}}_X((\mathcal{O}_X/\mathcal{I})_{|U}, \mathcal{F}_{|U})$ and $\underline{\mathrm{Hom}}_X(\mathcal{O}_X/\mathcal{I}, \mathcal{F})_{|U}$, we conclude that $\mathcal{F}_{|U} = 0$. The proof follows now from the fact that X admits a covering by such (affine) open subschemes U's.

(ii) It is now convenient to switch to ring theoretic notation: let $X = \mathrm{Spec}(R)$, $Z = \mathrm{Spec}(S)$, and $\phi \colon R \twoheadrightarrow S$ the surjective k-algebra homomorphism corresponding to the closed immersion $Z \hookrightarrow X$. Let us write I for the kernel of ϕ, $\widehat{R}$ for the completion of R at I, T for the graded symmetric algebra $\mathrm{Sym}_S(I/I^2)$, and $\widehat{T}$ for the completion of T at the augmentation ideal $T_{>0}$. Thanks to Proposition 6.12 below, I/I^2 is a finitely generated projective $S = R/I$-module and there exists an isomorphism $\widehat{T} \xrightarrow{\simeq} \widehat{R}$ compatible with ϕ and with the projection $T \to T_0 = S$.

Let us write $\mathcal{D}(R)$ for the derived category of R and $\mathcal{D}(R)_I$ for the full triangulated subcategory of those complexes of R-modules whose cohomology is locally annihilated by a power of I. Note that $\mathcal{D}(R)$, resp. $\mathcal{D}(R)_I$, agrees with the category $\mathcal{D}(\mathrm{Qcoh}(X))$, resp. $\mathcal{D}(\mathrm{Qcoh}(X))_Z$. As proved in [5, Thm. 5.1] (see also [6, Cor. 3.3.5]), we have an equivalence $\mathbf{R}\Gamma \colon \mathcal{D}_{\mathrm{Qcoh}}(X) \xrightarrow{\simeq} \mathcal{D}(\mathrm{Qcoh}(X))$. Since this equivalence preserves cohomology of complexes, it restricts to the equivalence

$$(6.10) \qquad \mathbf{R}\Gamma \colon \mathcal{D}_{\mathrm{Qcoh}}(X)_Z \xrightarrow{\simeq} \mathcal{D}(\mathrm{Qcoh}(X))_Z = \mathcal{D}(R)_I\,.$$

We also have the following equivalences of categories:

$$(6.11) \qquad \mathcal{D}_{\mathrm{Qcoh}}(X)_Z \xrightarrow{(6.10)} \mathcal{D}(R)_I \xrightarrow{(a)} \mathcal{D}(\widehat{R})_{\widehat{I}} \xrightarrow{(b)} \mathcal{D}(\widehat{T})_{\widehat{T}_{\geq 1}} \xrightarrow{(c)} \mathcal{D}(T)_{T_{\geq 1}}\,,$$

where (a) and (c) follow from (the ring theoretic version of) Theorem 4.25 and (b) from Proposition 6.12. Concretely, (a) is the base-change along the k-algebra homomorphism $R \to \widehat{R}$ and (b), resp. (c), the restriction along the k-algebra homomorphism $\widehat{T} \xrightarrow{\simeq} \widehat{R}$, resp. $T \to \widehat{T}$. Via the equivalences (6.11), the sheaf $\mathcal{O}_X/\mathcal{I} \in \mathcal{D}_{\mathrm{Qcoh}}(X)_Z$ corresponds to the T-module $S \in \mathcal{D}(T)_{T_{\geq 1}}$. Consequently, the dg algebra A becomes quasi-equivalent to the derived dg algebra of endomorphisms $A' := \mathbf{R}\mathrm{End}_T(S)$. The formality of A' follows now from Proposition 6.13 below.

