TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 2, February 2018, Pages 1321–1350
http://dx.doi.org/10.1090/tran/7055
Article electronically published on September 25, 2017

TOPOLOGICAL PROPERTIES OF A CLASS OF SELF-AFFINE TILES IN $\mathbb{R}^3$

GUOTAI DENG, CHUNTAI LIU, AND SZE-MAN NGAI

ABSTRACT. We construct a class of connected self-affine tiles in $\mathbb{R}^3$ and prove that it contains a subclass of tiles that are homeomorphic to a unit ball in $\mathbb{R}^3$. Our construction is obtained by generalizing a two-dimensional one by Deng and Lau. The proof of ball-likeness is inspired by the construction of a homeomorphism from Alexander's horned ball to a 3-ball.

1. INTRODUCTION

The main purpose of this paper is to construct a class of self-affine tiles in $\mathbb{R}^3$ that are homeomorphic to a unit ball in $\mathbb{R}^3$.

Let $d \geq 1$ be an integer and let A be a $d \times d$ expanding matrix (i.e., all of its eigenvalues have moduli greater than 1). It is well known (see [11, 16]) that for any finite set $\mathcal{D} \subset \mathbb{R}^d$ there exists a unique nonempty compact set $T = T(A, \mathcal{D})$ such that

$$T = \bigcup\nolimits_{d_i \in \mathcal{D}} A^{-1}(T + d_i).$$

The above set equation can be rewritten as $AT = T + \mathcal{D}$, and T can be expressed as

$$T = \Big\{ \sum\nolimits_{k \geq 1} A^{-k} d_k : d_k \in \mathcal{D} \Big\}. \tag{1.1}$$

We call $\mathcal{D}$ a *digit set*, $(A, \mathcal{D})$ a *self-affine pair*, and T a *self-affine set*. If $\#\mathcal{D} = |\det(A)|$ is an integer and the interior of T is nonempty, then T actually tiles $\mathbb{R}^d$ in the following sense: there exists a discrete set $\mathcal{L} \subset \mathbb{R}^d$ which satisfies (i) $T + \mathcal{L} = \mathbb{R}^d$ and (ii) $(T^\circ + \iota_1) \cap (T^\circ + \iota_2) = \emptyset$ for all distinct $\iota_1, \iota_2 \in \mathcal{L}$. We call such T a *self-affine tile*.

Extensive studies of self-affine tiles began in the late 1980s by Thurston, Kenyon, Bandt, Lagarias, Wang, and others (see [3, 13, 16, 21] and the references therein). Since then, analytic, number theoretic, as well as topological properties of tiles have been studied by many authors.

This paper concerns topological properties of self-affine tiles. Connectedness of self-affine tiles has been studied by Kirat and Lau [14], Kirat et al. [15], Akiyama and Gjini [1], Leung and Luo [18], and others. Disk-likeness of self-affine tiles in $\mathbb{R}^2$ has

Received by the editors November 8, 2014, and, in revised form, March 4, 2016, and July 19, 2016.

2010 *Mathematics Subject Classification.* Primary 28A80, 52C22; Secondary 05B45, 51M20.

Key words and phrases. Self-affine tile, connectedness, ball-like tiles.

The first author was supported by the China Scholarship Council. The second author was supported by the National Natural Science Foundation of China grant 11601403. The third author was supported in part by the National Natural Science Foundation of China grant 11271122, the Hunan Province Hundred Talents Program, and a Faculty Research Scholarly Pursuit Funding from Georgia Southern University.

been investigated by authors including Bandt and Wang [5], Luo et al. [19], Ngai and Tang [20], Leung and Lau [17], and Deng and Lau [8]. Topological properties of self-affine tiles related to number systems have been studied by Akiyama, Thuswaldner, and other authors (see [2] and the references therein).

The main motivation for this paper is the construction of ball-like tiles. Gelbrich [9] asked whether or not a lattice self-affine tile with two pieces in $\mathbb{R}^d$ ($d \geq 3$) is homeomorphic to the d-dimensional ball (this is true for $d = 2$). We refer the reader to Bandt [4] for some related conjectures on this question. Many tiles in $\mathbb{R}^2$ are disk-like. However, in $\mathbb{R}^d$ ($d \geq 3$), except for some trivial tiles such as a hypercube, the Cartesian product of a disk-like tile, and some interval, not much was known. Recently, by making use of deep theorems from geometric topology, Conner and Thuswaldner [7] formulated an algorithm which allowed them to prove that some nontrivial self-affine tiles in $\mathbb{R}^3$ are ball-like. Unlike the method in [7], our proof is direct analytic and topological in nature. After this work was completed, the authors were informed by Jun Luo that Kamae, Luo, and Tan [12] constructed a family of n-dimensional self-affine tiles that are homeomorphic to the unit cube $[0,1]^n$.

This paper considers tiles generated by digit sets that are noncollinear. More precisely, we consider the following family:

$$A := \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ -t & -s & r \end{pmatrix} \quad \text{and} \quad \begin{aligned} \mathcal{D} :=\big\{(i, j, k + a_i + b_j) :\ & 0 \leq i < |p|, \\ & 0 \leq j < |q|, 0 \leq k < |r|\big\}, \end{aligned} \tag{1.2}$$

where a_i, b_j are in $\mathbb{R}$. Note that $T(A^2, \widetilde{\mathcal{D}}) = T(A, \mathcal{D})$ if $\widetilde{\mathcal{D}} = \mathcal{D} + A\mathcal{D}$. So, by considering $(A^2, \widetilde{\mathcal{D}})$ instead of $(A, \mathcal{D})$ if necessary, we will assume $p, q, r \geq 2$. To study the set T, we consider the following iterated function system (IFS) that generates T:

$$\big\{S_{i,j,k}(x, y, z) = A^{-1}\big((x, y, z)+(i, j, k+a_i+b_j)\big) : 0 \leq i < p,\ 0 \leq j < q,\ 0 \leq k < r\big\}. \tag{1.3}$$

By using this IFS, we partition T in three different ways as follows. Let

$$G_{ij} = \bigcup_{k=0}^{r-1} S_{i,j,k}(T), \qquad E_i = \bigcup_{j=0}^{q-1} G_{ij}, \qquad F_j = \bigcup_{i=0}^{p-1} G_{ij}, \tag{1.4}$$

$$T = \bigcup_{i,j} G_{ij} = \bigcup_{i=0}^{p-1} E_i = \bigcup_{j=0}^{q-1} F_j. \tag{1.5}$$

We call each $G_{i,j}$ a *cylinder*. We say that two cylinders $G_{i,j}$, $G_{i',j'}$ are *adjacent* if $\max\{|i - i'|, |j - j'|\} \leq 1$ and are *diagonal* if $|i - i'| = |j - j'| = 1$ (see Figure 1).

We will use Lemma 2.3 from [10] to investigate the connectedness of T. For this purpose, we define $\delta_1, \delta_2, \delta_3$ as follows:

$$\begin{aligned} \delta_1(i) &:= \inf\big\{\big||\rho_1(i)| - r^{-n}|\rho_2(j)|\big| : 0 \leq j < q-1, n \geq 1\big\}, \quad 0 \leq i < p-1, \\ \delta_2(j) &:= \inf\big\{\big||\rho_2(j)| - r^{-n}|\rho_1(i)|\big| : 0 \leq i < p-1, n \geq 1\big\}, \quad 0 \leq j < q-1, \\ \delta_3(i,j) &:= \big||\rho_1(i)| - |\rho_2(j)|\big|, \end{aligned} \tag{1.6}$$

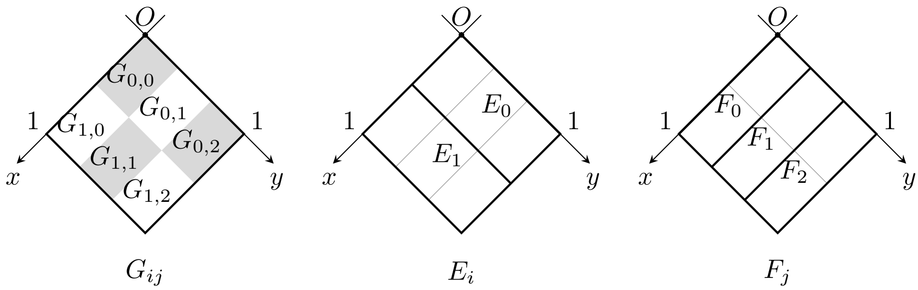


FIGURE 1. The projections on the plane $z = 0$ of T, G_{ij}, E_i, F_j, drawn with $p = 2$ and $q = 3$. Two adjacent cylinders marked in gray are diagonal.

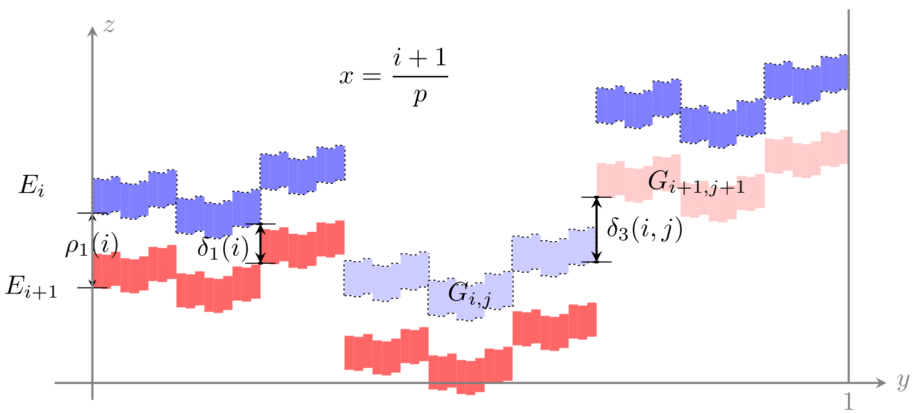


FIGURE 2. Geometric interpretation of δ_1, δ_3, and ρ_1. The figure is the vertical cross-section of T on the plane $x = (i+1)/p$. The set with a dotted boundary is the cross-section of E_i, and the rest is the cross-section of E_{i+1}. It is drawn with $p = 2$, $q = 3$, $r = 4$, $s = t = 1$, $a_0 = 0$, $a_1 = 13.7$, $b_0 = 0$, and $b_2 = -b_1 = 10$.

where

$$\rho_1(i) := \frac{a_{p-1} - a_0 + t}{r(r-1)} + \frac{a_i - a_{i+1}}{r}, \qquad \rho_2(j) := \frac{b_{q-1} - b_0 + s}{r(r-1)} + \frac{b_j - b_{j+1}}{r}. \tag{1.7}$$

We comment on these quantities. Let W_i^+, W_{i+1}^- be the restrictions of E_i, E_{i+1} to the plane $x = (i+1)/p$, respectively (see Figure 2). Then $\rho_1(i)$ is the amount of translation from W_i^+ to W_{i+1}^-, and $\delta_1(i)$ is the vertical distance between the bottoms of W_i^+ and W_{i+1}^-. The quantities ρ_2 and δ_2 have analogous geometric meanings (with respect to the sets F_j). The quantity $\delta_3(i,j)$ is the distance of the bottom of two diagonal cylinders restricted on the vertical line (see Figure 2):

$$x = \frac{i+1}{p}, \qquad y = \frac{j+1}{q}, \qquad z = t, \qquad t \in \mathbb{R}.$$

Our first main result concerns the connectedness of T and its interior.

Theorem 1.1. *Let $(A, \mathcal{D})$ be given as in* (1.2) *with $p, q, r \geq 2$ and let T be the corresponding self-affine set. Assume ρ_1, ρ_2 are defined as in* (1.7) *and $\delta_1, \delta_2, \delta_3$ are given as in* (1.6).

(a) *T is connected if* (i) *or* (ii) *below holds:*
  (i) *for all i, $\delta_1(i) \leq 1$, and for all j either $\delta_2(j) \leq 1$ or there exists i (depending only on j) such that $\delta_3(i, j) \leq 1$;*
  (ii) *for all j, $\delta_2(j) \leq 1$, and for all i, either $\delta_1(i) \leq 1$ or there exists j (depending only on i) such that $\delta_3(i, j) \leq 1$.*

Moreover, if a_i, b_j are zero for all i,j, then each of the above sufficient conditions is necessary; i.e., if T is connected, then (i) *or* (ii) *holds.*

(b) *T° is connected if and only if $|\rho_1(i)| < 1$ and $|\rho_2(j)| < 1$ for all i, j.*

We remark that the condition in Theorem 1.1(a) is sufficient but not necessary (see Example 3.4).

Next, we consider the question whether T is homeomorphic to a 3-ball. It is easy to get a necessary condition for T to be homeomorphic to a ball by considering the genus. However, it is not easy to obtain a sufficient condition. We do this by directly constructing a homeomorphism from T onto a 3-ball. This approach is inspired by the construction of a homeomorphism from the Alexander horned ball onto a 3-ball (see Bing [6]). The key to the construction is to decompose T in a way that allows us to define the homeomorphism inductively. Roughly speaking, we write T as a union of infinitely many levels (each level is a disjoint union of finitely many polyhedra) such that two levels intersect if and only if they are adjacent (for more details, see Section 4). The following is the main result of the paper.

Theorem 1.2. *Let $(A, \mathcal{D})$ be given as in* (1.2) *with $p, q, r \geq 2$ and let T be the corresponding self-affine set. Assume ρ_1, ρ_2 are defined as in* (1.7)*. If T is homeomorphic to a ball, then $|\rho_1(i)| + |\rho_2(j)| < 1$ for all i, j. The converse holds if all a_i, b_j are zero and $st \geq 0$.*

The rest of this paper is organized as follows. In Section 2, we establish some preliminary results that are needed in the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.1. Ball-likeness (Theorem 1.2) is proved in Section 4. Finally, we state an open question in Section 5.

2. Preparation for the proof of Theorem 1.1

In this section, we establish some results that will be used in the proofs of Theorems 1.1 and 1.2. For a set $E \subseteq \mathbb{R}^n$, let E°, ∂E, and $\overline{E}$ denote the interior, boundary, and closure of E respectively.

2.1. **The symbolic space.** For an integer $m \geq 2$, denote $\Sigma_m^k := \{0, 1, \ldots, m-1\}^k$, $\Sigma_m^* := \bigcup_{k \geq 0} \Sigma_m^k$ and $\Sigma_m^\infty := \{0, 1, \ldots, m-1\}^\infty$, where $\Sigma_m^0 := \{\emptyset\}$ ($\emptyset$ is the *empty word*). We equip the symbolic space Σ_m^n with the lexicographic order.

We call $\mathbf{i} \in \Sigma_m^k$ an *m-adic word* of length k, and denote its length by $|\mathbf{i}|$. For $\mathbf{i} = i_1 \cdots i_k \in \Sigma_m^k$ and $\mathbf{j} = j_1 j_2 \cdots \in \Sigma_m^* \bigcup \Sigma_m^\infty$, let $\mathbf{ij} = i_1 \cdots i_k j_1 i_2 \cdots$, $\mathbf{j}|n = j_1 \cdots j_n$, and $\mathbf{i}^\pm = i_1 \cdots i_{k-1}(i_k \pm 1)$ if $i_k \pm 1 \in \Sigma_m^1$. For $\mathbf{i} \in \Sigma_m^*$, we denote the infinite word $\mathbf{ii} \cdots$ by $\bar{\mathbf{i}}$.

The map $\varphi_m : \Sigma_m^\infty \to [0, 1]$ defined as

$$\varphi_m(\mathbf{i}) := \sum_{n \geq 1} \frac{i_n}{m^n}, \qquad \mathbf{i} = i_1 i_2 \cdots, \tag{2.1}$$

is surjective. We say $\mathbf{i} \in \Sigma_m^\infty$ is an *m-adic expansion* (or simply *expansion*) of $x \in [0,1]$ if $x = \varphi_m(\mathbf{i})$. A real number x can have one or two m-adic expansions. x has two expansions if and only if $x = \sum_{n=1}^{|\mathbf{i}|} i_n m^{-n}$ for some $\mathbf{i} \in \Sigma_m^*$ with $\mathbf{i}_{|\mathbf{i}|} \neq 0$; in this case the two expansions are

$$i_1 \cdots i_{|\mathbf{i}|}\overline{(m-1)} \qquad \text{and} \qquad i_1 \cdots i_{|\mathbf{i}|-1}(i_{|\mathbf{i}|}+1)\overline{0},$$

where $0 \le i_{|\mathbf{i}|} < m-1$. We also define $\varphi_m(\mathbf{i}) := \sum_{n=1}^{|\mathbf{i}|} m^{-n} i_n$ for $\mathbf{i} \in \Sigma_m^*$.

Denote $\Sigma_{p,q}^n := \Sigma_p^n \times \Sigma_q^n$ and $\Sigma_{p,q}^\infty := \Sigma_p^\infty \times \Sigma_q^\infty$. We also say $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^\infty$ is an *expansion* of (x,y) if $(x,y) = (\varphi_p(\mathbf{i}), \varphi_q(\mathbf{j}))$.

