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DRY TEN MARTINI PROBLEM FOR THE NON-SELF-DUAL EXTENDED HARPER'S MODEL

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Abstract. In this paper we prove the dry version of the Ten Martini problem: Cantor spectrum with all gaps open, for the extended Harper's model in the non-self-dual region for Diophantine frequencies.

1. Introduction

The study of independent electrons on a two-dimensional lattice exposed to a perpendicular magnetic field and periodic potentials can be reduced via an appropriate choice of gauge field to the study of discrete one-dimensional quasiperiodic Jacobi matrices. The most extensively studied case is the almost Mathieu operator (AMO) acting on $l^2(\mathbb{Z})$ defined by

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos 2\pi(\theta + n\alpha)u_n.$$

This is a one-dimensional tight-binding model with anisotropic nearest neighbor couplings in general. A more general model, called the extended Harper's model (EHM), is the operator acting on $l^2(\mathbb{Z})$ defined by:

$$(H_{\lambda,\alpha,\theta}u)_n = c(\theta + n\alpha)u_{n+1} + \tilde{c}(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n.$$

where $c(\theta) = \lambda_1 e^{-2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta+\frac{\alpha}{2})}$ and $\tilde{c}(\theta) = \lambda_1 e^{2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{-2\pi i(\theta+\frac{\alpha}{2})}$. It is obtained when both the nearest neighbor coupling (expressed through λ_2) and the next-nearest couplings (expressed through λ_1 and λ_3) are included. This model includes AMO as a special case (when $\lambda_1 = \lambda_3 = 0$).

For the AMO, it was proved in [5] that the spectrum is a Cantor set for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda \neq 0$. This is the *Ten Martini problem* dubbed by Barry Simon, after an offer of Mark Kac. A much more difficult problem, known as the dry version of the Ten Martini problem, is to prove that the spectrum is not only a Cantor set, but that all gaps predicted by the Gap-Labelling theorem [10], [15] are open. The first result was obtained for Liouvillean α [12], and later it was proved for a set (λ, α) of positive Lebesgue measure [16]. The most recent result is [6], in which they were able to deal with all Diophantine frequencies and $\lambda \neq 1$. A solution for all irrational frequencies and $\lambda \neq 1$ was also recently announced in [9].

Recently, there have been several important advances on the spectral theory of the EHM: purely point spectrum for Diophantine α and a.e. θ in the positive Lyapunov exponent region [13]; the exact formula for Lyapunov exponent for all

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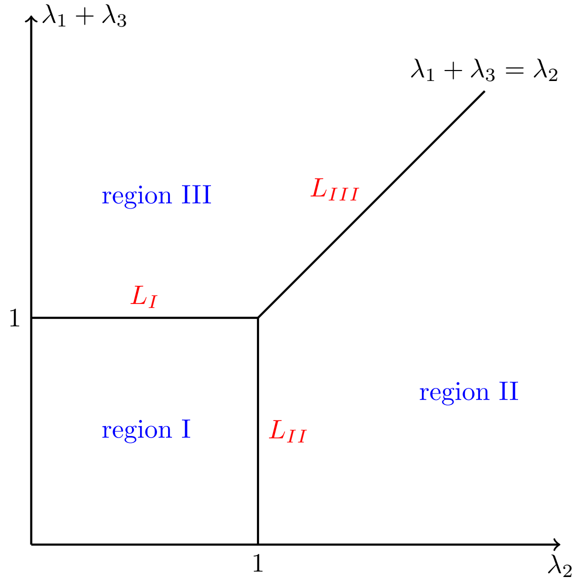
coupling constants [14]; the spectral decomposition for a.e. α [7]. However the results that study the spectrum as a set have not been obtained for the EHM.

For EHM, depending on the values of the parameters $\lambda_1, \lambda_2, \lambda_3$, we could divide the parameter space into three regions as shown in the picture below:



$$\textit{region I}: 0 < \max(\lambda_1 + \lambda_3, \lambda_2) < 1,$$
$$\textit{region II}: 0 < \max(\lambda_1 + \lambda_3, 1) < \lambda_2,$$
$$\textit{region III}: 0 < \max(1, \lambda_2) < \lambda_1 + \lambda_3.$$

According to the action of the duality transformation $\sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \to \hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2})$, region I and region II are dual to each other and region III is a self-dual region. Region I is the positive Lyapunov exponent region, which is a natural extension of the segment $\{\lambda_1 + \lambda_3 = 0, 0 < \lambda_2 < 1\}$ corresponding to the case $\lambda > 1$ in the AMO. Region II is the subcritical region, which is an extension of the segment $\{\lambda_1 + \lambda_3 = 0, 1 < \lambda_2\}$ corresponding to the case $\lambda < 1$ in the AMO.

In this paper we prove the dry version of the Ten Martini problem in region I and region II under the Diophantine condition.

Let p_n/q_n be the continued fraction appoximants of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.$$

If $\beta(\alpha) = 0$, we say α satisfies the Diophantine condition, denoted by $\alpha \in \mathrm{DC}$. It is easily seen that such α form a full measure subset of $\mathbb{T}$.

It is known that when E is in the closure of a spectral gap, the integrated density of states (IDS) $N(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ (refer to (2.5) for the definition of IDS) [10], [15]. Here we prove the inverse is true.

Theorem 1.1. *If $\alpha \in \mathrm{DC}$ and λ belongs to region I or region II, all possible spectral gaps are open.*

Remark 1.1. We note the Dry Ten Martini problem has not yet been solved for the self-dual AMO. In the self-dual region III, Cantor spectrum is known in the isotropic case (when $\lambda_1 = \lambda_3$); see Fact 2.1 in [7]. In fact one could prove the operator has zero Lebesgue measure spectrum for all frequencies.

Remark 1.2. In regions I and II, for Liouvillean α (where $\beta(\alpha)$ is large), it is not clear whether even the Cantor spectrum holds. The proof may require a non-trivial adjustment of the proof for AMO in [12].

We first establish almost localization (see section 3.1) in region I. Then a quantitative version of Aubry duality to obtain almost reducibility (see section 3.2) in region II which enables us to deal with all energies whose rotation numbers are α-rational.

Thus the strategy follows that of [6], but we need to extend the almost localization and quantitative duality, as well as the final argument to our Jacobi setting, which is non-trivial on a technical level. At the same time unlike [6], we only deal with a short-range dual operator, leading to a significant streamlining of some arguments of [6].

We organize the paper as follows: in section 2 we present some preliminaries, in section 3 we state our main results about almost localization and almost reducibility, relying on which we provide a proof of Theorem 1.1. In sections 4 and 5 we prove the main results that we present in section 3.

2. Preliminaries

2.1. **Cocycles.** Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $A \in C^0(\mathbb{T}, M_2(\mathbb{C}))$ measurable with $\log \|A(x)\| \in L^1(\mathbb{T})$. The quasiperiodic *cocycle* (α, A) is the dynamical system on $\mathbb{T} \times \mathbb{C}^2$ defined by $(\alpha, A)(x, v) = (x + \alpha, A(x)v)$. The *Lyapunov exponent* is defined by

$$L(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A_n(x)\| \mathrm{d}x = \inf_n \frac{1}{n} \int_{\mathbb{T}} \log \|A_n(x)\| \mathrm{d}x.$$

where

$$\begin{cases} A_n(x) = A(x + (n-1)\alpha) \cdots A(x) & \text{for } n \geq 0, \\ A_n(x) = A^{-1}(x + n\alpha) \cdots A^{-1}(x - \alpha) & \text{for } n < 0. \end{cases}$$

Lemma 2.1 (e.g. [6])**.** *Let (α, A) be a continuous cocycle; then for any $\delta > 0$ there exists $C_\delta > 0$ such that for any $n \in \mathbb{N}$ and $\theta \in \mathbb{T}$ we have*

$$\|A_n(\theta)\| \leq C_\delta e^{(L(\alpha,A)+\delta)n}.$$

We say that (α, A) is *uniformly hyperbolic* if there exists continuous splitting $\mathbb{C}^2 = E^s(x) \oplus E^u(x)$, $x \in \mathbb{T}$ such that for some constant $C, \eta > 0$ and all $n \geq 0$, $\|A_n(x)v\| \leq Ce^{-\eta n}\|v\|$ for $v \in E^s(x)$ and $\|A_{-n}(x)v\| \leq Ce^{-\eta n}\|v\|$ for $v \in E^u(x)$.

Given two complex cocycles $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$, we say they are *complex conjugate* to each other if there is $M \in C^0(\mathbb{T}, SL(2, \mathbb{C}))$ such that

$$M^{-1}(x + \alpha)A^{(1)}(x)M(x) = A^{(2)}(x).$$

We assume now that A is a real cocycle, $A \in C^0(\mathbb{T}, SL(2, \mathbb{R}))$. The notation of *real conjugacy* (between real cocycles) is the same as before, except that we look for $M \in C^0(\mathbb{T}, PSL(2, \mathbb{R}))$. A reason why we look for $M \in C^0(\mathbb{T}, PSL(2, \mathbb{R}))$ instead of $M \in C^0(\mathbb{T}, SL(2, \mathbb{R}))$ is given by the following well-known result.

Theorem 2.2. *Let (α, A) be uniformly hyperbolic, assume $\alpha \in \mathrm{DC}$ and A analytic. Then there exists $M \in C^\omega(\mathbb{T}, PSL(2,\mathbb{R}))$*[1] *such that $M^{-1}(x+\alpha)A(x)M(x)$ is constant.*

We say (α, A) is (analytically) *reducible* if it is real conjugate to a constant cocycle by an analytic conjugacy.

Let

$$R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Any $A \in C^0(\mathbb{T}, PSL(2,\mathbb{R}))$ is homotopic to $x \to R_{\frac{k}{2}x}$ for some $k \in \mathbb{Z}$ called the *degree* of A, denoted by $\deg A = k$.

Assume now that $A \in C^0(\mathbb{T}, SL(2,\mathbb{R}))$ is homotopic to identity. Then there exists $\phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ and $v : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+$ such that

$$A(x)\begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = v(x,y)\begin{pmatrix} \cos 2\pi(y+\phi(x,y)) \\ \sin 2\pi(y+\phi(x,y)) \end{pmatrix}.$$

The function ϕ is called a lift of A. Let μ be any probability on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ which is invariant under the continuous map $T : (x,y) \mapsto (x+\alpha, y+\phi(x,y))$, projecting over Lebesgue measure on the first coordinate. Then the number

$$\rho(\alpha, A) = \int \phi \, d\mu \mod \mathbb{Z}$$

is independent of the choices of ϕ and μ, and is called the *fibered rotation number* of (α, A).

It can be proved directly by the definition that

$$|\rho(\alpha, A) - \theta| < C\|A - R_\theta\|_0. \tag{2.1}$$

If $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$ are real conjugate, $M^{-1}(x+\alpha)A^{(2)}(x)M(x) = A^{(1)}(x)$, and $M : \mathbb{R}/\mathbb{Z} \to PSL(2,\mathbb{R})$ has degree k, then

$$\rho(\alpha, A^{(1)}) = \rho(\alpha, A^{(2)}) - k\alpha/2. \tag{2.2}$$

For uniformly hyperbolic cocycles there is the following well-known result.

