TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 3, March 2018, Pages 1823–1865
http://dx.doi.org/10.1090/tran/7135
Article electronically published on October 24, 2017

REDUCIBILITY OF 1-D SCHRÖDINGER EQUATION WITH TIME QUASIPERIODIC UNBOUNDED PERTURBATIONS. I

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ABSTRACT. We study the Schrödinger equation on $\mathbb{R}$ with a polynomial potential behaving as x^{2l} at infinity, $1 \le l \in \mathbb{N}$, and with a small time quasiperiodic perturbation. We prove that if the symbol of the perturbation grows at most like $(\xi^2 + x^{2l})^{\beta/(2l)}$, with $\beta < l + 1$, then the system is reducible. Some extensions including cases with $\beta = 2l$ are also proved. The result implies boundedness of Sobolev norms. The proof is based on pseudodifferential calculus and KAM theory

1. INTRODUCTION

In this paper we study the problem of reducibility of the time dependent Schrödinger equation

$$\mathrm{i}\dot{\psi} = H_\epsilon(\omega t)\psi \ , \ x \in \mathbb{R}, \tag{1.1}$$

$$H_\epsilon(\omega t) := -\partial_{xx} + V(x) + \epsilon W(x, -\mathrm{i}\partial_x, \omega t), \tag{1.2}$$

where V is a polynomial potential of degree $2l$, with $l \ge 1$, and $\mathbb{T}^n \ni \phi \mapsto W(x, \xi, \phi)$ is a C^∞ map from $\mathbb{T}^n$ to a space of symbols growing at infinity at most like $(\xi^2 + x^{2l})^{\beta/2l}$. We emphasize that the harmonic potential $l = 1$ is included.

We will prove that, if $\beta < l+1$, then, for sufficiently small ϵ, and for ω belonging to a set of large measure, there exists a unitary transformation which conjugates (1.1) to a time independent equation; the transformation depends on time in a smooth quasiperiodic way. We also deduce boundedness of the Sobolev norms and pure point spectrum of the Floquet operator. In the case where the average of the symbol W with respect to the flow of the classical Hamiltonian system $\xi^2 + V(x)$ vanishes, the result holds also for $\beta < (3l + 1)/2$. Finally we also prove reducibility in some cases with $\beta = 2l$.

The main limitation of the paper is that the allowed perturbations are of a quite particular type (it is the same as in [HR82b, HR82a]), as an example, in the case

$$W = -\mathrm{i}a_1(x, \omega t)\partial_x + a_0(x, \omega t) \tag{1.3}$$

the functions a_0 and a_1 must be polynomials in x. On the contrary the perturbation is allowed to grow at infinity (both in x and in the Fourier variable ξ) much faster than in all the preceding papers.

There is quite an extensive literature on the problem of reducibility of the time dependent Schrödinger equation and the related problems of growth of Sobolev norm and nature of the spectrum of the Floquet operator. We recall first the works [DŠ96, DLŠV02], in which pure point nature of the Floquet spectrum is obtained

Received by the editors June 14, 2016.
2010 *Mathematics Subject Classification.* Primary 35J10, 35S05, 37K55.

in the case in which the growth of V is superquadratic (and therefore the spectrum has increasing gaps) and the perturbations are bounded and time periodic. The first paper dealing with an unbounded time quasiperiodic perturbation is [BG01]. In [BG01] we assumed that the potential (not necessarily a polynomial) grows at infinity like x^{2l}, with a real $l > 1$ and the perturbation is bounded by $1 + |x|^{\beta}$ with $\beta < l - 1$; reducibility in the limiting case $\beta = l - 1$ was obtained in [LY10]. Concerning the case of harmonic potential we recall the pioneering work [Com87] in which reducibility is obtained in case of a perturbation which is smoothing and the works [Wan08] and [GT11] dealing with the case of a bounded perturbation. The present paper is the first one in which reducibility for an unbounded perturbation of the harmonic oscillator is obtained. We remark that the present result does not cover the results of [Wan08, GT11] since their perturbations are not in the class of symbols we use here. The technique of the present paper can be used also to obtain and improve [Wan08, GT11], but this requires quite heavy work and produces a bigger limitation for the allowed range of β. For this reason it will be developed in a future paper (paper II).

We also recall the interesting counterexamples in [GY00] and [Del14]. In particular we remark that the class of perturbation constructed in these papers is covered by the result of the present paper[1]. The main point is that in our case the frequencies fulfill a nonresonance relation which is violated in [GY00, Del14].

We recall that all the papers quoted above deal only with the one-dimensional case. The case of higher dimension is dealt with only in the papers [EK09] for the Schrödinger equation on $\mathbb{T}^d$ and in [GP16] for the case of the harmonic oscillator.

We remark that the problem of reducibility of linear equations is considered to be the main step for the proof of KAM type results in nonlinear PDEs, thus we think that the result of the present paper could be useful in this direction and in particular in order to construct quasiperiodic motions of a soliton in external potentials (in the spirit of [FGJS04, BM16a]).

The proof of the result of the present paper is based on a generalization of the ideas developed by Baldi, Berti, Montalto [BBM14] (see also [Mon14, FP15, BM16b]) in order to extend KAM theory to fully nonlinear equations, ideas which in turn are a development of those introduced by Plotnikov and Toland in [PT01] in order to study the water wave problem (see also [IPT05]). We recall that the idea is to proceed in two steps: first one uses pseudodifferential calculus in order to regularize the perturbation and then applies more or less standard KAM theory in order to conclude the proof. Actually an intermediate step is also required. This is due to the fact that, after the smoothing theorem the system is reduced to a smoothing perturbation of a time independent system, but the time independent system is not diagonal. So before developing KAM theory one has to diagonalize such a time independent system and to study its eigenvalues.

The main novelty of the present paper is that we deal here with the case of an equation on an unbounded domain, namely $\mathbb{R}$ so that a second source of unboundedness is the growth at infinity of symbols. In order to deal with the present case one has to develop in a quite careful way the regularization procedure, which is based on the strong connection existing between classical and quantum perturbation theory

[1]Actually in [GY00] it is shown that the instability exhibited in their counterexample is stable under a class of further perturbations not covered by the present paper. This class will be covered in paper II.

[GP87, BGP99]. The point is that, if one considers the classical Hamiltonian of the system and tries to eliminate order by order (in ϵ) the time dependence through classical normal form theory, then the quantization of the normalizing transformation conjugates the quantum system to a time independent system, *up to the quantum corrections*. But the quantum corrections are usually smoother than the original operators, and therefore one can expect the transformed quantum system to be a smoother perturbation of a time independent system. It turns out that this is the case. The framework (and the results) that we use here is the one developed by Hellfert and Robert in [HR82b].

After the regularization step one can use more or less standard KAM theory in order to reduce the regularized system to constant coefficients. However, there is an additional difficulty, namely that pseudodifferential calculus works well in the class of C^∞ functions, while the simplest formulation of KAM theory is that dealing with analytic functions. So one has to develop KAM theory in a C^∞ context. This is quite standard and indeed KAM theory is developed in C^∞ context, e.g., in the paper [BBM14]; however, we are here in a slightly different situation, thus we decided to insert in the paper also a proof of a KAM theorem with finite smoothness developed following the presentation of [Sal04]. We point out that the method of [Sal04] has already been applied to the problem of reducibility, in a slightly different context in [YZ13].

As anticipated above the main limitation of the present paper is that the symbols we consider here are of a quite particular type. The extension to more general symbols only fulfilling growth properties will be the goal of the paper II. The main point in order to get the extension is to introduce a different class of symbols; however, on the one hand some quite hard technical work is needed in order to deal with such a class, and on the other hand we only get the result under the strongest assumption $\beta < l$, which in particular rules out the case $\beta = 2l - 1$ which is very interesting in order to deal with the case of a soliton moving in an external potential.

The paper is organized as follows: in section 2 we state the results of the paper and give some examples and comments. The subsequent sections contain the corresponding proofs. Precisely, in section 3 we introduce and give the main properties of the unitary transformations generated by time dependent selfadjoint operators. Such transformations will be used in the rest of the paper: first at level of symbols and subsequently directly at the level of operators. In section 4 we prove the smoothing theorem. The section is split into a few subsections. In particular, in subsection 4.2 there is quite detailed description of the strategy used in order to prove the smoothing theorem. In section 5 we diagonalize the time independent part of the regularized system and study its eigenvalues. In section 6 we prove the analytic KAM theorem that constitutes the main step for the proof of the finite smoothness KAM theorem proved in section 7. Finally, the appendices contain some technical lemmas. They are grouped in different sections according to the role they have in the main part of the text.

2. Statement of the main result

Fix a positive integer $l \geq 1$ and define the weight

$$\lambda(x,\xi) := \left(1 + \xi^2 + x^{2l}\right)^{1/2l} . \tag{2.1}$$

Definition 2.1. The space S^m is the space of the symbols $g \in C^\infty(\mathbb{R})$ such that for all $k_1, k_2 \geq 0$ there exists C_{k_1,k_2} with the property that[2]

$$\left|\partial_\xi^{k_1}\partial_x^{k_2} g(x,\xi)\right| \leq C_{k_1,k_2}\left[\lambda(x,\xi)\right]^{m-k_1 l-k_2} . \tag{2.2}$$

The best constants C_{k_1,k_2} such that (2.2) holds form a family of seminorms for that space S^m.

Remark 2.2. Everything we do can be developed also for symbols with a finite, but large, differentiability.

In the following we will denote by $\mathcal{S}^m := C^\infty(\mathbb{T}^n, S^m)$ the space of C^∞ functions on $\mathbb{T}^n$ with values in S^m.

The frequencies ω will be assumed to vary in the set

$$\Omega := [1,2]^n ,$$

or in suitable closed subsets $\widetilde{\Omega}$.

To a symbol $g \in S^m$ we associate its Weyl quantization, namely the operator $g^w(x, D_x)$, $D_x := -\mathrm{i}\partial_x$, defined by

$$G\psi(x) \equiv g^w(x,D_x)\psi(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}(x-y)\cdot\xi} g\left(\frac{x+y}{2};\xi\right)\psi(y)dyd\xi. \tag{2.3}$$

We will often denote by a a capital letter the Weyl quantized of a symbol denoted with the corresponding lower case letter. As an exception, we will denote by W both the symbol of the perturbation and the corresponding operator.

We use the symbol $\lambda(x,\xi)$ to *define the spaces* $\mathcal{H}^s = D([\lambda^w(x,-\mathrm{i}\partial_x)]^{s(l+1)})$ *for* $s \geq 0$ *(domain of the* $(s(l+1))$*th-power of the operator operator* $\lambda^w(x,-\mathrm{i}\partial_x)$*) endowed by the graph norm.* For negative s, the space $\mathcal{H}^s$ is the dual of $\mathcal{H}^{-s}$.

We will denote by $B(\mathcal{H}^{s_1};\mathcal{H}^{s_2})$ the space of bounded linear operators from $\mathcal{H}^{s_1}$ to $\mathcal{H}^{s_2}$.

The potential V defining

$$H_0 := H_\epsilon\big|_{\epsilon=0} \equiv -\partial_{xx} + V$$

is assumed to be a polynomial of order $2l$, so that, in particular, it belongs to S^{2l}.

We also assume that

$$V'(x) \neq 0 \quad \forall x \neq 0 , \tag{2.4}$$

and normalize the potential by assuming $V(0) = 0$. The unperturbed Hamiltonian H_0 is the quantization of the classical Hamiltonian system with Hamiltonian function

$$h_0(x,\xi) := \xi^2 + V(x). \tag{2.5}$$

Remark 2.3. As a consequence of the assumptions above all the solutions of the Hamiltonian system h_0 are periodic with a period $T(E)$ which depends only on $E = h_0(x,\xi)$.

In the following we will denote by $\Phi^t_{h_0}$ the flow of the Hamiltonian system (2.5).

[2]This class of symbols coincides with the class introduced in [HR82b] and denoted by $S^m_{l,1}$.

We denote by λ_j^v the sequence of the eigenvalues of H_0 labeled in increasing order. It is well known that (see e.g. [HR82a])

$$\lambda_j \sim \frac{1}{c_l} j^d \ , \quad j \to \infty, \tag{2.6}$$

with $c_l > 0$ and

$$d = \frac{2l}{l+1} \tag{2.7}$$

(in concrete examples one can compute also a complete asymptotic expansion of the eigenvalues, see [HR82a]). In the harmonic case, whithout lack of generality, we assume $V(x) = x^2$.

In what follows we will identify L^2 with ℓ^2 by introducing the basis of the eigenvector of H_0. Similarly we will identify $\mathcal{H}^s$ with the space ℓ_s^2 of the sequences ψ_j s.t.

$$\sum_{j \geq 1} j^{2s} \left| \psi_j \right|^2 < \infty.$$

In order to state the assumptions on the perturbation we need a few notations. First we define the average with respect to the flow of h_0:

$$\langle W \rangle (x, \xi, \omega t) := \frac{1}{T(E)} \int_0^{T(E)} W\left(\Phi_{h_0}^\tau (x, \xi), \omega t\right) d\tau. \tag{2.8}$$

Concerning the perturbation, we assume that $W \in \mathcal{S}^\beta$ and we define

$$\tilde{\beta} := \begin{cases} 2\beta - 2l & \text{if} \quad \langle W \rangle \equiv 0, \\ \quad \beta & \quad\quad \text{otherwise.} \end{cases} \tag{2.9}$$

The main result of the paper is the following theorem.

Theorem 2.4. *Assume $\tilde{\beta} < l + 1$. Then there exist $\epsilon_* > 0$, C_λ and for all $|\epsilon| < \epsilon_*$ a closed set $\Omega(\epsilon) \subset \Omega$. For all $\omega \in \Omega(\epsilon)$ there exists a unitary (in L^2) time quasiperiodic operator $\Phi_\omega(\omega t)$ s.t. the function φ defined by $\Phi_\omega(\omega t)\varphi := \psi$ satisfies the equation*

$$\mathrm{i}\dot{\varphi} = H_\infty \varphi \ , \tag{2.10}$$

with $H_\infty = \mathrm{diag}(\lambda_j^\infty)$ and

$$\left| \lambda_j^\infty - \lambda_j^v \right| \leq C_\lambda \epsilon j^{\frac{\tilde{\beta}}{l+1}}. \tag{2.11}$$

Furthermore, one has

1. $\lim_{\epsilon \to 0} |\Omega - \Omega(\epsilon)| = 0$*;*
2. $\forall s$ $\exists \epsilon_s$ *s.t., if* $|\epsilon| < \epsilon_s$*, then* $\Phi_\omega(\omega t) \in B(\mathcal{H}^s; \mathcal{H}^s)$*;*
3. $\forall r > 0$ $\exists \epsilon_{s,r} > 0$ *s.t. if* $|\epsilon| < \epsilon_{s,r}$*, then* $\exists s_r$ *s.t. the map* $\phi \mapsto \Phi_\omega(\phi)$ *is of class* $C^r(\mathbb{T}^n; B(\mathcal{H}^{s+s_r}; \mathcal{H}^s))$*;*
4. $\exists a > 0$ *s.t.* $\|\Phi_\omega(\phi) - \mathbf{1}\|_{B(\mathcal{H}^{s+\beta}; \mathcal{H}^s)} \leq C_s \epsilon^a$.

Remark 2.5. Under the assumptions of the Theorem 2.4, the perturbation W is an unbounded operator; it is for this reason that Φ_ω is close to identity only as an operator decreasing smoothness.

Remark 2.6. With our technique we are not able to show that the sequence ϵ_r does not go to 0 as $r \to \infty$, thus we cannot guarantee that Φ_ω is actually a C^∞ function of the angles.

Remark 2.7. The dependence of Φ_ω on ω is Whitney smooth; however, for the sake of simplicity we did not work out a precise statement.

A consequence of the above theorem is that in the considered range of parameters all the Sobolev norms, i.e., the $\mathcal{H}^s$ norms of the solutions are bounded forever and the spectrum of the Floquet operator is pure point.

A couple of examples is useful in order to clarify the range of applicability of the result.

Example 2.8 (Duffing oscillators)**.** $l = 2$. The assumptions of Theorem 2.4 become $\beta < 3$ if $\langle W \rangle \neq 0$, otherwise $\beta < 7/2$. An example in which the assumptions are fulfilled is a singular version of the Duffing oscillator,

$$-\partial_{xx} + x^4 + \epsilon x^\beta f(\omega t) \ , \ \beta = 1, 2, 3, \tag{2.12}$$

where f is an arbitrary C^∞ function. (In this case one has that, for symmetry reasons, the average of x^3 is zero.) At the end of the section we will show that the method of the present paper can be extended to deal also with the case $\beta = 4$. The best previous result, due to [LY10], only allowed that we have $\beta = 1$.

One can also add a magnetic type term of the form

$$-(a_0(\omega t) + a_1(\omega t)x)\mathrm{i}\partial_x. \tag{2.13}$$

More general perturbations of the form of a pseudodifferential operator with symbol $W(x, \xi, \omega t)$ with W of class S^β, $\beta < 3$, are allowed.

Example 2.9 (Harmonic oscillator)**.** $l = 1$. In this case Theorem 2.4 applies when $\beta < 2$. Thus, for example we can deal with the case

$$-\partial_{xx} + x^2 + \epsilon x a_1(\omega t) - \mathrm{i} a_2(\omega t)\epsilon\partial_x. \tag{2.14}$$

The more general case of a perturbation quadratic in x and ξ will be covered at the end of the section.

Perturbations of the kind of those considered by Delort [Del14] belong to the class of symbols dealt with in Theorem 2.4. The same is true for the main term of the perturbation in [GY00].

Remark 2.10. If $W \in S^\beta$ is independent of ξ, namely $W = W(x)$, then it must be a polynomial. Indeed, if $k > \beta$, then $|\partial_x^k W(x)|$ must tend to zero as $\xi \to \infty$, and thus it must be identically zero.

As anticipated in the introduction, the extension of Theorem 2.4 to more general perturbations including the cases of the form (1.3) with a_0, a_1 nonpolynomial smooth functions will be obtained in paper II.

In order to give the extension to $\beta = 2l$ (and also for future use) it is useful to give the definition of quasihomogeneuos symbols.

Definition 2.11. We will say that a symbol f is quasihomogeneous of degree m if

$$f(\rho x, \rho^l \xi) = \rho^m f(x, \xi) \quad \forall \rho > 0. \tag{2.15}$$

The most general time dependent quasihomogeneous polynomial of degree $2l$ is given by

$$W_{2l}(x, \xi, \omega t) = a_1(\omega t)\xi^2 + a_2(\omega t)x^l\xi + a_3(\omega t)x^{2l}. \tag{2.16}$$

Theorem 2.12. *Consider the Schrödinger equation with Hamiltonian*

$$-\partial_{xx} + x^{2l} + \epsilon W_{2l}(x, D_x, \omega t) + \epsilon W(x, D_x, \omega t) \ , \tag{2.17}$$

with $l \geq 1$ a positive integer and $a_j \in C^\infty(\mathbb{T}^n)$, and $W \in \mathcal{S}^\beta$ fulfilling the assumptions of Theorem 2.4. *Then the same conclusions of Theorem* 2.4 *hold (in eq.* (2.11) *one has to put $\tilde{\beta} = l + 1$).*

3. Transformations of linear time dependent equations

In the following we will use in some different contexts transformations of the form $\psi = e^{-\mathrm{i}\epsilon X(\omega t)}\varphi$, with X a family of selfadjoint operators that in some sense depend smoothly on time. So, to start with, we study in a purely formal way how the Schrödinger equation is changed by such transformations. In the subsequent sections we will make all notions precise.