2.2. **Notation concerning tiles.** Let $T = T(A,\mathcal{D})$ be given as in Theorem 1.1. We introduce some notation that will simplify the proof of Theorem 1.1. First, by a simple calculation, the inverse of the expanding matrix A^n can be written as

$$A^{-n} = \begin{pmatrix} p^{-n} & 0 & 0 \\ 0 & q^{-n} & 0 \\ t_n & s_n & r^{-n} \end{pmatrix}, \tag{2.2}$$

where

$$t_n := \begin{cases} \dfrac{(p^{-n}-r^{-n})t}{r-p}, & r \neq p, \\ \dfrac{nt}{r^{n+1}}, & r = p, \end{cases} \qquad s_n := \begin{cases} \dfrac{(q^{-n}-r^{-n})s}{r-q}, & r \neq q, \\ \dfrac{ns}{r^{n+1}}, & r = q. \end{cases} \tag{2.3}$$

Using the expression for T (see (1.1)) and the above expression for A^{-n}, we may rewrite T as

$$\begin{aligned} T \;=\; \Big\{\Big(\sum_{n\ge1}\frac{i_n}{p^n},\, \sum_{n\ge1}\frac{j_n}{q^n},\, \sum_{n\ge1}\big(\frac{k_n}{r^n} + i_n t_n + \frac{a_{i_n}}{r^n} + j_n s_n + \frac{b_{j_n}}{r^n}\big)\Big) : 0 \le i_n < p, \\ 0 \le j_n < q,\ 0 \le k_n < r,\ n \ge 1\Big\}. \end{aligned}$$

To simplify this expression for T, we define

$$t(\mathbf{i}) := \sum_{n=1}^{|\mathbf{i}|} i_n t_n, \quad a(\mathbf{i}) := \sum_{n=1}^{|\mathbf{i}|} \frac{a_{i_n}}{r^n}, \quad s(\mathbf{j}) := \sum_{n=1}^{|\mathbf{j}|} j_n s_n, \quad b(\mathbf{j}) := \sum_{n=1}^{|\mathbf{j}|} \frac{b_{j_n}}{r^n}, \tag{2.4}$$

and we denote

$$d(\mathbf{i},\mathbf{j}) := t(\mathbf{i}) + a(\mathbf{i}) + s(\mathbf{j}) + b(\mathbf{j}),$$

for $\mathbf{i} \in \Sigma_p^* \bigcup \Sigma_p^\infty$ and $\mathbf{j} \in \Sigma_q^* \bigcup \Sigma_q^\infty$. Geometrically, $d(\mathbf{i},\mathbf{j})$ is on the bottom surface of T. For convenience, we set $t(\emptyset) = s(\emptyset) = a(\emptyset) = b(\emptyset) := 0$. Now T can be expressed as

$$T = \Big\{\big(\varphi_p(\mathbf{i}), \varphi_q(\mathbf{j}), d(\mathbf{i},\mathbf{j}) + \varphi_r(\mathbf{k})\big) : (\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^\infty,\ \mathbf{k} \in \Sigma_r^\infty\Big\}. \tag{2.5}$$

Consider the IFS $\{S_{i,j,k}\}$ in (1.3) that generates T. For $\mathbf{i} = i_1 \cdots i_n$, $\mathbf{j} = j_1 \cdots j_n$, and $\mathbf{k} = k_1 \cdots k_n$, define

$$S_{\mathbf{i},\mathbf{j},\mathbf{k}} := S_{i_1,j_1,k_1} \circ \cdots \circ S_{i_n,j_n,k_n} \quad \text{and} \quad G_{\mathbf{i},\mathbf{j}} := \bigcup_{\mathbf{k}\in\Sigma_r^n} S_{\mathbf{i},\mathbf{j},\mathbf{k}}(T).$$

It is easy to check that

$$\begin{aligned} S_{\mathbf{i},\mathbf{j},\mathbf{k}}(T) \;&=\; \big\{(\varphi_p(\mathbf{i}\mathbf{i}'), \varphi_q(\mathbf{j}\mathbf{j}'), d(\mathbf{i}\mathbf{i}',\mathbf{j}\mathbf{j}') + r(\mathbf{k}\mathbf{k}')) : (\mathbf{i}',\mathbf{j}') \in \Sigma_{p,q}^\infty,\ \mathbf{k}' \in \Sigma_r^\infty\big\}, \\ G_{\mathbf{i},\mathbf{j}} \;&=\; \big\{(\varphi_p(\mathbf{i}\mathbf{i}'), \varphi_q(\mathbf{j}\mathbf{j}'), d(\mathbf{i}\mathbf{i}',\mathbf{j}\mathbf{j}') + \varphi_r(\mathbf{k}')) : (\mathbf{i}',\mathbf{j}') \in \Sigma_{p,q}^\infty,\ \mathbf{k}' \in \Sigma_r^\infty\big\}. \end{aligned} \tag{2.6}$$

Let E_i and F_j be defined as in (1.4). We get from (1.5) and (2.6) that

$$T = \bigcup_{0\le i<p} E_i = \bigcup_{0\le j<q} F_j = \bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^n} G_{\mathbf{i},\mathbf{j}} \quad \text{for all } n \ge 1.$$

The components E_i, F_i, and $G_{\mathbf{i},\mathbf{j}}$ of T will be used to study the connectedness of T in Section 3.

2.3. **Some properties of the tile T.** In this subsection, we list some useful properties of the self-affine pair $(A, \mathcal{D})$ and T. Lemma 2.1 presents some formulas concerning $t_n, s_n, t(\mathbf{i}), s(\mathbf{j}), a(\mathbf{i}), b(\mathbf{j})$. Proposition 2.2 states some properties of the tile, where (a) and (b) are analogs of [8, Proposition 2.2], and (c) follows from (2.6). Theorem 2.3, due to Hata [10], is used to prove connectedness. Lemma 2.4 is a direct consequence of Theorem 2.3 and Proposition 2.2(b). Finally, Lemma 2.5 gives a complete description of the interior of T. We point out here that most of the computations in Sections 3 and 4 depend on (2.8) and (2.9).

Lemma 2.1. *Let $t_n, s_n, t(\mathbf{i}), s(\mathbf{j}), a(\mathbf{i}), b(\mathbf{j})$ be defined with respect to the self-affine pair $(A, \mathcal{D})$, and $\rho_1(i), \rho_2(j)$ be as in (1.7). Then the following equalities hold:*

$$pt_{n+1} - t_n = tr^{-n-1}, \qquad qs_{n+1} - s_n = sr^{-n-1}; \tag{2.7}$$

$$\sum_{k\ge n} t_k = \begin{cases} \dfrac{\big(p^{-n+1}(r-1) - r^{-n+1}(p-1)\big)t}{(p-1)(r-p)(r-1)}, & p \ne r, \\ \dfrac{(1+n(r-1))r^{-n}t}{(r-1)^2}, & p = r, \end{cases}$$
$$\sum_{k\ge n} s_k = \begin{cases} \dfrac{\big(q^{-n+1}(r-1) - r^{-n+1}(q-1)\big)s}{(q-1)(r-q)(r-1)}, & q \ne r, \\ \dfrac{(1+n(r-1))r^{-n}s}{(r-1)^2}, & q = r. \end{cases} \tag{2.8}$$

Moreover, for $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$, $0 \le i < p-1$ and $0 \le j < q-1$,
(2.9)
$$\begin{aligned} d(\mathbf{i}i\overline{p-1},\mathbf{j}) - d(\mathbf{i}(i+1)\overline{0},\mathbf{j}) &= t(\mathbf{i}i\overline{p-1}) - t(\mathbf{i}(i+1)\overline{0}) + a(\mathbf{i}i\overline{p-1}) - a(\mathbf{i}(i+1)\overline{0}) \\ &= r^{-n}\rho_1(i), \\ d(\mathbf{i},\mathbf{j}j\overline{q-1}) - d(\mathbf{i},\mathbf{j}(j+1)\overline{0}) &= s(\mathbf{j}j\overline{q-1}) - s(\mathbf{j}(j+1)\overline{0}) + b(\mathbf{j}j\overline{q-1}) - b(\mathbf{j}(j+1)\overline{0}) \\ &= r^{-n}\rho_2(j). \end{aligned}$$

Proof. Since (2.8) follows from (2.3), we show (2.7) and (2.9). Using symmetry, we only prove the equalities for p.

Using (2.3), we see that

$$pt_{n+1} - t_n = \begin{cases} \dfrac{p(p^{-n-1} - r^{-n-1})t}{r-p} - \dfrac{(p^{-n} - r^{-n})t}{r-p} = \dfrac{t}{r^{n+1}}, & p \ne r, \\ \dfrac{p(n+1)t}{r^{n+2}} - \dfrac{nt}{r^{n+1}} = \dfrac{t}{r^{n+1}}, & p = r. \end{cases}$$

Note that in both cases, $pt_{n+1} > t_n$.

As for (2.9), we first compute $\sigma := (p-1)\sum_{k\ge n+2} t_k - t_{n+1}$ by using (2.3) and (2.8). When $p = r$,

$$\sigma = \frac{\big(1+(n+2)(r-1)\big)r^{-n-2}t}{r-1} - \frac{(n+1)t}{r^{n+2}} = \frac{r^{-n}t}{r(r-1)}, \tag{2.10}$$

and when $p \neq r$,

$$\sigma = \frac{\big(p^{-n-1}(r-1) - r^{-n-1}(p-1)\big)t}{(r-p)(r-1)} - \frac{\big(p^{-n-1} - r^{-n-1}\big)t}{r-p} = \frac{r^{-n}t}{r(r-1)}. \tag{2.11}$$

Second, fix $\mathbf{i} = i_1 \cdots i_n \in \Sigma_p^n$ and $0 \le i < p-1$. From the definition of $t(\cdot)$, (2.10) and (2.11), we get

$$\begin{aligned} t(\mathbf{i}i\overline{(p-1)}) - t(\mathbf{i}(i+1)\overline{0}) &= it_{n+1} + (p-1)\sum_{k \ge n+2} t_k - (i+1)t_{n+1} \\ &= \sigma = \frac{r^{-n}t}{r(r-1)}. \end{aligned} \tag{2.12}$$

From the definition of $a(\cdot)$, we have

$$\begin{aligned} a(\mathbf{i}i\overline{(p-1)}) - a(\mathbf{i}(i+1)\overline{0}) &= \frac{a_i}{r^{n+1}} + a_{p-1}\sum_{k \ge n+2}\frac{1}{r^k} - \frac{a_{i+1}}{r^{n+1}} - a_0 \sum_{k \ge n+2}\frac{1}{r^k} \\ &= r^{-n}\Big(\frac{a_i - a_{i+1}}{r} + \frac{a_{p-1} - a_0}{r(r-1)}\Big). \end{aligned} \tag{2.13}$$

Finally, combining (2.12), (2.13) and the expression for $\rho_1(i)$ completes the proof. $\square$

To state the next lemma, we let

$$\ell_{x,y} := \{(x,y)\} \times \mathbb{R} \qquad \text{and} \qquad T_{x,y} := T \cap \ell_{x,y}$$

be, respectively, the vertical line passing through $(x,y,0)$ and the restriction of T to $\ell_{x,y}$.

Proposition 2.2. *Let $(A, \mathcal{D})$ be given as in Theorem 1.1; i.e.,*

$$A = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ -s & -t & r \end{pmatrix} \quad \textit{and} \quad \mathcal{D} = \{i, j, k + a_i + b_j : 0 \le i < p, 0 \le j < q, 0 \le k < r\},$$

where $p, q, r \ge 2$. Then the following statements hold.

(a) *The set $T = T(A, \mathcal{D})$ is a tile with Lebesgue measure 1, and for any sequence of real numbers $\{\alpha_{i,j} : (i,j) \in \mathbb{Z}^2\}$, $\mathcal{J} := \{(i, j, k+\alpha_{i,j}) : (i,j) \in \mathbb{Z}^2, k \in \mathbb{Z}\}$ is a tiling set for T in $\mathbb{R}^3$.*

(b) *For any $(i,j) \in \{0, \dots, p-1\} \times \{0, \dots, q-1\}$ and $k \in \{0, \dots, r-1\}$, $S_{i,j,k}(T) \cap S_{i,j,k+1}(T) \cap T^\circ \neq \emptyset$.*

(c) *$T_{x,y}$ can be written as a union of vertical unit intervals as*

$$T_{x,y} = \bigcup_{(\mathbf{i},\mathbf{j}) \in (\varphi_p^{-1}(\{x\}) \times \varphi_q^{-1}(\{y\})) \cap \Sigma_{p,q}^\infty} \{(x,y)\} \times \big([0,1] + d(\mathbf{i},\mathbf{j})\big).$$

Proof. (a) Let $\mathcal{D}_j = \{0, \dots, j-1\}$; then $T(j, \mathcal{D}_j) = [0,1]$ for $j \ge 2$. Fix $(x,y,z) \in \mathbb{R}^3$. We choose $(i,j) \in \mathbb{Z}^2$ such that $(x,y) - (i,j) \in [0,1]$, and hence there is $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^\infty$ such that $(x,y) - (i,j) = (\varphi_p(\mathbf{i}), \varphi_q(\mathbf{j}))$. Since $T(r, \mathcal{D}_r) = [0,1]$, there exists an integer k such that $z - d(\mathbf{i},\mathbf{j}) - k - \alpha_{i,j} \in [0,1]$. This implies that $z = d(\mathbf{i},\mathbf{j}) + \varphi_r(\mathbf{k}) + k + \alpha_{i,j}$ for some $\mathbf{k} \in \Sigma_r^\infty$. Therefore, $(x,y,z) \in T + (i,j,k+\alpha_{i,j})$, implying $T + \mathcal{J} = \mathbb{R}^3$.

It remains to show that $\{T + \mathbf{t} : \mathbf{t} \in \mathcal{J}\}$ are essentially disjoint. In fact, for almost all $(x,y) \in \mathbb{R}^2$, there is a unique pair $(i,j) \in \mathbb{Z}^2$ and $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^\infty$ such that $(x,y) - (i,j) = (\varphi_p(\mathbf{i}), \varphi_q(\mathbf{j}))$. When (i,j) and $(\mathbf{i},\mathbf{j})$ are fixed, for almost all $z \in \mathbb{R}$, the above $\mathbf{k}$ and k are unique. Hence the Lebesgue measure of $(T+\mathbf{t}_1)\cap(T+\mathbf{t}_2)$ is zero if $\mathbf{t}_1,\mathbf{t}_2\in\mathcal{J}$ with $\mathbf{t}_1\neq\mathbf{t}_2$. If we let all $\alpha_{i,j}$ be zero, then $\mathcal{J}=\mathbb{Z}^3$ is a lattice tiling set, completing the proof of (a).

(b) Let $\mathbf{e}_1=(0,0,1)$ be a unit vector in $\mathbb{R}^3$. Now we show that $T\cap(T+\mathbf{e}_1)$ contains an interior point of $(T\cup(T+\mathbf{e}_1))^\circ$, which implies the result since $S_{i,j,k}^{-1}\circ S_{i,j,k+1}(x,y,z)=(x,y,z)+\mathbf{e}_1$. Take $(x,y)\in(0,1)^2$ with a unique expansion $(\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^\infty$. Let

$$\mathbf{t}=\big(x,y,d(\mathbf{i},\mathbf{j})\big)+\mathbf{e}_1. \tag{2.14}$$

Then (2.5) implies $\mathbf{t}\in T\cap(T+\mathbf{e}_1)$.

Next we show that $\mathbf{t}$ is a desired interior point of $T\cup(T+\mathbf{e}_1)$. By (a), $\mathbb{Z}^3$ is a tiling set of T. Since (x,y) is an interior point of $[0,1]^2$ and $T\subset[0,1]^2\times\mathbb{R}$, that $\mathbf{t}\in T+(i,j,k)$ for some (i,j,k) implies $i=j=0$. On the other hand, if $\mathbf{t}\in T+(0,0,k)$, then the last coordinate of $\mathbf{t}$ must be of the form

$$d(\mathbf{i},\mathbf{j})+\varphi_r(\mathbf{k})+k$$

for some $\mathbf{k}\in\Sigma_r^\infty$. Considering (2.14) and (2.5), we have $\varphi_r(\mathbf{k})+k=1$ and hence $k=0$ or 1 since $\varphi_r(\mathbf{k})\in[0,1]$. Therefore $\mathbf{t}$ is contained only in $T\cup(T+\mathbf{e}_1)$ and not in other translates of T. Hence $\mathbf{t}$ does not lie in the boundary of $T\cup(T+\mathbf{e}_1)$. The conclusion follows.

(c) This is a consequence of the expression (2.5) for T, since $\varphi_r(\Sigma_r^\infty)=[0,1]$. □

Theorem 2.3 (Hata [10]). *Let $\{\psi_j\}_{j=1}^N$ be a family of contractions on $\mathbb{R}^d$ and let K be its attractor. Then K is connected if and only if for any $i\neq j\in\{1,\dots,N\}$, there exists a finite sequence of indices $j_1,\dots,j_n$ in $\{1,\dots,N\}$, with $j_1=i$ and $j_n=j$, such that $\psi_{j_k}(K)\cap\psi_{j_{k+1}}(K)\neq\emptyset$ for all $1\le k<n$.*

Applying the above lemma to the IFS $\{S_{i,j,k}\}$, we obtain the following lemma.

Lemma 2.4. *Assume the sequence $\{(i_1,j_1),\dots,(i_n,j_n)\}$ satisfies $\bigcup_{k=1}^n\{(i_k,j_k)\}=\{0,\dots,p-1\}\times\{0,\dots,q-1\}$. If $G_{i_k,j_k}\cap G_{i_{k+1},j_{k+1}}\neq\emptyset$ for $1\le k<n$, then T is connected.*

For $x,y\in(0,1)$, let

$$d_{\min}(x,y):=\min\big\{d(\mathbf{i},\mathbf{j}):\ \varphi_p(\mathbf{i})=x,\ \varphi_q(\mathbf{j})=y,\ (\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^\infty\big\},$$
$$d_{\max}(x,y):=\max\big\{d(\mathbf{i},\mathbf{j}):\ \varphi_p(\mathbf{i})=x,\ \varphi_q(\mathbf{j})=y,\ (\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^\infty\big\}.$$

Intuitively, $d_{\min}(x,y)$ and $d_{\max}(x,y)+1$ are the lowest and highest points of $T\cap\ell_{x,y}$, respectively.

