

HECKE ALGEBRA CORRESPONDENCES FOR THE METAPLECTIC GROUP

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ABSTRACT. Over a p -adic field of odd residual characteristic, Gan and Savin proved a correspondence between the Bernstein components of the even and odd Weil representations of the metaplectic group and the components of the trivial representation of the equal rank odd orthogonal groups. In this paper, we extend their result to the case of even residual characteristic.

INTRODUCTION

Fix a non-archimedean local field k of residue characteristic p and characteristic different from 2. Let W be a non-degenerate symplectic space over k of dimension $2n$ and $\widetilde{\mathrm{Sp}}(W)$ the 2-fold metaplectic cover of $\mathrm{Sp}(W)$. For an additive character ψ of k , let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(W)$, which decomposes into its even and odd constituents, $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$. In the category of genuine, smooth representations of $\widetilde{\mathrm{Sp}}(W)$, let \mathcal{G}_ψ^\pm be the Bernstein component containing ω_ψ^\pm .

Consider quadratic spaces V^\pm of dimension $2n + 1$ with trivial discriminant, where V^+ has the trivial Hasse invariant and V^- the non-trivial one. Then $\mathrm{SO}(V^+)$ is the split adjoint group of type B_n and $\mathrm{SO}(V^-)$ is its unique non-split inner form. In the category of smooth representations of $\mathrm{SO}(V^\pm)$, let \mathcal{S}_0^\pm be the Bernstein component containing the trivial representation of $\mathrm{SO}(V^\pm)$.

Let ϵ be $+$ or $-$. In [GS2], Gan and Savin proved an equivalence of categories between $\mathcal{G}_\psi^\epsilon$ and \mathcal{S}_0^ϵ assuming that $p \neq 2$. The aim of this paper is to extend their result to the case of even residual characteristic. We follow their general strategy of exploiting minimal types of the Weil representation to define a Hecke algebra H_ψ^ϵ , showing that the category $\mathcal{G}_\psi^\epsilon$ is equivalent to the category of H_ψ^ϵ -modules, and giving an isomorphism between H_ψ^ϵ and the standard Iwahori-Hecke algebra of $\mathrm{SO}(V^\epsilon)$.

A key ingredient for extending their result is an analysis of the K -types of the Weil representation in arbitrary residual characteristic which was carried out by Savin and the second-named author in [SW]. We also employ the machinery of Bushnell, Henniart, and Kutzko in [BHK] to compare the Plancherel measures induced from the respective Hecke algebras. More explicitly, the layout of the paper is as follows.

- §1. We introduce notation and summarize some relevant background material.
- §2. We describe a minimal type for an open compact subgroup and compute the corresponding spherical Hecke algebra H_ψ^ϵ . We give an isomorphism

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between H_ψ^ϵ and the standard Iwahori-Hecke algebra H^ϵ of $\mathrm{SO}(V^\epsilon)$. We show that the isomorphism $H_\psi^\epsilon \cong H^\epsilon$ is, in fact, an isomorphism of Hilbert algebras with involution, thus giving a coincidence of induced Plancherel measures under suitable normalization. A corollary of this result is that the correspondence of Hecke algebra modules preserves formal degree.

§3. We prove that the category of H_ψ^ϵ -modules is equivalent to the category $\mathcal{G}_\psi^\epsilon$, thus giving the desired equivalence of categories $\mathcal{G}_\psi^\epsilon \cong \mathcal{S}_0^\epsilon$. From the theory of Plancherel measures, we deduce that this equivalence preserves the temperedness and square-integrability of representations.

1. PRELIMINARIES

Throughout the paper, k is a non-archimedean local field with residual characteristic p ; we allow for arbitrary residual characteristic but assume that the characteristic of k is different from 2. Let \mathcal{O} be the ring of integers and ϖ a chosen uniformizer. Denote by q the cardinality of the residue field and by e the valuation of 2 in k . If $p = 2$, then e is the ramification index of 2; otherwise $e = 0$. Let ψ be a non-trivial additive character of k ; for convenience, we assume that ψ has conductor $2e$, i.e., that $4\mathcal{O}$ is the largest additive subgroup of \mathcal{O} on which ψ acts trivially.

For a vector space V over k , we denote by $S(V)$ the Schwartz space of smooth, compactly supported, \mathbb{C} -valued functions on V . We denote the subspaces of even and odd functions in $S(V)$ by $S(V)^+$ and $S(V)^-$, respectively.

1.1. The symplectic group $\mathrm{Sp}(W)$. Let W be a non-degenerate symplectic space over k of dimension $2n$ with basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_n\}$, where $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0 = \langle \mathbf{f}_i, \mathbf{f}_j \rangle$ for all i, j and $\langle \mathbf{e}_i, \mathbf{f}_j \rangle = \delta_{i,j}$. The symplectic group $\mathrm{Sp}(W)$ is the group of invertible transformations of W which preserve the symplectic form. The decomposition $W = X + Y$, where X is the span of the \mathbf{e}_i and Y is the span of the \mathbf{f}_i , is a polarization of W .

Let $W_{\mathbb{C}}$ be the \mathbb{C} -span of the symplectic basis and $\mathfrak{sp}(W_{\mathbb{C}})$ the symplectic Lie algebra, consisting of endomorphisms $T : W_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ such that $\langle Tu, v \rangle + \langle u, Tv \rangle = 0$ for all $u, v \in W_{\mathbb{C}}$. Let \mathfrak{h} be the diagonal Cartan subalgebra relative to the symplectic basis and $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ its linear dual. The roots of $\mathfrak{sp}(W_{\mathbb{C}})$ form a root system of type C_n , defined by

$$\Sigma = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n\} \cup \{\pm 2\epsilon_i : 1 \leq i \leq n\} \subset \mathfrak{h}^*,$$

where $\epsilon_i : \mathfrak{h} \rightarrow \mathbb{C}$ is given by

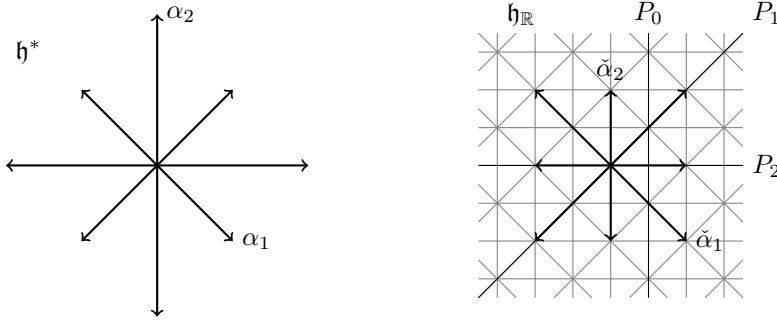
$$H = \begin{pmatrix} a & \\ & -a \end{pmatrix} \mapsto \epsilon_i(H) = a_i.$$

We take $\Delta = \{\alpha_1, \dots, \alpha_n\}$ as the set of simple roots, where $\alpha_n = 2\epsilon_n$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ otherwise. This choice of simple roots decomposes Σ into positive roots Σ^+ and negative roots Σ^- .

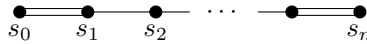
Each root $\alpha \in \mathfrak{h}^*$ has a corresponding coroot $\check{\alpha} \in \mathfrak{h}$ such that $\alpha(\check{\alpha}) = 2$; the coroots form a root system of type B_n . Denote by $\mathfrak{h}_{\mathbb{R}}$ the real span of the coroots.

Let $\Sigma_a = \{\alpha + m : \alpha \in \Sigma, m \in \mathbb{Z}\}$ be the set of affine roots, where $\alpha + m$ is the affine functional on $\mathfrak{h}_{\mathbb{R}}$ given by $(\alpha + m)(H) = \alpha(H) + m$. We take $\Delta_a = \Delta \cup \{\alpha_0\}$ to be the set of simple affine roots, where $\alpha_0 = -2\epsilon_1 + 1$.

For each affine root $\alpha + m$, define $s_{\alpha+m}$ to be the reflection across the affine hyperplane $P_{\alpha+m} = \{x \in \mathfrak{h}_{\mathbb{R}} : \alpha(x) + m = 0\}$. We write $s_i = s_{\alpha_i}$ for the simple affine reflections across the affine hyperplanes $P_i = P_{\alpha_i}$. The affine space $\mathfrak{h}_{\mathbb{R}}$ is an apartment for $\text{Sp}(W)$. The chambers of the apartment are the connected components of $\mathfrak{h}_{\mathbb{R}} \setminus \bigcup P_{\alpha+m}$. For $n = 2$, the root system, coroot system, and apartment are sketched below:



The Weyl group Ω is the group generated by the simple reflections s_1, \dots, s_n . The affine Weyl group Ω_a is the group generated by the affine simple reflections s_0, s_1, \dots, s_n ; it is the semi-direct product $\Omega_a = D\Omega$ of a translation group D and the Weyl group. Both Ω and Ω_a are Coxeter groups whose braid relations are given according to the following Coxeter diagram:



For each root $\alpha \in \Sigma$, we fix a map $\Phi_{\alpha} : \text{SL}_2(k) \rightarrow \text{Sp}(W)$ such that the images of the unipotent upper and lower triangular matrices in $\text{SL}_2(k)$ are the root subgroups of $\text{Sp}(W)$ corresponding to α and $-\alpha$, respectively. For $\alpha + m \in \Sigma_a$, we define the map $\Phi_{\alpha+m} : \text{SL}_2(k) \rightarrow \text{Sp}(W)$ by

$$\Phi_{\alpha+m} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_{\alpha} \begin{pmatrix} a & \varpi^m b \\ \varpi^{-m} c & d \end{pmatrix};$$

we write

$$\begin{aligned} x_{\alpha+m}(t) &= \Phi_{\alpha+m} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_{\alpha}(\varpi^m t) && (t \in k), \\ w_{\alpha+m}(t) &= \Phi_{\alpha+m} \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} = w_{\alpha}(\varpi^m t) && (t \in k^{\times}), \\ h_{\alpha+m}(t) &= \Phi_{\alpha+m} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = h_{\alpha}(t) && (t \in k^{\times}). \end{aligned}$$

We take the element $w_{\alpha_i}(1)$ as a representative in $\text{Sp}(W)$ of the simple affine reflection s_i . We will frequently use the same notation to refer to an element $w = s_{i_1} \cdots s_{i_r}$ in Ω_a and its representative $w = w_{\alpha_{i_1}}(1) \cdots w_{\alpha_{i_r}}(1)$ in $\text{Sp}(W)$.

