TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 2, February 2018, Pages 1251–1283
http://dx.doi.org/10.1090/tran/7014
Article electronically published on October 18, 2017

KOSZUL DUALITY AND SOERGEL BIMODULES FOR DIHEDRAL GROUPS

MARC SAUERWEIN

Abstract. Every Coxeter system (W, S) gives rise to a Hecke algebra $\mathbf{H}_{(W,S)}$ which can be categorified by the additive monoidal category of Soergel bimodules $\mathcal{SB}$. Under this isomorphism the Kazhdan-Lusztig basis $\{\underline{H}_x\}_{x\in W}$ corresponds to certain indecomposable Soergel bimodules $\{B_x\}_{x\in W}$ (up to shift). In this thesis we study the structure of the endomorphism algebra (of maps of all degrees) $\mathcal{A} := \mathrm{End}^{\bullet}_{\mathcal{SB}}\left(\bigoplus_{x\in W} B_x\right) \otimes_R \mathbb{R}$. Via category $\mathcal{O}$ it has been proven for all Weyl groups that $\mathcal{A}$ is a self-dual Koszul algebra. We extend this result to all dihedral groups by purely algebraic methods using representation theory of quivers and Soergel calculus.

Contents

1. Introduction

To any Coxeter system (W, S) one associates a Hecke algebra $\mathbf{H}_{(W,S)}$. The Hecke algebra may be categorified by Soergel bimodules $\mathcal{SB}$, an additive monoidal category of bimodules over a polynomial ring R. The indecomposable bimodules $\{B_x\}$ in $\mathcal{SB}$ are (up to grading shift) parametrised by the group W. The main object of interest in this paper is the endomorphism algebra (consisting of maps of all degrees) of $\mathbf{B} := \bigoplus_{x\in W} B_x$ where the action of polynomials of positive degree on the right is trivialised:

$$\mathcal{A} := \mathrm{End}^{\bullet}_{\mathcal{SB}}(\mathbf{B}) \otimes_R \mathbb{R}.$$

We prove the following result via purely algebraic methods.

Theorem 1. *For a dihedral group (W, S) the $\mathbb{R}$-algebra $\mathcal{A}$ is a self-dual Koszul algebra.*

Received by the editors June 15, 2015, and, in revised form, May 31, 2016, and June 27, 2016.
2010 *Mathematics Subject Classification.* Primary 16S37; Secondary 20F55.

1.1. **Motivation.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. It turns out that the category $\mathfrak{g}$- Mod of all $\mathfrak{g}$-modules is far too large to be understood algebraically. The introduction of category $\mathcal{O}$ by Bernstein, Gelfand and Gelfand (see [BGG76]) was seminal for the further study of the representation theory of $\mathfrak{g}$.

Fix a Borel $\mathfrak{b}$, a Cartan $\mathfrak{h}$ in $\mathfrak{g}$ and define $\mathcal{O} := \mathcal{O}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ to be the full subcategory of $\mathfrak{g}$- Mod whose objects M are finitely generated over $\mathfrak{g}$, $\mathfrak{h}$-semisimple and locally $\mathfrak{b}$-finite. In particular, all finite dimensional modules and Verma (= standard) modules $\Delta(\lambda)$ ($\lambda \in \mathfrak{h}^*$) lie in $\mathcal{O}$.

This restriction made it easier to handle the category and led to beautiful new results such as BGG reciprocity [BGG76, Prop. 1] and the Kazhdan-Lusztig conjectures [KL79, Conj. 1.5]. Within the principal block $\mathcal{O}_0 \subset \mathcal{O}$ let $L := \bigoplus_{w\in W} L(w\cdot 0)$ be the direct sum of the simple modules and $P := \bigoplus_{w\in W} P(w \cdot 0)$ the direct sum of their projective covers; i.e., P is a projective generator. We have the following result due to Soergel in [Soe90]:

Theorem 2 (Koszul self-duality for the principal block $\mathcal{O}_0$)**.** *There exists an isomorphism of finite dimensional $\mathbb{C}$-algebras*

$$A := \operatorname{End}_{\mathcal{O}_0}(P) \cong \operatorname{Ext}^\bullet_{\mathcal{O}_0}(L, L),$$

where the right-hand side is a ring via the cup product. Furthermore, $\operatorname{Ext}^\bullet_{\mathcal{O}_0}(L, L)$ *is a Koszul algebra.*

Although $\operatorname{End}_{\mathcal{O}_0}(P)$ is not obviously graded, it inherits a grading from the naturally graded Ext-algebra. The first glimpse of Koszul duality was discovered earlier when mathematicians were investigating composition series of Verma modules in category $\mathcal{O}_0$ and found formulas of the form

$$[\Delta(x \cdot 0) : L(y \cdot 0)] = \sum_i \dim \operatorname{Ext}^i(\Delta(w_0x \cdot 0), L(w_0y \cdot 0)).$$

These formulas can be explained by Koszul self-duality on the level of derived categories (see [BGS96, Theorem 1.2.6]). The existing proofs of Koszul self-duality are difficult and rely heavily on geometric techniques.

Using $\mathcal{O}_0 \cong$ Mod-A one obtains a $\mathbb{Z}$-graded version of $\mathcal{O}_0$ as $\mathcal{O}_0^{\mathbb{Z}} :=$ gMod- A. Soergel's combinatorial functor $\mathbb{V}$ induces an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-modules (see [Soe90] and [Str03a, Theorem 7.1]):

$$\mathcal{K}_0\left(\mathcal{O}_0^{\mathbb{Z}}\right) \xrightarrow{\sim} \mathcal{K}_0^s\left(\mathcal{S}\right),$$
$$P(x \cdot 0)\langle i\rangle \longmapsto \mathbb{V}P(x \cdot 0)\langle i\rangle = \overline{B_x}\langle i\rangle,$$

where $\overline{B} := B \otimes_R \mathbb{R}$ is the Soergel module corresponding to the Soergel bimodule B and $\mathcal{S}$ denotes the category of graded Soergel modules. Translating Theorem 2 into the setting of Soergel (bi)modules via the above identification frees the result from geometry and yields

$$\mathcal{A} \overset{\text{Thm. 2}}{\cong} \operatorname{Ext}^\bullet_{\mathcal{O}_0}(L, L) \cong E(\mathcal{A}),$$

where $E(\mathcal{A})$ denotes the Koszul dual of a Koszul algebra.

1.2. **Structure of the paper.** This paper contains two parts.

Part 1: In the first four sections we provide the necessary background on Hecke algebras, Soergel bimodules, Soergel calculus and Koszul algebras.

Part 2: In the last two sections we realise the endomorphism ring of Soergel modules as the path algebra of a quiver and show that it is Koszul self-dual.

2. Preliminaries

2.1. **Basic definitions.** Let (W, S) be a Coxeter system. Recall that W is equipped with the Bruhat order $\leq$ and the length function $\ell : W \to \mathbb{N}_0$ which counts the number of simple reflections in a reduced expression. For an arbitrary sequence $\underline{w} = (s_1, s_2, \dots, s_n)$ in S we denote the product $s_1 s_2 \cdots s_n$ by w, viewed as an element in W. For such a sequence $\underline{w}$ its length is defined by $\ell(\underline{w}) = n$. Observe that we have $\ell(\underline{w}) \geq \ell(w)$ and equality holds if and only if $\underline{w}$ is a reduced expression for w. By abuse of notation we write $\underline{w} = s_1 s_2 \cdots s_n$. It is crucial to distinguish between w and $\underline{w}$, since the latter denotes a distinct sequence of simple reflections whereas w is their product in W.

Following Soergel's normalisation as in [Soe97] we define the Hecke algebra $\mathbf{H} = \mathbf{H}_{(W,S)}$ as the unital, associative $\mathbb{Z}[v^{\pm 1}]$-algebra generated by $\{H_s\}_{s \in S}$ subject to the relations

$$H_s^2 = 1 + (v^{-1} - v) H_s, \tag{1}$$

$$\underbrace{H_s H_t H_s \dots}_{m_{st} \text{ factors}} = \underbrace{H_t H_s H_t \dots}_{m_{st} \text{ factors}}, \tag{2}$$

for all $s \neq t \in S$. Here v is just an indeterminant.

Given a reduced expression $\underline{w} = s_1 s_2 \cdots s_n$ we set $H_w := H_{s_1} \cdots H_{s_n}$, which is well-defined by the Lemma of Matsumoto (see [Mat64]). The elements $\{H_w\}_{w \in W}$ form the *standard basis* of $\mathbf{H}$ as a $\mathbb{Z}[v^{\pm 1}]$-module. An easy calculation shows the following.

Lemma 2.1. *Let $w \in W$ and $s \in S$. For a basis element H_w and a generator H_s we have the following multiplication rule:*

$$H_w H_s = \begin{cases} H_{ws} & \text{if } ws > w, \\ H_{ws} + (v^{-1} - v) H_w & \text{if } ws < w. \end{cases} \tag{3}$$

Each H_s for $s \in S$ is invertible with inverse $H_s + (v - v^{-1})$, and thus all standard basis elements are units. There is a unique $\mathbb{Z}$-linear involution $\bar{\cdot} : \mathbf{H} \to \mathbf{H}$ such that $v \mapsto v^{-1}$ and $H_s \mapsto H_s^{-1}$. This involution is called *duality* and it is easily checked that $H_w \mapsto H_{w^{-1}}^{-1}$ holds for $w \in W$.

Theorem 2.2 ([KL79])**.** *There exists a unique $\mathbb{Z}[v^{\pm 1}]$-basis $\{\underline{H}_w\}_{w \in W}$ of $\boldsymbol{H}$ consisting of self-dual elements such that*

$$\underline{H}_w = H_w + \sum_{x < w} h_{x,w} H_x,$$

where $h_{x,w} \in v\mathbb{Z}[v]$.

This basis is called the *Kazhdan-Lusztig basis* and the $h_{x,w}$ are the *Kazhdan-Lusztig polynomials*. Note that the $h_{x,w}$ are not the originally defined Kazhdan-Lusztig polynomials $p_{x,w}$ (see [KL79]), but there is the following relation (see [Soe97, Rem. 2.6.]):

$$h_{x,w}(v) = v^{\ell(w) - \ell(x)} p_{x,w}(v^{-2}).$$

Mimicking the notion of a trace form from linear algebra we define a trace on $\mathbf{H}$ to be a $\mathbb{Z}[v^{\pm 1}]$-linear map $\varepsilon : \mathbf{H} \to \mathbb{Z}[v^{\pm 1}]$ satisfying $\varepsilon(h_1 h_2) = \varepsilon(h_2 h_1)$ for all $h_1, h_2 \in \mathbf{H}$. The *standard trace* on $\mathbf{H}$ is defined via $\varepsilon(H_w) := \delta_{w,e}$.

Lemma 2.3. *For w and $w' \in W$ we have $\varepsilon(H_w H_{w'}) = \delta_{w,(w')^{-1}}$.*

Proof. Induction over the length of w' combined with the multiplication rule from Lemma 2.1. □

For a sequence $\underline{w} = s_1 s_2 \cdots s_n$ we call $\mathbf{e} = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_n$ with $\mathbf{e}_i \in \{0,1\}$ a 01-*(sub)sequence* of $\underline{w}$ which picks out the subsequence $\underline{w}^{\mathbf{e}} := s_1^{\mathbf{e}_1} s_2^{\mathbf{e}_2} \cdots s_n^{\mathbf{e}_n}$. Given such a 01-sequence its *Bruhat stroll* is the sequence $e, x_1, \dots, x_n = \underline{w}^{\mathbf{e}}$ where

$$x_i := s_1^{\mathbf{e}_1} s_2^{\mathbf{e}_2} \cdots s_i^{\mathbf{e}_i}. \tag{4}$$

This stroll allows us to decorate each step of the 01-sequence $\mathbf{e}$ with either U(p) or D(own) encoding the path in the Bruhat graph. We assign U to the index i if $x_{i-1} s_i > x_{i-1}$ and D if $x_{i-1} s_i < x_i$.

There is a $\mathbb{Z}[v^{\pm 1}]$-linear anti-involution ι satisfying $\iota(H_s) = H_s$ for $s \in S$. It is easily checked that $\iota(H_w) = H_{w^{-1}}$ for $w \in W$. Note that ι and $\bar{\cdot}$ commute; we denote their composition as ω. It follows that ω is a $\mathbb{Z}$-linear anti-involution on $\mathbf{H}$ satisfying $\omega(v) = v^{-1}$ and $\omega(H_w) = H_w^{-1}$ for $w \in W$. Using the aforementioned standard trace we can define the *standard pairing* $\mathbf{H} \times \mathbf{H} \to \mathbb{Z}[v^{\pm 1}]$ by $(h, h') := \varepsilon(\omega(h) h')$.

2.2. **Notation (following** [Eli16]**).** Let (W, S) be a *dihedral group of type $I_2(m)$* $(m \geq 3)$, that is, a Coxeter system $(W, \{s, t\})$ where $(st)^m = e$. The elements in $S = \{s, t\}$ are called *simple reflections* or *colours*. As before, we denote a sequence of simple reflections by $\underline{w}$ and shorten expressions of length ≥ 1 by

$${}_s\underline{k} := \underbrace{sts\dots}_{k \text{ factors}}, \qquad \underline{k}_s := \underbrace{\dots sts}_{k \text{ factors}},$$

similarly for t. Omitting the underline means the corresponding product in W. We write $e = {}_s0 = {}_t0$ for the identity and $w_0 = {}_sm = {}_tm$ for the longest element in W. The restriction to dihedral groups makes it possible to obtain a closed formula for all Kazhdan-Lusztig basis elements simultaneously.

Proposition 2.4. *Let $(W, \{s, t\})$ be a dihedral group and $w \in W$; then*

$$\underline{H}_w = \sum_{x \leq w} v^{\ell(w) - \ell(x)} H_x.$$

Proof. Induction over the length of w (see [Her99, Bsp. 2.3]). □

For arbitrary Coxeter groups there is no such formula, and the computation is carried out inductively following for example the proof of Theorem 2.2 in [Soe97, Thm. 2.1]. However, the Kazhdan-Lusztig basis element $\underline{H}_{w_0}$ of a finite Coxeter group with longest element w_0 can always be computed using the formula in Proposition 2.4 (see [KL79]). From Proposition 2.4 we can easily deduce the following lemma.