Lemma 2.5. *Let $x,y\in(0,1)$. A point $P=(x,y,z)\in T$ belongs to T° if and only if*

$$d_{\max}(x,y)<z<d_{\min}(x,y)+1. \tag{2.15}$$

Proof. Using Proposition 2.2(a), we let $\mathcal{L}=\mathbb{Z}^3$ be a tiling set for T. Assume (2.15) holds. If $P\in T+\mathbf{t}$ for some $\mathbf{t}=(t_1,t_2,t_3)\in\mathcal{L}$, then $t_1=t_2=0$. In this case, $P\in T_{x,y}+t_3$, i.e., $z-t_3\in[d_{\min}(x,y),d_{\max}(x,y)+1]$. Comparing this with (2.15), we get $t_3=0$. Hence P does not lie in any neighbor of T in the tiling and is hence an interior point of T. If (2.15) fails, then $z\in[0,1]+d_{\max}(x,y)-k$ or $z\in[0,1]+d_{\min}(x,y)+k$ for some integer $k\neq0$, implying $P\in T+(0,0,k)$. Hence $P\in T\cap(T+(0,0,k))$ and thus $P\notin T^\circ$. The proof is complete. □

3. Proof of Theorem 1.1

In this section, we study the self-affine set $T = T(A, \mathcal{D})$ in $\mathbb{R}^3$ and prove Theorem 1.1. To do this, we first establish some properties of the intersections of the $G_{i,j}$ (Lemma 3.1) and of the $G_{\mathbf{i},\mathbf{j}} \cap T^o$ (Lemma 3.3) by using ρ_1, ρ_2, δ_1, δ_2 and δ_3, which are defined as in Section 1. Then by using these properties and Lemma 2.4, we give a detailed proof of Theorem 1.1. Finally, we present an example to show that the condition in Theorem 1.1(a) is not necessary.

We first observe by definition that $\delta_1(i) \le |\rho_1(i)|$ and $\delta_2(j) \le |\rho_2(j)|$. Also, if $\rho_1(i) \ne 0$, then, since $r^{-n}|\rho_2(j)| \to 0$ as $n \to \infty$, we have

$$\delta_1(i) = \min_{0\le j\le q-1, n\ge 1} \big||\rho_1(i)| - r^{-n}|\rho_2(j)|\big|. \tag{3.1}$$

Similarly, if $\rho_2(j) \ne 0$, then

$$\delta_2(j) = \min_{0\le i\le p-1, n\ge 1} \big||\rho_2(j)| - r^{-n}|\rho_2(i)|\big|. \tag{3.2}$$

Lemma 3.1. *If $G_{i,j} \cap G_{i',j'} \ne \emptyset$, then $G_{i,j}$ and $G_{i',j'}$ are adjacent, i.e., $\max\{|i - i'|, |j - j'|\} \le 1$. Moreover,*

(a) *$G_{i,j} \cap G_{i+1,j} \ne \emptyset$ for any j if and only if $\delta_1(i) \le 1$;*
(b) *$G_{i,j} \cap G_{i,j+1} \ne \emptyset$ for any i if and only if $\delta_2(j) \le 1$;*
(c) *$G_{i,j} \cap G_{i+1,j+1} \ne \emptyset$ if and only if $|\rho_1(i) - \rho_2(j)| \le 1$, and $G_{i+1,j} \cap G_{i,j+1} \ne \emptyset$ if and only if $|\rho_1(i) + \rho_2(j)| \le 1$. In particular, if either of the inequalities is indeed an equality, then the corresponding intersection contains no interior points of T.*

Proof. The first part of the conclusion holds since $G_{i,j}$ is contained in the region

$$\left[\frac{i}{p}, \frac{i+1}{p}\right] \times \left[\frac{j}{q}, \frac{j+1}{q}\right] \times \mathbb{R}.$$

To show (a), assume $G_{i,j} \cap G_{i+1,j} \ne \emptyset$. Take any point $P = (x, y, z)$ contained in the intersection. Note that $G_{i,j} \cap G_{i+1,j}$ lies on the plane $x = (i+1)/p$, and so x has two p-adic expansions $\mathbf{i}_1 = i\overline{(p-1)}$ and $\mathbf{i}_2 = (i+1)\overline{0}$. Suppose $y \in [j/q, (j+1)/q]$ for some j. Recall that $T_{x,y}$, which contains P, is a union of vertical unit intervals (see Proposition 2.2(c)).

If y has only one expansion $\mathbf{j}$, then by the definition of $T_{x,y}$, the last coordinates of $T_{x,y}$ form a union of two unit intervals $I_1 \cup I_2$, where $I_m = [0,1] + d(\mathbf{i}_m, \mathbf{j})$ for $m = 1, 2$ (see Figure 3(a1)). Clearly, $\{(x, y)\} \times I_1 \subset G_{i,j}$ and $\{(x, y)\} \times I_2 \subset G_{i+1,j}$. That is, $T_{x,y}$ is a line segment, and so is $I_1 \cup I_2$. Noting that I_1 and I_2 are two unit intervals and putting $n = 0$ in (2.9), we get

$$\big|\rho_1(i)\big| = \big|d(\mathbf{i}_1, \mathbf{j}) - d(\mathbf{i}_2, \mathbf{j})\big| \le 1.$$

If y has two expansions $\mathbf{j}_1 = \mathbf{j}j'\overline{(q-1)}$, $\mathbf{j}_2 = \mathbf{j}(j'+1)\overline{0}$, where $\mathbf{j} \in \Sigma_q^n$, then the last coordinates of $T_{x,y}$ form a union of four unit intervals $\bigcup_{l,m=1}^2 I_{l,m}$, where $I_{l,m} = [0,1] + d(\mathbf{i}_l, \mathbf{j}_m)$ (see Figure 3(a2)). Clearly, $\{(x,y)\} \times (I_{1,1} \cup I_{1,2}) \subset G_{i,j}$, $\{(x,y)\} \times (I_{2,1} \cup I_{2,2}) \subset G_{i+1,j}$. Since $(x,y,z) \in T_{x,y} \cap G_{i,j} \cap G_{i+1,j} \ne \emptyset$, either $I_{2,1} \cap (I_{1,1} \cup I_{1,2}) \ne \emptyset$ or $I_{2,2} \cap (I_{1,1} \cup I_{1,2}) \ne \emptyset$. Hence either one of the following holds:

$$\min\big\{|d(\mathbf{i}_2, \mathbf{j}_1) - d(\mathbf{i}_1, \mathbf{j}_1)|,\ |d(\mathbf{i}_2, \mathbf{j}_1) - d(\mathbf{i}_1, \mathbf{j}_2)|\big\} \le 1 \quad \text{or}$$
$$\min\big\{|d(\mathbf{i}_2, \mathbf{j}_2) - d(\mathbf{i}_1, \mathbf{j}_1)|,\ |d(\mathbf{i}_2, \mathbf{j}_2) - d(\mathbf{i}_1, \mathbf{j}_2)|\big\} \le 1.$$

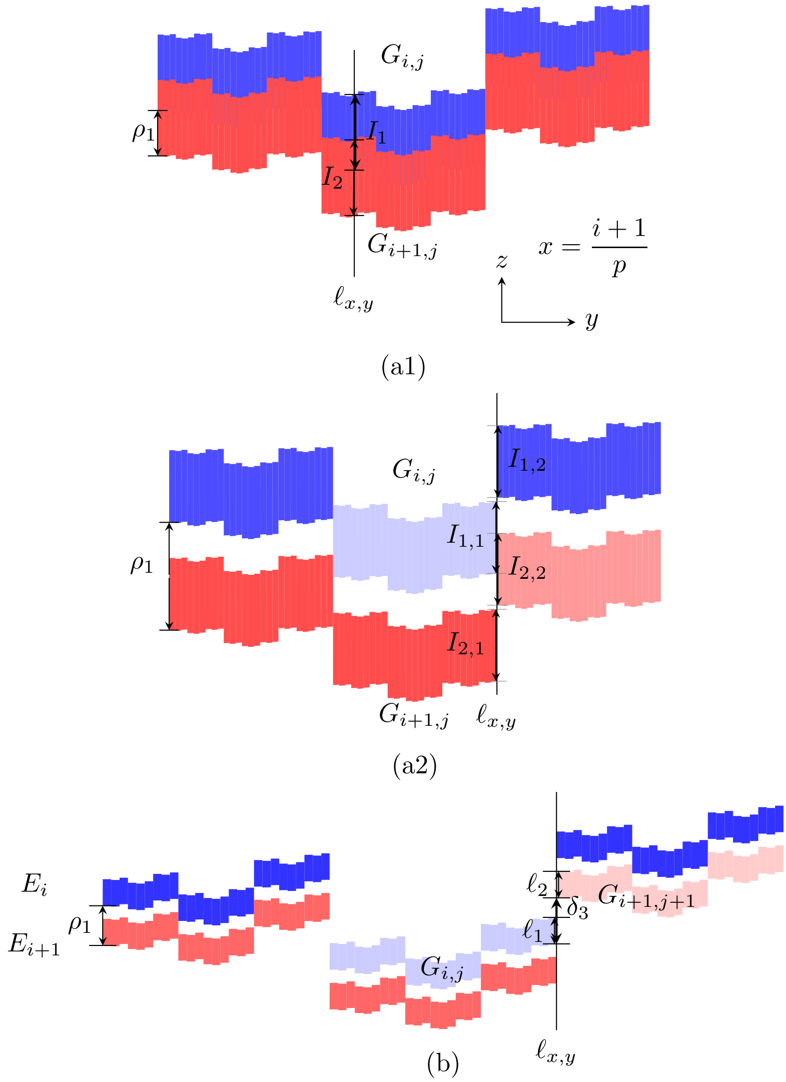


FIGURE 3. Figures for Lemma 3.1. This is on the plane $x = (i+1)/p$.

This implies, in view of (2.9) and the definition of $d(\mathbf{i},\mathbf{j})$,

$$\big|\rho_1(i)\big| \le 1 \qquad \text{or} \qquad \delta_1(i) \le \Big|\big|\rho_1(i)\big| \pm \frac{1}{r^n}\big|\rho_2(j')\big|\Big| \le 1.$$

Thus the necessity follows from the fact $\delta_1(i) \le |\rho_1(i)|$.

To show the sufficiency, we notice that $\rho_1(i) = 0$ implies $d(\mathbf{i}_1,\mathbf{j}) = d(\mathbf{i}_2,\mathbf{j})$ for all $\mathbf{j}$. That is, $G_{i,j} \cap G_{i+1,j} \supset \{(\varphi_p(\mathbf{i}_1), \varphi_q(\mathbf{j}))\} \times ([0,1] + d(\mathbf{i}_1,\mathbf{j}))$. So $\rho_1(i) = 0$ implies $G_{i,j} \cap G_{i+1,j} \neq \emptyset$. Now we suppose $\rho_1(i) \neq 0$. By (3.1), we let n, j' be chosen such that $\delta_1(i) = ||\rho_1(i)| - r^{-n}|\rho_2(j')||$. Set $\mathbf{j}_1 = j0^{n-1}j'\overline{(q-1)}$ and $\mathbf{j}_2 = j0^{n-1}(j'+1)\overline{0}$. Then either $|d(\mathbf{i}_1,\mathbf{j}_1) - d(\mathbf{i}_2,\mathbf{j}_2)|$ or $|d(\mathbf{i}_1,\mathbf{j}_2) - d(\mathbf{i}_2,\mathbf{j}_1)|$ is the minimum $\delta_1(i)$. It follows that $I_{1,1} \cup I_{1,2}$ intersects $I_{2,1} \cup I_{2,2}$, where $I_{l,m}$, $l, m = 1, 2$, are defined as above with respect to $\mathbf{i}_1, \mathbf{i}_2, \mathbf{j}_1, \mathbf{j}_2$. The argument above shows that $G_{i,j} \cap G_{i+1,j}$ is nonempty.

Similar reasoning yields (b). To prove (c), we only consider the set $G_{i,j} \cap G_{i+1,j+1}$ (see Figure 3(b)), the other case being similar. The real number $x = (i+1)/p$ has two p-adic expansions $\mathbf{i}_1 = i\overline{(p-1)}$, $\mathbf{i}_2 = (i+1)\overline{0}$, and the real number $y = (j+1)/q$ has two q-adic expansions $\mathbf{j}_1 = j\overline{(q-1)}$, $\mathbf{j}_2 = (j+1)\overline{0}$. The intersection $G_{i,j} \cap G_{i+1,j+1}$ is a subset of $T_{x,y}$ and the restrictions of $G_{i,j}$ and $G_{i+1,j+1}$ on $T_{x,y}$ are the line segments $\ell_1 = \{(x,y)\} \times ([0,1] + d(\mathbf{i}_1, \mathbf{j}_1))$ and $\ell_2 = \{(x,y)\} \times ([0,1] + d(\mathbf{i}_2, \mathbf{j}_2))$ respectively. Hence the intersection is not empty if and only if $\ell_1 \cap \ell_2 \neq \emptyset$ and the intersection contains exactly one point if and only if $\ell_1 \cap \ell_2$ contains exactly one point. Note that ℓ_1 and ℓ_2 are unit intervals. Hence $\ell_1 \cap \ell_2 \neq \emptyset$ if and only if

$$\big|d(\mathbf{i}_1, \mathbf{j}_1) - d(\mathbf{i}_2, \mathbf{j}_2)\big| \leq 1. \tag{3.3}$$

Moreover, (3.3) is an equality if and only if $\ell_1 \cap \ell_2$ contains a unique point, denoted by $P = (x, y, z)$. Because the last coordinate z does not satisfy the inequality in (2.15), we know $P \notin T^\circ$ by Lemma 2.5. Now, it follows from (2.9) and a direct calculation that the left side of (3.3) is equal to $|\rho_1(i) - \rho_2(j)|$, and the conclusion follows. $\square$

Notice that the expressions for $\delta_1(i)$ and $\delta_2(j)$ are independent of j and i respectively. Hence we obtain the following result from Lemma 3.1.

Corollary 3.2. *If $G_{i,j} \cap G_{i+1,j} \neq \emptyset$ holds for some j, then it holds for all j; if $G_{i,j} \cap G_{i,j+1} \neq \emptyset$ holds for some i, then it holds for all i.*

Lemma 3.3. *Let $n, m \geq 0$, $(\mathbf{i}, \mathbf{j}) \in \Sigma_{p,q}^n$, and $(\mathbf{i}', \mathbf{j}') \in \Sigma_{p,q}^{n+m+1}$. Then $G_{\mathbf{i}i(p-1)^m, \mathbf{j}'} \cap G_{\mathbf{i}(i+1)0^m, \mathbf{j}'}$ intersects T° if $|\rho_1(i)| < 1$ and $G_{\mathbf{i}', \mathbf{j}j(q-1)^m} \cap G_{\mathbf{i}', \mathbf{j}(j+1)0^m}$ intersects T° if $|\rho_2(j)| < 1$. When $n = m = 0$, the necessity is also sufficient; i.e., $(G_{i,j} \cap G_{i+1,j}) \cap T^\circ \neq \emptyset$ only if $|\rho_1(i)| < 1$ and $(G_{i,j} \cap G_{i,j+1}) \cap T^\circ \neq \emptyset$ only if $|\rho_2(j)| < 1$.*

Proof. Using the symmetry of i, j, we only check the value of $\rho_1(i)$. Recall that $J_{\mathbf{j}'} = \varphi_q(\mathbf{j}') + [0, q^{-n-m-1}]$. Take an irrational number $y_0 \in J_{\mathbf{j}'}$, which has a unique q-adic expansion (denoted by $\mathbf{j}''$), and let $\mathbf{i}_1 = \mathbf{i}i\overline{(p-1)}$ and $\mathbf{i}_2 = \mathbf{i}(i+1)\overline{0}$ be the two p-adic expansions of $x_0 = \varphi_p(\mathbf{i}(i+1))$. Suppose $|\rho_1(i)| < 1$. From the first equation in (2.9), we see that

$$\big|d(\mathbf{i}_1, \mathbf{j}'') - d(\mathbf{i}_2, \mathbf{j}'')\big| = \big|t(\mathbf{i}_1) + a(\mathbf{i}_1) - t(\mathbf{i}_2) - a(\mathbf{i}_2)\big| = r^{-n}\big|\rho_1(i)\big| < 1.$$

Then

$$d_{\min}(x_0, y_0) + 1 - d_{\max}(x_0, y_0) = 1 - r^{-n}|\rho_1(i)| > 0.$$

So, we can take $z_0 \in (d_{\max}(x_0, y_0), d_{\min}(x_0, y_0) + 1)$. Therefore, Lemma 2.5 shows that $P = (x_0, y_0, z_0) \in G_{\mathbf{i}i(p-1)^m, \mathbf{j}'} \cap G_{\mathbf{i}(i+1)0^m, \mathbf{j}'} \in T^\circ$.