Theorem 2.3. *Let (α, A) be a uniformly hyperbolic cocycle, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $2\rho(\alpha, A) \in \alpha\mathbb{Z} + \mathbb{Z}$.*

2.2. **Extended Harper's model.** We consider the extended Harper's model $\{H_{\lambda,\theta}\}_{\theta \in \mathbb{T}}$. The formal solution to $H_{\lambda,\theta}u = Eu$ can be reconstructed via the following equation:

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{\lambda,E}(\theta + n\alpha)\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

where $A_{\lambda,E}(\theta) = \frac{1}{c(\theta)}\begin{pmatrix} E - 2\cos 2\pi\theta & -\tilde{c}(\theta-\alpha) \\ c(\theta) & 0 \end{pmatrix}$. Notice that since $A_{\lambda,E}(\theta) \notin SL(2,\mathbb{R})$, we introduce the following matrix (see Lemma A.2):

$$\begin{aligned} \tilde{A}_{\lambda,E}(\theta) &= \frac{1}{\sqrt{|c|(\theta)|c|(\theta-\alpha)}}\begin{pmatrix} E - 2\cos 2\pi\theta & -|c|(\theta-\alpha) \\ |c|(\theta) & 0 \end{pmatrix} \\ &= Q_\lambda(\theta+\alpha)A_{\lambda,E}(\theta)Q_\lambda^{-1}(\theta), \end{aligned}$$

[1]In general one cannot take $M \in C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$.

where $|c|(\theta) = \sqrt{c(\theta)\tilde{c}(\theta)}$ (which is not the same as $|c(\theta)| = \sqrt{c(\theta)\overline{c(\theta)}}$ when $\theta \notin \mathbb{T}$) and $Q_\lambda(\theta)$ is analytic on $|\mathrm{Im}\theta| \leq \frac{\epsilon_1}{2\pi}$.

The spectrum of $H_{\lambda,\theta}$ denoted by Σ_λ, does not depend on θ [8], and it is the set of E such that $(\alpha, \tilde{A}_{\lambda,E})$ is not uniformly hyperbolic.

The Lyapunov exponent is defined by $L_\lambda(E) = L(\alpha, A_{\lambda,E}) = L(\alpha, \tilde{A}_{\lambda,E})$.

For a matrix-valued function $M(\theta)$, let $M_\epsilon(\theta) = M(\theta + i\epsilon)$ be the phase-complexified matrix.

In [4], Avila divides all the energies in the spectrum into three catagories: supercritical, namely the energy with positive Lyapunov exponent; subcritical, namely the energy whose Lyapunov exponent of the phase-complexified cocycle is identically equal to zero in a neighborhood of $\epsilon = 0$; critical, otherwise.

The following theorem is shown in [14] (see also the appendix):

Theorem 2.4. *Extended Harper's model is supercritical in region I and subcritical in region II. Indeed*

- *when λ belongs to region II, $L_\lambda(E) = L(\alpha, A_{\lambda,E,\epsilon}) = L(\alpha, \tilde{A}_{\lambda,E,\epsilon}) = 0$ on $|\epsilon| \leq \frac{1}{2\pi}\epsilon_1(\lambda)$,*
- *when λ belongs to region II, we have $\hat{\lambda} = (\frac{\lambda_3}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_1}{\lambda_2})$ belongs to region I and*

$$L_{\hat{\lambda}}(E) = \epsilon_1(\lambda), \tag{2.3}$$

where

$$\epsilon_1(\lambda) = \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{\max(\lambda_1 + \lambda_3, 1) + \sqrt{\max(\lambda_1 + \lambda_3, 1)^2 - 4\lambda_1\lambda_3}} > 0. \tag{2.4}$$

Fix a θ and $f \in l^2(\mathbb{Z})$. Let $\mu^f_{\lambda,\theta}$ be the *spectral measure* of $H_{\lambda,\theta}$ corresponding to f,

$$\langle (H_{\lambda,\theta} - z)^{-1} f, f\rangle = \int_{\mathbb{R}} \frac{1}{E - z} \mathrm{d}\mu^f_{\lambda,\theta}(E)$$

for z in the resolvent set $\mathbb{C} \setminus \Sigma_\lambda$.

The *integrated density of states* (IDS) is the function $N_\lambda : \mathbb{R} \to [0,1]$ defined by

$$N_\lambda(E) = \int_{\mathbb{T}} \mu^f_{\lambda,\theta}(-\infty, E]\mathrm{d}\theta, \tag{2.5}$$

where $f \in l^2(\mathbb{Z})$ is such that $\|f\|_{l^2(\mathbb{Z})} = 1$. It is a continuous non-decreasing surjective funtion.

Notice that $\tilde{A}_{\lambda,E}(\theta) \in SL(2,\mathbb{R})$ is homotopic to identity in $C^0(\mathbb{T}, SL(2,\mathbb{R}))$, in fact just consider

$$H_t(\lambda, E, \theta) = \frac{1}{\sqrt{|c|(\theta)|c|(\theta - t\alpha)}} \begin{pmatrix} t(E - v(\theta)) & -|c|(\theta - t\alpha) \\ |c|(\theta) & 0 \end{pmatrix}$$

which establishes a homotopy of $\tilde{A}_{\lambda,E}(\theta)$ to $R_{\frac{1}{4}}$ and hence to the identity. Therefore we can define the rotation number $\rho(\alpha, \tilde{A}_{\lambda,E})$. Let $\rho_\lambda(E) = \rho(\alpha, \tilde{A}_{\lambda,E})$. Notice that $\rho_\lambda(E)$ is associated to the operator

$$(\tilde{H}_{\lambda,\theta}u)_n = |c|(\theta + n\alpha)u_{n+1} + |c|(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n.$$

It is easily seen that for each θ, $\tilde{H}_{\lambda,\theta}$ and $H_{\lambda,\theta}$ differ by a unitary operator, thus they share the same spectrum and integrated density of states, $\tilde{N}_\lambda(E) = N_\lambda(E)$. The relation between the integrated density of states and rotation number of $\tilde{H}_{\lambda,\theta}$ yields

$$N_\lambda(E) = \tilde{N}_\lambda(E) = 1 - 2\rho_\lambda(E). \tag{2.6}$$

2.3. **The dual model.** It turns out the spectrum Σ_λ of $H_{\lambda,\theta}$ is related to the spectrum $\Sigma_{\hat{\lambda}}$ of $H_{\hat{\lambda},\theta}$ in the following way:

$$\Sigma_\lambda = \lambda_2 \Sigma_{\hat{\lambda}}$$

by Aubry duality. This map $\sigma : \lambda \to \hat{\lambda}$ establishes the duality between region I and region II. The IDS $N_\lambda(E)$ of $H_{\lambda,\theta}$ coincide with the IDS $N_{\hat{\lambda}}(E/\lambda_2)$ of $H_{\hat{\lambda},\theta}$. Since $\Sigma_\lambda = \lambda_2\Sigma_{\hat{\lambda}}$, we have the following

Theorem 2.5 ([11], [17]). *For any λ, θ, there exists a dense set of $E \in \Sigma_\lambda$ such that there exists a non-zero solution of $H_{\hat{\lambda},\theta} u = \frac{E}{\lambda_2} u$ with $|u_k| \leq 1 + |k|$.*

2.4. **Bounded eigenfunction for every energy.** The next result from [6] allows us to pass from a statement of every θ to every E.

Theorem 2.6 ([6]). *If $E \in \Sigma_\lambda$, then there exists $\theta(E) \in \mathbb{T}$ and a bounded solution of $H_{\hat{\lambda},\alpha,\theta} u = \frac{E}{\lambda_2} u$ with $u_0 = 1$ and $|u_k| \leq 1$.*

2.5. **Localization and reducibility.**

Theorem 2.7. *Given α irrational, $\theta \in \mathbb{R}$ and λ in region II, fix $E \in \Sigma_\lambda$, and suppose $H_{\hat{\lambda},\theta} u = \frac{E}{\lambda_2} u$ has a non-zero exponentially decaying eigenfunction $u = \{u_k\}_{k\in\mathbb{Z}}$, $|u_k| \leq e^{-c|k|}$ for k large enough. Then the following hold:*

- (A) *If $2\theta \notin \alpha\mathbb{Z} + \mathbb{Z}$, then there exists $M : \mathbb{R}/\mathbb{Z} \to SL(2, \mathbb{R})$ analytic, such that*
$$M^{-1}(x + \alpha)\tilde{A}_{\lambda,E}(x)M(x) = R_{\pm\theta}.$$
In this case $\rho(\alpha, \tilde{A}_{\lambda,E}) = \pm\theta + \frac{m}{2}\alpha \bmod \mathbb{Z}$, where $m = \deg M$ (here since $M \in SL(2, \mathbb{R})$, we have that m is an even number) and $2\rho(\alpha, \tilde{A}_{\lambda,E}) \notin \alpha\mathbb{Z} + \mathbb{Z}$.
- (B) *If $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ and $\alpha \in \mathrm{DC}$, then there exists $M : \mathbb{R}/\mathbb{Z} \to PSL(2, \mathbb{R})$ analytic, such that*
$$M^{-1}(x + \alpha)\tilde{A}_{\lambda,E}(x)M(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$$
with $a \neq 0$. In this case $\rho(\alpha, \tilde{A}_{\lambda,E}) = \frac{m}{2}\alpha \bmod \mathbb{Z}$, where $m = \deg M$, i.e. $2\rho(\alpha, \tilde{A}_{\lambda,E}) \in \alpha\mathbb{Z} + \mathbb{Z}$.

Proof. Let $u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{2\pi i k x}$, $U(x) = \begin{pmatrix} e^{2\pi i\theta} u(x) \\ u(x - \alpha) \end{pmatrix}$. Then

$$A_{\lambda,E}(x)U(x) = e^{2\pi i\theta} U(x + \alpha),$$

$$\tilde{A}_{\lambda,E}(x)\tilde{U}(x) = e^{2\pi i\theta}\tilde{U}(x + \alpha).$$

Notice $\tilde{U}(x) = Q_\lambda(x)U(x)$ is analytic in $|\mathrm{Im} x| < \frac{\tilde{c}}{2\pi}$, where $\tilde{c} = \min(\epsilon_1, c)$, ϵ_1 as in (2.4) and Q_λ as in Lemma A.2. Define $\overline{\tilde{U}(x)}$ to be the complex conjugate of

$\tilde{U}(x)$ on $\mathbb{T}$ and its analytic extension to $|\mathrm{Im}x| < \frac{\tilde{c}}{2\pi}$. Let $M(x)$ be the matrix with columns $\tilde{U}(x)$ and $\overline{\tilde{U}(x)}$. Then,

$$\tilde{A}_{\lambda,E}(x)M(x) = M(x+\alpha)\begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta}\end{pmatrix} \quad \text{on } \mathbb{T}.$$

Then since $\det M(x+\alpha) = \det M(x)$, we know $\det M(x)$ is a constant on $\mathbb{T}$.