1.2. Open compact subgroups of $\text{Sp}(W)$. For $0 \leq i \leq n$, we define the lattice

$$\mathcal{L}_i = \text{Span}_{\mathcal{O}}\{\mathbf{e}_1, \dots, \mathbf{e}_n, \varpi \mathbf{f}_1, \dots, \varpi \mathbf{f}_i, \mathbf{f}_{i+1}, \dots, \mathbf{f}_n\}.$$

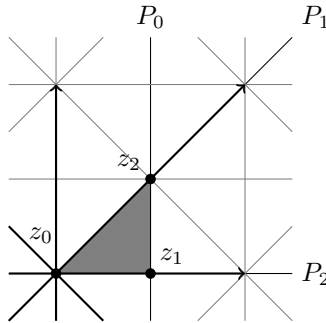
The stabilizer $K_i = \{g \in \text{Sp}(W) : g\mathcal{L}_i \subseteq \mathcal{L}_i\}$ is a maximal open compact subgroup of $\text{Sp}(W)$; it is the group generated by those $\Phi_{\alpha_j}(\mathcal{O})$ for which $j \neq i$; it is also the

stabilizer of the point z_i in the apartment, where

$$z_0 = (0, 0, \dots, 0), \quad z_1 = (\frac{1}{2}, 0, \dots, 0), \quad \dots, \quad z_n = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}).$$

In this way, each K_i corresponds to the vertex z_i in the ‘standard’ apartment; hence, every maximal open compact subgroup of $\mathrm{Sp}(W)$ is conjugate to one of the K_i ; cf. [Ti, §3.2].

The intersection of K_0, \dots, K_n is an Iwahori subgroup I . The unipotent radical of I is generated by the simple affine root groups $\Phi_{\alpha_i}(\mathrm{SL}_2(\mathcal{O}))$. In this way, the Iwahori subgroup I corresponds to the chamber in the apartment which is bounded by the hyperplanes P_0, \dots, P_n ; in addition, the vertices of the chamber are precisely z_0, \dots, z_n . The rank 2 picture is as follows:



The double cosets in $I \backslash \mathrm{Sp}(W) / I$ are parameterized by the affine Weyl group; namely, each I -double coset is of the form IwI for some $w \in \Omega_a$. The number of I -single cosets in IwI is

$$[IwI : I] = q^{\ell(w)},$$

where ℓ is the length function on Ω_a .

1.3. Metaplectic group and the Weil representation. For a polarization $W = X + Y$, the Schwartz space $S(Y)$ realizes the unique (up to isomorphism) representation ρ_ψ of the Heisenberg group with the central character ψ . Via the action of $\mathrm{Sp}(W)$ on the Heisenberg group, ρ_ψ gives a projective representation of $\mathrm{Sp}(W)$ which lifts to a linear representation ω_ψ , called the Weil representation, of the central extension $\mathrm{Sp}'(W)$ of $\mathrm{Sp}(W)$ given by

$$1 \rightarrow \mathbb{C}^\times \rightarrow \mathrm{Sp}'(W) \rightarrow \mathrm{Sp}(W) \rightarrow 1.$$

It is a theorem of Weil that the derived group of $\mathrm{Sp}'(W)$ is a 2-fold cover $\widetilde{\mathrm{Sp}}(W)$ of $\mathrm{Sp}(W)$ and that ω_ψ is a faithful representation of $\widetilde{\mathrm{Sp}}(W)$; cf. [We, IV.42-43], [MVW, 2.II.1].

For a subgroup $H \subseteq \mathrm{Sp}(W)$, we denote its preimage in $\widetilde{\mathrm{Sp}}(W)$ by \widetilde{H} . For each root α , the element $x_\alpha(t)$ canonically lifts to an element $\widetilde{x}_\alpha(t)$ in $\widetilde{\mathrm{Sp}}(W)$. We may therefore define lifts of $w_\alpha(t)$ and $h_\alpha(t)$ via the formulas

$$\begin{aligned} \widetilde{w}_\alpha(t) &= \widetilde{x}_\alpha(t) \widetilde{x}_{-\alpha}(-t^{-1}) \widetilde{x}_\alpha(t), \\ \widetilde{h}_\alpha(t) &= \widetilde{w}_\alpha(t) \widetilde{w}_\alpha(-1). \end{aligned}$$

We will take $\widetilde{w}_{\alpha_i}(1)$ for a representative in $\widetilde{\mathrm{Sp}}(W)$ of the affine simple reflection s_i . We will continue to abuse notation when referring to an element of Ω_a or its representatives in either $\mathrm{Sp}(W)$ or $\widetilde{\mathrm{Sp}}(W)$.

Explicitly, the Weil representation on $S(Y)$ is given by

$$\begin{aligned} [\tilde{x}(a)\phi](y) &= \psi({}^t y a y)\phi(y), \\ [\tilde{h}(a)\phi](y) &= \beta_a |\det a|^{1/2} \phi({}^t y), \\ [\tilde{w}\phi](y) &= \gamma_1 \widehat{\phi}(y). \end{aligned}$$

Here, $\tilde{x}(a)$, $\tilde{h}(a)$, and \tilde{w} are respective lifts of

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

the Fourier transform of ϕ is defined by

$$\widehat{\phi}(y) = \int_Y \psi(2{}^t u y)\phi(u)du;$$

and β_a and γ_1 are specific 8th roots of unity, whose precise value plays no role in the current investigation.

Under the Weil representation ω_ψ , the (positive) root groups act as follows:

$$\begin{aligned} [\tilde{x}_{\epsilon_i - \epsilon_j}(t)\phi](y) &= \phi(y + t y_i \mathbf{f}_j), \\ [\tilde{x}_{\epsilon_i + \epsilon_j}(t)\phi](y) &= \psi(2t y_i y_j)\phi(y), \\ [\tilde{x}_{2\epsilon_i}(t)\phi](y) &= \psi(t y_i^2)\phi(y). \end{aligned}$$

1.4. Minimal types of the Weil representation. Realized as a representation of $S(Y)$, the Weil representation ω_ψ decomposes into the sum of even and odd functions, $\omega_\psi^+ \oplus \omega_\psi^-$. We consider the lattices $L_i = \mathcal{L}_i \cap Y$. As computed in [SW], \tilde{K}_i acts on $\tau_i = S(L_0/2L_i)$, viewed naturally as a subspace of $S(Y)$. The space τ_0 consists entirely of even functions and is an irreducible \tilde{K}_0 -module. Otherwise, as a \tilde{K}_i -module, τ_i decomposes as $\tau_i^+ \oplus \tau_i^-$. Each τ_i^\pm admits a tensor product structure,

$$\begin{aligned} \tau_i^\pm &= S(\mathcal{O}\mathbf{f}_1/2\varpi\mathcal{O}\mathbf{f}_1)^\pm \otimes \cdots \otimes S(\mathcal{O}\mathbf{f}_i/2\varpi\mathcal{O}\mathbf{f}_i)^\pm \\ &\quad \otimes S(\mathcal{O}\mathbf{f}_{i+1}/2\mathcal{O}\mathbf{f}_{i+1}) \otimes \cdots \otimes S(\mathcal{O}\mathbf{f}_n/2\mathcal{O}\mathbf{f}_n), \end{aligned}$$

hence the dimension of τ_i^\pm is $\frac{1}{2}q^{en}(q^i \pm 1)$. We note that $\tau_i \subseteq \tau_{i+1}$ for $0 \leq i < n$.

We also note that, for $1 \leq i \leq n - 1$, the simple affine reflection s_i essentially acts on $\phi \in S(Y)$ by interchanging the i th and $(i + 1)$ th components; the reflection s_n acts essentially via Fourier transform on the n th component; the reflection s_0 acts on the first component ϕ_1 of ϕ as

$$[s_0\phi_1](y_1) = c\widehat{\phi}_1(\varpi^{-1}y_1),$$

for some constant c .

Lastly, we note that the Iwahori group $\tilde{I} = \tilde{K}_0 \cap \cdots \cap \tilde{K}_n$ is contained in each \tilde{K}_i , hence it preserves each of the minimal types τ_i^\pm . Similarly, the group $\tilde{J} = \tilde{K}_1 \cap \cdots \cap \tilde{K}_n$ is contained in \tilde{K}_i for $i \neq 0$, so it preserves the minimal type τ_i^\pm for $i \neq 0$.

We record these observations in the following lemma.