Now suppose $n = m = 0$ and $|\rho_1(i)| \geq 1$. Let $S = \bigcup_{y \in [j/q, (j+1)/q] \cap \mathbb{Q}^c} T_{x_0, y}$ which is dense in $G_{i,j} \cap G_{i+1,j}$. Note that for any $P = (x_0, y_0, z_0) \in S$, the second coordinate has exactly one q-adic expansion, which implies $d_{\max}(x_0, y_0) - d_{\min}(x_0, y_0) = |\rho_1(i)| \geq 1$. So (2.15) cannot hold. This implies $S \cap T^\circ = \emptyset$ by Lemma 2.5. Therefore, the intersection $G_{i,j} \cap G_{i+1,j}$ contains no interior points of T. $\square$

Now we prove Theorem 1.1.

Proof of Theorem 1.1. (a) First, notice that $\delta_3(i,j) \leq 1$ implies either $G_{i,j} \cap G_{i+1,j+1} \neq \emptyset$ or $G_{i,j+1} \cap G_{i+1,j} \neq \emptyset$. That is, $F_j \cap F_{j+1} \neq \emptyset$. If $\delta_2(j) \leq 1$

holds for some j, then $F_j \cap F_{j+1} \neq \emptyset$. The assertion (a)(i) implies $F_j \cap F_{j+1} \neq \emptyset$ for $0 \leq j < q-1$; consequently, there exist $\{(i_m, m)\}_{m=0}^{q-2}$ and $\{(i'_m, m+1)\}_{m=0}^{q-2}$ such that $G_{i_m,j_m} \in F_m$, $G_{i'_m,j'_m} \in F_{m+1}$, and $G_{i_m,j_m} \cap G_{i'_m,j'_m} \neq \emptyset$. Notice that the assertion $\delta_1(i) \leq 1$ implies that any two sets $G_{i,j}$ and $G_{i+1,j}$ contained in F_j have a nonempty intersection. So the sequence of indices

$$
\begin{array}{cc}
\{(0,0),(1,0),\ldots,(p-2,0),(p-1,0),(p-2,0),\ldots,(i_0+1,0),(i_0,0), & F_0\\
(i'_0,1),(i'_0-1,1),\ldots,(0,1),(1,1),\ldots,(p-1,1),(p-2,1),\ldots,(i_1,1), & F_1\\
(i'_1,2),(i'_1-1,2),\ldots,(0,2),(1,2),\ldots,(p-1,2),(p-2,2),\ldots,(i_2,2), & F_2\\
\vdots & \vdots\\
(i'_{q-2},q-1),(i'_{q-2}-1,q-1),\ldots,(0,q-1),(1,q-1),\ldots,(p-1,q-1)\}, & F_{q-1}
\end{array}
$$

satisfies $G_{i,j} \cap G_{i',j'} \neq \emptyset$ if $(i,j),(i',j')$ are two consecutive terms of the sequence. Thus, the connectedness of T follows from Lemma 2.4. The proof for (a)(ii) is similar.

To show that the sufficient condition is also necessary when $a_i = b_j = 0$ for all i,j, we suppose, without loss of generality, that $|t| \geq |s|$. The conditions $\delta_1(i) \leq 1$, $\delta_2(j) \leq 1$, and $\delta_3(i,j) \leq 1$ are equivalent, respectively, to

$$|t| - |s/r| \leq r(r-1); \tag{3.4}$$

$$\inf_{n\geq 1} \left\{ \big| |s| - |tr^{-n}| \big| \right\} \leq r(r-1); \tag{3.5}$$

$$|t| - |s| \leq r(r-1). \tag{3.6}$$

We point out here that (a)(i) is equivalent to (3.4) and (3.6) and (a)(ii) is equivalent to (3.5) and (3.6).

We prove by contradiction. Assume (a)(i) and (a)(ii) fail. Then at least two of (3.4)–(3.6) fail. First we notice that (3.4) must fail, for otherwise (3.6) would hold because $|t| - |s| < |t| - |s/r|$, which implies that (a)(i) is satisfied. If (3.5) fails (in this case, $\delta_1, \delta_2 > 1$), Lemma 3.1 says that if $G_{i,j} \cap G_{i',j'} \neq \emptyset$, then $|i-i'| = |j-j'| = 1$. In other words, $G_{i,j} \cap G_{i',j'} \neq \emptyset$ implies that $i+j$ and $i'+j'$ have the same parity and thus $(\bigcup\{G_{i,j} : i+j \text{ is odd}\}) \cap (\bigcup\{G_{i,j} : i+j \text{ is even}\}) = \emptyset$. Therefore T is disconnected. If (3.6) fails, we can conclude (note that (3.4) fails) that $G_{0,j} \cap G_{1,j'} = \emptyset$ for any j,j'. This implies $E_0 \cap E_1 = \emptyset$. So T is also disconnected. The necessity follows.

(b) Suppose $|\rho_1(i)| \geq 1$ for some i. By Lemma 3.3, $G_{i,j} \cap G_{i+1,j}$ contains no interior points of T for all j. Hence all interior points in $E_i \cap E_{i+1}$ belong to

$$\Big(\bigcup_{j=0}^{q-2} (G_{i,j} \cap G_{i+1,j+1}) \Big) \bigcup \Big(\bigcup_{j=1}^{q-1} (G_{i,j} \cap G_{i+1,j-1}) \Big).$$

However, the set stated above, consisting of finitely many line segments and finitely many points, contains no interior points of T. This implies $E_i \cap E_{i+1} \cap T^\circ = \emptyset$. So, we divide T into two parts, namely, $T_1 = \bigcup_{l=0}^{i} E_l$ and $T_2 = \bigcup_{l=i+1}^{p-1} E_l$ so that $T_1 \cap T_2 = E_i \cap E_{i+1}$ contains no interior points of T. The disconnectedness of T° follows. Similarly, that $|\rho_2(j)| \geq 1$ for some j forces T° to be disconnected.

To show the sufficiency, we choose n large enough so that

$$p \sum_{k>n} |t_n| + q \sum_{k>n} |s_n| + c \sum_{k>n} r^{-k} < \frac{1}{4},$$

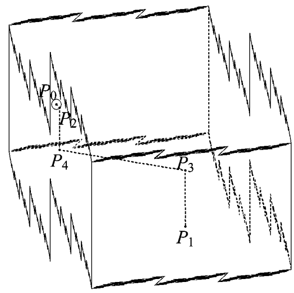


FIGURE 4. Figure for the proof of Theorem 1.1(b).

where $c = 2\sum_{i=0}^{p-1}|a_i| + 2\sum_{j=0}^{q-1}|b_j|$. So, for each $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$ and any two pairs $(\mathbf{i}_1,\mathbf{j}_1) = (i_{n+1}i_{n+2}\cdots, j_{n+1}j_{n+2}\cdots)$, $(\mathbf{i}_2,\mathbf{j}_2) = (i'_{n+1}i'_{n+2}\cdots, j'_{n+1}j'_{n+2}\cdots)$,

$$\begin{aligned}&\big|d(\mathbf{i}\mathbf{i}_1,\mathbf{j}\mathbf{j}_1) - d(\mathbf{i}\mathbf{i}_2,\mathbf{j}\mathbf{j}_2)\big|\\ &\quad\le \sum_{k>n}\left(\big|(i_k - i'_k)t_k\big| + \big|(j_k - j'_k)s_k\big| + \left|\frac{a_{i_k} - a_{i'_k}}{r^k}\right| + \left|\frac{b_{j_k} - b_{j'_k}}{r^k}\right|\right)\\ &\quad\le p\sum_{k>n}|t_k| + q\sum_{k>n}|s_k| + c\sum_{k>n}r^{-k} < \frac{1}{4}.\end{aligned}$$

Hence, for any two pairs $(x,y),(x',y') \in I_{\mathbf{i}}^\circ \times J_{\mathbf{j}}^\circ$,

$$\begin{aligned}&\max\big\{|d_{\min}(x,y) - d_{\min}(x',y')|, |d_{\max}(x,y) - d_{\max}(x',y')|\big\}\\ &\le\big|d_{\max}(x,y) - d_{\min}(x',y')\big| < \frac{1}{4}.\end{aligned} \tag{3.7}$$

Now we claim that if $P_0 \in T^\circ \cap \partial G_{\mathbf{i},\mathbf{j}}$ and $P_1 = (x_1,y_1,z_1) \in G_{\mathbf{i},\mathbf{j}}^\circ$, then there exists a piecewise linear curve $\ell \subset T^\circ$ connecting P_0 and P_1.

We show the claim by constructing such a curve ℓ (see Figure 4). Since $P_0 \in G_{\mathbf{i},\mathbf{j}}$ is an interior point of T, there exists $P_2 \in G_{\mathbf{i},\mathbf{j}}^\circ$ such that the line segment P_0P_2 is contained in T°. Set

$$P_{i+2} = (x_{i+2}, y_{i+2}, z_{i+2}) := \Big(x_i, y_i, 2^{-1}\big(d_{\min}(x_i,y_i) + d_{\max}(x_i,y_i) + 1\big)\Big), \quad i = 1,2,$$

and let $\ell = P_0P_2P_4P_3P_1$ be a piecewise linear curve. To show $\ell \subset T^\circ$, we need only check that the line segment P_3P_4 is contained in T°, since P_2P_4 and P_3P_1 are contained in T° by Lemma 2.5. Without loss of generality, suppose $z_4 \ge z_3$. For each $P = (x,y,z)$ on the line segment P_3P_4, by applying (3.7) twice, we see that

$$\begin{aligned} z - d_{\max}(x,y) &\ge z_3 - d_{\max}(x_3,y_3) - \big(d_{\max}(x,y) - d_{\max}(x_3,y_3)\big)\\ &\ge 2^{-1}\big(d_{\min}(x_3,y_3) + 1 - d_{\max}(x_3,y_3)\big) - 4^{-1}\\ &\ge 8^{-1} > 0.\end{aligned}$$

Similarly, $z - d_{\min}(x,y) - 1 < 0$. Lemma 2.5 says that $P \in T^\circ$. Hence, the claim follows.

Now we suppose $|\rho_1(i)| < 1$ and $|\rho_2(j)| < 1$ for all i,j. Let $P \in G_{0^n,0^n}^\circ$ be fixed. We will use the claim to show that for any $P' \in T^\circ$ there exists a piecewise linear curve $\ell \subset T^\circ$ connecting P and P'. This also shows the connectedness of T°. Suppose $P' \in G_{\mathbf{i},\mathbf{j}}$, where $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$. Recall that Σ_p^n and Σ_q^n are equipped with the lexicographic order. List the elements in $\{\mathbf{i}' \in \Sigma_p^n : \mathbf{i}' \le \mathbf{i}\}$ and $\{\mathbf{j}' \in \Sigma_q^n : \mathbf{j}' \le \mathbf{j}\}$ in the following order:

$$\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_{n_1}, \qquad \mathbf{j}_0, \mathbf{j}_1, \ldots, \mathbf{j}_{n_2}, \quad \text{with} \quad \mathbf{i}_0 = 0^n,\ \mathbf{i}_{n_1} = \mathbf{i},\ \mathbf{j}_0 = 0^n,\ \mathbf{j}_{n_2} = \mathbf{j}.$$

Now, from Lemma 3.3, the intersection of any two consecutive sets in

$$\{G_{\mathbf{i}_0,\mathbf{j}_0}, G_{\mathbf{i}_1,\mathbf{j}_0}, \ldots, G_{\mathbf{i}_{n_1},\mathbf{j}_0}, G_{\mathbf{i}_{n_1},\mathbf{j}_1}, \ldots, G_{\mathbf{j}_{n_1},\mathbf{j}_{n_2}}\} := \{G_{\mathbf{i}'_m,\mathbf{j}'_m}\}_{m=1}^{n_1+n_2+1}$$

contains a point $P_m \in T^\circ$. Let $P'_m \in G^\circ_{\mathbf{i}'_m,\mathbf{j}'_m}$ for $m = 1, \ldots, n_1 + n_2 + 1$, where $P'_1 = P$ and $P'_{n_1+n_2+1} = P'$ if $P' \in G^\circ_{\mathbf{i},\mathbf{j}}$. Denote $P_{n_1+n_2+2} = P'$ if $P' \in \partial G_{\mathbf{i},\mathbf{j}}$ and $P_{n_1+n_2+2} = P_{n_1+n_2+1}$ if $P' \notin \partial G_{\mathbf{i},\mathbf{j}}$. Let ℓ_m and ℓ'_m be two piecewise linear curves as in the claim above such that ℓ_m connects P'_m, P_m and ℓ'_m connects P_m, P'_{m+1}. Now the piecewise linear curves

$$\ell_1 \ell'_1 \ell_2 \ell'_2 \cdots \ell_{n_1+n_2+1} \ell'_{n_1+n_2+1} \subset T^\circ$$

connects P and P'. The connectedness of T° is proved.

We remark that the condition in Theorem 1.1(a) is sufficient but not necessary. We give a counterexample below.

Example 3.4. Let $t = s = 5$, $b_0 = 0,\ b_1 = -5,\ b_2 = -9,\ b_3 = -14,$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ -t & -s & 2 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \big\{(i, j, k + b_j) : 0 \le i, k \le 1, 0 \le j \le 3\big\}.$$

Then $T = T(A, \mathcal{D})$ is connected, but condition (a) in Theorem 1.1 fails.

Proof. We will use Lemma 3.1 to check whether or not $G_{i,j} \cap G_{i',j'}$ is empty for neighbors (i, j) and (i', j') by estimating $\delta_1(i)$ and $\delta_2(j)$. By assumption, we know that $p = r = 2$, $q = 4$, and $a_0 = a_1 = 0$. Thus $\rho_1(0) = 2^{-1}t > 0$. Note that $p - 1 = r - 1 = 1$. We have the following:

$$\begin{aligned} 2\big|\rho_1(0) - 2^{-n}|\rho_2(j)|\big| &= \Big|t - 2^{-n}\big|(b_3 - b_0 + s) + (b_j - b_{j+1})\big|\Big| \\ &= \Big|5 - 2^{-n}\big|9 - (b_j - b_{j+1})\big|\Big| \\ &\ge 5 - 2^{-1}5 > 2, \end{aligned}$$

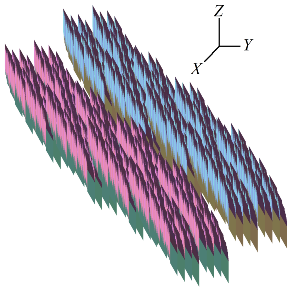
$$2\big||\rho_2(1)| - 2^{-n}\rho_1(0)\big| = \Big|\big|(b_3 - b_0 + s) + (b_1 - b_2)\big| - 2^{-n}t\Big| = 5 - 2^{-n}5 \ge 2^{-1}5 > 2.$$

We conclude that both $\delta_1(0)$ and $\delta_2(1)$ are larger than one. Lemma 3.1 and Corollary 3.2 yield $G_{0,j} \cap G_{1,j} = \emptyset$ for all j and $G_{i,1} \cap G_{i,2} = \emptyset$ for all i. Thus E_i and F_j are disconnected for all i, j. Hence condition (a) in Theorem 1.1 fails.

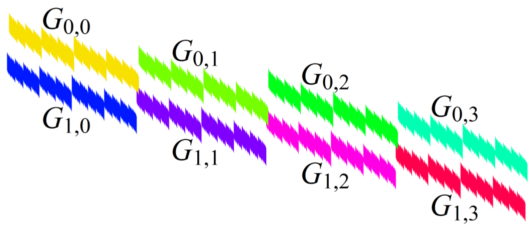
Since $\rho_2(0) = 2 \ne 0$,

$$2\delta_2(0) = \min_{n \ge 1}\Big\{\Big|\big|(b_3 - b_0 + s) + (b_0 - b_1)\big| - 2^{-n}t\Big|\Big\} = 4 - 2^{-1} \cdot 5 < 2.$$

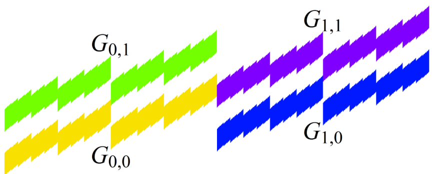
As $b_0 - b_1 = b_2 - b_3$, the above inequality shows that $\delta_2(2) = \delta_2(0) < 1$. Lemma 3.1 implies $G_{0,0} \cap G_{0,1} \ne \emptyset$ and $G_{0,2} \cap G_{0,3} \ne \emptyset$, while Corollary 3.2 yields $G_{1,0} \cap G_{1,1} \ne \emptyset$ and $G_{1,2} \cap G_{1,3} \ne \emptyset$. Notice that $2|\rho_1(0) + \rho_2(j)| = |t + (b_3 - b_0 + s + b_j - b_{j+1})|$ equals zero if $j = 1$ and equals 1 if $j = 0$ or 2. It follows from Lemma 3.1 that $G_{0,j+1} \cap G_{1,j} \ne \emptyset$ for $0 \le j \le 2$. The above nonempty intersections imply that any



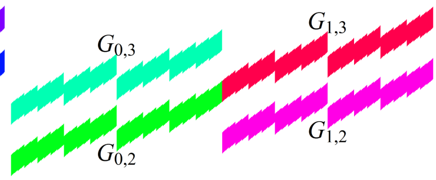
(a) The tile



(b) The cross section at $x = 2^{-1}$



(c) The cross section at $y = 4^{-1}$



(d) The cross section at $y = 2^{-1}$

FIGURE 5. The figure for Example 3.4.

two consecutive sets in the following finite sequence have nonempty intersection (see Figure 5):

$$G_{0,0},\ G_{0,1},\ G_{1,0},\ G_{1,1},\ G_{0,2},\ G_{0,3},\ G_{1,2},\ G_{1,3}.$$

It now follows from Lemma 2.4 that T is connected. □

4. A THEOREM ON HOMEOMORPHISM

This section is devoted to the proof of Theorem 1.2. The difficult part is to show the necessity, i.e., to find, under the given conditions, a homeomorphism between T and a 3-ball. The idea comes from the construction of a homeomorphism between an Alexander horned sphere and a 2-sphere [6, Chapter IV.3]: divide each of these spheres into infinitely many parts in such a way that two corresponding parts are homeomorphic, which leads to a homeomorphism between them. However, this result is not needed in our proofs; we only use a similar idea to construct a required homeomorphism.

