SOME COMMENTS ON MOTIVIC NILPOTENCE

JENS HORNBOSTEL, WITH AN APPENDIX BY MARCUS ZIBROWIUS

ABSTRACT. We discuss some results and conjectures related to the existence of the non-nilpotent motivic maps η and μ_9. To this purpose, we establish a theory of power operations for motivic H_∞-spectra. Using this, we show that the naive motivic analogue of the unstable Kahn-Priddy theorem fails. Over the complex numbers, we show that the motivic T-spectrum $S[\eta^{-1}, \mu_9^{-1}]$ is closely related to higher Witt groups, where S is the motivic sphere spectrum and η and μ_9 are explicit elements in $\pi_{**}(S)$.

1. INTRODUCTION

Let us fix a base field k, and consider the motivic stable homotopy category $\mathcal{SH}(k)$ of Morel-Voevodsky. In this article, we investigate possible motivic generalizations of three famous theorems in classical homotopy theory: Nishida's nilpotence theorem (see [N] and also [M1], [M2]), the fact that complex cobordism MU_* detects nilpotent self maps [DHS] and the classification of thick ideals in $\mathcal{SH}_{fin}$ resp. its p-localizations via Morava K-theories $K(n)$ [HS]. These theorems lay the foundations of the chromatic approach to stable homotopy theory. The theorems are closely related classically, and motivic counterparts would be as well: We have a decomposition of Bousfield classes $\langle MU_{(p)} \rangle = \bigvee_{n \geq 0} \langle K(n) \rangle$, and in the motivic case Joachimi [Jo, Theorem 9.5.1] has shown a slightly weaker decomposition theorem over $\mathbb{C}$ for p odd. Moreover, any non-nilpotent map $f \in \pi_{**}(S)$ produces a thick ideal $thickid(Cf)$ which consists precisely of those objects X for which $f^n \wedge id_X$ is zero for $n \gg 0$, as shown in [Ba, Theorem 2.15]. Besides elements in bidegree $(0, 0)$, the first non-nilpotent motivic self map is the *motivic Hopf map* $\eta \in \pi_{1,1}(S)$. It has been studied by Morel who has shown that unlike its topological counterpart it is *not* nilpotent, but is not detected by algebraic cobordism **MGL** either [Mo1]. Hence η shows that at least two of the three above theorems do not hold motivically. Applying the above result to η, one obtains a thick ideal $thickid(C\eta)$ which Joachimi [Jo, chapter 7] shows to be different from the ones arising from classical or $\mathbb{Z}/2$-equivariant stable homotopy theory, thus disproving a motivic version of the third theorem above. Recall that one may define motivic Morava K-theories by killing elements in **MGL**$_{**}$, as done first by Borghesi and Hu. See also [Ho2] for motivic versions of Johnson-Wilson spectra and [Jo] for more Bousfield class decompositions over $\mathbb{C}$.

Similar to the study of localizations of the topological sphere spectrum with respect to given integers in $\pi_0(S)$, one is naturally led to study the η-localization

of the motivic sphere spectrum S over k. For $k = \mathbb{C}$, Andrews and Miller [AM] recently proved the following, confirming a conjecture of [GI1]:

Theorem 1.1 (Andrews-Miller). *For $k = \mathbb{C}$, the motivic ANSS induces an isomorphism*

$$\pi_{**}(S)[\eta^{-1}] \cong \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2),$$

with $\mu_9 \in \langle 8\sigma, 2, \eta \rangle \subset \pi_{9,5}(S)$ and $\sigma \in \pi_{7,4}(S)$ the motivic Hopf element of [DI3].

Our notation follows the usual motivic grading convention in which the simplicial one-dimensional sphere is $S^{1,0}$ and the punctured affine line is $S^{1,1}$.

The element μ_9 considered in [AM] is originally defined only after motivic 2-completion. However, it can be shown to exist integrally; see Lemma 2.10 below. It is detected by Ph_1 in the motivic ASS and satisfies $2\mu_9 = 0$ [DI2, p. 1010], [Is, table 8]. We refer to [Is, table 23] for the indeterminacy of $\langle 8\sigma, 2, \eta \rangle$. Let us emphasize the importance of the huge amount of computations by Isaksen (and coauthors) at the prime 2; see in particular [Is].