Case 1. If $\det M(x) \neq 0$, then let $M(x) = \tilde{M}(x)\begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix}$. Then

$$\tilde{M}^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)\tilde{M}(x) = R_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta\end{pmatrix}.$$

Case 2. If $\det M(x) = 0$, then if we denote $\tilde{U}(x) = \begin{pmatrix} u_1(x) \\ u_2(x)\end{pmatrix}$, then $\det M(x) = 0$ means there exists $\eta(x)$ such that $u_1(x) = \eta(x)\overline{u_1(x)}$ and $u_2(x) = \eta(x)\overline{u_2(x)}$. This implies that $\eta(x) \in \mathbb{C}^\omega(\mathbb{T},\mathbb{C})$, and $|\eta(x)| = 1$ on $\mathbb{T}$. Therefore there exists $\phi(x) \in \mathbb{C}^\omega(\mathbb{R}/2\mathbb{Z},\mathbb{C})$ such that $\phi^2(x) = \eta(x)$ and $|\phi(x)| = 1$. It is easy to see $\overline{\phi(x)}u_1(x) = \phi(x)\overline{u_1(x)}$ and $\overline{\phi(x)}u_2(x) = \phi(x)\overline{u_2(x)}$. Then we define $W(x) = \begin{pmatrix} \overline{\phi(x)}u_1(x) \\ \overline{\phi(x)}u_2(x)\end{pmatrix}$, it is a real vector on $\mathbb{R}/2\mathbb{Z}$ with $W(x+1) = \pm W(x)$, and $\tilde{U}(x) = \phi(x)W(x)$. Now let us define $\tilde{M}(x)$ to be the matrix with columns $W(x)$ and $\frac{1}{\|W(x)\|^{-2}}R_{\frac{1}{4}}W(x)$; then $\det\tilde{M}(x) = 1$ and $\tilde{M}(x) \in PSL(2,\mathbb{R})$. Since

$$\tilde{A}_{\lambda,E}(x)W(x) = \frac{e^{2\pi i\theta}\phi(x+\alpha)}{\phi(x)}W(x+\alpha)$$

we have

$$\tilde{A}_{\lambda,E}(x)\tilde{M}(x) = \tilde{M}(x+\alpha)\begin{pmatrix} d(x) & \tau(x) \\ 0 & d(x)^{-1}\end{pmatrix}$$

where $d(x) = \frac{e^{2\pi i\theta}\phi(x+\alpha)}{\phi(x)}$, $|d(x)| = 1$ and $d(x)$ being a real number, therefore $d(x) = \pm 1$. Also $\tau(x) \in \mathbb{C}^\omega(\mathbb{R}/2\mathbb{Z},\mathbb{C})$. But in fact $\tilde{M}^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)\tilde{M}(x)$ is well defined on $\mathbb{T}$. Therefore $\tau(x) \in \mathbb{C}^\omega(\mathbb{T},\mathbb{C})$. Now since we assumed $\alpha \in \mathrm{DC}$, we can further reduce $\tau(x)$ to the constant $\tau = \int_{\mathbb{T}}\tau(x)\mathrm{d}x$. In fact there exists $\psi(x) \in \mathbb{C}^\omega(\mathbb{T},\mathbb{C})$ such that $-\psi(x+\alpha) + \psi(x) + \tau(x) = \int_{\mathbb{T}}\tau(x)\mathrm{d}x$. This implies

$$\begin{pmatrix} 1 & -\psi(x+\alpha) \\ 0 & 1\end{pmatrix}\tilde{M}^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)\tilde{M}(x)\begin{pmatrix} 1 & \psi(x) \\ 0 & 1\end{pmatrix} = \begin{pmatrix} \pm 1 & \tau \\ 0 & \pm 1\end{pmatrix}.$$

In fact if $\det M(x) = 0$, then $\frac{e^{2\pi i\theta}\phi(x+\alpha)}{\phi(x)} = \pm 1$, which implies that $2\theta \in \alpha\mathbb{Z}+\mathbb{Z}$. Therefore if $2\theta \notin \alpha\mathbb{Z}+\mathbb{Z}$, we must be in case (A). If on the other hand, $2\theta \in \alpha\mathbb{Z}+\mathbb{Z}$, $2\theta = k\alpha + n$, suppose $\tilde{M}^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)\tilde{M}(x) = R_\theta$; then

$$R_{-\frac{k}{2}(x+\alpha)}\tilde{M}^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)\tilde{M}(x)R_{\frac{k}{2}x} = R_{\frac{n}{2}} = \pm I$$

leading to a contradiction. Therefore if $2\theta \in \alpha\mathbb{Z}+\mathbb{Z}$, we must be in case (B). □

2.6. **Continued fractions.** Let $\{q_n\}$ be the denominators of the continued fraction approximants of α. We recall the following properties:

$$\|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} = \inf_{1\le|k|\le q_{n+1}-1} \|k\alpha\|_{\mathbb{R}/\mathbb{Z}},$$

$$\frac{1}{2q_{n+1}} \le \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \le \frac{1}{q_{n+1}}.$$

Recall that the Diophantine condition of α is $\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n} = 0$. Thus for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge C_\xi e^{-\xi|k|} \quad \text{for any } k \neq 0. \tag{2.7}$$

Lemma 2.8 ([5]). *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ and $0 \le l_0 \le q_n - 1$ be such that*

$$|\sin\pi(x + l_0\alpha)| = \inf_{0\le l\le q_n-1} |\sin\pi(x + l\alpha)|;$$

then for some absolute constant $C_1 > 0$,

$$-C_1 \ln q_n \le \sum_{0\le l\le q_n-1, l\neq l_0} \ln|\sin\pi(x + l\alpha)| + (q_n - 1)\ln 2 \le C_1 \ln q_n.$$

Lemma 2.9 ([6]). *Let $1 \le r \le [q_{n+1}/q_n]$. If $p(x)$ has essential degree at most $k = rq_n - 1$ and $x_0 \in \mathbb{R}/\mathbb{Z}$, then for some absolute constant C_2,*

$$\|p(x)\|_0 \le C_2 q_{n+1}^{C_2 r} \sup_{0\le j\le k} |p(x_0 + j\alpha)|.$$

3. Main estimates and proof of Theorem 1.1

3.1. Almost localization for every θ.

Definition 3.1. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $\theta \in \mathbb{R}$, $\epsilon_0 > 0$. We say that k is an ϵ_0-resonance of θ if $\|2\theta - k\alpha\| \le e^{-\epsilon_0|k|}$ and $\|2\theta - k\alpha\| = \min_{|l|\le|k|} \|2\theta - l\alpha\|$.

Definition 3.2. Let $0 = |n_0| < |n_1| < \ldots$ be the ϵ_0-resonances of θ. If this sequence is infinite, we say θ is ϵ_0-resonant, otherwise we say it is ϵ_0-non-resonant.

Definition 3.3. We say the extended Harper's model $\{H_{\lambda,\alpha,\theta}\}_\theta$ exhibits almost localization if there exists $C_0, C_3, \epsilon_0, \tilde{\epsilon}_0 > 0$, such that for every solution ϕ to $H_{\lambda,\alpha,\theta}\phi = E\phi$ satisfying $\phi(0) = 1$ and $|\phi(m)| \le 1+|m|$, and for every $C_0(1+|n_j|) < |k| < C_0^{-1}|n_{j+1}|$, we have $|\phi(k)| \le C_3 e^{-\tilde{\epsilon}_0|k|}$ (where n_j are the ϵ_0-resonances of θ).

Theorem 3.1. *If λ belongs to region II, $\{H_{\hat{\lambda},\alpha,\theta}\}_\theta$ is almost localized for every $\alpha \in$ DC.*

Remark 3.1. It is clear from Theorem 3.1 that almost localization implies localization for non-resonant θ.

We will actually prove the following explicit lemma:

Lemma 3.2. *Let λ be in region II. Let C_4 be the absolute constant in Lemma 4.3, $\epsilon_1 = \epsilon_1(\lambda)$ be as in (2.4); then for any $0 < \epsilon_0 < \frac{\epsilon_1}{100C_4}$, there exists constant $C_3 > 0$, which depends on λ, α and ϵ_0, so that for every solution u of $H_{\hat{\lambda},\alpha,\theta}u = Eu$ satisfying $u(0) = 1$ and $|u_k| \le 1 + |k|$, if $3(|n_j| + 1) < |k| < \frac{1}{3}|n_{j+1}|$, then $|u_k| \le C_3 e^{-\frac{\epsilon_1}{5}|k|}$, where $\{n_j\}$ are the ϵ_0-resonances of θ.*

The proof of Lemma 3.2 (and thus of Theorem 3.1) is given in section 4.

3.2. **Almost reducibility.** Let λ be in region II. For every $E \in \Sigma_\lambda$, let $\theta(E) \in \mathbb{T}$ be given in Theorem 2.6. Let $0 < \epsilon_0 < \frac{\epsilon_1}{100C_4}$ and $\{n_j\}$ be the set of ϵ_0-resonances of $\theta(E)$. Then for some positive constants N_0, C and c, independent of E and θ, we have the following theorem.

Theorem 3.3. *For any fixed j, with $N_0 < n = |n_j|+1 < \infty$, let $N = |n_{j+1}|$, $L^{-1} = \|2\theta - n_j\alpha\|$. Then there exists $W : \mathbb{T} \to SL(2,\mathbb{R})$ analytic such that $|\deg W| \le Cn$, $\|W\|_0 \le CL^C$ and $\|W^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)W(x) - R_{\mp\theta}\| \le Ce^{-cN}$.*

Remark 3.2. Notice that this theorem requires $n > N_0$, which is not always ensured when $\theta(E)$ is non-resonant, however in that case we have localization for $H_{\hat{\lambda},\alpha,\theta}$ instead of almost localization. We will prove Theorem 3.3 in section 5.

3.3. **Spectral consequences of almost reducibility.** Let $\epsilon_1 = \epsilon_1(\lambda)$ and C_4 be as in Lemma 3.2.

