ON NORMAL APPROXIMATIONS
TO SYMMETRIC HYPERGEOMETRIC LAWS

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ABSTRACT. The Kolmogorov distances between a symmetric hypergeometric law with standard deviation σ and its usual normal approximations are computed and shown to be less than $1/(\sqrt{8\pi}\,\sigma)$, with the order $1/\sigma$ and the constant $1/\sqrt{8\pi}$ being optimal. The results of Hipp and Mattner (2007) for symmetric binomial laws are obtained as special cases.

Connections to Berry-Esseen type results in more general situations concerning sums of simple random samples or Bernoulli convolutions are explained.

Auxiliary results of independent interest include rather sharp normal distribution function inequalities, a simple identifiability result for hypergeometric laws, and some remarks related to Lévy's concentration-variance inequality.

CONTENTS

1. INTRODUCTION AND MAIN RESULT

1.1. **Aim.** This paper generalizes the error bound in the central limit theorem for symmetric binomial laws of Hipp and Mattner [11], which up to now was the only nontrivial example of a Berry-Esseen type inequality with an optimal constant known to the present authors, to a still optimal bound covering also symmetric hypergeometric laws. These solutions of special cases of the Berry-Esseen problem are of some particular interest for more general situations, as we attempt to explain in subsection 1.2 below, and are also remarkable in view of the apparent difficulty of determining merely close to optimal Berry-Esseen type inequalities in related special situations, as witnessed by the recent investigations of arbitrary binomial laws by Nagaev et al. [21, 22] and of arbitrary Bernoulli convolutions, which include in particular all hypergeometric laws as is known from [31], by Neammanee [23].

1.2. **Background: Berry-Esseen for sampling with or without replace-
ment.** Throughout this paper, let Φ denote the distribution function of the stan-
dard normal law. In this subsection, let $g : [1, \infty[\to]0, \infty]$ denote the pointwise
smallest function such that

$$
(1) \qquad \|F - \Phi\|_\infty \quad \leq \quad \frac{g\left(\frac{\beta}{\sigma^3}\right)}{\sqrt{n}}
$$

holds whenever $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and F is the distribution function of the
standardized sum of n i.i.d. random variables with law P on the real line $\mathbb{R}$ with
mean μ, variance $\sigma^2 > 0$, and finite third centred absolute moment $\beta =
\int |x - \mu|^3 \, dP(x)$. Further let $C \in]0, \infty]$ denote the smallest constant such that
$g(\varrho) \leq C\varrho$ holds for every $\varrho \in [1, \infty[$. Then the classical Berry-Esseen theorem
for sums of i.i.d. random variables states that $C < \infty$. More recent investigations
aim, among other goals, to obtain rather sharp upper bounds on the function g,
and here the best result announced so far appears to be Shevtsova's [30] bound
$\min\{0.4690\varrho, 0.3322(\varrho + 0.429), 0.3031(\varrho + 0.646)\}$ for each ϱ, which, when com-
bined with a classical lower bound for C due to Esseen [5], yields in particular
$0.4097 < (\sqrt{10} + 3)/(6\sqrt{2\pi}) \leq C < 0.4690$, and $g(1) < 0.4690$. However, as, by a
discussion of equality in Lyapunov's moment inequality, $\beta/\sigma^3 = 1$ iff P is a uniform
law on two points, without loss of generality 0 and 1, the special Berry-Esseen theo-
rem for symmetric binomial laws [11, Corollary 1.2] yields $g(1) = 1/\sqrt{2\pi} < 0.3990$.
Although, unfortunately, we do not yet know whether g is continuous at 1, the cited
special result suggests the possibility of an improvement of Shevtsova's bound for
ϱ close to 1.

Analogously, the Berry-Esseen type theorem for sampling without replacement
from a finite population due to Höglund [12] can be stated as follows: Let $h :
[1, \infty[\to]0, \infty]$ denote the pointwise smallest function such that

$$
(2) \qquad \|F - \Phi\|_\infty \quad \leq \quad \frac{h\left(\frac{\beta}{\sigma^3}\right)}{\sqrt{n\left(1 - \frac{n}{N}\right)}}
$$

holds whenever $N \in \mathbb{N}$ and $x \in \mathbb{R}^N$ are such that the law $P := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ has
mean $\mu = \frac{1}{N} \sum_{i=1}^N x_i$, variance $\sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2 > 0$, and the third centred
absolute moment $\beta = \frac{1}{N} \sum_{i=1}^N |x_i - \mu|^3$, and whenever $n \in \{1, \ldots, N - 1\}$ and F is
the distribution function of

$$
(3) \qquad \frac{S - n\mu}{\sqrt{n\left(1 - \frac{n}{N}\right)}\,\sigma}
$$

with S being the sum of a simple random sample of size n from x. Further let
$D \in]0, \infty]$ denote the smallest constant such that $h(\varrho) \leq D\varrho$ holds for every
$\varrho \in [1, \infty[$. Then Höglund's theorem states that $D < \infty$. With g and C as in the
previous paragraph, we have the simple Lemma 1.1 below, and hence $C \leq D$, but
we are not aware of any published explicit upper bounds for h or D. However, using
again that $\beta/\sigma^3 = 1$ iff P is a uniform law on two points, we see that the special
Berry-Esseen theorem for symmetric hypergeometric laws Theorem 1.3 below and

the formula for σ_0 in (37) (where F and σ have different meanings) yield

$$
\begin{aligned}
h(1) &= \sup\left\{\sqrt{n\left(1-\tfrac{n}{N}\right)}\,d \,:\, d, n, N \text{ as in Theorem 1.3(a)}\right\} \\
&= \sup\left\{2\sigma_0 d \,:\, d, \sigma_0 \text{ as in Theorem 1.3(a)}\right\} = \frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

by Remark 1.4(b) with $\tau = \sigma_0$, and by using the optimality of $\frac{1}{\sqrt{8\pi}}$ from Theorem 1.3(a), or by using $g(1) = \frac{1}{\sqrt{2\pi}}$ and Lemma 1.1. Hence $h(1) = g(1)$, suggesting that any effective upper bounds for $h(\varrho)$ which might become available in the future should be close to $1/\sqrt{2\pi}$ for ϱ close to one, and perhaps even close to $g(\varrho)$ in any case. Again, unfortunately, we do not yet know whether h is continuous at 1.

Lemma 1.1. *The functions g and h introduced above satisfy $g \le h$.*

Proof. Given $\varrho \in [1, \infty[$ and any $\gamma \in \mathbb{R}$ with $\gamma < g(\varrho)$, the definition of $g(\varrho)$ as a supremum yields an $n \in \mathbb{N}$ and a law P on $\mathbb{R}$ with third standardized absolute moment $\beta/\sigma^3 = \varrho$ and, using a reflection argument if necessary, an $s \in \mathbb{R}$ with $\Delta := \sqrt{n}\left(P^{*n}\left(]0, s[\right) - \Phi\left(\frac{s-n\mu}{\sqrt{n}\,\sigma}\right)\right) > \gamma$. Using the denseness with respect to weak convergence of the laws with finite support and rational point masses following from [1, Theorem 15.10] together with a simple truncation argument, we can take $x_N \in \mathbb{R}^N$ for $N > n$ such that $P_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_{N,i}}$ converges to P weakly and together with its moments and absolute moments up to the third order, for $N \to \infty$. Since the law Q_N of the sum of a simple random sample of size n from P_N differs from P_N^{*n} in the supremum distance by at most $\frac{n(n-1)}{2N}$ (see [9]), and since P_N^{*n} tends weakly to P^{*n} for $N \to \infty$, we get

$$
\begin{aligned}
\frac{h(\varrho)}{\sqrt{n}} &= \lim_{N\to\infty} \frac{h(\varrho)}{\sqrt{n\left(1-\tfrac{n}{N}\right)}} \\
&\ge \lim_{N\to\infty} \left(Q_N\left(]-\infty, s[\right) - \Phi\left(\frac{s-n\mu}{\sqrt{n\left(1-\tfrac{n}{N}\right)}\,\sigma}\right)\right) \ge \frac{\Delta}{\sqrt{n}}
\end{aligned}
$$

and hence $h(\varrho) > \gamma$. $\qquad\square$

1.3. **Hypergeometric laws.** Let us formally define hypergeometric and a few related laws on $\mathbb{R}$ and collect some standard properties of them. For $a \in \mathbb{R}$, we write δ_a for the *Dirac* measure concentrated at a. For $\alpha \in \mathbb{R}$, we write $\alpha^{\underline{k}} := \prod_{j=1}^{k}(\alpha - j + 1)$ and $\binom{\alpha}{k} := \alpha^{\underline{k}}/k!$ for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and, with the exception of the proof of Lemma 2.3, we put in this paper $\binom{\alpha}{k} := 0$ if $k \notin \mathbb{N}_0$. Then, for $n \in \mathbb{N}_0$ and $p \in [0, 1]$, the *binomial* law $\mathrm{B}_{n,p}$ can be defined by $\mathrm{B}_{n,p}(\{k\}) := \mathrm{b}_{n,p}(k) := \binom{n}{k}p^k(1-p)^{n-k}$ for $k \in \mathbb{Z}$, and a law P is *Bernoulli* if $P = \mathrm{B}_{1,p}$ for some $p \in [0, 1]$. For $r, b \in \mathbb{N}_0$ and $n \in \{0, \ldots, r+b\}$, we let $\mathrm{H}_{n,r,b}$ denote the *hypergeometric* law of the number of red balls drawn in a simple random sample of size n from an urn containing r red and b blue balls (red and blue, and not for example black and white, since the present choice of the colours leads to the same initial letters in several languages the authors happen to know), so that we have

$$
(4) \qquad \mathrm{H}_{n,r,b}(\{k\}) =: \mathrm{h}_{n,r,b}(k) = \frac{\binom{r}{k}\binom{b}{n-k}}{\binom{r+b}{n}} \qquad \text{for } k \in \mathbb{Z},
$$

which may also be used to define $H_{n,r,b}$ to avoid reference to a sampling model. No confusion of the notation $h_{n,r,b}$ with the letter h used for various objects in this paper seems likely. We write $x_+ := x \vee 0 := \max\{x, 0\}$ for the positive part of $x \in \mathbb{R}$, and we use the convention $\frac{0}{0} := 0$, relevant for example in (6) below if $r + b \in \{0, 1, 2\}$. Except for the trivial cases of $n = 0$ or $p = 0$, a binomial law $B_{n,p}$ uniquely determines its parameters n and p and is symmetric about its mean iff $p = \frac{1}{2}$, in which case the mean is $\frac{n}{2}$. The following lemma collects analogous or related simple facts for hypergeometric laws used below but apparently not easily available from the literature.