We now fix the element $\mu_9 \in \pi_{9,5}(S)$ which is detected by the motivic α_5 in the motivic ANSS; see [AM, section 7]. More than a decade after Morel's study of η, the existence of μ_9 yields the second example of an element – at least over $\mathbb{C}$ – contradicting the naive motivic analogue of the classical three nilpotence theorems above:

Corollary 1.2. *Let $k = \mathbb{C}$.*

(i) The above element $\mu_9 \in \langle 8\sigma, 2, \eta \rangle$ is not nilpotent in $\pi_{**}(S)$.

(ii) The motivic algebraic cobordism spectrum $\mathbf{MGL}$ does not detect μ_9, i.e., $\mathbf{MGL}(\mu_9) = 0$.

(iii) The thick ideal $thickid(C\mu_9)$ in $\mathcal{SH}(\mathbb{C})$ is "new", that is, it is not induced by some thick ideal of $\mathcal{SH}$.

Proof. (i) Follows immediately from Theorem 1.1. (ii) This is an easy consequence of the computation in [HKO1, Theorem 7]. (As Andrews pointed out to the author, this also follows because μ_9 has Novikov filtration one.) (iii) This is similar to [Jo, Proposition 7.1.4 (2),(3)], using [Ba, Corollary 2.15] (which implies that $thickid(C\mu_9)$ is strictly smaller than $\mathcal{SH}(\mathbb{C})_{fin}$) and that the complex realization of μ_9 is nilpotent. □

The description of μ_9 as a Toda bracket $\langle 8\sigma, 2, \eta \rangle$ of motivic Hopf elements together with the fact $(1 - \epsilon)\eta = 0$ for all fields shows that similar elements exist over other fields as well, provided that $2^t \cdot (1 - \epsilon)^{4-t} \cdot \sigma = 0$ for some t. As Isaksen points out, computations in [DI4] and subsequent work imply that this product is non-zero for $k = \mathbb{R}$. Hence μ_9 is not defined for subfields $k \subset \mathbb{R}$. If μ_9 is defined for some subfield $k \subset \mathbb{C}$, then it is a non-zero non-nilpotent element over k. (It remains to compare $thickid(C\eta)$ and $thickid(C\mu_9)$, of course.)

After the appearance of μ_9, more non-nilpotent self maps have been recently discovered by Isaksen and his coauthors. We refer the reader to the beginning of the next section for a short discussion of this work in progress. Let us however mention the following here: Andrews conjectures that $\eta = w_0$ is just the beginning of a new chromatic motivic family $w_0, w_1, w_2...$ for $k = \mathbb{C}$ at the prime 2. He has constructed self maps $w_1^4 : S/\eta \to S/\eta$ and w_2^{32}. Inspired by his construction, Gheorghe [Gh] has constructed a new infinite family $K(w_n)$ of motivic Morava K-theory spectra.

The very recent article [Th] computes the homogenous spectrum of $K_*^{MW}(k)$. This together with work in progress by Heller and Ormsby (the article is now available, see [HO]) showing that the graded map [Ba] to the homogenous spectrum of $K_*^{MW}(k)$ is surjective leads to a refinement of the results above about thick (prime) ideals in $\mathcal{SH}(k)$ including $C\eta$, but not $C\mu_9$. (The case of a finite base field has been considered slightly earlier by [K].)

This introductory section contains mostly recollections of known recent results and some rather easy consequences thereof. The main results appear in the following section 2. There is a very close and precise relationship between the failure of motivic Nishida nilpotence and the motivic unstable Kahn-Priddy theorem; see Theorem 2.7. This has very explicit consequences; see Theorem 2.8. For instance, for $k = \mathbb{C}$ the motivic Hopf element $\eta \in \pi_{9,3}(S^{8,2})$ does not lift to $\pi_{9,3}((E\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} S^{4,1})$, although the topological Hopf η_{top} does lift to $\pi_9((E\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} S^4)$. In other words, a certain homotopical symmetry of η_{top} does not lift to the motivic Hopf map η.

The proofs of these results rely heavily on the study of *extended powers* and *power operations* for *motivic H_∞-spectra*, which generalizes the classical work of J. P. May et al. and represents a topic of interest for its own sake.