In view of the complexity of the proof, we give a sketch here. First, we prove some elementary properties of T such as symmetry, which allows us to focus on the case where s, t are positive (Proposition 4.1). We also use planes π_n^c to divide T into infinitely many parts (Lemma 4.6), each of which is homeomorphic to a 3-ball (Lemma 4.3). We also prove some basic geometric properties of these parts (Lemma 4.4 and Proposition 4.5). Second, we introduce the definition of path separation and prove a property of such separation (Lemma 4.8). Third, we use a cut-and-paste technique to give another construction of T. We give the details of the construction of the desired homeomorphism when $s \le t < rs$ (Lemma 4.10) and sketch the proof for the case $t \ge rs$ (Lemma 4.11). Finally, we complete the proof of Theorem 1.2.

4.1. **Preparation.** We begin this subsection with some properties of T. We first point out that when a_i, b_j are all zero, the inequality $|\rho_1(i)+\rho_2(j)|<1$ can be rewritten as

$$|t+s|<r(r-1),$$

and $d(\mathbf{i},\mathbf{j})$ is equal to $t(\mathbf{i})+s(\mathbf{j})$.

Proposition 4.1. *Assume all of a_i, b_j are zero. Then the following assertions hold.*

(a) *T is symmetric with respect to the point $\mathbf{t}=(1/2)(1,1,1+(t+s)/(r-1))$, i.e.,*

$$T=2\mathbf{t}-T=\big\{2\mathbf{t}-(x,y,z):\,(x,y,z)\in T\big\}.$$

(b) *Replace t,s by $-s,-t$ in A and denote the resulting matrix by A'. Then the new tile $T'=T(A',D)$ is a reflection of T with respect to the plane $z=1/2$.*

Proof. (a) Note from (2.8) that

$$\frac{t+s}{r-1}=(p-1)\sum_{n\ge 1}t_n+(q-1)\sum_{n\ge 1}s_n.$$

As $1=\varphi_m(\overline{m-1})$, using the above equation, we get

$$\begin{aligned}2\mathbf{t}-T &= \Big\{\Big(1-\varphi_p(\mathbf{i}),1-\varphi_q(\mathbf{i}),1-\varphi_r(\mathbf{k})+\frac{t+s}{r-1}-t(\mathbf{i})-s(\mathbf{j})\Big):\\ &\qquad\qquad (\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^{\infty},\ \mathbf{k}\in\Sigma_r^{\infty}\Big\}\\ &= \{(\varphi_p(\mathbf{i}'),\varphi_q(\mathbf{i}'),\varphi_r(\mathbf{k}')+t(\mathbf{i}')+s(\mathbf{j}')):\,(\mathbf{i}',\mathbf{j}')\in\Sigma_{p,q}^{\infty},\ \mathbf{k}'\in\Sigma_r^{\infty}\}\\ &= T,\end{aligned}$$

where $\mathbf{i}'=(p-1-i_1)(p-1-i_2)\cdots$, $\mathbf{j}'=(q-1-i_1)(q-1-j_2)\cdots$, and $\mathbf{k}'=(r-1-k_1)(r-1-k_2)\cdots$.

(b) Define $d'(\mathbf{i},\mathbf{j})=\sum_{n\ge 1}i_n(-t_n)+\sum_{n\ge 1}j_n(-s_n)$. Then T' can be rewritten as

$$T'=\{(\varphi_p(\mathbf{i}),\varphi_q(\mathbf{j}),d'(\mathbf{i},\mathbf{j})+\varphi_r(\mathbf{k})):\,(\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^{\infty},\ \mathbf{k}\in\Sigma_r^{\infty}\}.$$

The reflection of T with respect to the plane $z=1/2$ has the form

$$\big\{(\varphi_p(\mathbf{i}),\varphi_q(\mathbf{j}),1-d(\mathbf{i},\mathbf{j})-\varphi_r(\mathbf{k})):\,(\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^{\infty},\ \mathbf{k}\in\Sigma_r^{\infty}\big\}.$$

These two sets are in fact the same, since $d'(\mathbf{i},\mathbf{j})=-d(\mathbf{i},\mathbf{j})$ and both $1-\varphi_r(\mathbf{k})$ and $\varphi_r(\mathbf{k})$ fill the unit interval when $\mathbf{k}$ runs over Σ_r^{∞}. $\square$

Remark 4.2. Using Proposition 4.1, we may assume the parameters r,s,t in (1.2) satisfy

$$s,t>0 \qquad\text{and}\qquad t+s<r(r-1). \tag{4.1}$$

In the following, we will give another construction of T which allows us to construct the desired homeomorphism. Let π_n^c be the plane

$$-p^{n+1}t_{n+1}x-q^{n+1}s_{n+1}y+z-1+c=0.$$

Let $z_n^c=z_n^c(x,y)=p^{n+1}t_{n+1}x+q^{n+1}s_{n+1}y+1-c$. Then $(x,y,z_n^c(x,y))$ lies in π_n^c. When $c=0$, we also denote π_n^c and z_n^c by π_n and z_n respectively. The next lemma shows some properties of π_n^c and the relationship between π_n^c and T. Recall that

$$T=\big\{\big(\varphi_p(\mathbf{i}),\varphi_q(\mathbf{j}),\varphi_r(\mathbf{k})+t(\mathbf{i})+s(\mathbf{j})\big):\,(i,j)\in\Sigma_{p,q}^{\infty},\ \mathbf{k}\in\Sigma_r^{\infty}\big\}.$$

Lemma 4.3. (a) *For each n, the plane π_{n+1}^c lies above the plane π_n^c in the first quadrant. Consequently, $\{z_n^c\}$ is increasing.*

(b) *Let $\bar{c} = 1/2 - (t+s)/(2r(r-1))$. Then the plane $\pi_0^{\bar{c}}$ divides T into two symmetric parts.*

(c) *For each pair $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^{n-1}$, all pq points $(\varphi_p(\mathbf{i}i), \varphi_q(\mathbf{j}j), 1 + t(\mathbf{i}i) + s(\mathbf{j}j))$, $(i,j) \in \Sigma_{p,q}^1$ lie in the plane $\pi_n^{c_n}$, where*

$$c_n = c_n(\mathbf{i},\mathbf{j}) := p^{n+1}t_{n+1}\varphi_p(\mathbf{i}) + q^{n+1}s_{n+1}\varphi_q(\mathbf{j}) - t(\mathbf{i}) - s(\mathbf{j}) + 1. \tag{4.2}$$

Proof. (a) Fix a point (x,y) in the first quadrant. Let (x,y,ξ_n), (x,y,ξ_{n+1}) be two points belonging to π_n^c and π_{n+1}^c, respectively. We draw the conclusion by showing that $\xi_{n+1} > \xi_n$. In fact

$$\begin{aligned} \xi_{n+1} - \xi_n &= p^{n+2}t_{n+2}x + q^{n+2}s_{n+2}y - p^{n+1}t_{n+1}x - q^{n+1}s_{n+1}y \\ &= p^{n+1}x(pt_{n+2} - t_{n+1}) + q^{n+1}y(qs_{n+2} - s_{n+1}) > 0. \end{aligned}$$

The last inequality follows from (2.7) and the fact that $x, y > 0$. Hence π_{n+1}^c lies above π_n^c in the first quadrant and thus $z_{n+1}^c > z_n^c$.

(b) We show the conclusion by checking that the plane $\pi_0^{\bar{c}}$ passes through the point of symmetry $\mathbf{t}$ of T. In fact,

$$-pt_1 \cdot \frac{1}{2} - qs_1 \cdot \frac{1}{2} + \Big(\frac{1}{2} + \frac{t+s}{2(r-1)}\Big) - 1 + \bar{c} = \frac{t+s}{2r(r-1)} - \frac{1}{2} + \frac{1}{2} - \frac{t+s}{2r(r-1)} = 0,$$

which implies that $\mathbf{t}$ lies on the plane $\pi_0^{\bar{c}}$.

(c) For each $0 \le i \le p-1$,

$$\big(-p^{n+1}t_{n+1}\varphi_p(\mathbf{i}i) + t(\mathbf{i}i)\big) + \big(p^{n+1}t_{n+1}\varphi_p(\mathbf{i}) - t(\mathbf{i})\big) = -it_{n+1} + it_{n+1} = 0.$$

Similarly, for each $0 \le j \le q-1$,

$$\big(-q^{n+1}s_{n+1}\varphi_q(\mathbf{j}j) + s(\mathbf{j}j)\big) + \big(q^{n+1}s_{n+1}\varphi_q(\mathbf{j}) - s(\mathbf{j})\big) = -js_{n+1} + js_{n+1} = 0.$$

So the point $(\varphi_p(\mathbf{i}i), \varphi_q(\mathbf{j}j), 1 + t(\mathbf{i}i) + s(\mathbf{j}j))$ lies in the plane $\pi_n^{c_n}$. $\square$

For convenience, we set $c_0 = c_0(\emptyset,\emptyset) = 0$. For $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$, define

$$I_{\mathbf{i}} := [0, p^{-n}] + \varphi_p(\mathbf{i}), \qquad J_{\mathbf{j}} := [0, q^{-n}] + \varphi_q(\mathbf{j}).$$

Lemma 4.4. *Suppose $(\mathbf{i},\mathbf{j}) = (i_1 \cdots i_n, j_1 \cdots j_n) \in \Sigma_{p,q}^n$ and $(x,y) \in (I_{\mathbf{i}} \times J_{\mathbf{j}})^\circ$. Let $(\mathbf{i}',\mathbf{j}') = (i_{n+1}\cdots, j_{n+1}\cdots) \in \Sigma_{p,q}^\infty$ such that $(\mathbf{i}\mathbf{i}',\mathbf{j}\mathbf{j}')$ is an expansion of (x,y).*

(a) *$z_n^{c_n}$ can be expressed as $z_n^{c_n}(x,y) = 1 + d(\mathbf{i}\mathbf{i}',\mathbf{j}\mathbf{j}') - \varepsilon_{n+1}$, where*

$$\varepsilon_{n+1} := \sum_{k\ge1}\Big(i_{n+k}\big(t_{n+k} - \frac{t_{n+1}}{p^{k-1}}\big) + j_{n+k}\big(s_{n+k} - \frac{s_{n+1}}{q^{k-1}}\big)\Big) \tag{4.3}$$

decreases (not necessarily strictly) to zero as $n \to \infty$. Consequently, $z_n^{c_n} \le 1 + d(\mathbf{i}\mathbf{i}',\mathbf{j}\mathbf{j}')$.

(b) *Let $\bar{c}$ be given as in Lemma 4.3. Then $d(\mathbf{i}\mathbf{i}',\mathbf{j}\mathbf{j}') < z_0^{\bar{c}}(x,y)$.*

Proof. (a) Combining (2.3) (formulas for t_n, s_n) and (4.2) (expression for c_n) and using the definition of $z_n^{c_n}$, we obtain

$$\begin{aligned} z_n^{c_n}(x,y) &= 1+p^{n+1}t_{n+1}x+q^{n+1}s_{n+1}y-c_n \\ &= 1+p^{n+1}t_{n+1}\big(\varphi_p(\mathbf{i}\mathbf{i}')-\varphi_p(\mathbf{i})\big) \\ &\quad +q^{n+1}s_{n+1}\big(\varphi_q(\mathbf{j}\mathbf{j}')-\varphi_q(\mathbf{j})\big)+t(\mathbf{i})+s(\mathbf{j}) \\ &= 1+pt_{n+1}\varphi_p(\mathbf{i}')+qs_{n+1}\varphi_q(\mathbf{j}')+t(\mathbf{i})+s(\mathbf{j}) \\ &= 1+t(\mathbf{i}\mathbf{i}')+s(\mathbf{j}\mathbf{j}')-\big(t(0^n\mathbf{i}')-pt_{n+1}\varphi_p(\mathbf{i}')+s(0^n\mathbf{j}')-qs_{n+1}\varphi_q(\mathbf{j}')\big). \end{aligned}$$

Hence $\varepsilon_{n+1}=(t(0^n\mathbf{i}')-pt_{n+1}\varphi_p(\mathbf{i}'))+(s(0^n\mathbf{j}')-qs_{n+1}\varphi_q(\mathbf{j}')) =: B_1+B_2$. The definitions of $t(\cdot)$, $s(\cdot)$, $\varphi_p(\cdot)$, and $\varphi_q(\cdot)$ yield

$$B_1=\sum_{k\ge1} i_{n+k}\Big(t_{n+k}-\frac{t_{n+1}}{p^{k-1}}\Big), \qquad B_2=\sum_{k\ge1} j_{n+k}\Big(s_{n+k}-\frac{s_{n+1}}{q^{k-1}}\Big). \tag{4.4}$$

Now (4.3) is a direct consequence of (4.4).

Notice that $i_{n+k}<p$ for each $k\ge1$. By the first equality in (4.4) and the first equality in (2.8),

$$0\le B_1\le\sum_{k\ge1}(p-1)t_{n+k}=\begin{cases}\dfrac{p^{-n}(r-1)-r^{-n}(p-1)}{(r-p)(r-1)}<p^{-n}r+r^{-n}p, & p\ne r,\\[2ex] \dfrac{(1+(n+1)(r-1))r^{-n-1}}{r-1}<2nr^{-n}, & p=r.\end{cases}$$

In both cases, we get $\lim_{n\to\infty}B_1=0$. Similarly, $\lim_{n\to\infty}B_2=0$. Hence $\lim_{n\to\infty}\varepsilon_{n+1}=0$. To show the remaining part of (a), we notice that $pt_{n+1}-t_n=t\cdot r^{-n-1}>0$ from (2.7) and hence $t_{n+k}-p^{1-k}t_{n+1}>0$ for all $k\ge2$. Similarly, $s_{n+k}-q^{1-k}s_n>0$ for all $k\ge2$. Now applying (4.3) yields

$$\begin{aligned}\varepsilon_n-\varepsilon_{n+1} &= \sum_{k\ge1}\Big(i_{n+k}\big(\frac{t_{n+1}}{p^{k-1}}-\frac{t_n}{p^k}\big)+j_{n+k}\big(\frac{s_{n+1}}{p^{k-1}}-\frac{s_n}{q^k}\big)\Big)\\ &= (pt_{n+1}-t_n)\sum_{k\ge1}\frac{i_{n+k}}{p^{n+k}}+(qs_{n+1}-s_n)\sum_{k\ge1}\frac{j_{n+k}}{q^{n+k}}\ge0.\end{aligned}$$

This implies $\{\varepsilon_{n+1}\}$ is a nonincreasing sequence.

(b) Notice that part (a) also holds when $n=0$ since $t(\emptyset)=s(\emptyset)=0$. Using the expressions for $\sum t_n$ and $\sum s_n$ in (2.8), we know

$$\varepsilon_1\le\sum_{k\ge1}\Big((p-1)\big(t_{k+1}-\frac{t_1}{p^k}\big)+(q-1)\big(s_{k+1}-\frac{s_1}{q^k}\big)\Big)=\frac{t+s}{r(r-1)}. \tag{4.5}$$

Hence the second assumption in (4.1) implies that $\varepsilon_1+\bar c\le 1/2+(t+s)/(2r(r-1))<1$. Now it follows from (a) that $z_0^{\bar c}=z_0-\bar c=1+t(\mathbf{i}')+s(\mathbf{j}')-\varepsilon_1-\bar c>t(\mathbf{i}')+s(\mathbf{j}')$, proving (b). $\square$

We use $T_n(\mathbf{i},\mathbf{j})$ to denote the part of $I_\mathbf{i}\times I_\mathbf{j}\times\mathbb{R}$ lying between the planes $\pi_{n-1}^{c_{n-1}}$ and $\pi_n^{c_n}$, i.e.,

$$T_n(\mathbf{i},\mathbf{j}):=\big\{(x,y,z):\, x\in I_\mathbf{i}, y\in J_\mathbf{j}, z_{n-1}^{c_{n-1}}\le z\le z_n^{c_n}\big\}. \tag{4.6}$$

We call $T_n(\mathbf{i},\mathbf{j})$ *an nth order basic block.* Clearly, each basic block is homeomorphic to a 3-ball. We remark that each $T_n(\mathbf{i},\mathbf{j})$ is a subset of T, which will be shown later. The following proposition follows from the definition of basic block in (4.6) and the formulas for t_n, s_n in (2.3); we omit the proof.