Lemma 1.1. *The Iwahori group \tilde{I} preserves τ_i^\pm for $0 \leq i \leq n$, and the group \tilde{J} preserves τ_i^\pm for $1 \leq i \leq n$. Moreover,*

- (1) *the elements s_1, \dots, s_n preserve τ_0 while s_0 inflates τ_0 to τ_1^+ ;*
- (2) *the elements s_2, \dots, s_n preserve τ_1^- while s_1 inflates τ_1^- to τ_2^- .*

1.5. Spherical Hecke algebras. We summarize some generalities on Hecke algebras, most of which may be found in [GS2].

Let G be a totally disconnected topological group and $K \subseteq G$ an open compact subgroup; fix a Haar measure dg on G . For an irreducible, finite-dimensional representation (σ, V_σ) of K , let (σ^*, V_σ^*) be its contragredient representation and define the σ -spherical Hecke algebra by

$$\mathcal{H}(G//K; \sigma) = \left\{ f : G \rightarrow \text{End}(V_\sigma^*) : \begin{array}{l} f \text{ is smooth and compactly supported,} \\ f(k_1 g k_2) = \sigma^*(k_1) f(g) \sigma^*(k_2), \text{ for } k_i \in K, g \in G \end{array} \right\};$$

it is an algebra under convolution with an identity element which we denote 1_σ .

For a smooth representation (π, V_π) of G , consider the space $(V_\pi \otimes V_\sigma^*)^K$ of K -fixed vectors in $V_\pi \otimes V_\sigma^*$; this space admits a natural action of $\mathcal{H}(G//K; \sigma)$ by

$$\pi(f)(v \otimes e) = \int_G \pi(g)v \otimes f(g)e \, dg,$$

where $v \otimes e \in (V_\pi \otimes V_\sigma^*)^K$ and $f \in \mathcal{H}(G//K; \sigma)$.

Let Γ be an open compact subgroup of G containing K ; assume that the index $[\Gamma : K]$ is finite. We consider $\mathcal{H}(\Gamma//K; \sigma)$ as a finite-dimensional subalgebra of $\mathcal{H}(G//K; \sigma)$ via

$$\mathcal{H}(\Gamma//K; \sigma) = \{f \in \mathcal{H}(G//K; \sigma) : \text{supp}(f) \subseteq \Gamma\}.$$

We have a natural isomorphism $L : \mathcal{H}(\Gamma//K; \sigma) \xrightarrow{\sim} \text{End}_\Gamma(\text{Ind}_K^\Gamma(\sigma^*))$ given by

$$(L(f)\phi)(g) = \int_\Gamma f(h)\phi(h^{-1}g) \, dh$$

for $f \in \mathcal{H}(\Gamma//K; \sigma)$, $\phi \in \text{Ind}_K^\Gamma(\sigma^*)$, and $g \in \Gamma$.

Suppose that (π, V_π) is an irreducible, smooth, finite-dimensional representation of Γ such that $(V_\pi \otimes V_\sigma^*)^K \neq 0$. Then $(V_\pi \otimes V_\sigma^*)^K$ is a simple $\mathcal{H}(\Gamma//K; \sigma)$ -module via the action of $\mathcal{H}(G//K; \sigma)$. Now assume $\mathcal{H}(\Gamma//K; \sigma)$ is commutative. Then $(V_\pi \otimes V_\sigma^*)^K$ is 1-dimensional, and the action of $\mathcal{H}(\Gamma//K; \sigma)$ factors through a maximal ideal $\mathfrak{m} \subseteq \mathcal{H}(\Gamma//K; \sigma)$. Moreover

$$\text{Ind}_K^\Gamma(\sigma^*) / (L(\mathfrak{m}) \cdot \text{Ind}_K^\Gamma(\sigma^*)) \cong \pi^*.$$

Therefore if $(\pi_1, V_1), \dots, (\pi_l, V_l)$ are the irreducible representations (up to isomorphism) of Γ such that $(V_i \otimes V_\sigma^*)^K \neq 0$, then we have

$$\text{Ind}_K^\Gamma(\sigma^*) \cong \pi_1^* \oplus \dots \oplus \pi_r^*.$$

For each $f \in \mathcal{H}(\Gamma//K; \sigma)$, the trace of $L(f)$ is $\lambda_1 d_1 + \dots + \lambda_r d_r$, where $d_i = \dim V_i$ and $\lambda_i = \pi_i(f)$. If f is not supported on K , then the trace of $L(f)$ is 0. The case of $r = 2$ is summarized by the following lemma.

Lemma 1.2. *Suppose that $\dim \mathcal{H}(\Gamma//K; \sigma) = 2$ with $T \in \mathcal{H}(\Gamma//K; \sigma)$ not supported on K . Let (π_i, V_i) , for $i = 1, 2$, be the two irreducible representations (up to isomorphism) of Γ such that $(V_i \otimes V_\sigma^*)^K \neq 0$. Write $d_i = \dim V_i$ and $\lambda_i = \pi_i(T)$. Then*

- (1) $\lambda_1 d_1 + \lambda_2 d_2 = 0$;
- (2) the dimension of $\text{Ind}_K^\Gamma(\sigma^*)$ is $d = d_1 + d_2$;
- (3) the minimal polynomial of T is $(T - \lambda_1)(T - \lambda_2) = 0$.

For the groups and representations we will consider (specifically, representations on \mathbb{C} -vector spaces of connected reductive k -groups and their central extensions), there is additional structure on $\mathcal{H}(G // K; \sigma)$, namely the $*$ -operation, $f^*(g) = \overline{f(g^{-1})}$, and the trace operation, $\text{tr}(f) = f(1)$. Following [BHK, §4.1], $\mathcal{H}(G // K; \sigma)$ is a normalized Hilbert algebra with involution $f \mapsto f^*$ and scalar product

$$[f_1, f_2] = \frac{\text{vol}(K)}{\dim \sigma} \text{tr}(f_1^* f_2).$$

This structure yields a Plancherel formula on $\mathcal{H}(G // K; \sigma)$: there is a positive Borel measure μ_σ on the C^* -algebra completion $C^*(K, \sigma)$ of $\mathcal{H}(G // K; \sigma)$ such that

$$[f, 1_\sigma] = \int_{\widehat{C^*(K, \sigma)}} \text{tr} \pi(f) \, d\hat{\mu}_\sigma(\pi).$$

Note that μ_σ depends on the chosen Haar measure of G .

We now consider this situation for two such groups, G_1, G_2 . For $i = 1, 2$, fix an open compact subgroup $K_i \subseteq G_i$, an irreducible smooth representation σ_i of K_i , and a Haar measure μ_i of G_i . Let $\hat{\mu}_i$ be the Plancherel measure on \widehat{G}_i with respect to the Haar measure μ_i ; following the notation of [BHK] we denote by ${}_r\widehat{G}_i$ the support of $\hat{\mu}_i$. We write ${}_r\widehat{G}_i(\sigma_i)$ for the subspace of ${}_r\widehat{G}_i$ consisting of the representations π for which $(\pi \otimes \sigma_i^*)^{K_i} \neq 0$.

From [BHK, §5.2], if we have an isomorphism of Hecke algebras

$$\alpha : \mathcal{H}(G_1 // K_1; \sigma_1) \rightarrow \mathcal{H}(G_2 // K_2; \sigma_2)$$

such that, for all $f \in \mathcal{H}(G_1 // K_1; \sigma_1)$,

- (1) $\alpha(f^*) = \alpha(f)^*$, and
- (2) $\text{tr}(f) = 0$ implies $\text{tr}(\alpha(f)) = 0$,

then it is an isomorphism of Hilbert algebras. We then apply [BHK, Cor. C, p. 57].

Lemma 1.3. *An isomorphism*

$$\alpha : \mathcal{H}(G_1 // K_1; \sigma_1) \rightarrow \mathcal{H}(G_2 // K_2; \sigma_2)$$

of Hilbert algebras induces a homeomorphism

$$\hat{\alpha} : {}_r\widehat{G}_2(\sigma_2) \rightarrow {}_r\widehat{G}_1(\sigma_1)$$

such that

$$\frac{\mu_1(K_1)}{\dim \sigma_1} \hat{\mu}_1(\hat{\alpha}(S)) = \frac{\mu_2(K_2)}{\dim \sigma_2} \hat{\mu}_2(S)$$

for any Borel subset S of ${}_r\widehat{G}_2(\sigma_2)$.

In the later sections, we will apply this lemma with $G_1 = \widetilde{\text{Sp}}(W)$. Strictly speaking, the groups considered in [BHK] are connected, reductive k -groups; however, there is no obstruction in extending this result to the metaplectic group.

2. HECKE ALGEBRA ISOMORPHISMS

In this section, we define our Hecke algebras H_ψ^\pm of $\widetilde{\text{Sp}}(W)$ and show that they are isomorphic to the affine Hecke algebras H^\pm of $\text{SO}(V^\pm)$.