(iii) The proof follows from Proposition 6.13 below. $\square$

Proposition 6.12 (Affine tubular neighborhoods). *Let $\phi \colon R \twoheadrightarrow S$ be a surjective morphism between smooth k-algebras, with kernel I. Then the $S = R/I$-module I/I^2 is finitely generated projective. Let $\widehat{R}$ be the I-adic completion of R at I, T the graded symmetric algebra $\mathrm{Sym}_S(I/I^2)$ (with $T_n := \mathrm{Sym}_S^n(I/I^2)$), and $\widehat{T}$ the completion of T at the ideal $T_{\geq 1}$. Under this notation, there exists an isomorphism $\tau \colon \widehat{T} \xrightarrow{\simeq} \widehat{R}$ such that $\widehat{\phi} \circ \tau$ agrees with the projection onto $T_0 = S$.*

Proof. Since the k-algebras R and S are formally smooth (see [33, Prop. E.2]), the proof follows from [13, Cor. 0.19.5.4] applied to $A := k$, $B := \widehat{R}$, and $C := S$. The property that $\widehat{\phi} \circ \tau$ agrees with the projection onto $T_0 = S$ is not explicitly stated in loc. cit., but it follows immediately from the proof. $\square$

Proposition 6.13 (Koszul duality). *Let S be a commutative ring, P a finitely generated projective S-module, and $T := \mathrm{Sym}_S(P)$. Then the dg algebra $A := \mathbf{R}\mathrm{End}_T(S)$ is formal and its cohomology as a graded algebra is given by $\bigwedge_S^* P^\vee$.*

Proof. We may compute the cohomology $H^*(A) = \mathrm{Ext}_T^*(S, S)$ via the standard Koszul resolution of S given by

$$\mathbf{K} := (\mathrm{Sym}_S(\Sigma P) \otimes_S T, \eta \cap -),$$

where η is the unit element in $P^\vee \otimes_S P$. Since the differential in $\mathbf{K}$ becomes zero after applying $\mathrm{Hom}_T(-, S)$, we have abelian group isomorphisms

$$(6.14) \qquad \mathrm{Ext}_T^*(S, S) \simeq H^*(\mathrm{Hom}_T(\mathbf{K}, S)) \simeq \mathrm{Sym}_S \Sigma^{-1}(P^\vee).$$

Next, we observe that if $\omega \in \mathrm{Sym}_S^n \Sigma^{-1}(P^\vee)$, then $\omega \cap -$ (super-)commutes with $\eta \cap -$. This implies that $\omega \cap -$ defines a morphism of complexes $\mathbf{K} \to \Sigma^n \mathbf{K}$. We obtain in this way a dg algebra morphism $\mathrm{Sym}_S \Sigma^{-1}(P^\vee) \to \mathrm{End}_T(\mathbf{K}), \omega \mapsto \omega \cap -$, such that the composition $\mathrm{Sym}_S \Sigma^{-1}(P^\vee) \to H^*(\mathrm{End}_T(\mathbf{K})) \to H^*(\mathrm{Hom}_T(\mathbf{K}, S))$ is the inverse of (6.14). This concludes the proof. $\square$

We may now conclude the proof of Theorem 6.3, and hence of Theorem 1.9, in the affine case. Thanks to Theorem 6.8, the dg category $\mathrm{perf}_{\mathrm{dg}}(X)_Z$ is Morita equivalent to $H^*(A)$ and the dg functor $i_*\colon \mathrm{perf}_{\mathrm{dg}}(Z) \to \mathrm{perf}_{\mathrm{dg}}(X)_Z$ identifies with the inclusion of $H^0(A)$ into $H^*(A)$. Using Lemma 6.6 and Remark 6.4, we hence conclude that the induced morphism (6.2) is invertible.