Definition 3.1. Let X be a selfadjoint operator; we will say that

$$(Lie_{\epsilon X} F) := e^{\mathrm{i}\epsilon X} F e^{-\mathrm{i}\epsilon X} \tag{3.1}$$

is the quantum Lie transform of F generated by ϵX.

Note that the quantum Lie transform fulfills the equation

$$\frac{d}{d\epsilon} Lie_{\epsilon X} F = -\mathrm{i}\, [Lie_{\epsilon X} F; X] = e^{\mathrm{i}\epsilon X} \mathrm{i}[X; F] e^{-\mathrm{i}\epsilon X} \ ,$$

from which one immediately gets (formally!)

$$Lie_{\epsilon X} F = \sum_{k \geq 0} \frac{1}{k!} \epsilon^k F_k \ , \tag{3.2}$$

$$F_0 = F \ ; \quad F_k := -\mathrm{i}[F_{k-1}; X]. \tag{3.3}$$

Note also that one has

$$\frac{d^k}{d\epsilon^k} Lie_{\epsilon X} F = e^{\mathrm{i}\epsilon X} F_k e^{-\mathrm{i}\epsilon X}. \tag{3.4}$$

In the following we will meet situations where the above series are either convergent or asymptotic.

We will use the same terminology also when X depends on time and/or on ω (which in this case play the role of parameters).

Lemma 3.2. *Let F be a selfadjoint operator, and let $X(t)$ be a family of selfadjoint operators. Assume that $\psi(t)$ fulfills the equation*

$$\mathrm{i}\dot{\psi} = F\psi \ . \tag{3.5}$$

Then φ defined by

$$\varphi = e^{\mathrm{i}\epsilon X(t)}\psi \tag{3.6}$$

fulfills the equation

$$\mathrm{i}\dot{\varphi} = F_\epsilon(t)\varphi \tag{3.7}$$

with

$$F_\epsilon := Lie_{\epsilon X} F - Y_X \ , \tag{3.8}$$

$$Y_X := \int_0^\epsilon (Lie_{(\epsilon - \epsilon_1)X} \dot{X}) d\epsilon_1. \tag{3.9}$$

Proof. One has

$$\mathrm{i}\frac{d\varphi}{dt} = \mathrm{i}\frac{de^{\mathrm{i}\epsilon X}}{dt}\psi + e^{\mathrm{i}\epsilon X}Fe^{-\mathrm{i}\epsilon X}\varphi = \left(\mathrm{i}\frac{de^{\mathrm{i}\epsilon X}}{dt}e^{-\mathrm{i}\epsilon X} + e^{\mathrm{i}\epsilon X}Fe^{-\mathrm{i}\epsilon X}\right)\varphi.$$

So, the second term in the bracket is already $Lie_{\epsilon X}F$. Define

$$\tilde{Y}_X := \frac{de^{\mathrm{i}\epsilon X}}{dt}e^{-\mathrm{i}\epsilon X} \ ,$$

and compute

$$\begin{aligned}\frac{d\tilde{Y}_X}{d\epsilon} &= \frac{d}{dt}\left(\mathrm{i}Xe^{\mathrm{i}\epsilon X}\right)e^{-\mathrm{i}\epsilon X} - \mathrm{i}\frac{de^{\mathrm{i}\epsilon X}}{dt}e^{-\mathrm{i}\epsilon X}X \\ &= \mathrm{i}\dot{X} + \mathrm{i}X\frac{de^{\mathrm{i}\epsilon X}}{dt}e^{-\mathrm{i}\epsilon X} - \mathrm{i}\frac{de^{\mathrm{i}\epsilon X}}{dt}e^{-\mathrm{i}\epsilon X}X \\ &= \mathrm{i}\dot{X} - \mathrm{i}\left[\tilde{Y}_X, X\right].\end{aligned}$$

It follows that $\tilde{Y}_X$ solves the Cauchy problem

$$\frac{d\tilde{Y}_X}{d\epsilon} = \mathrm{i}\dot{X} - \mathrm{i}[\tilde{Y}_X; X] \ , \quad \tilde{Y}_X(0) = 0 \ ,$$

whose solution is easily computed by Duhamel's formula getting (3.9). □

Definition 3.3. Given X, we will say that

$$T_{\epsilon X}F := Lie_{\epsilon X}F - Y_X \tag{3.10}$$

is the transformation of F through ϵX. Note that

$$T_{\epsilon X}(F+G) = T_{\epsilon X}F + Lie_{\epsilon X}G.$$

Remark 3.4. In the following we will be interested in expansions either in ϵ or in operators which are more and more regularizing; in this second case, as usual, the key property that we use is that the commutator of two operators is more regularizing than the product of the original operators. Thus, up to higher order corrections, either in ϵ or in smoothness, we will have that if F has the structure $F = H_0 + \epsilon P$ with P more smoothing (or "less unbounded") than H_0, then, up to higher order corrections, one has

$$T_{\epsilon X}F = H_0 + \epsilon P - \mathrm{i}\epsilon[H_0; X] - \epsilon\dot{X} + \cdots . \tag{3.11}$$

4. Smoothing the perturbation

4.1. **Some symbolic calculus.** First we recall that, from the Calderon Vaillencourt Theorem, the following lemma holds.

Lemma 4.1. *Let $f \in S^m$. Then one has*

$$f^w(x, D_x) \in B(\mathcal{H}^{s_1+s}; \mathcal{H}^s) \qquad \forall s,\ \forall s_1 \geq m. \tag{4.1}$$

We emphasize that the result holds also for negative values of the indices m, s_1.

Given a symbol $g \in S^m$ we will write

$$g \sim \sum_{j\geq 0} g_j \ , \quad g_j \in S^{m_j} \ , \quad m_j \leq m_{j-1} \ , \tag{4.2}$$

if for all κ there exist N and $r_N \in S^{-\kappa}$ s.t.

$$g = \sum_{j=0}^{N} g_j + r_N .$$

The following result is standard.

Lemma 4.2. *Given a couple of symbols $a \in S^m$ and $b \in S^{m'}$, denote by $a^w(x, D_x)$ and $b^w(x, D_x)$ the corresponding Weyl operators. Then there exists a symbol c, denoted by $c = a\sharp b$ such that*

$$(a\sharp b)^w(x, D_x) = a^w(x, D_x) b^w(x, D_x) \ .$$

Furthermore, one has

$$(a\sharp b) \sim \sum_j c_j \tag{4.3}$$

with

$$c_j = \sum_{k_1+k_2=j} \frac{1}{k_1!k_2!} \left(\frac{1}{2}\right)^{k_1} \left(-\frac{1}{2}\right)^{k_2} (\partial_\xi^{k_1} D_x^{k_2} a)(\partial_\xi^{k_2} D_x^{k_1} b) \in S^{m+m'-(l+1)j}.$$

In particular, denoting[3]

$$\{a; b\}^q := -\mathrm{i}(a\sharp b - b\sharp a) \ ,$$

we have

$$\{a; b\}^q = \{a; b\} + S^{m+m'-3(l+1)} \ , \tag{4.4}$$

where

$$\{a; b\} := -\partial_\xi a \partial_x b + \partial_\xi b \partial_x a \in S^{m+m'-(l+1)}$$

is the Poisson Bracket between a and b, while (4.4) *means that $\{a; b\}^q = \{a; b\}$ +some quantity belonging to $\in S^{m+m'-3(l+1)}$. Similar notations will be systematically used in the following.*

Sometimes we will deal with symbols having finite differentiability. We will denote by S^m_N the space of symbols which are only N times differentiable and fulfill the inequality (2.2) only for $k_1 + k_2 \leq N$. This is a Banach space with the norm

$$\|g\|_{S^m_N} := \sum_{k_1+k_2 \leq N} \sup_{(x,\xi)\in\mathbb{R}^2} \frac{\left|\partial_x^{k_1} \partial_\xi^{k_2} g(x, \xi)\right|}{[\lambda(x, \xi)]^{m-lk_1-k_2}}. \tag{4.5}$$

We remark that for the space $\mathcal{S}^m$ a family of seminorms is given by the standard norms of $C^M(\mathbb{T}^n; S^m_N)$ as M and N vary.

Finally, we will deal with Whitney smooth functions of the frequencies[4]. To this end we recall (following [Ste70]) the definition of smooth functions on a closed set $\widetilde{\Omega} \subset \Omega$. Fix an integer k and a ρ fulfilling $k < \rho \leq k + 1$; let $\mathcal{B}$ be a Banach space, and $f : \widetilde{\Omega} \to \mathcal{B}$ a map. The map f is said to be of class $Lip_\rho(\widetilde{\Omega}; \mathcal{B})$, if there exist maps $f^{(j)}$, $0 \leq |j| \leq k$ defined on $\widetilde{\Omega}$, such that $f^{(0)} = f$ and so that, if

$$f^{(j)}(\omega) = \sum_{|j+l| \leq k} \frac{f^{(j+l)}(\nu)}{l!} (\omega - \nu)^l + R_j(\omega, \nu), \tag{4.6}$$

[3] Sometimes $\{.; .\}^q$ is called the Moyal Bracket.

[4] This will be needed only for the proof of Lemma 5.2. For the rest of KAM theory, Lipschitz dependence on the frequencies is enough.

then

$$\left\| f^{(j)}(\omega) \right\| \le M \ , \quad \| R_j(\omega,\nu) \| \le M \, |\omega - \nu|^{\rho - k} \qquad \forall \omega, \nu \in \widetilde{\Omega} \ , \quad |j| \le k. \tag{4.7}$$

Here we used a standard vector notation: $j = (j_1, \dots, j_n)$ and $\omega^j \equiv \omega_1^{j_1} \cdots \omega_n^{j_n}$. The minimum of the constants M for which (4.7) holds is a norm on the space $Lip_\rho(\widetilde{\Omega}; \mathcal{B})$.

Definition 4.3. We will say that a function $f : \widetilde{\Omega} \to \mathcal{S}^m$ is of class $Lip^m_\rho(\widetilde{\Omega})$ if for all N_1, N_2 it is of class $Lip_\rho(\widetilde{\Omega}; C^{N_1}(\mathbb{T}^n; S^m_{N_2}))$.

Definition 4.4. An operator F will be said to be a pseudodifferential operator of class O^m if there exists a sequence $f_j \in S^{m_j}$ with $m_j \le m_{j-1}$ and, for any κ there exist N and an operator $R_N \in B(\mathcal{H}^{s-\kappa}; \mathcal{H}^s)$ for all s such that

$$F = \sum_{j \ge 0}^{N} f_j^w + R_N. \tag{4.8}$$

In this case we will write $f \sim \sum_{j \ge 0} f_j$ and f will be said to be the symbol of F.

Concerning maps we will use the following definition.

Definition 4.5. A map $\mathbb{T}^n \ni \phi \mapsto F(\phi) \in O^m$ will be said to be of class $\mathcal{O}^m$ if the functions of the sequence f_j also depend smoothly on ϕ, namely $f_j \in \mathcal{S}^{m_j}$ and the operator valued map $\phi \mapsto R_N(\phi)$ has the property that for any $K \ge 1$ there exists $a_K \ge 0$ s.t. for any N one has

$$R_N(.) \in C^K(\mathbb{T}^n; B(\mathcal{H}^{s-\kappa+a_K}; \mathcal{H}^s)) \ \forall s. \tag{4.9}$$

Definition 4.6. A map $\tilde{\Omega} \ni \omega \mapsto F \in \mathcal{O}^m$ will be said to be of class $\mathcal{L}ip^m_\rho(\widetilde{\Omega})$ if the functions $f_j \in Lip^{m_j}_\rho(\widetilde{\Omega})$ and if the map $\omega \mapsto R_N$ has the property that there exists $b \ge 0$ s.t.

$$R_N \in Lip_\rho(\tilde{\Omega}; C^K(\mathbb{T}^n; B(\mathcal{H}^{s-\kappa+a_K+b}; \mathcal{H}^s))) \ \forall s. \tag{4.10}$$

We want now to study the quantum Lie transform generated by a symbol $\chi \in S^m$. First, applying Proposition A.2 of [MR16] we have the following lemma.

Lemma 4.7. *Let $\chi \in S^m$ with $m \le l+1$. Then $X := \chi^w(x, D_x)$ is selfadjoint and $e^{-\mathrm{i}\epsilon X}$ leaves invariant all the spaces $\mathcal{H}^s$.*

Proof. According to [MR16], the thesis holds if there exists a positive selfadjoint operator K such that both the operators XK^{-1} and $[X,K]K^{-1}$ are bounded. To this end we take K to be the Weyl operator of the symbol λ^m. From symbolic calculus it follows that $XK^{-1} \in O^0$ and $[X,K]K^{-1} \in O^{2m-(l+1)-m}$. Thus they are bounded under the assumption of the lemma. □

One can rewrite formulae (3.2) and (3.3) in terms of symbols. Thus, if f and χ are symbols and χ fulfills the assumptions of Lemma 4.7 one can define

$$f_0^q := f \ , \quad f_k^q := \left\{ f_{k-1}^q ; \chi \right\}^q \ , \tag{4.11}$$

and one can expect that the symbol of $Lie_{\epsilon X} F$ is $\sum_{k \ge 0} \epsilon^k f_k^q / k!$. This is ensured by the following lemma.

Lemma 4.8. *Let $\chi \in S^m$ and $f \in S^{m'}$ be symbols, and assume that $m < l+1$. Then $Lie_{\epsilon X} F \in O^m$, and furthermore its symbol, denoted by $lie_{\epsilon\chi} f$, fulfills*

$$lie_{\epsilon\chi} f \sim \sum_{k\geq 0} \frac{\epsilon^k f_k^q}{k!}. \tag{4.12}$$

Proof. First note that, by induction, one has $f_k^q \in S^{m'+k(m-l-1)}$. From (3.4) and the formula of the remainder of the Taylor expansion one also has

$$Lie_{\epsilon X} F(\epsilon) = \sum_{k=0}^{N} \frac{F_k}{k!}\epsilon^k + \frac{\epsilon^{N+1}}{N!}\int_0^1 (1+u)^J e^{-\mathrm{i}u\epsilon X} F_{N+1} e^{\mathrm{i}u\epsilon X} du \ ,$$

so that, by defining R_N to be the integral term of the previous formula, we have $R_N \in B(\mathcal{H}^{s-\kappa}, \mathcal{H}^s)$ with $\kappa = m' + (m-l-1)N$, which diverges as $N \to \infty$ and thus shows that the expansion (4.12) is asymptotic in the sense of Definition 4.4. $\square$

In the following, by abuse of language, we will call $lie_{\epsilon\chi} f$ the quantum Lie transform of f through χ.

Remark 4.9. Denote by Φ_χ^ϵ the flow of the Hamilton equations of χ. Then one has

$$f \circ \Phi_\chi^\epsilon \sim \sum_{k\geq 0} \frac{\epsilon^k f_k}{k!}, \tag{4.13}$$

$$f_0 := f \ , \quad f_k := \{f_{k-1}; \chi\} \ , \tag{4.14}$$

thus one has

$$lie_{\epsilon\chi} f = f \circ \Phi_\chi^\epsilon + S^{m+m'-3(l+1)} \tag{4.15}$$

$$= f + \{f; \chi\} + S^{m'+2(m-l-1)}. \tag{4.16}$$

In the following we will need also a result valid in the limit case $\chi \in S^{l+1}$. This is covered by the following lemma, which is a variant of Theorem 7.1 of [HR82b].

Theorem 4.10. *Let $\chi \in S^{l+1}$ and $f \in S^{m'}$, and assume that $f \circ \Phi_\chi^\epsilon \in S^{m'}$. Then equation* (4.15) *holds.*

The proof is obtained exactly as in [HR82b] and is omitted.

Remark 4.11. Let $\chi \in S^m$. Then the operator Y_X defined by (3.9) is a pseudodifferential operator with symbol

$$y_x := \int_0^\epsilon (lie_{(\epsilon-\epsilon_1)\chi}\dot{\chi}) d\epsilon_1 = \epsilon\dot{\chi} + \epsilon S^{2m-(l+1)}. \tag{4.17}$$

4.2. **Symbol of the transformed Hamiltonian and formal description of the smoothing algorithm.** The idea is to use the quantization of a time dependent symbol $\chi(\omega t)$ in order to transform the original Hamiltonian

$$h := h_0 + \epsilon W \tag{4.18}$$

into a new one with a more regular perturbation.

According to (3.11), written at the level of symbols, one has that the tranformed Hamiltonian has a symbol which, at highest order, is given by

$$h_\epsilon = h_0 + \epsilon W + \epsilon \{h_0; \chi\}^q - \epsilon\dot{\chi} + \cdots = h_0 + \epsilon W + \epsilon \{h_0; \chi\} - \epsilon\dot{\chi} + \cdots . \tag{4.19}$$

So, in order to increase the order of the perturbation one has to choose χ in such a way to eliminate the terms of order ϵ or to transform them into smoother objects. To explain the procedure one has to distinguish between the case $l > 1$ and the case $l = 1$.

Consider first $l > 1$. In this case it turns out that $\dot{\chi}$ is more regular than $\{h_0; \chi\}$ (see Lemma 4.17), so in that case one determines χ by solving the homological equation

$$p + \{h_0; \chi\} = \langle p \rangle, \tag{4.20}$$

with $p = W$ (this will be done in Lemma 4.17). Using such a χ to transform the Hamiltonian, one gets a new Hamiltonian with a symbol which is a perturbation of

$$h_0 + \epsilon \langle W \rangle (h_0, \omega t) \ . \tag{4.21}$$

Note that $\langle W \rangle$ is a function of the phase variables only through h_0 (since it is Poisson commutes with it), but it is also time dependent.