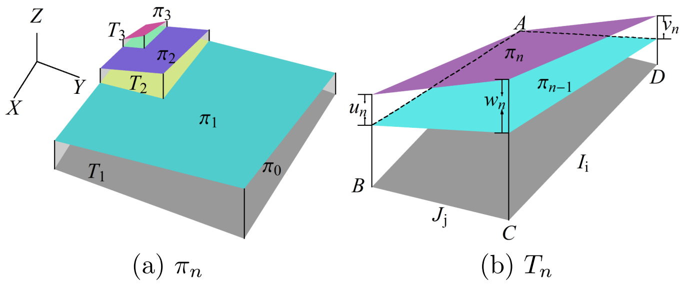


FIGURE 6. Figure for π_n and T_n. The coordinates of A, B, C, D are $(0,0,1)$, $(p^{-n},0,1)$, $(p^{-n},q^{-n},1)$, and $(0,q^{-n},1)$, respectively. The figure is drawn with $p=2, q=3, r=2$, $t=8/15$, $s=2/5$.

Proposition 4.5. *Suppose T_n is defined as in (4.6). Then:*

(a) *Each $T_n(\mathbf{i},\mathbf{j})$ is a translate of the set $T_n(0^n,0^n)$, and for each pair $(i,j) \in \Sigma^1_{p,q}$, the bottom of $T_{n+1}(\mathbf{i}i,\mathbf{j}j)$ is on the top of $T_n(\mathbf{i},\mathbf{j})$.*

(b) *Let u_n, v_n, w_n be the lengths of the three vertical sides of an nth order basic block, as shown in Figure 6(b). Then*

$$u_n = tr^{-n-1}, \qquad v_n = sr^{-n-1}, \qquad w_n = u_n + v_n = (t+s)r^{-n-1}. \tag{4.7}$$

(c) *Suppose $0 < r^{N_1}s < t < r^{N_1+1}s$ for some integer $N_1 \geq 0$. Then there exists $N > 0$ such that*

$$\sum_{k=N_1+n}^{N_1+N+n} u_k - \sum_{k\geq n} v_k > 0, \qquad \sum_{k=n}^{N+n} v_k - \sum_{k>N_1+n} u_k > 0. \tag{4.8}$$

Moreover, if $t = r^{N_1}s > 0$, the second inequality in (4.8) still holds when we set $N=1$.

Now we divide T as follows. Let $\pi_0^{\bar{c}}$ be the plane passing through the point of symmetry $\mathbf{t} = 1/2(1,1,1+(t+s)/(r-1))$. Define a sequence of sets $\{X_n\}_{n\geq 0}$ as follows:

$$\begin{aligned} X_0 &:= \big\{(x,y,z) : 0 \leq x, y \leq 1, z_0^{\bar{c}}(x,y) \leq z \leq z_0(x,y)\big\}, \\ X_n &:= X_{n-1} \cup \Big(\bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma^n_{p,q}} T_n(\mathbf{i},\mathbf{j})\Big), \qquad n \geq 1, \end{aligned} \tag{4.9}$$

and let

$$X := \overline{\bigcup_{n\geq 0} X_n}.$$

Roughly speaking, the set X_n is obtained from X_{n-1} by stacking $(pq)^n$ small basic blocks $T_n(\mathbf{i},\mathbf{j})$, $(\mathbf{i},\mathbf{j}) \in \Sigma^n_{p,q}$, onto the top of X_{n-1} (see Figure 7). The following lemma describes the relation between X and T.

Lemma 4.6. *Let $\bar{c}, \pi_0^{\bar{c}}$, X and X_n be defined as above. Then $T = X \cup (2\mathbf{t} - X)$. Furthermore, $X \cap (2\mathbf{t} - X)$ is a parallelogram lying on the plane $\pi_0^{\bar{c}}$.*

Proof. The fact that T is the closure of its interior (see, e.g., [16]) will be used twice in the proof, i.e.,

$$\overline{T^\circ} = T. \tag{4.10}$$

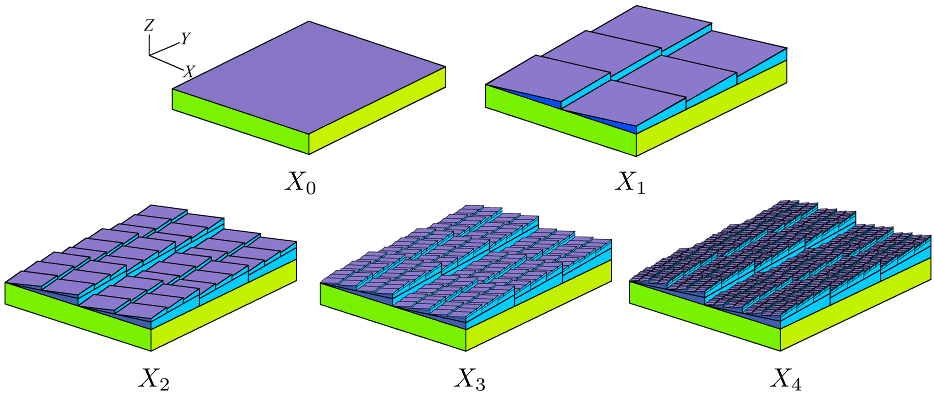


FIGURE 7. The first five steps of the construction of X_n. This is drawn with the same parameters as Figure 6, but with a different viewpoint.

Let $Y = X \cup (2\mathbf{t} - X)$. We prove the conclusion by showing that $Y \subset T$ and $T \subset Y$. Let B denote the part of T lying between the planes π_0 and $\pi_0^{\bar{c}}$, i.e.,

$$B = \{(x, y, z) \in T : z_0^{\bar{c}}(x, y) \le z \le z_0(x, y),\ 0 \le x, y \le 1\}.$$

Since $\bar{c} > 0$, we see that B is not empty.

We claim that $B = X_0$. First, the definition of X_0 yields $B \subset X_0$. On the other hand, for any $P = (x, y, z) \in X_0^\circ$, using Lemma 4.4(a,b) (with $n = 0$) we have

$$d(\mathbf{i}, \mathbf{j}) < z_0^{\bar{c}}(x, y) \le z \le z_0 \le d(\mathbf{i}, \mathbf{j}) + 1,$$

where $(\mathbf{i}, \mathbf{j})$ is an expansion of (x, y). Hence there exists $\mathbf{k} \in \Sigma_r^\infty$ such that $z = d(\mathbf{i}, \mathbf{j}) + \varphi_r(\mathbf{k})$. This implies $P \in T$. So $X_0 \subset T$ by (4.10). We get $B = X_0$ as claimed.

For any $n \ge 1$, any pair $(\mathbf{i}, \mathbf{j}) \in \Sigma_{p,q}^n$, and any $P = (x, y, z) \in (T_n(\mathbf{i}, \mathbf{j}))^\circ$, Lemma 4.4(a) yields

$$1 + d(\mathbf{i}\mathbf{i}', \mathbf{j}\mathbf{j}') - \varepsilon_n = z_{n-1}^{c_{n-1}}(x, y) < z < z_n^{c_n} = 1 + d(\mathbf{i}\mathbf{i}', \mathbf{j}\mathbf{j}') - \varepsilon_{n+1},$$

where $(\mathbf{i}\mathbf{i}', \mathbf{j}\mathbf{j}')$ is any expansion of (x, y). Since $\varepsilon_n \le \varepsilon_1 < 1/2$, there exists $\mathbf{k} \in \Sigma_r^\infty$ such that $z = d(\mathbf{i}\mathbf{i}', \mathbf{j}\mathbf{j}') + \varphi_r(\mathbf{k}) \in T$. So, we conclude that $X \subset T$. Notice that $T = 2\mathbf{t} - T$ by Proposition 4.1. We see that $X \cup (\mathbf{t} - X) \subset T \cup (\mathbf{t} - T) = T$. This yields $Y \subset T$.

Now we show $T \subset Y$. By the symmetry of T, we only show that the part of T lying above $\pi_0^{\bar{c}}$ is a subset of X. Using the claim $B = X_0$ above, we only need to check that all points in T lying above π_0 belong to X. Then, by (4.10) again, the conclusion follows if we can prove that each interior point of T lying above π_0 is in X.

Assume $P = (x, y, z) \in T^\circ$ lies above π_0. Then $z > z_0(x, y) = 1 + d(\mathbf{i}, \mathbf{j}) - \varepsilon_1$, where $(\mathbf{i}, \mathbf{j}) \in \Sigma_{p,q}^\infty$ is any expansion of (x, y). On the other hand, Lemma 2.5 yields $z < d_{\min}(x, y) + 1$. So $z < d(\mathbf{i}, \mathbf{j}) + 1$. Notice that $\{\varepsilon_n\}$ decreases to zero (Lemma 4.4(a)). There is $n > 0$ such that $1 + d(\mathbf{i}, \mathbf{j}) - \varepsilon_n \le z < 1 + d(\mathbf{i}, \mathbf{j}) - \varepsilon_{n+1}$, namely,

$$z_{n-1}^{c_{n-1}}(x, y) \le z < z_n^{c_n}(x, y).$$

Therefore $P \in T_n(\mathbf{i}|n, \mathbf{j}|n)$, proving that $T \subset Y$.

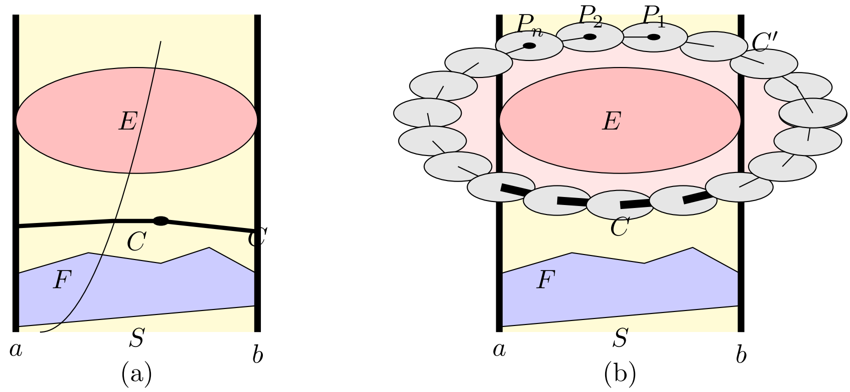


FIGURE 8. (a) Two bridges E, F and their path separation C. (b) Figure for the proof of Lemma 4.8. The biggest disk is $[E]_\delta$, whose boundary is C'. E and F are path separated by C, the widened curve.

The last part of the assertion is a direct consequence of Proposition 4.1(a). $\square$

By induction, we see that each X_n is homeomorphic to the 3-ball. We will show that X is homeomorphic to a 3-ball. To this end, we introduce the definition of path separation as well as some notation.

For two sets $E, F \subseteq \mathbb{R}^d$, denote their *distance* by

$$d(E,F) := \inf\big\{\|e-f\| : e \in E, f \in F\big\},$$

where $\|\cdot\|$ is the usual Euclidian norm in $\mathbb{R}^d$. For convenience, define $d(e,F) := d(\{e\},F)$. The closed *δ-neighborhood* of a set F is defined by $[F]_\delta := \{e \in \mathbb{R}^d : d(e,F) \le \delta\}$. Let $B^\circ(x,r)$ and $B(x,r)$ be, respectively, the open and closed balls in $\mathbb{R}^d$ with radius r and center x.

Definition 4.7. For $a < b$, let $S = [a,b] \times \mathbb{R}$ be an infinite band. A connected subset of S is called a *bridge* (of S) if it has nonempty intersections with both lines $x = a$ and $x = b$. Let $E, F \subset S$ be two bridges. We say a curve $C \subset S$ with $\min\{d(E,C), d(F,C)\} > 0$ *path separates* E and F if any curve in S intersecting both E and F must also intersect C (see Figure 8(a)).

Lemma 4.8. *Let E, F be two compact bridges of $S = [a,b] \times \mathbb{R}$ with E lying above F. If $d(E,F) > 0$, then for any $\varepsilon > 0$, there exists a piecewise linear curve $C \subset S$, with $d(E,C) < \varepsilon$, that path separates E and F.*

Proof. Take $\delta = 4^{-1}\min\{d(E,F), \varepsilon\}$. Consider $[E]_\delta$, the closed δ-neighborhood of E. Define the *exterior boundary* of $[E]_\delta$, denoted by $\tilde{\partial}[E]_\delta$, to be the boundary of the unbounded component of $\mathbb{R}^2 \setminus [E]_\delta$. Note that $\tilde{\partial}[E]_\delta$ is compact and is not a subset of S. Let $\{B^\circ(P,\delta) : P \in \tilde{\partial}[E]_\delta\}$ be an open cover of $\tilde{\partial}[E]_\delta$ by δ-balls. Then there exists a finite subcover which can be rearranged as $\{B^\circ(P_k,\delta)\}_{k=1}^n$ such that $d(P_k, P_{k+1}) < \delta$ for $1 \le k \le n$ and $P_{n+1} = P_1$. The closed piecewise linear curve connecting the centers of the balls, $C' = P_1P_2\cdots P_nP_1$, satisfies $0 < d(E,C') < \varepsilon$ and $d(F,C') > 0$. Note that $C' \cap S$ consists of finitely many piecewise linear curves. The conclusion holds by letting C be the bottom piecewise linear curve. $\square$

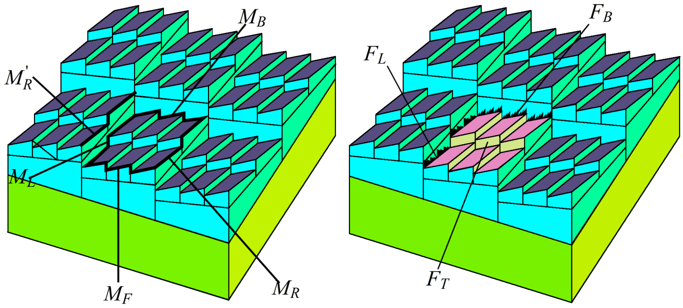


FIGURE 9. Boundaries and surfaces. The left one shows the four boundaries M_L, M_R, M_B, M_F of $F_T(\mathbf{i},\mathbf{j})$ (the top surface of $U_n(\mathbf{i},\mathbf{j})$) and the right boundary $M_R' = M_R(\mathbf{i},\mathbf{j}^-)$ of $F_T(\mathbf{i},\mathbf{j}^-)$ (the top surface of $U_n(\mathbf{i},\mathbf{j}^-)$). The right one shows the three surfaces of $R_n(\mathbf{i},\mathbf{j})$ and $U_n(\mathbf{i},\mathbf{j})$. This figure is drawn with the same parameters as those for Figure 2 (hence $N = 1$).

4.2. **Construction of homeomoprhism.** In this subsection, we will use Lemma 4.8 to give another partition of T, which allows us to define a homeomorphism between T and a 3-ball in a natural way. Let $\mathbf{i}^{\pm}$, $\mathbf{j}^{\pm}$ be defined as in Section 2. For $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$, divide the closure of $\bigcup_{k\geq 1}\bigcup_{(\mathbf{i}',\mathbf{j}')\in\Sigma_{p,q}^k} T_{n+k}(\mathbf{ii}',\mathbf{jj}')$ into two parts. One is $U_n^1(\mathbf{i},\mathbf{j})$, which contains all basic blocks with order no more than $N+n$; the other is $R_n(\mathbf{i},\mathbf{j})$, the remaining portion. More precisely, we define

$$L_{n+k}(\mathbf{i},\mathbf{j}) := \bigcup_{(\mathbf{i}',\mathbf{j}')\in\Sigma_{p,q}^k} T_{n+k}(\mathbf{ii}',\mathbf{jj}'), \tag{4.11}$$

$$U_n^1(\mathbf{i},\mathbf{j}) := \bigcup_{k=1}^{N} L_{n+k}(\mathbf{i},\mathbf{j}), \qquad R_n(\mathbf{i},\mathbf{j}) := \overline{\bigcup_{k>N} L_{n+k}(\mathbf{i},\mathbf{j})}. \tag{4.12}$$

In particular, $L_{N+n+1}(\mathbf{i},\mathbf{j})$ is the union of all basic blocks lying at the bottom of $R_n(\mathbf{i},\mathbf{j})$ with order $N+n+1$. From the equations (4.11) and (4.12), we see that $R_n(\mathbf{i},\mathbf{j})$ is on the top of $U_n(\mathbf{i},\mathbf{j})$ and

$$X = X_n \cup \Big(\bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^n} \big(R_n(\mathbf{i},\mathbf{j}) \cup U_n^1(\mathbf{i},\mathbf{j})\big)\Big) \quad \text{for all } n \geq 1. \tag{4.13}$$

Keeping this in mind, one can better understand our setting in the rest of this section.

The top exterior surface of $U_n^1(\mathbf{i},\mathbf{j})$, including the vertical polygons inside the interior of $I_{\mathbf{i}} \times J_{\mathbf{j}} \times \mathbb{R}$, is denoted by $F_T(\mathbf{i},\mathbf{j})$. We denote the left, right, back, and front exterior boundaries of $F_T(\mathbf{i},\mathbf{j})$ by $M_L(\mathbf{i},\mathbf{j})$, $M_R(\mathbf{i},\mathbf{j})$, $M_B(\mathbf{i},\mathbf{j})$ and $M_F(\mathbf{i},\mathbf{j})$ respectively.

Let $F_L(\mathbf{i},\mathbf{j})$ be the union of $M_L(\mathbf{i},\mathbf{j})$ and the left exterior vertical surface of $R_n(\mathbf{i},\mathbf{j})$. The sets $F_R(\mathbf{i},\mathbf{j})$, $F_B(\mathbf{i},\mathbf{j})$ and $F_F(\mathbf{i},\mathbf{j})$ are defined similarly. For an illustration of these sets, we refer the reader to Figure 9. We point out that all the surfaces and curves $F_T, F_L, F_R, F_B, F_F, M_L, M_R, M_B, M_F$ depend on U_n and R_n — or, more precisely, on the number N.