Finally, over the complex numbers there is a beautiful relationship between the motivic sphere spectrum and the spectrum **KT** representing Witt groups (see Theorem 3.2) for a more precise statement):

Theorem 1.3. *For $k = \mathbb{C}$, the unit map of the hermitian K-theory spectrum induces an epimorphism*

$$\pi_{**}(S[\eta^{-1}, \mu_9^{-1}]) \to \pi_{**}\mathbf{KT}$$

with the kernel being isomorphic to $\pi_{**}\mathbf{KT}$.

This theorem is an incarnation of the following general philosophy: quadratic forms and hermitian K-groups detect a lot of interesting motivic homotopy theory not visible by cohomology or classical homotopy theory. (In the classical case, this is reflected, e.g., by the Moebius stripe detecting the topological Hopf map.) This philosophy has been confirmed before, e.g., by [Mo1], [RSO] and [ALP]. Our theorem may be considered as an integral refinement of the latter over the complex numbers.

The appendix of Marcus Zibrowius provides a complete answer to motivic Nishida nilpotence in simplicial degree zero, that is, for Milnor-Witt K-theory. The main result is the following (see Proposition A.1):

Theorem 1.4 (Zibrowius)**.** *Nishida nilpotence holds in all non-negative degrees of the Milnor-Witt K-ring $K_*^{MW}(F)$. Nishida nilpotence holds in all degrees of $K_*^{MW}(F)$ if and only if F is formally real.*

2. On the failure of motivic Nishida nilpotence

We now concentrate on the first classical theorem, that is, Nishida nilpotence. Its proof relies on the following three key ingredients: the fact – due to Serre – that all elements in $\pi_*(S)$ are torsion for $* \neq 0$, a careful study of power operations for H_∞-ring spectra and the Kahn-Priddy theorem [KP]. (We do not discuss the slightly different proof for those elements annihilated by some prime p already,

which provides a sharper bound.) One might ask if there is a motivic generalization of Nishida's theorem:

Question 2.1. For which fields k and in which bidegrees is the following true for every $f \in \pi_{**}(S)$: If $r \cdot f = 0$ for some $r \in \mathbb{Z}$, then f is nilpotent?

If this was true for some field in all bidegrees, then every non-nilpotent element $f \in \pi_{**}(S)$ would be non-torsion, that is, $r \cdot f \neq 0$ for all $r \in \mathbb{Z}$. But for $k = \mathbb{C}$ both η and μ_9 are in the τ-local region as described in [GhI], so they must be torsion by Levine's comparison theorem [L] and Serre's classical finiteness result. Hence for $k = \mathbb{C}$ the answer is no in arbitrary bidegrees, and yes when restricted to weight zero. Next, one might look at $k = \mathbb{R}$ where we only have $(1 - \epsilon) \cdot \eta = 0$ rather than $2 \cdot \eta = 0$. This does not contradict motivic Nishida nilpotence, except if we generalize from $\mathbb{Z}$- to $GW(k)$-torsion. (Note that there is a $\mathbb{Z}/2$-equivariant version of Nishida's theorem; see [Ir]. This might be a hint that everything that goes wrong with nilpotence goes wrong over $\mathbb{C}$ in some sense.) Also, the elements η and μ_9 are 2-torsion, so the conjecture might be true for elements annihilated by (products of) powers of odd primes. Indeed, the recent computations by Stahn [St] at odd primes for $k = \mathbb{C}, \mathbb{R}$ have not yet led to new non-nilpotent self maps.

According to Guillou and Isaksen [GI1], over $\mathbb{C}$ completed at 2, besides the units in $\pi_{0,0}(S)$ there is an infinite family of non-nilpotent elements $\mu_{8k+1} \in \pi_{8k+1,4k+1}(S)$ detected by $P^k h_1$, and starting with $\mu_1 = \eta$ and μ_9. They further claim (compare also [GhI]) that there are other families of non-nilpotent elements, e.g., one starting with an element $d_1 \in \pi_{32,18}(S)$. According to Isaksen, the element d_1 even lifts to an element over $\mathbb{R}$, which consequently is non-nilpotent as well, and we have $4d_1 = 0$ over $\mathbb{R}$. The element d_1 lives in the motivic four-fold Toda bracket $\langle \eta, \sigma^2, \eta, \sigma^2 \rangle$, which is non-empty by Corollary 2.11. We do not include these unpublished results in our statements and proofs, except for Corollary 2.11 on d_1 below.