Lemma 2.5. *For $w \in W$ we have $\omega(\underline{H}_w) = \underline{H}_{w^{-1}}$.*

3. Soergel bimodules

For a Coxeter system (W, S) of type $I_2(m), m \geq 3$, we define the *geometric representation* $\mathfrak{h} = \mathbb{R}\alpha_s^\vee \oplus \mathbb{R}\alpha_t^\vee$ with its Cartan matrix (see [Hum90]):

$$\begin{pmatrix} 2 & -2\cos\left(\frac{\pi}{m}\right) \\ -2\cos\left(\frac{\pi}{m}\right) & 2 \end{pmatrix}. \tag{5}$$

We fix this realisation once and for all. Consider $R := S(\mathfrak{h}^*) = \bigoplus_{i\geq 0} S^i(\mathfrak{h}^*)$, the symmetric algebra on $\mathfrak{h}^*$, a graded $\mathbb{R}$-algebra with $\deg(\mathfrak{h}^*) = 2$. By construction, W acts on $\mathfrak{h}$ and hence it acts on $\mathfrak{h}^*$ via the contragredient representation. Extending this action by a grading preserving automorphism yields an action of W on R. The ring of invariants of a single simple reflection $s \in S$ is denoted by $R^s \subseteq R$.

The two main module categories in this thesis are R- Bim and R- gBim, the category of finitely generated R-bimodules and graded R-bimodules respectively (the latter with grading preserving morphisms). The category R- gBim is considered as a graded category with the grading shift down denoted by $\langle n\rangle$. For $M = \bigoplus M_i$ we define $M\langle n\rangle_i := M_{i+n}$. Moreover, for two graded bimodules M and N we write $\mathrm{Hom}^\bullet(M, N) := \bigoplus_{n\in\mathbb{Z}}(M, N\langle n\rangle)$ for the bimodule homomorphisms between M and N of all degrees.

Remark 3.1. Realisations of Coxeter groups can be defined in more generality as modules over a commutative domain and do not have to be symmetric either. For the more general case see [EW13].

For $s \in S$ define the graded R-bimodule $B_s := R \otimes_{R^s} R\langle 1\rangle$ which is a fixed graded lift of the R-bimodule $R \otimes_{R^s} R$. We often write $\otimes_s := \otimes_{R^s}$, and the tensor product structure $\otimes_R$ is denoted as juxtaposition. For a given sequence $\underline{w} = s_1 s_2 \cdots s_n$ define the corresponding *Bott-Samelson bimodule* by

$$B_{\underline{w}} := B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_n} = B_{s_1} B_{s_2} \cdots B_{s_n}.$$

The full monoidal subcategory $\mathcal{BS}$ of R- Bim generated by B_s for $s \in S$ is called the *category of Bott-Samelson bimodules*. Since we chose a fixed graded lift for $R \otimes_{R^s} R$, we have a graded lift for every Bott-Samelson bimodule. Finally, the *category of Soergel bimodules* $\mathcal{SB}$ is defined to be the Karoubian envelope of the additive closure of this graded version of $\mathcal{BS}$. It is crucial to distinguish that $\mathcal{BS}$ is a subcategory of R- Bim, whilst $\mathcal{SB}$ is a subcategory of R- gBim, and therefore morphisms between Soergel bimodules are grading preserving. Observe that $\mathcal{SB}$ is additive but not abelian. Soergel classified the indecomposable objects in $\mathcal{SB}$ (see [Soe97, Thm. 6.14]).

Theorem 3.2 (Classification of indecomposable Soergel bimodules)**.** *Given any reduced expression $\underline{w}$ of $w \in W$, the Bott-Samelson $B_{\underline{w}}$ contains up to isomorphism a unique indecomposable summand B_w which does not occur in $B_{\underline{y}}$ for any expression $\underline{y}$ of $y \in W$ with $\ell(y) < \ell(w)$. In addition, up to isomorphism B_w does not depend on the reduced expression $\underline{w}$. A complete set of representatives of the isomorphism classes of all indecomposable Soergel bimodules is given by*

$$\{B_w\langle m\rangle \mid w \in W \text{ and } m \in \mathbb{Z}\}.$$

Note that the split Grothendieck group $\mathcal{K}_0^s(\mathcal{C})$ of an additive, monoidal and graded category $\mathcal{C}$ inherits a $\mathbb{Z}[v^{\pm 1}]$-algebra structure. Soergel proved that the category of Soergel bimodules $\mathcal{SB}$ categorifies the Hecke algebra $\mathbf{H}$ (see [Soe07, Thm. 1.10]) for certain reflection faithful realisations over infinite fields of characteristic $\neq 2$. However, Libedinsky showed in [Lib08a] that this categorification works for the geometric representation as well.

Theorem 3.3 (Soergel's Categorification Theorem)**.** *For the geometric realisation* $\mathfrak{h}$ *there is a unique isomorphism of* $\mathbb{Z}[v^{\pm 1}]$*-algebras given by*

$$\begin{aligned} \varepsilon : \boldsymbol{H} &\xrightarrow{\sim} \mathcal{K}_0^s(\mathcal{SB}), \\ \underline{H}_s &\mapsto [B_s]. \end{aligned}$$

Using the standard pairing on $\mathbf{H}$ it is possible to describe the graded rank of the homomorphism space between two Soergel bimodules (see [Soe07, Thm. 5.15]).

Theorem 3.4 (Soergel's Hom-Formula)**.** *Given any two Soergel bimodules* B *and* B'*, the homomorphism space* $\operatorname{Hom}^\bullet_{\mathcal{SB}}(B, B')$ *is free as a left (resp. right)* R*-module, and its graded rank is given by* $(\varepsilon^{-1}[B], \varepsilon^{-1}[B'])$ *where* $(-,-)$ *denotes the standard pairing on the Hecke algebra.*

Soergel constructed an inverse map to ε which he called the character map ch. However, the construction is not explicit, and he conjectured what the pre-images of the indecomposable Soergel bimodules are for an arbitrary Coxeter group.

Conjecture 3.5 (Soergel's Conjecture)**.** *If* k *is a field of characteristic* 0*, then* $\operatorname{ch}(B_w) = \underline{H}_w$*.*

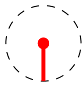
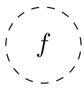
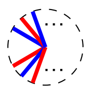
Soergel himself proved this for Weyl groups and dihedral groups (see [Soe98, Thm. 2]). The case of universal Coxeter groups was shown by Fiebig in [Fie08]. This conjecture is a very deep result and implies the Kazhdan-Lusztig positivity conjectures ([KL79]). Elias and Williamson recently gave the first algebraic proof for an arbitrary Coxeter group with a fixed reflection-faithful representation over $\mathbb{R}$ (see [EW14]). By a result of Libedinsky in [Lib08a] this includes every finite Coxeter group with its geometric realisation. In this paper we consider only dihedral groups and their geometric realisations over $\mathbb{R}$, and thus we can use Soergel's Conjecture for our calculations to determine the dimensions of homomorphism spaces between Soergel bimodules.

Recall that we denote the maximal ideal in $R = \bigoplus_{i \geq 0} S^i(\mathfrak{h}^*)$ by $S^+ := \bigoplus S^i(\mathfrak{h}^*)$. Therefore we can view $\mathbb{R} \cong R/S^+$ as an R-bimodule. To each Soergel bimodule M we can associate the *Soergel module* $\overline{M} := M \otimes \mathbb{R}$ where the action of R on the right is trivialised. Let $\mathcal{S}$ denote the *category of Soergel modules.* For a Weyl group Soergel proved that there is an isomorphism

$$\operatorname{Hom}^\bullet_{\mathcal{S}}(\overline{B}, \overline{B}') \cong \operatorname{Hom}^\bullet_{\mathcal{SB}}(B, B') \otimes_R \mathbb{R} \tag{6}$$

for $B, B' \in \mathcal{SB}$ (see [Soe98, Thm. 2, Part 4]). Moreover, he conjectured this isomorphism for every finite Coxeter group (see [Soe98, Thm. 2, Part 5]), which he proved recently ([Soe14]). For simplicity we write by abuse of notation $\operatorname{Hom}_{\mathcal{S}}(B, B')$ instead of $\operatorname{Hom}_{\mathcal{S}}(\overline{B}, \overline{B}')$ for Soergel bimodules B, B'. The important results for this

TABLE 1. Generating morphisms and their degrees in $\mathcal{D}$

	deg 1	$B_s \longrightarrow R$	$a \otimes b \mapsto ab$
	deg 1	$R \longrightarrow B_s$	$1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$
	deg -1	$B_s B_s \longrightarrow B_s$	$1 \otimes g \otimes 1 \mapsto \partial_s(g) \otimes 1$
	deg -1	$B_s \longrightarrow B_s B_s$	$1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1$
	deg$f$	$R \longrightarrow R$	$1 \mapsto f$
	deg 0	$\underbrace{B_s B_t B_s \cdots}_{m_{st} \text{ factors}} \longrightarrow \underbrace{B_t B_s B_t \cdots}_{m_s t \text{ factors}}$	

paper are summarised in the following theorem, which is an immediate consequence of the above:

Theorem 3.6. *Given a dihedral group W, we have for $x, y \in W$:*

- $\operatorname{grdim}_{\mathbb{R}} \operatorname{Hom}^\bullet_{\mathcal{S}}(B_x, B_y) = (\underline{H}_x, \underline{H}_y) = \sum_{a \leq x,y} v^{\ell(x)+\ell(y)-2\ell(a)}$,
- $\dim_{\mathbb{R}} \operatorname{Hom}^\bullet_{\mathcal{S}}(B_x, B_y) = |W_{\leq x} \cap W_{\leq y}|$

where $W_{\leq x} := \{w \in W \mid w \leq x\}$. In particular $\operatorname{Hom}^\bullet_{\mathcal{S}}(B_x, B_y)$ is concentrated in non-negative degrees d such that $0 \leq d \leq \ell(y) - \ell(x)$ for $x \leq y$ with $\dim_{\mathbb{R}} \operatorname{Hom}_{\mathcal{S}}(B_x, B_y) = \delta_{x,y}$.

4. Soergel calculus (in the dihedral case)

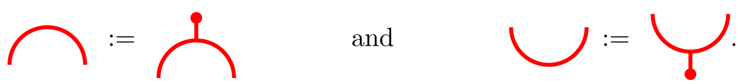
By construction the category of Soergel bimodules $\mathcal{SB}$ is the Karoubian envelope of $\mathcal{BS}$, and therefore it is enough to describe $\mathcal{BS}$ by planar graphs and identify the idempotents. In this section we introduce a diagrammatic approach to the category of Bott-Samelson bimodules $\mathcal{BS}$ following [EW13]. These techniques are what we refer to as *Soergel calculus.*

4.1. **Generators.** Recall that we fixed the geometric realisation $\mathfrak{h}$ for our given dihedral group $(W, \{s, t\})$ of type $I_2(m)$ for $m \geq 3$. The Bott-Samelson bimodule $B_{\underline{w}} = B_s \otimes B_t \otimes \cdots \otimes B_s$ for $\underline{w} = st \cdots s$ is completely determined by an ordered sequence of colours (or coloured dots on a horizontal line). A morphism between two Bott-Samelson bimodules from $B_{\underline{w}}$ to $B_{\underline{w'}}$ is given by a linear combination of isotopy classes of decorated graphs with coloured edges in the planar stripe $\mathbb{R} \times [0, 1]$ such that in each summand the edges induce sequences of coloured dots on the bottom boundary $\mathbb{R} \times \{0\}$ (resp. the top boundary $\mathbb{R} \times \{1\}$) corresponding to $\underline{w}$ (resp. $\underline{w'}$). In particular, these diagrams represent morphisms from the bottom sequence to the top sequence and therefore should be read from bottom to top.

Definition 4.1. For a dihedral group (W, S) define $\mathcal{D} = \mathcal{D}_{(W,S)}$ to be the $\mathbb{R}$-linear monoidal category as follows: The objects are sequences $\underline{w}$ in S (which are denoted sometimes by $B_{\underline{w}}$). The empty sequence $\emptyset$ is often denoted by $\mathbf{1}$. The Hom-spaces are $\mathbb{Z}$-graded $\mathbb{R}$-vector spaces generated by the diagrams in Table 1 modulo local relations. The monoidal structure is the concatenation of sequences.

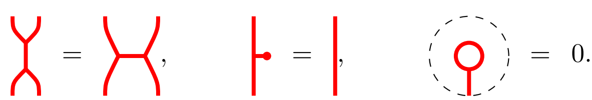
For $s \in S$ the *Demazure operator* $\partial_s : R \to R^s$ in Table 1 is defined as $\partial_s(f) := \frac{f - sf}{\alpha_s}$. The first two morphisms in Table 1 are called *dots*, whereas the next two morphisms are called *trivalent vertices* and the last morphism is called the $2m_{s,t}$-*valent vertex*. The explicit formula for the $2m_{s,t}$-valent vertex is very difficult. Therefore we only explain what the morphism does. In the case of a dihedral group of type $I_2(m)$ the longest element w_0 can be expressed as $st \cdots = ts \cdots$ with m factors on each side, and by Theorem 3.2 both $B_s B_t \cdots$ and $B_t B_s, \cdots$ contain B_{w_0} as summand with multiplicity 1. The $2m_{s,t}$-vertex is the projection and inclusion of this summand and therefore uniquely determined up to a scalar.

Libedinsky showed in [Lib08b] that the morphisms from Table 1 generate all morphisms in $\mathcal{BS}$. For the compositions of a trivalent vertex with a dot we define *caps* and *cups* as follows:

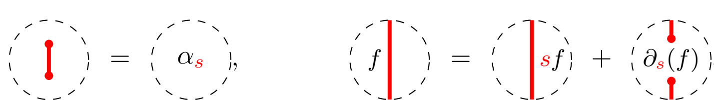
(7) 

4.2. **Relations.** Since we work with dihedral groups, only one- and two-colour relations can occur. For Coxeter groups of rank ≥ 3, there exist the three-colour or *Zamolodzhikov* relations which are more difficult (see [EW13]).