Theorem 3.4. *Assume $\alpha \in$ DC. For λ in region II, fix $E \in \Sigma_\lambda$. Assume $\theta(E) \in \mathbb{T}$ is such that $H_{\hat{\lambda},\alpha,\theta}u = \frac{E}{\lambda_2}u$ has solution satisfying $u_0 = 1$ and $|u_k| \le 1$. Let C be the constant in Theorem 3.3. Then $\theta(E)$ and $\rho(\alpha, \tilde{A}_{\lambda,E})$ have the following relation:*

- (A) *If θ is ϵ_0-non-resonant for some $\frac{\epsilon_1}{100C_4} > \epsilon_0 > 0$, then $2\theta \in \mathbb{Z}\alpha + \mathbb{Z}$ if and only if $2\rho(\alpha, \tilde{A}_{\lambda,E}) \in \mathbb{Z}\alpha + \mathbb{Z}$.*
- (B) *If θ is ϵ_0-resonant for some $\frac{\epsilon_1}{100C_4} > \epsilon_0 > 0$, then $\rho(\alpha, \tilde{A}_{\lambda,E})$ is $\frac{\epsilon_0}{C+2}$-resonant.*

Proof. (A) When θ is ϵ_0-non-resonant for some $\frac{\epsilon_1}{100C_4} > \epsilon_0 > 0$, Theorem 3.1 implies $H_{\hat{\lambda},\alpha,\theta}$ has exponentially decaying eigenfunction. Then applying Theorem 2.7 we get $2\theta \in \mathbb{Z}\alpha + \mathbb{Z}$ if and only if $2\rho(\alpha, \tilde{A}_{\lambda,E}) \in \mathbb{Z}\alpha + \mathbb{Z}$.

(B) Assume θ is ϵ_0-resonant for some $\frac{\epsilon_1}{100C_4} > \epsilon_0 > 0$. Fix any $\xi < \frac{\epsilon_0}{2C+2}$; then there exists $C_\xi > 0$ such that for any $k \ne 0$ we have $\|k\alpha\| \ge C_\xi e^{-\xi|k|}$. Now take an ϵ_0-resonance n_j of θ such that $n = |n_j| > \max\left(\frac{-\ln C_\xi/2}{\epsilon_0-(2C+2)\xi}, N_0\right)$. Then there exists $|m| \le Cn$ such that $2\rho(\alpha, \tilde{A}_{\lambda,E}) - m\alpha = -2\theta$. Then

$$\|2\rho(\alpha, \tilde{A}_{\lambda,E}) - (m-n_j)\alpha\| = \|2\theta - n_j\alpha\| < e^{-\epsilon_0 n} \le e^{-\frac{\epsilon_0}{C+2}|m-n_j|}.$$

Take any $|l| \le |m-n_j|$, $l \ne m-n_j$. Then

$$\|(l-(m-n_j))\alpha\| \ge C_\xi e^{-2\xi|m-n_j|} > 2e^{-\epsilon_0 n} > 2\|2\rho(\alpha, \tilde{A}_E) - (m-l_0)\alpha\|.$$

Thus $\|2\rho(\alpha, \tilde{A}_E) - l\alpha\| > \|2\rho(\alpha, \tilde{A}_E) - (m-n_j)\alpha\|$ for any $|l| \le |m-n_j|$, $l \ne m-n_j$. This by definition means $\rho(\alpha, \tilde{A}_{\lambda,E})$ is $\frac{\epsilon_0}{C+2}$-resonant. □

Now based on Theorem 3.4, we can complete the proof of the dry version of the Ten Martini problem for extended Harper's model in regions I and II.

Proof of Theorem 1.1. It is enough to consider λ in region II. Let $E \in \Sigma_\lambda$ be such that $N_\lambda(E) \in \mathbb{Z}\alpha + \mathbb{Z}$. We are going to show E belongs to the boundary of a component of $\mathbb{R} \setminus \Sigma_\lambda$. Now by (2.6) we have $2\rho(\alpha, \tilde{A}_{\lambda,E}) \in \alpha\mathbb{Z} + \mathbb{Z}$, thus by Theorem 3.4, $2\theta(E) \in \alpha\mathbb{Z} + \mathbb{Z}$. By Theorem 2.7, this means there exist $M(x) \in C_h^\omega(\mathbb{T}, PSL(2,\mathbb{R}))$ such that $M^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)M(x) = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$. Without loss of generality, we assume $M^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)M(x)=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$. Let $\tilde{M}(x)=\frac{M(x)}{\sqrt{|c|(x-\alpha)}}$; then

$$\tilde{M}^{-1}(x+\alpha)\begin{pmatrix}\frac{E-v(x)}{|c|(x)} & -\frac{|c|(x-\alpha)}{|c|(x)}\\ 1 & 0\end{pmatrix}\tilde{M}(x)=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}.$$

Now let $\tilde{M}(x)=\begin{pmatrix}M_{11}(x) & M_{12}(x)\\ M_{21}(x) & M_{22}(x)\end{pmatrix}$. Then $M_{21}(x)=M_{11}(x-\alpha)$ and $M_{22}(x)=M_{12}(x-\alpha)-aM_{11}(x-\alpha)$ and

$$\begin{aligned}&\tilde{M}^{-1}(x+\alpha)\begin{pmatrix}\frac{E+\epsilon-v(x)}{|c|(x)} & -\frac{|c|(x-\alpha)}{|c|(x)}\\ 1 & 0\end{pmatrix}\tilde{M}(x)\\ &\quad=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}+\epsilon\begin{pmatrix}M_{11}(x)M_{12}(x)-aM_{11}^2(x) & M_{12}^2(x)-aM_{11}(x)M_{12}(x)\\ -M_{11}^2(x) & -M_{11}(x)M_{12}(x)\end{pmatrix}.\\ &\quad\triangleq M_0+\epsilon M_1(x).\end{aligned}$$

Now we look for $Z_\epsilon(x)$ of the form $e^{\epsilon Y(x)}$ such that

$$Z_\epsilon^{-1}(x+\alpha)(M_0+\epsilon M_1(x))Z_\epsilon(x)=M_0+\epsilon[M_1]+O(\epsilon^2).$$

We then just need to solve the equation:

$$\begin{aligned}&(I-\epsilon Y(x+\alpha)+O(\epsilon^2))(M_0+\epsilon M_1(x))(I+\epsilon Y(x)+O(\epsilon^2))\\ &\quad=M_0+\epsilon[M_1]+O(\epsilon^2).\end{aligned}$$

It is sufficient to solve the coholomogical equation:

$$Y(x+\alpha)M_0-M_0Y(x)=M_1(x)-[M_1],$$

which is guaranteed by the Diophantine condition on α. Thus

$$\begin{aligned}&(M(x+\alpha)Z_\epsilon(x+\alpha))^{-1}\tilde{A}_{\lambda,E}(x)(M(x)Z_\epsilon(x))\\ &\quad=\begin{pmatrix}1+\epsilon[M_{11}M_{12}]-a\epsilon[M_{11}^2] & a+\epsilon[M_{12}^2]-a\epsilon[M_{11}M_{12}]\\ -\epsilon[M_{11}^2] & 1-\epsilon[M_{11}M_{12}]\end{pmatrix}+O(\epsilon^2)\\ &\quad\triangleq M_\epsilon+O(\epsilon^2).\end{aligned}$$

Notice that $\tilde{A}_{\lambda,E}$ is uniformly hyperbolic iff $\mathrm{Trace}(M_\epsilon)>2$ which is fulfilled when $-a\epsilon[M_{11}^2]>0$. Thus for ϵ small, satisfying $-a\epsilon[M_{11}^2]>0$, $E+\epsilon\notin\Sigma_\lambda$, which means this spectral gap is open. □

4. Almost localization in region I

In this section we will prove Lemma 3.2. For fixed λ in region II and E, let $D_{\hat{\lambda},E}(\theta)=c_{\hat{\lambda}}(\theta)A_{\hat{\lambda},E}(\theta)$, where $c_{\hat{\lambda}}(\theta)=\frac{\lambda_3}{\lambda_2}e^{-2\pi i(\theta+\frac{\alpha}{2})}+\frac{1}{\lambda_2}+\frac{\lambda_1}{\lambda_2}e^{2\pi i(\theta+\frac{\alpha}{2})}$. Regarding the Lyapunov exponent, we recall the following result in [14]:

$$L(\alpha,A_{\hat{\lambda},E})=L(\alpha,D_{\hat{\lambda},E})-\int_{\mathbb{T}}\ln|c_{\hat{\lambda}}(\theta)|\mathrm{d}\theta\triangleq\tilde{L}-\int\ln|c_{\hat{\lambda}}|>0,$$

where $\tilde{L}=\ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_2}$ and $\int\ln|c_{\hat{\lambda}}|=\ln\frac{\max(\lambda_1+\lambda_3,1)+\sqrt{\max(\lambda_1+\lambda_3,1)^2-4\lambda_1\lambda_3}}{2\lambda_2}$.

Proof of of Lemma 3.2. Suppose u is a solution satisfying the condition of Lemma 3.2. For an interval $I = [x_1, x_2]$, let Γ_I be the coupling operator between I and $\mathbb{Z} \setminus I$:

$$\Gamma_I(i,j) = \begin{cases} \tilde{c}(\theta + (x_1 - 1)\alpha), & (i,j) = (x_1, x_1 - 1), \\ c(\theta + (x_1 - 1)\alpha), & (i,j) = (x_1 - 1, x_1), \\ \tilde{c}(\theta + x_2\alpha), & (i,j) = (x_2 + 1, x_2), \\ c(\theta + x_2\alpha), & (i,j) = (x_2, x_2 + 1), \\ 0, & \text{otherwise}. \end{cases}$$

Let $H_I = R_I H_{\hat{\lambda},\theta} R_I^*$ be the restricted operator of $H_{\hat{\lambda},\theta}$ to I. Then for $x \in I$, we have $(H_I + \Gamma_I - E)u(x) = 0$. Thus $u(x) = G_I \Gamma_I u(x)$, where $G_I = (E - H_I)^{-1}$. By matrix multiplication,

$$\begin{aligned} u(x) &= \sum_{y \in I, (y,z) \in \Gamma_I} G_I(x,y)\Gamma_I(y,z)u(z) \\ &= \tilde{c}(\theta + (x_1 - 1)\alpha)G_I(x, x_1)u(x_1 - 1) + c(\theta + x_2\alpha)G_I(x, x_2)u(x_2 + 1). \end{aligned}$$

Let us denote $P_k(\theta) = \det(E - H_{[0,k-1]}(\theta))$. Then the k-step matrix $D_{\hat{\lambda},E,k}(\theta)$ satisfies

$$D_{\hat{\lambda},E,k}(\theta) = \begin{pmatrix} P_k(\theta) & -\tilde{c}(\theta - \alpha)P_{k-1}(\theta + \alpha) \\ c(\theta + (k-1)\alpha)P_{k-1}(\theta) & -\tilde{c}(\theta - \alpha)c(\theta + (k-1)\alpha)P_{k-2}(\theta + \alpha) \end{pmatrix}.$$

This relation between $P_k(\theta)$ and $D_{\hat{\lambda},E,k}(\theta)$ gives a general upper bound of $P_k(\theta)$ in terms of $\tilde{L}$. Indeed by Lemma 2.1, for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ so that

$$|P_n(\theta)| \le C(\epsilon)e^{(\tilde{L}+\epsilon)n} \quad \text{for any } n \in \mathbb{N}.$$

By Cramer's rule,

$$\begin{aligned} |G_I(x_1, y)| &= \prod_{j=x_1}^{y-1} |c(\theta + j\alpha)| \left| \frac{\det(E - H_{[y+1,x_2]}(\theta))}{\det(E - H_I(\theta))} \right| \\ &= \prod_{j=x_1}^{y-1} |c(\theta + j\alpha)| \left| \frac{P_{x_2-y}(\theta + (y+1)\alpha)}{P_k(\theta + x_1\alpha)} \right|, \\ |G_I(y, x_2)| &= \prod_{j=y+1}^{x_2} |c(\theta + j\alpha)| \left| \frac{\det(E - H_{[x_1,y-1]}(\theta))}{\det(E - H_I(\theta))} \right| \\ &= \prod_{j=y+1}^{x_2} |c(\theta + j\alpha)| \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|. \end{aligned}$$

Notice that $P_k(\theta)$ is an even function about $\theta + \frac{k-1}{2}\alpha$, it can be written as a polynomial of degree k in $\cos 2\pi(\theta + \frac{k-1}{2}\alpha)$. Let $P_k(\theta) = Q_k(\cos 2\pi(\theta + \frac{k-1}{2}\alpha))$. Let $M_{k,r} = \{\theta \in \mathbb{T},\ |Q_k(\cos 2\pi\theta)| \le e^{(k+1)r}\}$.