We also note that higher powers of ν and σ lie in the "not understood"-region. However, Isaksen's computations show that they are nilpotent over $\mathbb{C}$. Based on this and his computations over $\mathbb{R}$, Isaksen conjectures that $\nu^4 = 0$ and $\sigma^4 = 0$ over any base scheme. When restricting to simplicial degree zero, we have to study nilpotence for $K_*^{MW}(k)$ by Morel's theorem [Mo1, Theorem 6.2.1], at least if k is perfect and $char(k) \neq 2$. In this case, the appendix of Marcus Zibrowius shows that Nishida nilpotence holds in non-positive degrees – that is, non-negative degrees in the indexing of $K_*^{MW}(k)$ – and in all degrees if and only if k is formally real.

So what goes wrong when trying to translate the classical proof for Nishida nilpotence to the motivic case? Constructing extended powers and power operations in stable motivic homotopy theory is possible. For this, it is convenient to use motivic symmetric spectra as introduced by Hovey and Jardine. In what follows, we will work with motivic strictly commutative ring spectra which are in particular motivic H_∞-spectra. That is, for any motivic strictly commutative ring spectrum $\mathbf{E}$ we have maps

$$\xi_j : D_j \mathbf{E} := (E\Sigma_j)_+ \wedge_{\Sigma_j} \mathbf{E}^{\wedge j} \to \mathbf{E}$$

for all $j \geq 1$ such that the diagrams of [M2, Definition I.3.1] commute in $\mathcal{SH}(k)$. These maps are given by $\xi_j = \beta_{j,0}$ with [M2, Remark I.2.6] applied to the motivic setting. Virtually everything else in [M2, sections I.2+3] and the definitions of power operations, that is, [M2, Definitions I.4.1 and I.4.2], then easily extends to the motivic setting. We are mainly interested in the case $\mathbf{E} = S$, of course.

Having constructed these motivic power operations, we wish to proceed similarly to [M1] and [M2, sections II.1+2]. For this, we need to prove a motivic version of the Kahn-Priddy theorem (see below). The original proof of [KP] uses the Barratt-Priddy-Quillen theorem $B\Sigma_\infty^+ \simeq QS^0$ and various computations for (co-)homology with finite coefficients. For "geometric" classifying spaces BG, Voevodsky [V, section 6] has computed $H^{**}(BG, \mathbb{Z}/l)$ for $G = \Sigma_t, \mu_t$, which could be useful here. In an unpublished draft some time ago, Morel conjectured a motivic version of the Barratt-Priddy-Quillen theorem and pointed out a possible relationship with Serre's splitting principle for étale algebras.

Definition 2.2. We say that the unstable motivic Kahn-Priddy theorem holds for a field k at the prime p if the motivic version of the map $\tau_p = h_p$ of [M2, Definition II.1.4], [M1, Lemma 1.6], see Definition 2.4 below, induces an epimorphism

$$(\tau_p)_* : \pi_{**}(D_p S^{q,w}) \to \pi_{**}(S^{pq,pw})$$

in bidegrees $(r, *)$, provided r lies in the classical Kahn-Priddy range of [M2, Theorem II.2.8].

Similar to the classical case [KP], this is an unstable variant which is related to a stable one. The stable variant would predict epimorphisms (everything localized at p)

$$(\tau_p)_* : \pi_{**}(\Sigma_T^\infty B\Sigma_p) \to \pi_{**}(S)$$

in a certain range. There are many variants of the Kahn-Priddy theorem, both for the statement and for the proof. See, e.g., Segal [Se], Caruso-Cohen-May-Taylor [CCMT], and Löffler-Ray [LR].

Let us explain the map τ_p and some of its properties in the motivic setting. It is possible to prove the following two propositions for arbitrary monoidal model categories tensored over pointed simplicial sets. In particular, they are true for "global" model structures, that is, before carrying out motivic localizations as in [Hov], [Ja].

