TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 1, January 2018, Pages 695–726
http://dx.doi.org/10.1090/tran/7019
Article electronically published on August 3, 2017

BREUIL $\mathcal{O}$-WINDOWS AND π-DIVISIBLE $\mathcal{O}$-MODULES

CHUANGXUN CHENG

ABSTRACT. Let $p > 2$ be a prime number. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let π be a uniformizer of $\mathcal{O}$. We prove that, for any complete Noetherian regular local $\mathcal{O}$-algebra R with perfect residue field of characteristic p, the category of Breuil $\mathcal{O}$-windows over R is equivalent to the category of π-divisible $\mathcal{O}$-modules over R. We also prove that the category of Breuil $\mathcal{O}$-modules over R is equivalent to the category of commutative finite flat $\mathcal{O}$-group schemes over R which are kernels of isogenies of π-divisible $\mathcal{O}$-modules. As an application of these equivalences, we then prove a boundedness result on Barsotti-Tate groups and deduce some corollaries. The results generalize some earlier results of Zink, Vasiu-Zink, and Lau.

1. Introduction

The theory of displays is a powerful tool to study p-divisible groups. The aim of this paper is to generalize this theory and to study π-divisible $\mathcal{O}$-modules. We first review the main results from the theory of displays. Let p be a prime number and let R be a commutative ring. Assume that p is nilpotent in R. Following the notation in [23], we have a functor

$$\mathrm{BT} : \{\text{nilpotent displays over } R\} \to \{p\text{-divisible formal groups over } R\}.$$

Zink [24, Theorem 9] proved that this functor is an equivalence of categories if R is an excellent local ring or a ring such that R/pR is an algebra of finite type over a field k. Then Lau [14, Theorem 1.1] proved the equivalence for all R in which p is nilpotent.

Let R be a complete Noetherian local ring with perfect residue field of characteristic p. For $p = 2$, we assume that $pR = 0$. Zink [23] defined a category of Dieudonné displays over R and extended the functor BT to an equivalence

$$\mathrm{BT} : \{\text{Dieudonné displays over } R\} \to \{p\text{-divisible groups over } R\}.$$

Moreover, Lau [13] showed that this equivalence is compatible with duality.

The above results have been generalized to π-divisible formal $\mathcal{O}$-modules and the π-divisible $\mathcal{O}$-modules case. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F}_q$. Let R be an $\mathcal{O}$-algebra. A *π-divisible (formal) $\mathcal{O}$-module over R* is a p-divisible (formal) group G over R with an action of $\mathcal{O}$ given by $\iota : \mathcal{O} \to \mathrm{End}(G)$, such that the induced action of $\mathcal{O}$ on $\mathrm{Lie}(G)$ via ι coincides with the action through $\mathcal{O} \to R$. Here we use Zink's definition of formal groups ([24, Definition 80] and [2, Section 1.2.3]).

Received by the editors January 13, 2015 and, in revised form, September 15, 2015 and May 2, 2016.

2010 *Mathematics Subject Classification.* Primary 14L05, 14K10.

Key words and phrases. $\mathcal{O}$-frames, $\mathcal{O}$-windows, Breuil modules, p-divisible groups, π-divisible $\mathcal{O}$-modules, group schemes, regular rings.

Assume that $p > 2$. In his thesis [1], Ahsendorf defined a category of $\mathcal{O}$-displays over R and by adapting the method of Drinfeld [7], proved that the category of nilpotent $\mathcal{O}$-displays over R is equivalent to the category of π-divisible formal $\mathcal{O}$-modules over R, if π is nilpotent in R. This result was then extended to an equivalence between the category of Dieudonné $\mathcal{O}$-displays over R and the category of π-divisible $\mathcal{O}$-modules over R, for R a complete Noetherian local ring with perfect residue field of characteristic p. More concretely, we have an equivalence

$$\mathrm{BT}_{\mathcal{O}} : \{\text{Dieudonné } \mathcal{O}\text{-displays over } R\} \to \{\pi\text{-divisible } \mathcal{O}\text{-modules over } R\},$$

which is compatible with duality. See [2, Section 1] for more details of these results.

In [15, 21], the authors introduced frames and windows, which are generalizations of the notion of displays, to study p-divisible groups. In particular, an equivalence between the category of Breuil windows over R and the category of p-divisible groups over R is established for R, if R is a complete Noetherian regular local ring with perfect residue field of characteristic p. As an application, Vasiu and Zink [22] proved some boundedness results for commutative finite flat group schemes over a discrete valuation ring of mixed characteristic $(0, p)$. Similar results and generalizations are also obtained in Breuil [5], Bondarko [4], Kisin [12], Savitt [19], Liu [16, 17], Cais-Liu [6], etc. See the paper [22, Section 1] for a detail introduction on the history of earlier results. The main goal of this paper is to generalize the results in [15, 21, 22]. We explain our main results more precisely in what follows.

In Section 2.1, we define $\mathcal{O}$-frames and $\mathcal{O}$-windows (Definitions 2.1 and 2.3). Let R be a complete Noetherian regular local $\mathcal{O}$-algebra with perfect residue field k of characteristic p. On one hand, there is an $\mathcal{O}$-frame attached to R given by

$$\mathcal{D}_R = (\widehat{W}_{\mathcal{O}}(R), \widehat{I}_{\mathcal{O}}(R), R, {}^F, {}^{V^{-1}}).$$

Here $\widehat{W}_{\mathcal{O}}(R)$ is a subring of the ring of ramified Witt vectors $W_{\mathcal{O}}(R)$, ^{F} and ^{V} are the Frobenius and Verschibung morphisms, respectively, $\widehat{I}_{\mathcal{O}}(R) = {}^V\widehat{W}_{\mathcal{O}}(R)$. See Section 3.1 for a detailed construction. For the definition of the functor $W_{\mathcal{O}}$ and its properties, we refer to [2, Section 1.2.1], [9, Section 1.2] and [11]. Note that Hazewinkel [11] used a different set of notations. In particular, the functor $W_{\mathcal{O}}$, the Frobenius map ^{F}, the Verschiebung map ^{V}, the Cartier map Δ, the n-th Witt polynomial w_n in [2] and this paper are denoted by $W^F_{q,\infty}$, f, V, E, $w^F_{q,n}$, respectively, in [11, Theorem 6.17]. On the other hand, we may choose a ring epimorphism

$$\mathfrak{S} := W_{\mathcal{O}}(k)[[x_1, \ldots, x_r]] \xrightarrow{\mathfrak{h}} R$$

such that $x_i \mapsto t_i$ for $1 \leq i \leq r$, where $(t_i \in \mathfrak{m}_R \mid 1 \leq i \leq r)$ is a regular system of parameters of R. There exists $f(x_1, \ldots, x_r) \in (x_1, \ldots, x_r)\mathfrak{S}$, such that $E = \pi - f(x_1, \ldots, x_r) \in \mathrm{Ker}(\mathfrak{h})$. Then there is another $\mathcal{O}$-frame attached to R given by

$$\mathcal{B}_R = (\mathfrak{S}, E\mathfrak{S}, R, \sigma, \sigma_1).$$

Here $\sigma : \mathfrak{S} \to \mathfrak{S}$ is the morphism that extends the Frobenius on $W_{\mathcal{O}}(k)$ and $\sigma(x_i) = x_i^q$ for $1 \leq i \leq r$, where q is the cardinality of $\mathcal{O}/\pi\mathcal{O}$, $\sigma_1 : E\mathfrak{S} \to \mathfrak{S}$ is defined by $\sigma_1(Ef) = \sigma(f)$. See Section 3.3 for more details.

A *Breuil $\mathcal{O}$-window relative to* $\mathfrak{S} \to R$ is a pair (Q, ϕ), where Q is a free $\mathfrak{S}$-module of finite rank, $\phi : Q \to Q^{(\sigma)} := Q \otimes_{\mathfrak{S},\sigma} \mathfrak{S}$ is an $\mathfrak{S}$-linear map with cokernel annihilated by E. With the above notation, we have the following result.

Theorem 1.1. *Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let π be a uniformizer of $\mathcal{O}$. Let R be a complete Noetherian regular local $\mathcal{O}$-algebra with perfect residue field of characteristic p. Then the following categories are equivalent:*

(1) *the category of π-divisible $\mathcal{O}$-modules over R;*
(2) *the category of $\mathcal{O}$-windows over the frame $\mathcal{D}_R$;*
(3) *the category of $\mathcal{O}$-windows over the frame $\mathcal{B}_R$;*
(4) *the category of Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$.*

By a special $\mathcal{O}$-group, we mean a finite flat group scheme which is the kernel of an isogeny of π-divisible $\mathcal{O}$-modules. To study these objects, we define Breuil $\mathcal{O}$-modules. A *Breuil $\mathcal{O}$-module relative to* $\mathfrak{S} \to R$ is a pair (M, ϕ), where M is an $\mathfrak{S}$-module of projective dimension at most one and annihilated by a power of π, $\phi : M \to M^{(\sigma)}$ is an $\mathfrak{S}$-linear map whose cokernel is annihilated by E. Following from Theorem 1.1, we prove the following result in Section 3.5.

Theorem 1.2. *Let $\mathcal{O}$ and R be as in Theorem* 1.1*. Then the following two categories are equivalent:*

(1) *the category of special $\mathcal{O}$-groups over R;*
(2) *the category of Breuil $\mathcal{O}$-module relative to $\mathfrak{S} \to R$.*

As an application of Theorem 1.2, we generalize the boundedness result in [22] and obtain the following result.

Theorem 1.3. *Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π. Let $R \in \mathrm{Alg}_{\mathcal{O}}$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with fraction field K and residue field k. There exists a nonnegative integer s that depends only on R and that has the following property. Let G and H be two special $\mathcal{O}$-groups over R. For each homomorphism $f : G \to H$ whose generic fiber $f_K : G_K \to H_K$ is an isomorphism, there exists a homomorphism $f' : H \to G$ such that $f' \circ f = \pi^s \,\mathrm{id}_G$ and $f \circ f' = \pi^s \,\mathrm{id}_H$. Therefore the special fiber homomorphism $f_k : G_k \to H_k$ has a kernel and a cokernel annihilated by π^s.*

This result has interesting consequences. In particular, we prove the following results in Section 4.5.

Corollary 1.4. *Let R and K be as in Theorem* 1.3*. The following two claims hold.*

(1) *Let G and H be special $\mathcal{O}$-groups over R. Assume that the ramification degree of R over $\mathcal{O}$ is less than or equal to $(q-2)$. If G_K and H_K are isomorphic, then G and H are isomorphic.*
(2) *Let X and Y be π-divisible $\mathcal{O}$-modules over R. Then the natural map*

$$\mathrm{Hom}_{\mathcal{O}}(X, Y) \to \mathrm{Hom}_{\mathcal{O}}(X_K, Y_K)$$

is a bijection.

The first claim generalizes a result of Raynaud [18] and the second claim generalizes a result of Tate [20].

The content of the paper is as follows. In Section 2, we introduce $\mathcal{O}$-frames and $\mathcal{O}$-windows (Definitions 2.1 and 2.3) and prove some basic properties of these objects. In particular, in Theorems 2.12 and 2.15, we prove that a morphism of frames $\alpha : \mathcal{F} \to \mathcal{F}'$ is (nil)-crystalline under some conditions, i.e., it induces an equivalence between the category of (nilpotent) $\mathcal{F}$-windows and the category of (nilpotent) $\mathcal{F}'$-windows. This allows us to translate properties between different bases.

In Section 3, we introduce various $\mathcal{O}$-frames with special properties. The Dieudonné $\mathcal{O}$-frame $\mathcal{D}_R$ attached to R is defined in Section 3.1. The windows over $\mathcal{D}_R$ are the same as Dieduonné $\mathcal{O}$-displays over R. Then by [2, Theorem 1.5], the first category and the second category in Theorem 1.1 are equivalent. The Breuil $\mathcal{O}$-frame $\mathcal{B}_R$ attached to R is then defined in Section 3.3. A key property of $\mathcal{B}_R$ is that it is a κ-$\mathcal{O}$-frame (Definition 3.8). Thus there exists a morphism of $\mathcal{O}$-frames $\kappa : \mathcal{B}_R \to \mathcal{D}_R$. It turns out that this morphism κ is crystalline (Theorem 3.13). Then Theorem 1.1 follows by combining Theorem 3.13 and Proposition 3.18.

In Section 4, we prove Theorem 1.3. An explicit description of s is given at the beginning of Section 4.4, following from the computations in Section 4.3. In Section 4.5, we deduce some corollaries from Theorem 1.3.

2. $\mathcal{O}$-frames and $\mathcal{O}$-windows

2.1. **Definitions.** In this section, we introduce $\mathcal{O}$-frames and $\mathcal{O}$-windows following [15, Section 2], [1, Section 3.1], and [2, Section 3]. Most of the notions are generalizations from the paper [15]. Let $\mathcal{O}$ be a commutative unitary ring, $0 \neq \pi \in \mathcal{O}$ not a zero divisor, and q a power of p. We call the triple $(\mathcal{O}, \pi, q)$ a *ramification ring structure*, for short *RRS*, if $p \in \pi\mathcal{O}$ and $x \equiv x^q \pmod{\pi}$ for all $x \in \mathcal{O}$.

Definition 2.1. Let $(\mathcal{O}, \pi, q)$ be an RRS. An *$\mathcal{O}$-frame* is a quintuple $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$, where S is an $\mathcal{O}$-algebra, $I \subset S$ is an ideal, $R = S/I$, $\sigma : S \to S$ is an $\mathcal{O}$-algebra homomorphism, and $\sigma_1 : I \to S$ is a σ-linear map of S-modules, such that the following conditions hold:

(1) $I + \pi S \subset \mathrm{Rad}(S)$.
(2) $\sigma(a) \equiv a^q \pmod{\pi S}$ for all $a \in S$.
(3) $\sigma_1(I)$ generates S as an S-module.

Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ and $\mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be two $\mathcal{O}$-frames. A *morphism of $\mathcal{O}$-frames* $\alpha : \mathcal{F} \to \mathcal{F}'$ is an $\mathcal{O}$-algebra homomorphism $\alpha : S \to S'$, such that $\alpha(I) \subset I'$, $\sigma'\alpha = \alpha\sigma$, $\sigma_1'\alpha = u \cdot \alpha\sigma_1$ for a unit $u \in S'$. In order to specify u, we also call α a *u-homomorphism.* If $u = 1$, then α is called *strict.*

Note that in the definition, R is determined by S and I. We take R as part of the data because it serves as the base of the objects that we consider later and it is convenient to include it in the quintuple. Let R be an $\mathcal{O}$-algebra. A simple example is the so-called *Witt $\mathcal{O}$-frame attached to R* given by

$$\mathcal{W}_{\mathcal{O},R} := (W_{\mathcal{O}}(R), I_{\mathcal{O}}(R) := {}^{V}W_{\mathcal{O}}(R), R = W_{\mathcal{O}}(R)/I_{\mathcal{O}}(R), {}^{F}, {}^{V^{-1}}).$$

Remark 2.2.

(1) If $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ is an $\mathcal{O}$-frame, then there exists a unique element $\theta = \theta_{\mathcal{F}} \in S$ such that $\sigma(a) = \theta\sigma_1(a)$ for all $a \in I$ ([15, Lemma 2.2] and [1, Lemma 3.1.2]). Indeed, by definition, the map $\sigma_1^\sharp : I^{(\sigma)} \to S$ is surjective. Here $\sigma_1^\sharp$ is the linearization of σ_1. Choose $b \in I^{(\sigma)}$ such that $\sigma_1^\sharp(b) = 1$ and define $\theta = \sigma^\sharp(b)$, then for all $a \in I$, we have $\sigma(a) = \sigma_1^\sharp(b)\sigma(a) = \sigma_1^\sharp(ba) = \sigma^\sharp(b)\sigma_1(a) = \theta\sigma_1(a)$.
(2) Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a u-homomorphism of $\mathcal{O}$-frames. Let $\mathcal{F}''$ be the frame obtained from $\mathcal{F}'$ by replacing σ_1' by $u^{-1}\sigma_1'$. Then $\alpha : S \to S'$ induces a strict morphism of $\mathcal{O}$-frames $\mathcal{F} \to \mathcal{F}''$.

Definition 2.3. Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ be an $\mathcal{O}$-frame. An *$\mathcal{O}$-window* over $\mathcal{F}$, or an *$\mathcal{F}$-window*, is a quadruple $\mathcal{P} = (P, Q, F, F_1)$, where P is a finitely generated projective S-module, $Q \subset P$ is a submodule, $F : P \to P$ and $F_1 : Q \to P$ are σ-linear maps of S-modules, such that the following conditions hold:

(1) There is a decomposition $P = T \oplus L$ with $Q = IT \oplus L$. Such a decomposition is called a *normal decomposition* of $\mathcal{P}$.
(2) $F_1(ax) = \sigma_1(a)F(x)$ for all $a \in I$ and $x \in P$.
(3) $F_1(Q)$ generates P as an S-module.

If $\mathcal{P} = (P, Q, F, F_1)$ is an $\mathcal{F}$-window, define a morphism of S-modules $V^\sharp : P \to S \otimes_{S,\sigma} P$ by $V^\sharp(F_1 y) = 1 \otimes y$ for all $y \in Q$ and $V^\sharp(Fx) = \theta \otimes x$ for all $x \in P$. Here $\theta = \theta_\mathcal{F} \in S$ is the element in Remark 2.2. Let $(V^N)^\sharp$ be the composition of the following maps:

$$P \xrightarrow{V^\sharp} S \otimes_{S,\sigma} P \xrightarrow{\mathrm{id} \otimes V^\sharp} S \otimes_{S,\sigma} (S \otimes_{S,\sigma} P) \to \cdots \to S \otimes_{S,\sigma^N} P.$$

We say that $\mathcal{P}$ is *nilpotent* if $(V^N)^\sharp \equiv 0 \pmod{I + \pi S}$ for some $N \in \mathbb{Z}_{>0}$.