So, the second step consists in looking for a second generating function $\chi_1 = \chi_1(h_0, \omega t)$ in order to eliminate the time dependence from $\langle W \rangle$ (at the main order). Taking into account that in such a case $\{h_0; \chi_1\} \equiv 0$, the main term of the Hamiltonian transformed through such a χ_1 is simply given by

$$h_0 - \epsilon \dot{\chi}_1 + \epsilon \langle W \rangle (h_0, \omega t) + \cdots$$

and this leads to the second kind of homological equation that we need to solve, namely

$$-\omega \cdot \frac{\partial \chi_1}{\partial \phi} = p - \bar{p} \ , \tag{4.22}$$

where $p \equiv \langle W \rangle$, while $\bar{p}$ is defined by

$$\bar{p}(x, \xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} p(x, \xi, \phi) d\phi. \tag{4.23}$$

Using such a χ_1 one transforms the Hamiltonian into a perturbation of

$$h_0 + \epsilon \overline{\langle W \rangle}(h_0) \tag{4.24}$$

which is a function of h_0 only. Thus the idea is to repeat the procedure with h_0 replaced by the function (4.24). As a consequence, at the subsequent steps, we will have to solve homological equations of the form of (4.20) with h_0 replaced by a function of h_0, and this will lead to the homological equation

$$p + \{h_1; \chi\} = \langle p \rangle \tag{4.25}$$

with

$$h_1 := h_0 + \epsilon f(h_0) \ , \tag{4.26}$$

which will be solved thanks to Remark 4.18. Then one can proceed iteratively until the perturbation is reduced to a smoothing operator of arbitrary order. Actually the procedure we use is slightly modified in order to be able to deal with a singularity related to the singularity of the action variables at the origin and in order to get a better result when the average of W vanishes (see the proof of Theorem 4.22).

In the case $l = 1$ the situation is different since in this case $\dot{\chi}$ and $\{h_0; \chi\}$ belong to the same smoothness class. So in this case we consider again equation (4.19). In order to reduce all the terms of order ϵ one has to solve the homological equation

$$\{h_0, \chi\} - \dot{\chi} + W = \overline{\langle W \rangle} \ , \tag{4.27}$$

and in this case the original Hamiltonian is directly transformed into a new one of the form

$$h_0 + \epsilon\overline{\langle W\rangle}(h_0) + \epsilon p(x, \xi) \tag{4.28}$$

with p which is the symbol of a more smoothing operator. We remark that equation (4.27) can only be solved in the case where the period of the orbits of h_0 does not depend on the energy, and therefore only in the case where V is exactly quadratic. Now there is a difficulty: one cannot include $\overline{\langle W\rangle}$ in the main part of the Hamiltonian in order to iterate since this would eliminate the above property. However, it turns out that this is not needed, since in the harmonic case one has $\{h_0, f\}^q = \{h_0, f\}$. This allows us to proceed as in classical normal form theory and to conjugate the Hamiltonian to a very smoothing symbol.

4.3. **Solution of the homological equations.** From now on we will use the notation

$$a \preceq b \tag{4.29}$$

to mean "there exists a constant C independent of all the relevant quantities, such that $a \leq Cb$".

In the following we will meet functions which depend on the phase space variables only through h_0, namely functions p such that there exist a $\tilde{p}$ with the property that

$$p(x, \xi) = \tilde{p}(h_0(x, \xi)).$$

For such functions we introduce a new class of symbols.

Definition 4.12. A function $\tilde{p} \in \mathbb{C}^\infty$ will be said to be of class $\widetilde{S}^m$ if one has

$$\left| \frac{\partial^k \tilde{p}}{\partial E^k}(E) \right| \preceq \langle E^{\frac{m}{2l} - k} \rangle. \tag{4.30}$$

We will also need to use functions from $\mathbb{T}^n$ to $\widetilde{S}^m$ which may also depend in a (Whitney) smooth way on the frequencies. For these classes we will use the same notation we already introduced, simply we will put a tilde on the letter denoting the corresponding class. Furthermore, by abuse of notation, we will say that $p \in \widetilde{S}^m$ if there exists $\tilde{p} \in \widetilde{S}^m$ s.t. $p(x, \xi) = \tilde{p}(h_0(x, \xi))$.

As one can see in the case of a homogeneous potential $V(x) = x^{2l}$, $l > 1$, the period as a function of the energy has a singularity at zero. In order to avoid this problem, before starting the procedure, it is useful to modify the perturbation making a cutoff close to the origin.

Let η be a C^∞ function such that

$$\eta(E) = \begin{cases} 1 & \text{if} \quad |E| > 2, \\ 0 & \text{if} \quad |E| < 1, \end{cases} \tag{4.31}$$

and split

$$W = W_0 + W_\infty , \quad W_\infty(x, \xi) = W(x, \xi)(1 - \eta(h_0(x, \xi))) , \tag{4.32}$$
$$W_0(x, \xi) = W(x, \xi)\eta(h_0(x, \xi)) .$$

Then $W_\infty \in \mathcal{S}^{-\kappa}$ for any κ and $W_1 \in S^\beta$ is the actual perturbation that has to be regularized.

Remark 4.13. All of the smoothing procedure is based on the solution of the homological equation and computation of Moyal brackets, which (up to operators which are smoothing of all orders) are operations preserving the property of symbols of being zero in the region $E < 1$.

Lemma 4.14. *Consider the period $T(E)$. Then the function $\bar{T}(E) := \eta(E)T(E)$ is a symbol and one has $\bar{T} \in \widetilde{S}^{1-l}$.*

Proof. Consider the function

$$A(E) := \eta(2E) \int_{\{h_0(x,\xi) \leq E\}} dx d\xi .$$

According to Lemma (1-3) of [HR82a] this is a symbol of class $\tilde{S}^{l+1}$. But this function, when $E > 1$ is the classical action of the Hamiltonian system h_0. Thus, in the region $E > 1$, one has $T(E) = 2\pi \partial A / \partial E$. Now, $\bar{T}$ coincides with this function in the considered region and is regular and bounded in the other region, and thus the thesis follows. □

Remark 4.15. The function $A(E)$ is particularly important since the period of the orbits of the Hamiltonian system $A(h_0)$ is 2π whenever $E > 1$. Furthermore, exploiting the fact that, in the region $E > 1$, $A(h_0)$ admits an expansion in quasi-homogeneous polynomials (see the Appendix of [HR82b]), one can see that given a symbol $f \in S^m$ then $f \circ \phi^t_{A(h_0)} \in S^m$.

Lemma 4.16. *Let $p \in S^m$ be a symbol supported in the region $h_0(x,\xi) > 1$. Then $\langle p \rangle \in \widetilde{S}^m$.*

Proof. Consider the function $A(h_0)$. It is easy to see that, in the region $E > 1$,

$$\Phi^t_{h_0} = \Phi^{\frac{t2\pi}{T}}_{A(h_0)} . \tag{4.33}$$

Therefore one has

$$\frac{1}{T(h_0)} \int_0^{T(h_0)} p \circ \Phi^t_{h_0} dt = \frac{1}{\bar{T}(h_0)} \int_0^{\bar{T}(h_0)} p \circ \Phi^t_{h_0} dt = \frac{1}{\bar{T}(h_0)} \int_0^{\bar{T}(H_0)} p \circ \Phi^{2\pi t/\bar{T}}_{A(h_0)} dt$$
$$= \frac{1}{2\pi} \int_0^{2\pi} p \circ \Phi^t_{A(h_0)} dt \ ,$$

but $p \circ \Phi^t_{A(h_0)} \in S^m$, so that the result immediately follows. □

Concerning the solution of the homological equation (4.20) we have the following lemma.

Lemma 4.17. *Let $p \in S^m$ be a symbol which vanishes in the region $h_0 < 1$. Then the homological equation* (4.20) *has a solution χ which is a symbol of class $\chi \in S^{m-l+1}$.*

Proof. First, following Lemma 5.3 of [BG93], we have that χ is given by the formula

$$\chi = \frac{1}{T(E)} \int_0^{T(E)} t \check{p} \circ \Phi^t_{h_0} dt \ , \tag{4.34}$$

with $\check{p} := p - \langle p \rangle$. To see this, fix a value of E and compute

$$\begin{aligned}\{\chi; h_0\}(\zeta) &= \frac{d}{dt}\Big|_{t=0} \chi\left(\Phi^t_{h_0}(\zeta)\right) = \frac{d}{dt}\Big|_{t=0} \frac{1}{T}\int_0^T \check{p}\left(\Phi^{t+s}_{h_0}(\zeta)\right) s ds \\ &= \frac{1}{T}\int_0^T s\ \frac{d}{ds}\check{p}\left(\Phi^{t+s}_{h_0}(\zeta)\right) ds\Big|_{t=0} \\ &= \frac{1}{T}\check{p}\left(\Phi^s_{h_0}(\zeta)\right) s\Big|_0^T - \frac{1}{T}\int_0^T \check{p}\left(\Phi^s_{h_0}(\zeta)\right) ds = \check{p}(\zeta)\ ,\end{aligned}$$

where $\zeta = (x, \xi)$. Now, exploiting again (4.33), one has

$$\chi = \frac{1}{T(E)}\int_0^{T(E)} t\check{p}\circ\Phi^t_{h_0} dt = \frac{1}{\bar{T}(E)}\int_0^{\bar{T}(E)} t\check{p}\circ\Phi^{\frac{t2\pi}{\bar{T}(E)}}_{A(h_0)} dt = \frac{\bar{T}(E)}{4\pi^2}\int_0^{2\pi} t\check{p}\circ\Phi^t_{A(h_0)} dt\ ,$$

from which, exploiting Lemma 4.14, one immediately gets the result. $\square$

Remark 4.18. From the above proof one gets that the above technique also allows us to solve the homological equation (4.25) and to show that the solution also belongs to S^{m-l+1}.

Remark 4.19. In the above lemmas p can also depend on the angles ϕ and on the frequencies ω, but they only play the role of parameters, so in that case the result is still valid substituting the classes $\mathcal{S}$ or Lip_ρ to the classes S with the same index.

We come now to equation (4.22).

First, fix $\tau > n - 1$ and denote

$$\Omega_{0\gamma} := \left\{\omega \in \Omega\ :\ |k \cdot \omega| \geq \gamma |k|^{-\tau}\right\}\ . \tag{4.35}$$

Then it is well known that

$$|\Omega - \Omega_{0\gamma}| \preceq \gamma. \tag{4.36}$$

Lemma 4.20. *Let* $p \in \widetilde{Lip}^m_\rho(\Omega_{0\gamma})$*. Then there exists a solution* $\chi \in \widetilde{Lip}^m_\rho(\Omega_{0\gamma})$ *of* (4.22)*. Furthermore, in this case* $\bar{p} \in \widetilde{Lip}^m_\rho(\Omega_{0\gamma})$.

Proof. We proceed as usual expanding p in Fourier series. First we consider the case where p does not depend explicitly on ω. Define

$$p_k(E) := \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} p(E, \phi) e^{-\mathrm{i}k\phi} d\phi\ , \tag{4.37}$$

and note that, since for all MN the map $\phi \mapsto p(., \phi)$ is of class $C^M(\mathbb{T}^n; S^m_N)$, one has $p_k \in S^m_N$ and

$$\|p_k\|_{S^m_N} \leq \frac{\|p\|_{C^M(\mathbb{T}^n; S^m_N)}}{|k|^M}\ , \quad k \neq 0.$$

Thus, defining

$$\chi(E, \phi, \omega) := \sum_{k \neq 0} \frac{p_k(E) e^{\mathrm{i}k\cdot\phi}}{\mathrm{i}\omega \cdot k}\ , \tag{4.38}$$

for any $M_1 < M - \tau - n$ one has

$$\|\chi\|_{C^{M_1}(\mathbb{T}^n, S^m_N)} \leq \|p\|_{C^M(\mathbb{T}^n; S^m_N)} \sum_{k \neq 0} \frac{|k|^{M_1+\tau}}{\gamma |k|^M}\ ,$$

which is convergent. From the arbitrariness of M it follows that M_1 can also be chosen arbitrarily and therefore, for fixed $\omega \in \Omega_\gamma$, the symbol $\chi \in \mathcal{S}^m$. Furthermore, since

$$\frac{\partial^n}{\partial \omega_j^n} \frac{1}{\mathrm{i}\omega \cdot k} = \frac{(-\mathrm{i})^n n! k_j^n}{(-\mathrm{i}\omega \cdot k)^n} \ , \tag{4.39}$$

and similarly for the other derivatives, one has that for all ρ, the symbol $\chi \in \widetilde{Lip}_\rho^m(\Omega_{0\gamma})$.

Exploiting this remark it is easy to obtain the conclusion also for the case of p which depends on ω in a Whitney smooth way. □

In order to solve equation (4.27) we now define the set

$$\Omega_{1\gamma} := \left\{ \omega \in \Omega \ : \ \left| \omega \cdot k + k_0 \right| \geq \frac{\gamma}{1 + |k|^\tau} \right| \ , \ (k_0, k) \in \mathbb{Z}^{n+1} - \{0\} \right\} . \tag{4.40}$$

Lemma 4.21. *Let $p \in Lip_\rho^m(\Omega_{1\gamma})$. Then there exists a solution $\chi \in Lip_\rho^m(\Omega_{1\gamma})$ of (4.22). Furthermore, in this case $\overline{\langle p \rangle} \in \widetilde{Lip}_\rho^m(\Omega_{0\gamma})$.*

Proof. Following [Bam97], we prove that the solution of the homological equation (4.27) is given by

$$\chi(x, \xi, \phi) := \sum_{k \in \mathbb{Z}^n} \chi_k(x, \xi) e^{\mathrm{i}k \cdot \phi} \ ,$$

where

$$\chi_0 = \frac{1}{T} \int_0^T t(\overline{p} - \overline{\langle p \rangle}) \circ \Phi_{h_0}^t dt, \tag{4.41}$$

$$\chi_k(x, \xi) = \frac{1}{e^{\mathrm{i}\omega \cdot kT} - 1} \int_0^T e^{\mathrm{i}\omega \cdot kt} p_k(\Phi_{h_0}^t(x, \xi)) dt \ , \tag{4.42}$$

and p_k are the functions defined by (4.37). To this end, consider (4.27) and take first its k-th Fourier coefficient (in ϕ); we thus get

$$\{\chi_k; h_0\} + \mathrm{i}\omega \cdot k \chi_k = p_k - \delta_{k,0} p_0. \tag{4.43}$$

For $k = 0$ this reduces to (4.20) and thus we have already studied it. For $k \neq 0$ equation (4.43) is the value at $t = 0$ of the equation

$$\frac{d}{dt} \chi_k \circ \Phi_{h_0}^t + \mathrm{i}\omega \cdot k \chi_k \circ \Phi_{h_0}^t = p_k \circ \Phi_{h_0}^t \ ; \tag{4.44}$$

denoting the r.h.s. by $p_k(t)$ we solve such an equation as an ordinary differential equation for $\chi_k(t)$. The general solution is given by

$$a_k e^{-\mathrm{i}\omega \cdot kt} + \int_0^t e^{\mathrm{i}\omega \cdot ks} p_k(s) ds.$$

The value of the constant is determined by the requirement that the solution must be periodic of period T. Thus one gets the formula (4.42). Then it is immediate to use the Diophantine condition in (4.40) in order to estimate χ and its derivatives (both in ϕ and in ω). □

4.4. **The smoothing theorem.** We are now ready to state and prove the main result of the section.

Theorem 4.22. *Fix $\gamma > 0$ small, $\rho > 2$ and an arbitrary $\kappa > 0$. Assume*

$$\tilde\beta < l+1. \tag{4.45}$$

Then there exists a (finite) sequence of symbols $\chi_1, \dots, \chi_N$ with $\chi_j \in Lip_\rho^{m_j}(\Omega_{0\gamma})$, $m_j \geq m_{j+1}$ for all j, s.t., defining

$$X_j := \chi_j^w(x, D_x, \omega t) \ , \quad \omega \in \Omega_{0\gamma} \ , \tag{4.46}$$

such operators are selfadjoint and the transformation

$$\psi = e^{-\mathrm{i}\epsilon X_1(\omega t)} \cdots e^{-\mathrm{i}\epsilon X_N(\omega t)}\varphi \tag{4.47}$$

transforms $H_\epsilon(\omega t)$ (cf. (1.2)) into a pseudodifferential operator $H^{(reg)} = A_0 + \epsilon R$, where

$$A_0 := H_0 + \epsilon Z + \epsilon \tilde Z \tag{4.48}$$

has symbol

$$h_0 + \epsilon z(h_0) + \epsilon \tilde z(h_0, \omega) \ ; \tag{4.49}$$

here $z \in \widetilde{S}^{\tilde\beta}$ is a function of h_0 independent of ω, $\tilde z \in \widetilde{Lip}_\rho^{2\tilde\beta - l - 1}(\Omega_{0\gamma})$ is an ω dependent function of h_0, and $R \in \mathcal{L}ip_\rho^{-\kappa}(\Omega_{0\gamma})$ depends on (ϕ, ω). In the case $l = 1$ the set $\Omega_{0\gamma}$ must be substituted by the set $\Omega_{1\gamma}$.

Remark 4.23. Actually in order to develop the KAM part of the proof of Theorem 2.4 we only need the existence of a positive κ s.t. the above results hold.

Proof of Theorem 4.22 *in the case $l > 1$.* We only study $h_0 + \epsilon W_0$. We will transform it through a unitary (in L^2) operator leaving invariant the spaces $\mathcal{H}^s$; therefore under such transformations W_∞ remains a smoothing operator of arbitrary order.

Consider $h_0 + \epsilon W_0$; we transform it using the operator X_1 with symbol χ_1 obtained by solving the homological equation (4.20) with $p = W$, so that $\chi_1 \in \mathcal{S}^{\beta - l + 1}$, so that by Lemma 4.7 the corresponding Weyl operator is selfadjoint provided

$$\beta - l + 1 \leq l + 1 \iff \beta \leq 2l \tag{4.50}$$

and Lemma 4.8 applies provided the inequality is strict. Then the symbol of the transformed Hamiltonian is given by

$$h^{(1)} := h_0 + \epsilon(\langle W_0\rangle - W_0) + \epsilon^2 \mathcal{S}^{\beta - (l+1-m_1)} + \epsilon \mathcal{S}^{\beta - 2l - 2} \tag{4.51}$$

$$+ \epsilon W_0 + \epsilon^2 \{W_0; \chi_1\} + \epsilon^2 \mathcal{S}^{\beta - 2(l+1-m_1)} \tag{4.52}$$

$$- \epsilon \dot\chi_1 + \epsilon^2 \mathcal{S}^{2(\beta - l + 1) - (l+1)} \tag{4.53}$$

$$= h_0 + \epsilon \langle W_0 \rangle - \epsilon \dot\chi_1 + \epsilon P_1 \ , \tag{4.54}$$

with $P_1 \in \mathcal{S}^{\beta - (2l - \beta)}$ and $m_1 = \beta - l + 1$.

Consider first the case where $\langle W\rangle \equiv 0$, which implies $\langle W_0\rangle = 0$. In this case we iterate the procedure with $P_1' := -\dot\chi_1 + P_1 \in \mathcal{S}^{\beta_1}$ in place of W_0 and $\beta_1 := \max\{\beta - l + 1; 2\beta - 2l\}$. Note that

$$\langle \dot\chi_1 \rangle \equiv 0 \ ,$$

so that, after the second transformation generated by $\chi_2 \in S^{\beta_1 - l+1}$, one gets a Hamiltonian of the form

$$h^{(1_2)} = h_0 + \epsilon\langle P_1\rangle - \epsilon\dot{\chi}_2 + \epsilon\mathcal{S}^{\beta_1-(2l-\beta_1)} . \tag{4.55}$$

Then, if $\beta_1 - l + 1 > 2\beta - 2l$, we iterate again until we get

$$\tilde{h}^{(1)} = h_0 + \epsilon\langle P_1\rangle + \epsilon S^{\beta_2} \ ,$$

with some $\beta_2 < 2\beta - 2l$.