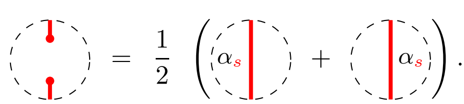
4.2.1. *The one-colour relations.* The object B_s is a Frobenius object in R-Bim (see [EK10]) where the dots correspond to unit and counit while the trivalent vertices correspond to the multiplication and comultiplication. Altogether we obtain that any one-coloured diagram is isotopy invariant and we have the following three non-polynomial relations:

(8) 

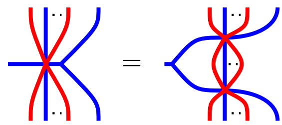
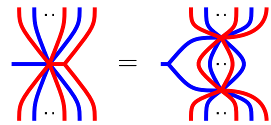
We refer to these relations as *general associativity*, *general unit* and *the needle*. Furthermore, there are two relations involving polynomials:

(9) 

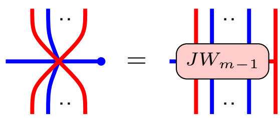
As a direct consequence we have

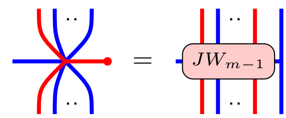
(10) 

4.2.2. *The two-colour relations.* There are three two-colour relations. The first is the cyclicity of the $2m_{s,t}$-vertex. The other two relations specify how the $2m_{s,t}$-vertex interacts with trivalent vertices and dots. For $m = m_{s,t}$ we have the *two-colour associativity* depending on the parity of m:

(11) m even:  m odd: 

The other relation allows us to express a diagram involving a $2m_{s,t}$-vertex and a dot as a linear combination of diagrams without the $2m_{s,t}$-vertex. This procedure depends on the parity of the integer m as well:

(12) m even: 

m odd: 

Note that the Jones-Wenzl morphism JW_{m-1} is an $\mathbb{R}$-linear combination of graphs consisting only of dots and trivalent vertices. This morphism will be discussed briefly in the next paragraph.

4.3. **Jones-Wenzl morphisms.**

4.3.1. *Gauss's q-numbers.* Before we can define the Jones-Wenzl projector it is helpful to recall Gauss's q-numbers. We use them to give formulas for the Jones-Wenzl projector for all dihedral groups simultaneously. Gauss's q-numbers are defined (see e.g. [Jan96]) by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} +^{-n+3} + \cdots + q^{n-3} + q^{n-1} \in \mathbb{Z}[q^{\pm 1}].$$

It is convenient to define $[0] := 0$. In most cases we omit the index and write $[n]$ instead of $[n]_q$. Observe that $[n]$ is the character of $L(n-1)$, the simple $\mathfrak{sl}_2(\mathbb{C})$-module of dimension n. Via the Clebsch-Gordon formula we obtain two out of many useful identities for q-numbers:

$$[2][n] = [n+1] + [n-1], \tag{13}$$

$$[n]^2 = [n-1][n+1] + [1]. \tag{14}$$

We can specialise q to a value ζ and denote the specialisation of $[n]_q$ by $[n]_\zeta$. If $\zeta = e^{2\pi i/2m} \in \mathbb{C}$, i.e., a primitive $2m$-th root of unity, we obtain algebraic integers $[n]_\zeta \in \mathbb{R}$. By choice ζ is primitive and therefore we have $\zeta^m = -1$. Thus, we get the following identities:

$$[m]_\zeta = 0, \qquad [m-i]_\zeta = [i]_\zeta, \qquad [m+i]_\zeta = -[i]_\zeta. \tag{15}$$

Recall the geometric representation of a dihedral group as in (5) and note that $[2]_\zeta = \zeta + \zeta^{-1} = 2\cos\left(\frac{\pi}{m}\right) = -a_{s,t}$. Hence, the geometric representation can be encoded in the Cartan matrix

$$\begin{pmatrix} 2 & -[2] \\ -[2] & 2 \end{pmatrix} \tag{16}$$

simultaneously for all dihedral groups. For a certain dihedral group of type $I_2(m)$ we only have to specialise q to a primitive $2m$-th root of unity.

Lemma 4.2. *For $i \geq 1$ we have $i_s = \cdots ts \in W$ of length i, and thus*

$$i_s(\alpha_t) = \begin{cases} [i]_\zeta \alpha_t + [i+1]_\zeta \alpha_s & \text{if } i \text{ is odd,} \\ [i+1]_\zeta \alpha_t + [i]_\zeta \alpha_s & \text{if } i \text{ is even.} \end{cases}$$

Proof. We prove this by induction on i. For $i = 1$ we have

$$s(\alpha_t) = \alpha_t - \langle \alpha_t, \alpha_s^\vee \rangle \; \alpha_s = \alpha_t + [2]_\zeta \alpha_s.$$

Now let $i > 1$. There are two cases to distinguish depending on the parity of i. If i is odd, then

$$\begin{aligned} (i+1)_s(\alpha_t) \;=\; t(i_s(\alpha_t)) &= t([i]_\zeta \alpha_t + [i+1]_\zeta \alpha_s) \\ &= (-[i]\zeta + [2]_\zeta [i+1]_\zeta)\alpha_t + [i+1]_\zeta \alpha_s \\ &= (-[i]_\zeta + [i]_\zeta + [i+2]_\zeta)\alpha_t + [i+1]_\zeta \alpha_s \\ &= [i+2]_\zeta \alpha_t + [i+1]_\zeta \alpha_s. \end{aligned}$$

The other case is similar, which finishes the proof. □

4.3.2. *Jones-Wenzl projectors.* We give a short introduction to Jones-Wenzl projectors with a recursive computation formula in the Temperley-Lieb algebra setting where its origins lie (see [Jon86, Wen87]). For a detailed discussion we refer the reader to [CK12] or [Eli16].

The *Temperley-Lieb algebra* TL_n on n strands is a diagram algebra over $\mathbb{Z}[\delta]$. It has a basis consisting of crossingless matchings with n points on bottom and top. The multiplication is given by vertical concatenation of diagrams such that circles evaluate to the scalar $-\delta$. The crossingless matching on n strands is the unit in TL_n and is denoted by $\mathbf{1}_n$. This algebra is contained in the *Temperley-Lieb category*, which is closely related to the quantum group $U := U_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ via the base change $\delta \mapsto [2]_q = q + q^{-1}$ (see [Eli16] for technical details).

Let V_k be the irreducible representation of highest weight q^k and let $V = V_1$. The highest non-zero projection from $V^{\otimes n}$ to V_n is known as the *Jones-Wenzl projector* $JW_n \in TL_n$.

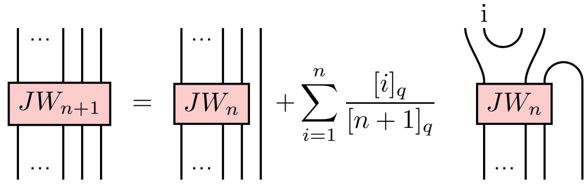
Remark 4.3. This definition of the Jones-Wenzl projector only works over the complex numbers. The categorification over the integers is harder and more subtle and was carefully treated in [CK12, FSS12, Roz14].

The following proposition states some important properties of JW_n (see [KL94, Prop. 3.2.2], [Lic97, Lem. 13.2] and [CK12, Sec. 2.2]).

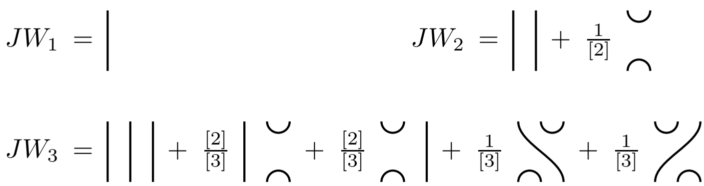
Proposition 4.4. *The Jones-Wenzl projector JW_n satisfies the following properties:*

- *JW_n is the unique map which is killed when any cap is attached on top or any cup on bottom, and for which the coefficient of $\mathbf{1}_n$ is 1.*
- *JW_n is invariant under horizontal/vertical reflection.*
- *The ideal $\langle JW_n \rangle \trianglelefteq TL_n$ has rank 1.*
- *Any element $x \in TL_n$ acts on JW_n by its coefficient of $\mathbf{1}_n$.*

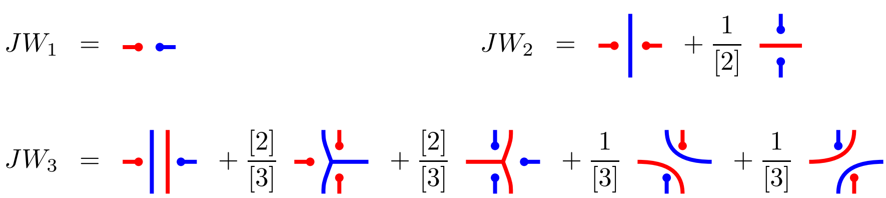
Note that the first property gives an alternative way of defining JW_n. For our computations we need the following recursive formula for JW_n (see [FK97, Thm. 3.5]):

(17)


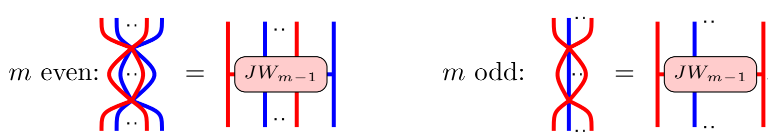
In the following we state the Jones-Wenzl projectors for $n = 1, 2, 3$:



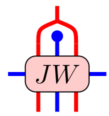
Any crossingless matching in TL_{m-1} divides the planar stripe in m regions, which we can colour alternatingly with red and blue (for s and t). This results in the definition of the *two-coloured Temperley-Lieb algebra*, which we omit here; details can be found in [Eli16, EW13]. Since our chosen realisation is symmetric we can treat a blue circle surrounded by red just as a red circle surrounded by blue and thus evaluate both to the same value. For each diagram there are two possible colourings. Each of those coloured crossingless matchings yields a coloured graph. Deformation retract each region into a tree consisting of trivalent and univalent vertices; colour those resulting trees according to the region. In this way we can associate to each coloured Jones-Wenzl projector a *Jones-Wenzl morphism* JW_n living on the disc in the place. We state the Jones-Wenzl morphisms (with red appearing in the far left region) for $n = 1, 2, 3$:



The Jones-Wenzl morphism above is not yet a morphism in $\mathcal{D}$ but can be plugged into another diagram to obtain a graph in the planar stripe (see (12); for technical details see [Eli16]). We often write JW instead of JW_{m-1} when specialised to ζ. Using the two-colour relations (see (11) (12)) we obtain the following relation between the $m_{s,t}$-valent vertex and the Jones-Wenzl projector JW_{m-1}. Since the latter is an idempotent in the Temperley-Lieb algebra (see [Eli16]), the right-hand sides in (18) are also idempotents. Thus, the $m_{s,t}$-valent vertex can be used to construct idempotents:

(18)


The two defining properties of JW_n (see Proposition 4.4 or [Eli16, Claim 4.4]) are crucial, and we use them repeatedly in our category $\mathcal{D}$. The first one is that the coefficient of one single graph is 1, namely the graph which becomes the identity morphism if trivalent vertices are attached to both sides (precisely the first summand in the examples above). The second and more important property is that JW is killed by "cups" and "caps":

(19) 

$= 0.$

We refer to this property as *death by pitchfork*.

4.4. **Light leaf and double leaf morphism.** The concept of light leaves and double leaves was introduced by Libedinsky for Soergel bimodules in [Lib08b]. Elias and Williamson transferred those morphisms into the combinatorial and diagrammatic setting of $\mathcal{D}$ (see [EW13]), where they used the double leaves to prove the equivalence in Theorem 4.5. For all (technical) details and results we refer to [EW13]. The set of light leaf morphisms $LL_{\underline{x},e}$ forms a (left) R-basis of $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \mathbf{1})$. This basis can be constructed inductively and is parametrised by the subexpressions e of $\underline{x}$ which expresses the neutral element in W. Composing these light leaf morphisms we obtain the set of double leaf morphisms $\mathbb{LL}_{\underline{x},\underline{y}}$, which forms a (left) R-basis of $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$ (see [EW13, Thm. 6.11]). Consequently, $\mathbb{LL}_{\underline{x},\underline{y}}$ is constructed inductively, too, and its elements are indexed by certain subexpressions of both $\underline{x}$ and $\underline{y}$. Hence all Hom-spaces in $\mathcal{D}$ are free graded (left) R-modules.

4.5. **Equivalence of categories.** Following [Eli16] we define an $\mathbb{R}$-linear monoidal functor $\mathcal{F} : \mathcal{D}_{(W,S)} \to \mathcal{BS}$ which maps a sequence $\underline{w}$ to the Bott-Samelson bimodule $B_{\underline{w}}$. The images of the morphisms under $\mathcal{F}$ are shown in Table 1. The next result is due to Elias (see [Eli16]):

Theorem 4.5. *The $\mathbb{R}$-linear monoidal functor $\mathcal{F}$ is well-defined and yields an equivalence of monoidal categories:*

$$\mathcal{F} : \mathcal{D}_{(W,S)} \to \mathcal{BS}.$$

Moreover, $\mathcal{F}$ induces an equivalence of monoidal additive categories:

$$\mathcal{F} : \mathrm{Kar}(\mathcal{D}_{(W,S)}) \to \mathcal{SB}.$$

Remark 4.6. The results in Theorem 4.5 were preceded by results by Elias and Khovanov in type A (see [EK10]) and by Libedinsky (see [Lib10]) for the right-angled case. Furthermore, the above results were generalised by Elias and Williamson (see [EW13, Thm. 6.28]) to all Coxeter groups with a fixed reflection-faithful realisation.

5. Koszul and quasi-hereditary algebras

Definition 5.1 ([BGS96])**.** A graded ring $A = \bigoplus_{i \geq 0} A_i$ is called *Koszul* if A_0 is semisimple and A_0 admits a *linear resolution* as a graded left A-module, i.e.,

$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow A_0,$$

where the maps are grading preserving and P^i is generated by $(P^i)_i$ as a left A-module.

Remark 5.2. Equivalently one could ask that every simple A-module admit a linear resolution.

Definition 5.3 ([BGS96])**.** A graded ring $A = \bigoplus_{i\geq 0} A_i$ is called *quadratic* if A_0 is semisimple and A is generated over A_0 by A_1 with relations in degree 2.

For a given k-vector space V the tensor algebra $T_k V$, the symmetric algebra SV and the exterior algebra ΛV are quadratic rings (with the usual grading, that is, $\deg X = 1$). In particular $k[X]$ and $k[X]/(X^2)$ are quadratic. If A satisfies some finiteness conditions we can define a quadratic dual of A:

Definition 5.4 ([BGS96])**.** A graded ring $A = \bigoplus_{i\geq 0} A_i$ is called *left finite* if all A_i are finitely generated as left A_0-modules. For a left finite quadratic ring $A = T_{A_0}A_1/(R)$ its *quadratic dual* is defined as $A^! = T_{A_0}A_1^*/(R^\perp)$ where $R^\perp \subset A_1^* \otimes_{A_0} A_1^* \cong (A_1 \otimes_{A_0} A_1)^*$ with respect to the standard pairing. Here, $T_{A_0}A_1$ denotes the free tensor algebra of the A_0-bimodule A_1. The ring A is *self-dual* if $A \cong A^!$.