Definition 4.1. Fix $m > 0$. A point $y \in \mathbb{Z}$ is called (k, m)-regular if there exists an interval $[x_1, x_2]$ containing y, where $x_2 = x_1 + k - 1$ such that

$$|G_I(y, x_i)| \le e^{-m|y-x_i|} \text{ and } \mathrm{dist}(y, x_i) \ge \frac{1}{3}k \text{ for } i = 1, 2,$$

otherwise y is called (k, m)-singular.

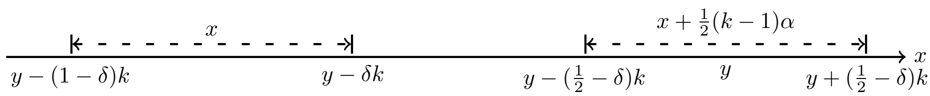
Lemma 4.1. *Suppose $y \in \mathbb{Z}$ is $(k, \tilde{L} - \int \ln |c_{\hat{\lambda}}| - \rho)$-singular. Then for any $\epsilon > 0$ and any $x \in \mathbb{Z}$ satisfying $y - \frac{2}{3}k \leq x \leq y - \frac{1}{3}k$, we have $\theta + (x + \frac{1}{2}(k-1))\alpha$ belongs to $M_{k, \tilde{L} - \frac{1}{3}\rho + \epsilon}$ for $k > k(\lambda, \epsilon, \rho)$.*

Proof. Suppose there exists $\epsilon > 0$ and x_1: $y - (1-\delta)k \leq x_1 \leq y - \delta k$, such that $\theta + (x_1 + \frac{1}{2}(k-1))\alpha$ does not belong to $M_{k, \tilde{L} - \frac{1}{3}\rho + \epsilon}$, that is, $|P_k(\theta + x_1\alpha)| > e^{(k+1)(\tilde{L} - \rho\delta + \epsilon)}$,

$$
\begin{aligned}
|G_I(x_1, y)| &\leq \prod_{j=x_1}^{y-1} |c_{\hat{\lambda}}(\theta + j\alpha)| e^{(k - |x_1 - y|)(\tilde{L} + \epsilon)} e^{-(k+1)(\tilde{L} - \frac{1}{3}\rho + \epsilon)} \\
&< e^{-(\tilde{L} - \int \ln |c_{\hat{\lambda}}| - \rho)|y - x_1|} \quad \text{for } k > k(\lambda, \epsilon, \rho).
\end{aligned}
$$

Similarly

$$
|G_I(x_2, y)| \leq e^{-(\tilde{L} - \int \ln |c_{\hat{\lambda}}| - \rho)|y - x_2|}.
$$



□

Definition 4.2. We say that the set $\{\theta_1, ..., \theta_{k+1}\}$ is γ-uniform if

$$
\max_{x \in [-1,1]} \max_{i=1,\ldots,k+1} \prod_{j=1, j \neq i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{k\gamma}.
$$

Lemma 4.2. *Let $\gamma_1 < \gamma$. If $\theta_1, ..., \theta_{k+1} \in M_{k, \tilde{L} - \gamma}$, then $\{\theta_1, ..., \theta_{k+1}\}$ is not γ_1-uniform for $k > k(\gamma, \gamma_1)$.*

Proof. Otherwise, using Lagrange interpolation form we can get $|Q_k(x)| < e^{k\tilde{L}}$ for all $x \in [-1, 1]$. This implies $|P_k(x)| < e^{k\tilde{L}}$ for all x. But by Herman's subharmonic function argument, $\int_{\mathbb{R}/\mathbb{Z}} \ln |P_k(x)| \mathrm{d}x \geq k\tilde{L}$. This is impossible. □

Now take ξ and ϵ_0 such that $0 < 1000\xi < \epsilon_0$. Then for $|n_{j+1}| > N(\xi)$ we have

$$
\begin{aligned}
2e^{-4\xi|n_{j+1}|} &\leq C_\xi e^{-2\xi|n_{j+1}|} \leq \|(n_{j+1} - n_j)\alpha\| \\
&= \|n_{j+1}\alpha - 2\theta + 2\theta - n_j\alpha\| \leq 2\|2\theta - n_j\alpha\| \leq 2e^{-\epsilon_0|n_j|},
\end{aligned}
$$

which yields that

$$
|n_{j+1}| > \frac{\epsilon_0}{4\xi}|n_j| > 250|n_j|. \tag{4.1}
$$

Without loss of generality, assume $3(|n_j| + 1) < y < \frac{|n_{j+1}|}{3}$ and $y > N(\xi)$. Select n such that $q_n \leq \frac{y}{8} < q_{n+1}$ and let s be the largest positive integer satisfying $sq_n \leq \frac{y}{8}$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows:

$$
\begin{aligned}
I_1 &= [1 - 2sq_n, 0] \text{ and } I_2 = [y - 2sq_n + 1, y + 2sq_n], \text{ if } n_j < 0, \\
I_1 &= [0, 2sq_n - 1] \text{ and } I_2 = [y - 2sq_n + 1, y + 2sq_n], \text{ if } n_j \geq 0.
\end{aligned}
$$

Lemma 4.3. *Let $\theta_j = \theta + j\alpha$; then set $\{\theta_j\}_{j \in I_1 \cup I_2}$ is $C_4\epsilon_0 + C_4\xi$-uniform for some absolute constant C_4 and $y > y(\alpha, \epsilon_0, \xi)$.*

Proof. Without loss of generality, we assume $n_j > 0$. Take $x = \cos 2\pi a$. Now it suffices to estimate

$$\sum_{j\in I_1\cup I_2,\ j\neq i} (\ln|\cos 2\pi a - \cos 2\pi\theta_j| - \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|) \triangleq \sum_1 - \sum_2 .$$

Lemma 2.8 reduces this problem to estimating the minimal terms.

First we estimate $\sum_1$:

$$\begin{aligned}\sum_1 &= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \\ &= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a+\theta_j)| + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a-\theta_j)| + (6sq_n - 1)\ln 2 \\ &\triangleq \sum_{1,+} + \sum_{1,-} + (6sq_n-1)\ln 2.\end{aligned}$$

We cut $\sum_{1,+}$ or $\sum_{1,-}$ into $6s$ sums and then apply Lemma 2.8. We get that for some absolute constant C_1:

$$\sum_1 \leq -6sq_n \ln 2 + C_1 s \ln q_n.$$

Next, we estimate $\sum_2$:

$$\begin{aligned}\sum_2 &= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi\theta_j - \cos 2\pi\theta_i| \\ &= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(2\theta + (i+j)\alpha)| \\ &\quad + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(i-j)\alpha| + (6sq_n - 1)\ln 2 \\ &\triangleq \sum_{2,+} + \sum_{2,-} + (6sq_n - 1)\ln 2.\end{aligned}$$

We need to carefully estimate the minimal terms. For $\sum_{2,+}$, we use the property of resonant set; and for $\sum_{2,-}$, we use the Diophantine condition on α.

For any $0 < |j| < q_{n+1}$, we have $\|j\alpha\| \geq \|q_n\alpha\| \geq C_\xi e^{-\xi q_n}$. Therefore

$$\max(\ln|\sin x|, \ln|\sin(x + \pi j\alpha)|) \geq -2\xi q_n \quad \text{for } y > y(\alpha,\xi).$$

This means in any interval of length sq_n, there can be at most one term which is less than $-2\xi q_n$. Then there can be at most 6 such terms in total.

For the part $\sum_{2,-}$, since $\|(i-j)\alpha\| \geq C_\xi e^{-\xi|i-j|} \geq e^{-20\xi sq_n}$, these 6 smallest terms must be bounded by $-20\xi sq_n$ from below. Hence $\sum_{2,-} \geq -6sq_n \ln 2 - C\xi sq_n - Cs\ln q_n$ for $y > y(\xi)$ and some absolute constant C.

For the part $\sum_{2,+}$, notice $|i+j| \leq 2y + 4sq_n < 3y < |n_{j+1}|$ and $i + j > 0 > -n_j$. Suppose $\|2\theta + k_0\alpha\| = \min_{j\in I_1\cup I_2} \|2\theta + (i+j)\alpha\| \leq e^{-100\epsilon_0 sq_n} < e^{-\epsilon_0|k_0|}$. Then for any $|k| \leq |k_0| \leq 40sq_n$ (including $|n_j|$),

$$\|2\theta - k\alpha\| \geq \|(k+k_0)\alpha\| - \|2\theta + k_0\alpha\| > \|2\theta + k_0\alpha\| \quad \text{for } y > y(\alpha,\epsilon_0,\xi).$$

This means $-k_0$ must be a ϵ_0-resonance, therefore $|k_0| \le |n_{j-1}|$. Then

$$\begin{aligned}\|2\theta - n_j\alpha\| &\ge \|(n_j + k_0)\alpha\| - \|2\theta + k_0\alpha\| \\ &\ge C_\xi e^{-12\xi s q_n} - e^{-100\epsilon_0 s q_n} > e^{-100\epsilon_0 s q_n} \ge \|2\theta + k_0\alpha\|\end{aligned}$$

leads to a contradiction. Thus the smallest terms must be greater than $-100\epsilon_0 s q_n$. We can bound $\sum_{2,+}$ by $-6sq_n \ln 2 - 600\epsilon_0 s q_n - 12\xi s q_n - Cs \ln q_n$ from below. Therefore $\sum_2 \ge -6sq_n \ln 2 - C\epsilon_0 s q_n - C\xi s q_n - Cs \ln q_n$. Thus the set $\{\theta_j\}_{j\in I_1 \cup I_2}$ is $C_4\epsilon_0 + C_4\xi$-uniform for $y > y(\alpha, \epsilon_0, \xi)$ and some absolute constant C_4. □