Denote by $\mathrm{Win}_\mathcal{F}$ (respectively $\mathrm{NilpWin}_\mathcal{F}$) the category of $\mathcal{F}$-windows (respectively the category of nilpotent $\mathcal{F}$-windows).

Remark 2.4. The operator F is determined by F_1. Indeed, assume that $\sigma_1^\sharp(b) = 1$ with $b \in I^{(\sigma)}$. Then $F(x) = F_1^\sharp(bx)$ for all $x \in P$. In particular, $F(x) = \theta F_1(x)$ if $x \in Q$.

2.2. **Structure equation.** Let $\mathcal{P} = (P, Q, F, F_1)$ be a window over the $\mathcal{O}$-frame $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$. By definition, we may write $P = T \oplus L$, $Q = IT \oplus L$. Thus $P/Q = T/IT$. Let $F^\sharp : S \otimes_{S,\sigma} T \to T$ be the morphism defined by $s \otimes t \mapsto s \cdot F(t)$ for all $s \in S$ and $t \in T$. Let $F_1^\sharp$ be the linearization of F_1. We obtain a morphism

$$F^\sharp \oplus F_1^\sharp : (S \otimes_{S,\sigma} T) \oplus (S \otimes_{S,\sigma} L) \to P.$$

We may write $F^\sharp \oplus F_1^\sharp = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, with $A : S \otimes_{S,\sigma} T \to T$, $B : S \otimes_{S,\sigma} L \to T$, $C : S \otimes_{S,\sigma} T \to L$, $D : S \otimes_{S,\sigma} L \to L$. In the case that T and L are free S-modules, the morphisms A, B, C, D may be represented by matrices.

Define two morphisms $\sigma : P \to S \otimes_{S,\sigma} P$ by $t \mapsto 1 \otimes t$ (for all $t \in P$) and $\sigma_1 : I \otimes_S T \to S \otimes_{S,\sigma} T$ by $a \otimes t \mapsto \sigma_1(a) \otimes t$ (for all $a \in I$ and $t \in T$). Let $y \in I \otimes_S T \subset Q = (I \otimes_S T) \oplus L$, $l \in L$, $t \in T$. The following is called the *structure equation* of $\mathcal{P}$:

$$\begin{cases} F_1 \begin{pmatrix} y \\ l \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma_1(y) \\ \sigma(l) \end{pmatrix}, \\ F \begin{pmatrix} t \\ l \end{pmatrix} = \begin{pmatrix} A & \theta B \\ C & \theta D \end{pmatrix} \begin{pmatrix} \sigma(t) \\ \sigma(l) \end{pmatrix}. \end{cases} \tag{2.1}$$

Conversely, the equation (2.1) defines an $\mathcal{F}$-window if and only if $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : S \otimes_{S,\sigma} P \to P$ is an isomorphism. In other words, we have the following result (cf. [15, Lemma 2.6]).

Lemma 2.5. *Let $\mathcal{F}$ be an $\mathcal{O}$-frame. Let $P = T \oplus L$ be a finitely generated projective S-module and $Q = IT \oplus L$. Then the set of $\mathcal{F}$-window structures (P, Q, F, F_1) on these modules is bijective to the set of σ-linear isomorphisms $\Psi : T \oplus L \to P$.*

2.3. **Base changes.**

Definition 2.6 (Cf. [15, Definition 2.9])**.** Let $\mathcal{P}$ (respectively $\mathcal{P}'$) be an $\mathcal{F}$-window (respectively $\mathcal{F}'$-window). Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a u-homomorphism of $\mathcal{O}$-frames. A *homomorphism of* $\mathcal{O}$*-windows* $g : \mathcal{P} \to \mathcal{P}'$ over α, also called an α*-homomorphism*, is an S-linear map $g : P \to P'$ with $g(Q) \subset Q'$, such that $F'g = gF$ and $F_1'g = u \cdot gF_1$. A homomorphism of $\mathcal{F}$-windows is an $\mathrm{id}_{\mathcal{F}}$-homomorphism in the previous sense.

Definition 2.7. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a u-homomorphism of $\mathcal{O}$-frames. Let $\mathcal{P} = (P, Q, F, F_1)$ be a window over $\mathcal{F}$ with structure equation defined by Ψ (Lemma 2.5). The *base change* $\alpha_*\mathcal{P}$ of $\mathcal{P}$ with respect to α is the $\mathcal{F}'$-window defined by $(\alpha_*L, \alpha_*T, \Psi')$, where $\alpha_*L = S' \otimes_S L$, $\alpha_*T = S' \otimes_S T$, and $\Psi'(s' \otimes l) = u\sigma'(s') \otimes \Psi(l)$, $\Psi'(s' \otimes t) = \sigma'(s') \otimes \Psi(t)$, for all $s' \in S'$, $t \in T$, $l \in L$.

Similarly to [15, Lemma 2.10], we have the following result.

Lemma 2.8. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of $\mathcal{O}$-frames. Let $\mathcal{P}$ be an $\mathcal{F}$-window and $\alpha_*\mathcal{P}$ the base change of $\mathcal{P}$ with respect to α. Then the α-homomorphism of $\mathcal{O}$-windows $\mathcal{P} \to \alpha_*\mathcal{P}$ induces a bijection $\mathrm{Hom}_{\mathcal{F}'}(\alpha_*\mathcal{P}, \mathcal{P}') = \mathrm{Hom}_\alpha(\mathcal{P}, \mathcal{P}')$ for any $\mathcal{F}'$-window $\mathcal{P}'$.*

2.4. **Limits.** In the following, we define the limits of $\mathcal{O}$-frames, limits of $\mathcal{O}$-windows, and dual $\mathcal{O}$-windows. We follow the corresponding parts in [15, Section 2].

Assume that for each positive integer n we have an $\mathcal{O}$-frame

$$\mathcal{F}_n = (S_n, I_n, R_n, \sigma_n, \sigma_{1n})$$

and a strict morphism of $\mathcal{O}$-frames $\pi_n : \mathcal{F}_{n+1} \to \mathcal{F}_n$ such that the maps $S_{n+1} \to S_n$ and $I_{n+1} \to I_n$ are surjective and $\mathrm{Ker}(\pi_n) \subset \mathrm{Rad}(S_{n+1})$. Define $\varprojlim \mathcal{F}_n = (S, I, R, \sigma, \sigma_1)$ by letting $S = \varprojlim S_n$, $I = \varprojlim_n I_n$, $R = S/I$, $\sigma = \varprojlim \sigma_n$, $\sigma_1 = \varprojlim \sigma_{1n}$. It is easy to check that $\varprojlim \mathcal{F}_n$ is an $\mathcal{O}$-frame. An $\mathcal{F}_*$*-window* is a system $\mathcal{P}_*$ of $\mathcal{F}_n$-windows $\mathcal{P}_n$ together with isomorphisms $\pi_{n*}\mathcal{P}_{n+1} \cong \mathcal{P}_n$.

Lemma 2.9. *The category of $(\varprojlim \mathcal{F}_n)$-windows is equivalent to the category of $\mathcal{F}_*$-windows.*

Proof. This is entirely similar to [15, Lemma 2.12]. In particular, from the proof of [15, Lemma 2.12], for any $\mathcal{F}_*$-window $\mathcal{P}_*$, the corresponding $(\varprojlim \mathcal{F}_n)$-window is given by $(\varprojlim \mathcal{P}_n) = (P, Q, F, F_1)$ with $P = \varprojlim P_n$, etc. □

2.5. **Dual $\mathcal{O}$-windows.** Let $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{P}''$ be windows over an $\mathcal{O}$-frame $\mathcal{F}$. A *bilinear form* of $\mathcal{F}$-windows $\beta : \mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$ is an S-bilinear map $\beta : P \times P' \to P''$ such that $\beta(Q \times Q') \subset Q''$ and

$$\beta(F_1(x), F_1'(x')) = F_1''(\beta(x, x'))$$

for all $x \in Q$ and $x' \in Q'$. Let $\mathcal{F}$ denote the $\mathcal{F}$-window (S, I, σ, σ_1) and let $\mathrm{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{F})$ be the set of all bilinear forms. For every $\mathcal{F}$-window $\mathcal{P}$, there is a unique dual $\mathcal{F}$-window $\mathcal{P}^t$ that represents the functor $\mathrm{Bil}(\mathcal{P} \times -, \mathcal{F})$, i.e., $\mathrm{Bil}(\mathcal{P} \times \mathcal{P}', \mathcal{F}) \cong \mathrm{Hom}(\mathcal{P}', \mathcal{P}^t)$ for any $\mathcal{F}$-window $\mathcal{P}'$. Indeed, $\mathcal{P}^t$ can be described as follows. Let $\mathcal{P} = (P, Q, F, F_1)$, then

$$\mathcal{P}^t = (P^\vee, \tilde{Q}, F^t, F_1^t),$$

where $\tilde{Q} = \{x \in P^\vee \mid x(Q) \subset I\}$ and $M^\vee = \mathrm{Hom}_S(M, S)$ for any S-module M . If $P = T \oplus L$ is a normal decomposition and $Q = IT \oplus L$, then $P^\vee = L^\vee \oplus T^\vee$ and $\tilde{Q} = IL^\vee \oplus T^\vee$. There is a natural isomorphism $\mathcal{P}^{tt} \cong \mathcal{P}$. See [15, Section 2] and the references there for more details. We have the following result.

Lemma 2.10. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a u-homomorphism of $\mathcal{O}$-frames. Let $c \in S'$ be a unit such that $c^{-1}\sigma'(c) = u$. For any $\mathcal{F}$-window $\mathcal{P}$ there is a natural isomorphism (depending on c)*

$$\alpha_*(\mathcal{P}^t) \cong (\alpha_*\mathcal{P})^t.$$

Proof. This is entirely similar to [15, Lemma 2.14]. The given bilinear form $\mathcal{P} \times \mathcal{P}^t \to \mathcal{F}$ induces a bilinear form $\alpha_*\mathcal{P} \times \alpha_*(\mathcal{P}^t) \to \mathcal{F}''$, where $\mathcal{F}'' = (S', I', u\sigma', u\sigma_1')$ is considered as an $\mathcal{F}'$-window. Moreover, the multiplication by c induces an isomorphism of $\mathcal{F}'$-windows $\mathcal{F}'' \cong \mathcal{F}'$. The composition gives us a bilinear form $\alpha_*\mathcal{P} \times \alpha_*(\mathcal{P}^t) \to \mathcal{F}'$, which induces the isomorphism $\alpha_*(\mathcal{P}^t) \cong (\alpha_*\mathcal{P})^t$. The lemma follows. □

2.6. **Crystalline homomorphisms.**

Definition 2.11 (Cf. [15, Definition 3.1]). A morphism of $\mathcal{O}$-frames $\alpha : \mathcal{F} \to \mathcal{F}'$ is called *crystalline* if the base change functor $\alpha_* : \mathrm{Win}_{\mathcal{F}} \to \mathrm{Win}_{\mathcal{F}'}$ is an equivalence of categories. It is called *nil-crystalline* if the base change functor $\alpha_* : \mathrm{NilpWin}_{\mathcal{F}} \to \mathrm{NilpWin}_{\mathcal{F}'}$ is an equivalence of categories.

Corresponding to [15, Theorems 3.2 and 10.3], we have the following results. The proofs here are similar to the proofs in [15], which are variations of the proofs of [24, Theorem 44] and [23, Theorem 3].

Theorem 2.12. *Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ and $\mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be two $\mathcal{O}$-frames. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of $\mathcal{O}$-frames such that $\alpha : S \to S'$ is surjective. Let $C = \mathrm{Ker}(S \to S')$. Assume that $R = R'$, $\sigma_1(C) \subset C$, $\sigma(C) = 0$, and σ_1 is elementwise nilpotent on C. Assume further that finitely generated projective S'-modules lift to projective S-modules. Then the morphism α is crystalline.*

Note that $C \subset I$ since $R = R'$. Thus $\sigma_1(C)$ makes sense.

Proof. The functor α_* is essentially surjective since normal representations (T, L, Ψ) can be lifted from $\mathcal{F}'$ to $\mathcal{F}$. It suffices to show that α_* is fully faithful. Since a homomorphism $g : \mathcal{P} \to \mathcal{P}'$ can be encoded by the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ g & 1 \end{smallmatrix}\right)$ on $\mathcal{P} \oplus \mathcal{P}'$, it suffices to show that α_* is fully faithful on automorphisms. This follows from the following lemma. □

Lemma 2.13. *In the situation of the theorem with $\mathcal{F} = \mathcal{F}'$, assume that $\mathcal{P} = (P, Q, F, F_1)$ and $\mathcal{P}' = (P, Q, F', F_1')$ are two $\mathcal{F}$-windows such that $F \equiv F' \pmod{C}$ and $F_1 \equiv F_1' \pmod{C}$. Then there is a unique $\mathcal{F}$-window isomorphism $g : \mathcal{P} \to \mathcal{P}'$ with $g \equiv \mathrm{id} \pmod{C}$.*

Proof. By assumption, we may write $F' = F + \epsilon$ and $F_1' = F_1 + \eta$, where $\epsilon : P \to CP$ and $\eta : Q \to CP$ are σ-linear maps. Let $g = 1 + \omega$, where $\omega : P \to CP$ is an arbitrary S-linear map. By Remark 2.4, g induces an isomorphism of $\mathcal{F}$-windows if and only if $gF_1 = F_1'g$ on Q, which is equivalent to

$$\eta = \omega F_1 - F_1'\omega. \tag{2.2}$$

Fix a normal decomposition $P = L \oplus T$, $Q = L \oplus IT$. Let $l + at \in Q$ with $l \in L$, $t \in T$, and $a \in I$. Then

$$\begin{aligned} \eta(l+at) &= \eta(l) + \sigma_1(a)\epsilon(t), \\ \omega(F_1(l+at)) &= \omega(F_1(l)) + \sigma_1(a)\omega(F(t)), \\ F_1'(\omega(l+at)) &= F_1'(\omega(l)) + \sigma_1(a)F'(\omega(t)). \end{aligned} \tag{2.3}$$

If $c \in C$ and $x \in P$, then $F'(cx) = \sigma(c)F'(x) = 0$ since $\sigma(C) = 0$. Therefore, $F'\omega = 0$. The equation (2.2) is equivalent to

$$\begin{cases} \epsilon = \omega F & \text{on } T, \\ \eta = \omega F_1 - F_1'\omega & \text{on } L. \end{cases} \tag{2.4}$$

By definition, $\Psi := F_1 + F : L \oplus T \to P$ is a σ-linear isomorphism. To give ω is equivalent to giving a pair of σ-linear maps

$$\omega_L = \omega F_1 : L \to CP, \quad \omega_T = \omega F : T \to CP.$$

Let $\lambda : L \to L^{(\sigma)}$ be the composition $L \subset P \xrightarrow{(\Psi^\sharp)^{-1}} L^{(\sigma)} \oplus T^{(\sigma)} \xrightarrow{\text{projection}} L^{(\sigma)}$ and $\tau : L \to T^{(\sigma)}$ be the composition $L \subset P \xrightarrow{(\Psi^\sharp)^{-1}} L^{(\sigma)} \oplus T^{(\sigma)} \xrightarrow{\text{projection}} T^{(\sigma)}$. Then $\omega|_L = \omega_L^\sharp\lambda + \omega_T^\sharp\tau$. Thus equation (2.4) is equivalent to

$$\begin{cases} \omega_T = \epsilon|_T, \\ \omega_L - F_1'\omega_L^\sharp\lambda = \eta|_L + F_1'\omega_T^\sharp\tau. \end{cases} \tag{2.5}$$

By assumption, σ_1 is elementwise nilpotent on C. Thus the endomorphism F_1' on CP is elementwise nilpotent since $F_1'(cx) = \sigma_1(c)F'(x)$ for all $c \in C$ and $x \in P$. Let $\mathcal{H}$ be the abelian group of σ-linear maps $L \to CP$. Define $U \in \operatorname{End}\mathcal{H}$ by $U(\omega_L) = F_1'\omega_L^\sharp\lambda$. Since L is finitely generated, U is also elementwise nilpotent, which implies that $(1 - U)$ is bijective. Therefore, equation (2.5) has a unique solution in (ω_L, ω_T). The lemma follows. □

Corollary 2.14. *Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ and $\mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be two $\mathcal{O}$-frames. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of $\mathcal{O}$-frames such that $\alpha : S \to S'$ is surjective and $R = R'$. Let $C = \operatorname{Ker}(S \to S')$. Assume that there is a finite filtration $C = C_0 \supset \cdots \supset C_n = 0$ with $\sigma(C_i) \subset C_{i+1}$ and $\sigma_1(C_i) \subset C_i$ such that σ_1 is elementwise nilpotent on C_i/C_{i+1}. Assume further that finitely generated projective S'-modules lift to projective S-modules. Then the morphism α is crystalline.*

Proof. The morphism α factors into $\mathcal{F} \to \mathcal{F}'' \to \mathcal{F}'$ where $\mathcal{F}''$ is determined by $S'' = S/C_1$. By induction, we may assume that $\sigma(C) = 0$. The corollary follows immediately. □

Theorem 2.15. *Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ and $\mathcal{F}' = (S', I', R', \sigma', \sigma_1')$ be two $\mathcal{O}$-frames. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of $\mathcal{O}$-frames such that $\alpha : S \to S'$ is surjective. Let $C = \operatorname{Ker}(S \to S')$. Assume that $R = R'$, $\sigma_1(C) \subset C$, and $\sigma(C) = 0$. Assume further that finitely generated projective S'-modules lift to projective S-modules. Let $J = (I, \pi)$. If $J^nC = 0$ for some large integer n, then the morphism α is nil-crystalline.*

Proof. The proof is the same as the proof of Theorem 2.12. In this case, $\mathcal{P}$ is nilpotent. Thus in the last paragraph of the proof of Lemma 2.13, λ is nilpotent modulo J^m for any $m \geq 1$. Since $J^n C = 0$, the endomorphism U is nilpotent. The theorem follows. □

Corollary 2.16. *In Theorem* 2.15*, the condition* $\sigma(C) = 0$ *is not necessary.*

Proof. Let $C_0 = C, C_1 = I^q + \pi C, \dots, C_n = I^{qn}C + \pi I^{q(n-1)}C + \dots + \pi^{n-1}IC + \pi^n C$. We claim that $\sigma_1(C_n) \subset C_n$, $\sigma(C_n) \subset C_{n+1}$.