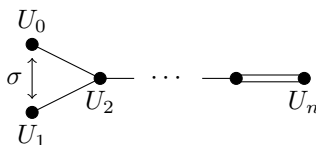
2.1. Hecke algebra of $SO(V^+)$. Let V^+ be a quadratic space of dimension $2n + 1$ with trivial discriminant and trivial Hasse invariant; then $SO(V^+)$ is a split, adjoint, orthogonal group of type B_n . Let I^+ and Ω_a^+ denote its Iwahori subgroup and affine Weyl group, respectively. The standard Iwahori-Hecke algebra is the set of smooth, compactly-supported I^+ -bi-invariant functions on $SO(V^+)$,

$$H^+ = \mathcal{H}(SO(V^+) // I^+; \mathbf{1}).$$

For each $w \in \Omega_a^+$, take U_w to be the characteristic function on the double coset I^+wI^+ . The collection $\{U_w\}$ forms a basis of H^+ as a vector space. As an algebra, H^+ is generated by elements U_0, \dots, U_n , and σ , where $U_i = U_{w_i}$ for w_i a simple affine reflection in Ω_a^+ , and σ is the outer automorphism which exchanges the nodes on the Coxeter diagram corresponding to U_0 and U_1 . The quadratic relations for the U_i are

$$(U_i + 1)(U_i - q) = 0,$$

and the braid relations are given by the affine diagram of type B_n :

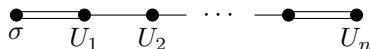


For details, see [IM, §3].

Noting that $\sigma^2 = 1$ and $\sigma U_1 \sigma = U_0$, we see that U_0 is abstractly unnecessary as a generator. Hence, H^+ is generated by σ, U_1, \dots, U_n subject to the quadratic relations,

$$(\sigma + 1)(\sigma - 1) = 0 \quad \text{and} \quad (U_i + 1)(U_i - q) = 0,$$

and the braid relations given by the affine diagram of type C_n :



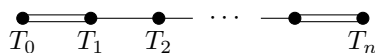
2.2. τ_0 -spherical Hecke algebra of $\widetilde{Sp}(W)$. The restriction of the minimal type τ_0 from \widetilde{K}_0 to the Iwahori subgroup \widetilde{I} remains irreducible, as shown in [SW]. In this section, we compute the τ_0 -spherical Hecke algebra

$$H_\psi^+ = \mathcal{H}(\widetilde{Sp}(W) // \widetilde{I}; \tau_0).$$

Theorem 2.1. *The Hecke algebra H_ψ^+ is generated by invertible elements T_0, T_1, \dots, T_n , satisfying the quadratic relations*

$$(T_0 + 1)(T_0 - 1) = 0 \quad \text{and} \quad (T_i + 1)(T_i - q) = 0 \quad \text{for } i \neq 0,$$

and the braid relations of the affine diagram of type C_n :



In particular, H_ψ^+ is abstractly isomorphic to H^+ .

Furthermore, this isomorphism is an isomorphism of Hilbert algebras; and, if the Haar measures on $\widetilde{Sp}(W)$ and $SO(V^+)$ are respectively normalized by

$$\text{vol}(\widetilde{I}) = \dim(\tau_0) = q^{en} = |2|^{-n} \quad \text{and} \quad \text{vol}(I^+) = 1,$$

then the Plancherel measures on H_ψ^+ and H^+ coincide.

where $\pi_1^* = \tau_{0,i}^*$ has dimension d_1 and π_2^* has dimension

$$d_2 = d - d_1 = \begin{cases} q^{en+1} & \text{if } i \neq 0, \\ \frac{1}{2}q^{en}(q+1) & \text{if } i = 0. \end{cases}$$

We normalize T_i to act by $\lambda_2 = -1$ on π_2^* and by λ_1 on π_1^* . Using Lemma 1.2, we have

$$\lambda_1 = \frac{d_2}{d_1} = \begin{cases} q & \text{if } i \neq 0, \\ 1 & \text{if } i = 0, \end{cases}$$

giving the desired quadratic relation $(T_i + 1)(T_i - \lambda_1) = 0$. The invertibility of T_i follows from its quadratic relation; explicitly,

$$T_0^{-1} = T_0 \quad \text{and} \quad T_i^{-1} = q^{-1}(T_i - q + 1) \quad \text{for } i \neq 0.$$

Suppose that we have a braid relation

$$s_i s_j \cdots = s_j s_i \cdots$$

in Ω_a . Then each of the Hecke algebra elements $T_i T_j \dots$ and $T_j T_i \dots$ is supported on the same \tilde{I} -double coset. From the normalization of the T_i , each of these elements must act on $(\tau_0 \otimes \tau_0^*)^{\tilde{I}}$ in the same way. Whence

$$T_i T_j \cdots = T_j T_i \cdots .$$

Therefore, the braid relations for the T_i are the same as those for the s_i , so any minimal expression $w = s_{i_1} \cdots s_{i_r}$ defines a Hecke algebra element $T_w = T_{i_1} \cdots T_{i_r}$, supported on $\tilde{I}w\tilde{I}$. From the quadratic and braid relations, we have an explicit isomorphism $H_\psi^+ \rightarrow H^+$ given by

$$T_0 \mapsto \sigma \quad \text{and} \quad T_i \mapsto U_i \quad \text{for } i \neq 0.$$

We now show that $H_\psi^+ \cong H^+$ is an isomorphism of Hilbert algebras. As each Hecke algebra is supported on its respective affine Weyl group, we have that

$$\text{tr}(T_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w \neq 1, \end{cases} \quad \text{and} \quad \text{tr}(U_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w \neq 1, \end{cases}$$

so the trace-zero property is clearly preserved. For $w \in \Omega_a^+$, the I^+ -double cosets of w and w^{-1} are equal, so the $*$ -operation in H^+ satisfies $U_i^* = U_i$, and hence, $U_w^* = U_{w^{-1}}$. In H_ψ^+ , we have that T_i^* and T_i are both supported on $\tilde{I}s_i\tilde{I}$, so T_i^* acts on τ_0^* by a constant. For $\phi \in \tau_0^*$, $[\phi, T_i^*\phi] = [T_i\phi, \phi]$, so T_i^* and T_i must act by the same constant. Thus, $T_i^* = T_i$ and $T_w^* = T_{w^{-1}}$, so the $*$ -operation is also preserved.

From the normalization given in the statement of the theorem, the preservation of the Plancherel measures follows immediately from Lemma 1.3. □

Corollary 2.2. *The isomorphism $H_\psi^+ \cong H^+$ preserves the formal degrees of the Steinberg representations of the respective Hecke algebras.*

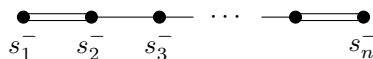
Remark. If $p \neq 2$, then the proof for the isomorphism $H_\psi^+ \cong H^+$ is essentially the one given in [GS2]. One notable difference is in the specific normalization of Hecke operators, which is always a delicate issue. In [GS2], they work in the central extension $\widetilde{\text{Sp}}(W)_8$ of $\text{Sp}(W)$ by the 8th roots of unity and normalize the generating Hecke operators to act on certain lifts of affine reflections in a specified way. We

have opted to normalize the generating Hecke operator T_i to act by -1 on the irreducible representation not containing the minimal type τ_0^* .

Remark. The Steinberg representation of a Hecke algebra is defined by having each of the generating Hecke operators act by -1 . In [GS2], they show directly that the formal degrees of the respective Steinberg representations coincide. This computation is avoided here because it follows from the more general coincidence of the induced Plancherel measures. Indeed, the implementation of the theory of induced Plancherel measures is the other notable difference between this proof and that of [GS2].

Remark. Assuming $k = \mathbb{Q}_2$, an isomorphic Hecke algebra is constructed in [Wo] by finding a 1-dimensional type for a subgroup of \tilde{I} . This construction extends to the case where k is an unramified extension of \mathbb{Q}_2 but does not appear to work for ramified extensions.

2.3. Hecke algebra of $SO(V^-)$. For the remainder of the section, we suppose that $n \geq 2$. Let V^- be a quadratic space of dimension $2n + 1$ with trivial discriminant and non-trivial Hasse invariant; then $SO(V^-)$ is the non-split inner form of $SO(V^+)$. Let I^- be the Iwahori subgroup of $SO(V^-)$, which is the pointwise stabilizer of a fundamental chamber in its Bruhat-Tits building, and Ω_a^- its affine Weyl group, which is generated by reflections s_1^-, \dots, s_n^- subject to the braid relations of the affine diagram of type C_{n-1} :



The standard Iwahori-Hecke algebra is the set of smooth, compactly-supported I^- -bi-invariant functions on $SO(V^-)$,

$$H^- = \mathcal{H}(SO(V^-) // I^-; \mathbf{1}).$$

For each $w \in \Omega_a^-$, let U_w be the characteristic function on the double coset $I^- w I^-$. The collection $\{U_w\}$ forms a basis of H^- as a vector space. As an algebra, H^- is generated by U_1, \dots, U_n , where $U_i = U_{s_i^-}$. These generators satisfy the quadratic relations,

$$(U_1 + 1)(U_1 - q^2) = 0 \quad \text{and} \quad (U_i + 1)(U_i - q) = 0 \quad \text{for } i \neq 1,$$

and the same braid relations as the s_i^- . See [GS2] or [Ti] for details.