For any positively graded algebra $A = \bigoplus_{i\geq 0} A_i$ the degree 0 part A_0 is an A-module. Consider the graded ring $E(A) := \operatorname{Ext}^\bullet_A(A_0, A_0)$ of self-extensions of A_0. We call $E(A)$ the *Koszul dual* of A. Beilinson, Ginzburg and Soergel showed that there exist isomorphisms relating a Koszul algebra A to its Koszul dual $E(A)$ and quadratic dual $A^!$ (see [BGS96, Cor. 2.3.3, Thm. 2.10.1, Thm. 2.10.2]).

Theorem 5.5 ([BGS96])**.** *Let $A = \bigoplus_{i\geq 0} A_i$ be a Koszul ring. Then A is quadratic. If additionally A is left finite, then there are canonical isomorphisms $E(A) \cong (A^!)^{op}$ and $E(E(A)) \cong A$.*

Remark 5.6. If A is a positively graded algebra there exists a quadratic duality functor on the bounded derived categories

$$K : \mathcal{D}^b(A\text{-}\operatorname{gMod}) \to D^b(A^!\text{-}\operatorname{gMod})$$

which is an equivalence of categories if and only if A is Koszul. This is treated completely and more generally in [BGS96, Thm. 1.2.6] and [MOS09, Thm. 30]. In conclusion Koszul algebras are certain quadratic algebras with additional nice homological properties. For example, under the above equivalence the standard t-structure maps to the non-standard t-structure on the dual side given by linear complexes of graded projective modules ([MOS09, Thm. 12]). If A is even Koszul self-dual, then this yields a second interesting t-structure in $\mathcal{D}^b(A\text{-}\operatorname{gMod})$.

The algebras we are interested in admit a quasi-hereditary structure which simplifies the later proofs in a crucial way.

Definition 5.7 ([Don98])**.** Let A be a finite dimensional algebra over a field k with a finite partially ordered set $(\Lambda, \leq)$ indexing the simple left A-modules $\{L(\lambda)\}_{\lambda\in\Lambda}$ and let $P(\lambda)$ be the indecomposable projective cover of $L(\lambda)$. A collection of left A-modules $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ defines a quasi-hereditary structure on $(A, (\Lambda, \leq))$ if:

- for $\lambda \in \Lambda$ there exists a surjective A-module homomorphism $\pi : P(\lambda) \twoheadrightarrow \Delta(\lambda)$ such that $\ker \pi$ has a Δ-filtration with subquotients isomorphic to some $\Delta(\mu_i)$ where $\mu_i > \lambda$,
- for $\lambda \in \Lambda$ there exists a surjective A-module homomorphism $\pi' : \Delta(\lambda) \twoheadrightarrow L(\lambda)$ such that $\ker \pi'$ has a composition series with composition factors only isomorphic to some $L(\mu_i)$ where $\mu_i < \lambda$.

The $\Delta(\lambda)$ are called (left) *standard modules*. (Right) standard modules can be defined similarly.

The following theorem is due to Ágoston, Dlab and Lukaács (see [ÁDL03, Thm. 1]).

Theorem 5.8. *Let $(A, (\Lambda, \leq))$ be a graded quasi-hereditary algebra. If both left and right standard modules admit linear resolutions (i.e., A is standard Koszul), then A is Koszul (i.e., all simple modules admit linear resolutions).*

6. Realisation as a path algebra of a quiver with relations

Recall that $(W, \{s, t\})$ is a Coxeter system of type $I_2(m), m \geq 3$, with its geometric representation $\mathfrak{h} = \mathbb{R}\alpha_s^v \oplus \mathbb{R}\alpha_t^v$. In this section we analyse the structure of the graded endomorphism algebra $\mathcal{A} := \mathrm{End}^\bullet_{\mathcal{S}}(\mathbf{B}) \cong \mathrm{End}^\bullet_{\mathcal{SB}}(\mathbf{B}) \otimes_R \mathbb{R}$ where $\mathbf{B} := \bigoplus_{x\in W} B_x$. Define $\zeta := e^{2\pi i/2m} \in \mathbb{C}$.

Definition 6.1. Let $Q_m := (Q_0, Q_1)$ be the directed quiver with the following vertex and arrow sets:

$$Q_0 := W = \{e, {}_sk, {}_tk, w_0 \mid 1 \leq k \leq m-1\},$$
$$Q_1 := \{w \to w' \mid |l(w) - l(w')| = 1\}.$$

We write (w, w') for an arrow $w \to w'$. For the fixed representation $\mathfrak{h}$ of $I_2(m)$ define the set $R_m^{\mathfrak{h}} \subset \mathbb{R}Q_m$ consisting of the following relations for all $2 \leq i \leq m-1$ and $0 \leq j \leq m-2$ (plus the ones with the roles of s and t switched):

$$(e, s, e) = 0, \tag{20}$$
$$(s, st, s) = 0, \tag{21}$$
$$({}_si, {}_s(i+1), {}_si) = \tfrac{[i-1]_\zeta}{[i]_\zeta}({}_si, {}_s(i-1), {}_si), \tag{22}$$
$$(s, ts, s) = -[2]_\zeta(s, e, s), \tag{23}$$
$$({}_si, {}_t(i+1), {}_si) = ([i-1]_\zeta - [i+1]_\zeta)({}_si, {}_s(i-1), {}_si) - \tfrac{[i+1]_\zeta}{[i]_\zeta}({}_si, {}_t(i-1), {}_si), \tag{24}$$
$$(s, st, t) = (s, e, t), \tag{25}$$
$$(s, ts, t) = (s, e, t), \tag{26}$$
$$({}_si, {}_s(i+1), {}_ti) = \tfrac{1}{[i]_\zeta}({}_si, {}_s(i-1), {}_ti) + ({}_si, {}_t(i-1), {}_ti), \tag{27}$$
$$({}_si, {}_t(i+1), {}_ti) = ({}_si, {}_s(i-1), {}_ti) + \tfrac{1}{[i]_\zeta}({}_si, {}_t(i-1), {}_ti), \tag{28}$$
$$({}_sj, {}_s(j+1), {}_s(j+2)) = ({}_sj, {}_t(j+1), {}_s(j+2)), \tag{29}$$
$$({}_sj, {}_s(j+1), {}_t(j+2)) = ({}_sj, {}_t(j+1), {}_t(j+2)), \tag{30}$$
$$({}_s(j+2), {}_s(j+1), {}_sj) = ({}_s(j+2), {}_t(j+1), {}_sj), \tag{31}$$
$$({}_s(j+2), {}_s(j+1), {}_tj) = ({}_s(j+2), {}_t(j+1), {}_tj). \tag{32}$$

We refer to Q_m as the Hasse graph of type $I_2(m)$ (cf. Figure 1), and the set $R_m^{\mathfrak{h}}$ is called the set of dihedral relations. Define $\mathbf{P}_m$ to be the $\mathbb{R}$-algebra $\mathbb{R}Q_m/(R_m^{\mathfrak{h}})$.

The algebra $\mathbf{P}_m$ inherits the natural grading of $\mathbb{R}Q_m$ by path length since all relations are homogenous (of degree 2).

Remark 6.2. While relation (21) is a special case of relation (22) for $i = 1$, this is not the case for relations (23) and (24). Moreover, for $i = m-1$ relations (22) and (24) agree as well as relations (27) and (28).

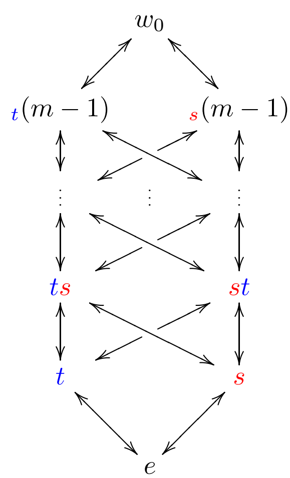


FIGURE 1. The Hasse graph of type $I_2(m)$

.

Remark 6.3. The above formulas are exactly the relations which hold for the morphisms between the Soergel modules (see Theorem 6.8). The coefficients are integral in type A_2, but in general they are real numbers. Stroppel showed in [Str03b] that for types B_2 and G_2 one may obtain rational (or even integral coefficients) by choosing different bases for the morphism spaces. However, those relations are not symmetric any more.

By definition the vertices of Q_m are indexed by the dihedral group of type $I_2(m)$. Hence we have the Bruhat order on the vertex set. For a vertex $x \in Q_0$ define $V_{\leq x} := \{y \in Q_0 \mid y \leq x\}$. Analysing the relations carefully we obtain the following lemma and propositions:

Lemma 6.4. *The path algebra* $\boldsymbol{P}_m$ *has (finite) dimension* $\sum_{x,y} |V_{\leq x} \cap V_{\leq y}|$.

Remark 6.5. Using the relations we can rewrite any path as a composition of a descending path followed by an ascending path. Lemma 6.4 shows that two paths (once simplified in this way) cannot coincide in $\mathbf{P}_m$ if the lowest vertices they pass through are different.

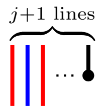
Proposition 6.6. *The isomorphism classes of simple graded* $\boldsymbol{P}_m$*-modules up to grading shift are in bijection with the vertices of* Q_m *via* $w \mapsto L(w)$ *where* $L(w)_{w'} := \begin{cases} \mathbb{R} & \text{if } w = w', \\ 0 & \text{if } w \neq w' \end{cases}$ *and the obvious maps.*

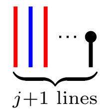
Proposition 6.7. *There exists an isomorphism* $\boldsymbol{P}_m \cong \boldsymbol{P}_m^{op}$.

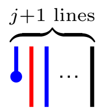
We can now state the main result of this section.

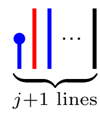
Theorem 6.8. *There is an isomorphism of graded algebras* $\boldsymbol{P}_m \cong \mathcal{A}$.

6.1. **Proof of Theorem 6.8.** Consider the assignment $w \mapsto B_{\underline{w}}$ for $w \in Q_0 \setminus \{w_0\}$ and $w_0 \mapsto B_{\underline{{}_s m}} \oplus B_{\underline{{}_t m}}$. We extend this assignment to the arrows for $0 \le j \le m-1$ as follows:

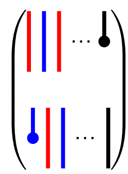
$({}_s j \to {}_s(j+1)) \longmapsto$ 

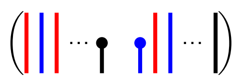
and $({}_s(j+1) \to {}_s j) \longmapsto$ 

$({}_s j \to {}_t(j+1)) \longmapsto$ 

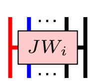
and $({}_t(j+1) \to {}_s j) \longmapsto$ 

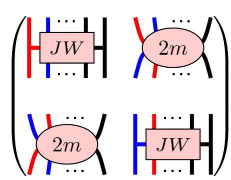
The colour of the black line depends on the parity of j and on the colour of the line on the far left. Swapping colours yields the other half of the assignment. Arrows adjacent to w_0 are treated differently since B_{w_0} is embedded diagonally in $B_{\underline{{}_s m}} \oplus B_{\underline{{}_t m}}$:

$({}_s(m-1) \to w_0) \longmapsto$  ,

$(w_0 \to {}_s(m-1)) \longmapsto \frac{1}{2}$  .

The assignment is similar with the roles of s and t swapped. Note that the images of the arrows are only morphisms between Bott-Samelson bimodules. In order to get a morphism between the corresponding Soergel bimodules we have to pre-/post-compose with the idempotents associated to the Soergel bimodules. That is, for e, s, t the idempotents are the identity since the corresponding Bott-Samelson bimodules are indecomposable. The idempotent corresponding to $B_{{}_s i}$ for $2 \le i \le m-1$ is the Jones-Wenzl morphism JW_i where the quantum numbers must be specialised to the appropriate $2m$-th root of unity; hence the coefficients depend on the dihedral group. In other words,

$2 \le i \le m-1: \quad e_{{}_s i} \mapsto$ 

$\qquad e_{w_0} \mapsto \frac{1}{2}$ 

$=: JW^{\Delta}$

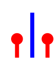
where e_x denotes the trivial path at vertex x. Similarly for t with all colours swapped. The idempotent of B_{w_0} is a 2×2-matrix since we consider B_{w_0} as a submodule of $B_{\underline{{}_s m}} \oplus B_{\underline{{}_t m}}$. By the classification in Theorem 3.2 the indecomposable bimodule B_{w_0} is a direct summand of $B_{\underline{{}_s m}}$ and $B_{\underline{{}_t m}}$ with the Jones-Wenzl projector as idempotent. A direct diagrammatic calculation shows that JW^{Δ} is indeed an idempotent and thus $e_{w_0} \mapsto JW^{\Delta}$ is well-defined. This assignment yields a homomorphism of graded $\mathbb{R}$-algebras $\varphi' : \mathbb{R}Q_m \to \mathcal{A}$. The proof of Theorem 6.8 is divided into three steps:

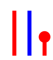
(I) The map φ' is surjective.
(II) $R_m^{\mathfrak{h}} \subseteq \ker \varphi'$, so φ' induces a surjection $\varphi : \mathbf{P}_m \to \mathcal{A}$.
(III) By dimension arguments we can deduce that φ is an isomorphism of graded algebras over $\mathbb{R}$.

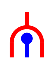
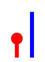
Step I. In order to show that φ' is surjective, it suffices to show that for $x, y \in W$ every morphism in $\mathrm{Hom}^\bullet_{\mathcal{S}}(B_x, B_y) \subseteq \mathcal{A}$ is generated as an element in the algebra $\mathcal{A}$ by elements of degree 1. By Theorem 3.6 we know that between B_x and B_y there exists a morphism of degree 1 if and only if $|l(x) - l(y)| = 1$. Moreover, the homogeneous part of degree 1 is at most one-dimensional; hence $\operatorname{im} \varphi'$ contains all maps of degree 1 by construction. The double leaves $\mathbb{LL}_{\underline{x},\underline{y}}$ form an R-basis for $\mathrm{Hom}_{\mathcal{D}}(\underline{x}, \underline{y})$. Using the equivalence in Theorem 4.5, we see that every element in $\mathrm{Hom}_{\mathcal{BS}}(B_{\underline{x}}, B_{\underline{y}})$ is a sum of maps factoring through various $B_{\underline{w}}$ where $\underline{w}$ is a reduced subexpression of both $\underline{x}$ and $\underline{y}$. We say *modulo lower terms* if we only consider maps which do not factor through $\overline{w}$ such that $w < x$ and $w < y$. Consequently the next proposition implies the surjectivity of φ'.