In both cases we thus get (maybe after the second group of transformations) a Hamiltonian of the form

$$h^{(1')} := h_0 + \epsilon f(h_0, \omega t) + \epsilon P_2 \ , \tag{4.56}$$

with $f(h_0, .) \in \mathcal{S}^{\tilde{\beta}}$ and $P_2 \in \mathcal{S}^{\tilde{\beta}_1}$, with

$$\tilde{\beta}_1 < \tilde{\beta}.$$

We now continue by eliminating the time dependence from f. To this end we take χ_3 to be the solution of (4.22) with $p = f(h_0)$, so that $\chi_3 \in \widetilde{Lip}_\rho^{\tilde{\beta}}(\Omega_{0\gamma})$. Provided

$$\tilde{\beta} < l + 1 \ ,$$

one gets that the corresponding Weyl operator is selfadjoint and the quantum Lie transform it generates transforms symbols into symbols. Then the symbol of the transformed Hamiltonian takes the form

$$\begin{aligned} h^{(2)} &= h_0 + \epsilon Lip_\rho^{2l+\tilde{\beta}-3(l+1)} \\ &\quad - \epsilon\dot{\chi}_3 + \epsilon^2 Lip_\rho^{2\tilde{\beta}-3(l+1)} \\ &\quad + \epsilon f(h_0) + \epsilon^2 Lip_\rho^{2\tilde{\beta}-3(l+1)} \\ &\quad + \epsilon P_2 + \epsilon^2 Lip_\rho^{\tilde{\beta}_1+\tilde{\beta}-(l+1)} \\ &= h_0 + \epsilon\overline{f(h_0)} + \epsilon P_2 + \epsilon Lip_\rho^{\tilde{\beta}_1+\tilde{\beta}-(l+1)}, \end{aligned}$$

where all the functions are defined on $\Omega_{0\gamma}$. In particular the perturbation $Lip_\rho^{\tilde{\beta}_1+\tilde{\beta}-(l+1)}$ is the lowest order term with a nontrivial dependence on ω.

Now denote

$$h_1 := h_0 + \epsilon f(h_0)$$

so that the Hamiltonian takes the form

$$h_1 + \epsilon P_2 + \epsilon Lip_\rho^{\tilde{\beta}_1+\tilde{\beta}-(l+1)} \ ;$$

we can now iterate the above construction (with h_1 in place of h_0, thus exploiting the homological equation (4.25)) until we get a Hamiltonian of the form

$$h^{(2')} := h_2 + \epsilon P_2' \ , \quad P_2' \in Lip_\rho^{\beta_2} \ , \quad h_2 := h_1 + \epsilon f_1(h_0) \ , \quad \beta_2 := \tilde{\beta}_1 + \tilde{\beta} - (l+1). \tag{4.57}$$

We are now in the position of concluding the proof of the theorem. We proceed in a quite explicit way. First we construct χ_4 by solving (4.25) (with h_2 in place of h_1 and P_2' in place of p). We transform the Hamiltonian getting

$$h_2 + \epsilon\langle P_2'\rangle + Lip_\rho^{\beta_2-a_1} \ , \quad a_1 := \min\{l-1; 2l-\beta_2\} \ . \tag{4.58}$$

We now construct $\chi_5 \in Lip_\rho^{\beta_2}$ by solving the homological equation (4.22) and transforming the Hamiltonian to get

$$h^{(3)} = h_3 + \epsilon Lip_\rho^{\beta_2 - a_1} \ , \quad h_3 := h_2 + \epsilon\overline{\langle P_2' \rangle} \ ; \tag{4.59}$$

we remark that the correction in h_3 has a nontrivial dependence on ω.

We also remark that the gain in the order of the remainder does not decrease as β_2 decreases. So one can iterate the construction lowering by a finite quantity at each step the order of the perturbation. In this way, by a finite number of steps one gets the order $-\kappa$. Finally we remark that in the considered range of the parameters the condition $\tilde{\beta} < l+1$ implies also condition (4.50). □

Proof of Theorem 4.22 *in the case* $l = 1$. First we remark that in this case the condition $\tilde{\beta} < l+1$ is equivalent to $\beta < l+1$. We prove that for any positive N there exists $\{\chi_j\}_{j=1}^N$, $\chi_j \in Lip^{\beta-(j-1)(2-\beta)}(\Omega_{1\gamma})$, s.t. the symbol $h^{(N)}$ of the Hamiltonian obtained after the transformation $\psi = e^{\mathrm{i}\epsilon X_N} \cdots e^{\mathrm{i}\epsilon X_1}\phi$ has the structure

$$h^{(N)} = h_0 + \epsilon\overline{\langle W \rangle} + \epsilon^2 \widetilde{z^{(N)}} + \epsilon^N r_N \ , \tag{4.60}$$

with $\widetilde{z^{(N)}} = \widetilde{z^{(N)}}(h_0, \omega)$, $\widetilde{z^{(N)}} \in \widetilde{S}^{\beta-1}$ and

$$r_N \in \mathcal{L}ip^{\beta - N(2-\beta)}(\Omega_{1\gamma}). \tag{4.61}$$

We prove this by induction. Of course it is true for $N = 0$ with $r_0 = W - \overline{\langle W \rangle}$. Assume it is true for N. We now transform $h^{(N)}$ using $\epsilon^{N+1}\chi_{N+1} \in Lip_\rho^{\beta_N}$, $\beta_N := \beta - N(2-\beta)$, which solves (4.27) with $p = r_N$. Remarking that in this case, for any symbol f, one has

$$\{h_0, f\}^q = \{h_0, f\} \ .$$

It follows that

$$\begin{aligned} h^{(N+1)} &= h_0 + \epsilon^N \{h_0, \chi_{N+1}\} + \epsilon^{2N}\frac{1}{2}\{\{h_0; \chi_{N+1}\}; \chi_{N+1}\} + l.o.t. \\ &+ \epsilon\overline{\langle W \rangle} + \epsilon^2\widetilde{z^{(N)}} + \epsilon^{N+1}\left\{\overline{\langle W \rangle}; \chi_{N+1}\right\} + l.o.t. \\ &+ \epsilon^N r_N + \epsilon^{2N}\{r_N; \chi_{N+1}\} + l.o.t. \\ &- \epsilon^N \dot{\chi}_{N+1} - \frac{1}{2}\epsilon^{2N}\{\dot{\chi}_{N+1}; \chi_{N+1}\} + l.o.t. \\ &= h_0 + \epsilon\overline{\langle W \rangle} + \epsilon^2\widetilde{z^{(N+1)}} + \epsilon^{2N} Lip_\rho^{2\beta_N - 2} + \epsilon^{N+1} Lip_\rho^{\beta+\beta_N-2} \ , \end{aligned}$$

where we put

$$\widetilde{z^{(N+1)}} := \widetilde{z^{(N)}} + \epsilon^{N-1}\overline{\langle r_N \rangle}.$$

□

5. Diagonalization of the time independent part

In this section we diagonalize the operator

$$A_0 := H_0 + \epsilon Z + \epsilon\tilde{Z} \tag{5.1}$$

associated to the time independent part of the Hamiltonian:

$$h_0 + \epsilon z(h_0) + \epsilon\tilde{z}(h_0, \omega) \ .$$

First write it in the basis $\mathbf{e}_j$ of the normalized eigenvectors of H_0 and fix a positive s identifying the order of the space $\mathcal{H}^s$ in which we will control the norm of the operators and a positive ρ larger than 2 controlling the smoothness in ω of the various objects.

We will denote $\Delta f := f(\omega') - f(\omega)$ and $\Delta\omega := \omega' - \omega$.

Lemma 5.1. *There exists a positive ϵ_* s.t., if $|\epsilon| < \epsilon_*$, then there exists a unitary (in L^2) operator U_1, Whitney smooth in ω, with*

$$\|U_1 - \mathbf{1}\|_{Lip_\rho(\Omega_{0\gamma};B(\mathcal{H}^{s-\delta};\mathcal{H}^s))} \preceq \epsilon \tag{5.2}$$

and $\delta := \tilde\beta - (l+1)$ s.t.

$$U_1^* A_0 U_1 = A^{(0)} \ , \tag{5.3}$$

where

$$A^{(0)} := \mathrm{diag}(\lambda_j^{(0)}) \ , \tag{5.4}$$

with $\lambda_j^{(0)}$ given by

$$\lambda_j^{(0)} = \lambda_j^v + \epsilon z(\lambda_j^v) + \epsilon\tilde z(\lambda_j^v,\omega) + \epsilon\frac{\nu_j(\omega)}{j^\delta} \ ; \tag{5.5}$$

here $\nu_j(\omega)$ are Whitney smooth functions which fulfill

$$|\nu_j(\omega)| \preceq 1 \ , \tag{5.6}$$

$$\left|\frac{\Delta\nu_j(\omega)}{\Delta\omega}\right| \preceq 1 \tag{5.7}$$

uniformly on $\Omega_{0\gamma}$ (or on $\Omega_{1\gamma}$) and in j.

Proof. Denote by $Z + \tilde Z$ the Weyl quantization of $z(h_0) + \tilde z(h_0,\omega)$, then from functional calculus one has that

$$R_a := z(H_0) + \tilde z(H_0,\omega) - Z - \tilde Z \in Lip_\rho^{\tilde\beta-(l+1)}(\Omega_{0\gamma});$$

since $l+1 > \tilde\beta$, the operator R is smoothing of order $\delta = l+1-\tilde\beta$. So we rewrite

$$A_0 = \Lambda + \epsilon R_a \ , \quad \Lambda := H_0 + \epsilon z(H_0) + \epsilon\tilde z(H_0,\omega). \tag{5.8}$$

Then we diagonalize the system by a series of transformations which are constructed in a way similar to the transformations that we will use in section 6 to prove Theorem 6.6 in order to develop the KAM part of the proof. Here the situation is much simpler since this procedure does not involve small denominators. In order to develop the procedure we need to control the differences between the eigenvalues. Denote

$$\lambda_j' := \lambda_j^v + \epsilon z(\lambda_j^v) + \epsilon\tilde z(\lambda_j^v,\omega) \ ; \tag{5.9}$$

then we have to estimate from below $|\lambda_i' - \lambda_j'|$. To this end consider first $z(\lambda_i^v) - z(\lambda_j^v)$, $i > j$. From the mean value theorem there exists $\bar E \in (\lambda_j^v, \lambda_i^v)$ s.t.

$$|z(\lambda_i^v) - z(\lambda_j^v)| = \left|\frac{\partial z}{\partial E}(\bar E)\right| |\lambda_i^v - \lambda_j^v| \succeq (\lambda_j^v)^{\frac{\tilde\beta}{l+1}-1}|\lambda_i^v - \lambda_j^v| \succeq |\lambda_i^v - \lambda_j^v| \ ,$$

so that (repeating the argument for $\tilde z$) one has

$$|\lambda_i' - \lambda_j'| \succeq |\lambda_i^v - \lambda_j^v| - \epsilon|\lambda_i^v - \lambda_j^v| \succeq |\lambda_i^v - \lambda_j^v| \succeq \left|i^d - j^d\right| . \tag{5.10}$$

Now define an operator X with matrix elements

$$X_{ij} := -\mathrm{i}\frac{R_{a,ij}}{\lambda_i' - \lambda_j'} \ , \quad i \neq j,$$

so that $-\mathrm{i}[\Lambda; X] = \overline{R}_a$, with $\overline{R}_a = \mathrm{diag}(R_{a,ii}) - R_a$. By Lemma C.2, X has the same boundedness properties of R_a, so it is smoothing of order δ. Furthermore its norm is estimated by

$$\|X\| \preceq \|F\| ,$$

where the norm is the norm in $Lip_\rho(\Omega_{0\gamma}; B(\mathcal{H}^{s-\delta}; \mathcal{H}^s))$, and the constant depends only on the indices of the norm and on the constant in inequality (5.10). In this proof we will use only such norm.

It follows from Lemma A.2 that the series defining $Lie_{\epsilon X} R_a$ is convergent and one has

$$\|Lie_{\epsilon X} F - F\| \preceq \epsilon \|X\| \|F\| \preceq \epsilon \|F\|^2 .$$

Furthermore exploiting the definition of X, one has

$$Lie_{\epsilon X}\Lambda = \Lambda + \epsilon \overline{R}_a + \sum_{k\geq 2} \epsilon^k \frac{\Lambda_k}{k!} \ , \tag{5.11}$$

$$\Lambda_1 := \overline{R}_a \ , \quad \Lambda_k = -\mathrm{i}[\Lambda_{k-1}; X] \ , \tag{5.12}$$

so that

$$Lie_{\epsilon X}(\Lambda + \epsilon R_a) = \Lambda^{(1)} + \epsilon^2 R^{(1)}, \tag{5.13}$$

where $\Lambda^{(1)} := \Lambda + \epsilon \, \mathrm{diag}(R_{a,ii})$ and $R^{(1)}$ is a suitable operator fulfilling

$$\left\| R^{(1)} \right\| \preceq \|R_a\|^2 \ , \tag{5.14}$$

again with a constant which depends only on the indices of the norm and on the constant in inequality (5.10).

It is easy to see that the eigenvalues of $\Lambda^{(1)}$ again fulfill inequality (5.10) with a constant which is decreased by $O(\epsilon)$, so that one can iterate the argument and get the existence of the operator U_1 claimed in the statement. □

We now study the properties of the eigenvalues (5.5). Before doing that, it is useful to introduce a few notations. First we denote

$$\langle m \rangle := \max\{1; |m|\} . \tag{5.15}$$

Then, given a closed set $\tilde{\Omega} \subset \Omega$, consider a sequence $\lambda = \{\lambda_j(\omega)\}_{j\geq 1}$ of functions of ω defined on $\tilde{\Omega}$. We denote

$$s_{ijk}(\lambda, \omega) := \lambda_i - \lambda_j + \omega \cdot k \ , \tag{5.16}$$

$$\mathcal{R}_{ijk}(\lambda, \alpha) := \left\{ \omega \in \tilde{\Omega} \ : \ |s_{ijk}(\lambda, \omega)| < \alpha \langle i^d - j^d \rangle \right\} . \tag{5.17}$$

The next lemma gives the properties of the eigenvalues. We emphasize that in its proof we use the property that both z and $\tilde{z}$ are Whithney smooth in the frequencies.

Moreover, in the case $d = 1$, we exploit the fact that $\delta > 0$, which is implied by $\tilde{\beta} < l + 1 = 2$ (strictly). In the case $d > 1$ this is not needed.

Lemma 5.2. *There exist $\epsilon_* > 0$ and $\tau > 0$ s.t. for any $|\epsilon| < \epsilon_*$ there exist $a > 0$ and a closed set $\Omega_\gamma^{(0)} \subset \Omega_{0\gamma}$ or $\Omega_\gamma^{(0)} \subset \Omega_{1\gamma}$ with the following properties:*

$$\left| \Omega - \Omega_\gamma^{(0)} \right| \preceq \gamma^a . \tag{5.18}$$

For any $\omega\in\Omega^{(0)}_\gamma$ *the following inequalities hold:*

$$\left|\lambda_j^{(0)}-\lambda_j^v\right|\preceq j^{\frac{\tilde\beta}{l+1}}\ , \tag{5.19}$$

$$\left|\lambda_i^{(0)}-\lambda_j^{(0)}\right|\succeq\left|i^d-j^d\right|\ , \tag{5.20}$$

$$\left|\frac{\Delta(\lambda_i^{(0)}-\lambda_j^{(0)})}{\Delta\omega}\right|\preceq\epsilon|i^d-j^d|, \tag{5.21}$$

$$\left|\lambda_i^{(0)}-\lambda_j^{(0)}+\omega\cdot k\right|\geq\frac{\gamma\langle i^d-j^d\rangle}{1+|k|^\tau}\ ,\quad |i-j|+|k|\neq 0. \tag{5.22}$$

Remark 5.3. In the case $d>1$ one can choose $a=1$ and $\tau>n+2/(d-1)$. In the case $d=1$ one can also compute such numbers, but they are more complicated.

Proof. Equations (5.19) and (5.20) immediately follow from the previous proof. To get (5.21) compute

$$\Delta(\lambda_i-\lambda_j)=\epsilon\Delta[\tilde z(\lambda_i^0)-\tilde z(\lambda_j^0)]+\epsilon\frac{\Delta\nu_i}{i^\delta}-\epsilon\frac{\Delta\nu_j}{j^\delta}\ . \tag{5.23}$$

To estimate the first term we use the mean value theorem (for Whitney smooth function); to simplify the notation we denote

$$(\omega,\omega')=\{\nu\in\Omega_{0\gamma}\ :\ \exists t\in(0,1)\text{ with }\nu=t\omega+(1-t)\omega'\}.$$

So we have

$$\begin{aligned}\left|\Delta[\tilde z(\lambda_i^v)-\tilde z(\lambda_j^v)]\right| &\leq\sup_{\nu\in(\omega,\omega')}\left|(\omega'-\omega)\cdot\frac{\partial}{\partial\omega}\left(\tilde z(\lambda_i^v,\nu)-\tilde z(\lambda_j^v,\nu)\right)\right|\\ &=\sup_{\nu\in(\omega,\omega')}\left|(\omega'-\omega)\cdot\left[\frac{\partial\tilde z}{\partial\omega}(\lambda_i^v,\nu)-\frac{\partial\tilde z}{\partial\omega}(\lambda_j^v,\nu)\right]\right|\\ &\leq|\Delta\omega|\left|\lambda_i^v-\lambda_j^v\right|\sup_{\nu\in(\omega,\omega')}\sup_{\lambda\in(\lambda_j^v,\lambda_i^v)}\left|\frac{\partial^2\tilde z}{\partial\lambda\partial\omega}(\lambda,\nu)\right|\\ &\preceq|\Delta\omega|\left|\lambda_i^v-\lambda_j^v\right|.\end{aligned}$$

Adding the estimate of the other two terms one gets (5.21).

We come to (5.22). Define

$$\Omega^{(0)}_\gamma:=\Omega_{0\gamma}-\bigcup_{ijk}\mathcal{R}_{ijk}(\lambda^{(0)},\gamma/(1+|k|^\tau)).$$

In order to estimate the above set we separate the case $d>1$ and the case $d=1$. Consider first $d>1$; then by Lemma A.4, the measure of $\mathcal{R}_{ijk}(\lambda^{(0)},\gamma/(1+|k|^\tau))$ is estimated by (A.6) with $\alpha=\gamma/(1+|k|^\tau)$. We fix k and estimate the cardinality of the i,j's such that the set $\mathcal{R}_{ijk}$ is not empty. By (A.4), exploiting the fact that

$$\left|i^d-j^d\right|\succeq\left(i^{d-1}+j^{d-1}\right)|i-j|\geq(i^{d-1}+j^{d-1})\ ,$$

such a cardinality is estimated by $|k|^{2/(d-1)}$, so we have

$$\left|\bigcup_{ijk}\mathcal{R}_{ijk}(\lambda^{(0)},\gamma/(1+|k|^\tau))\right|\preceq\sum_{k\in\mathbb{Z}^n}\frac{\gamma|k|^{2/(d-1)}}{1+|k|^\tau}\preceq\gamma\ ,$$

which concludes the proof in the case $d>1$.