Indeed, since the image of $\sigma_1 : I \to S$ generates S, we may write $1 = \sum_k s_k \sigma_1(a_k)$ for some $s_k \in S$ and $a_k \in I$. Then the number θ in Remark 2.2 is given by $\theta = \sum_k s_k \sigma(a_k)$. Since $\sigma(a_k) \equiv a_k^q \pmod{\pi S}$, θ is an element in $I^q + \pi S$. The claims follow by induction.

Let N be large enough such that $C_N = 0$. Consider the chain of morphisms

$$S = S/C_N \to S/C_{N-1} \to \dots \to S/C = S'.$$

Each map $S/C_i \to S/C_{i-1}$ induces a morphism of $\mathcal{O}$-frames which satisfies the assumptions in Theorem 2.15. The corollary follows easily. □

2.7. **Hodge filtration.**

Definition 2.17. Let $\mathcal{P} = (P, Q, F, F_1)$ be a window over $\mathcal{F}$. The *Hodge filtration* of $\mathcal{P}$ is the submodule

$$Q/IP \subset P/IP.$$

The following result is entirely similar to [15, Lemma 4.2].

Lemma 2.18. *Let* $\alpha : \mathcal{F} \to \mathcal{F}'$ *be a strict morphism of* $\mathcal{O}$*-frames such that* $S = S'$*. Hence* $R \to R'$ *is surjective and* $I \subset I'$*. Then the category* $\mathrm{Win}_{\mathcal{F}}$ *of windows over* $\mathcal{F}$ *is equivalent to the category of pairs* $(\mathcal{P}', V)$*, where* $\mathcal{P}'$ *is an* $\mathcal{F}'$*-window and* $V \subset P'/IP'$ *is a direct summand and is a lift of the Hodge filtration of* $\mathcal{P}'$*.*

Proof. The equivalence is given by the functor $\mathcal{P} = (P, Q, F, F_1) \mapsto (\alpha_* \mathcal{P}, Q/IP)$. In our case, if $\mathcal{P} = (P, Q, F, F_1)$, then $\alpha_*(\mathcal{P}) = (P, I'P + Q, F, F_1)$, where $F_1(ax) = \sigma_1(a)F(x)$ for all $a \in I'$ and $x \in P$. It is easy to see that this functor is fully faithful. We show that it is also essentially surjective. Let $(\mathcal{P}' = (P', Q', F', F_1'), V \subset P'/IP')$ be such a pair. Let $P = P'$, $Q \subset P$ the preimage of V of the map $P = P' \to P'/IP'$. Then $IP = IP' \subset Q \subset Q'$. Let $F_1 : Q \to P$ be the restriction of $F_1' : Q' \to P'$. We check that $\mathcal{P} = (P, Q, F = F', F_1)$ is an $\mathcal{F}$-window. It suffices to verify that $F_1 : Q \to P$ is a σ-linear epimorphism. Let $P' = L \oplus T$ be a normal decomposition. Thus $Q' = L \oplus I'T$. By changing the decomposition by $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ with a morphism $c : L \to I'T$, we may assume that $V = L/IL$. Therefore $Q = L \oplus IT$. To check that $F_1 : Q \to P$ is σ-linear epimorphic, it is equivalent to proving that

$$F \oplus F_1 : T \oplus L \to P$$

is a σ-linear isomorphism. This is true since (P, Q', F, F_1) is an $\mathcal{F}'$-window. The lemma follows. □

Remark 2.19. Assume that $\alpha : \mathcal{F} \to \mathcal{F}'$ is a strict morphism of $\mathcal{O}$-frames such that $S \to S'$ is surjective and $I' = IS'$. If we may factor α into strict $\mathcal{O}$-frame morphisms

$$(S, I, R, \sigma, \sigma_1) \xrightarrow{\alpha_1} (S, I'', R', \sigma, \sigma_1'') \xrightarrow{\alpha_2} (S', I', R', \sigma', \sigma_1'),$$

such that α_2 is crystalline, then the category of $\mathcal{F}$-windows is equivalent to the category of $\mathcal{F}'$-windows equipped with a certain lift of Hodge filtration. We explain this idea with an explicit example in Section 2.8.

2.8. **$\mathcal{O}$-pd-thickenings.** We recall the definition and basic properties of $\mathcal{O}$-pd-structures following [8, Section 7] and [10, Section B.5.1].

Definition 2.20. Let R be an $\mathcal{O}$-algebra and $\mathfrak{a} \subset R$ an ideal. An *$\mathcal{O}$-pd-structure on* $\mathfrak{a}$ is a map $\gamma : \mathfrak{a} \to \mathfrak{a}$, such that

(1) $\pi \cdot \gamma(x) = x^q$,
(2) $\gamma(r \cdot x) = r^q \cdot \gamma(x)$,
(3) $\gamma(x+y) = \gamma(x) + \gamma(y) + \sum_{0<i<q} \frac{1}{\pi} \binom{q}{i} \cdot x^i \cdot y^{q-i}$

hold for all $r \in R$ and $x, y \in \mathfrak{a}$. Let γ^n be the n-fold iteration of γ. We call γ *nilpotent* if $\mathfrak{a}^{[n]} = 0$ for all $n \gg 0$, where $\mathfrak{a}^{[n]} \subset \mathfrak{a}$ is generated by all products $\prod \gamma^{a_i}(x_i)$ with $x_i \in \mathfrak{a}$ and $\sum q^{a_i} \geq n$.

For each n, define

$$\alpha_n = \pi^{q^{n-1}+q^{n-2}+\cdots+q+1-n} \cdot \gamma^n : \mathfrak{a} \to \mathfrak{a}$$

and

$$\begin{aligned} w'_n : W_{\mathcal{O}}(\mathfrak{a}) &\to \mathfrak{a} \\ (x_0, x_1, \ldots, x_n, \ldots) &\mapsto \alpha_n(x_0) + \alpha_{n-1}(x_1) + \cdots + \alpha_1(x_{n-1}) + x_n. \end{aligned}$$

We call w'_n the *n-th divided Witt polynomial.* The main application of this structure is as follows (cf. [10, Lemma B.5.8]). Define on $\mathfrak{a}^{\mathbb{N}}$ a $W_{\mathcal{O}}(R)$-module structure by setting

$$\xi[a_0, a_1, \ldots] = [w_0(\xi)a_0, w_1(\xi)a_1, \ldots]$$

for all $\xi \in W_{\mathcal{O}}(R)$ and $[a_0, a_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$. Then we have an isomorphism of $W_{\mathcal{O}}(R)$-modules

$$\begin{aligned} \log : W_{\mathcal{O}}(\mathfrak{a}) &\to \mathfrak{a}^{\mathbb{N}} \\ \underline{a} = (a_0, a_1, \ldots) &\mapsto [w'_0(\underline{a}), w'_1(\underline{a}), \ldots]. \end{aligned}$$

Moreover, if γ is nilpotent, the above isomorphism induces an isomorphism

$$\log : \widehat{W}_{\mathcal{O}}(\mathfrak{a}) \to \mathfrak{a}^{\oplus \mathbb{N}}.$$

Here $\widehat{W}_{\mathcal{O}}(\mathfrak{a})$ is the object defined in Section 3.1. We may view $\mathfrak{a}$ as an ideal of $W_{\mathcal{O}}(\mathfrak{a})$ via the map $a \mapsto \tilde{a} = \log^{-1}([a, 0, \ldots])$. Since ^{F} acts on the right-hand side by

$${}^F[a_0, a_1, \ldots] = [\pi a_1, \pi a_2, \ldots, \pi a_i, \ldots]$$

for all $[a_0, a_1, \ldots] \in \mathfrak{a}^{\mathbb{N}}$, we obtain that, for the ideal $\mathfrak{a} \subset W_{\mathcal{O}}(\mathfrak{a})$, ${}^F\mathfrak{a} = 0$.

In this paper, as in [23], an *$\mathcal{O}$-pd-thickening* is a triple (S, R, γ), where S and R are $\mathcal{O}$-algebras with a surjection $S \to R$, γ is a nilpotent $\mathcal{O}$-pd-structure on $\mathrm{Ker}(S \to R)$.

Let (S, R, γ) be an $\mathcal{O}$-pd-thickening such that π is nilpotent in S and R. For R and S, we have the Witt $\mathcal{O}$-frames

$$\mathcal{W}_R = (W_{\mathcal{O}}(R), I_{\mathcal{O}}(R) = {}^V W_{\mathcal{O}}(R), R, {}^F, {}^{V^{-1}}),$$

$$\mathcal{W}_S = (W_{\mathcal{O}}(S), I_{\mathcal{O}}(S) = {}^V W_{\mathcal{O}}(S), S, {}^F, {}^{V^{-1}}),$$

where ^{F} and ^{V} are the corresponding Frobenius and Verschiebung, respectively.

The natural map $W_{\mathcal{O}}(S) \to W_{\mathcal{O}}(R)$ gives us a strict morphism of $\mathcal{O}$-frames $\mathcal{W}_S \to \mathcal{W}_R$. Let $J = I_{\mathcal{O}}(S) + W_{\mathcal{O}}(\mathfrak{a}) \subset W_{\mathcal{O}}(S)$. Since $W_{\mathcal{O}}(\mathfrak{a}) \setminus (I_{\mathcal{O}}(S) \cap W_{\mathcal{O}}(\mathfrak{a})) = \{[a, 0, \ldots, 0, \ldots] \mid a \in \mathfrak{a}\} \cong \mathfrak{a}$, we may extend ${}^{V^{-1}} : I_{\mathcal{O}}(S) \to W_{\mathcal{O}}(S)$ to $\sigma_1 : J \to W_{\mathcal{O}}(S)$ by setting

$$\sigma_1({}^{V}\eta) = \eta \quad (\text{for all } \eta \in W_{\mathcal{O}}(S)) \text{ and } \sigma_1(\tilde{a}) = 0 \quad (\text{for all } a \in \mathfrak{a}).$$

Thus we obtain a third $\mathcal{O}$-frame

$$\mathcal{W}_{S/R} = (W_{\mathcal{O}}(S), J, R, {}^{F}, \sigma_1).$$

The morphism $\mathcal{W}_S \to \mathcal{W}_R$ factors as $\mathcal{W}_S \to \mathcal{W}_{S/R} \xrightarrow{\alpha} \mathcal{W}_R$ in the obvious way.

Proposition 2.21. *The morphism α is nil-crystalline, that is, the categories* $\mathrm{NilpWin}_{\mathcal{W}_{S/R}}$ *and* $\mathrm{NilpWin}_{\mathcal{W}_R}$ *are equivalent.*

Proof. It suffices to check that $\alpha : \mathcal{W}_{S/R} \to \mathcal{W}_R$ factors through a finite chain of morphisms of $\mathcal{O}$-frames such that each morphism satisfies the conditions in Theorem 2.15.

For t big enough, we have

$$S = S/\pi^t\mathfrak{a} \to S/\pi^{t-1}\mathfrak{a} \to \cdots \to S/\mathfrak{a} = R.$$

This induces a chain of morphisms of $\mathcal{O}$-frames

$$\mathcal{W}_{S/R} = \mathcal{W}_{(S/\pi^t\mathfrak{a})/R} \to \mathcal{W}_{(S/\pi^{t-1}\mathfrak{a})/R} \to \cdots \to \mathcal{W}_{(S/\mathfrak{a})/R} = \mathcal{W}_R.$$

Note that $\mathrm{Ker}(W_{\mathcal{O}}(S/\pi^{i+1}\mathfrak{a}) \to W_{\mathcal{O}}(S/\pi^i\mathfrak{a})) = W_{\mathcal{O}}(\pi^i\mathfrak{a}/\pi^{i+1}\mathfrak{a})$. Using logarithmic coordinates, since $\pi^i\mathfrak{a}/\pi^{i+1}\mathfrak{a}$ is π-torsion, it is easy to see that ${}^{F}(W_{\mathcal{O}}(\pi^i\mathfrak{a}/\pi^{i+1}\mathfrak{a})) = 0$. The claim follows. ☐

Definition 2.22. Let $\mathcal{P} = (P, Q, F, F_1)$ be a nilpotent $\mathcal{W}_R$-window. The *Dieudonné crystal* $\mathbb{D}_{\mathcal{P}}$ is the functor that sends an $\mathcal{O}$-pd-thickening $S \to R$ to the finitely generated S-module $\tilde{P}/I_{\mathcal{O}}(S)\tilde{P}$, where $(\tilde{P}, \tilde{Q}, F, F_1)$ is the unique $\mathcal{W}_{S/R}$-window lifting $\mathcal{P}$.

Let $\mathrm{NilpWin}^S_{\mathcal{W}_R}$ be the category of pairs $(\mathcal{P}, V)$, where $\mathcal{P}$ is a $\mathcal{W}_R$-window and V is a lift of the Hodge filtration in $\mathbb{D}_{\mathcal{P}}(S)$. By Lemma 2.18, we have the following result.

Corollary 2.23. *The two categories* $\mathrm{NilpWin}_{\mathcal{W}_S}$ *and* $\mathrm{NilpWin}^S_{\mathcal{W}_R}$ *are equivalent.*

3. Breuil $\mathcal{O}$-frames

3.1. **Dieudonné $\mathcal{O}$-frames.** Let R be a local $\mathcal{O}$-algebra. Assume that R is an Artinian local ring with perfect residue field k. Let $\mathfrak{m} \subset R$ be the maximal ideal of R. Then we have the following exact sequence

$$0 \to W_{\mathcal{O}}(\mathfrak{m}) \to W_{\mathcal{O}}(R) \xrightarrow{\tau} W_{\mathcal{O}}(k) \to 0.$$

It admits a canonical section $\delta : W_{\mathcal{O}}(k) \xrightarrow{\Delta} W_{\mathcal{O}}(W_{\mathcal{O}}(k)) \to W_{\mathcal{O}}(R)$, which is a ring homomorphism commuting with F. Here Δ is the unique natural morphism (Cartier morphism) of $\mathcal{O}$-algebras

$$\Delta : W_{\mathcal{O}}(-) \longrightarrow W_{\mathcal{O}}(W_{\mathcal{O}}(-))$$

such that $\mathcal{W}(\Delta(x)) = [{}^{F^n}x]_{n \geq 0}$, where $\mathcal{W} = (w_0, w_1, \ldots)$. The Cartier morphism is the morphism E in [11, Theorem 6.17].

Since $\mathfrak{m}$ is nilpotent, we have a subalgebra of $W_{\mathcal{O}}(\mathfrak{m})$:

$$\widehat{W}_{\mathcal{O}}(\mathfrak{m}) = \{(x_0, x_1, \ldots) \in W_{\mathcal{O}}(\mathfrak{m}) \mid x_i = 0 \text{ for all but finitely many } i\}.$$

Note that $\widehat{W}_{\mathcal{O}}(\mathfrak{m})$ is stable under ^{F} and ^{V}.

Definition 3.1. In the case R is Artinian, we define the subring $\widehat{W}_{\mathcal{O}}(R) \subset W_{\mathcal{O}}(R)$ by

$$\widehat{W}_{\mathcal{O}}(R) = \{\xi \in W_{\mathcal{O}}(R) \mid \xi - \delta\tau(\xi) \in \widehat{W}_{\mathcal{O}}(\mathfrak{m})\}.$$

Again we have an exact sequence

$$0 \to \widehat{W}_{\mathcal{O}}(\mathfrak{m}) \to \widehat{W}_{\mathcal{O}}(R) \xrightarrow{\tau} W_{\mathcal{O}}(k) \to 0$$

with a canonical section δ of τ.

In the case R is Noetherian, we define $\widehat{W}_{\mathcal{O}}(R) := \varprojlim \widehat{W}_{\mathcal{O}}(R/\mathfrak{m}_R^n)$, where $\mathfrak{m}_R \subset R$ is the maximal ideal.

We also define $\widehat{I}_{\mathcal{O}}(R) = {}^V(\widehat{W}_{\mathcal{O}}(R))$.

The following result is proved in [2, Lemma 1.8].

Lemma 3.2. *$\widehat{W}_{\mathcal{O}}(R)$ is stable under ^{F} and ^{V}.*

Definition 3.3. The *Dieudonné $\mathcal{O}$-frame* attached to R is the frame

$$\mathcal{D}_R = (\widehat{W}_{\mathcal{O}}(R), \widehat{I}_{\mathcal{O}}(R), R, {}^F, {}^{V^{-1}}).$$

Remark 3.4.

(1) For the $\mathcal{O}$-frame $\mathcal{D}_R$, $\theta_{\mathcal{D}_R} = \pi$.
(2) Windows over $\mathcal{D}_R$ are Dieudonné $\mathcal{O}$-displays over R in the sense of [2, Section 5.1]. Note that $\widehat{W}_{\mathcal{O}}(R)$ is a local ring, therefore the normal decompositions exist automatically.
(3) The inclusion $\widehat{W}_{\mathcal{O}}(R) \to W_{\mathcal{O}}(R)$ induces a strict $\mathcal{O}$-frame morphism $\mathcal{D}_R \to \mathcal{W}_R$.
(4) Let S be another Noetherian local $\mathcal{O}$-algebra. A local $\mathcal{O}$-algebra homomorphism $S \to R$ induces a strict $\mathcal{O}$-frame morphism $\mathcal{D}_S \to \mathcal{D}_R$.