2.4. τ_1^- -spherical Hecke algebra of $\widetilde{Sp}(W)$. We define the open compact subgroup $\tilde{J} \subseteq \widetilde{Sp}(W)$ to be the full inverse image of

$$J = \bigcap_{j \neq 0} K_j = I \cup I s_0 I,$$

and consider the restriction of τ_1^- to \tilde{J} . The group \tilde{J} contains the metaplectic preimage of the subgroup

$$\Phi_{-2\epsilon_1+1}(\mathrm{SL}_2(\mathcal{O})) \times I_{n-1},$$

where I_{n-1} is an Iwahori subgroup of the symplectic group of type C_{n-1} . From [SW], each component of this direct product acts irreducibly on the corresponding component of the tensor product

$$\tau_1^- = S(\mathcal{O}/2\varpi\mathcal{O})^- \otimes S(\mathcal{O}^{n-1}/2\mathcal{O}^{n-1}),$$

hence the restriction of τ_1^- to \tilde{J} must remain irreducible. In this section, we compute the τ_1^- -spherical Hecke algebra

$$H_\psi^- = \mathcal{H}(\widetilde{\text{Sp}}(W) // \tilde{J}; \tau_1^-).$$

We define $\Omega'_a = \langle s'_1, \dots, s'_n \rangle \subseteq \Omega_a$, where

$$s'_i = \begin{cases} s_i & \text{if } i \neq 1, \\ s_1 s_0 s_1 & \text{if } i = 1. \end{cases}$$

The reflection s'_1 corresponds to the affine reflection $s_{-2\epsilon_2+1}$, hence Ω'_a is isomorphic to the affine Weyl group of type C_{n-1} , i.e., to Ω_a^- ; explicitly, Ω'_a acts as the affine Weyl group of type C_{n-1} on the hyperplane P_0 .

The proof of the following lemma is a slight variation on that of [GS2, Lemma 10].

Lemma 2.3. *The support of H_ψ^- is contained in $\tilde{J}\Omega'_a\tilde{J}$.*

Proof. Fix $f \in H_\psi^-$ and $\sigma \in \Omega_a$. Write $\sigma = as$, where $s \in \Omega$ and a is translation by (a_1, \dots, a_n) . As $J = I \cup Is_0I$, we have that σ and $s_0\sigma$ represent the same \tilde{J} -double coset, so it suffices to show that $f(\sigma) \neq 0$ implies that either σ or $s_0\sigma$ is in Ω'_a .

The element σ conjugates the root group of α_0 to the root group of $\beta + m$, where $\beta = s^{-1}(-2\epsilon_1)$ is a long root and $m = 1 - 2a_1$; in particular, $\sigma^{-1}\tilde{x}_{\alpha_0}(4)\sigma = \tilde{x}_{\beta+m}(t)$ for some $t \in 4\mathcal{O}$.

From the description of the Weil representation in section 1.3, we derive the following criteria for long roots α :

$$\tilde{x}_\alpha(u) \in \ker \tau_1^- \quad \text{if and only if} \quad \begin{cases} u \in 4\mathcal{O} & \text{if } \alpha \neq -2\epsilon_1, \\ u \in 4\varpi^2\mathcal{O} & \text{if } \alpha = -2\epsilon_1. \end{cases}$$

Therefore, σ or $s_0\sigma$ is in Ω'_a if and only if $\beta + m \neq \pm\alpha_0$.

Let $t \in 4\mathcal{O}$ be such that

$$\sigma^{-1}\tilde{x}_{\alpha_0}(4)\sigma = \tilde{x}_{\beta+m}(t).$$

First suppose that $m > -1$ and $\beta \neq -2\epsilon_1$ or that $m > 1$ and $\beta = -2\epsilon_1$. Then $\tilde{x}_{\beta+m}(t) \in \ker \tau_1^-$ and $\tilde{x}_{\alpha_0}(4) \notin \ker \tau_1^-$, hence

$$\begin{aligned} f(\sigma) &= f(\sigma)(\tau_1^-)^*(\tilde{x}_{\beta+m}(t)) = f(\sigma\tilde{x}_{\beta+m}(t)) \\ &= f(\tilde{x}_{\alpha_0}(4)\sigma) = (\tau_1^-)^*(\tilde{x}_{\alpha_0}(4))f(\sigma), \end{aligned}$$

giving that $f(\sigma) = 0$.

Now suppose that $m < 1$ and $\beta \neq 2\epsilon_1$ or that $m < -1$ and $\beta = 2\epsilon_1$. Then $\tilde{x}_{-\beta-m}(t) \in \ker \tau_1^-$ and $\tilde{x}_{-\alpha_0}(4) \notin \ker \tau_1^-$, hence

$$\begin{aligned} f(\sigma) &= f(\sigma)(\tau_1^-)^*(\tilde{x}_{-\beta-m}(t)) = f(\sigma\tilde{x}_{-\beta-m}(t)) \\ &= f(\tilde{x}_{-\alpha_0}(4)\sigma) = (\tau_1^-)^*(\tilde{x}_{-\alpha_0}(4))f(\sigma), \end{aligned}$$

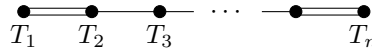
giving that $f(\sigma) = 0$.

In sum, if $f(w) \neq 0$, then $\beta + m$ must equal $\pm\alpha_0$, and the lemma is proved. \square

Theorem 2.4. *The Hecke algebra H_ψ^- is generated by invertible elements T_1, \dots, T_n , satisfying the quadratic relations*

$$(T_1 + 1)(T_1 - q^2) = 0 \quad \text{and} \quad (T_n + 1)(T_n - q) = 0 \quad \text{for } i \neq 0,$$

and the braid relations of the affine diagram of type C_{n-1} :



In particular, H_{ψ}^- is abstractly isomorphic to H^- .

Furthermore, this isomorphism is an isomorphism of Hilbert algebras and, if the Haar measures on $\widetilde{\text{Sp}}(W)$ and $\text{SO}(V^-)$ are respectively normalized by

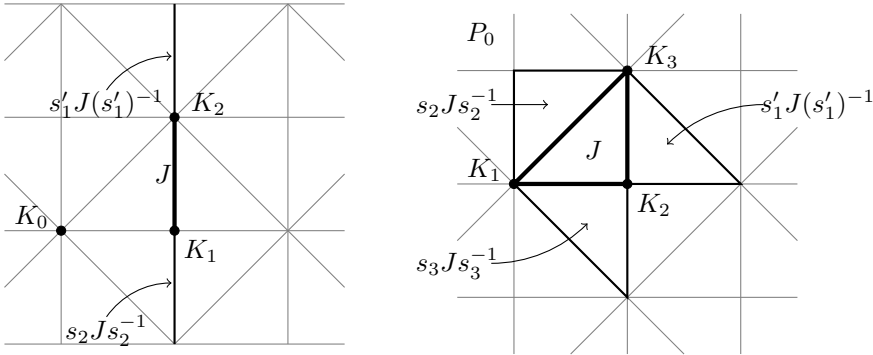
$$\text{vol}(\widetilde{J}) = \dim \tau_1^- = \frac{1}{2}q^{en}(q - 1) \quad \text{and} \quad \text{vol}(I^-) = 1,$$

then the Plancherel measures on H_{ψ}^- and H^- coincide.

Proof. This proof is similar to the proof of Theorem 2.1. We investigate the structure of some 2-dimensional Hecke subalgebras in order to see that H_{ψ}^- is supported exactly on $\widetilde{J}\Omega'_a\widetilde{J}$. For $1 \leq i \leq n$, we define \widetilde{J}_i to be the group generated by \widetilde{J} and $\widetilde{J}s'_i\widetilde{J}$; in particular, \widetilde{J}_i is the full inverse image of

$$J_i = \bigcap_{j \neq 0, i} K_j.$$

The group J corresponds to the facet of the fundamental chamber with vertices K_1, \dots, K_n , i.e., the facet that lies in the hyperplane P_0 . The conjugate $s'_i J (s'_i)^{-1}$ corresponds to a facet in the same hyperplane. The figure on the left depicts the apartment in rank 2; the figure on the right depicts the hyperplane P_0 in the rank 3 case.



If $i \neq 1$, then $\widetilde{J}_i = \widetilde{J} \cup \widetilde{J}s_i\widetilde{J}$, hence $[\widetilde{J}_i : \widetilde{J}] = q + 1$; for the case $i = 1$, we note that \widetilde{J}_1 is the union of those $\widetilde{I}w\widetilde{I}$ for which w is in the group generated by s_0 and s_1 , hence

$$[\widetilde{J}_1 : \widetilde{J}] = \frac{[\widetilde{J}_1 : \widetilde{I}]}{[\widetilde{J} : \widetilde{I}]} = \frac{1 + 2q + 2q^2 + 2q^3 + q^4}{1 + q} = 1 + q + q^2 + q^3.$$

We take $H_{\psi, i}^-$ to be the subalgebra of H_{ψ}^- consisting of elements supported on \widetilde{J}_i ; that is,

$$H_{\psi, i}^- = \mathcal{H}(\widetilde{J}_i // \widetilde{J}; \tau_1^-).$$

This subalgebra is at most 2-dimensional and is isomorphic to $\text{End}_{\widetilde{J}_i}(\text{Ind}_{\widetilde{J}}^{\widetilde{J}_i}(\tau_1^-)^*)$.

Let $\tau_{1, i}^-$ be the subspace of $S(Y)$ generated by the action of \widetilde{J}_i on τ_1^- . We use Lemma 1.1 repeatedly to compute $\tau_{1, i}^-$. First, we note that \widetilde{J} preserves τ_i^- for

$1 \leq i \leq n$, so it suffices to consider the action of s'_i on τ_1^- . Next, if $i \geq 2$, then $s'_i = s_i$ preserves τ_1^- , hence $\tau_{1,i}^- = \tau_1^-$.