The case $d=1$ is slightly more complicated. In this case we have $\lambda_j^v = j+\frac{1}{2}$, so that the cardinalty above is infinite.

First we write $i=j+m$, so that one has

$$\lambda_i^{(0)} - \lambda_j^{(0)} = m + \epsilon(f(j+m+1/2) - f(j+1/2)) + \epsilon\left(\frac{\nu_{j+m}}{(j+m)^\delta} - \frac{\nu_j}{j^\delta}\right) , \tag{5.24}$$

where $f = z+\tilde{z}$. Now, by the mean value theorem, there exists $\bar{E} \in (j+1/2, j+m+1/2)$ s.t.

$$|f(j+m+1/2) - f(j+1/2)| = |f'(\bar{E})m| \le \frac{2}{j^{1-\tilde{\beta}/2}} m.$$

Let C be the constant in (5.6), and define

$$\delta_j := \frac{2}{j^{1-\tilde{\beta}/2}} + 2\frac{C}{j^\delta}$$

so that $|s_{ijk} - m - \omega \cdot k| \le m\delta_j$. Now define the sets

$$\mathcal{Q}_{mjk} = \left\{\omega \in \Omega_{1\gamma} \ : \ |m+\omega\cdot k| < \frac{\gamma m}{1+|k|^\tau} + m\delta_j\right\} , \tag{5.25}$$

and remark that $\mathcal{R}_{ijk} \subset \mathcal{Q}_{mjk}$ and also $\mathcal{Q}_{mjk} \subset \mathcal{Q}_{mj'k}$ if $j > j'$. Exploiting this remark we take some j_*, fix k and proceed as follows:

$$\bigcup_{ij} \mathcal{R}_{ijk}(\lambda^{(0)}, \alpha) \subset \left(\bigcup_{i-j=m, j<j_*} \mathcal{R}_{ijk}\right) \cup \left(\bigcup_m \mathcal{Q}_{mj_*k}\right) ; \tag{5.26}$$

then (by Lemma A.4), remarking that (A.4) implies $|m| \preceq |k|$ one has

$$\left|\left(\bigcup_{i-j=m, j<j_*} \mathcal{R}_{ijk}\right)\right| \preceq \alpha j_* |k|.$$

Furthermore, one has that the set $\mathcal{Q}_{mjk}$ is just the set $\mathcal{R}_{ijk}$ with $\lambda_j = j$ and $\alpha = \gamma/(1+|k|^\tau) + \delta_j$. It follows that

$$\left|\bigcup_m \mathcal{Q}_{mj_*k}\right| \preceq |k| \left(\frac{\gamma}{1+|k|^\tau} + \delta_{j_*}\right) ,$$

and therefore the measure of (5.26) is estimated by

$$|k|\left(\frac{\gamma(j_*+1)}{1+|k|^\tau} + \delta_{j_*}\right) \simeq |k|\left(\frac{\gamma j_*}{1+|k|^\tau} + \frac{1}{j_*^{\tilde{\delta}}}\right),$$

where $\tilde{\delta} = \min\left\{\delta; 1-\tilde{\beta}/2\right\}$; choosing $j_* = \left(\frac{1+|k|^\tau}{\gamma}\right)^{1/(\tilde{\delta}+1)}$, inserting in the above estimate and summing over k one gets the thesis. □

Corollary 5.4. *The transformation U_1 transform $A_0 + \epsilon R$ into*

$$A^{(0)} + \epsilon R_0 , \tag{5.27}$$

where

$$R_0 := U_1^{-1} R U_1 \in \mathcal{L}ip_\rho(\Omega_\gamma^{(0)}; C^\ell(\mathbb{T}^n; B(\mathcal{H}^{s-\kappa}; \mathcal{H}^s))) \quad \forall \ell. \tag{5.28}$$

6. Analytic KAM theory

In this section we prove the KAM theorem for analytic perturbations of $A^{(0)}$. The procedure is essentially identical to the one developed in [BG01] (which is actually a small modification of [Pös96]), except that here we take advantage of the fact that the perturbation is smoothing, so everything is slightly simpler.

In the previous section we fixed a positive arbitrary s; now we also fix a positive (large) κ, then we define the following norms of operators and of operator valued functions of $\omega \in \tilde{\Omega}$ (with $\tilde{\Omega}$ closed), and of $\phi \in \mathbb{T}^n_r$. Here and below $\mathbb{T}^n_r$ is the set of the angles belonging to the complexified torus and fulfilling $|\mathrm{Im}\phi_j| < r$.

Let $F : \mathbb{T}^n_r \mapsto B(\mathcal{H}^{s-\kappa}, \mathcal{H}^s)$ be an analytic map. We define

$$\|F\|_r := \sup_{\phi\in\mathbb{T}^n_r} \|F(\phi)\|_{B(\mathcal{H}^{s-\kappa},\mathcal{H}^s))} \, . \tag{6.1}$$

If F depends also in a Lipschitz way on $\omega \in \widetilde{\Omega}$, we still denote

$$\|F\|_r := \sup_{\omega\in\tilde{\Omega}} \sup_{\phi\in\mathbb{T}^n_r} \|F(\phi,\omega)\|_{B(\mathcal{H}^{s-\kappa},\mathcal{H}^s))} \ , \tag{6.2}$$

and we define

$$\|F\|^{\mathcal{L}}_r := \sup_{\omega\neq\omega'\in\tilde{\Omega}} \frac{\|F(\omega) - F(\omega')\|_r}{|\omega - \omega'|} . \tag{6.3}$$

Definition 6.1. An analytic map F which is Lipschitz dependent on $\omega \in \widetilde{\Omega}$ will be said to be Lipschitz analytic.

6.1. **Squaring the order of the perturbation.** Consider a Hamiltonian of the form

$$H = A + P(\omega t, \omega) \ , \quad A = \mathrm{diag}(\lambda_j(\omega)) \ , \quad \|P\|_r \, , \|P\|^{\mathcal{L}}_r \ll 1. \tag{6.4}$$

We look for a selfadjoint operator $X = X(\omega t)$ with the property that the transformation T_X that it generates according to Definition 3.3 transforms H into

$$H^+ = A^+ + P^+(\omega t, \omega) \ , \quad A^+ = \mathrm{diag}(\lambda^+_j(\omega)) \ , \tag{6.5}$$

with P^+ having a size which is essentially the square of that of P.

By Lemma 3.2 one has

$$H^+ = A \tag{6.6}$$
$$- \mathrm{i}[A; X] - \dot{X} + P \tag{6.7}$$
$$+ Lie_X A - (A - \mathrm{i}[A; X]) \tag{6.8}$$
$$+ Lie_X P - P \tag{6.9}$$
$$+ Y_X + \dot{X}. \tag{6.10}$$

So, we look for an X solving the "quantum homological equation":

$$-\mathrm{i}[A; X] - \dot{X} + P = [P] \ , \quad [P] := \mathrm{diag}(P_{jj}) \ , \tag{6.11}$$

in order to get the wanted result with

$$A^+ = A + [P] \ , \quad P^+ = (6.8) + (6.9) + (6.10). \tag{6.12}$$

Lemma 6.2. *Fix positive constants $\Gamma, \mathcal{K}_1$. Assume that there exists a set $\Omega_\gamma \subset \widetilde{\Omega}$ s.t. for all $\omega \in \Omega_\gamma$ one has*

$$|\lambda_i(\omega) - \lambda_j(\omega) + \omega \cdot k| \geq \frac{\gamma \langle i^d - j^d \rangle}{1 + |k|^\tau}\ , \quad |i-j| + |k| \neq 0, \tag{6.13}$$

$$\left| \frac{\Delta(\lambda_i - \lambda_j)}{\Delta\omega} \right| \leq \mathcal{K}_{1\lambda} \left| i^d - j^d \right| , \tag{6.14}$$

with some

$$\gamma \geq \Gamma\ , \quad \mathcal{K}_{1\lambda} < \mathcal{K}_1\ . \tag{6.15}$$

Then (6.11) *has an analytic Lipschitz solution X defined on Ω_γ and fulfilling*

$$\|X\|_{r-\sigma} \preceq \frac{\|P\|_r}{\sigma^{n+\tau}}\ , \tag{6.16}$$

$$\|X\|^{\mathcal{L}}_{r-\sigma} \preceq \frac{\|P\|^{\mathcal{L}}_r}{\sigma^{n+\tau}} + \frac{\|P\|_r}{\sigma^{n+2\tau+1}}, \tag{6.17}$$

where the constants depend on Γ, τ, n.

Proof. The proof is standard. We insert it only for the sake of completeness. Expanding the equation (6.11) in Fourier series

$$X_k := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} X(\phi) e^{-\mathrm{i}k\cdot\phi} d\phi$$

one gets

$$\mathrm{i}[A, X_k] - \mathrm{i}\omega \cdot k X_k = P_k - [P_k]\ ;$$

taking the ij element of the matrix one gets that X can be defined by

$$X_{kij} := \frac{P_{kij}}{\mathrm{i}(\lambda_i - \lambda_j + \omega \cdot k)}\ . \tag{6.18}$$

Note now that one has

$$\|P_k\|_{B(\mathcal{H}^{s-\kappa};\mathcal{H}^s)} \leq \|P\|_r\, e^{-|k|r}\ , \quad \frac{\|\Delta P_k\|_{B(\mathcal{H}^{s-\kappa};\mathcal{H}^s)}}{|\Delta\omega|} \leq \|P\|^{\mathcal{L}}_r\, e^{-|k|r}\ , \tag{6.19}$$

so that

$$|X_{kij}| \preceq \frac{|P_{kij}|(1+|k|^\tau)}{\gamma\langle i^d - j^d\rangle}.$$

Applying Lemma C.1, one gets

$$\|X_k\|_{r-\sigma} \preceq \frac{\|P_k\|\,(1+|k|^\tau)}{\gamma} \preceq \frac{\|P\|_r\,(1+|k|^\tau)e^{-r|k|}}{\gamma} \tag{6.20}$$

and

$$\|X\|_{r-\sigma} \preceq \frac{\|P\|_r}{\gamma} \frac{1}{\sigma^{n+\tau}} \left[\sigma^n \sum_{k\in\mathbb{Z}^n} (1 + |\sigma k|^\tau) e^{-\sigma|k|} \right]\ , \tag{6.21}$$

and remarking that the term in square bracket is just the Riemann sum for the integral of the function $(1+|y|^\tau)e^{-|y|}$ one gets (6.16).

To get (6.17) write

$$|\Delta X_{kij}| \leq \left| \Delta \frac{P_{kij}}{s_{ijk}} \right| \leq \left| \frac{\Delta P_{kij}}{s_{ijk}} \right| + |P_{kij}| \left| \Delta \frac{1}{s_{ijk}} \right|\ . \tag{6.22}$$

The first addendum is estimated exactly as before. Concerning the second one, one has

$$\left|\Delta\frac{1}{s_{ijk}}\right| = \left|\frac{s_{ijk}(\omega')-s_{ijk}(\omega)}{s_{ijk}(\omega')s_{ijk}(\omega)}\right| \leq \frac{|k||\Delta\omega| + |\Delta(\lambda_i'-\lambda_j')|}{s_{ijk}(\omega')s_{ijk}(\omega)} \tag{6.23}$$

$$\leq |\Delta\omega|(1+|k|^\tau)^2\frac{|k|+\mathcal{K}_{1\lambda}\left|i^d-j^d\right|}{\gamma^2\langle i^d-j^d\rangle^2} \tag{6.24}$$

$$\leq |\Delta\omega|(1+|k|^\tau)^2(1+|k|)\left|i^d-j^d\right|\frac{(1+\mathcal{K}_{1\lambda})}{\gamma^2\langle i^d-j^d\rangle^2} \tag{6.25}$$

$$\leq |\Delta\omega|\frac{(1+|k|^\tau)^2(1+|k|)}{\gamma^2\langle i^d-j^d\rangle}. \tag{6.26}$$

Then, proceeding exactly as in the previous case one gets the thesis. $\square$

Exploiting the Lemmas B.2, B.3 and B.4, it is immediate to get the following result:

Lemma 6.3. *Under the assumptions of Lemma* 6.2, *there exists a constant C_* s.t. if*

$$\|X\|_{r-\sigma} \leq C_*,$$

then one has

$$\left\|P^+\right\|_{r-\sigma} \preceq \frac{\|P\|_r}{\sigma^{2n+2\tau+1}}\|P\|_r \ , \quad \left\|P^+\right\|^{\mathcal{L}}_{r-\sigma} \preceq \frac{\|P\|_r}{\sigma^{2n+2\tau+1}}\left(\|P\|^{\mathcal{L}}_r + \frac{\|P\|_r}{\sigma^{\tau+1}}\right) \ . \tag{6.27}$$

Furthermore one has

$$\left\|\mathbf{1}-e^{-iX}\right\|_{r-\sigma} \preceq \frac{\|P\|_r}{\sigma^{n+\tau}} \ , \quad \left\|\mathbf{1}-e^{-iX}\right\|^{\mathcal{L}}_{r-\sigma} \preceq \frac{\|P\|^{\mathcal{L}}_r}{\sigma^{n+\tau}} + \frac{\|P\|_r}{\sigma^{n+2\tau+1}}. \tag{6.28}$$

In order to be able to iterate the construction we still have to show that the new eigenvalues also fulfill a Diophantine inequality.

Lemma 6.4. *Fix two constants $\mathcal{K}_0$ and $\mathcal{K}_1$ fulfilling*

$$\mathcal{K}_1 \leq \frac{\mathcal{K}_0}{8}. \tag{6.29}$$

Assume $\kappa > 1$ and $\tau > \left(n-\kappa^{-1}\right)/(1-\kappa^{-1})$. Assume also that the eigenvalues λ_j fulfill the estimates

$$|\lambda_i - \lambda_j| \geq \mathcal{K}_{0\lambda}\left|i^d-j^d\right| \ , \tag{6.30}$$

$$\left|\frac{\Delta(\lambda_i-\lambda_j)}{\Delta\omega}\right| \leq \mathcal{K}_{1\lambda}\left|i^d-j^d\right| \ , \tag{6.31}$$

and (6.13). *Fix some K fulfilling $1+K^\tau \leq \gamma/\|P\|_r$; then the eigenvalues λ_j^+ fulfill* (6.30) *and* (6.31) *with new constants given by*

$$\mathcal{K}^+_{0\lambda} := \mathcal{K}_{0\lambda} - 2\|P\|_r \ , \tag{6.32}$$

$$\mathcal{K}^+_{1\lambda} := \mathcal{K}_{1\lambda} + 2\|P\|^{\mathcal{L}}_r \ . \tag{6.33}$$

Assume furthermore that

$$\mathcal{K}^+_{0\lambda} \geq \mathcal{K}_0 \ , \quad \mathcal{K}^+_{1\lambda} \leq \mathcal{K}_1.$$

Then there exists a measurable set Ω^+_γ *and a positive constant* d_1 *such that for any* $\omega \in \Omega^+_{\gamma^+}$ *one has*

$$\left|\lambda_i^+(\omega) - \lambda_j^+(\omega) + \omega \cdot k\right| \geq \frac{\gamma^+ \langle i^d - j^d \rangle}{1 + |k|^\tau} \ , \quad |i-j| + |k| \neq 0, \tag{6.34}$$

with

$$|\Omega_\gamma - \Omega^+_{\gamma^+}| \preceq \gamma^+ K^{-d_1} \tag{6.35}$$

and

$$\gamma^+ = \gamma - 2(1 + K^\tau) \left\|P\right\|_r . \tag{6.36}$$

The constant in (6.35) *depends on* τ, κ, $\mathcal{K}_0$, $\mathcal{K}_1$ *and* d_1*.*

Proof. Denote $\epsilon_1 := \left\|P\right\|_r$ and $\epsilon_2 := \left\|P\right\|_r^{\mathcal{L}}$. We have

$$\lambda_j^+ = \lambda_j + \frac{\nu_j(\omega)}{j^\kappa} \ , \quad |\nu_j| \leq \epsilon_1 \ , \quad \left|\frac{\Delta \nu_j}{\Delta\omega}\right| \leq \epsilon_2.$$

Therefore (6.32) and (6.33) immediately follow. Furthermore one has

$$\begin{aligned} \left|\lambda_i^+(\omega) - \lambda_j^+(\omega) + \omega \cdot k\right| &\geq |\lambda_i(\omega) - \lambda_j(\omega) + \omega\cdot k| - \frac{2\epsilon_1}{j^\kappa} \\ &\geq \frac{\langle i^d - j^d\rangle}{1+|k|^\tau}\left(\gamma - \frac{2\epsilon_1}{j^\kappa}\frac{1+|k|^\tau}{\langle i^d - j^d\rangle}\right) \ , \end{aligned} \tag{6.37}$$

which is automatically larger then the r.h.s. of (6.34) if

$$\frac{1+|k|^\tau}{j^\kappa \langle i^d - j^d\rangle} \leq 1 + K^\tau. \tag{6.38}$$

In turn this is automatic if $|k| \leq K$. So we now consider the case $|k| > K$. In that case (6.38) is again automatic if $j^\kappa > |k|^{\tau-1}\frac{\mathcal{K}_{0\lambda}}{4}$ (exploiting (A.4)).

So, fix a value of k with $|k| > K$ and consider the case

$$j \leq |k|^{\frac{\tau-1}{\kappa}} \left(\frac{\mathcal{K}_{0\lambda}}{4}\right)^{\frac{1}{\kappa}} . \tag{6.39}$$

Fix a value of j fulfilling (6.39), then (from (A.4)) the set

$$\mathcal{R}^+_{ijk} := \mathcal{R}_{ijk}(\lambda^+, \gamma^+/(1+|k|^\tau))$$

is not empty only for a set of i's which has at most a cardinality proportional to $|k|$. We have (by Lemma A.4)

$$\left|\bigcup_{i,j}\mathcal{R}^+_{ijk}\right| \preceq \sum_{i,j}\frac{1}{\mathcal{K}^+_{0\lambda}}\frac{\gamma^+}{1+|k|^\tau} \preceq \frac{1}{\mathcal{K}^+_{0\lambda}}\frac{\gamma^+}{1+|k|^\tau}|k||k|^{\frac{\tau-1}{\kappa}} \ , \tag{6.40}$$

where the sum is restricted to the j's fulfilling (6.39) and the i's for which $\mathcal{R}^+_{ijk}$ is not empty. Summing over k with $|k| > K$ one gets the result. □

6.2. **Iterative lemma and the analytic KAM theorem.** We are now in the position of stating the iterative lemma which is a direct consequence of the results of the above subsection. Such a lemma yields the analytic KAM result that we need.