Step III: General case. In order to bootstrap the proof of Theorem 6.3 from the affine case to the general case, we use the following induction principle:

Proposition 6.15 (See [6, Prop. 3.3.1]). *Given a property $\mathcal{P}$, assume the following:*

(A1) *The property $\mathcal{P}$ holds for all affine k-schemes X.*

(A2) *Let $V_1 \cup V_2 = X$ be a Zariski open cover of a scheme X; such that X, V_1, V_2, V_{12} are quasi-compact quasi-separated (see §5). If the property $\mathcal{P}$ holds for $V_1, V_2,$ and V_{12}, then it also holds for X.*

Under the assumptions (A1)-(A2), *the property $\mathcal{P}$ holds for all quasi-compact quasi-separated k-schemes.*

Let $\mathcal{P}$ be the following property: "If X is a smooth k-scheme, then (6.2) is invertible for every smooth closed subscheme Z". As proved in Step II, the assumption (A1) of Proposition 6.15 is satisfied. Let us now verify assumption (A2). Given a smooth k-scheme X and a smooth closed subscheme $i\colon Z \hookrightarrow X$, consider the

commutative diagram

$$(6.16)$$

in the triangulated category $\mathcal{T}$, where $Z_i := Z \cap V_i$ and $Z_{12} := Z_1 \cap Z_2$. In order to prove assumption (A2) we need to show that if the morphisms $E(i_*^1)$, $E(i_*^2)$, and $E(i_*^{12})$ are invertible, then $E(i_*)$ is also invertible. Thanks to Theorem 5.3, (6.16) gives rise to the following morphism between LES-triangles:

Making use of the 5-lemma, we conclude that $E(i_*)$ becomes invertible after applying $\mathrm{Hom}_{\mathcal{T}}(b, -)$, for every object b of $\mathcal{T}$. The Yoneda lemma hence implies that $E(i_*)$ is invertible. This finishes the proof of Theorem 6.3, and hence of Theorem 1.9.

7. Proof of the generalizations (G1)-(G2)

Generalization (G1). Consider the cartesian square of algebraic spaces

$$(7.1)$$

where p_1 is an open immersion and p_2 is an étale map inducing an isomorphism on reduced algebraic spaces $p_2^{-1}(\mathcal{X} \backslash \mathcal{V}_1)_{\mathrm{red}} \simeq (\mathcal{X} \backslash \mathcal{V}_1)_{\mathrm{red}}$. Let $i \colon \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed algebraic subspace, $\mathcal{Z}_1 := \mathcal{Z} \cap \mathcal{V}_1$, $\mathcal{Z}_2 := p_2^{-1}(\mathcal{Z})$, and $\mathcal{Z}_{12} := \mathcal{Z}_1 \times_{\mathcal{Z}} \mathcal{Z}_2$.

Remark 7.2. Theorem 5.3 holds similarly with (5.1) replaced by (7.1) and Z_1, Z_2, and Z_{12}, replaced by $\mathcal{Z}_1$, $\mathcal{Z}_2$, and $\mathcal{Z}_{12}$, respectively.

In the case of algebraic spaces, Proposition 6.15 admits the following variant:

Proposition 7.3 (See [21, Tag 08GL, Lem. 62.8.3]). *Given a property $\mathcal{P}$, assume:*

(A1) *The property $\mathcal{P}$ holds for all affine k-schemes $\mathcal{X}$.*

(A2) *If the property $\mathcal{P}$ holds for the quasi-compact quasi-separated algebraic spaces $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_{12}$ in (7.1), with $\mathcal{V}_2$ being an affine k-scheme, then it also holds for $\mathcal{X}$.*

Under the assumptions (A1)-(A2), *the property* $\mathcal{P}$ *holds for all quasi-compact quasi-separated algebraic spaces.*

Let $\mathcal{X}$ be a smooth algebraic space, $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ a smooth closed algebraic space, and $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ the open complement of $\mathcal{Z}$. Similarly to (6.1), we have the following short exact sequence of dg categories:

$$0 \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathcal{X})_{\mathcal{Z}} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathcal{X}) \xrightarrow{j^*} \mathrm{perf}_{\mathrm{dg}}(\mathcal{U}) \longrightarrow 0\,.$$

Therefore, in order to establish (G1), it suffices to prove the generalization of Theorem 6.3 obtained by replacing (6.2) with $E(i_*)\colon E(\mathcal{Z}) \to E(\mathcal{X}; \mathcal{Z})$. Steps I-II hold mutatis mutandis. For Step III, simply replace Proposition 6.15 by Proposition 7.3 and run the same argument using Remark 7.2.