Let (S, R, γ) be an $\mathcal{O}$-pd-thickening and $\mathfrak{a}$ the kernel of $S \to R$. The discussion in Section 2.8 remains true if we replace the Witt $\mathcal{O}$-frames $\mathcal{W}_S$ and $\mathcal{W}_R$ by the Dieudonné $\mathcal{O}$-frames $\mathcal{D}_S$ and $\mathcal{D}_R$, respectively. More precisely, as $\mathcal{W}_{S/R}$, define

$$\mathcal{D}_{S/R} = (\widehat{W}_{\mathcal{O}}(S), \widehat{J}, R, {}^F, \sigma_1),$$

where $\widehat{J} = \widehat{I}_{\mathcal{O}}(S) + \widehat{W}_{\mathcal{O}}(\mathfrak{a})$. We have the following result.

Proposition 3.5. *Let (S, R, γ) be an $\mathcal{O}$-pd-thickening with π nilpotent in S and R. The following claims hold.*

(1) *The two categories* $\mathrm{NilpWin}_{\mathcal{D}_{S/R}}$ *and* $\mathrm{NilpWin}_{\mathcal{D}_R}$ *are equivalent.*
(2) *The two categories* $\mathrm{Win}_{\mathcal{D}_{S/R}}$ *and* $\mathrm{Win}_{\mathcal{D}_R}$ *are equivalent.*

Definition 3.6. Let $\mathcal{P} = (P, Q, F, F_1)$ be a $\mathcal{D}_R$-window. The *Dieudonné crystal* $\mathbb{D}_{\mathcal{P}}$ is the functor that sends an $\mathcal{O}$-pd-thickening $S \to R$ to the finitely generated S-module $\tilde{P}/I_{\mathcal{O}}(S)\tilde{P}$, where $(\tilde{P}, \tilde{Q}, F, F_1)$ is the unique $\mathcal{D}_{S/R}$-window lifting $\mathcal{P}$.

Let $\mathrm{Win}^S_{\mathcal{D}_R}$ (resp. $\mathrm{NilpWin}^S_{\mathcal{D}_R}$) be the category of pairs $(\mathcal{P}, V)$, where $\mathcal{P}$ is a $\mathcal{D}_R$-window (resp. nilpotent $\mathcal{D}_R$-window) and V is a lift of the Hodge filtration in $\mathbb{D}_{\mathcal{P}}(S)$. We have the following result.

Proposition 3.7. *Let (S, R, γ) be an $\mathcal{O}$-pd-thickening with π nilpotent in S and R. The following claims hold.*

(1) *The two categories* $\mathrm{NilpWin}_{\mathcal{D}_S}$ *and* $\mathrm{NilpWin}^S_{\mathcal{D}_R}$ *are equivalent.*
(2) *The two categories* $\mathrm{Win}_{\mathcal{D}_S}$ *and* $\mathrm{Win}^S_{\mathcal{D}_R}$ *are equivalent.*

3.2. **κ-$\mathcal{O}$-frames.**

Definition 3.8. A *κ-$\mathcal{O}$-frame* is an $\mathcal{O}$-frame $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ such that S and $W_{\mathcal{O}}(R)$ have no π-torsion, and $\sigma(\theta) - \theta^q \in \pi S^\times$.

Lemma 3.9. *Let R be an $\mathcal{O}$-algebra with $\pi \in \mathrm{Rad}(R)$. Let $u \in W_{\mathcal{O}}(R)$ and $r \in \mathbb{Z}_{\geq 0}$. Assume that $\pi^r u = (a_0, a_1, a_2, \ldots)$. Then the element u is a unit in $W_{\mathcal{O}}(R)$ if and only if a_r is a unit in R.*

Proof. Since $\pi(b_0, b_1, \ldots) = (0, b_0^q, b_1^q, \ldots)$, by replacing R by $R/\pi R$, it suffices to prove the claim for $r = 0$, i.e., $u = (a_0, a_1, \ldots)$ is a unit in $W_{\mathcal{O}}(R)$ if and only if a_0 is a unit in R. Since $W_{\mathcal{O}}(R) = \varprojlim W_{\mathcal{O},n}(R)$ where $W_{\mathcal{O},n}(R) = W_{\mathcal{O}}(R)/({}^{V^n}W_{\mathcal{O}}(R))$, it suffices to show that an element $u \in W_{\mathcal{O},n+1}(R)$ that maps to 1 in $W_{\mathcal{O},n}(R)$ is a unit. Using the formula of multiplications of Witt vectors, this is the same as saying that, for any $x \in R$, $x + y + \pi^n xy = 0$ has a solution. This is true since $\pi \in \mathrm{Rad}(R)$. The lemma follows. □

Proposition 3.10. *Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ be a κ-$\mathcal{O}$-frame with $\pi \in \mathrm{Rad}(R)$. Then there exists a u-homomorphism of $\mathcal{O}$-frames $\kappa : \mathcal{F} \to \mathcal{W}_R$ lying over id_R for some unit $u \in R$. The element u and the morphism κ are functorial with respect to strict $\mathcal{O}$-frame morphisms.*

Proof. There exists a morphism $\delta : S \to W_{\mathcal{O}}(S)$ such that $w_n(\delta(s)) = \sigma^n(s)$, where w_n is the n-th Witt polynomial attached to $W_{\mathcal{O}}$. Let $\kappa : S \to W_{\mathcal{O}}(R)$ be the composition

$$S \xrightarrow{\delta} W_{\mathcal{O}}(S) \to W_{\mathcal{O}}(R).$$

We show that κ induces a morphism of $\mathcal{O}$-frames. First, by the following commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\delta} & W_{\mathcal{O}}(S) & \longrightarrow & W_{\mathcal{O}}(R) \\ \text{inclusion}\uparrow & & w_0 \downarrow & & \downarrow w_0 \\ I & \xrightarrow{\text{inclusion}} & S & \longrightarrow & R \end{array} \tag{3.1}$$

it is easy to see that $\kappa(I) \subset I_{\mathcal{O}}(R)$. Note that $w_n(\delta(\sigma(s))) = w_n({}^F(\delta(s))) = \sigma^{n+1}(s)$ for all $s \in S$, we see that $\delta \circ \sigma = {}^F \circ \delta$, and thus $\kappa \circ \sigma = {}^F \circ \sigma$. Next, we check that there exists $u \in W_{\mathcal{O}}(R)^\times$ such that ${}^{V^{-1}}\kappa = u \cdot \kappa\sigma_1$. For any $a \in I$,

$$\begin{aligned} \kappa(\theta)\kappa(\sigma_1(a)) &= \kappa(\theta\sigma_1(a)) = \kappa(\sigma(a)) \\ &= {}^F(\kappa(a)) = \pi \cdot {}^{V^{-1}}(\kappa(a)). \end{aligned} \tag{3.2}$$

It suffices to show that $\pi^{-1}\kappa(\theta)$ is a unit in $W_{\mathcal{O}}(R)$. Then we can take $u = \pi^{-1}\kappa(\theta)$. Let $\kappa(\theta) = (x_0, x_1, \ldots)$ and $\delta(\theta) = (\tilde{x}_0, \tilde{x}_1, \ldots)$. By Lemma 3.9, it suffices to check that $\tilde{x}_1$ is a unit in S. Using the following two identities

$$\begin{cases} \sigma(\theta) = w_1(\delta(\theta)) = \tilde{x}_0^q + \pi\tilde{x}_1, \\ \theta = w_0(\delta(\theta)) = \tilde{x}_0, \end{cases} \tag{3.3}$$

we obtain $\pi\tilde{x}_1 = \sigma(\theta) - \theta^q$. Note that $\sigma(\theta) \equiv \theta^q \pmod{\pi S^\times}$, thus $\tilde{x}_1 \in S^\times$. From the construction, κ and u are functorial. $\square$

Corollary 3.11. *Let $\mathcal{F}=(S, I, R, \sigma, \sigma_1)$ be a κ-$\mathcal{O}$-frame with $S = W_{\mathcal{O}}(k)[[x_1, \dots, x_r]]$ for a perfect field k of characteristic p. Assume that σ extends the Frobenius automorphism of $W_{\mathcal{O}}(k)$ by $\sigma(x_i) = x_i^q$ for $1 \le i \le r$. Then u is a unit in $\widehat{W}_{\mathcal{O}}(R)$, and κ induces a u-homomorphism of $\mathcal{O}$-frames $\kappa : \mathcal{F} \to \mathcal{D}_R$.*

Proof. By the construction in the proof of the proposition, it suffices to show that $\delta(S) \subset \widehat{W}_{\mathcal{O}}(S)$. Since $w_n(\delta(x_i)) = w_n([x_i]) = x_i^{q^n}$ for all n, for each monomial $\prod_i x_i^{e_i}$, $\delta(\prod_i x_i^{e_i}) = [\prod_i x_i^{e_i}] \in \widehat{W}(S)$. Let $\mathfrak{m}_S$ be the maximal ideal of S. Then S has image in $\widehat{W}_{\mathcal{O}}(S/\mathfrak{m}_S^n)$ under the composition $S \to W_{\mathcal{O}}(S) \to W_{\mathcal{O}}(S/\mathfrak{m}_S^n)$. Indeed, there are only finitely many terms of an element of S with degree less that n. Therefore, $\delta(S) \in \varprojlim \widehat{W}_{\mathcal{O}}(S/\mathfrak{m}_S^n) = \widehat{W}_{\mathcal{O}}(S)$. The claim follows. $\square$

3.3. **Breuil $\mathcal{O}$-frames.** Let R be a complete regular local $\mathcal{O}$-algebra with perfect residue field k of characteristic p. We choose a ring homomorphism

$$\mathfrak{S} := W_{\mathcal{O}}(k)[[x_1, \dots, x_r]] \xrightarrow{\mathfrak{h}} R$$

such that $x_i \mapsto t_i$, where $(t_i \in \mathfrak{m}_R)$ is a regular system of parameters of R. In R, $\pi = \sum a_i t_i \in \mathfrak{m}_R$ with $a_i \in R$. There exists $f(x_1, \dots, x_r) \in (x_1, \dots, x_r)\mathfrak{S}$, such that $E = \pi - f(x_1, \dots, x_r) \in \operatorname{Ker}(\mathfrak{h})$. Note that $E \notin \mathfrak{m}_{\mathfrak{S}}^2$ since $\pi, x_1, \dots, x_r$ form a basis of $\mathfrak{m}_{\mathfrak{S}}/\mathfrak{m}_{\mathfrak{S}}^2$. Also, $\mathfrak{S}/(E)$ is a regular local ring. Thus $\mathfrak{S}/(E) \cong R$.

Let $\sigma : \mathfrak{S} \to \mathfrak{S}$ be the morphism that extends the Frobenius on $W_{\mathcal{O}}(k)$ and $\sigma(x_i) = x_i^q$ $(1 \le i \le r)$. Define $\sigma_1 : E\mathfrak{S} \to \mathfrak{S}$ by $\sigma_1(Ef) = \sigma(f)$ (for all $f \in \mathfrak{S}$). Then we have the following result.

Lemma 3.12. *The quintuple $\mathcal{B}_R = (\mathfrak{S}, E\mathfrak{S}, R, \sigma, \sigma_1)$ is a κ-$\mathcal{O}$-frame. We call it the Breuil $\mathcal{O}$-frame over R.*

Proof. It is easy to check that $\mathcal{B}_R$ is an $\mathcal{O}$-frame. Since $\sigma(Ef) = \sigma(E)\sigma(f) = \sigma(E)\sigma_1(Ef)$ for all $f \in \mathfrak{S}$, we see that $\theta = \theta_{\mathcal{B}_R} = \sigma(E)$. By assumption, $\mathfrak{S}$ and $W_{\mathcal{O}}(R)$ have no π-torsion. Moreover, $\sigma(\theta) - \theta^q$ has constant term $\pi - \pi^q$. So $(\sigma(\theta) - \theta^q) \in \pi\mathfrak{S}^\times$. The lemma follows. $\square$

An immediate consequence of the lemma is the existence of an $\mathcal{O}$-frame u-homomorphism

$$\kappa : \mathcal{B}_R \to \mathcal{D}_R,$$

where $u \in \widehat{W}_{\mathcal{O}}(R)$ is determined by the equation $\pi u = \kappa(\sigma(E))$. The following result corresponds to [15, Theorem 7.2].

Theorem 3.13. *Let R be a complete regular local $\mathcal{O}$-algebra with perfect residue field k of characteristic p. The $\mathcal{O}$-frame morphism $\kappa : \mathcal{B}_R \to \mathcal{D}_R$ is crystalline. i.e., it induces an equivalence between the categories $\mathrm{Win}_{\mathcal{B}_R}$ and $\mathrm{Win}_{\mathcal{D}_R}$.*

To prove the theorem, we introduce more objects. Let $J \subset \mathfrak{S}$ be the ideal $(x_1, \dots, x_r)$, $\mathfrak{m}_R$ the maximal ideal of R. For each positive integer a, let $\mathfrak{S}_a = \mathfrak{S}/J^a\mathfrak{S}$ and $R_a = R/\mathfrak{m}_R^a$. Then $R_a = \mathfrak{S}_a/E\mathfrak{S}_a$. Note that E is not a zero divisor of $\mathfrak{S}_a$, because the leading term of E is $\pi \in \operatorname{Gr}_0^J(\mathfrak{S}_a) = W_{\mathcal{O}}(k)$ and is not a zero divisor in $\operatorname{Gr}^J \mathfrak{S}_a$. Since $\sigma(J) \subset J$, $\sigma : \mathfrak{S} \to \mathfrak{S}$ induces a morphism $\sigma_a : \mathfrak{S}_a \to \mathfrak{S}_a$. Define $\sigma_{1a} : E\mathfrak{S}_a \to \mathfrak{S}_a$ by $\sigma_{1a}(Ey) = \sigma_a(y)$ for all $y \in \mathfrak{S}_a$. Then we have the following result.

Lemma 3.14. *The quintuple* $\mathcal{B}_{R_a} = (\mathfrak{S}_a, E\mathfrak{S}_a, R_a, \sigma_a, \sigma_{1a})$ *is a* κ*-*$\mathcal{O}$*-frame. The projection* $\mathfrak{S} \to \mathfrak{S}_a$ *induces a strict* $\mathcal{O}$*-frame morphism* $\mathcal{B}_R \to \mathcal{B}_{R_a}$*.*

Let u *denote the image of* u *in* $W_{\mathcal{O}}(R_a)$*. Then the* u*-homomorphism* $\kappa : \mathcal{B}_R \to \mathcal{D}_R$ *induces a* u*-homomorphism*

$$\kappa_a : \mathcal{B}_{R_a} \to \mathcal{D}_{R_a}.$$

Proof. The claims follow from the construction. □

Proposition 3.15. *For each positive integer* a*, the morphism* $\kappa_a : \mathcal{B}_{R_a} \to \mathcal{D}_{R_a}$ *is crystalline.*

Proof. Let $\tilde{\mathcal{B}}_{a+1}$ be the quintuple $(\mathfrak{S}_{a+1}, \tilde{I}_{a+1}, R_a, \sigma_{a+1}, \tilde{\sigma}_{1(a+1)})$, where

- $\tilde{I}_{a+1} = E\mathfrak{S}_{a+1} + J^a/J^{a+1}$;
- $\tilde{\sigma}_{1(a+1)} : \tilde{I}_{a+1} \to \mathfrak{S}_{a+1}$ is the extension of $\sigma_{1(a+1)} : E\mathfrak{S}_{a+1} \to \mathfrak{S}_{a+1}$ by sending J^a/J^{a+1} to zero.

The map $\tilde{\sigma}_{1(a+1)}$ is well-defined. Indeed, $E\mathfrak{S}_{a+1} \cap J^a/J^{a+1} = E(J^a/J^{a+1})$. For any $x \in J^a/J^{a+1}$, $\sigma_{1(a+1)}(Ex) = \sigma_{a+1}(x)$, which is zero in J^a/J^{a+1}.

It is easy to check that $\tilde{\mathcal{B}}_{a+1}$ is a κ-$\mathcal{O}$-frame. The homomorphism $\kappa_{a+1} : \mathfrak{S}_{a+1} \to \widehat{W}_{\mathcal{O}}(R_{a+1})$ induces a morphism of $\mathcal{O}$-frames $\tilde{\kappa}_{a+1} : \tilde{\mathcal{B}}_{a+1} \to \mathcal{D}_{R_{a+1}/R_a}$. We claim that $\tilde{\kappa}_{a+1}$ is a u-homomorphism. Indeed, it suffices to check that $\sigma_1\kappa_{a+1} = u \cdot \kappa_{a+1}\tilde{\sigma}_{1(a+1)}$ on $\tilde{I}_{a+1}$. For this, it suffices to check that $\sigma_1\kappa_{a+1} = 0$ on J^a/J^{a+1}. This follows from the identity $\sigma_1([x]) = 0$, where $x \in J^a$ is a monomial of degree a.

Summing up the above construction, we obtain the following commutative diagram of $\mathcal{O}$-frames:

$$\begin{array}{ccccc}
\mathcal{B}_{R_{a+1}} & \xrightarrow{\iota} & \tilde{\mathcal{B}}_{a+1} & \xrightarrow{\mathfrak{P}} & \mathcal{B}_{R_a} \\
\downarrow{\scriptstyle \kappa_{a+1}} & & \downarrow{\scriptstyle \tilde{\kappa}_{a+1}} & & \downarrow{\scriptstyle \kappa_a} \\
\mathcal{D}_{R_{a+1}} & \xrightarrow{\iota'} & \mathcal{D}_{R_{a+1}/R_a} & \xrightarrow{\mathfrak{P}'} & \mathcal{D}_{R_a}
\end{array} \tag{3.4}$$

Here the lower line is obtained from the $\mathcal{O}$-pd-thickening $R/\mathfrak{m}_R^{a+1} \to R/\mathfrak{m}_R^a$ with trivial $\mathcal{O}$-pd-structure on $\mathfrak{m}_R^a/\mathfrak{m}_R^{a+1}$.