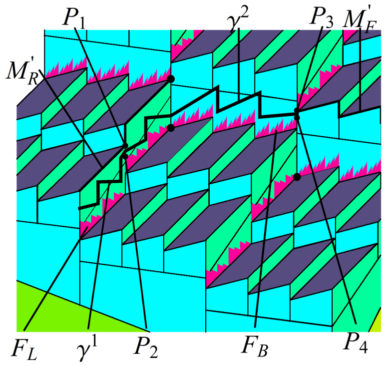


FIGURE 10. Figure for Lemma 4.9. Here P_0 is the junction of $M_L(\mathbf{i},\mathbf{j})$ and $M_B(\mathbf{i},\mathbf{j})$, $M_R' = M_R(\mathbf{i},\mathbf{j}^-)$ and $M_F' = M_F(\mathbf{i}^-,\mathbf{j}^+)$. This figure is drawn using the same parameters as those for Figure 2 and $(\mathbf{i},\mathbf{j}) = (1,2)$.

Clearly, the construction stated above yields:

(1) $R_n(\mathbf{i},\mathbf{j})$ is a translate of $R_n(0^n,0^n)$;
(2) $R_m(\mathbf{i}',\mathbf{j}') \subset R_n(\mathbf{i},\mathbf{j})$ if $(\mathbf{i},\mathbf{j})$ is a prefix of $(\mathbf{i}',\mathbf{j}')$;
(3) $F_L(\mathbf{i},\mathbf{j})$ and $F_R(\mathbf{i},\mathbf{j}^-)$ lie in the same vertical plane, as do $F_B(\mathbf{i},\mathbf{j})$ and $F_F(\mathbf{i}^-,\mathbf{j})$ (see Figure 10). This fact is useful in proving the following separation lemma.

Recall that $\ell_{x,y} := \{(x,y)\} \times \mathbb{R}$.

Lemma 4.9. *Assume that $s \le t < rs$. Let $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$ with $\mathbf{i} = i_1 \cdots i_n$ and $\mathbf{j} = j_1 \cdots j_n$, and let N be determined by Proposition 4.5 for $N_1 = 0$. The following statements hold.*

(a) *If $j_n > 0$, then there exists a piecewise linear curve $\gamma^1 = \gamma^1(\mathbf{i},\mathbf{j}) \subset I_{\mathbf{i}} \times \{\varphi_j(\mathbf{j})\} \times \mathbb{R}$ path separating $F_R(\mathbf{i},\mathbf{j}^-)$ and $F_L(\mathbf{i},\mathbf{j})$.*
(b) *If $i_n > 0$, then there exists a piecewise linear curve $\gamma^2 = \gamma^2(\mathbf{i},\mathbf{j}) \subset \{\varphi_i(\mathbf{i})\} \times J_{\mathbf{j}} \times \mathbb{R}$ path separating $F_F(\mathbf{i}^-,\mathbf{j})$ and $F_B(\mathbf{i},\mathbf{j})$.*
(c) *We can require that the piecewise linear curves in* (a) *and* (b) *satisfy:*
   (i) *if $i_n j_n > 0$, then $\gamma^1(\mathbf{i},\mathbf{j}) \cup \gamma^2(\mathbf{i},\mathbf{j})$ is connected;*
   (ii) *if $s = t$, then $\gamma^2(\mathbf{i},\mathbf{j}) \cup \gamma^1(\mathbf{i}^-,\mathbf{j}^+)$ is connected;*
   (iii) *if $s \ne t$, then the right end-point of $\gamma^2(\mathbf{i},\mathbf{j})$ lies below $F_F(\mathbf{i}^-,\mathbf{j}^+)$ for $j_n < q-1$.*

Proof. We apply Lemma 4.8. Denote $y_0 = \varphi_q(\mathbf{j})$. Since each complex $T_n(\mathbf{i},\mathbf{j})$ is a translate of $T_n(0^n,0^n)$, and the bottom of $T_n(0^n,0^n)$ is a parallelogram, $M_L(\mathbf{i},\mathbf{j})$ is a piecewise linear curve which is a vertical translate of $M_R(\mathbf{i},\mathbf{j}^-)$ (see Figure 10). In fact,

$$M_R(\mathbf{i},\mathbf{j}^-) = M_L(\mathbf{i},\mathbf{j}) + \Big(0, 0, \sum_{k=n}^{N+n} v_k\Big).$$

So there exist i, $0 \le i < p$, and two points $P_1 \in M_R(\mathbf{i},\mathbf{j}^-), P_2 \in F_L(\mathbf{i},\mathbf{j})$, both lying in $\ell_{\varphi_p(\mathbf{i}i),y_0}$, such that the distance between $M_R(\mathbf{i},\mathbf{j}^-)$ and $F_L(\mathbf{i},\mathbf{j})$ is equal to

$Cd(P_1, P_2)$, where C is a positive constant depending only on $M_L(\mathbf{i}, \mathbf{j})$. Since (see Figure 10)

$$d(P_1, P_2) = \sum_{k=n}^{N+n} v_k - \sum_{k>n} u_k > 0, \tag{4.14}$$

$d(F_R(\mathbf{i}, \mathbf{j}^-), F_L(\mathbf{i}, \mathbf{j})) > 0$. Notice that $F_R(\mathbf{i}, \mathbf{j}^-)$ and $F_L(\mathbf{i}, \mathbf{j})$ are connected. This, together with (4.14), implies the existence of γ^1 by Lemma 4.8. This proves part (a).

A similar discussion yields (b). As for (c), we see that

(1) $i_n j_n > 0$ implies that $M_L(\mathbf{i}, \mathbf{j})$ and $M_B(\mathbf{i}, \mathbf{j})$ are joined at the left end-point of $M_B(\mathbf{i}, \mathbf{j})$ (independent of the relation of s, t);
(2) $i_n(q - 1 - j_n) > 0$ implies that $M_B(\mathbf{i}, \mathbf{j})$ and $M_F(\mathbf{i}^-, \mathbf{j}^+)$ are joined at the right end-point of $M_B(\mathbf{i}, \mathbf{j})$ when $s = t$, as $u_n = v_n$ in this case.

Using these facts, together with the arbitrariness of ε as given in Lemma 4.8, we can change the end-points of those piecewise linear curves, with the required separation properties, such that (c)(i) and (c)(ii) hold. Now we show (c)(iii). Suppose $j_n < q - 1$ and $s < t$. Let $\ell = \ell_{\varphi_p(\mathbf{i}), \varphi_q(\mathbf{j}^+)}$. Denote the unique point in $M_F(\mathbf{i}^-, \mathbf{j}^+) \cap \ell$ by P_3 and the top point in $F_B(\mathbf{i}, \mathbf{j}) \cap \ell$ by P_4. The difference of the last coordinates of P_3, P_4 is equal to

$$\sum_{k=n}^{N+n} u_k - \sum_{k \geq n} v_k,$$

which is positive by Proposition 4.5. This implies that P_4 lies below P_3 and the conclusion holds when we set $\varepsilon < 2^{-1} d(P_3, P_4)$ in Lemma 4.8. □

Lemma 4.10. *If $s \leq t < rs$, then X is ball-like.*

Proof. In the following proof, we first present another construction of X by a cut-and-paste technique that reveals a similarity between the structure of X and that of a horned ball. The new construction establishes a homeomorphism between X and the 3-ball $B(0, 1)$. We divide the proof into three steps.

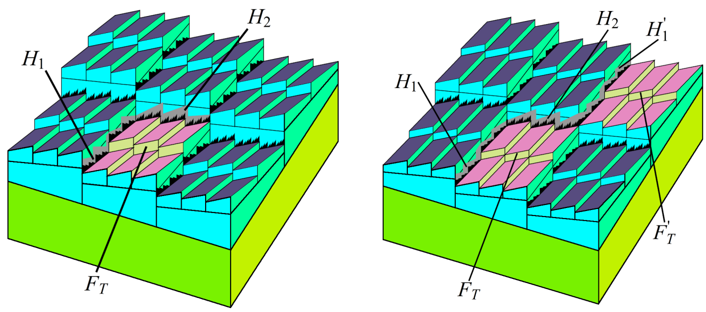
Step 1. The definition of $U_n(\mathbf{i}, \mathbf{j})$. Let N be given as in Proposition 4.5, and let $U_n^1(\mathbf{i}, \mathbf{j})$, $R_n(\mathbf{i}, \mathbf{j})$, $L_{n+N+1}(\mathbf{i}, \mathbf{j})$ be defined as in (4.11) and (4.12). Also, let γ^1, γ^2 be given as in Lemma 4.9 for the pair $(\mathbf{i}, \mathbf{j}) \in \Sigma_{p,q}^n$. If $j_n > 0$, connect the end-points of γ^1 and $M_L(\mathbf{i}, \mathbf{j})$ with two vertical line segments, and denote the resulting polygon by $H_1 = H_1(\mathbf{i}, \mathbf{j})$. If $i_n > 0$, connect the end-points of γ^2 and $M_B(\mathbf{i}, \mathbf{j})$ with two vertical line segments and denote the resulting polygon by $H_2 = H_2(\mathbf{i}, \mathbf{j})$. We set $H_1 = \emptyset$ if $j_n = 0$ and $H_2 = \emptyset$ if $i_n = 0$. Let

$$Q(\mathbf{i}, \mathbf{j}) := \big[H_1 \cup H_2 \cup F_T(\mathbf{i}, \mathbf{j})\big]_{2^{-n}\delta} \cap X_{N+n},$$

where δ will be given in the next step (see Figure 11).

Remove all polyhedra $Q(i, j)$ from X_{N+1}, and denote the closure of the resulting set by $U_0(\emptyset, \emptyset) := U_0$, i.e.,

$$U_0 = \overline{X_{N+1} \backslash \Big(\bigcup_{(i,j) \in \Sigma_{p,q}^1} Q(i, j) \Big)}.$$



(a) $s < t < rs$ (b) $s = t$

FIGURE 11. Figures for H_1, H_2, F_T and H_1', F_T'. The set $Q(\mathbf{i},\mathbf{j})$ is the closed $2^{-n}\delta$-neighborhood of the shaded area $H_1 \cup H_2 \cup F_T$ restricted to X_{N+n}.

For $n \geq 1$, we define $U_n(\mathbf{i},\mathbf{j})$ as follows. We put $Q(\mathbf{i},\mathbf{j})$, which is removed in the $(n-1)$th step, back to the set $L_{n+N+1}(\mathbf{i},\mathbf{j})$, and remove all the polyhedra $Q(\mathbf{i}i,\mathbf{j}j)$, $(i,j) \in \Sigma_{p,q}^1$, from the union. Finally, denote the closure of the resulting set by $U_n(\mathbf{i},\mathbf{j})$, i.e.,

$$U_n(\mathbf{i},\mathbf{j}) = \overline{L_{N+n+1}(\mathbf{i},\mathbf{j}) \cup Q(\mathbf{i},\mathbf{j}) \setminus \Big(\bigcup_{(i,j)\in\Sigma_{p,q}^1} Q(\mathbf{i}i,\mathbf{j}j) \Big)}. \tag{4.15}$$

Step 2. The new construction of X. Fix n. For $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$, let

$$\begin{aligned}\big[(\mathbf{i},\mathbf{j})\big] \quad := \quad \big\{(\mathbf{i}',\mathbf{j}') \in \Sigma_{p,q}^n : \ &\text{there exist } (\mathbf{i}_1,\mathbf{j}_1) = (\mathbf{i},\mathbf{j}),\dots,(\mathbf{i}_k,\mathbf{j}_k) = (\mathbf{i}',\mathbf{j}') \\ &\text{such that } U_n(\mathbf{i}_m,\mathbf{j}_m) \cap U_n(\mathbf{i}_{m+1},\mathbf{j}_{m+1}) \neq \emptyset, 1 \leq m < k\big\},\end{aligned}$$

and denote $[U_n(\mathbf{i},\mathbf{j})] := \bigcup_{(\mathbf{i}',\mathbf{j}')\in[\mathbf{i},\mathbf{j}]} U_n(\mathbf{i}',\mathbf{j}')$. We choose $\delta > 0$ small enough such that, when $s = t$, $U_n(\mathbf{i},\mathbf{j}) \cap U_n(\mathbf{i}',\mathbf{j}') \neq \emptyset$ if and only if $(\mathbf{i}',\mathbf{j}') = (\mathbf{i}^-,\mathbf{j}^+)$ or $(\mathbf{i}',\mathbf{j}') = (\mathbf{i}^+,\mathbf{j}^-)$ and, when $s < t$, $[U_n(\mathbf{i},\mathbf{j})]$ contains only $U_n(\mathbf{i},\mathbf{j})$. The choice of δ yields:

(1) the sets $[U_n(\mathbf{i},\mathbf{j})]$, where $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$, are pairwise disjoint and homeomorphic to the 3-ball;
(2) if $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$ is a prefix of $(\mathbf{i}',\mathbf{j}') \in \Sigma_{p,q}^m$, then $U_n(\mathbf{i},\mathbf{j}) \cap U_m(\mathbf{i}',\mathbf{j}') \neq \emptyset$ if and only if $m = n+1$.

Denote $\bar{X}_{N+1} = U_0$ and for $n \geq 1$ define

$$\bar{X}_{N+n+1} = \bar{X}_{N+n} \cup \Big(\bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^n} U_n(\mathbf{i},\mathbf{j}) \Big).$$

Notice that the difference of $\bar{X}_{N+n}$ and X_{N+n} is in the set $\bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma_{p,q}^n} Q(\mathbf{i},\mathbf{j})$, and moreover, the thickness of each $Q(\mathbf{i},\mathbf{j})$, $(\mathbf{i},\mathbf{j}) \in \Sigma_{p,q}^n$, tends to zero as $n \to \infty$. So $\bigcup_{n\geq N+1} X_n$ and $\bigcup_{n\geq N+1} \bar{X}_n$ have the same closure. Hence X coincides with $\bigcup_{n\geq N+1} \bar{X}_n$.

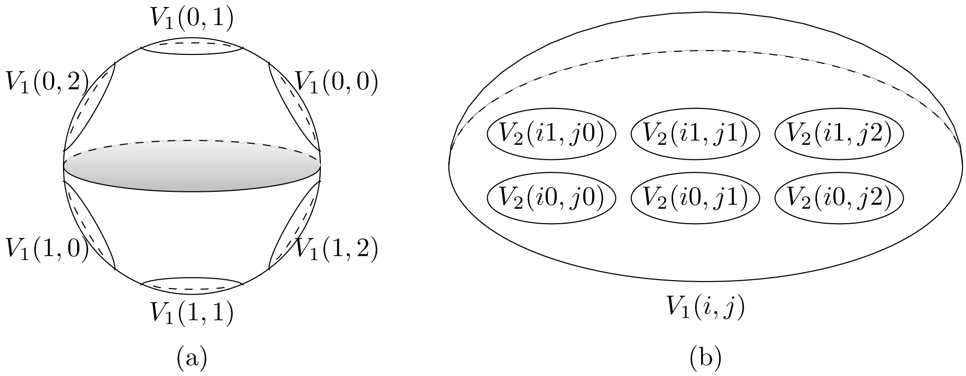


FIGURE 12. The homeomorphism h for $s < t$. The figures are drawn with $p = 2$ and $q = 3$.