Generalization (G2). Given a dg category $\mathcal{A}$, Drinfeld proved in [11, Prop. 1.6.3] that the functor $- \otimes \mathcal{A}$ preserves short exact sequences of dg categories. Consequently, in order to establish (G2), it suffices to prove the generalization of Theorem 6.3 obtained by replacing (6.2) with the morphism

$$(7.4) \qquad E(i_* \otimes \mathrm{id})\colon E(Z; \mathcal{A}) \longrightarrow E(X; Z; \mathcal{A})\,.$$

Step I holds mutatis mutandis. For Step II, recall from Theorem 6.8 that the dg category $\mathrm{perf}_{\mathrm{dg}}(X)_Z$ is Morita equivalent to $H^*(A)$ and that the dg functor $i_*\colon \mathrm{perf}_{\mathrm{dg}}(X) \to \mathrm{perf}_{\mathrm{dg}}(X)_Z$ identifies with the inclusion of $H^0(A)$ into $H^*(A)$. Thanks to Remarks 6.4-6.5, we hence conclude from Lemma 6.6 that the morphism (7.4) is also invertible. For Step III, run the same argument using Remark 5.7. Finally, the second claim of Generalization (G2) follows from Lemma 4.26.

8. Proof of Theorem 2.1

We start with the following birationality result:

Proposition 8.1. *Let k be a perfect field, X and Y two birational smooth connected k-schemes, and $E\colon \mathrm{dgcat}(k) \to \mathcal{T}$ a functor which satisfies conditions* (C1)-(C2). *Let $\mathcal{U}$ be a triangulated subcategory of $\mathcal{T}$. Assume that all the objects $E(W)$, with W a smooth k-scheme of dimension strictly inferior to $\dim(X)$, belong to $\mathcal{U}$. Under these assumptions, if $E(X)$ or $E(Y)$ belongs to $\mathcal{U}$, so does the other one.*

Proof. It is sufficient to consider the case where Y is an open subscheme of X. Let $Z \hookrightarrow X$ be the closed complement of Y. Since by assumption k is a perfect field, there exists a stratification of Z into closed subschemes

$$\varnothing = Z_{-1} \hookrightarrow Z_0 \hookrightarrow \cdots \hookrightarrow Z_r \hookrightarrow \cdots \hookrightarrow Z_{n-1} \hookrightarrow Z_n = Z$$

such that $Z_r \backslash Z_{r-1}$ is smooth for every $0 \leq r \leq n$. Consider the Gysin triangles

$$(8.2) \qquad E(Z_r \backslash Z_{r-1}) \longrightarrow E(X \backslash Z_{r-1}) \longrightarrow E(X \backslash Z_r) \xrightarrow{\partial} \Sigma E(Z_r \backslash Z_{r-1})$$

provided by Theorem 1.9. Since Y is an open *dense* subscheme of X, the dimension of $Z_r \backslash Z_{r-1}$ is strictly inferior to $\dim(X)$ for every $0 \leq r \leq n$. Therefore, the proof follows recursively from the Gysin triangles (8.2). $\square$

Item (i). Let $\mathcal{U}$ be the triangulated category $\mathcal{T}^{\mathrm{sp}}$. Without loss of generality we may assume that X is connected. Furthermore, using induction on $\dim(X)$ we may assume that all the objects $E(W)$, with W a smooth k-scheme of dimension strictly inferior to $\dim(X)$, belong to $\mathcal{U}$. Since by assumption k admits resolution of singularities, X can be realized as the open complement of a strict normal crossing divisor D inside a smooth projective k-scheme Y. Using the fact that $E(Y)$ belongs to $\mathcal{U}$, the proof of item (i) of Theorem 2.1 follows now from Proposition 8.1.