Lemma 1.2. *Let $r, b \in \mathbb{N}_0$ and $n \in \{0, \dots, r + b\}$.*
  (a) **Some basic descriptive properties.** $H_{n,r,b}$ *has the support*

$$(5) \qquad \{k \in \mathbb{Z} : h_{n,r,b}(k) > 0\} \quad = \quad \{(n - b)_+, \dots, n \wedge r\}$$

and the first three cumulants (mean, variance, third centred moment)
$$(6)$$
$$\mu = \frac{nr}{r + b}, \quad \sigma^2 = \frac{nrb(r + b - n)}{(r + b)^2(r + b - 1)}, \quad \kappa_3 = \frac{nrb(b - r)(r + b - n)(r + b - 2n)}{(r + b)^3(r + b - 1)(r + b - 2)}.$$

  (b) **(Non-)identifiability of parameters.** *We have*

$$(7) \qquad\qquad H_{n,r,b} = H_{r,n,r+b-n}$$

so that $H_{n,r,b}$ is already determined by $\{n, r\}$ together with $r + b$. Conversely and more precisely, we have:

  (i) $H_{n,r,b} = \delta_a$ *for some a iff $n \wedge r \wedge b \wedge (r + b - n) = 0$ and $n \wedge r = a$;*
  (ii) $H_{n,r,b} = B_{1,p}$ *for some $p \in {]0, 1]}$ iff $n \wedge r = 1$ and $\frac{n \vee r}{r+b} = p$;*
  (iii) *in all other cases, $H_{n,r,b}$ is not a binomial law and determines $\{n, r\}$ and $r + b$; that is, $H_{n,r,b} = H_{n',r',b'}$ for some $r', b' \in \mathbb{N}_0$ and $n' \in \{0, \dots, r' + b'\}$ holds iff $\{n, r\} = \{n', r'\}$ and $r + b = r' + b'$.*

  (c) **Reflections.** $h_{n,r,b}(k) = h_{n,b,r}(n - k)$ *for $k \in \mathbb{Z}$.*
  (d) **Symmetries.** $H_{n,r,b}$ *is symmetric about its mean μ iff $n \wedge r \wedge b \wedge (r+b-n) = 0$ or $\frac{r+b}{2} \in \{n, r\}$, which is the case iff $\kappa_3 = 0$, and which implies that $\mu \in \{\frac{n}{2}, \frac{r}{2}\}$.*

Proof. (a) Claim (5) is obvious from (4). The formulas for μ and σ^2 in (6) are proved in several textbooks as in [3] by considering a sum of indicator variables indicating "red" at each of the n draws, and this method works for κ_3 as well. Alternatively one may use (4) and the differential equation for hypergeometric functions as in [13, §5.14].
  (b) With $\alpha := n \wedge r$, $\beta := n \vee r$, and $N := r + b$, a computation starting from (4) yields

$$(8) \qquad h_{n,r,b}(k) \quad = \quad \frac{n^{\underline{k}} r^{\underline{k}} b^{\underline{n-k}}}{k!(r + b)^{\underline{n}}} \quad = \quad \frac{\alpha^{\underline{k}} \beta^{\underline{k}} (N - \beta)^{\underline{\alpha - k}}}{k! N^{\underline{\alpha}}} \qquad \text{for } k \in \{0, \dots, n\},$$

hence (7).
  (i) follows from (5) and the formula for σ^2 in (6).
  (ii) The "if" claim is clear by (8) with $k \in \{0, 1\}$. Conversely, if $H_{n,r,b} = B_{1,p}$ with $p \in {]0, 1]}$, then $n \wedge r = 1$ by (5), and $p = \mu = \frac{n \vee r}{r+b}$ in view of (6).
  (iii) Assume that $H_{n,r,b}$ is not as in (i) or (ii) and, without loss of generality in view of (7), that $n \leq r$. Then $r \wedge b > 0$ and $n > 1$, hence also $0 < \mu < n$, and (6)

yields

$$(9) \qquad \sigma^2 \;=\; \mu\left(1 - \frac{\mu}{n}\right)\frac{r+b-n}{r+b-1} \;<\; \mu\left(1 - \frac{\mu}{n}\right).$$

The identity in (9) yields $r+b$ as a function of the mean μ, the variance σ^2, and the right endpoint $n = n \wedge r$ of $H_{n,r,b}$, and then $r = (r+b)\mu/n$ and hence $\{n,r\}$ as a function of quantities already determined by $H_{n,r,b}$. The inequality $\sigma^2 < \mu\left(1 - \frac{\mu}{n}\right)$, as a relation between the mean, the variance, and the right endpoint of a law, would instead be an equality if $H_{n,r,b}$ were binomial.

(c) Trivial using (4).

(d) If $H_{n,r,b}$ is symmetric about its mean, then $\kappa_3 = 0$, as for any law with existing third moment. If $\kappa_3 = 0$, then (6) yields the stated condition for the parameters. If the latter holds, then $\sigma^2 = 0$ and symmetry is trivial, or $\frac{r+b}{2} \in \{n,r\}$ and then (c) yields for $k \in \mathbb{Z}$ either $r = b$ and hence

$$h_{n,r,b}(k) \;=\; h_{n,b,r}(n-k) \;=\; h_{n,r,b}(n-k),$$

or $n = r + b - n$ and hence, using also (7) at the first and at the last step below, $h_{n,r,b}(k) = h_{r,n,r+b-n}(k) = h_{r,r+b-n,n}(r-k) = h_{r,n,r+b-n}(r-k) = h_{n,r,b}(r-k)$, and hence in either case the symmetry of $H_{n,r,b}$, necessarily about its mean. The final claim about μ is obvious using (6). $\qquad\square$

Let P be a binomial or a hypergeometric law. We then call $N \in \mathbb{N} \cup \{\infty\}$ a *population size parameter* of P if $N = \infty$ and P is binomial or if $P = H_{n,r,b}$ for some $r, b \in \mathbb{N}_0$ and $n \in \{0, \dots, r+b\}$ with $r + b = N$. By Lemma 1.2(b), N is uniquely determined by P unless P is a Dirac or a Bernoulli law. Given a population size parameter N of P, we let σ_0^2 denote the *usual approximate variance* of P, with respect to N, namely, with σ^2 denoting the true variance of P,

$$(10) \qquad \sigma_0^2 \;:=\; \begin{cases} 0 & \text{if } N = 0, \\ \frac{N-1}{N}\sigma^2 & \text{if } N \in \mathbb{N}, \\ \sigma^2 & \text{if } N = \infty, \end{cases}$$

which is uniquely determined by P, and hence may then be denoted by $\sigma_0^2(P)$, unless $P = B_{1,p}$ with $p \in \left]0, 1\right[$. The customary but somewhat illogical dependence of σ_0^2 not only on P in this last case is a source of the slightly awkward "except" proviso at the end of Theorem 1.3(a) below.

1.4. The main result.

Theorem 1.3. (a) *Let F and f be the distribution function and the density of a symmetric hypergeometric or symmetric binomial law, with mean $\frac{n}{2}$, standard deviation $\sigma > 0$, population size parameter N, and the usual approximate standard deviation σ_0. Let G be the distribution function of a normal law with mean $\frac{n}{2}$ and standard deviation $\tau \in [\sigma_0, \sigma]$. Then, for $s \in \mathbb{R}$,*

$$(11) \quad |F(s) - G(s)| < d \ \text{ if } s \neq \left\lfloor \tfrac{n}{2} \right\rfloor \quad \text{ and } \quad |F(s-) - G(s-)| < d \ \text{ if } s \neq \left\lceil \tfrac{n}{2} \right\rceil$$

holds with

$$(12) \quad d \;\; := \;\; F\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) - G\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) \;\; = \;\; G\left(\left\lceil \tfrac{n}{2} \right\rceil -\right) - F\left(\left\lceil \tfrac{n}{2} \right\rceil -\right) \;\; = \;\; \|F - G\|_\infty$$

$$(13) \qquad = \quad \begin{cases} \Phi\left(\tfrac{1}{2\tau}\right) - \tfrac{1}{2} & \text{if } n \text{ is odd,} \\ \tfrac{1}{2} f\left(\tfrac{n}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

$$(14) \qquad \in \quad \frac{1}{\sigma} \cdot \left[\frac{\Phi(1) - \tfrac{1}{2}}{2}, \; \frac{1}{\sqrt{8\pi}} \right[\;\; = \;\; \left[\frac{0.17\ldots}{\sigma}, \; \frac{0.19\ldots}{\sigma} \right[$$

except that the upper bound claim $d < \frac{1}{\sigma\sqrt{8\pi}}$ in (14) is false if we have both $N = 2$ and $\tau/\sigma \le c = 0.78\ldots$, with c defined by $\sqrt{2\pi}\left(\Phi\left(\tfrac{1}{c}\right) - \tfrac{1}{2}\right) = 1$.

(b) *The interval $\left[\frac{\Phi(1) - \tfrac{1}{2}}{2}, \; \frac{1}{\sqrt{8\pi}} \right[$ in part (a) is the least possible, even if we assume there in addition that $N = \infty$ (binomial case) and hence $\tau = \sigma = \sigma_0 = \tfrac{1}{2}\sqrt{n}$.*

Theorem 1.3 and the supplements stated in the following Remark 1.4 are proved at the end of this paper.

Remark 1.4. (a) If τ is restricted to be σ in Theorem 1.3, then the formulation obviously simplifies a bit, and in particular the "except" proviso concerning (14) becomes redundant.

(b) Under the assumptions of Theorem 1.3(a) as stated, and if $\frac{N}{N-1}$ is read as 1 in case of $N = \infty$, we have without any exception

$$(15) \qquad \frac{\Phi\left(\sqrt{2}\right) - \tfrac{1}{2}}{\sqrt{8}\,\tau} \;\; \le \;\; \frac{\sqrt{\frac{N-1}{N}}\left(\Phi\left(\sqrt{\frac{N}{N-1}}\right) - \tfrac{1}{2}\right)}{2\tau} \;\; \le \;\; d \;\; < \;\; \frac{1}{\tau\sqrt{8\pi}}.$$

(c) In the special case of $\tau = \sigma_0$, the upper bound for d in (15) can be refined to

$$(16) \qquad d \;\; \le \;\; \Phi\left(\frac{1}{2\sigma_0}\right) - \frac{1}{2} \;\; < \;\; \frac{1}{\sqrt{8\pi}\,\sigma_0},$$

with equality in the first inequality iff n is odd.