We now prove the proposition by induction on a. If $a = 1$, then κ_1 is an isomorphism. There is nothing to prove. Assume that κ_a is crystalline. Since the filtration on J^a/J^{a+1} is trivial and both $\mathfrak{P}$ and $\mathfrak{P}'$ are crystalline, the morphism $\tilde{\kappa}_{a+1}$ is crystalline. Since to lift a Hodge filtration in the upper case and in the lower case are the same, we conclude that κ_{a+1} is crystalline. The proposition follows. □

Proof of Theorem 3.13. Because $\mathcal{B}_R$-windows (respectively $\mathcal{D}_R$-windows) are equivalent to compatible systems of $\mathcal{B}_{R_a}$-windows (respectively $\mathcal{D}_{R_a}$-windows), the theorem follows immediately from Proposition 3.15. □

3.4. **Breuil $\mathcal{O}$-windows and Breuil $\mathcal{O}$-modules.** Let $(\mathfrak{S}, R, E)$ be as in Section 3.3.

Definition 3.16. A *Breuil* $\mathcal{O}$*-window relative to* $\mathfrak{S} \to R$ is a pair (Q, ϕ), where Q is a free $\mathfrak{S}$-module of finite rank, $\phi : Q \to Q^{(\sigma)} := Q \otimes_{\mathfrak{S},\sigma} \mathfrak{S}$ is an $\mathfrak{S}$-linear map with cokernel annihilated by E.

We denote by $\mathrm{BrWin}_{\mathfrak{S}/R}$ the category of Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$.

Lemma 3.17. *Let (Q,ϕ) be a Breuil $\mathcal{O}$-window relative to $\mathfrak{S}\to R$. Then ϕ is injective and* $\operatorname{Coker}\phi$ *is a free R-module.*

Proof. The first claim follows from the surjectivity of the morphism $\phi\otimes\operatorname{Frac}\mathfrak{S}$. By this claim, the cohomological dimension of $\operatorname{Coker}\phi$ is 1. Thus the height of $\operatorname{Coker}\phi$ is r. Therefore, $\operatorname{Coker}\phi$ is a free R-module because R is regular of dimension r. $\square$

Proposition 3.18. *The categories* $\mathrm{BrWin}_{\mathfrak{S}/R}$ *and* $\mathrm{Win}_{\mathcal{B}_R}$ *are equivalent.*

Proof. Let $\mathcal{P}=(P,Q,F,F_1)$ be an object in $\mathrm{Win}_{\mathcal{B}_R}$. Let $\mathcal{Q}=(Q,\phi)$, where $\phi: Q\to Q^{(\sigma)}$ is the composition $Q\xrightarrow{\text{inclusion}}P\xrightarrow{(F_1^\sharp)^{-1}}Q^{(\sigma)}$. Then $\mathcal{Q}$ is a Breuil $\mathcal{O}$-window relative to $\mathfrak{S}\to R$. Conversely, for a Breuil $\mathcal{O}$-window (Q,ϕ), define a quadruple $\mathcal{P}=(P,Q,F,F_1)$, where $P=Q^{(\sigma)}$, $F_1: Q\to Q^{(\sigma)}$ is given by $x\mapsto x\otimes 1$ for all $x\in Q$, $F: P\to P$ is given by $1\otimes x\mapsto F_1(Ex)$ for all $x\in Q$. Then $\mathcal{P}$ is an $\mathcal{O}$-window over $\mathcal{B}_R$. The two functors are inverse to each other. The proposition follows. $\square$

Definition 3.19. A *Breuil $\mathcal{O}$-module relative to* $\mathfrak{S}\to R$ is a pair (M,ϕ), where M is an $\mathfrak{S}$-module of projective dimension at most one and annihilated by a power of π, $\phi: M\to M^{(\sigma)}$ is an $\mathfrak{S}$-linear map whose cokernel is annihilated by E.

Following the strategy in [21, Section 6], we prove some properties of Breuil $\mathcal{O}$-modules.

Lemma 3.20. *Let (M,ϕ) be a Breuil $\mathcal{O}$-module relative to $\mathfrak{S}\to R$. Then ϕ is injective.*

Proof. Let $x\in\mathfrak{S}$ such that $x\notin\pi\mathfrak{S}$. We claim that $x: M\to M$ is injective. Indeed, let

$$0\to P'\xrightarrow{\alpha}P\to M\to 0$$

be a resolution of M, where P and P' are finitely generated free $\mathfrak{S}$-modules of the same rank. In this case, $\det(\alpha)=\pi^n\cdot\text{unit}$ for some integer n. There exists $\beta: P\to P'$, such that $\alpha\circ\beta=\pi^n$. Thus the induced morphism $P'/xP'\to P/xP$ is injective since π is not a zero divisor in $\mathfrak{S}/x\mathfrak{S}$. The claim follows by the Snake lemma.

By the claim, the map $M\to M_{(\pi)}$ from M to its localization at the prime ideal (π) is injective. The localization $M_{(\pi)}$ is of finite length over the discrete valuation ring $\mathfrak{S}_{(\pi)}$. Let $\sigma:\mathfrak{S}_{(\pi)}\to\mathfrak{S}_{(\pi)}$ be the extension of σ on $\mathfrak{S}$ by setting $\sigma(\pi)=\pi$. Then we see that $M_{(\pi)}$ and $M_{(\pi)}^{(\sigma)}$ have the same length. Because $E\notin(\pi)$, the induced morphism $\phi_{(\pi)}: M_{(\pi)}\to M_{(\pi)}^{(\sigma)}$ is surjective. Therefore it is an isomorphism. Thus ϕ is an injection. $\square$

Corollary 3.21. *Let (M,ϕ) be a Breuil $\mathcal{O}$-module relative to $\mathfrak{S}\to R$. There exists a unique $\mathfrak{S}$-linear morphism $\psi: M^{(\sigma)}\to M$, such that $\phi\psi=\psi\phi=E$.*

Remark 3.22. Let P' and P be projective modules of the same rank over ring S. Let $\alpha: P'\to P$ be a homomorphism. Then there is a well-defined ideal $\vartheta(\alpha):=\det(\alpha)S$, which generalizes the usual $\det(\alpha)$ in the free modules case.

Lemma 3.23. *Let $\mathcal{F}=(S,I,R,\sigma,\sigma_1)$ be an $\mathcal{O}$-frame. Let $\alpha:\mathcal{P}\to\mathcal{P}'$ be a morphism of $\mathcal{F}$-windows. Assume that* $\operatorname{Rank}_S P=\operatorname{Rank}_S P'$, $\operatorname{Rank}_R(P/Q)=\operatorname{Rank}_R(P'/Q')$. *Then $\sigma(\vartheta(\alpha))S=\vartheta(\alpha)$.*

Proof. Since the question is local, we may assume that all modules are free. By assumption, we may assume that P and P' have normal decompositions $P = T \oplus L$, $P' = T' \oplus L'$, where $T \cong S^d \cong T'$ and $L \cong S^c \cong L'$ for some integers c and d. Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}_{c+d}(S)$ and $\left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) \in \mathrm{GL}_{c+d}(S)$ be morphism matrices of $F_1 : IT \oplus L \to T \oplus L$ and $F_1' : IT' \oplus L' \to T' \oplus L'$, respectively. Let $\left(\begin{smallmatrix} X & Y \\ U & Z \end{smallmatrix}\right) \in \mathrm{GL}_{c+d}(S)$ be the matrix that defines $\alpha : T \oplus L \to T' \oplus L'$. Since α defines a morphism of windows, Y has entries in I and

$$\begin{pmatrix} X & Y \\ U & Z \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \sigma(X) & \sigma_1(Y) \\ \theta\sigma(U) & \sigma(Z) \end{pmatrix}.$$

Let $t = \det \left(\begin{smallmatrix} X & Y \\ U & Z \end{smallmatrix}\right)$ and $t' = \det \left(\begin{smallmatrix} \sigma(X) & \sigma_1(Y) \\ \theta\sigma(U) & \sigma(Z) \end{smallmatrix}\right)$. Then $\vartheta(\alpha) = tS = t'S$. On the other hand, $\det(\sigma(\alpha)) = \det \left(\begin{smallmatrix} \sigma(X) & \sigma_1(Y) \\ \theta\sigma(U) & \sigma(Z) \end{smallmatrix}\right)$. The lemma follows. $\square$

Lemma 3.24. *In the same situation as in Lemma* 3.23*, assume that π is not a zero divisor in S and $\bigcap_{n\geq 1} \pi^n S = 0$. Assume further that* $\operatorname{Spec} S/\pi S$ *is connected. Then $\vartheta(\alpha) = \pi^h S$ or $\vartheta(\alpha) = 0$.*

In the case that $\vartheta(\alpha) = \pi^h S$, we call α *an isogeny of $\mathcal{O}$-height h.*

Proof. Since the situation is locally principal, we may assume that $\vartheta(\alpha) = \xi S$ for some $\xi \in S$. Assume that $\sigma(\xi) = \tau\xi$ for $\tau \in S^\times$. Then $\tau\xi \equiv \xi^q \pmod{\pi S}$. Because ξ and $\xi^{q-1} - \tau$ are relatively prime, i.e., $(\xi, \xi^{q-1} - \tau) = S$, we have

$$\operatorname{Spec}(S/\pi S) = D(\xi) \cup D(\xi^{q-1} - \tau).$$

By assumption, $D(\xi) = \operatorname{Spec}(S/\pi S)$ or $D(\xi) = \emptyset$. If ξ is a unit, we are done. If $\xi \in \pi S$, assume that $\xi = \pi\xi'$. Applying the above argument repeatedly, we either obtain a unit $\tilde{\xi}$ with $\xi = \pi^h\tilde{\xi}$, or $\xi \in \bigcap_{n\geq 1} \pi^n S = 0$. In either case, the lemma holds. $\square$

Lemma 3.25. *Each Breuil $\mathcal{O}$-module relative to $\mathfrak{S} \to R$ is the cokernel of an isogeny of Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$.*

Proof. Let (M, ϕ) be a Breuil $\mathcal{O}$-module. Let J and L be finitely generated free $\mathfrak{S}$-modules and $\tau : J \oplus L \to M^{(\sigma)}$ be an $\mathfrak{S}$-linear epimorphism which maps $EJ \oplus L$ surjectively to $\phi(M)$. Then there exists a unique $\mathfrak{S}$-linear map $\tau_1 : J \oplus L \to M$ such that the following diagram is commutative:

$$\begin{array}{ccc} J \oplus L & \xrightarrow{\tau_1} & M \\ \Big\downarrow{\scriptstyle E\cdot\mathrm{id}_J + \mathrm{id}_L} & & \Big\downarrow{\scriptstyle \phi} \\ EJ \oplus L & \xrightarrow{\tau} & \mathrm{Im}(\phi) \end{array} \tag{3.5}$$

Furthermore, there exists an $\mathfrak{S}$-linear isomorphism $\gamma : J \oplus L \to J^{(\sigma)} \oplus L^{(\sigma)}$ which makes the following diagram commutative:

$$\begin{array}{ccc} J \oplus L & \xrightarrow{\quad\tau\quad} & M^{(\sigma)} \\ {\scriptstyle \gamma}\Big\downarrow & \nearrow{\scriptstyle \tau_1^{(\sigma)}} & \\ J^{(\sigma)} \oplus L^{(\sigma)} & & \end{array} \tag{3.6}$$

Indeed, let N be a finitely generated module over a local ring A, and let F_1 and F_2 be two finitely generated free A-modules of the same rank equipped with A-linear epimorphisms $\tau_i : F_i \to N$ $(i = 1, 2)$. Then there exists an isomorphism $\gamma_{12} : F_1 \to F_2$ such that $\tau_2 \circ \gamma_{12} = \tau_1$. Applying this general property to our case, the existence of γ follows.

Let $Q := J \oplus L$ and $\phi := \gamma \circ (E \cdot \mathrm{id}_J + \mathrm{id}_L) : J \oplus L \to J^{(\sigma)} \oplus L^{(\sigma)}$. Then the pair (Q, ϕ) is a Breuil $\mathcal{O}$-window relative to $\mathfrak{S} \to R$ and we have a commutative diagram:

$$\begin{array}{ccc} Q & \xrightarrow{\tau_1} & M \\ \downarrow \phi & & \downarrow \phi \\ Q^{(\sigma)} & \xrightarrow{\tau_1^{(\sigma)}} & M^{(\sigma)} \end{array} \tag{3.7}$$

Hence τ_1 is a surjection from (Q, ϕ) to (M, ϕ). It is clear that the kernel (Q', ϕ') is a Breuil $\mathcal{O}$-window relative to $\mathfrak{S} \to R$. The lemma follows. □

Corollary 3.26. *If (M, ϕ) is a Breuil $\mathcal{O}$-module relative to $\mathfrak{S} \to R$, then the quotient $M^{(\sigma)}/\phi(M)$ is an R-module of projective dimension at most one.*

Proof. From Lemma 3.25, (M, ϕ) is the cokernel of an isogeny of Breuil $\mathcal{O}$-windows, i.e., we have a short exact sequence

$$0 \to (Q', \phi') \to (Q, \phi) \to (M, \phi) \to 0.$$

This induces an exact sequence

$$\mathrm{Coker}(\phi'_{Q'}) \to \mathrm{Coker}(\phi_Q) \to \mathrm{Coker}(\phi_M) \to 0.$$

The claim follows. □

Lemma 3.27. *If $(M, \phi) \to (\tilde{M}, \tilde{\phi})$ is a morphism of Breuil $\mathcal{O}$-modules relative to $\mathfrak{S} \to R$, then it is the cokernel of a morphism of two exact complexes $0 \to (Q', \phi') \to (Q, \phi)$ and $0 \to (\tilde{Q}', \tilde{\phi}') \to (\tilde{Q}, \tilde{\phi})$ of Breuil $\mathcal{O}$-windows.*

Proof. From Lemma 3.25, there exists a Breuil $\mathcal{O}$-window $(\tilde{Q}, \tilde{\phi})$ relative to $\mathfrak{S} \to R$ with a surjection $(\tilde{Q}, \tilde{\phi}) \to (\tilde{M}, \tilde{\phi})$. Let $Q = M \times_{\tilde{M}} \tilde{Q}$ be the fiber product. The functor $L \mapsto L^{(\sigma)}$ from $\mathfrak{S}$-modules to $\mathfrak{S}$-modules is exact and therefore respects fibre products. We obtain the following commutative diagram:

$$\begin{array}{ccc} (Q, \phi) & \longrightarrow & (M, \phi) \\ \downarrow & & \downarrow \\ (\tilde{Q}, \tilde{\phi}) & \longrightarrow & (\tilde{M}, \tilde{\phi}) \end{array} \tag{3.8}$$

As in the proof of Lemma 3.25, the kernels of the horizontal arrows are Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$. The lemma follows. □

3.5. **π-divisible $\mathcal{O}$-modules and special $\mathcal{O}$-group schemes.** In the rest of this paper, we assume that $p > 2$ and $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π. First, we have the following result.

Theorem 3.28. *Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π. Let R be a local complete regular Noetherian $\mathcal{O}$-algebra with perfect residue field of characteristic p. The category of π-divisible $\mathcal{O}$-modules over R is equivalent to the category of Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$.*

Proof. This is an immediate consequence of Theorem 3.13, Proposition 3.18 and [2, Theorem 1.5]. □

Definition 3.29. A *special $\mathcal{O}$-group scheme* over R is a finite flat commutative group scheme which is the kernel of an isogeny of π-divisible $\mathcal{O}$-modules over R. Write sGr_R for the category of special $\mathcal{O}$-group schemes over R.

From the definition, a special $\mathcal{O}$-group scheme is annihilated by a power of π and is of q-power order.

Remark 3.30. By [3, Theorem 3.1.1], for a Noetherian local ring R with perfect residue field of characteristic p, every finite flat commutative group scheme of p-power order over R is the kernel of an isogeny of p-divisible groups over R. Hence, in the case $\mathcal{O} = \mathbb{Z}_p$, every finite flat commutative group scheme of p-power order over R is special.

Theorem 3.31. *With the same setting as in Theorem* 3.28*, the category of special $\mathcal{O}$-group schemes over R is equivalent to the category of Breuil $\mathcal{O}$-modules relative to $\mathfrak{S} \to R$.*

Proof. Let H be a special $\mathcal{O}$-group scheme over R. By definition, H is the kernel of an isogeny of π-divisible $\mathcal{O}$-modules over R,

$$0 \to H \to G' \to G \to 0.$$

Let (Q', ϕ') and (Q, ϕ) be the Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$ which corresponds to G' and G, respectively. Let $(Q', \phi') \to (Q, \phi)$ be the morphism corresponding to the isogeny $G' \to G$. The cokernel of this map is annihilated by a power of π. Therefore it is an isogeny and the cokernel $M_{G'}(H) = (M, \phi)$ is a Breuil $\mathcal{O}$-module relative to $\mathfrak{S} \to R$.

Assume that $h : H \to H_1$ is a homomorphism of special $\mathcal{O}$-groups. Write H_1 as the kernel of an isogeny of π-divisible $\mathcal{O}$-modules

$$0 \to H_1 \to G_1' \to G_1 \to 0.$$

Let (Q_1', ϕ_1') and (Q_1, ϕ_1) be the Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$ which correspond to G_1' and G_1, respectively. Let $(Q_1', \phi_1') \to (Q_1, \phi_1)$ be the morphism corresponding to the isogeny $G_1' \to G_1$. Embed H into $G_2' = G' \oplus G_1'$ by $(1, h)$ and define $G_2 = G_2'/H$. We obtain two morphisms $G' \leftarrow G_2' \to G_1'$. They induce morphisms of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H & \longrightarrow & G_2' & \longrightarrow & G_2 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_1 & \longrightarrow & G_1' & \longrightarrow & G_1 & \longrightarrow & 0 \end{array} \tag{3.9}$$

The upper half of the diagram is a quasi-isomorphism and induces an isomorphism $M_{G'}(H) \cong M_{G_2'}(H)$. This shows that $M_{G'}(H)$ is independent of the isogeny, and we denote it by $\mathbb{M}(H)$. Moreover, the diagram induces a morphism $\mathbb{M}(H) \to \mathbb{M}(H_1)$. It is easy to see that $\mathbb{M}$ is an additive functor.