Remark 8.3. Let D_i be the irreducible components of D. One may easily show that the object $E(X)$ belongs to the smallest triangulated subcategory of $\mathcal{T}$ containing the objects $E(Y)$ and $\{E(D_{r_1} \cap \cdots \cap D_{r_i}) \,|\, 1 \le r_1, \ldots, r_i \le m\}$.

Item (ii). Let $\mathcal{U}$ be the triangulated category $\overline{\mathcal{T}^{\mathrm{sp}}}$. Without loss of generality we may assume that X is connected. Furthermore by induction on $\dim(X)$ we may assume that all the objects $E(W)$, with W a smooth k-scheme of dimension strictly inferior to $\dim(X)$, belong to $\mathcal{U}$. We start with the following auxiliary results:

Proposition 8.4. *For each prime $l \ne p$, there exists an open dense subscheme $V \hookrightarrow X$ and a finite étale cover $g_V : V' \to V$ such that $g_{V*}(\mathcal{O}_{V'}) \simeq \mathcal{O}_V^{\oplus d}$ with d coprime to l. Moreover, $E(V')$ belongs to $\mathcal{U}$.*

Proof. Gabber's refined version of de Jong's theory [20] of alterations (see [17, Thms. 3(i) and 3.2.1]) allows us to construct for each prime $l \ne p$ a diagram

$$
\begin{array}{ccccc}
V' := V \times_X X' & \xrightarrow{\ j'\ } & X' := X \times_{\overline{X}} Y & \longrightarrow & Y \\
{\scriptstyle g_V}\big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle g} \\
V & \xrightarrow[\ j\ ]{} & X & \longrightarrow & \overline{X} \,,
\end{array}
$$

where $\overline{X}$ is a compactification of X, g is an alteration, $j : V \hookrightarrow X$ is an open dense subscheme, and g_V is a finite étale surjective map of rank d prime to l. Shrinking V if necessary, we may assume that $(g_V)_*(\mathcal{O}_{V'}) \simeq \mathcal{O}_V^{\oplus d}$. Since Y is irreducible, the open subscheme $V' \hookrightarrow Y$ is dense. Using the fact that Y is smooth projective and $\dim(Y) = \dim(X)$, we conclude from Proposition 8.1 that $E(V')$ belongs to $\mathcal{U}$. $\square$

Lemma 8.5. *Let $f : X \to Y$ be a finite map between quasi-compact quasi-separated k-schemes such that $f_*(\mathcal{O}_X) \simeq \mathcal{O}_Y^{\oplus d}$. Then for every additive invariant F, the composition $F(f_*) \circ F(f^*) : F(Y) \to F(X) \to F(Y)$ is equal to d times the identity.*

Proof. Since F factors through the universal additive invariant $\mathrm{U}_{\mathrm{add}}$ (see §4.3), it is sufficient to prove Lemma 8.5 in the case $F = \mathrm{U}_{\mathrm{add}}$. Thanks to the projection formula (see [49, §3.17]), the dg functor $f_* f^* : \mathrm{perf}_{\mathrm{dg}}(Y) \to \mathrm{perf}_{\mathrm{dg}}(Y)$ is given by $f_*(\mathcal{O}_X) \otimes -$. Since by assumption $f_*(\mathcal{O}_X) \simeq \mathcal{O}_Y^{\oplus d}$, the $\mathrm{perf}_{dg}(Y)$-bimodule ${}_{f_* f^*}\mathrm{B}$ corresponding to $f_* f^*$ (see (4.1)) is isomorphic to $({}_{\mathrm{id}}\mathrm{B})^{\oplus d}$ in the triangulated category $\mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(Y), \mathrm{perf}_{\mathrm{dg}}(Y))$. Consequently, $[{}_{f_* f^*}\mathrm{B}] = d[{}_{\mathrm{id}}\mathrm{B}]$ in the Grothendieck group $K_0\mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(Y), \mathrm{perf}_{\mathrm{dg}}(Y))$. The proof follows now from the fact that $[{}_{\mathrm{id}}\mathrm{B}]$ is the identity of the object $\mathrm{U}_{\mathrm{add}}(Y) \in \mathrm{Hmo}_0(k)$. $\square$