Theorem 1.3 specialized to symmetric binomial laws with $N = \infty$ reduces to [11, Theorem 1.1 and Corollaries 1.1 and 1.2]. All other results in the literature related to Theorem 1.3 and known to us yield weaker or incomparable conclusions under more general hypotheses. Let us mention a few of these:

The central limit theorem for hypergeometric laws, namely "$\|F - G\|_\infty \to 0$ if $\sigma \to \infty$" with the notation of Theorem 1.3 extended to not necessarily symmetric laws, is proved by Rényi in [27, pp. 465–466] as a corollary to [4]. Rényi names S.N. Bernstein as the originator under the additional assumption "$\frac{r}{r+b}$ constant" in the notation of subsection 1.2. He also states that a direct proof of the general case "leads to tiresome calculations", which is refuted by Morgenstern's treatment in [19, pp. 62–63], where the appropriate local central limit theorem is elegantly derived from the corresponding one for binomial laws by writing

$$\mathrm{h}_{n,r,b}(k) = \mathrm{b}_{r,p}(k)\mathrm{b}_{b,p}(n-k)/\mathrm{b}_{r+b,p}(n)$$

with $p := \frac{n}{r+b}$ in the notation of subsection 1.3.

Now let C denote the optimal Berry-Esseen constant in the non-i.i.d. case, so that $0.4097 < (3 + \sqrt{10})/(6\sqrt{2\pi}) \leq C < 0.5583$ with the upper bound as announced in [30]. Further let F be the distribution function of a Bernoulli convolution $P = *_{j=1}^{n} B_{p_j}$ with $p \in [0,1]^n$, and let G be the distribution function of a normal law with the same mean $\mu = \sum_{j=1}^{n} p_j$ and variance $\sigma^2 = \sum_{j=1}^{n} p_j (1 - p_j)$. Then, since $\beta_j := p_j (1 - p_j)(p_j^2 + (1 - p_j)^2) \leq p_j (1 - p_j)$ is the third absolute moment of B_{p_j}, we have $\|F - G\|_\infty \leq C\sigma^{-3} \sum_{j=1}^{n} \beta_j$ and hence

$$(17) \qquad \frac{1}{2\sqrt{1 + 12\sigma^2}} \quad \leq \quad \|F - G\|_\infty \quad < \quad \frac{0.5583}{\sigma},$$

where the lower bound follows from the continuity of G and from the lower bound for the maximal jump size of F obtained from (18) below with $h = 1$. Now it is well known from [31, Corollary 5 with $n = 2$, hence F_2 generating function of $H_{s_1,s_2,N-s_2} = H_{s_2,s_1,N-s_1}$] that every hypergeometric law is a Bernoulli convolution as above, with certain in general not explicitly available p_j, but of course μ and σ^2 computable from (6). Thus, as already known from [31, Theorem 1 with $n = 2$, rewritten in terms of $\mu_0 + s_1 + s_2 - N$] in case of the upper bound, (17) directly applies to F and G as in the previous paragraph, and thus yields a result more explicit than the two theorems in [14] and with a simpler proof, but (17) is in the symmetric case of course weaker than (14) applied to $\|F - G\|_\infty$.

Höglund's theorem already mentioned in subsection 1.2 yields the upper bound in (17), in the general hypergeometric case, with an unspecified constant in place of 0.5583.

Some further related results and references can be found in the papers [18] concerning in particular sums of simple random samples, [23] concerning Bernoulli convolutions, and [15] concerning hypergeometric laws.

1.5. **On concentration-variance inequalities.** In deriving the lower bound in (17) above, we have used inequality (18) below, which is due to Paul Lévy in a sharper version.

Lemma 1.5. *Let P be a law on $\mathbb{R}$ with variance σ^2. Then we have*

$$(18) \qquad \sup_{x \in \mathbb{R}} P(]x, x + h[) \quad \geq \quad \frac{h}{\sqrt{h^2 + 12\sigma^2}} \quad \text{for } h \in]0, \infty[,$$

$$(19) \qquad \operatorname*{ess\,sup}_{x \in \mathbb{R}} f(x) \quad \geq \quad \frac{1}{\sqrt{12\sigma^2}} \quad \text{if } f \text{ is a Lebesgue density of } P.$$

Proof. For (19) we may assume that P has mean zero and $M := \text{L.H.S.}(19) < \infty$. With $c := \frac{1}{2M}$, we then have $\alpha := \int_{|x|>c} f(x)\,dx = \int_{|x| \leq c} (M - f(x))\,dx$, hence $\int_{|x|>c} x^2 f(x)\,dx \geq c^2\alpha \geq \int_{|x| \leq c} x^2 (M - f(x))\,dx$, and thus $\sigma^2 = \int x^2 f(x)\,dx \geq \int_{|x| \leq c} x^2 (f(x) + M - f(x))\,dx = \frac{2}{3}c^3 M = \frac{1}{12M^2}$.

To prove now (18), we apply (19) to the density $x \mapsto g(x) := \frac{1}{h} P(]x - h, x[)$ and the variance $\sigma^2 + h^2/12$ of the convolution of P with the uniform law on $]0, h[$ to get

$$(20) \qquad \text{L.H.S.}(18) \; = \; h \sup_{x \in \mathbb{R}} g(x) \; \geq \; h \operatorname*{ess\,sup}_{x \in \mathbb{R}} g(x) \; \geq \; h\frac{1}{\sqrt{12\left(\sigma^2 + \frac{h^2}{12}\right)}}. \qquad \square$$

Lévy [16, p. 149, Lemme 48,1] proved under the assumption of Lemma 1.5: If $p \in \mathbb{N}$ and $\lambda \in [0,1]$ are such that $c := \text{L.H.S.(18)} = \frac{\lambda}{p} + \frac{1-\lambda}{p+1}$, then

$$(21) \qquad\qquad 12\frac{\sigma^2}{h^2} \;\geq\; \lambda p^2 + (1-\lambda)(p+1)^2 - 1,$$

with equality for $P = \frac{\lambda}{p} \sum_{j=0}^{p-1} \delta_{jh} + \frac{1-\lambda}{p+1} \sum_{j=0}^{p} \delta_{(j-\frac{1}{2})h}$. Writing $p^2 = \left(\frac{1}{p}\right)^{-2}$ and $(p+1)^2 = \left(\frac{1}{p+1}\right)^{-2}$, and using convexity, (21) yields $12\frac{\sigma^2}{h^2} \geq \left(\frac{\lambda}{p} + \frac{1-\lambda}{p+1}\right)^{-2} - 1 = c^{-2} - 1$, hence (18), and (19) follows easily using $\frac{1}{h}P(]x, x+h[) \leq \operatorname{ess\,sup} f$. We refer to [10, p. 27] for a proof of (21) more formal than Lévy's and to [8] for generalizations.

The present proof of first (19) and then (18) is a slightly simplified and corrected version of an argument given by Bobkov and Chistyakov: Our first part is simpler, or at least more elementary, than [2, first 5 lines of Proof of Proposition 2.1]. To see the correction in the second part, let us first observe that we actually have equality at the second step in (20), since our g is lower semicontinuous, but that this could be wrong if we had closed intervals $[x, x+h]$ on the left in (18) and analogously also in the definition of g, as for example if $P = \frac{1}{2}(\delta_0 + \delta_1)$ and $h = 1$, contrary to [2, (2.1)], where hence $Q(X; \lambda)$ should be replaced by $Q(X; \lambda-)$.

Finally we have to mention that (19) also follows by letting $p \to \infty$ in Moriguti's sharp inequality [20, (3.4)] for L^p-norms, valid under the hypothesis of (19), namely

$$\|f\|_p \;\geq\; \left(\frac{2p}{3p-1}\right)^{\frac{1}{p}} \left(\sqrt{\frac{p-1}{3p-1}} \Big/ \left(\mathrm{B}\left(\frac{p}{p-1}, \frac{1}{2}\right)\sigma\right)\right)^{1-\frac{1}{p}} \qquad \text{for } p \in\,]1, \infty[.$$

1.6. **The method of proof.** The proof of Theorem 1.3 near the end of section 4 below rests on the following simple lemma, which was implicitly used also in [11].

Lemma 1.6. *Let F and G be distribution functions of laws P and Q on $\mathbb{R}$ with $P(\mathbb{Z}) = 1$, G continuous and strictly increasing, P and Q symmetric about $\frac{n}{2} \in \mathbb{R}$, and d defined by the first equality in (12). Then we have $n \in \mathbb{Z}$, the second equality in (12), and*

$$(22) \qquad\qquad d \;=\; \begin{cases} G\left(\frac{n+1}{2}\right) - \frac{1}{2} & \text{if } n \text{ is odd}, \\ \frac{1}{2}P\left(\left\{\frac{n}{2}\right\}\right) & \text{if } n \text{ is even}. \end{cases}$$

Further, (11) holds for every $s \in \mathbb{R}$ iff the following two conditions are satisfied:

$$(23) \qquad\qquad F(s) - G(s) \;<\; d \quad \text{for } s \in \mathbb{Z} \text{ with } s > \left\lfloor\frac{n}{2}\right\rfloor,$$

$$(24) \qquad\qquad G(s) - F(s-1) \;<\; d \quad \text{for } s \in \mathbb{Z} \text{ with } s > \left\lceil\frac{n}{2}\right\rceil.$$

Proof. The symmetry assumptions can be written as

$$(25) \; F(s) = 1 - F((n-s)-) \quad \text{and} \quad G(s) = 1 - G((n-s)-) \quad \text{for } s \in \mathbb{R}.$$

The assumption $P(\mathbb{Z}) = 1$ then yields $0 < P(\{k\}) = F(k) - F(k-1) = P(\{n-k\})$ for some $k \in \mathbb{Z}$, and hence $n \in \mathbb{Z}$. Next, (25) for $s = \left\lfloor\frac{n}{2}\right\rfloor$ yields

$$F\left(\left\lfloor\tfrac{n}{2}\right\rfloor\right) + F\left(\left\lceil\tfrac{n}{2}\right\rceil-\right) \;=\; 1 \;=\; G\left(\left\lfloor\tfrac{n}{2}\right\rfloor\right) + G\left(\left\lceil\tfrac{n}{2}\right\rceil-\right),$$

and we get the second equality in (12), and also $G\left(\left\lceil\frac{n}{2}\right\rceil-\right) - F\left(\left\lceil\frac{n}{2}\right\rceil-\right) = G\left(\frac{n+1}{2}\right) - \frac{1}{2}$ if n is odd, and $F\left(\left\lfloor\frac{n}{2}\right\rfloor\right) - G\left(\left\lfloor\frac{n}{2}\right\rfloor\right) = \frac{1}{2} + \frac{1}{2}P\left(\left\{\frac{n}{2}\right\}\right) - \frac{1}{2}$ if n is even, and thus (22).