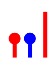
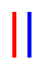
Proposition 6.9. *For $x \geq y \in W$ the morphism space $\mathrm{Hom}^\bullet_{\mathcal{SB}}(B_x, B_y)$ modulo lower terms is free of rank 1 as an R-left/right module. If x and y are not comparable the morphism space $\mathrm{Hom}^\bullet_{\mathcal{SB}}(B_x, B_y) = 0$ modulo lower terms.*

Proof. Before we prove the general case, consider the example $x = sts$ (which covers already all the important cases). Then there exist eight decorated 01-sequences of x with their corresponding light leaves:

$U0\,U0\,U0 \rightsquigarrow$  $\qquad U0\,U1\,U0 \rightsquigarrow$ 

(33) $U1\,U0\,D1 \rightsquigarrow$  $\qquad U1\,U1\,U0 \rightsquigarrow$ 

(34) $U1\,U0\,D0 \rightsquigarrow$  $\qquad U0\,U1\,U1 \rightsquigarrow$ 

$U0\,U0\,U1 \rightsquigarrow$  $\qquad U1\,U1\,U1 \rightsquigarrow$ 

The light leaves in (33) and (34) yield the zero map when pre-composed with the idempotent corresponding to sts due to death by pitchfork, and all other light leaves are generated by degree 1 maps. Now assume $l(x) \geq 4$ and set $j := l(y) \leq l(x) =: i$. This means that we have to extend the sequences above with 0's and 1's on the right-hand side and decorate the new entries properly with either D's or U's. Using successively the fact that the decorated sequences in (33) and (34) yield morphisms which get cancelled by the idempotents, we obtain that the only two possible sequences are

(35) $\underbrace{U0 \cdots U0}_{i-j \text{ times}} \underbrace{U1 \cdots U1}_{j \text{ times}} \rightsquigarrow$ ,

(36) $\underbrace{U0 \cdots U0}_{i-j-1 \text{ times}} \underbrace{U1 \cdots U1}_{i \text{ times}} U0 \rightsquigarrow$ .

Both of these remaining morphisms are generated by degree 1 elements, which proves the first statement. The second statement is clear since every proper subexpression is strictly smaller than both (this holds for dihedral groups but not in arbitrary Coxeter groups). □

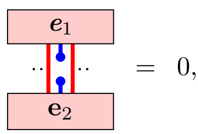
Step II. We check that every relation from $R_m^{\mathfrak{h}}$ holds for the morphisms between Soergel bimodules (with trivialised right action). For this we use the following properties:

(i) death by pitchfork,
(ii) the polynomial sliding relations in (9),
(iii) non-constant polynomials acting as 0 on the right,
(iv) isotopy invariance,
(v) two-colour associativity,
(vi) $JW := JW_{m-1}$ being rotation invariant.

It is important to note that all calculations and equations in this section are meant as morphisms between indecomposable Soergel bimodules (with trivialised right action). Therefore every morphism represented by a string diagram should be pre-/post-composed with the corresponding idempotent. We omit this composition in order to make the presentation clearer.

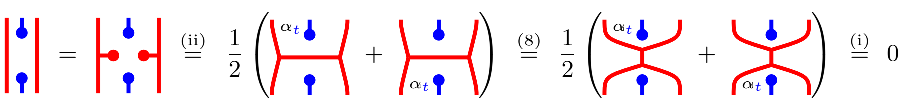
Before we start proving the relations one by one, we state two useful results.

Lemma 6.10. *For a Coxeter system of type $I_2(m)$ we have*



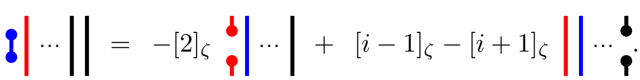
where e_1 and e_2 are idempotents corresponding to indecomposable Soergel bimodules.

Proof. Without loss of generality we can assume that the morphism is given by

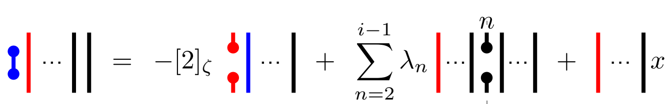


The general case follows easily from the above considerations. □

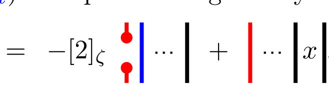
Proposition 6.11. *As a morphism $B_{s^i} \to B_{s^i}$ in the category of Soergel modules the following holds for $i \geq 2$:*



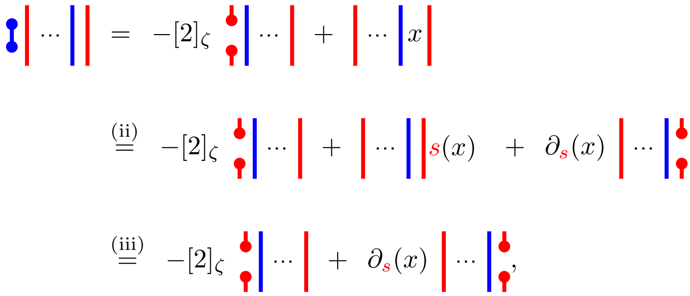
Proof. By property (ii) we can slide the polynomial α_t successively through the strings and we obtain for some λ_n:



where $x = (i-1)_s(\alpha_t)$ is of positive degree. By Lemma 6.10 we have



From now on assume that i is odd:

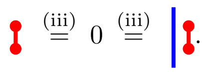


where we used that s acts grading preservingly and thus $s(x)$ acts trivially on the right since x was of positive degree. Recall that i is odd and hence by Lemma 4.2 we have

$$\partial_s(x) = \partial_s([i]_\zeta\alpha_t + [i-1]_\zeta\alpha_s) = -[2]_\zeta[i]_\zeta + 2[i-1]_\zeta \overset{13}{=} [i-1]_\zeta - [i+1]_\zeta.$$

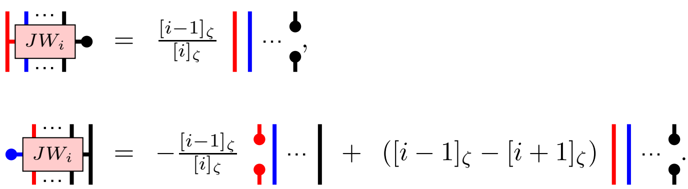
The other case (i even) can be treated similarly using ∂_t instead of ∂_s. ☐

6.1.1. *Relations* (20) *and* (21). Clearly we have

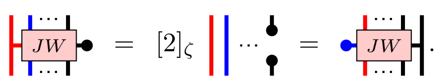


6.1.2. *Relations* (22) *and* (24). The desired relations for $i \in \{2, \dots, m-2\}$ are immediate consequences of the following lemma:

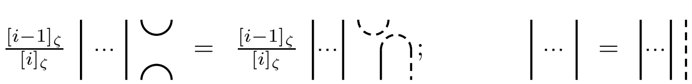
Lemma 6.12. *For $i \in \{2, \dots, m-1\}$ the following hold:*



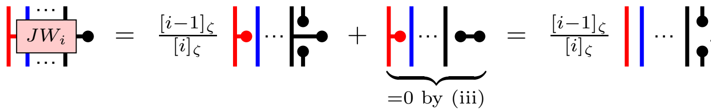
In particular,



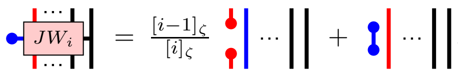
Proof. Recall the recursive formula for JW_i (see equation (17)) in the Temperley-Lieb algebra setting in which we specialise q to $\zeta = e^{2\pi i/2m}$. Transforming the Jones-Wenzl projector into a morphism in $\mathcal{D}$ and putting a trivalent vertex on the left side and a dot on the right side yields a morphism between Bott-Samelson bimodules. If we pre-/post-compose with the idempotents corresponding to the indecomposable Soergel bimodules every summand is cancelled by property (i) except the following two summands (the lines from the last inductive step are dashed):



The first summand is a cup attached to the identity of JW_{i-1} at position $i-1$, and thus its scalar is $\frac{[i-1]_\zeta}{[i]_\zeta}$. The second summand corresponds to the identity, and therefore its scalar is 1. Hence in $\mathcal{D}$ we have

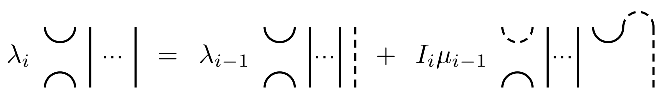


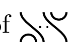
Similarly, we have



The scalar in front of the first summand can be computed using the recursive formula for JW_i:

(37)

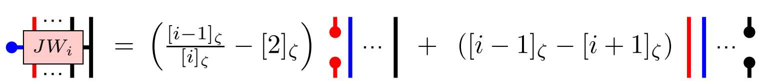


where λ_i is the coefficient of  in JW_i, μ_i is the coefficient of  in JW_i and I_i is the coefficient in the sum from the induction step.

Inductively it can be shown that $\mu_i = \frac{1}{[i]_\zeta}$ using the formula for JW_i. Clearly $I_i = \frac{1}{[i]}_\zeta$, and hence by another induction it follows that $\lambda_i = \frac{[i-1]_\zeta}{[i]_\zeta}$ using

$$\lambda_i = \lambda_{i-1} + I_i\mu_{i-1} = \frac{[(i-1)-1]_\zeta[(i-1)+1]_\zeta + 1}{[i]_\zeta[i-1]_\zeta} \overset{14}{=} \frac{[i-1]_\zeta^2}{[i]_\zeta[i-1]_\zeta} = \frac{[i-1]_\zeta}{[i]_\zeta}.$$

The second summand in (37) arises again from the identity and hence its scalar is 1. With Proposition 6.11 we obtain

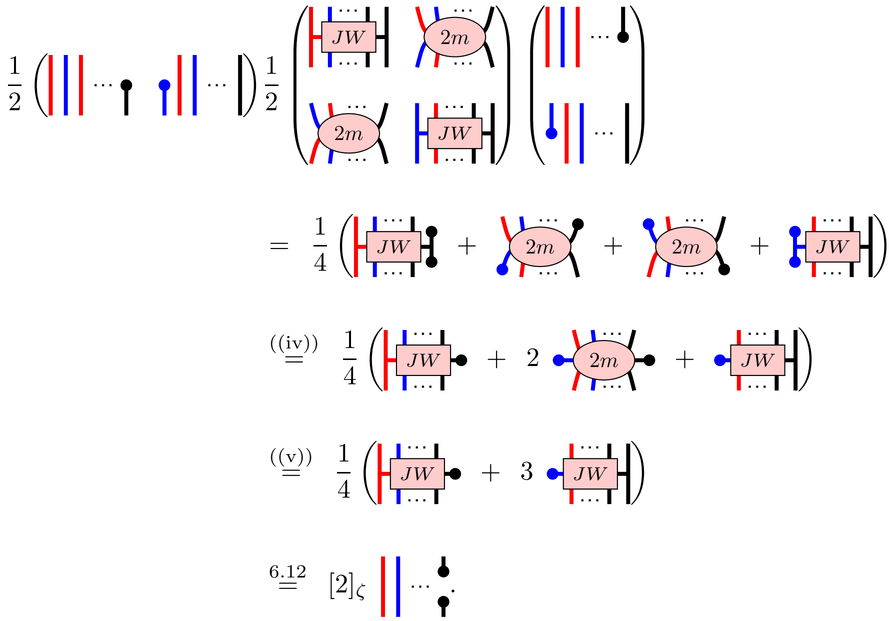


where the scalar in front of the first summand is

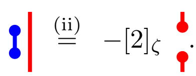
$$\left(\frac{[i-1]_\zeta}{[i]_\zeta} - [2]_\zeta\right) = \frac{[i-1]_\zeta - [2]_\zeta[i]_\zeta}{[i]_\zeta} \overset{13}{=} \frac{[i-1]_\zeta - [i+1]_\zeta - [i-1]_\zeta}{[i]_\zeta} = -\frac{[i+1]_\zeta}{[i]_\zeta}.$$

□

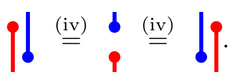
For $i = m - 1$ the Soergel bimodule is embedded diagonally, and therefore we have



6.1.3. *Relation* (23). Clearly, we have:

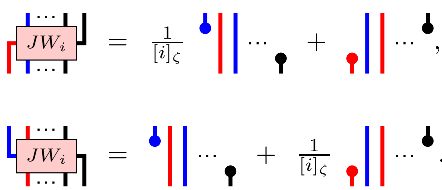


6.1.4. *Relations* (25) *and* (26). The bimodules $B_s B_t$ and $B_t B_s$ are indecomposable and therefore the corresponding idempotent is trivial; hence by isotopy invariance

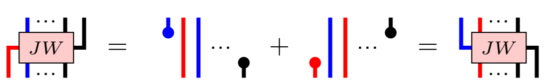


6.1.5. *Relations* (27) *and* (28). The relations (27) and (28) for $i \in \{2, \ldots, m-2\}$ are direct consequences of the following lemma:

Lemma 6.13. *For $i \in \{2, \ldots, m-2\}$ the following hold:*



In particular,



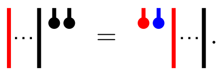
Proof. Similarly to the proof of Lemma 6.12. □

For $i = m - 1$ relations (27) and (28) agree (see Remark 6.2), and the relation is proven similarly to relations (22) and (24) since B_{w_0} is embedded diagonally.

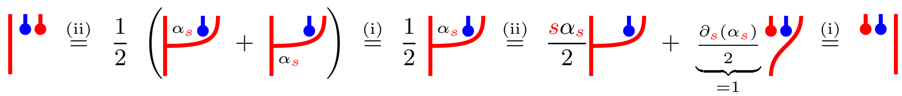
6.1.6. *Relations* (29) *and* (31). Inspecting both sides of the relation (29) gives us

since by property (i) every summand except the identity is cancelled (when pre-/post-composed with the idempotents). The right-hand sides are equal by the following lemma (which proves the desired relation).