Now let C_4 be the absolute constant in Lemma 4.3. Choose $0 < 1000\xi < \epsilon_0 < \frac{\epsilon_1}{100C_4}$. Combining Lemma 4.2 and Lemma 4.3, we know that when $y > y(\alpha, \epsilon_0, \xi)$, $\{\theta_j\}_{j\in I_1\cup I_2}$ cannot be inside the set $M_{6sq_n-1, \tilde{L}-2C_4\epsilon_0}$ at the same time. Therefore 0 and y cannot be $(6sq_n - 1, \tilde{L} - \int \ln|c_{\hat{\lambda}}| - 9C_4\epsilon_0)$ at the same time. However 0 is $(6sq_n - 1, \tilde{L} - \int \ln|c_{\hat{\lambda}}| - 9C_4\epsilon_0)$-singular given n large enough. Therefore

$$\{\theta_j\}_{j\in I_1} \subset M_{6sq_n-1, \tilde{L}-2C_4\epsilon_0}.$$

Thus y must be $(6sq_n - 1, \tilde{L} - \int \ln|c_{\hat{\lambda}}| - 9C_4\epsilon_0)$-regular. This implies

$$|u(y)| \le e^{-(\tilde{L}-\int \ln|c_{\hat{\lambda}}|-9C_4\epsilon_0)\frac{1}{4}|y|} < e^{-\frac{\epsilon_1}{5}|y|} \quad \text{for } |y| \ge y(\lambda, \alpha, \epsilon_0, \xi).$$

Thus there exists $C_3 = C_{\lambda,\alpha,\epsilon_0,\xi}$ such that $|u(y)| \le C_3 e^{-\frac{\epsilon_1}{5}|y|}$ for any $3|n_j| \le |y| \le \frac{1}{3}|n_{j+1}|$ and $j \in \mathbb{N}$.

5. Almost reducibility in region II

Proof of Theorem 3.3. For any $E \in \Sigma_\lambda$, take $\theta(E)$ and $\{u_k\}$ as in Theorem 2.6. Let ϵ_1 be as in (2.4), C_4 be the absolute constant from Lemma 4.3, and C_2 be the absolute constant from Lemma 2.9. Fix $\max(32C_2\xi, 1000\xi) < \epsilon_0 < \min(\frac{\epsilon_1}{200}, \frac{\epsilon_1}{100C_4})$. By Lemma 3.2, there exists C depending on λ and α such that for any $3|n_j| < |k| < \frac{1}{3}|n_{j+1}|$, we have $|u_k| \le Ce^{-\frac{\epsilon_1}{5}|k|}$.

For any n, $9|n_j| < n < \frac{1}{9}|n_{j+1}|$, of the form

$$n = rq_m - 1 < q_{m+1}.^2 \tag{5.1}$$

Let $u(x) = u^I(x) = \sum_{k\in I} u_k e^{2\pi i k x}$ with $I = [-[\frac{n}{2}], [\frac{n}{2}]] = [x_1, x_2]$. Define

$$U(x) = \begin{pmatrix} e^{2\pi i\theta} u(x) \\ u(x-\alpha) \end{pmatrix}.$$

Let $A(\theta) = A_{\lambda,E}(\theta)$. By direct computation:

$$A(x)U(x) = e^{2\pi i\theta}U(x+\alpha) + \begin{pmatrix} g(x) \\ 0 \end{pmatrix} \triangleq e^{2\pi i\theta}U(x+\alpha) + G(x).$$

The Fourier coefficients of $g(x)$ are possibly non-zero only at four points x_1, x_2, $x_1 - 1$ and $x_2 + 1$. Since $|u_k| \le C_1 e^{-\frac{\epsilon_1}{5}|k|}$ when $3|n_j| < |k| < \frac{1}{3}|n_{j+1}|$, we know that $\|G(x)\|_{\frac{\epsilon_1}{20\pi}} \le C_1 e^{-\frac{\epsilon_1}{20}n}$.

Combining Lemmas A.3 and 2.1, we have exponential control of the growth of the transfer matrix, for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|\tilde{A}_k(x)\|_{\frac{\epsilon_1}{2\pi}} \le C_\delta e^{\delta|k|}, \quad \text{for any } k.$$

[2]The existence of such n comes from (4.1).

With some effort we are able to get the following significantly improved upper bound.

Theorem 5.1. *For some $C > 0$ depending on λ and α,*

$$\|\tilde{A}_k(x)\|_{\mathbb{T}} \leq C(1 + |k|)^C.$$

Proof. Let $\tilde{U}(x) = Q(x)U(x)$, $\tilde{G}(x) = Q(x + \alpha)G(x)$, where $Q = Q_\lambda$ is given in Lemma A.2. Since

$$\max\left(\|Q(x)\|_{\frac{\epsilon_1}{20\pi}}, \|Q^{-1}(x)\|_{\frac{\epsilon_1}{20\pi}}\right) \leq C,$$

we have

$$\tilde{A}(x)\tilde{U}(x) = e^{2\pi i\theta}\tilde{U}(x + \alpha) + \tilde{G}(x),$$

where $\|\tilde{G}(x)\|_{\frac{\epsilon_1}{20\pi}} \leq Ce^{-\frac{\epsilon_1}{20}n}$.

Lemma 5.2. *Let C_2 be the constant from Lemma 2.9. Then for any δ, $2C_2\xi < \delta < \frac{\epsilon_0}{16}$, we have*

$$\inf_{|\mathrm{Im}(x)| \leq \frac{\epsilon_1}{20\pi}} \|\tilde{U}(x)\| \geq e^{-2\delta n},$$

for $n > n(\alpha, \delta)$.

Proof. We will prove the statement by contradiction. Suppose for some $x_0 \in \{|\mathrm{Im}(x)| \leq \frac{\epsilon_1}{20\pi}\}$ we have $\|\tilde{U}(x_0)\| < e^{-2\delta n}$. Notice that for any $l \in \mathbb{N}$,

$$e^{2\pi il\theta}\tilde{U}(x_0 + l\alpha) = \tilde{A}_l(x_0)\tilde{U}(x_0) - \sum_{m=1}^{l} e^{2\pi i(m-1)\theta}\tilde{A}_{l-m}(x_0 + m\alpha)\tilde{G}(x_0 + (m-1)\alpha).$$

This implies for $n > n(\delta)$ large enough and for any $0 \leq l \leq n$, $\|\tilde{U}(x_0 + l\alpha)\| \leq e^{-\delta n}$, thus $\|u(x_0 + l\alpha)\| \leq C_\delta e^{-\delta n}$. By Lemma 2.9, $\|u(x + i\mathrm{Im}(x_0))\|_{\mathbb{T}} \leq C_2 C_\delta e^{C_2\xi n}e^{-\delta n} \leq e^{-\frac{\delta}{2}n}$. This contradicts with $\int_{\mathbb{T}} u(x + i\mathrm{Im}(x_0))\mathrm{d}x = u_0 = 1$. □

Lemma 5.3 ([3])**.** *Let $V : \mathbb{T} \to \mathbb{C}^2$ be analytic in $|\mathrm{Im}(x)| < \eta$. Assume that $\delta_1 < \|V(x)\| < \delta_2^{-1}$ holds on $|\mathrm{Im}(x)| < \eta$. Then there exists $M : \mathbb{T} \to SL(2, \mathbb{C})$ analytic on $|\mathrm{Im}(x)| < \eta$ with first column V and $\|M\|_\eta \leq C\delta_1^{-2}\delta_2^{-1}(1 - \ln(\delta_1\delta_2))$.*

Applying Lemma 5.3, let $M(x)$ be the matrix with first column $\tilde{U}(x)$. Then $e^{-2\delta n} \leq \|\tilde{U}(x)\|_{\frac{\delta}{\pi}} \leq e^{\delta n}$ and hence $\|M(x)\|_{\frac{\delta}{\pi}} \leq Ce^{6\delta n}$. Therefore

$$M^{-1}(x + \alpha)\tilde{A}(x)M(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} \beta_1(x) & b(x) \\ \beta_3(x) & \beta_4(x) \end{pmatrix}$$

where $\|\beta_1(x)\|_{\frac{\delta}{\pi}}$, $\|\beta_3(x)\|_{\frac{\delta}{\pi}}$, $\|\beta_4(x)\|_{\frac{\delta}{\pi}} \leq Ce^{-\frac{\epsilon_1}{40}n}$, and $\|b(x)\|_{\frac{\delta}{\pi}} \leq Ce^{13\delta n}$. Let

$$\Phi(x) = M(x)\begin{pmatrix} e^{\frac{\epsilon_1}{160}n} & 0 \\ 0 & e^{-\frac{\epsilon_1}{160}n} \end{pmatrix}.$$

Then we would have

$$\Phi(x + \alpha)^{-1}\tilde{A}(x)\Phi(x) = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + H(x),$$

where $\|H(x)\|_{\frac{\delta}{\pi}} \leq Ce^{-\frac{\epsilon_1}{160}n}$, and $\|\Phi(x)\|_{\frac{\delta}{\pi}} \leq Ce^{\frac{\epsilon_1}{80}n}$. Thus

$$\sup_{0\leq s\leq e^{\frac{\epsilon_1}{320}n}} \|\tilde{A}_s(x)\|_{\mathbb{T}} \leq e^{\frac{\epsilon_1}{20}n}$$

for $n \geq n(\lambda,\alpha)$ satisfying (5.1). For s large, there always exists $9|n_j| < n < \frac{1}{9}|n_{j+1}|$ satisfying (5.1) such that $cn \leq \frac{320}{\epsilon_1}\ln s \leq n$ with some absolute constant c. Thus there exists C depending on λ and α such that $\|\tilde{A}_k(x)\|_{\mathbb{T}} \leq C(1+|k|)^C$. □

Now we come back to the proof of Theorem 3.3. Fix some $n = |n_j|$, and $N = |n_{j+1}|$. Let $u(x) = u^{I_2}(x)$ with $I_2 = [-[\frac{N}{9}],[\frac{N}{9}]]$ and $U(x) = \begin{pmatrix} e^{2\pi i\theta}u(x) \\ u(x-\alpha)\end{pmatrix}$. Then

$$A(x)U(x) = e^{2\pi i\theta}U(x+\alpha) + G(x) \quad \text{with} \quad \|G(x)\|_{\frac{\epsilon_1}{20\pi}} \leq Ce^{-\frac{\epsilon_1}{90}N}.$$

Define $U_0(x) = e^{\pi i n_j x}U(x)$. Notice that if n_j is even, then $U_0(x)$ is well-defined on $\mathbb{T}$, otherwise $U_0(x+1) = -U_0(x)$. Then

$$\tilde{A}(x)\tilde{U}_0(x) = e^{2\pi i\tilde{\theta}}\tilde{U}_0(x+\alpha) + H(x),$$

where $\tilde{\theta} = \theta - \frac{n_j}{2}\alpha$, $\tilde{U}_0(x) = Q(x)U_0(x)$ and $\|H(x)\|_{\frac{\epsilon_1}{20\pi}} \leq Ce^{-\frac{\epsilon_1}{100}N}$. Consider the matrix $W(x)$ with $\tilde{U}_0(x)$ and $\overline{\tilde{U}_0(x)}$ being its two columns. Then

$$\tilde{A}(x)W(x) = W(x+\alpha)\begin{pmatrix} e^{2\pi i\tilde{\theta}} & 0 \\ 0 & e^{-2\pi i\tilde{\theta}}\end{pmatrix} + \tilde{H}(x).$$

Theorem 5.4. *Let $L^{-1} = \|2\theta - n_j\alpha\|$. Then for $n > N_0(\lambda,\alpha)$ we have*

$$|\det W(x)| \geq L^{-4C} \quad \text{for any } x \in \mathbb{T},$$

where C is the constant appeared in Theorem 5.1.