Choose an arbitrary prime $l_0 \ne p$ and construct V, V', g_V, d as in Proposition 8.4 with $l = l_0$. Denote the result by V_0', V_0, g_{V_0}, d_0. Let $\{l_1, \ldots, l_t\}$ be the prime factors of d_0 distinct from p. Construct V', V, g_V, d for each $l = l_i$ and denote the result by

V_i', V_i, g_{V_i}, d_i. Let us write V for the intersection $\bigcap_{i=1}^{t} V_i$. Thanks to Proposition 8.1, we can assume that $V_0 = V_1 = \cdots = V_t = V$ and replace V_i' by $g_{V_i}^{-1}(V)$ without changing the statement of Proposition 8.4; in particular, $E(V_i')$ belongs to $\mathcal{U}$. Lemma 8.5 below hence implies that the composition $E(g_{V_i,*}) \circ E(g_{V_i}^*) \colon E(V) \to E(V_i') \to E(V)$ is equal to d_i times the identity.

Now, choose integers $(a_i)_i \in \mathbb{Z}$ such that $\sum_i a_i d_i = \gcd(d_0, \ldots, d_t)$. Under these choices, the (matrix) composition

$$E(V) \xrightarrow{[E(g_{V_i}^*)]_{t \times 1}} \bigoplus_{i=0}^{t} E(V_i') \xrightarrow{[a_i E(g_{V_i,*})]_{1 \times t}} E(V)$$

is equal to $e := \gcd(d_0, \ldots, d_n)$ times the identity. We claim that e is a power of p. Indeed, if $q \neq p$ is a prime which divides e, then q divides d_0. This implies that $q = l_i$ for some $1 \leq i \leq t$. But then we would conclude that q does not divides d_i, which is a contradiction! Since e is a power of p and, by assumption, the triangulated category $\mathcal{T}$ is $\mathbb{Z}[1/p]$-linear, the object $E(V)$ is a direct summand of $\bigoplus_{i=0}^{t} E(V_i')$. Using the fact that $E(V_i')$ belongs to $\mathcal{U}$ and that the triangulated category $\mathcal{U}$ is idempotent complete, we hence conclude that $E(V)$ also belongs to $\mathcal{U}$. The proof of item (ii) of Theorem 2.1 follows now from Proposition 8.1 applied to the open dense subscheme $j \colon V \hookrightarrow X$.

9. Proof of Theorem 2.7

We start with the following "invariance" result concerning affine fibrations:

Proposition 9.1. *Let $f \colon X \to Y$ be an affine fibration between quasi-compact quasi-separated k-schemes. For every functor $E \colon \mathrm{dgcat}(k) \to \mathcal{T}$ which satisfies conditions (C1)-(C2), we have an induced isomorphism $E(f^*) \colon E(Y) \overset{\simeq}{\to} E(X)$.*

Proof. Let us denote by d the relative dimension of f. Using an appropriate variant of Proposition 6.15, it suffices to verify the following conditions:

  (A1) The morphism $E(Y) \to E(Y \times \mathbb{A}^d)$, induced by the projection, is invertible.
  (A2) Let $V_1 \cup V_2 = Y$ be a Zariski cover of Y. If the morphisms $E(f_1^*)$, $E(f_2^*)$, and $E(f_{12}^*)$ are invertible, then $E(f^*)$ is also invertible.