Next, we construct an additive functor $M \mapsto \mathbb{H}(M)$ from Breuil $\mathcal{O}$-modules to special $\mathcal{O}$-groups. Each M is the cokernel of an isogeny of Breuil $\mathcal{O}$-windows

$Q' \to Q$, and $\mathbb{H}(M)$ is defined to be the kernel of the associated isogeny of π-divisible $\mathcal{O}$-modules. By a similar argument as above, $\mathbb{H}$ is a well-defined additive functor. Also, from the construction, it is easy to check that $\mathbb{H}$ and $\mathbb{M}$ are inverse of each other. The theorem follows. □

3.6. **Duality.** Let (Q, ϕ) be a Breuil $\mathcal{O}$-window relative to $\mathfrak{S} \to R$. The *dual* of (Q, ϕ) is the Breuil $\mathcal{O}$-window $(Q, \phi)^t = (Q^\vee, \psi^\vee)$, where $Q^\vee = \mathrm{Hom}_{\mathfrak{S}}(Q, \mathfrak{S})$ and $\psi : Q^{(\sigma)} \to Q$ is the unique $\mathfrak{S}$-linear map with $\psi\phi = E$. Here we identify $(Q^{(\sigma)})^\vee$ and $(Q^\vee)^{(\sigma)}$.

Let $\mathcal{G}$ be the $\mathcal{O}$-module attached to the $\mathcal{O}$-display $(W_{\mathcal{O}}(R), I_{\mathcal{O}}(R), {}^F, {}^{V^{-1}})$. Let $\mathcal{G}[\pi^n]$ be the π^n-torsion of $\mathcal{G}$. For a π-divisible $\mathcal{O}$-module G over R, the *Serre $\mathcal{O}$-dual* (or *special $\mathcal{O}$-dual*) $G^\vee$ of G is defined in the same way as the Serre dual of G, by using $\mathcal{G}$ and $\mathcal{G}[\pi^n]$ instead of $\mathbb{G}_m$ and $\mu_{p^n} = \mathbb{G}_m[p^n]$. Similarly, for H in sGr_R, the *Cartier $\mathcal{O}$-dual* $H^\vee$ of H is defined in the same way as the Cartier dual of H, by using $\mathcal{G}[\pi]$ instead of μ_p. Let $\mathbb{W}(G)$ be the Breuil $\mathcal{O}$-window attached to G via the equivalence in Theorem 3.28.

Proposition 3.32. *There is a functorial isomorphism* $\lambda_G : \mathbb{W}(G^\vee) \cong \mathbb{W}(G)^t$.

Proof. By [2, Theorem 1.5], the equivalence between π-divisible $\mathcal{O}$-modules over R and Dieudonné $\mathcal{O}$-displays over R is compatible with duality. The equivalence in Proposition 3.18 preserves duality. It suffices to show that the functors κ_* in Theorem 3.13 preserves duality. By Lemma 2.10, it suffices to show that there exists a unit $c \in \widehat{W}_{\mathcal{O}}(R)$, such that $c^{-1}({}^F c) = u$. Note that $u = \pi^{-1}\kappa(\sigma E)$ (Proposition 3.10 and Lemma 3.12) lies in $1 + \widehat{W}_{\mathcal{O}}(\mathfrak{m}_R)$, the element $u({}^F u)({}^{F^2} u)\cdots$ converges in $\widehat{W}_{\mathcal{O}}(R) = \varprojlim \widehat{W}_{\mathcal{O}}(R/\mathfrak{m}_R^n)$. Let $c^{-1} = u({}^F u)({}^{F^2} u)\cdots$, the claim follows. □

Let (M, ϕ) be a Breuil $\mathcal{O}$-module relative to $\mathfrak{S} \to R$. The *dual* $(M, \phi)^t$ of (M, ϕ) is the Breuil $\mathcal{O}$-module (M^*, ϕ^*), where $M^* = \mathrm{Ext}^1_{\mathfrak{S}}(M, \mathfrak{S})$, ϕ^* is determined by Corollary 3.21 (cf. [22, Section 2]). Then we have the following result.

Proposition 3.33. *Let H be an object in* sGr_R. *There is a functorial isomorphism* $\lambda_H : \mathbb{M}(H^\vee) \cong \mathbb{M}(H)^t$.

Proof. Let H be the kernel of the isogeny of π-divisible $\mathcal{O}$-modules $G' \to G$. Then $\mathbb{M}(H)$ is the cokernel of $Q' \to Q$, where Q' and Q are Breuil $\mathcal{O}$-windows corresponding to G and G', respectively. Thus $\mathbb{M}(H)^t$ is the cokernel of $Q^t \to (Q')^t$. On the other hand, $H^\vee$ is the kernel of $G^\vee \to (G')^\vee$. By Proposition 3.32, we have the isomorphism $\lambda_H : \mathbb{M}(H^\vee) \cong \mathbb{M}(H)^t$. This isomorphism is independent of the choice of G' and functorial in H. The proposition follows. □

For the application in the next section, it is convenient to use contravariant Breuil $\mathcal{O}$-windows and contravariant Breuil $\mathcal{O}$-modules ([22, Section 2]).

Definition 3.34. A *contravariant Breuil $\mathcal{O}$-window relative to* $\mathfrak{S} \to R$ is a pair (Q, ϕ), where Q is a free $\mathfrak{S}$-module of finite rank and $\phi : Q^{(\sigma)} \to Q$ is an $\mathfrak{S}$-linear map whose cokernel is annihilated by E.

A *contravariant Breuil $\mathcal{O}$-module relative to* $\mathfrak{S} \to R$ is a pair (M, ϕ), where M is a finitely generated $\mathfrak{S}$-module annihilated by a power of π and of projective dimension at most one, and $\phi : M^{(\sigma)} \to M$ is an $\mathfrak{S}$-linear map whose cokernel is annihilated by E.

The category of Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$ (respectively Breuil $\mathcal{O}$-modules relative to $\mathfrak{S} \to R$) is equivalent to the category of contravariant Breuil $\mathcal{O}$-windows relative to $\mathfrak{S} \to R$ (respectively contravariant Breuil $\mathcal{O}$-modules relative to $\mathfrak{S} \to R$) by taking dual objects.

4. An application

Using the theory of Breuil modules, Vasiu and Zink [22] proved some boundedness results for finite flat group schemes over discrete valuation rings of mixed characteristic. With the results proved in Section 3, we now generalize the results in [22] to the case of special $\mathcal{O}$-group schemes.

4.1. **Setup.** Let $p > 2$ be a prime number. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F}_q$. Let R be a complete regular discrete valuation ring of mixed characteristic $(0, p)$ with fraction field K and residue field k. Assume that R is an $\mathcal{O}$-algebra. We view canonically R as a $W_{\mathcal{O}}(k)$-algebra, which as a $W_{\mathcal{O}}(k)$-module is free of rank e. Here e is the ramification degree of R over $\mathcal{O}$.

Let $\mathfrak{S} = W_{\mathcal{O}}(k)[[u]]$ and $\mathfrak{S}_n = \mathfrak{S}/\pi^n\mathfrak{S}$. Let

$$E := E(u) = u^e + a_{e-1}u^{e-1} + \cdots + a_0 \in W_{\mathcal{O}}(k)[u]$$

be the Eisenstein polynomial associated with a uniformizer ρ of R. We have a $W_{\mathcal{O}}(k)$-epimorphism $\mathfrak{S} \to R$ with kernel $E\mathfrak{S}$ which maps u to the fixed uniformizer.

Let $\mathrm{BrMod}_{\mathfrak{S}/R}$ be the category of contravariant Breuil $\mathcal{O}$-modules relative to $\mathfrak{S} \to R$. Let $\mathrm{BrMod}^1_{\mathfrak{S}/R}$ be the full subcategory of $\mathrm{BrMod}_{\mathfrak{S}/R}$ whose objects are pairs (M, ϕ) with M annihilated by π. If (M, ϕ) is an object of $\mathrm{BrMod}^1_{\mathfrak{S}/R}$, then M is a free $\mathfrak{S}_1$-module of finite rank. In the following, a Breuil $\mathcal{O}$-module means a contravariant Breuil $\mathcal{O}$-module.

Let sGr_R be the category of special $\mathcal{O}$-groups over R. Let sGr^1_R be the full subcategory of sGr_R whose objects are annihilated by π. Applying the results in Section 3, we have the following proposition.

Proposition 4.1. *There exists a contravariant functor* $\mathbb{B} : \mathrm{sGr}_R \to \mathrm{BrMod}_{\mathfrak{S}/R}$ *which is an anti-equivalence of categories. It is $\mathcal{O}$-linear and takes short exact sequences (in the category of abelian sheaves in the faithfully flat topology of* $\mathrm{Spec}\, R$*) to short exact sequences (in the category of $\mathfrak{S}$-modules with Frobenius maps).*

The restriction of $\mathbb{B}$ *induces an anti-equivalence* $\mathbb{B} : \mathrm{sGr}^1_R \to \mathrm{BrMod}^1_{\mathfrak{S}/R}$.

Definition 4.2. For an object G of sGr_R, let $o(G) \in \mathbb{N}$ be such that $q^{o(G)}$ is the order of G.

For (M, ϕ) an object of $\mathrm{BrMod}^1_{\mathfrak{S}/R}$, the *rank* of (M, ϕ) is the rank of M as a free $\mathfrak{S}_1$-module.

Remark 4.3. If G is an object of sGr^1_R, then by definition, the rank of $\mathbb{B}(G)$ is $o(G)$.

Let H be an object of sGr_R. Assume that π^n annihilates H, then we have a chain of natural epimorphisms

$$H \to H/H[\pi] \to H/H[\pi^2] \to \cdots \to H/H[\pi^n] = \{0\}.$$

This induces a chain of Breuil $\mathcal{O}$-modules

$$0 = (M_n, \phi_n) \subset (M_{n-1}, \phi_{n-1}) \subset \cdots \subset (M_0, \phi_0) = (M, \phi),$$

whose quotient factors are objects of $\mathrm{BrMod}^1_{\mathfrak{S}/R}$. Then we can compute the order $q^{o(H)}$ of H via the formula

$$o(H) = o(M, \phi) := \sum_{i=1}^{n} \mathrm{Rank}_{\mathfrak{S}_1}(M_{i-1}/M_i) = \mathrm{Length}_{\mathfrak{S}_{(\pi)}}(M_{(\pi)}).$$

The following proposition corresponds to [22, Proposition 1]. The proof is similar. We give details here for completeness.

Lemma 4.4. *Let $f : G \to H$ be a morphism of special $\mathcal{O}$-group schemes. Let $g := \mathbb{B}(f) : \mathbb{B}(H) = (M, \phi) \to \mathbb{B}(G) = (N, \psi)$. Then the following claims hold.*

(1) *The morphism $f_K : G_K \to H_K$ is a closed embedding if and only if the cokernel of $g : M \to N$ is annihilated by some power of u.*
(2) *The morphism $f_K : G_K \to H_K$ is an epimorphism if and only if the map $g : M \to N$ is a monomorphism.*
(3) *The morphism $f_K : G_K \to H_K$ is an isomorphism if and only if the map $g : M \to N$ is injective and the cokernel of g is annihilated by some power of u.*

Proof. We prove the first statement. Let $\tilde{N} = \mathrm{Coker}(g)$. Assume that f_K is not a closed embedding, then there exists a nontrivial flat closed subgroup G_0 of G, which is contained in the kernel of f_K and which is annihilated by π. Let $\mathbb{B}(G_0) = (N_0, \psi_0)$. Then N_0 is free over $\mathfrak{S}_1$ with positive rank. On the other hand, $\mathbb{B}$ takes short exact sequences to short exact sequences, we have an epimorphism $\tilde{N} \to N_0$ and $\tilde{N}$ is not annihilated by a power of u.

Assume that $\tilde{N}$ is not annihilated by a power of u, then $N_1 = \tilde{N}/\pi\tilde{N}$ is not annihilated by a power of u. As $\mathfrak{S}_1 = k[[u]]$ is a principal ideal domain, we have a short exact sequence

$$0 \to N_2 \to N_1 \to N_0 \to 0,$$

where N_2 is the largest $\mathfrak{S}_1$-submodule of N_1 annihilated by a power of u and N_0 is a free $\mathfrak{S}_1$-submodule of positive rank. The map $\psi : N^{(\sigma)} \to N$ induces a σ-linear map $\psi_0 : N_0^{(\sigma)} \to N_0$. It is easy to see that the pair (N_0, ψ_0) is an object of $\mathrm{BrMod}^1_{\mathfrak{S}/R}$. Then by Proposition 4.1, there exists a nontrivial flat closed subgroup G_0 of G, which is contained in the kernel of f_K. Therefore, f_K is not a closed embedding.

The second statement follows by a similar argument. Assume that f_K is not an epimorphism, then there exists a nontrivial flat closed subgroup H_0 of H, which is not contained in the image of f_K and which is annihilated by π. The corresponding Breuil $\mathcal{O}$-module $\mathbb{B}(H_0)$ produces nontrivial elements in $\mathrm{Ker}(g)$ and the map $g : M \to N$ is not a monomorphism. On the other hand, assume that $g : M \to N$ is not a monomorphism, then the kernel $\mathrm{Ker}(g)$ produces a nontrivial flat closed subgroup H_0 of H which is not contained in the image of f_K. Thus f_K is not an epimorphism.

The third statement follows from the first and the second. □

4.2. **Truncations.** By a *special truncated Barsotti-Tate $\mathcal{O}$-group of level n over R*, we mean a Barsotti-Tate $\mathcal{O}$-group of level n over R, which is the π^n-torsion of a π-divisible $\mathcal{O}$-module. Let H be such a group. Let $(M, \phi) := \mathbb{B}(H)$. Then M is a free $\mathfrak{S}_n$-module of finite rank h.

Lemma 4.5. *There exist two bases $(e_1,\dots,e_h)$ and $(v_1,\dots,v_h)$ of M, such that*

$$\begin{cases}\phi(1\otimes e_i)=v_i, & i=d+1,d+2,\dots,h,\\ \phi(1\otimes e_j)\equiv Ev_j \pmod{v_{d+1},\dots,v_h}, & i=1,\dots,d,\end{cases}\tag{4.1}$$

for some integer d.

Proof. Since H is special, we may assume that $H=\mathcal{H}[\pi^n]$ for some π-divisible $\mathcal{O}$-module $\mathcal{H}$. The normal decomposition of the Breuil $\mathcal{O}$-window associated with $\mathcal{H}$ induces a direct sum decomposition $M=T\oplus L$ into free $\mathfrak{S}_n$-submodules, such that T is free of rank d and $\mathrm{Im}(\phi)=ET\oplus L$. Consider the composition

$$M\xrightarrow{1\otimes -}M^{(\sigma)}\xrightarrow{\phi}\phi(M^{(\sigma)})\xrightarrow{\text{projection}}M/T=L.$$

All the arrows are surjective after tensoring with the residue field k. By Nakayama's lemma, there exists a basis $(e_1,\dots,e_h)$ of M, such that the images of $\phi(1\otimes e_i)$ $(i=d+1,\dots,h)$ form a basis of $M/T=L$. Define $v_i=\phi(1\otimes e_i)$ for $i=d+1,\dots,h$. They form a basis of L. Note that $(\phi(1\otimes e_i):i=1,\dots,h)$ form a basis of $ET\oplus L$, there exists a basis $(v_1,\dots,v_d)$ of T, which satisfies the required conditions. □

Lemma 4.6. *Let $t\in\mathbb{Z}_{\geq 0}$. Let $x\in\frac{1}{u^t}M$ such that $\phi(1\otimes x)\in\frac{1}{u^t}M$. Using the basis $(e_1,\dots,e_h)$ in Lemma 4.5, write $x=\sum_{i=1}^h\frac{\alpha_i}{u^t}e_i$ with $\alpha_i\in\mathfrak{S}_n$. Then for each $i=1,\dots,h$, $E\sigma(\alpha_i)\in u^{t(q-1)}\mathfrak{S}_n$, $\alpha_i\in(\pi^{n-1},u)\mathfrak{S}_n$, and $\pi\alpha_i(0)=0\in\mathfrak{S}_n$.*

Proof. By definition,

$$\begin{aligned}\phi(1\otimes x)&=\sum_{i=1}^d\frac{\sigma(\alpha_i)}{u^{tq}}\phi(1\otimes e_i)+\sum_{i=d+1}^h\frac{\sigma(\alpha_i)}{u^{tq}}\phi(1\otimes e_i)\\&=\sum_{i=1}^d\frac{\sigma(\alpha_i)}{u^{tq}}Ev_i+\sum_{i=d+1}^h(\frac{\sigma(\alpha_i)}{u^{tq}}+\sum_{j=1}^d\lambda_{ij}\frac{\sigma(\alpha_j)}{u^{tq}})v_i\in\frac{1}{u^t}M,\end{aligned}\tag{4.2}$$

for some $\lambda_{ij}\in\mathfrak{S}_n$. Thus for $i\in\{1,\dots,d\}$, $E\sigma(\alpha_i)\in u^{t(q-1)}\mathfrak{S}_n$. Moreover, for $i>d$,

$$E\frac{\sigma(\alpha_i)}{u^{tq}}=E(\frac{\sigma(\alpha_i)}{u^{tq}}+\sum_{j=1}^d\lambda_{ij}\frac{\sigma(\alpha_j)}{u^{tq}})-\sum_{j=1}^d\lambda_{ij}E\frac{\sigma(\alpha_j)}{u^{tq}}.$$

Therefore, $E\sigma(\alpha_i)\in u^{t(q-1)}\mathfrak{S}_n$ for all $i=1,\dots,h$. The other two claims follow easily. □

Lemma 4.7. *Let $t\in\mathbb{Z}_{\geq 0}$. Let N be an $\mathfrak{S}_n$-submodule of $\frac{1}{u^t}M$ which contains M. Assume that ϕ induces an $\mathfrak{S}$-linear map $N^{(\sigma)}\to N$. Then $\pi^tN\subset M$.*

Proof. We prove this by induction on t. If $t=0$, the claim is trivial. Assume that the lemma is true for $t-1$. Let $x\in N$. Then by Lemma 4.6, $\pi x\in\frac{1}{u^{t-1}}M$. Thus $\pi N\subset\frac{1}{u^{t-1}}M$. Applying induction to $N':=\pi N+M\subset\frac{1}{u^{t-1}}M$, we get $\pi^{t-1}N'\subset M$. Therefore, $\pi^tN\subset M$. The lemma follows. □

4.3. **Some formulas.** In this section, we prove the results corresponding to those in [22, Section 3]. The motivations for these results are explained in [22]. Our arguments here are entirely similar to those in [22]. In many cases, to give the proofs, we may just replace the number $p \in \mathbb{Z}$ in [22] with the number $q \in \mathbb{Z}$ and the uniformizer $p \in \mathbb{Z}_p$ with the uniformizer $\pi \in \mathcal{O}$. For completeness, we give details in the following.