Trivially, (11) implies (23) and (24). Conversely, let us assume (23) and (24). If $s \in \mathbb{R} \setminus \mathbb{Z}$, then $F(s) - G(s) = F(\lfloor s \rfloor) - G(s) < F(\lfloor s \rfloor) - G(\lfloor s \rfloor)$ and $G(s) - F(s) = G(s) - F(\lceil s \rceil -) < G(\lceil s \rceil -) - F(\lceil s \rceil -)$; hence it is enough to prove (11) for $s \in \mathbb{Z}$. If $s \in \mathbb{Z}$ with $s > \lfloor \frac{n}{2} \rfloor$, then $F(s) - G(s) < d$ by (23), and $G(s) - F(s) \leq G(s+1) - F(s) < d$ by (24) as $s + 1 > \lceil \frac{n}{2} \rceil$; hence $|F(s) - G(s)| < d$. If $s \in \mathbb{Z}$ with $s < \lfloor \frac{n}{2} \rfloor$, then $t := n - s > \lceil \frac{n}{2} \rceil$, and (25), (23), (24) yield $F(s) - G(s) = G(t) - F(t-1) < d$ and $G(s) - F(s) = F(t-1) - G(t) \leq F(t) - G(t) < d$; hence again $|F(s) - G(s)| < d$. Thus the first part of (11) holds for $s \in \mathbb{Z}$, and the second follows by applying, for a given $s \neq \lceil \frac{n}{2} \rceil$, the first one to $t := n - s \neq \lfloor \frac{n}{2} \rfloor$. □

In the situation of Theorem 1.3, assumption (23) and part of assumption (24) are proved below in Lemmas 4.4 and 4.5 by monotonicity considerations, and the part of (24) not thus covered is proved by using lower bounds for d from Lemma 4.2 together with Lemma 4.6. The proofs of the lemmas of section 4 use various auxiliary inequalities from sections 2 and 3.

We do not know whether Theorem 1.3 could alternatively and perhaps even more efficiently be proved using, say, characteristic functions or Stein's method, and we are in fact as ignorant already for the earlier special case from [11].

2. SOME STANDARD ANALYTIC INEQUALITIES

Very elementary inequalities like $1 + x < e^x$ for $x \in \mathbb{R} \setminus \{0\}$ and $\frac{x}{1+x} < \log(1+x) < x$ for $x > -1$ will often be used without comment.

Lemma 2.1. *If $x, y \in \mathbb{R}$ satisfy $0 \leq y \leq |x|$ or $x \leq y \leq 0$ or $x - \frac{2}{3}x^2 \geq -y \geq 0$, then*

$$(26) \qquad (1 + x)e^{-x} \quad \leq \quad (1 + y)e^{-y},$$

and equality holds iff $x = y$. The constant $\frac{2}{3}$ in the assumption cannot be lowered.

Proof. See [11, Lemma 2.1]. □

Lemma 2.2. *Let $x \in \mathbb{R} \setminus \{0\}$. Then $\exp(\frac{x^2}{2} - \frac{x^4}{12}) < \cosh(x) < (1 + \frac{x^2}{3})\exp(\frac{x^2}{6})$.*

Proof. Analogously to [25, Part I, Problem 154 and solution, pp. 35, 212], the partial fraction expansion of the hyperbolic tangent function

$$\frac{\tanh(x)}{x} \quad = \quad \sum_{k=1}^{\infty} \frac{2}{((k+\frac{1}{2})\pi)^2 + x^2}$$

proved for example in [26, pp. 199, 294] implies that $\tanh(x)/x$ is enveloped by its power series around zero, namely

$$(-)^n \left(\frac{\tanh(x)}{x} - \sum_{k=0}^{n} (-)^k \alpha_k x^{2k} \right) \quad > \quad 0 \quad \text{ for } x \in \mathbb{R} \setminus \{0\}, n \in \mathbb{N}_0,$$

where $\alpha_0 = 1, \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{15}, \ldots$, and using $\log(\cosh(x)) = \int_0^x \tanh(t)\,dt$ then yields

$$(27) \quad (-)^n \left(\log(\cosh(x)) - \sum_{k=0}^{n} (-)^k \frac{\alpha_k}{2k+2} x^{2k+2} \right) < 0 \quad \text{ for } x \in \mathbb{R} \setminus \{0\}, n \in \mathbb{N}_0.$$

Taking $n = 1$ yields the first inequality claimed.

To prove the second one, which improves the case $n = 0$ of (27), we observe that the coefficients of x^{2k} in the power series of the two functions involved, namely $a_k := \frac{1}{(2k)!}$ and $b_k := \frac{2k+1}{6^k k!}$ for $k \in \mathbb{N}_0$, are all > 0, and their quotients $c_k := a_k/b_k$ satisfy $c_0 = c_1 = 1$ and $c_{k+1}/c_k = 3/(2k+3) < 1$ for $k \in \mathbb{N}$. $\qquad\square$

Lemma 2.3. *With* $w(x) := \frac{\Gamma(2x+1)}{\Gamma^2(x+1)} 2^{-2x} = \frac{\Gamma(x+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(x+1)}$, *we have*

$$(28) \qquad -\frac{1}{8x} \;<\; \log\left(\sqrt{\pi x}\,w(x)\right) \;<\; -\frac{1}{8x} + \frac{1}{192x^3} \qquad \text{for } x \in \,]0,\infty[$$

$$(29) \qquad\qquad\qquad\qquad\qquad\qquad \leq \; -\frac{23}{192x} \qquad\qquad \text{for } x \in [1,\infty[.$$

Two proofs. Inequality (29) is of course trivial in view of $\frac{1}{x^3} \leq \frac{1}{x}$. Concerning (28):

For integer x, and only this case will be needed in this paper, (28) is proved by Everett in [6, (10)], with W_n there being the present $(\sqrt{\pi n}\,w(n))^2]$.

For general x, Sasvári [28] presents the inequalities in (28) as special cases of a more general corollary to a theorem yielding the monotonicity in x of the error of each of the asymptotic expansions $\sum_{j=1}^{N} c_{r,j} x^{1-2j}$ of $\log\left(\sqrt{2\pi \frac{r-1}{r}} \left(\frac{(r-1)^{r-1}}{r^r}\right)^x \binom{rx}{x}\right)$ for $]0,\infty[\,\ni\, x \to \infty$, with $r \in \,]1,\infty[$ and $N \in \mathbb{N}_0$ fixed and here $\binom{rx}{x} := \Gamma(rx+1)/(\Gamma(x+1)\Gamma((r-1)x))$. Sasvári's proof is short and elegant, but to get just (28) and its analogues in Sasvári's corollary can even be shortened a bit by using in his formula (2) and in his notation just "$Q_s < 0$" rather than "Q_s increasing". $\qquad\square$

Although not needed here, let us remark that numerical calculations suggest that we have in fact $\sup_{x \in [1,\infty[} x \log\left(\sqrt{\pi x}\,w(x)\right) = \log(\frac{1}{2}\sqrt{\pi}) = -0.1207\ldots < -\frac{23}{192} = -0.1197\ldots$.

3. Normal distribution function inequalities

For comparing normal distribution function increments with their midpoint derivative approximations, we will need the rather sharp inequalities (30) below, which improve the ones in [7, p. 322, Lemma 1] and in [24, pp. 475–476, Lemma 1] in an optimal way.

Lemma 3.1. *For* $x, h \in \mathbb{R}$ *with* $h \neq 0$, *we have*
$$(30)$$
$$\exp\left(\frac{(x^2-1)h^2}{24} - \frac{x^4 h^4}{960}\right) \;<\; \frac{\Phi(x+\frac{h}{2}) - \Phi(x-\frac{h}{2})}{h\varphi(x)} \;<\; \exp\left(\frac{(x^2-1)h^2}{24} + \frac{h^4}{1440}\right),$$

and these inequalities are optimal for small h *in the sense that we have*
$$(31)$$
$$\log\left(\frac{\Phi(x+\frac{h}{2}) - \Phi(x-\frac{h}{2})}{h\varphi(x)}\right) \;=\; \frac{(x^2-1)h^2}{24} + \frac{(-x^4-4x^2+2)h^4}{2880} + O(h^6) \quad \text{for } x, h \text{ bounded},$$

with $\max_{x \in \mathbb{R}}(-x^4 - 4x^2 + 2) = 2$ *and* $\min_{x \in \mathbb{R}}(-x^4 - 4x^2 + 2)/x^4 = -3$.

Proof. For $x, y \in \mathbb{R}$, let

$$\varepsilon_1(x,y) \;:=\; -\frac{x^4 y^4}{60} \quad \text{and} \quad \varepsilon_2(x,y) \;:=\; \frac{y^4}{90}$$

and, for $i \in \{1, 2\}$,

$$f_i(x,y) \;:=\; \frac{\sqrt{2\pi}}{2}\left(\Phi(x+y) - \Phi(x-y)\right) - y\exp\left(-\frac{x^2}{2} + \frac{(x^2-1)y^2}{6} + \varepsilon_i(x,y)\right).$$

Noting that (30) is unaffected by sign changes of x or h, and writing y in place of $h/2$, we have to prove for $x \geq 0$ and $y > 0$ the inequalities

$$(32) \qquad f_1(x, y) \quad > \quad 0 \quad > \quad f_2(x, y).$$

Now $f_i(x, 0) = 0$ and, with a subscript y denoting the partial derivative with respect to that variable,

$$\frac{f_{i,y}(x, y)}{\exp\left(\frac{x^2 y^2}{6} - \frac{x^2 + y^2}{2}\right)}$$

$$= \quad \frac{\cosh(xy)}{\exp\left(\frac{x^2 y^2}{6}\right)} - \left(1 - \frac{y^2}{3} + y\varepsilon_{i,y}(x, y) + \frac{x^2 y^2}{3}\right) \exp\left(\frac{y^2}{3} + \varepsilon_i(x, y)\right)$$

$$=: \quad g_i(x, y).$$

For $i = 1$, we use the first inequality in Lemma 2.2 and $1 + t < e^t$ for $0 \neq t \in \mathbb{R}$ to get

$$g_1(x, y) \quad > \quad \exp\left(\frac{x^2 y^2}{3} - \frac{x^4 y^4}{12}\right) - \exp\left(y\varepsilon_{1,y}(x, y) + \frac{x^2 y^2}{3} + \varepsilon_1(x, y)\right) \quad = \quad 0,$$

considering the cases $x \neq 0$ and $x = 0$ separately to check the strict inequality, and hence the first half of (32).

For $i = 2$, the second inequality in Lemma 2.2 and $\frac{x^2 y^2}{3} \exp\left(\frac{y^2}{3} + \frac{y^4}{90}\right) \geq \frac{x^2 y^2}{3}$ yield

$$g_2(x, y) \quad \leq \quad 1 + \frac{x^2 y^2}{3} - \left(1 - \frac{y^2}{3} + \frac{2y^4}{45}\right) \exp\left(\frac{y^2}{3} + \frac{y^4}{90}\right) - \frac{x^2 y^2}{3} \quad = \quad 1 - \exp(g(y^2))$$

where, for $t \in \mathbb{R}$,

$$g(t) \quad := \quad \frac{t}{3} + \frac{t^2}{90} + \log\left(1 - \frac{t}{3} + \frac{2t^2}{45}\right)$$

is well-defined with $g(0) = 0$ and, for $t > 0$, satisfies

$$g'(t) \quad = \quad \frac{1}{3} + \frac{t}{45} + \frac{-\frac{1}{3} + \frac{4t}{45}}{1 - \frac{t}{3} + \frac{2t^2}{45}} \quad = \quad \frac{\frac{t^2}{135} + \frac{2t^3}{2025}}{1 - \frac{t}{3} + \frac{2t^2}{45}} \quad > \quad 0$$

and hence $g(t) > 0$, yielding $g_2(x, y) < 0$ and hence the second half of (32).