Proof. First, we fix $\xi_1 < \frac{\epsilon_0}{1600}$ so that $\|k\alpha\| \geq C_{\xi_1}e^{-\xi_1|k|}$ for any $k \neq 0$. We have the following estimate about L:

Lemma 5.5. $e^{\epsilon_0 n} \leq L \leq e^{4\xi_1 N}$. *This can be seen by the following inequality:*

$$e^{-2\xi_1 N} \leq \|(n_{j+1}-n_j)\alpha\| \leq 2\|n_j\alpha - 2\theta\| = 2L^{-1} \leq 2e^{-\epsilon_0 n} \quad \text{for } n \geq N(\xi_1).$$

Now we prove by contradiction. Suppose there exists κ and $x_0 \in \mathbb{T}$ such that $\|\tilde{U}_0(x_0) - \kappa\overline{\tilde{U}_0(x_0)}\| < L^{-4C}$. Then

$$\begin{aligned}
&\|\tilde{U}_0(x_0+l\alpha)e^{2\pi il\tilde{\theta}} - \kappa\overline{\tilde{U}_0(x_0+l\alpha)}e^{-2\pi il\tilde{\theta}}\| \\
&\leq \|\sum_{m=0}^{l-1}\tilde{A}_{l-m}(x_0+m\alpha)H(x_0+m\alpha) \\
&\quad - \kappa\sum_{m=0}^{l-1}\tilde{A}_{l-m}(x_0+m\alpha)\overline{H(x_0+m\alpha)}\| + \|A_l(x_0)\|L^{-4C} \\
&\leq CL^{2C}e^{-\frac{\epsilon_1}{100}N} + CL^{-2C} < L^{-C}
\end{aligned}$$

for $0 \leq |l| \leq L^2$. If we take $j = \frac{L}{4}$, then

$$\|\tilde{U}_0(x_0+\frac{L}{4}\alpha) + \kappa\overline{\tilde{U}_0(x_0+\frac{L}{4}\alpha)}\| < L^{-1}. \tag{5.2}$$

Next since $\|U_0(x)\|_{\mathbb{T}} \le n$, we have $\|\tilde{U}_0(x)\|_{\mathbb{T}} \le Cn$. Thus

$$\|\tilde{U}_0(x_0+l\alpha) - \kappa\overline{\tilde{U}_0(x_0+l\alpha)}\| < L^{-\frac{1}{3}} \quad \text{for } 0 \le |l| \le L^{\frac{1}{2}}.$$

For any analytic function $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{2\pi i k x}$, define

$$f_{[-m,m]}(x) = \sum_{|k|\le m} \hat{f}_k e^{2\pi i k x}.$$

For any column vector $V(x) = \begin{pmatrix} v^{(1)}(x) \\ v^{(2)}(x)\end{pmatrix}$, let $V_{[-m,m]}(x) = \begin{pmatrix} v^{(1)}_{[-m,m]}(x) \\ v^{(2)}_{[-m,m]}(x)\end{pmatrix}$. Now let us define $\tilde{U}_0^{[9n]}(x) = Q(x)e^{\pi i n_j x}U_{[-9n,9n]}(x)$. Then

$$\|\tilde{U}_0^{[9n]}(x) - \tilde{U}_0(x)\|_{\mathbb{T}} \le Ce^{-\frac{9}{5}\epsilon_1 n}.$$

Consider $[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}(x)]_{[-18n,18n]}(x)e^{\pi i n_j x}$. This function differs from a polynomial with essential degree $36n$ only by a multiple of $e^{\pi i n_j x}$. Notice that $Q(x)$ is analytic in $\{x : |\mathrm{Im}(x)| \le \frac{\epsilon_1}{4\pi}\}$, thus $|\hat{Q}(k)| \le Ce^{-\frac{\epsilon_1}{2}|k|}$. Then

$$|\widehat{e^{-\pi i n_j x}\tilde{U}_0^{[9n]}}(k)| \le \sum_{|m|\le 9n} |\hat{Q}(k-m)\hat{U}(m)| \le Cne^{-\frac{\epsilon_1}{2}(|k|-9n)} \quad \text{for } |k| \ge 18n.$$

Thus

$$\|e^{-\pi i n_j x}\tilde{U}_0^{[9n]}(x) - [e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)\|_{\mathbb{T}} \le e^{-4\epsilon_1 n},$$
$$\|\tilde{U}_0(x) - [e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{\pi i n_j x}\|_{\mathbb{T}} \le e^{-4\epsilon_1 n}.$$

Hence

$$\begin{aligned}\|[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x_0+l\alpha)e^{2\pi i n_j(x_0+l\alpha)} \\ - \kappa\overline{[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x_0+l\alpha)}\|_{\mathbb{T}} < 2L^{-\frac{1}{3}} + e^{-4\epsilon_1 n},\end{aligned}$$

for $|l| \le L^{\frac{1}{2}}$. Notice that

$$[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{2\pi i n_j x} - \kappa\overline{[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)}$$

is a polynomial whose essential degree is at most $37n$. Thus by Lemma 2.9, we would have

$$\begin{aligned}\|[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{\pi i n_j x} \\ - \kappa\overline{[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x)e^{\pi i n_j x}}\|_{\mathbb{T}} < L^{-\frac{1}{4}} + e^{-2\epsilon_1 n}.\end{aligned}$$

Hence $\|\tilde{U}_0(x) - \kappa\overline{\tilde{U}_0(x)}\|_{\mathbb{T}} < L^{-\frac{1}{4}} + 2e^{-2\epsilon_1 n}$. But combining with (5.2) we would get $\|\tilde{U}_0(x_0 + \frac{L}{4}\alpha)\| < 2L^{-\frac{1}{4}} + 2e^{-2\epsilon_1 n}$, but this contradicts with $\inf_{x\in\mathbb{T}} \|\tilde{U}_0(x)\| > e^{-2\delta n}$ since $\delta < \frac{\epsilon_0}{16}$. □

Now for $n > N_0(\lambda,\alpha)$, take $S(x) = \mathrm{Re}\tilde{U}_0(x)$ and $T(x) = \mathrm{Im}\tilde{U}_0(x)$. Let $W_1(x)$ be the matrix with columns $S(x)$ and $T(x)$. Notice that $\det W_1(x)$ is well-defined on $\mathbb{T}$ and $\det W_1(x) \ne 0$ on $\mathbb{T}$, hence without loss of generality we could assume $\det W_1(x) > 0$ on $\mathbb{T}$; otherwise we simply take $W_1(x)$ to be the matrix with columns $S(x)$ and $-T(x)$. Then

$$\|\tilde{A}(x)W_1(x) - W_1(x+\alpha)R_{-\tilde{\theta}}\|_{\mathbb{T}} \le Ce^{-\frac{\epsilon_1}{45}N}.$$

By taking determinant, we get

$$\det W_1(x) = \det W_1(x+\alpha) + O(e^{-\frac{\epsilon_1}{50}N}) \text{ on } \mathbb{T}.$$

Since $\det W_1(x)$ is analytic on $|\mathrm{Im} x| \le \frac{\epsilon_1}{20\pi}$, by considering the Fourier coefficients we could get

$$\det W_1(x) = w_0 + O(e^{-\frac{\epsilon_1}{100}N}) \text{ on } \mathbb{T},$$

where $w_0 \ge L^{-5C}$. Thus $\det W_1(x)$ is almost a positive constant.

Define $W_2(x) = \det W_1(x)^{-\frac{1}{2}} W_1(x)$. Then $W_2(x) \in C^\omega(\mathbb{T})$ and $\det W_2(x) = 1$. We have

$$W_2^{-1}(x+\alpha)\tilde{A}(x)W_2(x) = \frac{\det W_1(x+\alpha)^{\frac{1}{2}}}{\det W_1(x)^{\frac{1}{2}}} R_{-\tilde{\theta}} + O(e^{-\frac{\epsilon_1}{100}N}) \text{ on } \mathbb{T},$$
$$W_2^{-1}(x+\alpha)\tilde{A}(x)W_2(x) = R_{-\tilde{\theta}} + O(e^{-\frac{\epsilon_1}{200}N}) \text{ on } \mathbb{T}.$$

Now let's prove $\deg W_2(x) \le 36n$. $\deg W_2(x)$ is the same as the degree of its columns. For $M : \mathbb{R}/2\mathbb{Z} \to \mathbb{R}^2$, we say $\deg M = k$ if M is homotopic to $\begin{pmatrix} \cos k\pi x \\ \sin k\pi x \end{pmatrix}$.