Lastly, we claim that $\tau_{1,1}^- = \tau_2^-$. Since $s'_1 = s_1 s_0 s_1$, we need to consider the action of s_1 and s_0 . By the same lemma, s_1 inflates τ_1^- to τ_2^- . Hence, it remains to show that, if $\phi \in \tau_2^-$, then $s_0 \phi \in \tau_2^-$. For such ϕ , we recall from section 1.3 that the first component ϕ_1 is in $S(\mathcal{O}/2\varpi\mathcal{O})^-$ and that s_0 acts on ϕ_1 via

$$[s_0 \phi_1](y_1) = c \widehat{\phi}_1(\varpi^{-1} y_1).$$

Since the Fourier transform maps $S(\mathcal{O}/2\varpi\mathcal{O})^-$ to $S(\varpi^{-1}\mathcal{O}/2\mathcal{O})^-$, the first component of $s_0 \phi$ remains in $S(\mathcal{O}/2\varpi\mathcal{O})^-$ and the claim is proved.

To see that $H_{\psi,i}^-$ is exactly 2-dimensional, we again work in the dual setting and note that

$$d_1 = \dim \tau_{1,i}^- = \begin{cases} \dim(\tau_1^-) & \text{if } i \neq 1, \\ \dim(\tau_2^-) & \text{if } i = 1, \end{cases} = \begin{cases} \frac{1}{2}q^{en}(q-1) & \text{if } i \neq 1, \\ \frac{1}{2}q^{en}(q^2-1) & \text{if } i = 1, \end{cases}$$

is strictly smaller than

$$d = \dim(\text{Ind}_{\widetilde{J}}^{\widetilde{J}_i}(\tau_1^-)) = \dim(\tau_1^-) \cdot [\widetilde{J}_i : \widetilde{J}] = \begin{cases} \frac{1}{2}q^{en}(q^2-1) & \text{if } i \neq 1, \\ \frac{1}{2}q^{en}(q^4-1) & \text{if } i = 1. \end{cases}$$

Hence, for $1 \leq i \leq n$, there exists $T_i \in H_{\psi}^-$ supported precisely on $\widetilde{J} s_i \widetilde{J}$. We consider the decomposition

$$\text{Ind}_{\widetilde{J}}^{\widetilde{J}_i}(\tau_1^-)^* = \pi_1^* \oplus \pi_2^*,$$

where $\pi_1^* = (\tau_{1,i}^-)^*$ has dimension d_1 and π_2^* has dimension

$$d_2 = d - d_1 = \begin{cases} \frac{1}{2}q^{en}(q^2 - q) & \text{if } i \neq 1, \\ \frac{1}{2}q^{en}(q^4 - q^2) & \text{if } i = 1. \end{cases}$$

We normalize T_i to act by $\lambda_2 = -1$ on π_2^* and by λ_1 on π_1^* . Using Lemma 1.2, we have

$$\lambda_1 = \frac{d_2}{d_1} = \begin{cases} q & \text{if } i \neq 1, \\ q^2 & \text{if } i = 1, \end{cases}$$

giving the desired quadratic relation $(T_i + 1)(T_i - \lambda_1) = 0$. The invertibility of T_i follows from its quadratic relation; explicitly, $T_i^{-1} = \lambda_1^{-1}(T_i - \lambda_1 + 1)$.

The proof of the braid relations mimics the proof of Theorem 2.1 with \widetilde{J} instead of \widetilde{I} , τ_1^- instead of τ_0 , and Ω'_a instead of Ω_a . We note that computations involving \widetilde{J} -double cosets involve a weighted length function ℓ' on Ω'_a , defined by setting $\ell'(s'_1) = 3$ and $\ell'(s'_i) = 1$ if $i \neq 1$. The details of this length function are contained in [GS2, Prop. 1]; it suffices to mention here that

- (1) $[JwJ : J] = q^{\ell'(w)}$ for all $w \in \Omega'_a$.
- (2) If $w_1, w_2 \in \Omega'_a$ satisfy $\ell'(w_1) + \ell'(w_2) = \ell'(w_1 w_2)$, then $Jw_1 J \cdot Jw_2 J = Jw_1 w_2 J$.

For a minimal expression $w = s'_{i_1} \cdots s'_{i_r}$ in Ω'_a , the braid relations in H_{ψ}^- allow us to define a canonical Hecke operator $T_w = T_{i_1} \cdots T_{i_r}$, supported precisely on the double coset $\tilde{J}w\tilde{J}$. From the quadratic and braid relations, we have the explicit isomorphism $H_{\psi}^- \rightarrow H^-$ given by $T_i \mapsto U_i$.

As in the proof of Theorem 2.1, one can show that $H_{\psi}^- \cong H^-$ as Hilbert algebras, hence Lemma 1.3 will give the coincidence of Plancherel measures, under the prescribed normalization. □

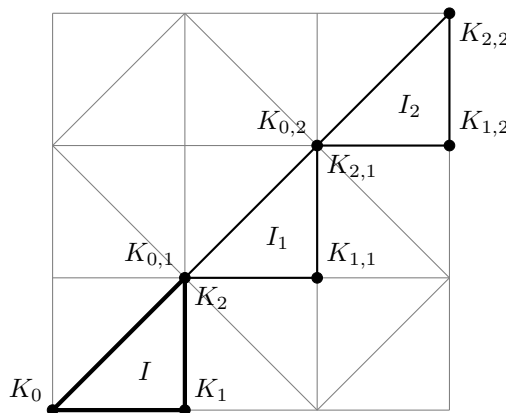
Corollary 2.5. *The isomorphism $H_{\psi}^- \cong H^-$ preserves the formal degrees of the Steinberg representations of the respective Hecke algebras.*

Remark. Let ϵ be $+$ or $-$. Even for a fixed character ψ , the isomorphism $H_{\psi}^{\epsilon} \cong H^{\epsilon}$ constructed in these two sections is far from unique: the Hecke algebras admit many inner automorphisms which preserve the quadratic and braid relations of the generators.

Remark. These two Hecke algebra isomorphisms may be constructed using additive characters of any conductor; we chose the conductor $2e$ for its convenience. For an even conductor, the method of construction would essentially go unchanged. For an odd conductor, different minimal types from [SW] would be employed. What follows is a very brief summary of the relevant details.

Consider an additive character ψ' given by $\psi'(t) = \psi(\varpi^{-c}t)$. This new character has conductor $2e + c$; that is, $4\varpi^c\mathcal{O}$ is the largest subgroup of k on which ψ' acts trivially. The parity of c is important, so we write $c = 2k + \delta$, where δ is 0 or 1, and we define the diagonal matrix g_c to have $\varpi^{k+\delta}$ in the first n entries and ϖ^{-k} in the last n entries.

The conjugation, $x \mapsto x_c = g_c^{-1}xg_c$, is an automorphism of $\text{Sp}(W)$. If $\delta = 0$, g_c is an element of the affine Weyl group and conjugation by g_c is an inner automorphism. If $\delta = 1$, g_c is not an element of the affine Weyl group (or even of the symplectic group!) and conjugation by g_c is an outer automorphism. We write $G_c = g_c^{-1}Gg_c$ for any subgroup G of $\text{Sp}(W)$ and \tilde{G}_c for its inverse image in $\tilde{\text{Sp}}(W)$. In the apartment, conjugation by g_c corresponds to translation by cz_n , so the fundamental chamber I with vertices K_0, \dots, K_n is translated to the chamber I_c with vertices $K_{0,c}, \dots, K_{n,c}$. If $\delta = 0$, then $K_{i,c}$ is conjugate in $\text{Sp}(W)$ to K_i . If $\delta = 1$, then $K_{i,c}$ is conjugate in $\text{Sp}(W)$ to K_{n-i} . For any δ , I_c is conjugate in $\text{Sp}(W)$ to I . The rank 2 picture is as follows:



If we follow conjugation by the Weil representation ω_ψ , we get a Weil representation $\omega_{\psi,c}$; in particular,

$$\begin{aligned} \omega_{\psi,c}(\tilde{x}(a)) &= \omega_\psi(\tilde{x}(\varpi^{-c}a)), \\ \omega_{\psi,c}(\tilde{h}(a)) &= \omega_\psi(\tilde{h}(a)), \\ \omega_{\psi,c}(\tilde{w}) &= \omega_\psi(\tilde{w}(\varpi^c)), \end{aligned}$$

where $\tilde{w}(\varpi^c)$ is a lift of $g_c^{-1}wg_c$. Some straightforward computations reveal that $\omega_{\psi,c} = \omega_{\psi'}$:

$$\begin{array}{ccc} \widetilde{\mathrm{Sp}}(W) & \xrightarrow{g_c} & \widetilde{\mathrm{Sp}}(W) \\ & \searrow \omega_{\psi'} & \swarrow \omega_{\psi,c} \\ & \mathrm{GL}(S(Y)) & \end{array}$$

We note that $x \in \mathrm{Sp}(W)$ stabilizes a lattice \mathcal{L} if and only if x_c stabilizes $g_c^{-1}\mathcal{L}$. For convenience, we write $\mathcal{L}_{i,c} = g_c^{-1}\mathcal{L}_i$ and $L_{i,c} = \mathcal{L}_{i,c} \cap Y$.

For $\delta = 0$, the space $\tau_{i,c} = S(L_{0,c}/2L_{i,c})$ is a type for $\widetilde{K}_{i,c}$. It is easily checked that $\tau_{i,c}$ is isomorphic to τ_i . The group $J_c = K_{1,c} \cap \cdots \cap K_{n,c}$ is isomorphic to $J = K_1 \cap \cdots \cap K_n$.

For $\delta = 1$, the space $\tau_{i,c} = S(L_{i,c}/2\varpi L_{0,c})$ is a type for $\widetilde{K}_{i,c}$. It is easily checked that $\tau_{i,c}$ is isomorphic to τ_{n-i} . The group $J_c = K_{1,c} \cap \cdots \cap K_{n,c}$ is isomorphic to $K_0 \cap \cdots \cap K_{n-1}$.