Assume that $\mathcal{O}$ is of degree rf over $\mathbb{Z}_p$, where r is the ramification degree and f is the residue degree. Then $q = p^f$ and $\mathrm{ord}_\pi(p) = r$. For $x \in \mathbb{R}$, $[x]$ denotes the maximal integer with the property $[x] \leq x$.

Define $m := \mathrm{ord}_q(e) = [\frac{\mathrm{ord}_p(e)}{f}]$. Let $a_e = 1$. Recall that $E = \sum_{i=0}^{e} a_i u^i$ is the Eisenstein polynomial of a uniformizer ρ of R. Define

$$E_0 := \sum_{q|i} a_i u^i \in W_{\mathcal{O}}(k)[u^q],$$

and $E_1 = E - E_0 \in W_{\mathcal{O}}(k)[u]$.

If $m = 0$, define $\tau(\rho) = 1$ and $\iota(\rho) = 0$.

If $m \geq 1$, define $\tau(\rho) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by

$$\tau(\rho) := \mathrm{ord}_\pi(E_1) = \min\{\mathrm{ord}_\pi(a_i) \mid i \in \{1, \ldots, e-1\} \backslash q\mathbb{Z}\}.$$

If $m \geq 1$ and $\tau(\rho) < \infty$, let $\iota(\rho) \in \{1, \ldots, e-1\} \backslash q\mathbb{Z}$ be the smallest number such that $\tau(\rho) = \mathrm{ord}_\pi(a_{\iota(\rho)})$.

For all $m \geq 0$, define

$$\tau = \tau_R := \min\{\tau(\rho) \mid \rho \text{ is a uniformizer of } R\}.$$

If $\tau < \infty$, which is always true as we show in next lemma, define

$$\iota = \iota_R := \min\{\iota(\rho) \mid \rho \text{ is a uniformizer of } R \text{ with } \tau(\rho) = \tau\}.$$

Lemma 4.8. *With the notation as above, $\tau < \infty$.*

Proof. If $m = 0$, then $\tau = 1$ by definition and the claim follows. Assume that $m \geq 1$. Note that ρ is a uniformizer with Eisenstein polynomial $E(u)$. Then another uniformizer $\rho' = \rho + \pi$ of R is the root of the Eisenstein polynomial $E'(u) = E(u - \pi) = \sum_{i=0}^{e} a'_i u^i$. Thus $a'_{e-1} = -\pi e + a_{e-1}$. At least one of a_{e-1} and a'_{e-1} is not divisible by $\pi^{2+\mathrm{ord}_\pi(e)}$. The lemma follows. $\square$

Lemma 4.9. *Let $n, t \in \mathbb{Z}_{>0}$. Assume that $q \mid e$. Let*

$$C = C(u) = u^d + c_{d-1}u^{d-1} + \cdots + c_1 u + c_0 \in W_{\mathcal{O}}(k)[u]$$

be a Weierstrass polynomial (i.e., $C - u^d \equiv 0 \pmod{\pi}$), such that $qd < t$ and $c_0 \notin \pi^n W_{\mathcal{O}}(k)$. Assume that $E_0\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$. Then $d = (n-1)e/q$ and for each $i \in \{0, 1, \ldots, n-1\}$, we have

$$\mathrm{ord}_\pi(c_{i\frac{e}{q}}) = n - i - 1 \quad \textit{and} \quad \mathrm{ord}_\pi(c_j) \geq n - i, \textit{ for } 0 \leq j < i\frac{e}{q}. \tag{4.3}$$

Moreover, $t \leq ne$.

Proof. Let $c_d = 1$. Let $\gamma_i = \sigma(c_i) \in W_{\mathcal{O}}(k)$ for $0 \leq i \leq d$. Then

$$\sigma(C) = \gamma_d u^{qd} + \gamma_{d-1}u^{q(d-1)} + \cdots + \gamma_1 u^q + \gamma_0 \in W_{\mathcal{O}}(k)[u].$$

For $i \notin \{0, \ldots, d\}$, define $c_i = \gamma_i = 0$. Note that $\mathrm{ord}_\pi(c_i) = \mathrm{ord}_\pi(\gamma_i)$. To prove equation (4.3), it suffices to prove it for γ_i. Moreover,

$$E_0\sigma(C) = \sum_{j=0}^{d+\frac{e}{q}} \beta_{jq} u^{jq},$$

where

$$\beta_{jq} = a_0\gamma_j + a_q\gamma_{j-1} + \cdots + a_e\gamma_{j-\frac{e}{q}}. \tag{4.4}$$

By assumption, $\mathrm{ord}_\pi(\beta_{jq}) \geq n$ for $jq < t$. In particular, $\mathrm{ord}_\pi(\beta_{jq}) \geq n$ for $j \leq d$. We prove that

$$\mathrm{ord}_\pi(\gamma_{i\frac{e}{q}}) = n - i - 1 \quad \text{and} \quad \mathrm{ord}_\pi(\gamma_j) \geq n - i, \text{ for } 0 \leq j < i\frac{e}{q}, \tag{4.5}$$

by induction on j. The case $j = 0$ is easy. The passage from $j - 1$ to j goes as follows. Assume first $(i-1)\frac{e}{q} < j < i\frac{e}{q}$ for some integer i. By equation (4.4), $a_0\gamma_j = \beta_{jq} - (a_q\gamma_{j-1} + \cdots + a_e\gamma_{j-\frac{e}{q}})$. Each term on the right-hand side has π-order strictly bigger than $n - i$. Thus, $\mathrm{ord}_\pi(\gamma_i) \geq n - i + 1 - \mathrm{ord}_\pi(a_0) = n - i$.

In the case $j = i\frac{e}{q}$, $a_0\gamma_{i\frac{e}{q}} + a_e\gamma_{(i-1)\frac{e}{q}} = \beta_{ie} - (a_q\gamma_{i\frac{e}{q}-1} + \cdots + \alpha_{e-q}\gamma_{1+(i-1)\frac{e}{q}})$. Each term on the right-hand side has π-order $\geq n - i + 1$. Since $\mathrm{ord}_\pi(a_e\gamma_{(i-1)\frac{e}{q}}) = n - i < n - i + 1$, we must have $\mathrm{ord}_\pi(a_0\gamma_{i\frac{e}{q}}) = n - i$. This ends the induction.

If d is of the form $i\frac{e}{q}$, then $0 = \mathrm{ord}_\pi(c_d) = n-i-1$ and $d = (n-1)\frac{e}{q}$. Suppose that d is not of the form $i\frac{e}{q}$. Assume that $(i-1)\frac{e}{q} < d < i\frac{e}{q}$, then $0 = \mathrm{ord}_\pi(c_d) \geq n - i$ and $i \geq n$. This implies $\mathrm{ord}_\pi(\gamma_{(i-1)\frac{e}{q}}) = n-i \leq 0$, which contradicts the assumption that C is a Weierstrass polynomial.

Finally, as $q \mid e$ and $d = (n-1)\frac{e}{q}$, $E_0\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$ is a monic polynomial of degree $e + qd = ne$. Thus we must have $t \leq ne$. The lemma follows. □

Corollary 4.10. *With the same notation as in Lemma* 4.9, *let* $l \in \{0, 1, \ldots, e-1\}$. *Let* $E_2 = E_2(u) = u^l + b_{l-1}u^{l-1} + \cdots + b_1u + b_0 \in W_{\mathcal{O}}(k)[u]$ *be a Weierstrass polynomial of degree* l. *If we have* $E_2\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$, *then* $l \geq t$.

Proof. If $n = 1$, then $d = 0$ and $C = c_0$ is a unit. The corollary follows. Assume that $n \geq 2$. Write

$$E_2\sigma(C) = \sum_{i=0}^{l+qd} \delta_i u^i,$$

where

$$\delta_l = \gamma_0 + b_{l-q}\gamma_1 + \cdots + b_{l-q[\frac{l}{q}]}\gamma_{[\frac{l}{q}]}.$$

For $i \in \{1, \ldots, [\frac{l}{q}]\}$, we have $\mathrm{ord}_\pi(\gamma_i) \geq n-1$ by equation (4.5). Thus $\mathrm{ord}_\pi(b_{l-iq}\gamma_i) \geq n$. On the other hand, $\mathrm{ord}_\pi(\gamma_0) = n - 1$. Thus $\mathrm{ord}_\pi(\delta_l) = n - 1$ and $l \geq t$. □

The following proposition corresponds to [22, Proposition 2], which is the key to the computation.

Proposition 4.11. *Let* n *and* t *be positive integers. Let* $C = C(u) \in \mathfrak{S}$ *be a power series whose constant term is not divisible by* π^n. *Assume that*

$$E\sigma(C) \in (u^t, \pi^n)\mathfrak{S}.$$

If $\tau(\rho) = \infty$, *then* $t \leq ne$. *If* $\tau(\rho) < \infty$, *then*

$$t \leq \min\{\tau(\rho)e + \iota(\rho), ne\}.$$

Moreover, if $m = 0$*, then we have* $\pi\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$*; if* $m \geq 1$*, then we have* $\pi^{\tau(\rho)+1}\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$*.*

In particular, if $m \geq 1$ *and the content of* C *is* 1*, then* $\tau(\rho) + 1 \geq n$*.*

Proof. Since $\sigma(u) = u^q$, without loss of generality, we may assume that C is a polynomial of degree d with $dq < t$. Each term of $\sigma(C)$ has degree divisible by q, $E_0\sigma(C)$ and $E_1\sigma(C)$ do not contain monomials of the same degree. Therefore, $E\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$ implies $E_0\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$ and $E_1\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$.

Consider the case $m = 0$, i.e., $q \nmid e$. In this case, $\pi^{-1}E_0$ is a unit in the ring $\mathfrak{S}$. Therefore, $\pi\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$. By our assumption on the degree of C, $\pi\sigma(C) \equiv 0 \pmod{\pi^n}$. Moreover, $E_1 - u^e \equiv 0 \pmod{\pi}$, $u^e\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$. As the constant term of C is not divisible by π^n, we must have $t \leq e = \min\{\tau(\rho)e + \iota(\rho), ne\}$.

Assume now that $q \mid e$. By Weierstrass preparation theorem, we may assume that C is a monic polynomial of degree d such that $C - u^d \equiv 0 \pmod{\pi}$. Indeed, if c, the content of C, is greater than 0, then we may just replace the pair (C, n) by the pair $(C', n - c)$, where $C' = \pi^{-c}C$, and prove the proposition for $(C', n - c)$. It suffices to show that $\tau(\rho) + 1 \geq n$. As in [22], assume that $\tau(\rho) + 1 \leq n$ and it suffices to show that $\tau(\rho) + 1 = n$.

As $\tau(\rho) = \mathrm{ord}_\pi(E_1)$, by Weierstrass preparation theorem, we may write

$$E_1 = \pi^{\tau(\rho)}E_2\theta,$$

where $\theta \in \mathfrak{S}$ is a unit and $E_2(u) \in W_{\mathcal{O}}(k)[u]$ is a Weierstrass polynomial of degree $\iota(\rho) < e$. The property $E_1\sigma(C) \in (u^t, \pi^n)\mathfrak{S}$ implies that

$$E_2\sigma(C) \in (u^t, \pi^{n-\tau(\rho)})\mathfrak{S}.$$

Note that $\tau(\rho)\frac{e}{q} \leq (n-1)\frac{e}{q} = d$ and $c_d = 1$, we consider the monic polynomial

$$C_1 = C_1(u) = u^d + c_{d-1}u^{d-1} + \cdots + c_{\tau(\rho)\frac{e}{q}}u^{\tau(\rho)\frac{e}{q}} \in W_{\mathcal{O}}(k)[u].$$

By Lemma 4.9, $\mathrm{ord}_\pi(c_j) \geq n - \tau(\rho)$ for $j < \tau(\rho)\frac{e}{q}$. Thus $C - C_1 \in \pi^{n-\tau(\rho)}\mathfrak{S}$. Therefore,

$$E_2\sigma(C_1) \in (u^t, \pi^{n-\tau(\rho)})\mathfrak{S}.$$

Write $C_1 = u^{\tau(\rho)\frac{e}{q}}C_2$. Then the constant term of C_2 is $c_{\tau(\rho)\frac{e}{q}}$, which is not divisible by $\pi^{n-\tau(\rho)}$. Therefore, as $t > qd = (n-1)e \geq \tau(\rho)e$,

$$E_2\sigma(C_2) \in (u^{t-\tau(\rho)e}, \pi^{n-\tau(\rho)})\mathfrak{S}.$$

Similarly, since $E_0\sigma(C_2) = E_0u^{-\tau(\rho)e}\sigma(C) - E_0u^{-\tau(\rho)e}\sigma(C - C_1)$,

$$E_0\sigma(C_2) \in (u^{t-\tau(\rho)e}, \pi^{n-\tau(\rho)})\mathfrak{S}.$$

Applying Corollary 4.10 to the quintuple $(t - \tau(\rho)e, C_2, E_0, E_2, n - \tau(\rho))$ instead of the quintuple (t, C, E_0, E_2, n), we obtain that $\iota(\rho) = \deg(E_2) \geq t - \tau(\rho)e$. Since $\iota(\rho) \leq e - 1$ and $n \geq \tau(\rho) + 1$, it is easy to see that $t \leq \tau(\rho)e + \iota(\rho) = \min\{\tau(\rho)e + \iota(\rho), ne\}$. The property $(n-1)e = qd < t$ implies $n \leq \tau(\rho) + 1$. Thus $n = \tau(\rho) + 1$ and the proposition follows. □

4.4. **The number s.** For a uniformizer ρ of R, define

$$t(\rho) := [\frac{\tau(\rho)e + \iota(\rho)}{q-1}] \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

By Lemma 4.8, there exists ρ, such that $t(\rho)$ is finite.

Let $\epsilon = 0$ (respectively $\epsilon = 1$) if $m = 0$ (respectively $m \geq 1$). For all nonnegative integers i, define

$$s_i := i(\tau + \epsilon) \text{ and } t_i := [\frac{\tau e + \iota}{(q-1)q^i}].$$

Thus $t_{i+1} = [\frac{t_i}{q}]$ (an easy computation) and

$$t_0 = \min\{t(\rho) \mid \rho \text{ is a uniformizer of } R\}.$$

Define the number $s \in \mathbb{Z}_{\geq 0}$ which only depends on R by

$$s = s_R := \min\{s_i + t_i \mid i \in \mathbb{Z}_{\geq 0}\}.$$

Let $z \in \mathbb{Z}_{\geq 0}$ be the smallest number with the property $s_z + t_z = s$.

With the above definition, we have

$$0 = s_0 < s_1 < \cdots < s_z \text{ and } t_0 > t_1 > \cdots > t_z \geq 0.$$

Theorem 4.12. *With the notation as above, let G and H be two special $\mathcal{O}$-groups over R. For each homomorphism $f : G \to H$ whose generic fiber $f_K : G_K \to H_K$ is an isomorphism, there exists a homomorphism $f' : H \to G$ such that $f' \circ f = \pi^s \operatorname{id}_G$ and $f \circ f' = \pi^s \operatorname{id}_H$. Therefore the special fiber homomorphism $f_k : G_k \to H_k$ has a kernel and a cokernel annihilated by π^s.*

Proof. If there exists $f' : H \to G$ such that $f \circ f' = \pi^s \operatorname{id}_H$, then $f' \circ f = \pi^s \operatorname{id}_G$ as this is true on the generic fiber. The claim on the special fiber homomorphism follows easily.

Choose an epimorphism $\xi_H : \tilde{H} \to H$ from a special truncated Barsotti-Tate $\mathcal{O}$-group $\tilde{H}$. Let $\tilde{G} = G \times_H \tilde{H}$ be the fiber product in the category sGr_R. Let $\tilde{f} : \tilde{G} \to \tilde{H}$ be the corresponding morphism. Then $\tilde{f}_K$ is an isomorphism. Assume that there exists a homomorphism $\tilde{f}' : \tilde{H} \to \tilde{G}$ such that $\tilde{f} \circ \tilde{f}' = \pi^s \operatorname{id}_{\tilde{H}}$. Then $\xi_G \circ \tilde{f}'$ is zero on $\operatorname{Ker}(\xi_H)$ because this is true for the generic fibers. Thus there exists $f' : H \to G$ such that $f' \circ \xi_H = \xi_G \circ \tilde{f}'$. Therefore $f \circ f' \circ \xi_H = f \circ \xi_G \circ \tilde{f}' = \xi_H \circ \tilde{f} \circ \tilde{f}' = \pi^s \xi_H$ and $f \circ f' = \pi^s \operatorname{id}_H$.