To start with take a positive r and consider a quantum Hamiltonian of the form

$$H^{(0)} = A^{(0)} + P^{(0)} \ , \tag{6.41}$$

with $A^{(0)}$ given by (5.4) and $P^{(0)}$ an analytic Lipschitz map fulfilling

$$\left\| P^{(0)} \right\|_r \leq \epsilon_1^{(0)} \ , \quad \left\| P^{(0)} \right\|_r^{\mathcal{L}} \leq \epsilon_2^{(0)} \ , \tag{6.42}$$

with some positive (small) $\epsilon_1^{(0)}$ and $\epsilon_2^{(0)}$.

The next lemma is a direct consequence of Lemmas 6.3 and 6.4 applied iteratively by taking $K = K^{(l)}$ as defined by the first of (6.45).

Lemma 6.5. *Fix $0 < \vartheta < 1$, $\mathcal{K}_0$, $\mathcal{K}_1$ and Γ with $\mathcal{K}_0 \geq 8\mathcal{K}_1$ and define*

$$\sigma_l := \frac{(1-\vartheta)r}{2^l} \ , \quad r_l := r - \sum_{i=1}^{l} \sigma_i . \tag{6.43}$$

Then there exist positive constants d_2, d_3 s.t. if one defines iteratively (for $l \geq 0$) the sequences of constants

$$\epsilon_1^{(l+1)} \simeq \frac{(\epsilon_1^{(l)})^2}{(\sigma^{(l)})^{2n+2\tau+1}} \ , \quad \epsilon_2^{(l+1)} \simeq \frac{\epsilon_1^{(l)}}{(\sigma^{(l)})^{2n+2\tau+1}} \left(\epsilon_2^{(l)} + \frac{\epsilon_1^{(l)}}{(\sigma^{(l)})^{\tau+1}} \right) \ , \tag{6.44}$$

$$K^{(l)} = (\epsilon_1^{(l)})^{-1/2\tau} \ , \quad \gamma^{(l+1)} = \gamma^{(l)} - (\epsilon_1^{(l)})^{d_2} \ , \quad \delta^{(l)} \simeq \gamma^{(l)} (\epsilon_1^{(l)})^{d_3} \ , \tag{6.45}$$

$$\mathcal{K}_{0\lambda}^{(l+1)} = \mathcal{K}_{0\lambda}^{(l)} - 2\epsilon_1^{(l)} \ , \quad \mathcal{K}_{1\lambda}^{(l+1)} = \mathcal{K}_{1\lambda}^{(l)} + 2\epsilon_1^{(l)}, \tag{6.46}$$

and for any $l \geq 0$ the inequalities

$$\frac{\epsilon_1^{(l)}}{(\sigma^{(l)})^{n+\tau}} \preceq 1 \ , \quad \frac{\epsilon_2^{(l)}}{(\sigma^{(l-1)})^{n+\tau}} + \frac{\epsilon_1^{(l)}}{(\sigma^{(l)})^{n+2\tau+1}} \preceq 1, \tag{6.47}$$

$$\gamma^{(l)} \geq \Gamma \ , \quad \mathcal{K}_{0\lambda}^{(l)} \geq \mathcal{K}_0 \ , \quad \mathcal{K}_{1\lambda}^{(l)} \leq \mathcal{K}_1 \tag{6.48}$$

hold, then the following holds true: for any l there exists a measurable set $\Omega_{\gamma^{(l)}}^{(l)}$ and a Lipschitz analytic map $X^{(l)}$ defined on $\Omega_{\gamma^{(l)}}^{(l)}$ with the property that $T_{X^{(l)}}$ is well defined and one has

$$T_{X^{(l)}} H^{(l-1)} = H^{(l)} = A^{(l)} + P^{(l)} \ , \quad l \geq 1 \ , \tag{6.49}$$

with $A^{(l)} = \mathrm{diag}(\lambda_j^{(l)})$*. Furthermore the following estimates hold:*

$$\left|\Omega^{(l-1)}_{\gamma^{(l-1)}} - \Omega^{(l)}_{\gamma^{(l)}}\right| \leq \delta^{(l)} \ , \tag{6.50}$$

$$\left\|P^{(l)}\right\|_{r_l} \leq \epsilon_1^{(l)} \ , \quad \left\|P^{(l)}\right\|^{\mathcal{L}}_{r_l} \leq \epsilon_2^{(l)} \ , \tag{6.51}$$

$$\left|\lambda_i^{(l)} - \lambda_j^{(l)}\right| \geq \mathcal{K}^{(l)}_{0\lambda}\left|i^d - j^d\right| \ , \quad \left|\frac{\Delta(\lambda_i^{(l)} - \lambda_j^{(l)})}{\Delta\omega}\right| \leq \mathcal{K}^{(l)}_{1\lambda}\left|i^d - j^d\right| \ , \tag{6.52}$$

$$\left|\lambda_i^{(l)} - \lambda_j^{(l)} + \omega\cdot k\right| \geq \frac{\gamma^{(l)}}{1+|k|^\tau}\langle i^d - j^d\rangle \ , \quad |i-j|+|k| \neq 0, \tag{6.53}$$

$$\left\|X^{(l)}\right\|_{r_l} \preceq \frac{\epsilon_1^{(l-1)}}{(\sigma^{(l-1)})^{n+\tau}} \ , \quad \left\|X^{(l)}\right\|^{\mathcal{L}}_{r_l} \preceq \frac{\epsilon_2^{(l-1)}}{(\sigma^{(l-1)})^{n+\tau}} + \frac{\epsilon_1^{(l-1)}}{(\sigma^{(l-1)})^{n+2\tau+1}}, \tag{6.54}$$

$$\left\|\mathbf{1} - e^{-\mathrm{i}X^{(l)}}\right\|_{r_l} \preceq \left\|X^{(l)}\right\|_{r_l} \ , \quad \left\|\mathbf{1} - e^{-\mathrm{i}X^{(l)}}\right\|^{\mathcal{L}}_{r_l} \preceq \left\|X^{(l)}\right\|^{\mathcal{L}}_{r_l}. \tag{6.55}$$

Theorem 6.6. *Consider the quantum Hamiltonian* (6.41) *defined and Lipschitz on a set* $\Omega^{(0)}_{\gamma^{(0)}}$ *s.t.*

$$\left|\Omega - \Omega^{(0)}_{\gamma^{(0)}}\right| \leq \Upsilon^{(0)} \tag{6.56}$$

with some positive $\Upsilon^{(0)}$*. Fix positive numbers* $\mathcal{K}_0$, $\mathcal{K}_1$, Γ, τ, ϑ *fulfilling*

$$\tau > \frac{n-\kappa^{-1}}{1-\kappa^{-1}} \ , \quad 0 < \vartheta < 1 \ , \quad \mathcal{K}_0 > 8\mathcal{K}_1. \tag{6.57}$$

Assume that, for some $0 < r \leq 1$ *and some positive* ς *one has*

$$\left\|P^{(0)}\right\|_r \leq \varsigma r^b \ , \quad r^{\tau+1}\left\|P^{(0)}\right\|^{\mathcal{L}}_r \leq \varsigma r^b \ , \quad b := 2n + 2\tau + 1. \tag{6.58}$$

Then there exist positive constants $\varsigma_*, C_\Gamma, C_0, C_\Omega, C_U$ *s.t., if* $|\varsigma| < \varsigma_*$ *and the eigenvalues* $\lambda_j^{(0)}$ *fulfill*

$$\left|\lambda_i^{(0)} - \lambda_j^{(0)}\right| \geq \mathcal{K}^{(0)}_{0\lambda}\left|i^d - j^d\right|, \tag{6.59}$$

$$\left|\frac{\Delta(\lambda_i^{(0)} - \lambda_j^{(0)})}{\Delta\omega}\right| \leq \mathcal{K}^{(0)}_{1\lambda}\left|i^d - j^d\right| \ , \tag{6.60}$$

$$\left|\lambda_i^{(0)} - \lambda_j^{(0)} + \omega\cdot k\right| \geq \frac{\gamma^{(0)}}{1+|k|^\tau}\langle i^d - j^d\rangle \ , \ |i-j|+|k| \neq 0, \tag{6.61}$$

with constants s.t.

$$\mathcal{K}^{(0)}_{0,\lambda} - C_0 r^b\varsigma > \mathcal{K}_0 \ , \quad \mathcal{K}^{(0)}_{1\lambda} + C_0 r^{b+\tau+1}\varsigma < \mathcal{K}_1 \ , \quad \gamma^{(0)} - C_\Gamma(r^b\varsigma)^{d_2} \geq \Gamma \ . \tag{6.62}$$

Then there exists a measurable set $\Omega^{(\infty)}_{\gamma^{(\infty)}}$ *and a Lipschitz analytic map* U *defined on* $\Omega^{(\infty)}_{\gamma^{(\infty)}}$*, with* $U(\phi,\omega)$*,* L^2 *unitary, s.t. the transformation* $U(\omega t,\omega)\psi' = \psi$ *transforms the system* (6.41) *into*

$$\mathrm{i}\dot{\psi}' = A^{(\infty)}\psi' \ , \quad A^{(\infty)} := \mathrm{diag}(\lambda_j^{(\infty)}). \tag{6.63}$$

Furthermore the following estimates hold:

$$\left|\lambda_i^{(\infty)} - \lambda_j^{(\infty)}\right| \geq (\mathcal{K}_{0\lambda}^{(0)} - C_0 r^b \varsigma)\left|i^d - j^d\right| , \tag{6.64}$$

$$\left|\frac{\Delta(\lambda_i^{(\infty)} - \lambda_j^{(\infty)})}{\Delta\omega}\right| \leq (\mathcal{K}_{1\lambda}^{(0)} + C_0 r^{b+\tau+1}\varsigma)\left|i^d - j^d\right| \ , \tag{6.65}$$

$$\left|\lambda_i^{(\infty)} - \lambda_j^{(\infty)} + \omega\cdot k\right| \geq \frac{\gamma^{(0)} - C_\Gamma (r^b\varsigma)^{d_2}}{1+|k|^\tau}\langle i^d - j^d\rangle \ , \ |i-j| + |k| \neq 0, \tag{6.66}$$

$$|\Omega_{\gamma^{(0)}}^{(0)} - \Omega_{\gamma^{(\infty)}}^{(\infty)}| \leq \Upsilon^{(0)}(1 + C_\Omega \gamma^{(0)}(\varsigma r^b)^{d_3}) \ , \tag{6.67}$$

$$\|\mathbf{1} - U\|_{\theta r} \leq C_U \varsigma r^{b-(n+\tau)} \ , \quad r^{\tau+1}\|\mathbf{1} - U\|_{\theta r}^{\mathcal{L}} \leq C_U \varsigma r^{b-(n+\tau)}. \tag{6.68}$$

Proof. We apply Lemma 6.5. To this end we define $\epsilon_1^{(l)} := \varsigma^{(l)} r^b$ and $\epsilon_2^{(l)} := \varsigma^{(l)} r^{b-\tau-1}$ with $\varsigma^{(0)} := \varsigma$. We fix $\vartheta = \frac{1}{2}$, then all the constants (6.44)–(6.46) are defined by the recursion. We first analyze (6.44) which take the form

$$\epsilon_1^{(l+1)} \preceq \frac{\epsilon_1^{(l)} 2^{bl}}{r^b}\epsilon_1^{(l)} \ , \quad \epsilon_2^{(l+1)} \preceq \frac{\epsilon_1^{(l)} 2^{bl}}{r^b}\left(\epsilon_2^{(l)} + \frac{\epsilon_1^{(l)} 2^{l(\tau+1)}}{r^{\tau+1}}\right) \ , \tag{6.69}$$

which in turn can be reformulated in terms of $\varsigma^{(l)}$:

$$\varsigma^{(l+1)} r^b \preceq \frac{\varsigma^{(l)} r^b 2^{bl}}{r^b}\varsigma^{(l)} r^b \iff \varsigma^{(l+1)} \preceq 2^{bl}\left(\varsigma^{(l)}\right)^2 ,$$

$$\varsigma^{(l+1)} r^{b-\tau-1} \preceq \frac{\varsigma^{(l)} r^b 2^{bl}}{r^b}\varsigma^{(l)}\left(\varsigma^{(l)} r^{b-(\tau+1)} + \frac{\varsigma^{(l)} 2^{l(\tau+1)} r^b}{r^{\tau+1}}\right)$$

$$\iff \varsigma^{(l+1)} \preceq 2^{(b+\tau+1)l}\left(\varsigma^{(l)}\right)^2 \ ,$$

which is solved, thanks to Lemma C.3, by defining

$$\varsigma^{(l)} := \frac{1}{c_3 2^{(b+\tau+1)l}}\left(2^{b+\tau+1} c_3 \varsigma\right)^{2^l} \ , \quad l \geq 1 \ , \tag{6.70}$$

with c_3 the nonwritten constant in the definition of the iterative estimates. Then $\varsigma^{(l)}$ tends to zero provided

$$\varsigma \leq \frac{1}{c_3 2^{b+\tau+1}}. \tag{6.71}$$

Then the inequalities (6.62) ensure that the assumptions (6.47) and (6.48) of Lemma 6.5 hold. By taking the limit $l \to \infty$ one gets the result. □

Remark 6.7. Consider a Hamiltonian of the form

$$H' = H^{(0)} + P' = A^{(0)} + P^{(0)} + P' \ . \tag{6.72}$$

Then the transformation $U\psi' = \psi$ transforms it into

$$A^{(\infty)} + U^{-1}P'U \ , \tag{6.73}$$

and by the estimate (6.68), one has

$$\left\|U^{-1}P'U - P'\right\|_{\vartheta r} \leq 2C_U \varsigma r^{b-(n+\tau)} \ , \tag{6.74}$$

and a similar estimate for the Lipschitz norm.

7. KAM with finite smoothness (end of the proof of Theorem 2.4)

First we define the standard C^ℓ (Hölder) norms of functions on $\mathbb{T}^n$ (we use here a definition slightly different from that used for Whitney smooth functions in order to use tools developed in [Sal04]).

Let $0 < \mu < 1$, and let F be a Hölder function from $\mathbb{T}^n$ to $B(\mathcal{H}^{s-\kappa}, \mathcal{H}^s)$ with Hölder exponent μ. Then we put

$$|F|_{C^\mu} := \sup_{0<|\phi-\phi'|<1} \frac{\|F(\phi) - F(\phi')\|}{|\phi-\phi'|^\mu} + \sup_{\phi\in\mathbb{T}^n} \|f\| \ , \tag{7.1}$$

$$|F|_{C^\ell} := \sum_{|\alpha|\le\ell} |\partial^\alpha F|_{C^\mu} \ , \quad \mu := \ell - \lfloor \ell \rfloor. \tag{7.2}$$

In order to extend a C^ℓ function to a complex neighborhood of $\mathbb{T}^n$ we will use the following polynomials:

$$P_{F,\ell}(\phi,\theta) := \sum_{|\alpha|\le\ell} \frac{1}{\alpha!} \partial^\alpha F(\phi)\theta^\alpha \ , \tag{7.3}$$

and remark that

$$\sup_{\substack{|\phi|\in\mathbb{T}^n \\ |\theta|\le r}} \|P_{F,\ell}(\phi,\theta)\| \le |F|_{C^\ell} \tag{7.4}$$

for any $0 < r \le 1$. Then the following smoothing lemma (from [Sal04]) holds.

Lemma 7.1 (Lemma 3 of [Sal04])**.** *There is a family of convolution operators*

$$S_r f(\phi) = \frac{1}{r^n} \int_{\mathbb{R}^n} K\left(\frac{\phi-\phi'}{r}\right) f(\phi') d^n\phi' \ , \quad 0 < r \le 1 \ , \tag{7.5}$$

from $C^0(\mathbb{R}^n)$ into the space of entire analytic functions on $\mathbb{C}^n$ with the following property. For any $\ell \ge 0$ there exists a constant $c(\ell, n) > 0$ such that, for every $\phi \in \mathbb{C}^n$, we have

$$|\mathrm{Im}\,\phi| \le r \implies \left|\partial^\alpha S_r f(\phi) - P_{\partial^\alpha f, \lfloor\ell\rfloor - |\alpha|}(\mathrm{Re}\,\phi; \mathrm{i}\mathrm{Im}\,\phi)\right| \le c\,|f|_{C^\ell}\, r^{\ell-|\alpha|}.$$

Moreover, if f is periodic in ϕ, then $S_r f$ is periodic in Re ϕ *and $S_r f$ is real valued whenever f is real valued. The result holds also for functions with values in Banach spaces.*

A converse of this lemma is given by the following.

Lemma 7.2 (Lemma 4 of [Sal04])**.** *Let $\ell \ge 0$ be real, and let n be a positive integer. Then there exists a constant $c = c(\ell, n) > 0$ with the following property. If $f : \mathbb{R}^n \to \mathbb{R}$ is the limit of a sequence of functions $f_\nu(\phi)$ real analytic in the strips $|\mathrm{Im}\,\phi| \le r_\nu := 2^{-\nu} r_0$, with $0 < r_0 \le 1$ and*

$$f_0 = 0 \ , \quad |f_\nu(\phi) - f_{\nu-1}(\phi)| \le A r_\nu^\ell$$

for $\nu \ge 1$ and $|\mathrm{Im}\,\phi| \le r_\nu$, then $f \in C^m(\mathbb{R}^n)$ for every $m \le \ell$ which is not an integer and moreover

$$|f|_{C^m} \le \frac{cA}{\mu(1-\mu)} r_0^{\ell-m} \ , \quad 0 < \mu := m - \lfloor m \rfloor < 1. \tag{7.6}$$

The proof of the KAM theorem with finite smoothness is based on the repeated application of Theorem 6.6 to the Hamiltonian (5.27). To describe the procedure we first fix the parameters that we will use.

Fix a positive m (which will control the smoothness of the reduction transformation) and some $\ell > m - b$. Define $\gamma = \ell - m - b$, $r_0 := \epsilon^{1/\ell}$, $r_\nu := r_0 2^{-\nu}$ and $R^{(\nu)} := \epsilon S_{r_\nu} R_0$ with S_r the smoothing operator of Lemma 7.1 and R_0 the perturbation defined in (5.27) (note that we inserted ϵ in the definition of $R^{(\nu)}$).

The scheme of the iteration is the following: first we construct the unitary transformation $U^{(0)}$ transforming $A^{(0)} + R^{(0)}$ into a time independent diagonal operator $\tilde{A}^{(1)}$. Then we use $U^{(0)}$ to transform $A^{(0)} + R^{(1)}$. According to Remark 6.7 it transforms such a system into $\tilde{A}^{(1)} + [U^{(1)}]^{-1}(R^{(1)} - R^{(0)})U^{(1)}$, which is a smaller perturbation of a time independent system. After $\nu - 1$ steps we have thus constructed a unitary transformation $\Phi^{(\nu-1)}$ which transforms $A^{(0)} + R^{(\nu-1)}$ into $\tilde{A}^{(\nu-1)} = \mathrm{diag}(\tilde{\lambda}_j^{(\nu-1)})$. Now use $\Phi^{(\nu-1)}$ to transform $A^{(0)} + R^{(\nu)}$. One gets the system

$$\tilde{A}^{(\nu-1)} + [\Phi^{(\nu-1)}]^{-1}(R^{(\nu)} - R^{(\nu-1)})\Phi^{(\nu-1)} , \tag{7.7}$$

to which we apply Theorem 6.6 again. Then one has to check the assumptions of such a theorem and to add estimates showing that the procedure converges.