Lemma 6.14. *In $\mathcal{SB}$ we have the following equality as morphisms $B_{s^i} \to B_{s^{(i+2)}}$ for $i \geq 0$:*



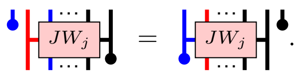
Proof. The case $i = 0$ is trivial, so let us assume $i = 1$:



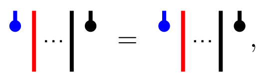
The general case follows immediately by using the above equation repeatedly. □

Relation (31) can be dealt with in the same way by flipping the diagrams horizontally.

6.1.7. *Relations* (30) *and* (32). Translating relation (30) into the Soergel bimodule setting yields



In JW_j on both sides above, every summand is cancelled by property (i) except the one corresponding to the identity with scalar 1 (after pre-/post-composing with the idempotents). Hence, the equation above becomes



which clearly holds. Relation (32) holds for the same reasons after flipping the above diagrams horizontally.

Step III. The path algebra $\mathbf{P}_m$ has by Lemma 6.4 dimension $\sum_{x,y} |V_{\leq x} \cap V_{\leq y}|$ over $\mathbb{R}$, whereas by Theorem 3.6,

$$\dim_{\mathbb{R}}(\mathcal{A}) = \sum_{x,y \in W} \dim_{\mathbb{R}} \operatorname{Hom}^{\bullet}_{\mathcal{S}}(B_x, B_y) = \sum_{x,y \in W} |W_{\leq x} \cap W_{\leq y}|.$$

Altogether we conclude that φ is an isomorphism. □

7. Proof of the Koszul self-duality

7.1. **Linear resolutions of the standard modules.** The indexing set Λ of isomorphism classes of simple graded $\mathbf{P}_m$-modules is in bijection with W by Proposition 6.6. Hence, the reverse Bruhat order $\leq_r$ turns Λ into a finite partially ordered set $(\Lambda, \leq_r)$. For $x \in W$ define the left modules

$$\Delta(x) := P(x)/M(x)$$

where $M(x) := \langle$all paths starting in x passing through $y, y \not\leq_r x$ of length $\geq 1\rangle$. Therefore we have the short exact sequence

$$0 \longrightarrow M(x) \longrightarrow P(x) \xrightarrow{\pi} \Delta(x) \longrightarrow 0\ . \tag{38}$$

Note that $e_y\Delta(x) = 0$ for all $y \not\leq_r x$, which means that no path in $\Delta(x)$ ends in such a y. The following lemma gives a precise formula for the dimensions of the involved modules in this short exact sequence.

Lemma 7.1. *For* $0 \leq i \leq m$ *we have*

$$\dim_{\mathbb{R}} P({}_si) = \begin{cases} 2m & \text{if } i = 0, \\ 2i(2m-i) & \text{if } 1 \leq i \leq m-1, \\ 1 + 2m^2 & \text{if } i = m; \end{cases} \tag{39}$$

$$\dim_{\mathbb{R}} \Delta({}_si) = \begin{cases} 2(m-i) & \text{if } 0 \leq i \leq m-1, \\ 1 & \text{if } i = m; \end{cases} \tag{40}$$

$$\dim_{\mathbb{R}} M({}_si) = \begin{cases} 0 & \text{if } i = 0, \\ 2i(2m-i) - 2(m-i) & \text{if } 1 \leq i \leq m-1, \\ 2m^2 & \text{if } i = m. \end{cases} \tag{41}$$

Proof. For fixed $x \in W$ the indecomposable projective cover $P(x)$ has a basis consisting of all paths starting in x. Hence by Lemma 6.4 and its proof we have

$$\dim_{\mathbb{R}} P(x) = \sum_{y \in W} \dim_{\mathbb{R}} e_y \mathbf{P}_m e_x = \sum_{y \in W} |V_{\geq_r y} \cap V_{\geq_r x}|. \tag{42}$$

Note that we reversed the order on the vertices. For $x = {}_si$ and $1 \leq i \leq m-1$ this becomes

$$\begin{aligned} &= \underbrace{1}_{y=e} + \underbrace{2\sum_{j=1}^{i-1} 2j}_{1\leq \ell(y)\leq i-1} + \underbrace{2i + (2i-1)}_{\ell(y)=i} + \underbrace{2\sum_{j=i+1}^{m-1} 2i}_{i+1\leq \ell(y)\leq m-1} + \underbrace{2i}_{y=w_0} \\ &= 2i(2m-i). \end{aligned} \tag{43}$$

Consider equation (43), which reduces for $i = m$ to $1 + \sum_{j=1}^{m-1} 4j + 2m = 1 + 2m^2$. The case $i = 0$ follows easily from equation (42). By construction of $\Delta(x)$ we have

$$\dim_{\mathbb{R}} \Delta(x) = \sum_{y \in W} |V_{\leq_r y} \cap \{x\}| = \sum_{y \leq_r x} 1 = \begin{cases} 2(m - \ell(x)) & \text{if } x \neq w_0, \\ 1 & \text{if } x = w_0, \end{cases}$$

which yields (40). The dimension of $M({}_si)$ follows directly from the calculations above combined with the short exact sequence in equation (38). □

Theorem 7.2. *The set* $\{\Delta(x)\}_{x\in W}$ *of left modules defines a quasi-hereditary structure on* $(\boldsymbol{P}_m, (W, \leq_r))$*; i.e., for* $x \in W$ *the (left) module* $\Delta(x)$ *is the* (left) standard *module.*

Proof. We prove that $P(x)$ has a Δ-filtration with subquotients isomorphic to $\Delta(y)$ for $y \geq_r x$ (each with multiplicity 1) and $x = {}_si$ via induction over i (this proves the first condition for being quasi-hereditary).

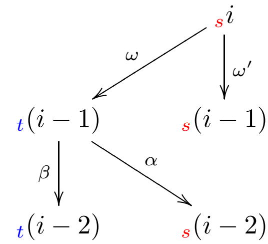


FIGURE 2. Exemplary setting of the proof of Theorem 7.2

For $x = e$, i.e., $i = 0$, there is nothing to show since $P(x) = \Delta(x)$ (by comparing dimensions via Lemma 7.1). For $i = 1$, that is, $x = s$, we have the short exact sequence

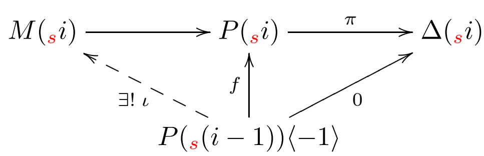
$$0 \longrightarrow P(e)\langle -1\rangle \xrightarrow{f} P(s) \longrightarrow \Delta(s) \longrightarrow 0$$

where f is given by pre-composition with (s, e). It is clear that $\operatorname{im} f \subseteq M(s)$ and equality follows from comparing dimensions (by Lemma 7.1).

Now let $i \geq 2$. Pre-composing with $\omega' := ({}_si, {}_s(i-1))$ (as indicated in Figure 2) gives us

$$f:\ P({}_s(i-1))\langle -1\rangle \longrightarrow P({}_si), \tag{44}$$

which turns out to be injective since basis elements are mapped to pairwise non-equivalent paths. Clearly $\pi f = 0$. Hence we obtain the following commutative diagram using equation (38):



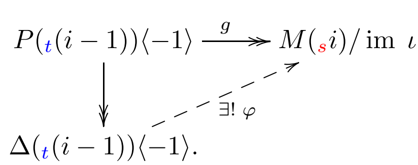
Consider the composition

$$g:\ P({}_t(i-1))\langle -1\rangle \xrightarrow{h} M({}_si) \longrightarrow M({}_si)/\operatorname{im}\ \iota$$

where h is given by pre-composition with $\omega := ({}_si, {}_t(i-1))$. The module $M({}_si)$ has generators ω and ω' as a left module. We can conclude that g is surjective since $\omega \in \operatorname{im}\ h$ and $\omega' \in \operatorname{im}\ \iota$. Similarly, the module $M({}_t(i-1))$ has generators $\alpha := ({}_t(i-1), {}_s(i-2))$ and $\beta := ({}_t(i-1), {}_t(i-2))$. Using the relations (31) and (32) we obtain the following identities:

$$\begin{aligned} \alpha\omega &= ({}_si, {}_t(i-1), {}_s(i-2)) = ({}_si, {}_s(i-1), {}_s(i-2)), \\ \beta\omega &= ({}_si, {}_t(i-1), {}_t(i-2)) = ({}_si, {}_s(i-1), {}_t(i-2)). \end{aligned}$$

Hence we can deduce that $M({}_t(i-1))\langle -1\rangle \subseteq \ker\ g$ and therefore



Since g is surjective φ is surjective, too. Using Lemma 7.1, we can compare dimensions of both sides and deduce that φ is an isomorphism. Thus we have the short exact sequence

$$0 \longrightarrow P({}_s(i-1))\langle -1\rangle \longrightarrow M({}_si) \longrightarrow \Delta({}_t(i-1))\langle -1\rangle \longrightarrow 0\,.$$

By our induction hypothesis $P({}_s(i-1))$ has Δ-filtration with subquotients isomorphic to $\Delta(y)$ with $y \geq_r {}_s(i-1)$ (each with multiplicity 1). Thus $M({}_si)$ has a Δ-filtration with subquotients isomorphic to $\Delta(y)$ with $y >_r {}_si$, and therefore $P({}_si)$ has the desired Δ-filtration.

For $x \in W$ there clearly exists a surjective map $\pi' : \Delta(x) \to L(x)$, and by construction $\ker \pi'$ has only composition factors isomorphic to $L(\mu_i)$ where $\mu_i \leq_r x$. Since every non-trivial path from x to x passes through a vertex $y >_r x$, we have $[\Delta(x) : L(x)] = 1$, and hence the second condition is satisfied. $\square$

Theorem 7.3. *$\boldsymbol{P}_m$ is standard Koszul; i.e., every left (resp. right) standard module admits a linear resolution.*

Proof. By Proposition 6.7 we have an isomorphism $\mathbf{P}_m \cong \mathbf{P}_m^{op}$, and thus left and right modules can be identified. Therefore it is enough to show that the (left) standard modules admit such resolutions. Since $e \in W$ is maximal we have $\Delta(e) = P(e)$ and there is nothing to show. For $x \in W \setminus \{e\}$ we construct a linear resolution $P_\bullet(x) \xrightarrow{\epsilon} \Delta(x) \to 0$ with $P_\bullet(x) := (P_i)_{i\geq 0}$ defined by

$$P_i := \left(\bigoplus_{\substack{w \geq_r x \\ \ell(x)-\ell(w)=i}} P(w) \right) \langle -i \rangle.$$

Note that $P_0 = P(x)$. The augmentation map $\epsilon : P_0 \to \Delta(x)$ is just the canonical projection $\pi : P(x) \to \Delta(x)$ (see (38)). For $x = {}_s\ell$ the boundary maps $p_i : P_i \to P_{i-1}$ for $i \geq 1$ are defined as follows:

$$p_i = \begin{cases} \cdot \begin{pmatrix} ({}_s\ell, {}_s(\ell-1)) & ({}_s\ell, {}_t(\ell-1)) \end{pmatrix} & \text{if } i = 1 < \ell, \\ \cdot \begin{pmatrix} (s, e) \end{pmatrix} & \text{if } i = 1 = \ell, \\ \cdot \begin{pmatrix} ({}_s(\ell-i+1), {}_s(\ell-i)) & (-1)^{i+1}\,({}_s(\ell-i+1), {}_t(\ell-i)) \\ (-1)^{i+1}\,({}_t(\ell-i+1), {}_s(\ell-i)) & ({}_t(\ell-i+1), {}_t(\ell-i)) \end{pmatrix} & \text{if } 2 \leq i < \ell, \\ \cdot \begin{pmatrix} (s, e) \\ (-1)^{i+1}(t, e) \end{pmatrix} & \text{if } i = \ell, \\ 0 & \text{if } i > \ell. \end{cases}$$

Recall that we compose paths from right to left (just as morphisms), and hence the defined p_i's are given by pre-composition with certain arrows. All indecomposable projective modules $P(w)$ are generated by the corresponding idempotent e_w which lies in degree 0. Thus, by construction each P_i is projective and generated by its degree i component. Clearly, the augmentation map ϵ is surjective.

The fact that the above sequence $P_\bullet(x)$ is a complex for fixed x is a direct consequence of relations (31) and (32). We only present the generic case $2 \leq i < \ell(x)$. Each $P(w)$ is generated by e_w, so it is enough to check that $p^2(e_w) = 0$. Let

$w = {}_{s}(i-2)$. Then we have

$$p(e_w) = \begin{pmatrix} ({}_{s}(i-1), {}_{s}(i-2)) \\ (-1)^{*}({}_{t}(i-1), {}_{s}(i-2)) \end{pmatrix},$$
$$p^2(e_w) = \begin{pmatrix} ({}_{s}i, {}_{s}(i-1), {}_{s}(i-2)) + (-1)^{*+\bullet}({}_{s}i, {}_{t}(i-1), {}_{s}(i-2)) \\ (-1)^{\bullet}({}_{t}i, {}_{s}(i-1), {}_{s}(i-2)) + (-1)^{*}({}_{t}i, {}_{t}(i-1), {}_{s}(i-2)) \end{pmatrix},$$

where $*$ and $\bullet$ depend on the parity of $\ell - i$ (resp. $\ell - i - 1$). Nevertheless, the parity of $*$ and $\bullet$ is always different and therefore $(-1)^{*+\bullet} = -1$ and $(-1)^{*} = -(-1)^{\bullet}$. Thus by relations (31) and (32) we can deduce that $p^2(e_w) = 0$. The other remaining cases are similar. Hence the sequence is a complex, and it remains to show that this complex is exact.