By the above discussion, to prove the existence of f', we may assume that $f = \tilde{f}$ and $H = \tilde{H}$ is a special truncated Barsotti-Tate $\mathcal{O}$-group of level $n > s$. We translate the problem in terms of Breuil $\mathcal{O}$-modules. Let $\mathbb{B}(H) = (M, \phi)$ and $\mathbb{B}(G) = (N, \psi)$. By Proposition 4.4, f induces an $\mathfrak{S}$-linear monomorphism $M \hookrightarrow N$ whose cokernel is annihilated by some power u^t. Assume that t is the smallest natural number with this property. If $t = 0$, then f is an isomorphism. Thus we assume that $t > 0$. The existence of $f' : H \to G$ is equivalent to the inclusion

$$\pi^s N \subset M.$$

Now we prove by induction that, for $j \in \{0, \ldots, z\}$, $\pi^{s_j} N \subset \frac{1}{u^{t_j}} M$. For the base case $j = 0$, it suffices to show that $t \leq t_0$. Choose $x \in N$ such that $u^{t-1}x \notin M$. Write

$$x = \sum_{i=1}^{h} \frac{\alpha_i}{u^t} e_i,$$

where $(e_1, \ldots, e_h)$ is an $\mathfrak{S}_n$-basis of M as in Lemma 4.5. Then by Lemma 4.6, $E\sigma(\alpha_i) \in u^{t(q-1)}\mathfrak{S}_n$ for $1 \leq i \leq h$. By the minimality of t, there exists $1 \leq i_0 \leq h$, such that α_{i_0} is not divisible by u. Let $C = C(u) \in \mathfrak{S}$ be such that its reduction modulo π^n is α_{i_0}. The constant term of C is not divisible by π^n and $E\sigma(C) \in (u^{t(q-1)}, \pi^n)\mathfrak{S}$. Applying Proposition 4.11, we see that $t(q-1) \leq \min\{\tau e + \iota, ne\}$. Thus $t \leq t_0$ by definition of t_0.

If $0 \leq j \leq z$, the passage from $j-1$ to j goes as follows. The induction hypothesis says that $\pi^{s_{j-1}}N \subset \frac{1}{u^{t_{j-1}}}M$. Let $l_{j-1} \in \{0, \ldots, t_{j-1}\}$ be the smallest number such that $\pi^{s_{j-1}}N \subset \frac{1}{u^{l_{j-1}}}M$. If $l_{j-1} = 0$, then $\pi^{s_{j-1}}N \subset M$. Thus $\pi^{s_j}N \subset \pi^{s_{j-1}}N \subset M \subset \frac{1}{u^{t_j}}M$. Assume now that $l_{j-1} \geq 1$. Choose $y \in \pi^{s_{j-1}}N$. Write

$$y = \sum_{i=1}^{h} \frac{\eta_i}{u^{n_i}} e_i,$$

where $\eta_i \in \mathfrak{S}_n \backslash u\mathfrak{S}_n$ and $n_i \in \{0, \ldots, l_{j-1}\}$. We want to show that $\pi^{\tau+\epsilon}y \in \frac{1}{u^{t_j}}M$. For this, it suffices to show that $\pi^{\tau+\epsilon}\frac{\eta_i}{u^{n_i}} \in \frac{1}{u^{t_j}}\mathfrak{S}_n$ for all i. If $n_i \leq t_j$, this is obvious. Assume that $n_i \geq t_j + 1$. The inequality

$$n_i \geq t_j + 1 = [\frac{t_{j-1}}{q}] + 1 \geq \frac{t_{j-1}+1}{q} \geq \frac{l_{j-1}+1}{q}$$

implies that $qn_i - l_{j-1} \geq 1$. Let $C_i = C_i(u) \in \mathfrak{S}$ be such that its reduction modulo π^n is η_i. Applying Lemma 4.6, we have $E\sigma(C_i u^{l_{j-1}-n_i}) \in (u^{(q-1)l_{j-1}}, \pi^n)\mathfrak{S}$. This implies that

$$E\sigma(C_i) \in (u^{qn_i - l_{j-1}}, \pi^n)\mathfrak{S} \subset (u, \pi^n)\mathfrak{S}.$$

The constant term of C_i is not divisible by π^n. Applying Proposition 4.11 to the pair $(C_i, qn_i - l_{j-1})$ instead of (C, t), we get $\sigma(\pi^{\tau+\epsilon}C_i) = \pi^{\tau+\epsilon}\sigma(C_i) \in (u^{qn_i-l_{j-1}}, \pi^n)\mathfrak{S}$. So we may write $\pi^{\tau+\epsilon}C_i = A_i + B_i$, where $A_i \in \pi^n\mathfrak{S}$ and $B_i \in u^{n_i - [\frac{l_{j-1}}{q}]}\mathfrak{S}$. Thus

$$\pi^{\tau+\epsilon}\frac{\eta_i}{u^{n_i}} \in \frac{1}{u^{[\frac{l_{j-1}}{q}]}}\mathfrak{S}_n \subset \frac{1}{u^{[\frac{t_{j-1}}{q}]}}\mathfrak{S}_n = \frac{1}{u^{t_j}}\mathfrak{S}_n.$$

This shows that $\pi^{s_j}N \subset \frac{1}{u^{t_j}}M$ and ends the induction.

Finally, applying Lemma 4.6, the inclusion $\pi^{s_j}N \subset \frac{1}{u^{t_j}}\mathfrak{S}$ implies $\pi^{s_j+t_j}N \subset M$. In particular, we may take $j = z$ and obtain the inclusion $\pi^s N \subset M$. This finishes the proof of the theorem. □

4.5. **Some corollaries.** In this section, we deduce several consequences of Theorem 4.12. The corresponding results for p-divisible groups appear in [22, Section 1]. We refer to that paper for more details on the history of these results. The idea of the proofs are the same as in [22, Section 5].

If G is a π-divisible $\mathcal{O}$-module over R, we denote by $G\lceil\pi^n\rceil$ the schematic closure of $G_K\lceil\pi^n\rceil$ in G.

Corollary 4.13. *With the same notation as in Theorem 4.12, if H is a special truncated Barsotti-Tate $\mathcal{O}$-group of level $n > s$, then the natural homomorphism $f\lceil\pi^{n-s}\rceil : G\lceil\pi^{n-s}\rceil \to H[\pi^{n-s}]$ is an isomorphism.*

Proof. Let $f' : H \to G$ be such that $f \circ f' = \pi^s \,\mathrm{id}_H$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{\;f'\;} & G \\ & \searrow^{\pi^s} & \downarrow f \\ & & H \end{array}$$

On generic fibers we have $H_K[\pi^s] \subset \mathrm{Ker}(f'_K)$, thus $H[\pi^s] \subset \mathrm{Ker}(f')$. We obtain a second commutative diagram:

$$\begin{array}{ccc} H/(H[\pi^s]) & \xrightarrow{\;f''\;} & G \\ & \searrow^{\pi^s} & \downarrow f \\ & & H \end{array}$$

In the diagram, f'' is a closed immersion because π^s is a closed immersion. Applying the functor $\lceil \pi^{n-s} \rceil$, we obtain a third commutative diagram:

$$\begin{array}{ccc} H/(H[\pi^s]) & \xrightarrow{\;f''\lceil \pi^{n-s}\rceil\;} & G\lceil \pi^{n-s}\rceil \\ & \searrow^{\pi^s} & \downarrow f\lceil \pi^{n-s}\rceil \\ & & H[\pi^{n-s}] \end{array}$$

In this diagram, $f''\lceil \pi^{n-s}\rceil$ is a closed immersion. It is then an isomorphism because the domain and the range are finite flat group schemes of the same order. The diagonal map is an isomorphism by the assumption on H. Therefore, the map $f\lceil \pi^{n-s}\rceil$ is an isomorphism. The corollary follows. □

Corollary 4.14 (Raynaud)**.** *Let G and H be special $\mathcal{O}$-groups over R. Assume that $e \le q-2$. If G_K and H_K are isomorphic, then G and H are isomorphic.*

Proof. Since $e \le q-2$, we have $m = \iota = 0$, $\tau = 1$. In this case

$$s \le [\frac{\tau e + \iota}{q-1}] = 0.$$

Thus $s = 0$. The corollary follows. □

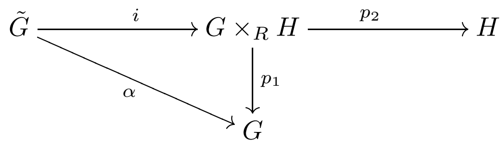
Remark 4.15. In the case $\mathcal{O} = \mathbb{Z}_p$, this is a classical result of Raynaud [18, Theorem 3.3.3]. It is proved by others with different methods. See [22, Section 1] for a detail list. The condition $e \le p-2$ is necessary, as we may see from the fact that μ_p and $\mathbb{Z}/p\mathbb{Z}$ have the same generic fiber over $\mathbb{Z}_p[\zeta_p]$, where ζ_p is a p-th root of unity.

On the other hand, in the case of higher ramification, (i.e., $e_R \ge p-1$), if the group G_K is endowed with a strict $\mathcal{O}$-action and $\mathcal{O}$ is large, then there is still at most one way to extend G_K to an integral model which also extends the $\mathcal{O}$-action.

Corollary 4.16. *Let $h : G_K \to H_K$ be a homomorphism over K. Then $\pi^s h$ extends to a homomorphism $G \to H$ over R. Moreover, the kernel of the natural homomorphism $\mathrm{Ext}^1(H,G) \to \mathrm{Ext}^1(H_K, G_K)$ is annihilated by π^s.*

Proof. Let $\tilde{G}$ be the schematic closure in $G \times_R H$ of the graph of the morphism h. Let $i : \tilde{G} \to G \times_R H$ be the corresponding closed embedding. We have a commutative diagram:

(4.6)



Let $\alpha' : G \to \tilde{G}$ be such that $\alpha' \circ \alpha = \pi^s \, \mathrm{id}_{\tilde{G}}$. Then the morphism $p_2 \circ i \circ \alpha' : G \to H$ is an extension of $\pi^s h$.

Let $\nu \in \mathrm{Ker}(\mathrm{Ext}^1(H, G) \to \mathrm{Ext}^1(H_K, G_K))$. Assume that it is represented by a short exact sequence

$$0 \to G \to J \to H \to 0, \tag{4.7}$$

whose generic fiber splits. Let $h : H_K \to J_K$ be a homomorphism that is a splitting of

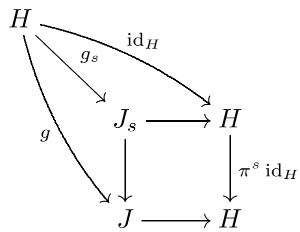
$$0 \to G_K \to J_K \to H_K \to 0.$$

Let $g : H \to J$ be an extension of $\pi^s h$. Let

$$0 \to G \to J_s \to H \to 0$$

be the pullback of (4.7) via $\pi^s \, \mathrm{id}_H$. Then by the universal property of pullback,

(4.8)



there exists a unique $g_s : H \to J_s$, such that its composite with $J_s \to J$ is g. Thus $\pi^s \nu = 0$. The corollary follows. $\square$

Corollary 4.17. *Assume that G and H are special truncated Barsotti-Tate $\mathcal{O}$-groups of level $n > s$. Let $h : G_K \to H_K$ be a homomorphism. Then the restriction homomorphism $h[\pi^{n-s}] : G_K[\pi^{n-s}] \to H_K[\pi^{n-s}]$ extends to a homomorphism $G[\pi^{n-s}] \to H[\pi^{n-s}]$.*

Proof. Let $h' : G \to H$ be an extension of $\pi^s h : G_K \to H_K$ as in Corollary 4.16. It induces a homomorphism $G[\pi^{n-s}] = G/G[\pi^s] \to H[\pi^{n-s}]$ whose generic fiber is $h[\pi^{n-s}]$. $\square$

Corollary 4.18. *Assume that $n > 2s$. Let H be a special truncated Barsotti-Tate $\mathcal{O}$-group of level n over R. Let G be a special $\mathcal{O}$-group such that we have an isomorphism $h : G_K \to H_K$. Then the quotient group scheme $G\lceil \pi^{n-s} \rceil / G\lceil \pi^s \rceil$ is isomorphic to $H[\pi^{n-2s}]$ and thus it is a truncated Barsotti-Tate $\mathcal{O}$-group of level $n - 2s$.*

Proof. The proof is exactly the same as the proof of [22, Corollary 4]. $\square$

Corollary 4.19 (Tate)**.** *Let X and Y be π-divisible $\mathcal{O}$-modules over R. Then the natural map*

$$\mathrm{Hom}_{\mathcal{O}}(X, Y) \to \mathrm{Hom}_{\mathcal{O}}(X_K, Y_K)$$

is a bijection.

Proof. Let $f \in \mathrm{Hom}(X_K, Y_K)$. For any integer $n > 0$, it induces a morphism $f[\pi^n] : X_K[\pi^n] \to Y_K[\pi^n]$. If $n > s$, the morphism $f[\pi^{n-s}] : X_K[\pi^{n-s}] \to Y_K[\pi^{n-s}]$ extends to a morphism $g_{n-s} : X[\pi^{n-s}] \to Y[\pi^{n-s}]$ by Corollary 4.17. Taking limit for $n > s$, we obtain a morphism $g : X \to Y$, which lifts the morphism $f : X_K \to Y_K$. The corollary follows. □

Acknowledgments

The author learned the theory of displays and the theory of frames and windows from lectures of Professor Zink, where Professor Zink explained most of the results in this paper in the case of p-divisible groups. The author would like to thank him for the great lectures and for helpful discussions. The author would like to thank the referee for numerous comments and suggestions.

References

[1] Tobias Ahsendorf, *$\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules*, Thesis 2012.

[2] Tobias Ahsendorf, Chuangxun Cheng, and Thomas Zink, *$\mathcal{O}$-displays and π-divisible formal $\mathcal{O}$-modules*, J. Algebra **457** (2016), 129–193, DOI 10.1016/j.jalgebra.2016.03.002. MR3490080

[3] Pierre Berthelot, Lawrence Breen, and William Messing, *Théorie de Dieudonné cristalline. II* (French), Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982. MR667344

[4] M. V. Bondarko, *The generic fiber of finite group schemes; a "finite wild" criterion for good reduction of abelian varieties* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **70** (2006), no. 4, 21–52, DOI 10.1070/IM2006v070n04ABEH002323; English transl., Izv. Math. **70** (2006), no. 4, 661–691. MR2261169

[5] Christophe Breuil, *Groupes p-divisibles, groupes finis et modules filtrés* (French, with French summary), Ann. of Math. (2) **152** (2000), no. 2, 489–549, DOI 10.2307/2661391. MR1804530

[6] Bryden Cais and Tong Liu, *On F-crystalline representations*, Doc. Math. **21** (2016), 223–270. MR3505135

[7] V. G. Drinfelʹd, *Coverings of p-adic symmetric domains* (Russian), Funkcional. Anal. i Priložen. **10** (1976), no. 2, 29–40. MR0422290

[8] Gerd Faltings, *Group schemes with strict $\mathcal{O}$-action*, Mosc. Math. J. **2** (2002), no. 2, 249–279. MR1944507

[9] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibré vectoriels en thérie de Hodge p-adique*, Preprint `https://webusers.imj-prg.fr/~laurent.fargues/Prepublications.html`.

[10] Laurent Fargues, Alain Genestier, and Vincent Lafforgue, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld* (French), Progress in Mathematics, vol. 262, Birkhäuser Verlag, Basel, 2008. MR2441311

[11] Michiel Hazewinkel, *Twisted Lubin-Tate formal group laws, ramified Witt vectors and (ramified) Artin-Hasse exponentials*, Trans. Amer. Math. Soc. **259** (1980), no. 1, 47–63, DOI 10.2307/1998143. MR561822

[12] Mark Kisin, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180, DOI 10.4007/annals.2009.170.1085. MR2600871

[13] Eike Lau, *A duality theorem for Dieudonné displays* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 2, 241–259. MR2518078

[14] Eike Lau, *Displays and formal p-divisible groups*, Invent. Math. **171** (2008), no. 3, 617–628, DOI 10.1007/s00222-007-0090-x. MR2372808

[15] Eike Lau, *Frames and finite group schemes over complete regular local rings*, Doc. Math. **15** (2010), 545–569. MR2679066

[16] Tong Liu, *Potentially good reduction of Barsotti-Tate groups*, J. Number Theory **126** (2007), no. 2, 155–184, DOI 10.1016/j.jnt.2006.11.008. MR2354925

[17] Tong Liu, *Torsion p-adic Galois representations and a conjecture of Fontaine* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 4, 633–674, DOI 10.1016/j.ansens.2007.05.002. MR2191528

[18] Michel Raynaud, *Schémas en groupes de type* $(p, \ldots, p)$ (French), Bull. Soc. Math. France **102** (1974), 241–280. MR0419467

[19] David Savitt, *On a conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197, DOI 10.1215/S0012-7094-04-12816-7. MR2137952

[20] J. T. Tate, *p-divisible groups*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 158–183. MR0231827

[21] Adrian Vasiu and Thomas Zink, *Breuil's classification of p-divisible groups over regular local rings of arbitrary dimension*, Algebraic and arithmetic structures of moduli spaces (Sapporo 2007), Adv. Stud. Pure Math., vol. 58, Math. Soc. Japan, Tokyo, 2010, pp. 461–479. MR2676165

[22] Adrian Vasiu and Thomas Zink, *Boundedness results for finite flat group schemes over discrete valuation rings of mixed characteristic*, J. Number Theory **132** (2012), no. 9, 2003–2019, DOI 10.1016/j.jnt.2012.03.010. MR2925859

[23] Thomas Zink, *A Dieudonné theory for p-divisible groups*, Class field theory—its centenary and prospect (Tokyo, 1998), Adv. Stud. Pure Math., vol. 30, Math. Soc. Japan, Tokyo, 2001, pp. 139–160. MR1846456

[24] Thomas Zink, *The display of a formal p-divisible group*, Astérisque **278** (2002), 127–248. MR1922825

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: cxcheng@nju.edu.cn