By Lemma 4.26, we have an isomorphism $E(Y \times \mathbb{A}^d) \simeq E(Y; \mathrm{perf}_{\mathrm{dg}}(\mathbb{A}^d))$. Therefore, condition (A1) follows from the Morita equivalence $k[t]^{\otimes d} \to \mathrm{perf}_{\mathrm{dg}}(\mathbb{A}^d)$ and from an iterated application of condition (C2). In order to prove condition (A2), consider the commutative diagram
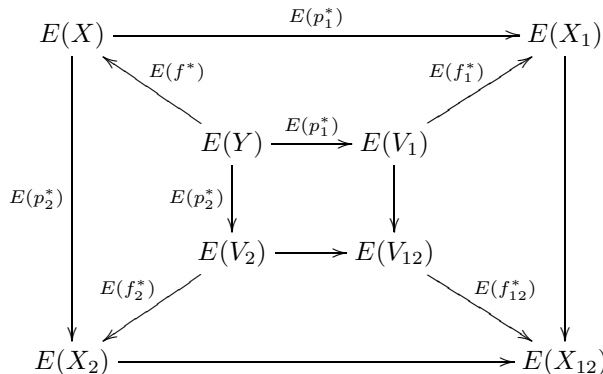
in the triangulated category $\mathcal{T}$, where $X_i := f^{-1}(V_i)$ and $X_{12} := f^{-1}(V_{12})$. Thanks to Theorem 5.1, the outer and inner squares give rise to "Mayer-Vietoris" LES-triangles. Hence, a proof similar to the one of (Step III: General case) shows that if $E(f_1^*)$, $E(f_2^*)$, and $E(f_{12}^*)$ are invertible, then $E(f^*)$ is also invertible.   □

The filtration (2.5), combined with the isomorphisms $\mathrm{U}(Y_i) \simeq \mathrm{U}(X_i \backslash X_{i-1})$ provided by Proposition 9.1, gives rise to the following Gysin triangles in $\mathrm{Mot}(k)$:

$$(9.2) \quad \mathrm{U}(Y_i) \longrightarrow \mathrm{U}(X \backslash X_{i-1}) \longrightarrow \mathrm{U}(X \backslash X_i) \xrightarrow{\partial} \Sigma \mathrm{U}(Y_i), \qquad 0 \leq i \leq n-1 \,.$$

We will prove by descending of induction on j for $-1 \leq j \leq n-1$ that $\mathrm{U}(X \backslash X_j) \simeq \bigoplus_{i=j+1}^n \mathrm{U}(Y_i)$. For $j = n-1$ this boils down to $\mathrm{U}(X \backslash X_{n-1}) = \mathrm{U}(Y_n)$ which has already been proved. Assume now that $\mathrm{U}(X \backslash X_j) \simeq \bigoplus_{i=j+1}^n \mathrm{U}(Y_i)$. Then the distinguished Gysin triangle (9.2) for $i = j$ becomes

$$(9.3) \qquad \mathrm{U}(Y_j) \longrightarrow \mathrm{U}(X \backslash X_{j-1}) \longrightarrow \bigoplus_{i=j+1}^n \mathrm{U}(Y_i) \xrightarrow{\partial} \Sigma \mathrm{U}(Y_j) \,.$$

Since the schemes Y_i's are smooth projective, Proposition 4.13 implies that $\partial = 0$. Therefore, the distinguished triangle (9.3) splits and gives rise to an isomorphism $\mathrm{U}(X \backslash X_{j-1}) \simeq \bigoplus_{i=j}^n \mathrm{U}(Y_i)$. This completes the proof of the induction step.

According to Lemma 4.15, the additive functor $\overline{\mathrm{U}}$ (see §4.3) restricts to an equivalence between the full subcategories of $\mathrm{Hmo}_0(k)$ and $\mathrm{Mot}(k)$ spanned by the objects $\mathrm{U}_{\mathrm{add}}(X)$ and $\mathrm{U}(X)$, respectively, where X runs through the smooth proper k-schemes. As a consequence, we also have an induced isomorphism $\mathrm{U}_{\mathrm{add}}(X) \simeq \bigoplus_{i=0}^n \mathrm{U}_{\mathrm{add}}(Y_i)$ in the additive category $\mathrm{Hmo}_0(k)$. The proof of Theorem 2.7 follows now from the fact that F, being additive, factors through $\mathrm{U}_{\mathrm{add}}$.

Acknowledgments

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