We only deal with the generic case $2 \le i \le \ell(x) - 2$, which covers all important arguments. Assume that we have $(y, x) \in \ker\ p_i$ where $p_i : P_i \to P_{i-1}$. Without loss of generality,

$$P_i = \Big[P({}_{s}j) \bigoplus P({}_{t}j)\Big] \langle -i \rangle, \qquad P_{i-1} = \Big[P({}_{s}(j+1)) \bigoplus P({}_{t}(j+1))\Big] \langle -i+1 \rangle$$

for some $2 \le j \le \ell(x) - 2$. From the definition of the P_i we know that y (resp. x) is a path starting in ${}_{s}j$ (resp. ${}_{t}j$). By assumption we have $p_i((y, x)) = 0$ and in particular $\pi_1(p_i((y, x))) = 0 \in P({}_{s}(j-1))\langle -i+1 \rangle$. Recall that $\pi_1 p_i$ is given by pre-composition with $\omega' = ({}_{s}(j+1)), {}_{s}j)$ in the first component and pre-composition with $\omega = ({}_{s}(j+1), {}_{t}j)$ in the second component. Since $(y, x) \in \ker\ p_i$ we can conclude that they must end in the same vertex, say z (see Figure 3). Note that it is not possible for either x or y to be the trivial path because then the other path would have to be of length ≥ 2 and hence they would be linearly independent. Assuming that both x and y pass only through vertices v with $v \le_r {}_{s}j$ we have $z \notin \{{}_{s}j, {}_{t}j\}$ (otherwise one path would pass through a vertex v with $v >_r {}_{s}j$). Moreover, $x\omega$ (resp. $y\omega'$) does not pass through ${}_{s}j$ (resp. ${}_{t}j$). In particular, the highest vertex $x\omega$ (resp. $y\omega'$) passes through is ${}_{t}j$ (resp. ${}_{s}j$). Thus $x\omega$ and $y\omega'$ are linearly independent by Remark 6.5 (which is formulated in terms of the Bruhat order). This is a contradiction to $(y, x) \in \ker\ p_i$. If only one path passes through a vertex v with $v >_r {}_{s}j$, then the same argument using Remark 6.5 again yields a contradiction to $(y, x) \in \ker\ p_i$. Therefore x and y pass only through vertices v such that $v >_r {}_{s}j$. Hence we can deduce that y starts either with α or β and x starts either with γ or δ. But that means that $(y, x) \in \mathrm{im}\ p_{i+1}$. Thus the complex is exact and indeed a linear resolution. ☐

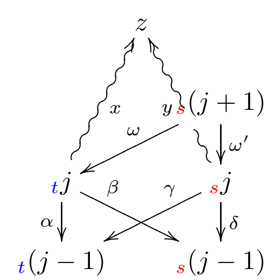


FIGURE 3. Exemplary setting of the proof of Theorem 7.3

By Theorem 5.8 we obtain the following corollary:

Corollary 7.4. *$\mathbf{P}_m$ is Koszul.*

7.2. **The quadratic dual of $\mathbf{P}_m$.** By construction $\mathbf{P}_m = \mathbb{R}Q_m/(R_m^\natural)$. Since $A := \mathbb{R}Q_m = \bigoplus_{i\geq 0} A^i$ is graded by path length and $R_m^\natural \subseteq A^2$ is homogeneous, $\mathbf{P}_m$ inherits the grading by path length. We note that $A = T_{A^0}A^1 = \bigoplus_{i\geq 0} A^i$, and thus $\mathbf{P}_m = T_{A^0}V/(R_m^\natural)$ is the desired quadratic structure for the A^0-bimodule $V = A^1$. V is a vector space spanned by the edges in Q_m on which we can define the standard scalar product

$$\langle \alpha, \beta\rangle := \begin{cases} 1 & \text{if } t(\alpha) = s(\beta) \text{ and } t(\beta) = s(\alpha), \\ 0 & \text{otherwise} \end{cases}$$

for edges $\alpha = (s(\alpha), t(\alpha))$ and $\beta = (s(\beta), t(\beta))$. Using this scalar product we can identify $V \cong V^*$. In the construction of the quadratic dual we have to consider

$$R^\perp := \{x \mid \forall v \in R_m^\natural : \langle v, x\rangle = 0\} \subseteq V^* \otimes_{A^0} V^* \cong (V \otimes_{A^0} V)^*.$$

By Lemma 6.4 the algebra $\mathbf{P}_m$ is finite dimensional, and with the above identification $R^\perp$ is just the usual orthogonal complement of $R_m^\natural$ inside A^2 with respect to the standard scalar product. The vector space A^2 has a basis consisting of all paths of length 2, and therefore it is of the form $A^2 = \bigoplus_{x,y\in W} e_y A^2 e_x$. Taking duals commutes with finite direct sums, so for fixed $x, y \in W$ the complement of all relations starting in x and ending in y lies inside $e_y A^2 e_x$, which has at most $\mathbb{R}$-dimension 4. With the above considerations it follows that $\mathbf{P}_m^! = T_{A^0}V/(R^\perp)$.

Recall that all defining relations in $\mathbf{P}_m$ are homogeneous of degree 2 and can be interpreted as linear combinations of paths of length 2. Using this identification there are three types of relations in $\mathbf{P}_m$: paths with the same starting and terminal point, paths with different starting and terminal points between vertices of the same length and paths with different starting and terminal points between vertices of different lengths. We only present the cases where the starting point x is of the form ${}_s i$ for $0 \leq i \leq m$. The cases where $x = {}_t i$ can be treated similarly since all relations are symmetric in s and t. For the sake of simplicity define $\bar{i} := m - i$ for $0 \leq i \leq m$.

7.2.1. *Relations* (20)–(24). These relations have in common that the starting point and terminal point coincide, say x. For fixed x all calculations in this section take place in $A(x) := e_x A^2 e_x$. We have to distinguish five different cases:

- $x = e$: We have $\dim_{\mathbb{R}} A(e) = 2$ and both basis elements (e, s, e) and (e, t, e) are in $R_m^\natural$; hence the orthogonal complement is 0 and there are no orthogonal relations.
- $x = s$: The vector space $A(s)$ has a basis consisting of

  $$u_1^s := (s, st, s), \qquad u_2^s := (s, ts, s), \qquad u_3^s := (s, e, s).$$

  The relations are $u_1^s = 0$ and $u_2^s + [2]_\zeta u_3^s = 0$. Thus the orthogonal complement is spanned by $[2]_\zeta u_2^s - u_3^s$.
- $x = {}_s i$ for some $2 \leq i \leq m-2$: The vector space $A({}_s i)$ has a basis consisting of

  $$v_1^{s,i} := ({}_s i, {}_s(i+1), {}_s i), \qquad v_2^{s,i} := ({}_s i, {}_t(i+1), {}_s i),$$
  $$v_3^{s,i} := ({}_s i, {}_s(i-1), {}_s i), \qquad v_4^{s,i} := ({}_s i, {}_t(i-1), {}_s i).$$

The relations are $v_1^{s,i} - \lambda_i v_3^{s,i} = 0$ and $v_2^{s,i} - \mu_i v_3^{s,i} + \nu_i v_4^{s,i} = 0$ where

$$\lambda_i := \frac{[i-1]_\zeta}{[i]_\zeta}, \qquad \mu_i := [i-1]_\zeta - [i+1]_\zeta, \qquad \nu_i := \frac{[i+1]_\zeta}{[i]_\zeta}.$$

The orthogonal complement is spanned by $\lambda_i v_1^{s,i} + \mu_i v_2^{s,i} + v_3^{s,i}$ and $\nu_i v_2^{s,i} - v_4^{s,i}$.

- $x = {}_s(m-1) = {}_s\overline{1}$: The vector space $A({}_s\overline{1})$ has a basis consisting of

$$w_1^s := ({}_s\overline{1}, w_0, {}_s\overline{1}), \qquad w_2^s := ({}_s\overline{1}, {}_s\overline{2}, {}_s\overline{1}), \qquad w_3^s := ({}_s\overline{1}, {}_t\overline{2}, {}_s\overline{1}).$$

The relation is $0 = w_1^s - \frac{[m-2]_\zeta}{[m-1]_\zeta} w_2^s = w_1^s - [2]_\zeta w_2^s$ (using equation (15)). Therefore the orthogonal complement is spanned by $[2]_\zeta w_1^s + w_2^s$ and w_3^s.

- $x = w_0$: We have $\dim_{\mathbb{R}} A(w_0) = 2$ and there are no relations. Hence the orthogonal complement is the complete subspace with basis $(w_0, {}_s\overline{1}, w_0)$ and $(w_0, {}_t\overline{1}, w_0)$.

7.2.2. *Relations* (25)–(28). All these relations are paths from x to y such that $x \neq y$ and $\ell(x) = \ell(y)$. All computations in this section take place in $A(x,y) := e_y A^2 e_x$, i.e., the vector subspace consisting of all paths of length 2 from x to y. For these relations there are three cases to consider:

- $x = s$ and $y = t$: The vector space $A(s,t)$ has a basis consisting of

$$\overline{u}_1^s := (s, st, t), \qquad \overline{u}_2^s := (s, ts, t), \qquad \overline{u}_3^s := (s, e, t).$$

The relations are $\overline{u}_1^s - \overline{u}_3^s = 0$ and $\overline{u}_2^s - \overline{u}_3^s = 0$. Thus the orthogonal complement is spanned by $\overline{u}_1^s + \overline{u}_2^s + \overline{u}_3^s$.

- $x = {}_s i$ and $y = {}_t i$ for some $2 \leq i \leq m-2$: The vector space $A({}_s i, {}_t i)$ has a basis consisting of

$$\overline{v}_1^{s,i} := ({}_s i, {}_s(i+1), {}_t i), \qquad \overline{v}_2^{s,i} := ({}_s i, {}_t(i+1), {}_t i),$$
$$\overline{v}_3^{s,i} := ({}_s i, {}_s(i-1), {}_t i), \qquad \overline{v}_4^{s,i} := ({}_s i, {}_t(i-1), {}_t i).$$

The relations are $\overline{v}_1^{s,i} - \alpha_i \overline{v}_3^{s,i} - \overline{v}_4^{s,i} = 0$ and $\overline{v}_2^{s,i} - \overline{v}_3^{s,i} - \alpha_i \overline{v}_4^{s,i} = 0$ where $\alpha_i := \frac{1}{[i]_\zeta}$. Hence the orthogonal complement is spanned by $\alpha_i \overline{v}_1^{s,i} + \overline{v}_2^{s,i} + \overline{v}_3^{s,i}$ and $\overline{v}_1^{s,i} + \alpha_i \overline{v}_2^{s,i} + \overline{v}_4^{s,i}$.

- $x = {}_s\overline{1}$ and $y = {}_t\overline{1}$: The vector space $A({}_s\overline{1}, {}_t\overline{1})$ has a basis consisting of

$$\overline{w}_1^s := ({}_s\overline{1}, w_0, {}_t\overline{1}), \qquad \overline{w}_2^s := ({}_s\overline{1}, {}_s\overline{2}, {}_t\overline{1}), \qquad \overline{w}_3^s := ({}_s\overline{1}, {}_t\overline{2}, {}_t\overline{1}).$$

The relation is $\overline{w}_1^s - \overline{w}_2^s - \overline{w}_3^s$ (using (15)), and therefore the orthogonal complement is spanned by $\overline{w}_1^s + \overline{w}_2^s$ and $\overline{w}_1^s + \overline{w}_3^s$.

7.2.3. *Relations* (29)–(32). For the relations for which the starting point x and the terminal point y differ in their length, it is immediately clear that $|\ell(x) - \ell(y)| = 2$. Hence, $\dim_{\mathbb{R}} A(x,y) = 2$, say with basis $\{p_1, p_2\}$. The relations in (29)–(32) can be reformulated as $p_1 - p_2 = 0$, and therefore their complements are spanned by $p_1 + p_2 = 0$.

TABLE 2. Relations and its orthogonal relations

Relations	Orthogonal Relations
(e, s, e) (e, t, e)	n/a
u_1^s $u_2^s + [2]_\varsigma u_3^s$	$[2]_\varsigma u_2^s - u_3^s$
$v_1^{s,i} - \lambda_i v_3^{s,i}$ $v_2^{s,i} - \mu_i v_3^{s,i} + \nu_i v_4^{s,i}$	$\lambda_i v_1^{s,i} + \mu_i v_2^{s,i} + v_3^{s,i}$ $\nu_i v_2^{s,i} - v_4^{s,i}$
$w_1^s - [2]_\varsigma w_2^s$	$[2]_\varsigma w_1^s + w_2^s$ w_3^s
n/a	$(w_0, {}_s(m-1), w_0)$ $(w_0, {}_t(m-1), w_0)$
$\overline{u}_1^s - \overline{u}_3^s$ $\overline{u}_2^s - \overline{u}_3^s$	$\overline{u}_1^s + \overline{u}_2^s + \overline{u}_3^s$
$\overline{v}_1^{s,i} - \alpha_i \overline{v}_3^{s,i} - \overline{v}_4^{s,i}$ $\overline{v}_2^{s,i} - \overline{v}_3^{s,i} - \alpha_i \overline{v}_4^{s,i}$	$\alpha_i \overline{v}_1^{s,i} + \overline{v}_2^{s,i} + \overline{v}_3^{s,i}$ $\overline{v}_1^{s,i} + \alpha_i \overline{v}_2^{s,i} + \overline{v}_4^{s,i}$
$\overline{w}_1^s - \overline{w}_2^s - \overline{w}_3^s$	$\overline{w}_1^s + \overline{w}_2^s$ $\overline{w}_1^s + \overline{w}_3^s$
$({}_sj, {}_s(j+1), {}_s(j+2))$ $- ({}_sj, {}_t(j+1), {}_s(j+2))$	$({}_sj, {}_s(j+1), {}_s(j+2))$ $+ ({}_sj, {}_t(j+1), {}_s(j+2))$
$({}_sj, {}_s(j+1), {}_t(j+2))$ $- ({}_sj, {}_t(j+1), {}_t(j+2))$	$({}_sj, {}_s(j+1), {}_t(j+2))$ $+ ({}_sj, {}_t(j+1), {}_t(j+2))$
$({}_s(j+2), {}_s(j+1), {}_sj)$ $- ({}_s(j+2), {}_t(j+1), {}_sj)$	$({}_s(j+2), {}_s(j+1), {}_sj)$ $+ ({}_s(j+2), {}_t(j+1), {}_sj)$
$({}_s(j+2), {}_s(j+1), {}_tj)$ $- ({}_s(j+2), {}_t(j+1), {}_tj)$	$({}_s(j+2), {}_s(j+1), {}_tj)$ $+ ({}_s(j+2), {}_t(j+1), {}_tj)$

Table 2 summarises all relations and orthogonal relations. Define $R^\perp \subseteq A^2$ to be the set of all orthogonal relations from Table 2. Then as aforementioned we have $\mathbf{P}_m^! = A/(R^\perp)$. Mimicking the proof of Lemma 6.4 we obtain the following lemma:

Lemma 7.5. *The quadratic dual $\boldsymbol{P}_m^!$ has the same dimension as $\boldsymbol{P}_m$ as $\mathbb{R}$-vector space.*

7.3. **Self-duality.** In this section we prove the following theorem.