With the Hermite polynomials H_n given by $H_n(x) = (-)^n e^{x^2/2} \partial_x^n e^{-x^2/2}$, in particular $H_0(x) = 1, H_2(x) = x^2 - 1, H_4(x) = x^4 - 6x^2 + 3$, a Taylor expansion around $h = 0$ shows that, for x, h bounded, we have

$$\frac{\Phi(x + \frac{h}{2}) - \Phi(x - \frac{h}{2})}{h\varphi(x)} \quad = \quad \sum_{j=0}^{\infty} \frac{H_{2j}(x) h^{2j}}{2^{2j}(2j + 1)!}$$

$$= \quad 1 + H_2(x)\frac{h^2}{24} + H_4(x)\frac{h^4}{1920} + O(h^6),$$

and hence an application of $\log(1 + y) = y - y^2/2 + O(y^3)$ for y near zero and a short computation yield (31). $\qquad \square$

Lemma 3.2. *Let $x > 0$. Then*

$$(33) \qquad x e^{-\frac{x^2}{6}} \quad < \quad \sqrt{2\pi}\left(\Phi(x) - \tfrac{1}{2}\right) \quad < \quad x e^{-\frac{x^2}{6} + \frac{x^4}{90}}.$$

Proof. The claim results if we apply (30) to $(0, 2x)$ in place of (x, h). $\qquad \square$

The following lemma often improves on [11, Lemma 2.2], which yields (34) with $\alpha = 1$ but with the upper bound replaced by $\exp\left(-\frac{y^2-x^2}{2} + \frac{|h|}{2}(|y|-|x|)\right)$, and it always improves on [29], where (34) with $\alpha = 1$ is only obtained for $h = y - x$ and with $\beta = 1 - \frac{h^2}{2} < 1 - \frac{h^2}{12}$.

Lemma 3.3. *Let $h \in \mathbb{R} \setminus \{0\}$. Then*
$$(34)$$
$$\exp\left(-\alpha\frac{y^2-x^2}{2}\right) \;<\; \frac{\Phi\left(y+\frac{h}{2}\right) - \Phi\left(y-\frac{h}{2}\right)}{\Phi\left(x+\frac{h}{2}\right) - \Phi\left(x-\frac{h}{2}\right)} \;<\; \exp\left(-\beta\frac{y^2-x^2}{2}\right) \quad \text{if } |x| < |y|$$

holds with the optimal constants

$$(35) \quad \alpha := 1 \quad and \quad \beta := \frac{\frac{h}{2}\exp\left(-\frac{h^2}{8}\right)}{\sqrt{2\pi}\left(\Phi(\frac{h}{2}) - \frac{1}{2}\right)} > \exp\left(-\frac{h^2}{12} - \frac{h^4}{1440}\right) > 1 - \frac{h^2}{12}.$$

Proof. Since (34) and (35) are unaffected by sign changes of x or y or h, we may and do always assume that $0 \le x < y$ and $h > 0$ in this proof. For $\gamma \in \mathbb{R}$, let

$$f_\gamma(x) := e^{\gamma\frac{x^2}{2}}\left(\Phi\left(x+\frac{h}{2}\right) - \Phi\left(x-\frac{h}{2}\right)\right) \quad \text{for } x \in [0, \infty[.$$

If $\alpha, \beta \in \mathbb{R}$ are arbitrary, then (34) holds iff f_α is strictly increasing and f_β is strictly decreasing. Now for $x \in]0, \infty[$, the derivative $f'_\gamma(x)$ has the same sign as $\gamma - g(x)$ where

$$g(x) := \frac{\varphi\left(x-\frac{h}{2}\right) - \varphi\left(x+\frac{h}{2}\right)}{x\left(\Phi(x+\frac{h}{2}) - \Phi(x-\frac{h}{2})\right)} = e^{-\frac{h^2}{8}} \cdot \frac{\sinh\left(\frac{xh}{2}\right)}{\frac{xh}{2}} \cdot \frac{h\varphi(x)}{\Phi(x+\frac{h}{2}) - \Phi(x-\frac{h}{2})}$$

$$= e^{-\frac{h^2}{8}} \frac{\frac{2}{h}\int_0^{\frac{h}{2}} \cosh(xt)\,dt}{\frac{2}{h}\int_0^{\frac{h}{2}} \cosh(xt)\exp(-\frac{t^2}{2})\,dt},$$

and the unattained supremum and infimum of $g(x)$ over $x \in]0, \infty[$ are α and β as defined in (35), by $\exp(-\frac{t^2}{2}) > \exp(-\frac{h^2}{8})$ and by considering $x \to \infty$, and by "Chebyshev's other inequality" [17, Chapter IX] for the integral of a product of two monotone functions applied to yield

$$\frac{2}{h}\int_0^{\frac{h}{2}}\cosh(xt)\exp(-\frac{t^2}{2})\,dt \;<\; \frac{2}{h}\int_0^{\frac{h}{2}}\cosh(xt)\,dt \cdot \frac{2}{h}\int_0^{\frac{h}{2}}\exp(-\frac{t^2}{2})\,dt$$

and by considering $x = 0$. This proves our claim except for the inequalities in (35), of which the first one follows from (33) and the second one is trivial if $u := \frac{h^2}{12} \ge 1$ and follows from $\log(1 - u) < -u - \frac{u^2}{2} < -u - \frac{u^2}{10}$ otherwise. $\qquad\square$

4. Lemmas on symmetric hypergeometric laws, proof of the main result

To avoid pedantic repetitions of assumptions below, let us agree that in this section $F, f, n, \sigma, N, \sigma_0, \tau, G$ are in principle fixed and as postulated in Theorem 1.3, but that we may nevertheless use reduction arguments as in the proof of Lemma 4.4, where the case of $N = \infty$ is reduced to the case of $N < \infty$. We have or put

$$G(s) = \Phi\left(\frac{s - \frac{n}{2}}{\tau}\right) \quad \text{and} \quad g(s) := G(s) - G(s-1) \quad \text{for } s \in \mathbb{R},$$

and we note the following corollary to Lemma 3.3.

Lemma 4.1. *Let* $s \in]\frac{n}{2}, \infty[$. *Then*

$$(36) \qquad \exp\left(-\frac{s - \frac{n}{2}}{\tau^2}\right) \quad < \quad \frac{g(s+1)}{g(s)} \quad < \quad \exp\left(-\left(1 - \frac{1}{12\tau^2}\right)\frac{s - \frac{n}{2}}{\tau^2}\right).$$

Proof. Lemma 3.3 applied to $h := \frac{1}{\tau}$, $x := (s - \frac{n}{2} - \frac{1}{2})/\tau$, $y := (s - \frac{n}{2} + \frac{1}{2})/\tau$. $\quad\square$

Let us note that $g(s+1)/g(s)$ in (36) may alternatively be bounded from above by $\exp\left(-(s - \frac{n}{2} - \frac{1}{2})/\tau^2\right)$, as in [11, Proof of Lemma 3.1], which however appears to be insufficient for proving Lemma 4.5 below.

Let $r := \frac{N}{2}$. If $N < \infty$, then N is even by Lemma 1.2(d), $f = h_{n,r,r}$ by Lemma 1.2(b) and (d); hence

$$(37) \qquad \sigma^2 = \frac{n(N-n)}{4(N-1)} \quad \text{and} \quad \sigma_0^2 = \frac{n(N-n)}{4N}$$

by (6) and (10), so that in particular $\sigma^2 > 0$ yields $n \in \{1, \ldots, N-1\}$ and thus $N \geq 2$ and $r \geq 1$, and we further have

$$(38) \quad \sigma_0^2 \geq \frac{3}{16} \text{ if } n \in \{1, \ldots, N-1\}, \qquad \sigma_0^2 \geq \frac{1}{4} \text{ if } n \in \{2, \ldots, N-2\},$$

by considering n extremal and N minimal. If $N = \infty$, then $f = b_{n,1/2}$ and $n \geq 1$.

Lemma 4.2. *Assume* $N < \infty$ *and* n *even. Then*

$$(39) \qquad\qquad \frac{\sqrt{2}}{4} \quad < \quad \sqrt{\frac{(N-2)N^2}{8(N-1)^3}} \quad \leq \quad \sigma f\left(\frac{n}{2}\right)$$

with equality in the second inequality iff $n = 2$ *or* $n = N - 2$.

Proof. We have $n \geq 2$ and hence $N \geq 4$. Let $a_k := \sigma^2\left(H_{2k,r,r}\right) \cdot \left(h_{2k,r,r}(k)\right)^2$ for $k \in \{1, \ldots, r-1\}$. Then, for $k \leq r - 2$, we have

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)(r-k-1)}{k(r-k)}\left(\frac{(r-k)^2(k+\frac{1}{2})(k+1)}{(k+1)^2(r-k)(r-k+\frac{1}{2})}\right)^2$$

$$= \frac{1 + \frac{1}{4k(k+1)}}{1 + \frac{1}{4(r-k)(r-k+1)}}$$

and hence $a_k \leq a_{k+1}$ iff $k(k+1) \leq (r-k)(r-k+1)$ iff $k \leq \frac{r-1}{2}$. Hence the sequence (a_k) can attain its minimal value only at $k = 1$ or at $k = r - 1$, and we have in fact $\sigma^2(H_{2,r,r}) = \sigma^2(H_{N-2,r,r}) = \frac{N-2}{2(N-1)}$ and $h_{2,r,r}(1) = h_{N-2,r,r}(r-1) = \frac{N}{2(N-1)}$ and thus $a_1 = a_{r-1} = \frac{(N-2)N^2}{8(N-1)^3}$, and the latter expression is strictly decreasing in $N \in [4, \infty[$, as $\left(\log\frac{(x-2)x^2}{(x-1)^3}\right)' = ((x-2)(x-1)x)^{-1}(4-x) < 0$ for $x \in]4, \infty[$ and hence $a_1 \geq \frac{4}{27} \geq \frac{2}{16}$. $\quad\square$

Lemma 4.3. *If* n *is even and* $N < \infty$, *then* $n = 2k$ *with* $k \in \{1, \ldots, r-1\}$, $r \geq 2$,

$$(40) \qquad\qquad\qquad \frac{1}{4} \quad \leq \quad \sigma_0^2 = \frac{k(r-k)}{2r} \quad \leq \quad \frac{r}{8},$$