Theorem 7.3. *Consider the quantum Hamiltonian* (5.27) *defined and Lipschitz on a set* $\Omega^{(0)}_{\gamma^{(0)}}$ *s.t.*

$$\left|\Omega - \Omega^{(0)}_{\gamma^{(0)}}\right| \le \Upsilon^{(0)} \tag{7.8}$$

with some positive $\Upsilon^{(0)} \simeq [\gamma^{(0)}]^a$. *Let* $\mathcal{K}_0$, $\mathcal{K}_1$, Γ, τ, *be the constants fixed in Theorem* 6.6. *Fix* m *in such a way that* $m + n + \tau$ *is not an integer and let* $\ell > m + b$. *Assume that* $R_0 \in Lip_1\left(\Omega^{(0)}_\gamma; C^\ell(\mathbb{T}^n; B(\mathcal{H}^{s-\kappa}, \mathcal{H}^s))\right)$ *(in the following* C^ℓ *Lipschitz, for short) and let* M *be a constant such that*

$$\|R_0\|_{C^\ell} \le M \ , \quad \|R_0\|^{\mathcal{L}}_{C^\ell} \le M. \tag{7.9}$$

Then there exist positive constants ϵ_*, C_1, C'_Ω *s.t. if* $|\epsilon| < \epsilon_*$ *and the eigenvalues* $\lambda_j^{(0)}$ *fulfill* (6.59)–(6.61) *with constants s.t.*

$$\begin{gathered} \mathcal{K}'_{0\lambda} := \mathcal{K}^{(0)}_{0\lambda} - 2C_0 \epsilon^{\frac{m+b}{\ell}} > \mathcal{K}_0 \ , \quad \mathcal{K}'_{1\lambda} := \mathcal{K}^{(0)}_{0\lambda} + 2C_0 \epsilon^{\frac{b+\tau+1+m}{\ell}} < \mathcal{K}_1 \ , \\ \gamma' := \gamma^{(0)} - C_\Gamma C_1 \epsilon^{\frac{b+m}{\ell} d_2} \ge \Gamma \ , \end{gathered} \tag{7.10}$$

then there exist a measurable set $\widetilde{\Omega}^{(\infty)}_{\gamma'}$ *and a* $C^{m+n+\tau}$ *Lipschitz map* $\Phi^{(\infty)}$ *defined on* $\widetilde{\Omega}^{(\infty)}_{\gamma'}$, *with* $\Phi^{(\infty)}(\phi, \omega)$ *unitary as a map on* L^2 *s.t. the transformation* $\Phi^{(\infty)}(\omega t, \omega)\psi' = \psi$ *transforms the system* (5.27) *into*

$$\mathrm{i}\dot{\psi}' = \widetilde{A}^{(\infty)}\psi' \ , \quad \widetilde{A}^{(\infty)} := \mathrm{diag}(\widetilde{\lambda}_j^{(\infty)}). \tag{7.11}$$

Furthermore the eigenvalues $\widetilde{\lambda}_j^{(\infty)}$ *fulfill the estimates* (6.59)–(6.61) *with the new constants defined in* (7.10) *and one has*

$$|\Omega^{(0)}_{\gamma^{(0)}} - \widetilde{\Omega}^{(\infty)}_{\gamma'}| \le C'_\Omega \gamma' \ , \tag{7.12}$$

$$\left\|\mathbf{1} - \Phi^{(\infty)}\right\|_{C^{m'}} \le 2C_U \epsilon^{\frac{m+n+\tau-m'}{\ell}} \tag{7.13}$$

for any $m' < m+n+\tau$ *s.t.* $\frac{m+n+\tau-m'}{\ell}$ *is not an integer.*

Proof. First we remark that (by Lemma 7.1)

$$\begin{aligned} &\sup_{\phi\in\mathbb{T}^n_{r_\nu}} \left\|R^{(\nu)}(\phi) - \epsilon P_{R_0;\ell}(\mathrm{Re}\,\phi)\mathrm{iIm}\,\phi\right\| \le c_1\epsilon M r_\nu^\ell \ , \\ &\sup_{\phi\in\mathbb{T}^n_{r_\nu}} \left\|R^{(\nu)}(\phi) - P_{R_0,\ell}(\mathrm{Re}\,\phi)\mathrm{iIm}\,\phi\right\|^{\mathcal{L}} \le c_1\epsilon M r_\nu^\ell \end{aligned} \tag{7.14}$$

so that

$$\left\|R^{(0)}\right\|_{r_0} \le c_1\epsilon M r_0^\ell + \epsilon M \le 2\epsilon M = 2Mr_0^\ell = 2Mr_0^\gamma r_0^m r_0^b \le r_0^m r_0^b \ , \tag{7.15}$$

provided $2Mr_0^\gamma < 1$ (which is a smallness assumption on ϵ). For the Lipschitz norm an equal estimate holds:

$$r_0^{\tau+1}\left\|R^{(0)}\right\|^{\mathcal{L}}_{r_0} \le c_1\epsilon M r_0^\ell + \epsilon M \le 2\epsilon M = 2Mr_0^\ell = 2Mr_0^\gamma r_0^m r_0^b \le r_0^m r_0^b \tag{7.16}$$

(of course one also has a better estimate, but we do not need it). We also have the following estimates:

$$\begin{aligned} \left\|R^{(\nu)} - R^{(\nu-1)}\right\|_{r_\nu} &\le \left\|R^{(\nu)} - P_{R_0;\ell}(\mathrm{Re}\,\phi)\mathrm{iIm}\,\phi\right\|_{r_\nu} \\ &\quad + \left\|R^{(\nu-1)} - P_{R_0;\ell}(\mathrm{Re}\,\phi)\mathrm{iIm}\,\phi\right\|_{r_{\nu-1}} \qquad (7.17) \\ &\le c_1\epsilon M r_\nu^\ell + c_1\epsilon M r_\nu^\ell 2^{2\ell} = c_1\epsilon M r_\nu^\ell(1+2^{2\ell}). \qquad (7.18) \end{aligned}$$

We now show that for any ν there exists a set $\widetilde{\Omega}^{(\nu)}_{\gamma^{(\nu)}}$ and a Lipschitz analytic transformation $U^{(\nu)}$, defined on it, unitary in L^2 such that, if one defines

$$\Phi^{(\nu)} := U^{(0)}\cdots U^{(\nu)} \ , \tag{7.19}$$

then it transforms $A^{(0)} + R^{(\nu)}$ into a time independent system $A^{(\nu)} = \mathrm{diag}(\tilde{\lambda}_j^{(\nu)})$ with eigenvalues fulfilling (6.59)–(6.61) with constants

(7.20)

$$\widetilde{\mathcal{K}}^{(\nu)}_{0\lambda} = \mathcal{K}^{(0)}_{0\lambda} - C_0 r_0^{b+m}\sum_{j=0}^{\nu}\frac{1}{2^{(b+m)j}} \ , \quad \widetilde{\mathcal{K}}^{(\nu)}_{1\lambda} = \mathcal{K}^{(0)}_{1\lambda} + C_0 r_0^{b+m+\tau+1}\sum_{j=0}^{\nu}\frac{1}{2^{(b+m+\tau+1)j}}$$

$$\widetilde{\gamma}^{(\nu)} = \gamma^{(0)} - C_\Gamma r_0^{(b+m)d_2}\sum_{j=0}^{\nu}\frac{1}{2^{(b+m)j}} \ , \tag{7.21}$$

and furthermore the following estimates hold:

$$\begin{aligned}&\left\|\Phi^{(\nu)}-\mathbf{1}\right\|_{r_{\nu+1}}\le 2C_U r_0^{b_1}\sum_{j=0}^{\nu}\frac{1}{2^{b_1 j}}<4C_U r_0^{b_1}\ ,\\ &r^{\tau+1}\left\|\Phi^{(\nu)}-\mathbf{1}\right\|^{\mathcal{L}}_{r_{\nu+1}}\le 2C_U r_0^{b_1}\sum_{j=0}^{\nu}\frac{1}{2^{b_1 j}}\ ,\end{aligned}\tag{7.22}$$

with $b_1=m+n+\tau$.

Consider the case $\nu=0$. We apply Theorem 6.6 with $r:=r_0$ and $\varsigma:=r_0^m$ to $A^{(0)}+R^{(0)}$. This is possible since the assumptions on the eigenvalues are verified by (7.10). Then, by (6.68) and (6.64)–(6.67) the equations (7.22) and (7.20) hold with $\nu=0$.

Assume now that the result is true for $\nu-1$. Then, as anticipated above, the transformation $\Phi^{(\nu-1)}$ transform $A^{(0)}+R^{(\nu)}$ into the system (7.7), to which we apply Theorem 6.6. To this end, we remark that the assumptions on the eigenvalues are satisfied by the iterative assumption. We just have to add an estimate of the new perturbation. In view of the iterative estimate (7.22) and of (7.18), one has

$$\left\|[\Phi^{(\nu-1)}]^{-1}(R^{(\nu)}-R^{(\nu-1)})\Phi^{(\nu-1)}\right\|_{r_\nu}\le c_1\epsilon M r_\nu^{\ell}(1+2^{2\ell})4\le r_\nu^m r_\nu^b\ ,$$

provided $4C_U r_0^{b_1}<1$ and $c_1\epsilon M r_\nu^{\gamma}(1+2^{2\ell})4<1$, which are smallness assumptions on ϵ. A similar estimate holds for the Lipschitz norm.

Applying Theorem 6.6 one gets the transformation $U^{(\nu)}$ that we need. In particular the iterative estimate of $\Phi^{(\nu)}$ follows from (6.68).

We now have to show that the sequence of transformations $\Phi^{(\nu)}$ converges. To this end we apply Lemma 7.2 to the sequence $f^{(\nu)}:=\Phi^{(\nu)}-\mathbf{1}$. Defining $f^{(-1)}:=\mathbf{1}$, the initial step is fulfilled and one has

$$\left\|f^{(\nu)}-f^{(\nu-1)}\right\|_{r_\nu}=\left\|\left[U^{(\nu)}-\mathbf{1}\right]\Phi^{(\nu-1)}\right\|_{r_\nu}\le 2C_U r_\nu^{b_1}\ ,\tag{7.23}$$

which implies the thesis. □

End of the proof of Theorem 2.4. In order to conclude the proof of Theorem 2.4 one has to show that the measure of the set of the allowed frequencies becomes full as $\epsilon\to 0$. To this end we remark that the statement implies the fact that (once all the other parameters are fixed) for any Γ there exists $\epsilon_*(\Gamma)>0$ s.t. for smaller ϵ the theorem holds. Denote $d_4:=\frac{\ell}{(b+m)d_2}$. For given Γ, define $\gamma^{(0)}:=3\Gamma/2$; then Theorem 7.3 applies provided $\epsilon<\min\left\{C\Gamma^{d_4};\epsilon_*(\Gamma)\right\}$ with a suitable C. Let $\Gamma(\epsilon)$ be the smallest Γ s.t.

$$2\epsilon=\min\left\{C\Gamma^{d_4};\epsilon_*(\Gamma)\right\}\ ;$$

we claim that $\Gamma(\epsilon)$ goes to zero. Indeed, assume by contradiction that this is false, then it means that $\epsilon_*(\Gamma(\epsilon))>0$ (strictly) for all $\epsilon>0$, but this contradicts Theorem 7.3. □

8. Proof of Theorem 2.12

Proof of Theorem 2.12 *in the case $l>1$.* Consider the case of $h=h_0+\epsilon W_{2l}$. The lower order corrections will be added after a first set of transformations.

We start by transforming h using the transformation generated by

$$\chi_1 := \frac{b_1(\omega t) x^{l+1}}{l+1} \ , \quad b_1(\omega t) := \frac{a_2(\omega t)}{2(1+\epsilon a_1(\omega t))}. \tag{8.1}$$

It is easy to see that the flow it generates is

$$\Phi^{\epsilon}_{\chi_1}(x,\xi) = (x, \xi - \epsilon b_1(\omega t) x^l) \ , \tag{8.2}$$

so that, by explicit computation

$$h^{(1)} := h \circ \Phi^{\epsilon}_{\chi_1} = (1+\epsilon a_1(\omega t))\xi^2 + (1+\epsilon c_1(\omega t)) x^{2l} \ , \tag{8.3}$$

with

$$c_1 = a_3 + \epsilon b_1^2 + \epsilon^2 a_1 b_1^2 - \epsilon a_2 b_1.$$

One also has

$$y_x^{(1)} := \int_0^{\epsilon} \dot{\chi}_1 \circ \Phi^{\epsilon - \epsilon_1}_{\chi_1} d\epsilon_1 + \epsilon \mathcal{S}^{-(l+1)} = \epsilon \dot{\chi}_1 + \epsilon \mathcal{S}^{-(l+1)}. \tag{8.4}$$

Note also that (again by explicit computation)

$$f \circ \Phi^{\epsilon}_{\chi_1} \in \mathcal{S}^m \ , \quad \text{whenever} \quad f \in \mathcal{S}^m \ , \tag{8.5}$$

so that, by Theorem 4.10, equation (4.15) holds.

In conclusion one has that the transformed Hamiltonian has the form

$$h^{(1)} - \epsilon \dot{\chi}_1 + \epsilon \mathcal{S}^{-2}. \tag{8.6}$$

We now make a new transformation using

$$\chi_2 := b_2(\omega t) x \xi \ , \quad b_2(\omega t) := \frac{1}{4(l+1)\epsilon} \ln\left(\frac{1+\epsilon a_1}{1+\epsilon c_1}\right) \tag{8.7}$$

(the ϵ in b_2 only plays the role of a parameter), whose flow is given by

$$\phi^{\epsilon}_{\chi_2}(x,\xi) = (e^{b_2 \epsilon} x, e^{-b_2 \epsilon} \xi);$$

thus equation (8.5) holds. One has

$$h^{(2)} := h^{(1)} \circ \phi^{\epsilon}_{\chi_2} = c_2(\omega t)(\xi^2 + x^{2l}) \ , \quad c_2 = (1+\epsilon c_1)^{\frac{1}{2(l+1)}} (1+\epsilon a_1)^{\frac{2l+1}{2l+2}} \ , \tag{8.8}$$

and

$$y_x^{(2)} = -\epsilon \dot{\chi}_2 + \epsilon \mathcal{S}^{-(l+1)}.$$

Thus, after these two transformations, $h_0 + \epsilon W_{2l}$ is transformed to

$$c_2(\omega t) h_0 - \epsilon p_{l+1} + \epsilon \mathcal{S}^{-2} \ ,$$

with

$$p_{l+1} := \dot{\chi}_1 \circ \phi^{\epsilon}_{\chi_2} + \dot{\chi}_2 \ ,$$

which is quasihomogeneous of degree $l+1$. The idea (following [BBM14]) is now to get rid of the time dependence of the main term by reparametrizing time, i.e., to pass to a new time τ such that

$$\frac{d\tau}{dt} = c_2(\omega t). \tag{8.9}$$

First we show that (8.9) defines a good reparametrization of time. Indeed, by making a Fourier expansion of c_2,

$$c_2(\phi) = (1+\epsilon c_{20}) + \epsilon \sum_{k \neq 0} c_{2k} e^{\mathrm{i} k \cdot \phi} \ ,$$

one has

$$\tau(t) = (1+\epsilon c_{20})t + \epsilon \sum_{k\neq 0} \frac{c_{2k}}{\mathrm{i}\omega\cdot k} e^{\mathrm{i}k\cdot\omega t} =: (1+\epsilon c_{20})t + \epsilon\tau_1(\omega t) \ , \tag{8.10}$$

which is well defined and C^∞ on $\Omega_{0\gamma}$. Then one can use the implicit function theorem in order to show that the inverse of the transformation (8.10) has the form

$$t(\tau) = a\tau - \epsilon t_1(\omega a\tau) \ , \quad a := (1+\epsilon c_{20})^{-1} \ , \tag{8.11}$$

and t_1 defined and smooth on $\mathbb{T}^n$. Precisely, this is obtained by applying the implicit function to the equation (that defines t_1)

$$G(\epsilon, t_1) := \tau_1(\phi - \epsilon\omega t_1(\phi)) - t_1(\phi) = 0 \ ,$$

where $G : \mathbb{R}\times C_0^K(\mathbb{T}^n) \to C_0^K(\mathbb{T}^n)$, with an arbitrary K and the index 0 means "with zero average".

After the introduction of the new time the system is reduced to the quantization of

$$h_0 + \epsilon p'_{l+1} + \mathcal{S}^{-2} \ ,$$

with $p'_{l+1} := p_{l+1}/c_2$ (and the frequencies are now $\omega' = a\omega$).

We now have to eliminate p'_{l+1}. To this end we proceed as explained in section 4.2, i.e., we solve equation (4.20) with $p = p'_{l+1}$, thus getting a $\chi_3 \in S^2$ which conjugates the Hamiltonian to

$$h_0 + \epsilon\langle p'_{l+1}\rangle(h_0, \omega'\tau)\eta(h_0) + S^2.$$

The last step is achieved by removing the time dependence from $\langle p'_{l+1}\rangle\eta$. To this end we look for a χ_4 solving (4.22) with $p = \langle p'_{l+1}\rangle\eta$. The main remark is that the function $\chi_4 \in S^{l+1}$ turns out to be quasihomogeneous (in the region $E > 2$), thus it is easy to see that it has the property that $f\circ\Phi^\epsilon_{\chi_4} \in S^m$ whenever $f \in S^m$, and therefore equation (4.15) holds. Using such a χ_4 one conjugates the Hamiltonian to

$$h_0 + \epsilon\overline{\langle p'_{l+1}\rangle} + \epsilon S^2.$$

At this point we can add the lower order corrections W and apply Theorem 2.4 getting the result. □

Proof of Theorem 2.12 *in the case* $l = 1$. The proof is a simple KAM type theorem in which, working at the level of symbols, one eliminates iteratively the time dependence from the Hamiltonian. The key remark is that, if χ is quadratic, then given a symbol $h_0 + \epsilon p$, one has that the symbol of $T_{\epsilon X}(H_0 + \epsilon P)$ is *exactly*

$$(h_0 + \epsilon p)\circ\Phi^\epsilon_\chi - y_x \simeq h_0 + \epsilon\{h_0;\chi\} - \epsilon\dot\chi \ , \tag{8.12}$$

$$y_x = \int_0^\epsilon \dot\chi\circ\Phi^{\epsilon-\epsilon_1}_\chi d\epsilon_1. \tag{8.13}$$

So, in order to establish the recursion one determines χ by solving the homological equation (4.27) (with p in place of W) and uses it in order to square the order of the time dependent part of the symbol.