Theorem 7.6. *The algebra $\boldsymbol{P}_m$ is Koszul self-dual, i.e., $\boldsymbol{P}_m \cong E(\boldsymbol{P}_m) \cong \boldsymbol{P}_m^!$.*

Proof. The algebra $\mathbf{P}_m$ is finite dimensional and Koszul by Corollary 7.4. Thus there is a canonical isomorphism $E(\mathbf{P}_m) \cong (\mathbf{P}_m^!)^{op}$. Since $\mathbf{P}_m \cong \mathbf{P}_m^{op}$ by Proposition 6.7 it is enough to show that $\mathbf{P}_m \cong \mathbf{P}_m^! = E(\mathbf{P}_m)$.

Define the map Θ on the vertices of Q_m via $x \mapsto x^{-1}w_0$ which extends to an A^0-bimodule homomorphism $\Theta : A \to A$ for $A = \mathbb{R}Q_m$ (see Figure 4). Note that the images of the arrows are only determined up to scalars, which we choose as ± 1 as indicated in Figure 5. Note that the scalar of an arrow only depends on the adjacent vertices and is independent of the direction. The pattern is highly regular except for the scalar for the edge $({}_t(m-1), w_0)$. Since all scalars are invertible, the map Θ is an isomorphism, and thus we have a surjection $\Phi : A \xrightarrow{\Theta} A \twoheadrightarrow \mathbf{P}_m^!$. It suffices to show that $R_m^\mathfrak{h} \subseteq \ker \Phi$ (or equivalently: $\Theta(R_m^\mathfrak{h}) \subseteq (R^\perp)$), which implies

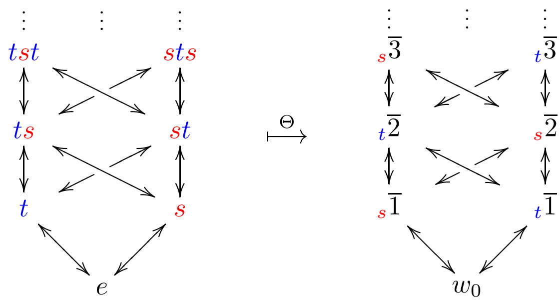


FIGURE 4. The Hasse graph Q_m and its image under Θ

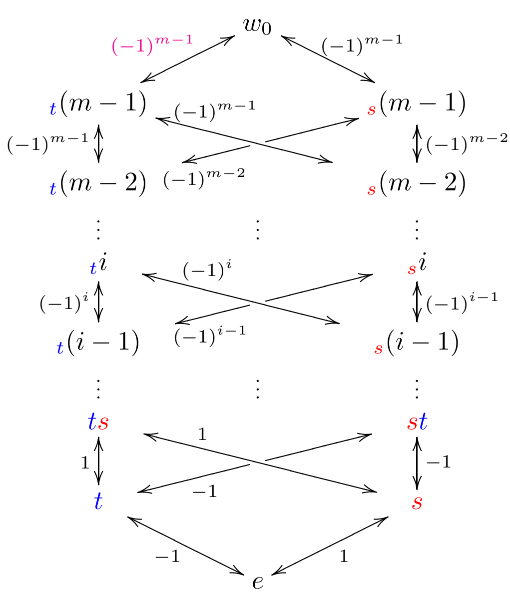


FIGURE 5. A choice of scalars yielding an isomorphism $\mathbf{P}_m \cong \mathbf{P}^!_m$

the existence of a surjection $\Psi : \mathbf{P}_m \twoheadrightarrow \mathbf{P}^!_m$, which is an isomorphism for dimension reasons (see Lemma 7.5).

In order to show that $R^\natural_m \subseteq \ker\, \Phi$ we check that each relation in $R^\natural_m$ is mapped to a relation in $R^\perp$ (up to a sign). The map $\Theta : W \to W$ is an isomorphism, and for all $w \in W$ we have $\ell(\Theta(w)) = m - \ell(w)$. Therefore we can treat the three types mentioned above separately again. Note that

$$\Theta({}_si) \quad = \quad \begin{cases} {}_s(m-i) = {}_s\overline{i} & \text{if } i \text{ is even,} \\ {}_t(m-i) = {}_t\overline{i} & \text{if } i \text{ is odd.} \end{cases} \tag{45}$$

Recall the identity in equation (15) for the quantum numbers $[i]_\zeta = [m-i]_\zeta = \left[\,\overline{i}\,\right]_\zeta$ when q is specialised to $\zeta = e^{2\pi i/2m}$. Therefore we have the following identities for $\alpha_i, \lambda_i, \mu_i$ and ν_i:

$$\alpha_{\bar{i}} = \alpha_i, \qquad \lambda_{\bar{i}} = \nu_i, \qquad \mu_{\bar{i}} = -\mu_i. \tag{46}$$

Since the calculations are not difficult but tedious we only compute the images of the relations (20)–(24) under Θ using equation (45). Note that all basis elements are loops and therefore all occurring scalars are $1 = (-1)^2$ since the scalar is independent of the direction of the arrow:

$$\begin{aligned}
(e, s, e) &\longmapsto (w_0, {}_t(m-1), w_0),\\
(e, t, e) &\longmapsto (w_0, {}_s(m-1), w_0),\\
u_1^s &\longmapsto w_3^t,\\
u_2^s + [2]_\varsigma u_3^s &\longmapsto w_2^t + [2]_\varsigma w_1^t,\\
v_1^{s,i} - \lambda_i v_3^{s,i} &\longmapsto \begin{cases} v_4^{s,\bar{i}} - \lambda_i v_2^{s,\bar{i}} \overset{46}{=} v_4^{s,\bar{i}} - \nu_{\bar{i}} v_2^{s,\bar{i}} & \text{if } i \text{ is even},\\ v_4^{t,\bar{i}} - \lambda_i v_2^{t,\bar{i}} \overset{46}{=} v_4^{t,\bar{i}} - \nu_{\bar{i}} v_2^{t,\bar{i}} & \text{if } i \text{ is odd},\end{cases}\\
v_2^{s,i} - \mu_i v_3^{s,i} + \nu_i v_4^{s,i} &\longmapsto \begin{cases} v_3^{s,\bar{i}} - \mu_i v_2^{s,\bar{i}} + \nu_i v_1^{s,\bar{i}} \overset{46}{=} v_3^{s,\bar{i}} + \mu_{\bar{i}} v_2^{s,\bar{i}} + \lambda_{\bar{i}} v_1^{s,\bar{i}} & \text{if } i \text{ is even},\\ v_3^{t,\bar{i}} - \mu_i v_2^{t,\bar{i}} + \nu_i v_1^{t,\bar{i}} \overset{46}{=} v_3^{t,\bar{i}} + \mu_{\bar{i}} v_2^{t,\bar{i}} + \lambda_{\bar{i}} v_1^{t,\bar{i}} & \text{if } i \text{ is odd},\end{cases}\\
w_1^s - [2]_\varsigma w_2^s &\longmapsto \begin{cases} u_3^t - [2]_\varsigma u_2^t & \text{if } m \text{ is even},\\ u_3^s - [2]_\varsigma u_2^s & \text{if } m \text{ is odd}.\end{cases}
\end{aligned}$$

The images of the relations agree with the right hand side in Table 2 up to a sign after swapping the roles of s and t and replacing i with $\bar{i}$. Therefore their image is contained in $(R^\perp)$. The other relations can be treated analogously as above. ☐

Remark 7.7. There is a certain degree of freedom in the choice of the scalars for this isomorphism. In types A_2 and B_2 there are 5 scalars which can be chosen freely. However, solving such a system of quadratic equations gets more and more complicated and is not very enlightening.

Remark 7.8. The basis for our approach in this paper was the equivalence of the diagrammatic category $\mathcal{D}$ and the category of Soergel bimodules $\mathcal{SB}$ (see Theorem 4.5). This equivalence holds in particular for all finite Coxeter groups with the geometric realisation (see [EW13]). With the diagrammatic presentation one can in principle write a complete description of the quiver; however outside small examples this seems prohibitively difficult. Note that in higher rank the difficult three-coloured Zamolodzhikov relations appear and there is no complete diagrammatic description of the idempotents.

Acknowledgments

This paper started as my master's thesis under the supervision of Geordie Williamson. I would like to thank him for his support and this very interesting project. Moreover, I would like to thank my second supervisor, Catharina Stroppel, for very helpful discussions. I am also grateful to Thorge Jensen for helpful remarks on preliminary versions of this paper.

References

[ÁDL03] István Ágoston, Vlastimil Dlab, and Erzsébet Lukács, *Quasi-hereditary extension algebras*, Algebr. Represent. Theory **6** (2003), no. 1, 97–117, DOI 10.1023/A:1022373620471. MR1960515

[BGG76] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand, *A certain category of $\mathfrak{g}$-modules* (Russian), Funkcional. Anal. i Priložen. **10** (1976), no. 2, 1–8. MR0407097

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527, DOI 10.1090/S0894-0347-96-00192-0. MR1322847

[CK12] Benjamin Cooper and Vyacheslav Krushkal, *Categorification of the Jones-Wenzl projectors*, Quantum Topol. **3** (2012), no. 2, 139–180, DOI 10.4171/QT/27. MR2901969

[Don98] S. Donkin, *The q-Schur algebra*, London Mathematical Society Lecture Note Series, vol. 253, Cambridge University Press, Cambridge, 1998. MR1707336

[EK10] Ben Elias and Mikhail Khovanov, *Diagrammatics for Soergel categories*, Int. J. Math. Math. Sci. (2010), Art. ID 978635, 58. MR3095655

[Eli16] Ben Elias, *The two-color Soergel calculus*, Compos. Math. **152** (2016), no. 2, 327–398, DOI 10.1112/S0010437X15007587. MR3462556

[EW13] Ben Elias and Geordie Williamson, *Soergel calculus*, Represent. Theory **20** (2016), 295–374, DOI 10.1090/ert/481. MR3555156

[EW14] Ben Elias and Geordie Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136, DOI 10.4007/annals.2014.180.3.6. MR3245013

[Fie08] Peter Fiebig, *The combinatorics of Coxeter categories*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 4211–4233, DOI 10.1090/S0002-9947-08-04376-6. MR2395170

[FK97] I. B. Frenkel and M. G. Khovanov, *Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$*, Duke Math. J. **87** (1997), no. 3, 409–480. MR1446615 (99a:17019)

[FSS12] Igor Frenkel, Catharina Stroppel, and Joshua Sussan, *Categorifying fractional Euler characteristics, Jones-Wenzl projectors and 3j-symbols*, Quantum Topol. **3** (2012), no. 2, 181–253, DOI 10.4171/QT/28. MR2901970

[Her99] I. Herrmann, *Untersuchungen zu den Kazhdan-Lusztig-Polynomen und zu dazupassenden Bimoduln*, Diplomarbeit, University of Freiburg, July 1999, available at http://immi.karimmi.de/paper/diplom.ps.

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460

[Jan96] Jens Carsten Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, RI, 1996. MR1359532

[Jon86] V. F. R. Jones, *Braid groups, Hecke algebras and type II_1 factors*, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., vol. 123, Longman Sci. Tech., Harlow, 1986, pp. 242–273. MR866500 (88k:46069)

[KL79] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184, DOI 10.1007/BF01390031. MR560412

[KL94] Louis H. Kauffman and Sóstenes L. Lins, *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, Annals of Mathematics Studies, vol. 134, Princeton University Press, Princeton, NJ, 1994. MR1280463

[Lib08a] Nicolas Libedinsky, *Équivalences entre conjectures de Soergel* (French, with French summary), J. Algebra **320** (2008), no. 7, 2695–2705, DOI 10.1016/j.jalgebra.2008.05.030. MR2441995

[Lib08b] Nicolas Libedinsky, *Sur la catégorie des bimodules de Soergel* (French, with French summary), J. Algebra **320** (2008), no. 7, 2675–2694, DOI 10.1016/j.jalgebra.2008.05.027. MR2441994

[Lib10] Nicolas Libedinsky, *Presentation of right-angled Soergel categories by generators and relations*, J. Pure Appl. Algebra **214** (2010), no. 12, 2265–2278, DOI 10.1016/j.jpaa.2010.02.026. MR2660912

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978

[Mat64] Hideya Matsumoto, *Générateurs et relations des groupes de Weyl généralisés* (French), C. R. Acad. Sci. Paris **258** (1964), 3419–3422. MR0183818

[MOS09] Volodymyr Mazorchuk, Serge Ovsienko, and Catharina Stroppel, *Quadratic duals, Koszul dual functors, and applications*, Trans. Amer. Math. Soc. **361** (2009), no. 3, 1129–1172, DOI 10.1090/S0002-9947-08-04539-X. MR2457393

[Roz14] Lev Rozansky, *An infinite torus braid yields a categorified Jones-Wenzl projector*, Fund. Math. **225** (2014), no. 1, 305–326, DOI 10.4064/fm225-1-14. MR3205575

[Soe90] Wolfgang Soergel, *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe* (German, with English summary), J. Amer. Math. Soc. **3** (1990), no. 2, 421–445, DOI 10.2307/1990960. MR1029692

[Soe97] Wolfgang Soergel, *Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules*, Represent. Theory **1** (1997), 83–114, DOI 10.1090/S1088-4165-97-00021-6. MR1444322

[Soe98] Wolfgang Soergel, *Combinatorics of Harish-Chandra modules*, Representation theories and algebraic geometry (Montreal, PQ, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, pp. 401–412. MR1653039

[Soe07] Wolfgang Soergel, *Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen*, J. Inst. Math. Jussieu **6** (2007), no. 3, 501–525, DOI 10.1017/S1474748007000023. MR2329762

[Soe14] Wolfgang Soergel, private communication, 2014.

[Str03a] Catharina Stroppel, *Category $\mathcal{O}$: gradings and translation functors*, J. Algebra **268** (2003), no. 1, 301–326, DOI 10.1016/S0021-8693(03)00308-9. MR2005290

[Str03b] Catharina Stroppel, *Category $\mathcal{O}$: quivers and endomorphism rings of projectives*, Represent. Theory **7** (2003), 322–345, DOI 10.1090/S1088-4165-03-00152-3. MR2017061

[Wen87] Hans Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada **9** (1987), no. 1, 5–9. MR873400

Department of Mathematics, University of Bonn, 53115 Bonn, Germany
E-mail address: `sauerwein@math.uni-bonn.de`