$\sigma = \sqrt{\frac{2r}{2r-1}}\,\sigma_0$, and

$$(41) \qquad f(k) \;>\; \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left(\frac{23}{192r}-\frac{1}{16\sigma_0^2}\right) \;>\; \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left(-\frac{1}{16\sigma_0^2}\right),$$

$$(42) \qquad f(k) \;<\; \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left(\frac{1}{8r}-\frac{23}{384\sigma_0^2}\right) \;<\; \frac{1}{\sigma_0\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{24\sigma_0^2}} \;<\; \frac{1}{\sigma\sqrt{2\pi}}.$$

Proof. Only the claims in (41) and (42) are not obvious. Writing the binomial coefficient occurring in $f(k)$ in terms of gamma functions and using the definition of the function w from Lemma 2.3 shows that

$$h(k) \quad := \quad \log\left(\sigma_0\sqrt{2\pi}\,f(k)\right)$$

admits the representation

$$h(k) \quad = \quad -\log\left(\sqrt{\pi r}\,w(r)\right) + \log\left(\sqrt{\pi k}\,w(k)\right) + \log\left(\sqrt{\pi(r-k)}\,w(r-k)\right),$$

so that Lemma 2.3 yields

$$h(k) \;>\; \frac{23}{192r}-\frac{1}{8k}-\frac{1}{8(r-k)} \;=\; \frac{23}{192r}-\frac{1}{16\sigma_0^2} \;>\; -\frac{1}{16\sigma_0^2}$$

and hence (41), and, also using (40),

$$h(k) \;<\; \frac{1}{8r}-\frac{23}{192}\left(\frac{1}{k}+\frac{1}{r-k}\right) \;=\; \frac{1}{8r}-\frac{23}{384\sigma_0^2} \;\leq\; \left(\frac{1}{64}-\frac{23}{384}\right)\frac{1}{\sigma_0^2}$$

$$=\; -\frac{17}{384\sigma_0^2} \;<\; -\frac{1}{24\sigma_0^2} \;\leq\; -\frac{1}{3r},$$

and since $\log\frac{2r}{2r-1} < \frac{1}{2r-1} \leq \frac{2}{3r}$ due to $r \geq 2$, we also get

$$\frac{1}{8r}-\frac{23}{384\sigma_0^2}+\log\frac{\sigma}{\sigma_0} \;<\; -\frac{1}{3r}+\frac{1}{3r} \;=\; 0$$

and hence (42). $\qquad\square$

Lemma 4.4. (a) f/g *is strictly decreasing on* $\{s \in \mathbb{Z} : \frac{n}{2} \leq s \leq (n \wedge r)+1\}$.
(b) *We have* $f(s) < g(s)$ *for* $s \in \mathbb{Z}$ *with* $s > \lfloor n/2 \rfloor$.
(c) *We have* $0 < F(s)-G(s) < F(\lfloor n/2 \rfloor)-G(\lfloor n/2 \rfloor)$ *for* $s \in \mathbb{Z}$ *with* $s > \lfloor n/2 \rfloor$.

Proof. (a) Let $s \in \mathbb{Z}$ with $\frac{n}{2} \leq s \leq n \wedge r$. Then we have

$$(43) \qquad \frac{f(s+1)}{f(s)} \;\leq\; \Theta\exp\left(-\frac{s-\frac{n}{2}}{\sigma_0^2}\right) \quad \text{with } \Theta := \frac{1-y}{1+y}\mathrm{e}^{2y} < 1,\; y := \frac{2s-n+1}{n+1},$$

where $\Theta < 1$ holds by Lemma 2.1 with $x := -y$ using $y > 0$. The other inequality claimed holds first in the case of $r < \infty$, as then Lemma 2.1 applied to $\tilde{y} := \frac{2s-n+1}{2r-n+1} > 0$ and $\tilde{x} := -\tilde{y}$ yields, using $s \geq n-r$ due to $n-r < \frac{n}{2}$ in the first step,

$$\frac{f(s+1)}{f(s)} \;=\; \frac{(r-s)(n-s)}{(s+1)(r-n+s+1)} \;=\; \frac{1-y}{1+y}\cdot\frac{1-\tilde{y}}{1+\tilde{y}} \;<\; \Theta\mathrm{e}^{-2(y+\tilde{y})}$$

with $2(y+\tilde{y}) = \frac{8(r+1)}{(n+1)(2r-n+1)}(s-\frac{n-1}{2})$, and as we have

$$\frac{s-\frac{n}{2}}{\sigma_0^2}\Big/\big(2(y+\tilde{y})\big) \;=\; \frac{r(n+1)(2r-n+1)}{(r+1)n(2r-n)}\cdot\frac{2s-n}{2s-n+1} \;\leq\; 1$$

by using $(2s-n)/(2s-n+1) \leq (2(n \wedge r)-n)/(2(n \wedge r)-n+1)$ and considering separately the cases $n \wedge r = r$ and $n \wedge r = n$, and then also for $r = \infty$, by taking

the limit for $r \to \infty$ in (43). Now (43) yields, using $\tau \geq \sigma_0$ and then Lemma 4.1 in case of $s > \frac{n}{2}$,

$$\frac{f(s+1)}{f(s)} \; < \; \exp\left(-\frac{s-\frac{n}{2}}{\sigma_0^2}\right) \; \leq \; \exp\left(-\frac{s-\frac{n}{2}}{\tau^2}\right) \; \leq \; \frac{g(s+1)}{g(s)}$$

and hence $f(s+1)/g(s+1) < f(s)/g(s)$.

(b) By part (a) and since $h(s) = 0$ for $s > n \wedge r$, we can and do assume that $s = \lfloor n/2 \rfloor + 1$. Let us first assume that $r < \infty$.

If n is even, then $n = 2k$ with $k \in \{1, \ldots, r-1\}$ and $s = k+1$, and we get

$$g(s) \;=\; g(k+1) - g(k) \;=\; \Phi\left(\tfrac{1}{\tau}\right) - \tfrac{1}{2} \;\geq\; \Phi\left(\tfrac{1}{\sigma}\right) - \tfrac{1}{2} \;>\; \tfrac{1}{\sigma\sqrt{2\pi}}\exp\left(-\tfrac{1}{6\sigma^2}\right)$$

by $\tau \leq \sigma$ and Lemma 3.2, and, using below several parts of Lemma 4.3, we have

$$\frac{f(k+1)}{f(k)} \;=\; \frac{k\,(r-k)}{(k+1)(r-k+1)} \;=\; \frac{k\,(r-k)}{1+r+k\,(r-k)}$$

$$\leq\; \frac{k\,(r-k)}{\frac{3}{2}r + k\,(r-k)} \;=\; \frac{1}{1+\frac{3}{4\sigma_0^2}}$$

since $r \geq 2$, and hence

$$\log\frac{f(k+1)}{f(k)} \;\leq\; -\log\left(1 + \frac{3}{4\sigma_0^2}\right) \;<\; \frac{-\frac{3}{4\sigma_0^2}}{1+\frac{3}{4\sigma_0^2}} \;\leq\; \frac{-3}{16\sigma_0^2}$$

by using $\sigma_0^2 \geq \frac{1}{4}$ in the last step, so that

$$(44) \qquad \log\frac{f(s)}{g(s)} \;=\; \log\frac{f(k+1)}{f(k)} - \log g(s) + \log f(k)$$

$$<\; \frac{-3}{16\sigma_0^2} + \frac{1}{6\sigma^2} + \log\left(\sigma\sqrt{2\pi}\,f(k)\right) \;<\; -\frac{1}{48\sigma^2}$$

using in the final step $\sigma_0 \leq \sigma$ for the first two terms and (42) for the last one.

Now let n be odd. Then $n = 2k-1$ with $k \in \{1, \ldots, r\}$ and $s = k$, and we get

$$(45) \quad g(s) \;=\; 2\left(\Phi\left(\tfrac{1}{2\tau}\right) - \tfrac{1}{2}\right) \;\geq\; 2\left(\Phi\left(\tfrac{1}{2\sigma}\right) - \tfrac{1}{2}\right) \;>\; \tfrac{1}{\sigma\sqrt{2\pi}}\exp\left(-\tfrac{1}{24\sigma^2}\right)$$

using $\tau \leq \sigma$ and Lemma 3.2. If $k \in \{1, r\}$, then in either case $\sigma^2 = \frac{1}{4}$ and $h(s) = \frac{1}{2}$, and (45) yields

$$(46) \qquad g(s) \;\geq\; 2\Phi(1) - 1 \;=\; 0.6827\ldots \;>\; \frac{1}{2} \;=\; f(s),$$

and we now assume that $2 \leq k \leq r-1$. We have

$$(47) \qquad f(s) \;=\; f(k) \;=\; \frac{r-k+\frac{1}{2}}{r-k+1}h_{2k,r,r}(k)$$

and, using $(k-\frac{1}{2})/k \geq 3/4$ due to $k \geq 2$ for the lower bound and writing $\sigma_{0,2k} := \sigma_0(H_{2k,r,r})$, we get

$$(48) \qquad \frac{\sigma^2}{\sigma_{0,2k}^2} \;=\; \frac{r(k-\frac{1}{2})(r-k+\frac{1}{2})}{(r-\frac{1}{2})k(r-k)} \;\in\; \left]\frac{3}{4}, \frac{r-k+\frac{1}{2}}{r-k}\right],$$

and then

(49)
$$
\log \frac{f(s)}{g(s)} = \log\left(\sigma_{0,2k}\sqrt{2\pi}\,\mathrm{h}_{2k,r,r}(k)\right) + \log \frac{r-k+\frac{1}{2}}{r-k+1} - \log\left(\sigma\sqrt{2\pi}\,g(s)\right) + \log \frac{\sigma}{\sigma_{0,2k}}
$$

$$
< \quad \frac{1}{8r} - \frac{23}{384\sigma_{0,2k}^2} + \log \frac{r-k+\frac{1}{2}}{r-k+1} + \frac{1}{24\sigma^2} + \frac{1}{2}\log \frac{r-k+\frac{1}{2}}{r-k+1}
$$

$$
< \quad \left(\frac{1}{24} - \frac{23}{384}\cdot\frac{3}{4}\right)\frac{1}{\sigma^2} + \frac{1}{8r} + \frac{1}{4}\log \frac{r-\frac{1}{2}}{r} + h(r,k)
$$

$$
< \quad -\frac{5}{1536\sigma^2}
$$

by using at the second step (42) with $n = 2k$, (45), and (48); at the third step (48), $k \geq 0$, and the definition of $h(r,k)$ given below; and at the final step three applications of $\log(1+x) < x$, one for $\log((r-\frac{1}{2})/r) < -1/(2r)$, and the other two contained in

$$
h(r,k) := \frac{3}{4}\log \frac{r-k+\frac{1}{2}}{r-k+1} + \frac{1}{2}\log \frac{r-k+\frac{1}{2}}{r-k}
$$

$$
< \quad -\frac{3}{4}\cdot\frac{1}{2(r-k+1)} + \frac{1}{2}\cdot\frac{1}{2(r-k)} = -\frac{r-k-2}{8(r-k-1)(r-k)},
$$

which yields $h(r,k) < 0$ always, namely, by the above if $k \leq r-2$, and by $h(r,r-1) = \frac{3}{4}\log\frac{3}{4} + \frac{1}{2}\log\frac{3}{2} = \frac{1}{4}\log\frac{3^3 3^2}{4^3 2^2} = \frac{1}{4}\log\frac{243}{256} < 0$ if $k = r-1$.