In either case, we build the two Hecke algebras

$$H_{\psi,c}^+ = \mathcal{H}(\widetilde{\mathrm{Sp}}(W) // \widetilde{I}_c; \tau_{0,c}) \quad \text{and} \quad H_{\psi,c}^- = \mathcal{H}(\widetilde{\mathrm{Sp}}(W) // \widetilde{J}_c; \tau_{1,c}^-).$$

The same sort of geometry of the apartment employed in the previous sections will yield the existence of generators of $H_{\psi,c}^\pm$ with the same quadratic and braid relations as H_ψ^\pm .

The curious reader is referred to [GS2], where Gan and Savin use an odd conductor in their computation of H_ψ^- under the assumption that $p \neq 2$.

3. EQUIVALENCE OF CATEGORIES BETWEEN \mathcal{G}_ψ^\pm AND \mathcal{S}_0^\pm

In the category of smooth genuine representations of $\widetilde{\mathrm{Sp}}(W)$, let \mathcal{G}_ψ^\pm be the Bernstein component containing the even/odd Weil representation ω_ψ^\pm . In the category of smooth representations of $\mathrm{SO}(V^\pm)$, let \mathcal{S}_0^\pm be the Bernstein component containing the trivial representation.

We will prove our main theorem, namely that there is an equivalence of categories between $\mathcal{G}_\psi^\epsilon$ and \mathcal{S}_0^ϵ , where ϵ is + or -. Our proof essentially follows that of [GS2].

3.1. Equivalence between \mathcal{G}_ψ^+ and \mathcal{S}_0^+ . Let U (resp. U^-) be the unipotent radical in $\mathrm{Sp}(W)$ generated by positive (resp. negative) root groups. Let $\widetilde{B} = \widetilde{T}U \subseteq \widetilde{\mathrm{Sp}}(W)$ be the preimage of the Borel subgroup $B = TU$ of $\mathrm{Sp}(W)$. (Recall that the unipotent radical U splits in $\widetilde{\mathrm{Sp}}(W)$.)

An element t of the maximal torus T may be expressed uniquely as

$$t = (t_1, \dots, t_n) = h_{2\epsilon_1}(t_1) \cdots h_{2\epsilon_n}(t_n),$$

hence we have a canonical lift of t given by

$$\tilde{t} = \tilde{h}_{2\epsilon_1}(t_1) \cdots \tilde{h}_{2\epsilon_n}(t_n).$$

With this convention, multiplication in \tilde{T} is given by

$$\tilde{t} \cdot \tilde{u} = (t, u) \tilde{t}u = \prod_{i=1}^n (t_i, u_i) \tilde{h}_{2\epsilon_i}(t_i u_i),$$

where the cocycle $(t, u) \in \{\pm 1\}$ is the product of Hilbert symbols (t_i, u_i) on k . Note that multiplication in T is commutative.

Recalling the action of T on Y by $ty = (t_1 y_1, \dots, t_n y_n)$, the action of \tilde{t} on $S(Y)$ is given by

$$\tilde{t}\phi(y) = \beta_t |\det t|^{1/2} \phi(ty),$$

where β_t is a 4th root of unity satisfying $\beta_t \beta_u = (t, u) \beta_{tu}$.

Given a character $\chi = (\chi_1, \dots, \chi_n)$ on T , we define a genuine character $\tilde{\chi}$ on \tilde{T} by

$$\tilde{\chi}(\tilde{t}) = \chi(t) \beta_t.$$

We extend this character trivially to all of \tilde{B} and define $I(\tilde{\chi})$ to be the normalized induced representation $\text{Ind}_{\tilde{B}}^{\tilde{\text{Sp}}(W)} \tilde{\chi}$. By Frobenius reciprocity,

$$(3.1) \quad \text{Hom}_{\tilde{\text{Sp}}(W)}(\pi, I(\tilde{\chi})) \cong \text{Hom}_{\tilde{T}}(\pi_U, \tilde{\chi}),$$

where π is any smooth representation of $\tilde{\text{Sp}}(W)$ and π_U is the normalized Jacquet module with respect to the Borel \tilde{B} .

Lemma 3.2. *The Bernstein component \mathcal{G}_{ψ}^+ is precisely the component whose irreducible representations are submodules of $I(\tilde{\chi})$ for some unramified character χ .*

Proof. The functional $l : S(Y)^+ \rightarrow \mathbb{C}$ defined by $l(\phi) = \phi(0)$ factors through the Jacquet module $(\omega_{\psi}^+)_U$ and gives a non-trivial element in $\text{Hom}_{\tilde{T}}((\omega_{\psi}^+)_U, \tilde{\chi})$ for some unramified χ , which in turn gives an embedding $\omega_{\psi}^+ \subseteq I(\tilde{\chi})$ via Frobenius reciprocity. \square

The Iwahori subgroup \tilde{I} admits a factorization

$$\tilde{I} = I_U \tilde{I}_T I_U,$$

where $I_{U^-} = \tilde{I} \cap U^-$, $\tilde{I}_T = \tilde{I} \cap \tilde{T}$, and $I_U = \tilde{I} \cap U$. (Note that I_U and I_{U^-} split in the central extension $\tilde{\text{Sp}}(W)$.)

Now let us define the ‘‘Jacquet module’’ $(\tau_0)_U$ of τ_0 with respect to I_U ; that is, $(\tau_0)_U$ is the quotient of $\tau_0 = S(L_0/2L_0)$ by

$$\langle \tau_0(u)\phi - \phi : u \in I_U, \phi \in \tau_0 \rangle,$$

which may be viewed as a representation of \tilde{I}_T .

Lemma 3.3. *The space $(\tau_0)_U$ is 1-dimensional and spanned by the image of the characteristic function of $2L_0$. Moreover each element $\tilde{t} \in \tilde{I}_T$ acts by β_t on $(\tau_0)_U$.*

Proof. Suppose that $\phi \in \tau_0$ is supported on $a + 2L_0$ for $a \in L_0 \setminus 2L_0$, and let i be such that $a_i \in \mathcal{O}^\times$. The element $\tilde{x}_{2\epsilon_i}(1)$ acts on ϕ by the constant $\psi(a_i^2) \neq 1$. Therefore, the image of ϕ in $(\tau_0)_U$ is trivial.

On the other hand, let ϕ be the characteristic function on $2L_0$. Using the formulas in section 1.3 for the action of the positive root groups, it is simple

to check that I_U acts trivially on ϕ . Moreover, we know that $\tilde{t} \in \tilde{I}_T$ acts by $\tilde{t}\phi(y) = \beta_t\phi(ty) = \beta_t\phi(y)$. \square

Theorem 3.4. *The functor from the category \mathcal{G}_ψ^+ to the category of H_ψ^+ -modules, given by*

$$\pi \mapsto (\pi \otimes \tau_0^*)^{\tilde{I}},$$

is an equivalence of categories. In particular, there is an equivalence of categories between \mathcal{G}_ψ^+ and \mathcal{S}_0^+ given by the isomorphism $H_\psi^+ \cong H^+$ of Hecke algebras. Furthermore, this equivalence preserves the temperedness and square integrability of representations.

Proof. We have the natural surjection

$$r : (\pi \otimes \tau_0^*)^{\tilde{I}} \rightarrow (\pi_U \otimes (\tau_0^*)_U)^{\tilde{I}_T},$$

which is a slight variant of what is called ‘‘Jacquet’s Lemma’’ in [B2, 64-65]. (To prove our version, one can follow the argument there. Also, see [B1, Prop. 3.5.2].)

We first show that r is an isomorphism. Suppose that $v \in \ker r$, i.e., that there exists an open compact subgroup U_v of U such that $\int_{U_v} \pi(u)v \, du = 0$.

For a translation $\lambda = (\lambda_1, \dots, \lambda_n)$ in $D \subseteq \Omega_a$, we write

$$\lambda = \tilde{h}_{2\epsilon_1}(\varpi^{\lambda_1}) \cdots \tilde{h}_{2\epsilon_n}(\varpi^{\lambda_n})$$

as its representative in \tilde{T} . Take $\lambda \in D$ such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ and $\lambda^{-1}I_U\lambda \supseteq U_v$. Then

$$\tilde{I}\lambda\tilde{I} = \bigcup_{i=1}^{q^{\ell(\lambda)}} \lambda u_i \tilde{I}$$

where the u_i are representatives of the \tilde{I}_U -cosets in $\lambda^{-1}I_U\lambda$. Let T_λ be the Hecke algebra element supported on $\tilde{I}\lambda\tilde{I}$ obtained by using a minimal expression for $\lambda \in \Omega_a$ as in Theorem 2.1. Then

$$\pi(T_\lambda)v = \pi(\lambda) \sum_{i=1}^{q^{\ell(\lambda)}} \pi(u_i)v = \pi(\lambda) \int_{\lambda^{-1}\tilde{I}_U\lambda} \pi(u)v \, du = 0.$$

The element T_λ is invertible, as it is the product of invertible elements, hence $v = 0$ and r is injective.