Then one has to add estimates and to prove an iterative lemma which allows to establish the convergence of the procedure. We remark that such an iterative lemma is actually a simple 2-dimensional version of Lemma 6.5. For this reason we omit the details of the proof. □

Appendix A. Some technical lemmas

We start with a couple of results which apply to operators depending in a C^∞ way on the angles. They are used in section 5.

Remark A.1. For a rough estimate of the commutator of two operators we remark that, having fixed a set $\tilde{\Omega}$ and indices ρ, s, δ, then there exists a constant C which depends on all these indices s.t.

(A.1)
$$\|[X,F]\|_{Lip_\rho\left(\tilde{\Omega};B(\mathcal{H}^{s-\delta};\mathcal{H}^s)\right)} \le C\,\|X\|_{Lip_\rho\left(\tilde{\Omega};B(\mathcal{H}^{s-\delta};\mathcal{H}^s)\right)}\,\|F\|_{Lip_\rho\left(\tilde{\Omega};B(\mathcal{H}^{s-\delta};\mathcal{H}^s)\right)}\,.$$

Exploiting such a remark it is immediate to get the following result whose proof is obtained just by estimating each term of the series defining the quantum Lie transform and summing up the series.

Lemma A.2. *Let F and X be two operators belonging to $Lip_\rho\left(\tilde{\Omega};B(\mathcal{H}^{s-\delta};\mathcal{H}^s)\right)$. Then $Lie_{\epsilon X}F$ also belongs to such a space and there exists a constant C which depends only on the indices of the norm, such that*

$$\|Lie_{\epsilon X}F - F\| \le C\,\|X\|\,\|F\|\,. \tag{A.2}$$

The norm is the norm in the above space.

We now prove some general properties of sequences λ_j, which have behaviors of that of the eigenvalues of the operators that we meet in the main part of the text.

Lemma A.3. *Assume that*

$$|\lambda_i(\omega)-\lambda_j(\omega)| \ge \mathcal{K}_{0\lambda}\left|i^d-j^d\right| \ , \quad i\ne j \ , \tag{A.3}$$

and $\alpha \le \mathcal{K}_{0\lambda}/2$. Then

$$\mathcal{R}_{ijk}(\lambda,\alpha)\ne\emptyset \implies |k| \ge \frac{\mathcal{K}_{0\lambda}}{4}\left|i^d-j^d\right|. \tag{A.4}$$

Proof. Since $\mathcal{R}_{ijk}(\lambda,\alpha)\ne\emptyset$ one has

$$2\,|k| \ge |\omega|_{\ell^\infty}|k| \ge |\lambda_i-\lambda_j| - \alpha\left|i^d-j^d\right| \ge (\mathcal{K}_{0\lambda}-\alpha)\left|i^d-j^d\right|.$$

□

Lemma A.4. *Assume* (A.3) *and*

$$\left|\frac{\Delta(\lambda_i-\lambda_j)}{\Delta\omega}\right| \le \mathcal{K}_{1\lambda}\left|i^d-j^d\right| \qquad \forall\omega\in\tilde{\Omega} \ , \tag{A.5}$$

with $\mathcal{K}_{1\lambda}\le\mathcal{K}_{0\lambda}/8$. Then one has

$$|\mathcal{R}_{ijk}(\lambda,\alpha)| \le \frac{4\alpha}{\mathcal{K}_{0\lambda}}n^{(n-1)/2}. \tag{A.6}$$

Proof. Assume that $\mathcal{R}_{ijk}$ is not empty, so that (A.4) holds. Let $\omega\in\mathcal{R}_{ijk}$; choose a vector $v\in\{-1,1\}^n$ such that $k\cdot v=|k|$ and write $\omega = rv+w$ with $w\in v^\perp$. We estimate the size by which one has to move r in order to go outside $\mathcal{R}$. Let $\omega' := r'v+w$ and compute

$$\left|\Delta s'_{ijk}\right| \ge |(r-r')||k| - |\Delta(\lambda_i-\lambda_j)| \ge |\Delta\omega|\left(|k|-\mathcal{K}_{1\lambda}\left|i^d-j^d\right|\right).$$

So, if such a quantity is larger than $\alpha\left|i^d-j^d\right|$, then ω' is outside $\mathcal{R}_{ijk}$. It follows that

$$|\mathcal{R}_{ijk}| \leq \frac{2\alpha\left|i^d-j^d\right|}{|k|-\mathcal{K}_{1\lambda}\left|i^d-j^d\right|}(\mathrm{diam}(\Omega))^{n-1}$$
$$\leq \frac{2\alpha\left|i^d-j^d\right|\mathcal{K}_{0\lambda}}{(\mathcal{K}_{0\lambda}-4\mathcal{K}_{1\lambda})|k|}n^{(n-1)/2} \leq \frac{4\alpha}{\mathcal{K}_{0\lambda}}n^{(n-1)/2}.$$

□

Appendix B. Estimates of analytic quantum Lie transform

Remark B.1. One has

$$\left\|\mathrm{i}[X;F]\right\|_{r-\sigma} \leq 2\left\|X\right\|_{r-\sigma}\left\|F\right\|_{r-\sigma}, \tag{B.1}$$

$$\left\|\mathrm{i}[X;F]\right\|^{\mathcal{L}}_{r-\sigma} \leq 2\left(\left\|X\right\|^{\mathcal{L}}_{r-\delta}\left\|F\right\|_{r-\sigma}+\left\|X\right\|_{r-\sigma}\left\|F\right\|^{\mathcal{L}}_{r-\sigma}\right). \tag{B.2}$$

Lemma B.2. *Provided*

$$\left\|X\right\|_{r_\sigma} < \frac{\ln 2}{2}, \tag{B.3}$$

one has

$$\left\|e^{\mathrm{i}X}Fe^{-\mathrm{i}X}-F\right\|_{r-\sigma} \leq 4\left\|X\right\|_{r-\sigma}\left\|F\right\|_r , \tag{B.4}$$

$$\left\|e^{\mathrm{i}X}Fe^{-\mathrm{i}X}-F\right\|^{\mathcal{L}}_{r-\sigma} \leq 4\left\|X\right\|_{r-\sigma}\left\|F\right\|^{\mathcal{L}}_r+2\left\|X\right\|^{\mathcal{L}}_{r-\sigma}\left\|F\right\|_r . \tag{B.5}$$

Proof. From the recursive formula (3.3) and Remark B.1, one immediately gets

$$\left\|F_k\right\|_{r-\sigma} \leq (2\left\|X\right\|_{r-\sigma})^k\left\|F\right\|_{r-\sigma} , \tag{B.6}$$

from which

$$\left\|e^{\mathrm{i}X}Fe^{-\mathrm{i}X}-F\right\|_{r-\sigma} \leq \sum_{k\geq 1}\frac{(2\left\|X\right\|_{r-\sigma})^k}{k!}\left\|F\right\|_r \leq (e^{2\left\|X\right\|_{r-\sigma}}-1)\left\|F\right\|_r ,$$

which, under the assumption (B.3), is smaller then the r.h.s. of (B.4).

We come to (B.5). From (B.2) one gets

$$\left\|F_k\right\|^{\mathcal{L}}_{r-\sigma} \leq 2\left(\left\|X\right\|_{r-\sigma}\left\|F_{k-1}\right\|^{\mathcal{L}}_{r-\sigma}+\left\|X\right\|^{\mathcal{L}}_{r-\sigma}\left\|F_{k-1}\right\|_{r-\sigma}\right) \tag{B.7}$$

$$\leq 2\left\|X_{r-\sigma}\right\|\left\|F_{k-1}\right\|^{\mathcal{L}}_{r-\sigma}+\left\|X\right\|^{\mathcal{L}}_{r-\sigma}(2\left\|X\right\|_{r-\sigma})^{k-1}\left\|F\right\|_r . \tag{B.8}$$

To write the formulae we need in a simpler way denote

$$\lambda := 2\left\|X\right\|_{r-\sigma} , \quad \mu := \left\|X\right\|^{\mathcal{L}}_{r-\sigma} , b := \left\|F\right\|_r , \tag{B.9}$$

and look for a sequence a_k such that $a_k \geq \left\|F_k\right\|^{\mathcal{L}}_{r-\sigma}$, then such a sequence can be defined by

$$a_k = \lambda a_{k-1}+\mu\lambda^{k-1}b , \tag{B.10}$$

which is easily solved by the discrete equivalent of Duhamel's formula, which is actually obtained by making the substitution $a_k=\lambda^k c_k$, so that c_k satisfies

$$c_k = c_{k-1}+\frac{\mu}{\lambda}b ,$$

which gives

$$c_k = c_0+k\frac{\mu}{\lambda}b , \quad a_k = \lambda^k a_0+\lambda^{k-1}k\mu b .$$

Now, the l.h.s. of (B.5) is estimated by

$$\sum_{k\geq 1}\frac{a_k}{k!}=(e^{\lambda}-1)a_0+\mu b e^{\lambda}\ ,$$

which again under (B.3) gives the result. □

Let X be the solution of (6.11). Then, if A is not bounded, its Lie transform with X has good properties. Indeed the following lemma holds.

Lemma B.3. *One has*

$$\left\|e^{\mathrm{i}X}Ae^{-\mathrm{i}X}-A-\mathrm{i}\,[X;A]\right\|_{r-2\sigma}\leq 4\left\|X\right\|_{r-\sigma}\left[\frac{1}{\sigma}\left\|X\right\|_{r-\sigma}+2\left\|P\right\|_{r-2\sigma}\right]\ , \tag{B.11}$$

$$\begin{aligned}&\left\|e^{\mathrm{i}X}Ae^{-\mathrm{i}X}-A-\mathrm{i}\,[X;A]\right\|^{\mathcal{L}}_{r-2\sigma}\\ &\quad\leq\frac{8}{\sigma}\left\|X\right\|_{r-\sigma}\left\|X\right\|^{\mathcal{L}}_{r-\sigma}+\frac{4}{\sigma}\left\|X\right\|^{2}_{r-\sigma}+8\left\|X\right\|_{r-\sigma}\left\|P\right\|^{\mathcal{L}}_{r-2\sigma}\\ &\qquad+8\left\|X\right\|^{\mathcal{L}}_{r-\sigma}\left\|P\right\|_{r-2\sigma}\,.\end{aligned} \tag{B.12}$$

Proof. We just remark that the recursion defining A_k can be generated starting from

$$A_1=-\mathrm{i}\,[A;X]=\dot{X}-P+[P]\ ,$$

which allows us to start with the estimate

$$\left\|A_1\right\|_{r-2\sigma}\leq\frac{|\omega||}{\sigma}\left\|X\right\|_{r-\sigma}+2\left\|P\right\|_{r-2\sigma}=:b\ ; \tag{B.13}$$

from this one gets $\left\|A_k\right\|_{r-2\sigma}\leq\lambda^{k-1}b$ with λ defined by (B.9), which gives that the l.h.s. of (B.11) is estimated by

$$\sum_{k\geq 2}\frac{\lambda^{k-1}b}{k!}=\frac{b}{\lambda}(e^{\lambda}-1-\lambda)\ ,$$

which gives the wanted estimate.

To estimate the Lipschitz norm we proceed as above: let a_k be a sequence estimating the Lipschitz norm of A_k, then we have

$$a_k=2b\mu\lambda^{k-2}+\lambda a_{k-1}=\frac{2b\mu}{\lambda}\lambda^{k-1}+\lambda b_{k-1}.$$

Proceeding again by the discrete Duhamel formula ($a_k=\lambda^{k-1}c_k$), one gets

$$c_k=\frac{2b\mu}{\lambda}+c_{k-1},$$

which gives

$$c_k=\frac{2b\mu}{\lambda}(k-1)+a_1,\quad a_k=\lambda^{k-1}\left(\frac{2b\mu}{\lambda}(k-1)+a_1\right).$$

It follows that

$$\sum_{k\geq 2}\frac{a_k}{k!}=\sum_{k\geq 2}\lambda^{k-1}\frac{2b\mu}{\lambda k!}-\sum_{k\geq 2}\left(\frac{2b\mu}{\lambda k!}\frac{\lambda^{k-1}}{\lambda}-\frac{a_1\lambda^{k-1}}{k!}\right)$$
$$=\frac{2b\mu}{\lambda}(e^\lambda-1)+\left(a_1-\frac{2b\mu}{\lambda}\right)\frac{e^\lambda-1-\lambda}{\lambda}$$
$$=\frac{2b\mu}{\lambda}\left(\frac{\lambda e^\lambda-e^\lambda+1}{\lambda}\right)+a_1\frac{e^\lambda-1-\lambda}{\lambda}$$
$$\leq\frac{2b\mu}{\lambda}\lambda^2e^\lambda+2a_1\lambda\leq 4b\mu+2a_1\lambda,$$

from which, taking

$$a_1:=\frac{1}{\sigma}\left\|X\right\|^{\mathcal{L}}_{r-\sigma}+\frac{1}{\sigma}\left\|X\right\|_{r-\sigma}+2\left\|P\right\|^{\mathcal{L}}_{r-2\sigma}$$

the thesis follows. □

Concerning Y_X we have the following lemma.

Lemma B.4. *Let Y_X be defined by* (3.9) *with $\epsilon=1$. Then we have the following estimates:*

$$\left\|Y_X+\dot{X}\right\|_{r-2\sigma}\leq\frac{4}{\sigma}\left\|X\right\|^2_{r-\sigma}, \tag{B.14}$$

$$\left\|Y_X+\dot{X}\right\|^{\mathcal{L}}_{r-2\sigma}\leq\frac{6}{\sigma}\left\|X\right\|_{r-\sigma}\left\|X\right\|^{\mathcal{L}}_{r-\sigma}. \tag{B.15}$$

Proof. Define Y_k by

$$Y_0=\dot{X},\quad Y_k:=\mathrm{i}[X;Y_{k-1}];$$

then

$$Y_X=-\int_0^1 d\epsilon_1 e^{\mathrm{i}(1-\epsilon_1)X}\dot{X}e^{-\mathrm{i}(1-\epsilon_1)X}=-\int_0^1 d\epsilon_1\sum_{k\geq 0}Y_k\frac{(1-\epsilon_1)^k}{k!}$$
$$=-\sum_{k\geq 0}Y_k\frac{1}{k!}(-)\frac{1}{k+1}(1-\epsilon_1)\Big|_0^1=-\sum_{k\geq 0}Y_k\frac{1}{(k+1)!}.$$

Thus, following the proof of Lemma B.2, one easily gets the thesis. □

Appendix C. A few more lemmas

Finally, we add two lemmas on the solution of the quantum homological equation (6.2) and a lemma allowing us to solve superexponential recursions.

Lemma C.1 (Lemma A.1 of [Pös96])**.** *If $P=(P_{ij})$ is (the matrix of) a bounded linear operator on ℓ^2, then also $X=(X_{ij})$ with*

$$X_{ij}=\frac{|P_{ij}|}{|i-j|},\quad i\neq j, \tag{C.1}$$

and $B_{ii}=0$ is a bounded linear operator on ℓ^2, and $\left\|X\right\|\leq\pi\left\|P\right\|/\sqrt{3}$, where the norm is the norm in $B(\ell^2;\ell^2)$.

Corollary C.2. *Let $P\in B(\mathcal{H}^{s_1};\mathcal{H}^{s_2})$ for some s_1,s_2 be a bounded operator with matrix P_{ij}. Define X by* (C.1)*; then also $X\in B(\mathcal{H}^{s_1};\mathcal{H}^{s_2})$ and one has*

$$\left\|X\right\|_{B(\mathcal{H}^{s_1};\mathcal{H}^{s_2})}\leq\frac{\pi}{\sqrt{3}}\left\|P\right\|_{B(\mathcal{H}^{s_1};\mathcal{H}^{s_2})}. \tag{C.2}$$

Proof. We remark that an operator P belongs to $B(\mathcal{H}^{s_1};\mathcal{H}^{s_2})$ if and only if the operator with matrix $i^{s_2}P_{ij}j^{-s_1}$ is bounded on ℓ^2, and apply Lemma C.1. $\square$

Lemma C.3. *For $\nu \geq 0$, define*

$$\varsigma_{\nu+1} := c_1 2^{a\nu}\varsigma_\nu^2. \tag{C.3}$$

Then one has

$$\varsigma_\nu = \frac{1}{c_1 2^{a\nu}}\left(2^{2a}c_1\varsigma_0\right)^{2^\nu}. \tag{C.4}$$

Assume also $2^{2a}c_1\varsigma_0 < 1$. Then one has

$$\sum_{\nu\geq k}\varsigma_\nu \leq \frac{(2^{2a}c_1\varsigma_0)^{2^k}2}{c_1 2^{ak}}, \tag{C.5}$$

and for any $b > 0$ there exists C_b independent of c_1 s.t.

$$\sum_{\nu\geq 0}\nu^b\varsigma_\nu \leq C_b\varsigma_0. \tag{C.6}$$

Proof. Make the substitution $\varsigma_\nu = k_1k_2^\nu\delta_\nu$, and rewrite formula (C.3) for the sequence δ_ν. One gets

$$k_1k_2^{\nu+1}\delta_{\nu+1} = 2^{a\nu}c_1k_1^2k_2^{2\nu}\delta_\nu^2,$$

which becomes particularly simple taking

$$k_2^\nu = 2^{a\nu}k_2^{2\nu} \iff k_2 = 2^{-a},$$
$$k_2k_1 = c_1k_1^2 \iff k_1 = (2^ac_1)^{-1},$$

so that we get

$$\delta_{\nu+1} = \delta_\nu^2 \iff \delta_\nu = \delta_1^{2^{\nu-1}}.$$

Substituting back in ς_ν one gets (C.4). To get (C.5) we remark that

$$\sum_{\nu\geq k}\varsigma_\nu = \frac{(2^{2a}c_1\varsigma_0)^{2^k}}{c_1 2^{ak}}\sum_{\nu\geq k}\frac{1}{2^{a(\nu-k)}}(2^{2a}c_1\varsigma_0)^{2^\nu-2^k}.$$

Note that

$$2^\nu - 2^k = 2^k(2^{\nu-k}-1) \geq 2^k(\nu-k),$$

so that the above sum is smaller than

$$\frac{(2^{2a}c_1\varsigma_0)^{2^k}}{c_1 2^{ak}}\sum_{\nu\geq k}\frac{1}{2^{a(\nu-k)}}(2^{2a}c_1\varsigma_0)^{2^k(\nu-k)} \leq \frac{(2^{2a}c_1\varsigma_0)^{2^k}}{c_1 2^{ak}}2.$$

The cases with $b > 0$ are estimated in the same way. $\square$

Acknowledgments

This paper originated from a series of discussions with many people about the methods of [BBM14, Mon14, FP15, BM16b] and about the possibility of extending them to the case of the Schrödinger equation. In particular, the author warmly thanks P. Baldi, R. Montalto and M. Procesi who explained their work in great detail. During the preparation of the present work the author benefited from many suggestions and discussions with A. Maspero and D. Robert. In particular,

D. Robert pointed out (and often explained) his papers in which the class of symbols that are used here are extensively studied. The author also thanks B. Grébert for some discussions on the harmonic case that helped to fix some points of the proof.

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