By (44), (46), (49), there is a constant $c > 0$ not depending on $r, n \in \mathbb{N}$ with $n < 2r$ satisfying $\log(f(s)/g(s)) \leq -c\sigma^{-2}$, and this remains true also for the limit case of $r = \infty$.

(c) By part (b), $F - G$ is strictly decreasing on $\{s \in \mathbb{Z} : s \geq \lfloor n/2 \rfloor\}$. Hence we get the second inequality claimed and, since $F(s) - G(s) = 1 - G(s) > 0$ for $s \geq n$, also the first one. $\qquad\square$

Lemma 4.5. *Let $\tau \in [\sigma_0, \sigma]$ and $M := \frac{n}{2} + 1 + \frac{3}{2}\sigma$.*
  (a) *$f(\cdot - 1)/g$ is strictly increasing on $\{s \in \mathbb{Z} : \frac{n+1}{2} \leq s \leq M\}$.*
  (b) *We have $g(s) < f(s-1)$ for $s \in \mathbb{Z}$ with $\lceil n/2 \rceil < s \leq M$.*
  (c) *We have $G(s) - F(s-1) < G(\lceil n/2 \rceil) - F(\lceil n/2 \rceil - 1)$ for $s \in \mathbb{Z}$ with $\lceil n/2 \rceil < s \leq M$.*

Proof. (a) Let $s \in \mathbb{Z}$ with $\frac{n+1}{2} \leq s \leq \frac{n}{2} + \frac{3}{2}\sigma$.

If $n = 1$ or N is finite and $n = N-1$, then $\sigma = \frac{1}{2}$ and hence $s = \frac{n+1}{2}$, and, using the unimodality of φ, we indeed get $\frac{f(s-1)}{g(s)} = \frac{1/2}{\Phi(\frac{1}{2\tau})-\Phi(-\frac{1}{2\tau})} < \frac{1/2}{\Phi(\frac{3}{2\tau})-\Phi(\frac{1}{2\tau})} = \frac{f(s)}{g(s+1)}$. Hence we can assume $1 < n < N-1$ and thus $N \geq 4$ and $\sigma_0^2 \geq \frac{1}{4}$ in what follows, by (38).

First let $N < \infty$. We have $n - r < s \leq n \wedge r$, for else we would have one of the inequalities $\frac{n+1}{2} \leq n - r$, $n+1 \leq \frac{n}{2} + \frac{3}{2}\sigma$, $r+1 \leq \frac{n}{2} + \frac{3}{2}\sigma$, which are easily checked to be false. Hence, putting $x_1 := \frac{2s-n}{n}$, $y_1 := -\frac{2s-n-2}{n}$, $x_2 := \frac{2s-n}{2r-n}$, and

$y_2 := -\frac{2s-n-2}{2r-n}$, we have

$$\tag{50} \frac{f(s)}{f(s-1)} = \frac{n-s+1}{s} \cdot \frac{r-s+1}{r-n+s} = \frac{1+y_1}{1+x_1} \cdot \frac{1+y_2}{1+x_2}$$

$$\geq e^{y_1-x_1+y_2-x_2} = \exp\left(-\frac{s-\frac{n+1}{2}}{\sigma_0^2}\right),$$

where the inequality is a trivial equality if $s = \frac{n+1}{2}$, and follows otherwise by two applications of Lemma 2.1, as $s \geq \frac{n}{2}+1$ yields $-y_i \geq 0$, and we also have $x_i - \frac{2}{3}x_i^2 \geq -y_i$ for $i \in \{1,2\}$, since $y_1+x_1-\frac{2}{3}x_1^2 = \frac{2}{n}-\frac{2}{3}\left(\frac{2s-n}{n}\right)^2 \geq \frac{2}{n}-\frac{2}{3}\left(\frac{3\sigma}{n}\right)^2 \geq \frac{2}{n}-\frac{6}{n^2}\cdot\frac{n}{4} \geq 0$ and $y_2+x_2-\frac{2}{3}x_2^2 = \frac{2}{2r-n}-\frac{2}{3}\left(\frac{2s-n}{2r-n}\right)^2 \geq \frac{2}{2r-n}-\frac{2}{3}\left(\frac{3\sigma}{2r-n}\right)^2 \geq \frac{2}{2r-n}-\frac{6}{(2r-n)^2}\cdot\frac{2r-n}{4} \geq 0$. On the other hand, putting $x := (s-\frac{n+1}{2})/\tau$ and applying below Lemma 4.1 at the first inequality, $\tau \leq \sigma$ and $\tau x \leq \frac{3}{2}\sigma - \frac{1}{2} \leq \frac{9}{8}\sigma^2 = \frac{9}{8}\frac{N}{N-1}\sigma_0^2 \leq \frac{3}{2}\sigma_0^2$ and $\tau^2 \geq \sigma_0^2 \geq \frac{1}{4}$ at the second, and $\frac{\tau^2}{N-1} \leq \frac{\sigma^2}{N-1} = \frac{n(N-n)}{4(N-1)^2} \leq \frac{N^2}{16(N-1)^2} \leq \frac{1}{9}$ at the third, we get

$$\exp\left(\frac{s-\frac{n+1}{2}}{\sigma_0^2}\right)\frac{g(s+1)}{g(s)} < \exp\left(\frac{\tau x}{\sigma_0^2}\right)\exp\left(-\left(1-\frac{1}{12\tau^2}\right)\left(\frac{x}{\tau}+\frac{1}{2\tau^2}\right)\right)$$

$$= \exp\left(\left(\frac{1}{\sigma_0^2}-\frac{1}{\tau^2}\right)\tau x + \frac{\tau x}{12\tau^4} - \frac{1}{2\tau^2} + \frac{1}{24\tau^4}\right)$$

$$\leq \exp\left(\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma^2}\right)\frac{9}{8}\sigma^2 + \frac{\frac{3}{2}\sigma_0^2}{12\tau^2\sigma_0^2} - \frac{1}{2\tau^2} + \frac{1}{6\tau^2}\right)$$

$$= \exp\left(\left(\frac{9\tau^2}{8(N-1)}-\frac{5}{24}\right)\frac{1}{\tau^2}\right)$$

$$\leq \exp\left(-\frac{1}{12\tau^2}\right).$$

Thus $(f(s)/g(s+1))/(f(s-1)/g(s)) \geq \exp\left(\frac{1}{12\tau^2}\right) > 1$, also if $N = \infty$, hence the claim.

(b) By part (a), we can and do assume that $s = \lceil\frac{n}{2}\rceil + 1$. First let $r < \infty$.

If $n = 2k$ is even, then $s = k+1$, so that Lemma 3.2, $\tau \geq \sigma_0$, and $\sigma_0^2 \geq \frac{1}{4}$ from Lemma 4.3 yield

$$g(s) = \Phi\left(\frac{1}{\tau}\right) - \frac{1}{2} \leq \Phi\left(\frac{1}{\sigma_0}\right) - \frac{1}{2}$$

$$< \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left(-\frac{1}{6\sigma_0^2}+\frac{1}{90\sigma_0^4}\right) \leq \frac{\exp\left(-\frac{11}{90\sigma_0^2}\right)}{\sigma_0\sqrt{2\pi}},$$

and hence an application of (41) and finally $\sigma_0 \leq \sigma$ yield

$$\tag{51} \log\frac{g(s)}{f(s-1)} < -\frac{11}{90\sigma_0^2} + \frac{1}{16\sigma_0^2} = -\frac{43}{720\sigma_0^2} \leq -\frac{43}{720\sigma^2}.$$

Now let n be odd. Then $n = 2k-1$ with $k \in \{1,\ldots,r\}$ and $s = k+1$. If also $k \leq r-1$ and thus $r \geq 2$, then we have, using Lemma 3.1 with $x := h := \frac{1}{\tau}$ at the

first inequality, $\tau^2 \geq \sigma_0^2 \geq 3/16$ by (38) at the second, and $\tau^2 \leq \sigma^2$ at the third,

$$
\begin{aligned}
g(s) \;=\; \Phi\left(\tfrac{3}{2\tau}\right) - \Phi\left(\tfrac{1}{2\tau}\right) \;&<\; \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{13}{24\tau^2} + \frac{61}{1440\tau^4}\right) \\
&\leq\; \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{13}{24\tau^2} + \frac{61}{270\tau^2}\right) \;=\; \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{341}{1080\tau^2}\right) \\
&\leq\; \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{341}{1080\sigma^2}\right).
\end{aligned}
$$

Further, using the second equality in (47) and then (41) for $\mathrm{h}_{2k,r,r}$ and writing $\sigma_{0,2k} := \sigma_0(\mathrm{H}_{2k,r,r})$,

$$
f(s-1) \;=\; f(k) \;=\; \mathrm{h}_{2k,r,r}(k) \cdot \frac{r-k+\frac{1}{2}}{r-k+1} \;>\; \frac{\exp\left(-\frac{1}{16\sigma_{0,2k}^2}\right)}{\sigma_{0,2k}\sqrt{2\pi}} \cdot \frac{r-k+\frac{1}{2}}{r-k+1},
$$

and together with $\frac{\sigma^2}{\sigma_{0,2k}^2} \leq \frac{r-k+\frac{1}{2}}{r-k} \leq \frac{3}{2}$ by (48) we get the first two inequalities below and recall $r \geq 2$ for the last:

$$
\begin{aligned}
\log\frac{g(s)}{f(s-1)} \;&\leq\; \log\frac{\sigma_{0,2k}}{\sigma_0} - \frac{341}{1080\sigma^2} + \frac{1}{16\sigma_{0,2k}^2} - \log\frac{r-k+\frac{1}{2}}{r-k+1} \\
&<\; -\frac{1}{\sigma^2}\left(\frac{341}{1080} - \frac{3}{32}\right) + \frac{1}{2}\log\frac{k(r-k)}{(k-\frac{1}{2})(r-k+\frac{1}{2})} + \log\frac{r-k+1}{r-k+\frac{1}{2}} \\
&=\; -\frac{959}{4320\sigma^2} + \frac{1}{2}\log\left(1 + \frac{1/2}{k-1/2}\right) + \frac{1}{2}\log\left(1 + \frac{1/2}{r-k+\frac{1}{2}}\right) \\
&\quad + \frac{1}{2}\log\frac{(r-k)\cdot(r-k+1)}{(r-k+\frac{1}{2})^2} \\
&\leq\; -\frac{959}{4320\sigma^2} + \frac{1}{2}\left(\frac{1/2}{k-\frac{1}{2}} + \frac{1/2}{r-k+\frac{1}{2}}\right) \\
&=\; -\frac{1}{\sigma^2}\left(\frac{959}{4320} - \frac{1}{8}\cdot\frac{r}{r-\frac{1}{2}}\right) \;\leq\; -\frac{1}{\sigma^2}\left(\frac{959}{4320} - \frac{1}{6}\right) \;=\; -\frac{239}{4320\sigma^2}.
\end{aligned}
$$

If, on the other hand, $k = r$, then $n = 2r-1$, $s = r+1$, and hence

$$
g(s) \;=\; G(s) - G(s-1) \;=\; \Phi\left(\tfrac{3}{2\tau}\right) - \Phi\left(\tfrac{1}{2\tau}\right) \;<\; 1 - \Phi(0) \;=\; \frac{1}{2} \;=\; f(s-1).
$$

Hence $g(s) < f(s-1)$ in every case, also if $r = \infty$.