For some constant $c > 0$, we obviously have

$$\int_{\mathbb{T}} \|S(x)\| \,\mathrm{d}x + \int_{\mathbb{T}} \|T(x)\| \,\mathrm{d}x \ge \int_{\mathbb{T}} \|S(x) + iT(x)\| \,\mathrm{d}x = \int_{\mathbb{T}} \|\tilde{U}_0(x)\| \,\mathrm{d}x \ge c.$$

Without loss of generality we could assume $\int_{\mathbb{T}} \|S(x)\| \,\mathrm{d}x > \frac{c}{2}$. Also

$$\tilde{A}(x)S(x) = S(x+\alpha)\cos 2\pi\tilde{\theta} - T(x+\alpha)\sin 2\pi\tilde{\theta} + O(e^{-\frac{\epsilon_1}{45}N}) \text{ on } \mathbb{T}.$$

Then since $\|2\tilde{\theta}\| = L^{-1}$,

$$\tilde{A}(x)S(x) = S(x+\alpha) + O(L^{-\frac{1}{2}}) \text{ on } \mathbb{T}.$$

First we prove $\inf_{x\in\mathbb{T}} \|S(x)\| \ge e^{-2\epsilon_1 n}$. Suppose otherwise. Then there exists $x_0 \in \mathbb{T}$, so that $\|S(x_0)\| < e^{-2\epsilon_1 n}$. Then $\|\mathrm{Re}\tilde{U}_0(x_0 + l\alpha)\| < e^{-\frac{\epsilon_0}{8}n}$ for $|l| < e^{\frac{\epsilon_0}{4C}n}$, where C is the constant that appeared in Theorem 5.1. We have already shown that

$$\|\tilde{U}_0(x) - [e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]} e^{\pi i n_j x}\|_{\mathbb{T}} < e^{-4\epsilon_1 n}.$$

Thus

$$\|\mathrm{Re}[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x_0 + l\alpha)\| < e^{-\frac{\epsilon_0}{16}n}$$

for $|l| < e^{\frac{\epsilon_0}{4C}n}$. However $\mathrm{Re}[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}$ is a polynomial with essential degree at most $36n$. Using Lemma 2.9 we are able to get

$$\|\mathrm{Re}[e^{-\pi i n x}\tilde{U}_0^{[9n]}]_{[-18n,18n]} e^{\pi i n_j x}\|_{\mathbb{T}} < e^{-\frac{\epsilon_0}{32}n},$$

and thus $\|\mathrm{Re}\tilde{U}_0(x)\|_{\mathbb{T}} < e^{-\frac{\epsilon_0}{64}n}$ which is a contradiction to $\int_{\mathbb{T}} \|\mathrm{Re}\tilde{U}_0(x)\| \,\mathrm{d}x > \frac{c}{2}$. In the meantime, we also get

$$\|S(x) - \mathrm{Re}[e^{-\pi i n_j x}\tilde{U}_0^{[9n]}]_{[-18n,18n]}(x) e^{\pi i n_j x}\|_{\mathbb{T}} \triangleq \|S(x) - h(x)\|_{\mathbb{T}} \le e^{-4\epsilon_1 n}.$$

The first column of $W_2(x)$ is $\det W_1(x)^{-\frac{1}{2}} S(x)$. We have

$$\begin{aligned}&\|\frac{S(x)}{\det W_1(x)^{\frac{1}{2}}} - \frac{h(x)}{w_0{}^{\frac{1}{2}}}\| \\ &\quad \le \frac{1}{|\det W_1(x)^{\frac{1}{2}}|}\|S(x) - h(x) + (1 - \frac{\det W_1(x)^{\frac{1}{2}}}{w_0{}^{\frac{1}{2}}})h(x)\| \\ &\quad \le L^{2C}(e^{-4\epsilon_1 n} + L^{8C} e^{-\frac{\epsilon_1}{100}N}) \\ &\quad \le e^{-3\epsilon_1 n} < \|\frac{S(x)}{\det W_1(x)^{\frac{1}{2}}}\| \quad \text{on } \mathbb{T}.\end{aligned}$$

Thus by Rouché's theorem $|\deg W_2(x)| = |\deg h(x)| \le 19n$. Notice that

$$|\rho(\alpha, W_2^{-1}\tilde{A}W_2) + \tilde{\theta}| < Ce^{-\frac{\epsilon_1}{200}N}.$$

Then, by (2.2) for some $|m| \le 19n$:

$$|\rho(\alpha, \tilde{A}) - \frac{m}{2}\alpha + \tilde{\theta}| < Ce^{-\frac{\epsilon_1}{200}N}.$$

Appendix A

When λ belongs to region II, let $\epsilon_2 = \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{\lambda_1+\lambda_3+\sqrt{(\lambda_1+\lambda_3)^2 - 4\lambda_1\lambda_3}} > \epsilon_1$. Then $c(x)$ is analytic and non-zero on $|\mathrm{Im}(x)| < \frac{\epsilon_2}{2\pi}$. Furthermore, the winding number of $c(\cdot + i\epsilon)$ is equal to zero when $|\epsilon| < \frac{\epsilon_2}{2\pi}$.

Lemma A.1. *When λ belongs to region II, we can find an analytic function $f(x)$ on $|\mathrm{Im}(x)| \le \frac{\epsilon_1}{2\pi}$ such that $c(x) = |c|(x)e^{f(x+\alpha)-f(x)}$ and $\tilde{c}(x) = |c|(x)e^{-f(x+\alpha)+f(x)}$.*

Proof. Since the winding numbers of $c(x)$ and $\tilde{c}(x)$ are 0 on $|\mathrm{Im}(x)| \le \frac{\epsilon_1}{2\pi}$, there exist analytic functions $g_1(x)$ and $g_2(x)$ on $|\mathrm{Im}(x)| \le \frac{\epsilon_1}{2\pi}$, such that $c(x) = e^{g_1(x)}$ and $\tilde{c}(x) = e^{g_2(x)}$. Notice that

$$\int_{\mathbb{T}} \ln|c(x)|\ \mathrm{d}x = \int_{\mathbb{T}} \ln|\tilde{c}(x)|\ \mathrm{d}x,$$
$$\int_{\mathbb{T}} \arg c(x)\ \mathrm{d}x = \int_{\mathbb{T}} \arg \tilde{c}(x)\ \mathrm{d}x,$$

so there exists an analytic function $f(x)$ such that $2f(x+\alpha) - 2f(x) = g_1(x) - g_2(x)$. Then $c(x) = |c|(x)e^{f(x+\alpha)-f(x)}$. □

Lemma A.2. *When λ belongs to region II, there exists an analytic matrix $Q_\lambda(x)$ defined on $|\mathrm{Im}(x)| \le \frac{\epsilon_1}{2\pi}$ such that*

$$Q_\lambda^{-1}(x+\alpha)\tilde{A}_{\lambda,E}(x)Q_\lambda(x) = A_{\lambda,E}(x).$$

Proof.

$$\begin{aligned}\tilde{A}_{\lambda,E}(x) =&\frac{1}{\sqrt{|c|(x)|c|(x-\alpha)}}\begin{pmatrix}1 & 0\\ 0 & \sqrt{\frac{\tilde{c}(x)}{c(x)}}\end{pmatrix}\begin{pmatrix}E-v(x) & -\tilde{c}(x-\alpha)\\ c(x) & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \sqrt{\frac{c(x-\alpha)}{\tilde{c}(x-\alpha)}}\end{pmatrix}\\ =&\frac{c(x)}{\sqrt{|c|(x)|c|(x-\alpha)}}\begin{pmatrix}1 & 0\\ 0 & \sqrt{\frac{\tilde{c}(x)}{c(x)}}\end{pmatrix}A(x)\begin{pmatrix}1 & 0\\ 0 & \sqrt{\frac{c(x-\alpha)}{\tilde{c}(x-\alpha)}}\end{pmatrix}\\ =&e^{f(x+\alpha)}\sqrt{|c|(x)}\begin{pmatrix}1 & 0\\ 0 & \sqrt{\frac{\tilde{c}(x)}{c(x)}}\end{pmatrix}A(x)\left\{e^{f(x)}\sqrt{|c|(x-\alpha)}\begin{pmatrix}1 & 0\\ 0 & \sqrt{\frac{\tilde{c}(x-\alpha)}{c(x-\alpha)}}\end{pmatrix}\right\}^{-1}\\ =&Q_\lambda(x+\alpha)A_{\lambda,E}(x)Q_\lambda^{-1}(x).\end{aligned}$$

□

Lemma A.3. *If α is irrational, λ belongs to region II, $E\in\Sigma(\lambda)$, then*

$$L(\alpha, A_{\lambda,E}(\cdot+i\epsilon)) = L(\alpha,\tilde{A}_{\lambda,E}(\cdot+i\epsilon)) = 0$$

for $|\epsilon|\le\frac{\epsilon_1}{2\pi}$.

Proof. $L(A(\cdot+i\epsilon)) = L(D(\cdot+i\epsilon)) - \int\ln|c(x+i\epsilon)|\mathrm{d}x$.

$$\begin{aligned}&D(x+i\epsilon)\\ &=\begin{pmatrix}E-e^{2\pi i(x+i\epsilon)}-e^{-2\pi i(x+i\epsilon)} & -\lambda_1e^{2\pi i(x-\frac{\alpha}{2}+i\epsilon)}-\lambda_2-\lambda_3e^{-2\pi i(x-\frac{\alpha}{2}+i\epsilon)}\\ \lambda_1e^{-2\pi i(x+\frac{\alpha}{2}+i\epsilon)}+\lambda_2+\lambda_3e^{2\pi i(x+\frac{\alpha}{2}+i\epsilon)} & 0\end{pmatrix}\\ &=e^{2\pi\epsilon}\begin{pmatrix}-e^{2\pi ix}+o(1) & -\lambda_3e^{-2\pi i(x-\frac{\alpha}{2})}+o(1)\\ \lambda_1e^{-2\pi i(x+\frac{\alpha}{2})}+o(1) & 0\end{pmatrix}.\end{aligned}$$

Thus the asymptotic behaviour of $L(D(\cdot+i\epsilon))$ is:

$$L(D(\cdot+i\epsilon)) = \ln|\frac{1+\sqrt{1-4\lambda_1\lambda_3}}{2}|+2\pi\epsilon \quad\text{when } \epsilon\to\infty,$$
$$L(D(\cdot+i\epsilon)) = \ln|\frac{1+\sqrt{1-4\lambda_1\lambda_3}}{2}|-2\pi\epsilon \quad\text{when } \epsilon\to-\infty.$$

Then it suffices to calculate $\int\ln|c(x+i\epsilon)|\mathrm{d}x$ in region II. We have

$$\begin{aligned}&\int\ln|c(x+i\epsilon)|\mathrm{d}x\\ &\qquad=\ln\lambda_3-2\pi\epsilon+\int\ln|e^{2\pi ix}-y_{1,\epsilon}|+\int\ln|e^{2\pi ix}-y_{2,\epsilon}|,\end{aligned}$$

where $y_{1,\epsilon}=\frac{-\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_3}e^{2\pi\epsilon}$ and $y_{2,\epsilon}=\frac{-\lambda_2-\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_3}e^{2\pi\epsilon}$.

$$\begin{aligned}&\int\ln|c(x+i\epsilon)|\mathrm{d}x\\ &=\begin{cases}2\pi\epsilon+\ln\lambda_1, & \epsilon>\frac{1}{2\pi}\ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1},\\ \ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2}, & \frac{1}{2\pi}\ln\frac{\lambda_2-\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1}\le\epsilon\le\frac{1}{2\pi}\ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1},\\ -2\pi\epsilon+\ln\lambda_3, & \epsilon<\frac{1}{2\pi}\ln\frac{\lambda_2-\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{2\lambda_1}.\end{cases}\end{aligned}$$

Thus $L(A(\cdot+i\epsilon))=0$ when $|\epsilon|\le\frac{1}{2\pi}\ln\frac{\lambda_2+\sqrt{\lambda_2^2-4\lambda_1\lambda_3}}{\max(1,\lambda_1+\lambda_3)+\sqrt{\max(1,\lambda_1+\lambda_3)^2-4\lambda_1\lambda_3}}=\frac{\epsilon_1}{2\pi}$.

Since $\tilde{A}_{\lambda,E}(x+i\epsilon) = Q_\lambda(x+\alpha+i\epsilon)A_{\lambda,E}(x+i\epsilon)Q_\lambda^{-1}(x+i\epsilon)$, the statement about $\tilde{A}_{\lambda,E}$ is also true. □

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