Let π be an irreducible representation of $\widetilde{\text{Sp}}(W)$ such that $(\pi \otimes \tau_0^*)^{\tilde{I}} \neq 0$. As r is an isomorphism, $(\pi_U \otimes (\tau_0^*)_U)^{\tilde{I}_T} \neq 0$. This implies that $\text{Hom}_{\tilde{T}}(\pi_U, \tilde{\chi}) \neq 0$ for some unramified χ , since $\tilde{t} \in \tilde{I}_T$ acts by β_t on $(\tau_0)_U$. Therefore, by Frobenius reciprocity, we have that π is a subrepresentation of $I(\tilde{\chi})$.

Conversely, let π be an irreducible submodule of $I(\tilde{\chi})$ for some unramified χ . By Frobenius reciprocity, we have that

$$0 \neq (\pi_U \otimes (\tau_0^*)_U)^{\tilde{I}_T} \cong (\pi \otimes \tau_0^*)^{\tilde{I}}.$$

Thus, condition (iii) of [BK, 3.11] is satisfied, which proves the equivalence of categories.

To complete the proof, we note that, as categories,

$$\mathcal{S}_0^+ \cong H^+\text{-modules} \cong H_\psi^+\text{-modules} \cong \mathcal{G}_\psi^+;$$

moreover, the trivial representation of $\mathrm{SO}(V^+)$ corresponds to the trivial module of $H^+ \cong H_\psi^+$, and hence to the even Weil representation ω_ψ^+ .

Finally, to show that the equivalence preserves temperedness and square integrability, let us note that the equivalence H_ψ^+ -modules $\cong \mathcal{G}_\psi^+$ implies that (\tilde{I}, τ_0) is an \mathfrak{s} -type in the sense of [BHK, 1.6]. (To see this, let $e \in H_\psi^+$ be the idempotent corresponding to τ_0 and $\mathfrak{R}_e(\widetilde{\mathrm{Sp}}(W))$ the full subcategory of the category of smooth genuine representations of $\mathrm{Sp}(W)$ as defined in [BHK, 1.4]. Then we have the functor $\mathfrak{R}_e(\widetilde{\mathrm{Sp}}(W)) \rightarrow H_\psi^+$ -modules given by $\pi \mapsto (\pi \otimes \tau_0^*)^{\tilde{I}}$. Clearly this functor composed with the equivalence H_ψ^+ -modules $\cong \mathcal{G}_\psi^+$ is the identity, which implies $\mathfrak{R}_e(\widetilde{\mathrm{Sp}}(W)) = \mathcal{G}_\psi^+$. Hence (\tilde{I}, τ_0) is an \mathfrak{s} -type with \mathfrak{s} being the inertial equivalence class represented by $(\tilde{B}, \mathbf{1})$.) Hence by the first paragraph of [BHK, 0.6], one can see that all the irreducible tempered representations in \mathcal{G}_ψ^+ are in ${}_r\tilde{G}(\tau_0)$ with $G = \widetilde{\mathrm{Sp}}(W)$. The same applies to the group $\mathrm{SO}(V^+)$. Furthermore, from [BHK, 5.1], the equivalence $\mathcal{S}_0^+ \cong \mathcal{G}_\psi^+$ restricts to the homeomorphism $\hat{\alpha}$ of Lemma 1.3 with $G_1 = \widetilde{\mathrm{Sp}}(W), \sigma_1 = \tau_0, G_2 = \mathrm{SO}(V^+), \sigma_2 = \mathbf{1}$. Hence, we see that the equivalence preserves temperedness and square integrability. \square

Remark. In [GS2], the preservation of temperedness and square integrability is shown by using Casselman’s criterion. In this paper, however, we invoke the theory of [BHK], which can be applied once the Hecke algebra isomorphism $H_\psi^+ \cong H^+$ is shown to be an isomorphism of Hilbert algebras. Indeed, this is one of the benefits of showing that $H_\psi^+ \cong H^+$ is not just an algebra isomorphism but a Hilbert algebra isomorphism.

3.2. Equivalence between \mathcal{G}_ψ^- and \mathcal{S}_0^- . Consider the partial flag

$$X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_2$$

where X_i is the k -span of $\mathbf{e}_i, \dots, \mathbf{e}_n$. Let $P = MN$ be the parabolic subgroup which is the stabilizer of this partial flag. Let W_1 be the symplectic subspace spanned by $\{\mathbf{e}_1, \mathbf{f}_1\}$ so that $\mathrm{Sp}(W_1) = \mathrm{SL}_2(k)$. Define $\omega_{\psi,1}$ to be the Weil representation of $\widetilde{\mathrm{Sp}}(W_1)$, realized as a representation in the space $S(k\mathbf{f}_1)$. This representation decomposes into even and odd parts; the odd part $\omega_{\psi,1}^-$ is supercuspidal.

Let $\tilde{P} = \tilde{M}N$ be the preimage of P in $\widetilde{\mathrm{Sp}}(W)$. Each element $m \in \tilde{M}$ is uniquely written as

$$m = m_1 \cdot \tilde{h}_{2\epsilon_2}(t_2) \cdots \tilde{h}_{2\epsilon_n}(t_n)$$

where $t_i \in k^\times$ and $m_1 \in \widetilde{\mathrm{Sp}}(W_1)$. Given a character $\chi = (\chi_2, \dots, \chi_n)$, we define a genuine representation $\omega_{\psi,1}^- \otimes \tilde{\chi}$ of \tilde{M} by

$$[\omega_{\psi,1}^- \otimes \tilde{\chi}](m) = \omega_{\psi,1}^-(m_1) \prod_{i=2}^n \chi_i(t_i) \beta_{t_i}.$$

We set

$$I(\omega_{\psi,1}^- \otimes \tilde{\chi}) = \mathrm{Ind}_{\tilde{P}}^{\widetilde{\mathrm{Sp}}(W)}(\omega_{\psi,1}^- \otimes \tilde{\chi})$$

to be the normalized induced representation. For a smooth representation π of $\widetilde{\mathrm{Sp}}(W)$ and π_N its normalized Jacquet module, we have Frobenius reciprocity:

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)}(\pi, I(\omega_{\psi,1}^- \otimes \tilde{\chi})) \cong \mathrm{Hom}_{\tilde{M}}(\pi_N, \omega_{\psi,1}^- \otimes \tilde{\chi}).$$

Lemma 3.5. *The Bernstein component \mathcal{G}_ψ^- is precisely the component whose irreducible representations are submodules of $I(\omega_{\psi,1}^- \otimes \tilde{\chi})$ for an unramified character χ .*

Proof. The functional $l : S(Y)^- \rightarrow S(k\mathfrak{f}_1)^-$, defined by restriction of functions from Y to $k\mathfrak{f}_1$, factors through the Jacquet module $(\omega_\psi^-)_N$. Therefore, there is a non-trivial element in $\text{Hom}_{\tilde{M}}((\omega_\psi^-)_N, \omega_{\psi,1}^- \otimes \tilde{\chi})$ for some unramified χ , which gives an embedding $\omega_\psi^- \subseteq I(\omega_\psi^- \otimes \tilde{\chi})$ via Frobenius reciprocity. \square

Theorem 3.6. *The functor from the category \mathcal{G}_ψ^- to the category of H_ψ^- -modules, given by*

$$\pi \mapsto (\pi \otimes (\tau_1^-)^*)^{\tilde{J}},$$

is an equivalence of categories. In particular, there is an equivalence of categories between \mathcal{G}_ψ^- and S_0^- given by the isomorphism $H_\psi^- \cong H^-$ of Hecke algebras. Furthermore, this equivalence preserves the temperedness and square integrability of representations.

Proof. Let $\tilde{J}_M = \tilde{J} \cap \tilde{M}$ and $\tilde{J}_N = \tilde{J} \cap N$. As in the previous subsection, we define the ‘‘Jacquet module’’ $(\tau_1^-)_N$ with respect to \tilde{J}_N , which we view as a representation of \tilde{J}_M . Recall that

$$\tau_1^- = S(\mathcal{O}/2\varpi\mathcal{O})^- \otimes S(\mathcal{O}^{n-1}/2\mathcal{O}^{n-1}),$$

and hence,

$$(\tau_1^-)_N = S(\mathcal{O}/2\varpi\mathcal{O})^- \otimes (\tau_{0,n-1})_{U_{n-1}},$$

where the second factor is the Jacquet module from the previous subsection in rank $n - 1$. Therefore, $(\tau_1^-)_N$ is an irreducible representation of \tilde{J}_M .

We have the natural surjection

$$r : (\pi \otimes (\tau_1^-)^*)^{\tilde{J}} \rightarrow (\pi_N \otimes (\tau_1^-)_N)^{\tilde{J}_M}.$$

Just as in Theorem 3.4, one can show that r is injective, which together with Frobenius reciprocity shows that $(\pi \otimes (\tau_1^-)^*)^{\tilde{J}} \neq 0$ if and only if π is a submodule of $I(\omega_{\psi,1}^- \otimes \tilde{\chi})$ for some unramified χ . Hence [BK, (3.11)] implies the equivalence of the categories.

Finally, this equivalence implies that (\tilde{J}, τ_1^-) is an \mathfrak{s} -type in the sense of [BHK], from which one can deduce the preservation of temperedness and square integrability just as in the previous section. \square

Remark. As a final remark, let us mention that in [GS2, Sec. 15 and 16] it is shown that the theta correspondence preserves unramified Langlands parameters, which relies on another work [GS1] of Gan and Savin. The only obstruction to remove the $p \neq 2$ assumption from [GS1], however, is the Howe duality conjecture, which was recently proven by Gan and the first-named author in [GT] for the case at hand. Hence, everything discussed in [GS2, Sec. 15 and 16] holds without the assumption $p \neq 2$.

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