(c) By part (b), $G - F(\cdot - 1)$ is strictly decreasing on $\{s \in \mathbb{Z} : \lceil n/2\rceil \leq s \leq M\}$. $\qquad\square$

Lemma 4.6. *Let $s \in \mathbb{Z}$ with $s \geq \frac{n}{2} + 1 + \frac{3}{2}\sigma$. Then $G(s) - F(s-1) < \frac{1}{\sigma}\varphi\left(\frac{3}{2}\right) = \frac{0.1295\ldots}{\sigma}$.*

Proof. By Lemma 4.4(c), applicable due to $s-1 > \lfloor\frac{n}{2}\rfloor$, and then since $[0,\infty[\ni x \mapsto \varphi(x)$ and $[1,\infty[\ni x \mapsto x\varphi(x)$ are strictly decreasing and since we have $\frac{3\sigma}{2\tau} \geq \frac{3}{2} \geq 1$, we get

$$
\begin{aligned}
G(s) - F(s-1) \;&=\; G(s-1) - F(s-1) + G(s) - G(s-1) \\
&<\; G(s) - G(s-1) \;<\; \tfrac{1}{\tau}\varphi\left(\tfrac{s-\frac{n}{2}-1}{\tau}\right) \\
&\leq\; \tfrac{1}{\tau}\varphi\left(\tfrac{3\sigma}{2\tau}\right) \;\leq\; \tfrac{1}{\sigma}\varphi\left(\tfrac{3}{2}\right). \qquad\square
\end{aligned}
$$

Proof of Theorem 1.3. Let d be defined by the first equality in (12). Then the second equality in (12) and the equality in (13) hold by the first part of Lemma 1.6.

Let us now consider the lower bound

$$(52) \qquad\qquad d \;\geq\; \frac{\Phi(1) - \frac{1}{2}}{2\sigma}$$

claimed in (14). If n is odd, then, using first $\tau \leq \sigma$, and then the concavity of Φ on $[0, \infty[$ and $\sigma \geq \frac{1}{2}$, we get

$$d \;=\; \Phi\left(\tfrac{1}{2\tau}\right) - \tfrac{1}{2} \;\geq\; \frac{\Phi\left(\frac{1}{2\sigma}\right) - \frac{1}{2}}{\frac{1}{2\sigma}} \cdot \frac{1}{2\sigma} \;\geq\; \frac{\Phi(1) - \frac{1}{2}}{2\sigma} \;=\; \frac{0.170672\ldots}{\sigma}$$

with equality throughout if $\tau = \sigma$ and $n = 1$. If n is even, then Lemma 4.2 yields

$$d \;=\; \frac{1}{2} f\left(\tfrac{n}{2}\right) \;>\; \frac{\sqrt{2}}{8\sigma} \;=\; \frac{0.176776\ldots}{\sigma}.$$

The above implies (52) and half of the optimality claim in part (b) of the theorem.

We now prove (11), using the second part of Lemma 1.6. We have (23) by Lemma 4.4(c). To prove (24), let $s \in \mathbb{Z}$ with $s > \left\lceil \frac{n}{2} \right\rceil$ be given. If $s \leq M$ from Lemma 4.5, then $G(s) - F(s-1) < d$ by part (c) of that lemma. If, on the other hand, $s > M$, then $G(s) - F(s-1) < d$ by Lemma 4.6 combined with (52). Hence for part (a) of the theorem it only remains to prove the claim involving the upper bound

$$(53) \qquad\qquad d \;<\; \frac{1}{\sigma\sqrt{8\pi}}$$

contained in (14). If n is even, then (53) follows from (42).

If n is odd and $N \neq 2$, then $N \geq 4$. If N is finite, then, using $\tau \geq \sigma_0$ in the first step, $\sigma_0^2 \leq \frac{N}{16}$ by (40) and the concavity of Φ on $[0, \infty[$ and $\frac{\sigma}{\sigma_0} = \sqrt{\frac{N}{N-1}}$ in the second, (33) in the third, $x := \frac{1}{N} \in \,]0, \frac{1}{4}]$ and $h(x) := -\frac{1}{2} \log(1-x) - \frac{2}{3} x + \frac{8}{45} x^2$ in the fourth, and the convexity of h on $[0, \frac{1}{4}]$ and $h(0) = 0$ and $h\left(\frac{1}{4}\right) = -\frac{1}{2} \log\left(\frac{3}{4}\right) - \frac{1}{6} + \frac{1}{90} < -0.0117145 < 0$ in the fifth, we get

$$\sqrt{8\pi}\,\sigma\left(\Phi\left(\tfrac{1}{2\tau}\right) - \tfrac{1}{2}\right) \;\leq\; \frac{\sigma}{\sigma_0} \sqrt{2\pi} \, \frac{\Phi\left(\frac{1}{2\sigma_0}\right) - \frac{1}{2}}{\frac{1}{2\sigma_0}}$$

$$\leq\; \sqrt{\frac{N}{N-1}} \sqrt{2\pi} \, \frac{\Phi\left(\frac{2}{\sqrt{N}}\right) - \frac{1}{2}}{\frac{2}{\sqrt{N}}}$$

$$<\; \sqrt{\frac{N}{N-1}} \exp\left(-\tfrac{1}{6}\left(\tfrac{2}{\sqrt{N}}\right)^2 + \tfrac{1}{90}\left(\tfrac{2}{\sqrt{N}}\right)^4\right)$$

$$=\; \exp\left(h(x)\right)$$

$$<\; 1.$$

If $N = \infty$, then $\tau = \sigma$ and hence $\sqrt{8\pi}\,\sigma\left(\Phi\left(\frac{1}{2\tau}\right) - \frac{1}{2}\right) < 1$ obviously by $\Phi'(x) < \frac{1}{\sqrt{2\pi}}$ for $0 \neq x \in \mathbb{R}$.

Finally if $N = 2$, then $n = 1$, $\sigma = \frac{1}{2}$, $\sigma_0 = \frac{1}{\sqrt{8}}$, and $\sqrt{8\pi}\,\sigma d = \sqrt{2\pi}\left(\Phi\left(\frac{1}{2\tau}\right) - \frac{1}{2}\right) = \varrho(\tau)$ is, as a function of $\varrho \in [\sigma_0, \sigma]$, strictly decreasing with $\varrho(\sigma_0) = 1.05616\ldots > 1$, and $\varrho(\tau) < 1$ iff $\tau > \tau_0$ with $\tau_0 = 0.391961\ldots$, and we have $\tau_0/\sigma = 0.783923\ldots$.

This proves part (a), and the remaining half of the optimality claim in (b) follows from $\lim_{\sigma\to\infty} \sigma \left(\Phi\left(\frac{1}{2\sigma}\right) - \frac{1}{2} \right) = \frac{1}{\sqrt{8\pi}}$. $\qquad\square$

Proof of Remark 1.4. (a) is trivial.

(b) The first inequality in (15) is trivial by $\frac{N}{N-1} \geq 2$ and the concavity of Φ on $[0, \infty[$. If n is odd, then (13), concavity again, and $\tau \geq \sigma_0 \geq \sqrt{\frac{N-1}{4N}}$ yield the second inequality through $2\tau d = \left(\Phi\left(\frac{1}{\tau}\right) - \frac{1}{2} \right) / \left(\frac{1}{2\tau} \right) \geq \left(\Phi\left(\sqrt{\frac{N}{N-1}}\right) - \frac{1}{2} \right) / \left(\sqrt{\frac{N}{N-1}} \right)$. If n is even, and first also $4 < N < \infty$, then Lemma 4.2 yields $2\tau d \geq 2\sigma_0 d = 2\sqrt{\frac{N-1}{N}}\sigma d \geq \sqrt{\frac{N-1}{N}}\sqrt{\frac{(N-2)N^2}{8(N-1)^3}} = \sqrt{\frac{1}{8}(1 - \frac{1}{N^2-2N+1})} \geq \sqrt{\frac{1}{8}(1 - \frac{1}{36-12+1})} = 0.3461\cdots \geq \Phi\left(1\right) - \frac{1}{2} \geq \sqrt{\frac{N-1}{N}}\left(\Phi\left(\sqrt{\frac{N}{N-1}}\right) - \frac{1}{2} \right)$. If $N = \infty$, then again $\sigma = \sigma_0$, and the claim follows from (14). It remains that $N = 4$, but in this case $2\sigma_0 d = \frac{1}{3} > 0.3255\cdots = \sqrt{\frac{3}{4}}\left(\Phi\left(\sqrt{\frac{3}{4}}\right) - \frac{1}{2} \right)$.

If n is even, then $N \neq 2$, and then the third inequality $d < \frac{1}{\tau\sqrt{8\pi}}$ follows trivially from Theorem 1.3 and $\tau \leq \sigma$. If n is odd, then $d = \Phi\left(\frac{1}{2\tau}\right) - \frac{1}{2} < \frac{1}{\tau\sqrt{8\pi}}$.

(c) The second inequality is again obvious. In the first inequality, we have equality if n is odd, and if n is even, then Lemma 4.3 and Lemma 3.2 yield

$$
\begin{aligned}
d \;=\; &\frac{1}{2}f\left(\frac{n}{2}\right) \;\leq\; \frac{1}{2\sigma_0\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{1}{24\sigma_0^2}} \\
=\; &\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\sigma_0}\mathrm{e}^{-\frac{1}{6(2\sigma_0)^2}} \;<\; \Phi\left(\frac{1}{2\sigma_0}\right) - \frac{1}{2}.
\end{aligned}
$$

$\qquad\square$

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