Step 3. A desired homeomorphism. We let $V_1(i,j)$, $(i,j) \in \Sigma^1_{p,q}$, be pq spherical caps of the 3-ball $B(0,1)$ such that the simply connected sets $[V_1(i,j)] := \bigcup_{(i',j')\in[(i,j)]} V_1(i',j')$ are pairwise disjoint, with the equivalence relation being defined in Step 2 (see Figure 12(a) for $s < t$). A homeomorphism h can be constructed by specifying that h sends $B(0,1) \setminus \bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma^1_{p,q}} V_1(i,j)$ to $\bar{X}_{N+1}$. Let $V_2(ii',jj')$, $(i',j') \in \Sigma^1_{p,q}$, be pq smaller spherical caps of $B(0,1)$ in $V_1(i,j)$ such that the simply connected sets $[V_2(ii',jj')] := \bigcup_{(\mathbf{i},\mathbf{j})\in[(ii',jj')]} V_2(\mathbf{i},\mathbf{j})$ are pairwise disjoint (see Figure 12(b) for $s < t$). We also assume that none of the $V_2(ii',jj')$ intersects the bottom of $V_1(i,j)$. Extend h to $\bar{X}_{N+2}$. This procedure can be continued ad infinitum by using the properties (1) and (2) of $\{U_n(\mathbf{i},\mathbf{j})\}$ in Step 2. Now one obtains a homeomorphism h from the major part of $B(0,1)$ to the union of $\bigcup_{n\geq 1} \bar{X}_{N+n}$. It is not yet defined on the set $\bigcap_{n\geq1}\bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma^n_{p,q}} V_n(\mathbf{i},\mathbf{j})$, which is either a Cantor set ($s < t$ or $p \neq q$; see Figure 13(a)) or a union of infinitely many curves ($s - t = p - q = 0$; see Figure 13(b)), but this set is sent by h onto

$$\bigcap_{n\geq 1} \bigcup_{(\mathbf{i},\mathbf{j})\in\Sigma^n_{p,q}} \big(R_n(\mathbf{i},\mathbf{j}) \cup Q(\mathbf{i},\mathbf{j})\big).$$

This finishes the construction of the homeomorphism h and completes the proof. $\square$

Lemma 4.11. *If $t \geq rs$, then X is ball-like.*

Proof. The proof is similar to that of Lemma 4.10; we provide only an outline. Let $N_1 \in \mathbb{Z}$ satisfy $r^{N_1}s \leq t < r^{N_1+1}s$ and let N be determined by Proposition 4.5. First, we let $U^1_n(\mathbf{i},\mathbf{j})$ and $R_n(\mathbf{i},\mathbf{j})$ be defined as in (4.11) and (4.12) with respect to the integer $N + N_1$. Then partition them into p^{N_1} parts as follows (see Figures 14 and 15):

$$U^{1;\mathbf{i}_1}_n(\mathbf{i},\mathbf{j}) = \bigcup_{\mathbf{j}_1\in\Sigma^{N_1}_q} \bigcup_{k=1}^{N} \bigcup_{(\mathbf{i}',\mathbf{j}')\in\Sigma^k_{p,q}} T_{n+N_1+k}(\mathbf{i}\mathbf{i}_1\mathbf{i}',\mathbf{j}\mathbf{j}_1\mathbf{j}'),$$

$$R^{\mathbf{i}_1}_n(\mathbf{i},\mathbf{j}) = \bigcup_{\mathbf{j}_1\in\Sigma^{N_1}_q} \bigcup_{k>N} \bigcup_{(\mathbf{i}',\mathbf{j}')\in\Sigma^k_{p,q}} T_{n+N_1+k}(\mathbf{i}\mathbf{i}_1\mathbf{i}',\mathbf{j}\mathbf{j}_1\mathbf{j}'),$$

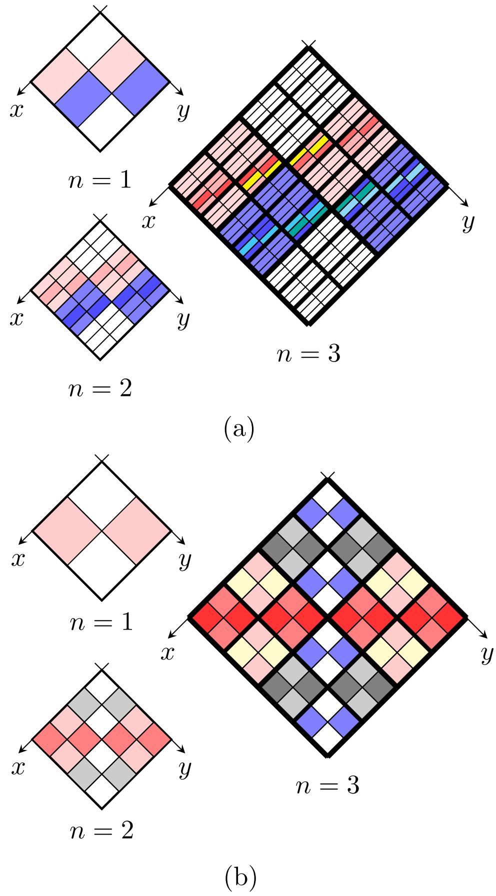
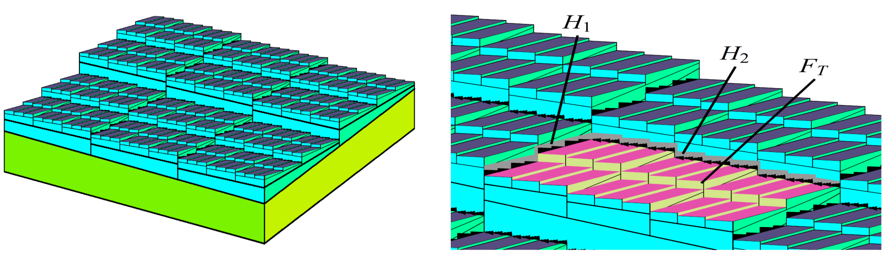
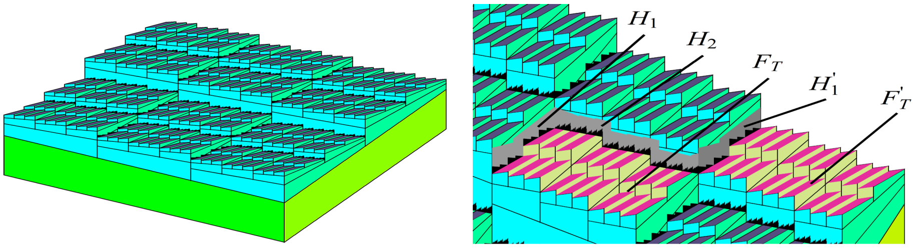


FIGURE 13. Bird's-eye views of $U_n(i,j)$ for $n = 1, 2, 3$ and $s = t$. (a) is for $p \neq q$ and (b) is for $p = q$. Each cube stands for a U_n. Two diagonal (from left to right) cubes can be connected only if they have the same color.



(a) X_{N+N_1+1} and some surfaces of $R_n^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})$ (b) Three polygons

FIGURE 14. The figures of X_{n+N_1+N} and the polygons H_1, H_2, F_T, determined by $U_n^{1;\mathbf{i}_1}(\mathbf{i},\mathbf{j})$. The figures are drawn with $p = r = 2$, $q = 3$ and $s = 0.3$, $t = rs$ (and hence $N = N_1 = 1$).



(a) X_{N+N_1+1} and some surfaces of $R_n^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})$ (b) Five polygons

FIGURE 15. Figures of X_{n+N_1+N} and the five polygons $H_1, H_2, F_T, H_1', F_T'$. The first three polygons H_1, H_2, F_T are determined by $U_n^{1,0^{N_1}}(\mathbf{i},\mathbf{j})$, while the last two polygons H_1', F_T' are induced by $U_n^{1;(p-1)^{N_1}}(\mathbf{i},\mathbf{j}^+)$. The figures are drawn with $p=r=2$, $q=3$, $s=0.1$, and $t=1.5rs$ (and hence $N=N_1=1$).

where $\mathbf{i}_1\in\Sigma_p^{N_1}$. The set $L^{\mathbf{i}_1}_{n+N_1+N+1}(\mathbf{i},\mathbf{j})$ is the union of all basic blocks in $R_n^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})$ with order $n+N_1+N+1$, i.e.,

$$L^{\mathbf{i}_1}_{n+N_1+N+1}(\mathbf{i},\mathbf{j}) = \bigcup_{\mathbf{j}_1\in\Sigma_q^{N_1}} \bigcup_{(\mathbf{i}',\mathbf{j}')\in\Sigma_{p,q}^{N+1}} T_{n+N_1+N+1}(\mathbf{i}\mathbf{i}_1\mathbf{i}',\mathbf{j}\mathbf{j}_1\mathbf{j}').$$

Second, applying Proposition 4.5 for such N_1, and using Lemma 4.9 (we replace $U_n(\mathbf{i},\mathbf{j})$, $R_n(\mathbf{i},\mathbf{j})$ by $U_n^{1;\mathbf{i}_1}(\mathbf{i},\mathbf{j}), R_n^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})$ respectively), we get piecewise linear curves $\gamma^{k;\mathbf{i}_1}(\mathbf{i},\mathbf{j})$, $k=1,2$. For these piecewise linear curves and the sets $L^{\mathbf{i}_1}_{n+N_1+1}$, we define $Q^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})$ as $Q(\mathbf{i},\mathbf{j})$. Then we let

$$U_0^{\mathbf{i}_1} = U_0^{\mathbf{i}_1}(\emptyset,\emptyset) = \overline{X_{N_1+N+1}\setminus \bigcup_{\mathbf{i}_1'\in\Sigma_p^{N_1}} \bigcup_{(i,j)\in\Sigma_{p,q}^1} Q^{\mathbf{i}_1'}(i,j)}$$

and

$$U_n^{\mathbf{i}_1}(\mathbf{i},\mathbf{j}) = \overline{L^{\mathbf{i}_1}_{n+N_1+N+1}(\mathbf{i},\mathbf{j}) \cup Q^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})\setminus\Big(\bigcup_{\mathbf{i}_1'\in\Sigma_p^{N_1}} \bigcup_{(i,j)\in\Sigma_{p,q}^1} Q^{\mathbf{i}_1'}(\mathbf{i}i,\mathbf{j}j)\Big)}.$$

Their definitions are similar to those of the sets $U_n(\mathbf{i},\mathbf{j})$. Third, we define subsets $V_n^{\mathbf{i}_1}$ of the 3-ball as V_n in Step 3 of the proof of Lemma 4.10 such that the interiors of $V_n^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})$ and $V_n^{\mathbf{i}_1}(\mathbf{i}',\mathbf{j}')$ are disjoint if and only if $U_n^{\mathbf{i}_1}(\mathbf{i},\mathbf{j})\cap U_n^{\mathbf{i}_1}(\mathbf{i}',\mathbf{j}')=\emptyset$. Finally, for the sets $\{U_n^{\mathbf{i}_1}\}$ and $\{V_n^{\mathbf{i}_1}\}$, we construct a homeomorphism h from the 3-ball to X, as described in Lemma 4.10. The proof is complete. $\square$

Theorem 4.12. *Assume that $s,t>0$ satisfy $s+t<r(r-1)$. Then T is ball-like.*

Proof. Without loss of generality, we assume $s\le t$. Combining Lemmas 4.10 and 4.11, we see X is ball-like. Now Lemma 4.6 says $T = X\cup(2\mathbf{t}-X)$ is also ball-like. $\square$

Proof of Theorem 1.2. Suppose $|\rho_1(i)+\rho_2(j)|\ge 1$ for some i and j. Let $x=(i+1)/p$ and $y=(j+1)/q$. Then x has two expansions $\mathbf{i}_1 = i\overline{p-1}$ and $\mathbf{i}_2=(i+1)\overline{0}$, while y has two expansions $\mathbf{j}_1 = j\overline{q-1}$ and $\mathbf{j}_2=(j+1)\overline{0}$. Now, from (2.9) and the definitions of $d_{\max}$ and $d_{\min}$, we have

$$d_{\max}(x,y)-d_{\min}(x,y)\ge |d(\mathbf{i}_1,\mathbf{j}_1)-d(\mathbf{i}_2,\mathbf{j}_2)| = |\rho_1(i)+\rho_2(j)|\ge 1,$$

which implies, by Lemma 2.5, that $T_{x,y}$ is a subset of ∂T. Therefore, either T° is disconnected or the genus of ∂T is at least one. Hence T is not homeomorphic to a ball.

For the sufficiency, we assume that $a_i = b_j = 0$ for all i, j, and that $|t + s| < r(r-1)$ with $ts \geq 0$. When $t = 0$ or $s = 0$, T is the Cartesian product of a unit interval and a disk-like self-affine tile in $\mathbb{R}^2$. For example, when $t = 0$,

$$T = \left\{\varphi_r(\mathbf{i}) : \mathbf{i} \in \Sigma_p^\infty\right\} \times \left\{(\varphi_q(\mathbf{j}), \varphi_r(\mathbf{k}) + s(\mathbf{j})) : \mathbf{j} \in \Sigma_q^\infty,\, \mathbf{k} \in \Sigma_r^\infty\right\}.$$

Thus, T is homeomorphic to a 3-ball.

Next, we assume $st \neq 0$. Theorem 4.12 shows that T is ball-like when s, t are positive. Proposition 4.1(b), together with Theorem 4.12, shows that the reflection of T with respect to the plane $z = 1/2$ is ball-like when s, t are negative, and thus T is also ball-like. This completes the proof. $\square$

5. A question

Theorem 1.2 gives a necessary and sufficient condition for T to be ball-like, under the assumption that $st \geq 0$ and $a_i = b_j = 0$ for all i, j. When $st < 0$, our cut-and-paste technique fails, as T cannot be expressed as a limit of the union of X_0 and all nth basic blocks. Our proof in the case $st > 0$ requires the added "horns" to be homeomorphic to a 3-ball and that there are no gaps between the horns and X_n. If $a_i = b_j = 0$, an added horn is a basic block with the bottom and some face of X_n lying in the same plane, and so the gap is zero. However, if a_i, b_j are not zero, it might still be possible to find new horns and new X_n, all being homeomorphic to a 3-ball, to make the gap zero. The bottom of the horns will be a surface in $\mathbb{R}^3$, not contained in a plane. So we have

Conjecture 5.1. *Condition* $|\rho_1(i)| + |\rho_2(j)| < 1$ *is sufficient for* T *to be ball-like.*

Acknowledgments

The major part of this work was carried out when the first two authors were visiting the Department of Mathematical Sciences of Georgia Southern University. They thank the department for its hospitality and support. The authors are indebted to the anonymous referee for some very valuable comments and suggestions.

References

[1] Shigeki Akiyama and Nertila Gjini, *On the connectedness of self-affine attractors*, Arch. Math. (Basel) **82** (2004), no. 2, 153–163, DOI 10.1007/s00013-003-4820-z. MR2047669

[2] Shigeki Akiyama and Jörg M. Thuswaldner, *A survey on topological properties of tiles related to number systems*, Geom. Dedicata **109** (2004), 89–105, DOI 10.1007/s10711-004-1774-7. MR2113188

[3] Christoph Bandt, *Self-similar sets. V. Integer matrices and fractal tilings of* $\mathbf{R}^n$, Proc. Amer. Math. Soc. **112** (1991), no. 2, 549–562, DOI 10.2307/2048752. MR1036982

[4] C. Bandt, Combinatorial topology of three-dimensional self-affine tiles, http://arxiv.org/abs/1002.0710.

[5] C. Bandt and Y. Wang, *Disk-like self-affine tiles in* $\mathbb{R}^2$, Discrete Comput. Geom. **26** (2001), no. 4, 591–601, DOI 10.1007/s00454-001-0034-y. MR1863811

[6] R. H. Bing, *The geometric topology of 3-manifolds*, American Mathematical Society Colloquium Publications, vol. 40, American Mathematical Society, Providence, RI, 1983. MR728227

[7] Gregory R. Conner and Jörg M. Thuswaldner, *Self-affine manifolds*, Adv. Math. **289** (2016), 725–783, DOI 10.1016/j.aim.2015.11.022. MR3439698

[8] Qi-Rong Deng and Ka-sing Lau, *Connectedness of a class of planar self-affine tiles*, J. Math. Anal. Appl. **380** (2011), no. 2, 493–500, DOI 10.1016/j.jmaa.2011.03.043. MR2794408

[9] Götz Gelbrich, *Self-affine lattice reptiles with two pieces in* $\mathbf{R}^n$, Math. Nachr. **178** (1996), 129–134, DOI 10.1002/mana.19961780107. MR1380707

[10] Masayoshi Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. **2** (1985), no. 2, 381–414, DOI 10.1007/BF03167083. MR839336

[11] John E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747, DOI 10.1512/iumj.1981.30.30055. MR625600

[12] Teturo Kamae, Jun Luo, and Bo Tan, *A gluing lemma for iterated function systems*, Fractals **23** (2015), no. 2, 1550019, 10, DOI 10.1142/S0218348X1550019X. MR3351926

[13] Richard Kenyon, *Self-replicating tilings*, Symbolic dynamics and its applications (New Haven, CT, 1991), Contemp. Math., vol. 135, Amer. Math. Soc., Providence, RI, 1992, pp. 239–263, DOI 10.1090/conm/135/1185093. MR1185093

[14] Ibrahim Kirat and Ka-Sing Lau, *On the connectedness of self-affine tiles*, J. London Math. Soc. (2) **62** (2000), no. 1, 291–304, DOI 10.1112/S002461070000106X. MR1772188

[15] Ibrahim Kirat, Ka-Sing Lau, and Hui Rao, *Expanding polynomials and connectedness of self-affine tiles*, Discrete Comput. Geom. **31** (2004), no. 2, 275–286, DOI 10.1007/s00454-003-2879-8. MR2060641

[16] Jeffrey C. Lagarias and Yang Wang, *Self-affine tiles in* $\mathbf{R}^n$, Adv. Math. **121** (1996), no. 1, 21–49, DOI 10.1006/aima.1996.0045. MR1399601

[17] King-Shun Leung and Ka-Sing Lau, *Disklikeness of planar self-affine tiles*, Trans. Amer. Math. Soc. **359** (2007), no. 7, 3337–3355, DOI 10.1090/S0002-9947-07-04106-2. MR2299458

[18] King-Shun Leung and Jun Jason Luo, *Connectedness of planar self-affine sets associated with non-consecutive collinear digit sets*, J. Math. Anal. Appl. **395** (2012), no. 1, 208–217, DOI 10.1016/j.jmaa.2012.05.034. MR2943615

[19] Jun Luo, Hui Rao, and Bo Tan, *Topological structure of self-similar sets*, Fractals **10** (2002), no. 2, 223–227, DOI 10.1142/S0218348X0200104X. MR1910665

[20] Sze-Man Ngai and Tai-Man Tang, *Topology of connected self-similar tiles in the plane with disconnected interiors*, Topology Appl. **150** (2005), no. 1-3, 139–155, DOI 10.1016/j.topol.2004.11.009. MR2133675

[21] W. P. Thurston, *Groups, tilings, and finite state automata*, AMS colloquium lecture notes, 1989.

School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, People's Republic of China
E-mail address: hilltower@163.com

School of Mathematics and Computer Science, Wuhan Polytechnic University, Wuhan 430023, People's Republic of China
E-mail address: lct984@163.com

College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, People's Republic of China – and – Department of Mathematical Sciences, Georgia Southern University, Statesboro, Georgia 30460-8093
E-mail address: smngai@georgiasouthern.edu