We start with a motivic variant of [M2, Theorem II.1.1]. We fix positive integers j and k and motivic (symmetric) T-spectra $Y_1, ..., Y_k$. Let $Z := Y_1 \vee ... \vee Y_k$, and let $\nu_i : Y_i \to Z$ be the inclusions. For a partition $J = (j_1, ..., j_k)$ of $j = j_1 + ... + j_k$, let f_J denote the composite

$$D_{j_1} Y_1 \wedge ... \wedge D_{j_k} Y_k \overset{D_J(\nu_J)}{\to} D_{j_1} Z \wedge ... \wedge D_{j_k} Z \overset{\alpha_J}{\to} D_j Z.$$

Here $D_J(\nu_J) := D_{j_1}\nu_1 \wedge ... \wedge D_{j_k}\nu_k$, and α_J is induced by the multiplication of the commutative ring spectrum, and is well-defined by the motivic analogue of [M2, Lemma I.2.8].

Proposition 2.3. *In the above situation, the wedge sum*

$$f_j : \bigvee_J D_{j_1} Y_1 \wedge ... \wedge D_{j_k} Y_k \to D_j Z$$

of the maps f_J is a stable motivic equivalence.

Proof. This is similar to [M2, Theorem II.1.1], using that motivic spectra are tensored over simplicial sets. The map $i \wedge 1$ corresponding to the one considered at the end of the proof of loc. cit. is a weak equivalence already for global model structures, that is, before motivic localization. □

For a fixed partition $J = (j_1, ..., j_k)$ of j, let $g_J : D_j(Z) \to D_{j_1}Y_1 \wedge ... \wedge D_{j_k}Y_k$ be the Jth component of f_j^{-1}. We now restrict to the special case $Y := Y_1 = ... = Y_k$, hence $Z = \bigvee_{i=1}^{k} Y$. Let $\Delta : Y \to \bigvee_{i=1}^{k} Y = \prod_{i=1}^{k} Y$ be the diagonal map.

Definition 2.4. For J and Y as above, we let τ_J be the composition

$$D_j Y \overset{D_j\Delta}{\to} D_j \bigvee_{i=1}^{k} Y \overset{g_J}{\to} D_{j_1}Y \wedge ... \wedge D_{j_k}Y.$$

If $J = (1, ..., 1)$ and hence $k = j$, we set

$$\tau_j := \tau_J : D_j Y \to D_1 Y \wedge ... D_1 Y = Y^{\wedge j}.$$

The following is a motivic generalization of [M1, Lemma 1.6], [M2, Corollary II.1.8]:

Proposition 2.5. *If $r = p^i v$ with p prime, $i \geq 1$, and v prime to p, then $D_p(r) : D_p Y \to D_p Y$ can be written as $p^i \lambda + (p, r - p)\iota_p \tau_p$ for some self map λ, where $\iota_p : Y^{\wedge p} \to D_p(Y)$ is the canonical inclusion, where $(p, r - p) := \binom{r}{p}$.*

Proof. Similar to [M2, Corollary II.1.8], using Proposition 2.3 and other results above. From this, we can deduce the following key result, which is essentially a motivic generalization of [M2, Corollary II.2.4] and [M1, Theorem 3.8]. (More precisely, setting $n = 1$ in the latter these two results are equivalent except that [M2] considers $\alpha \in E_r(D_p S^q)$ and [M1] considers $\tilde{y} \in \pi_{pr+t}(D_p S^r)$. However, we later restrict to $E = S$ anyway.) The proof of [M2] uses additivity formulae for power operations, which is some refinement of the classical formulae for powers of sums. We follow the proof hinted at in [M1] instead; see also the remark after [M2, Corollary II.2.5]. $\square$

Proposition 2.6. *Fix a strictly commutative (or H_∞) motivic p-local ring spectrum $\mathbf{E}$ and $x \in \pi_{q,w}\mathbf{E} = \mathbf{E}_{q,w}$. If $p^i \cdot x = 0$ for some positive integer i, then $p^{i-1} \cdot (\tau_p)_*(\alpha) \cdot x^{p+1} = 0$ for all $\alpha \in \pi_{p(q,w)+(s,t)}(D_p S^{q,w})$.*

Proof. As some details in the proof of the classical analogue [M1, Theorem 3.8] are omitted, we will include these here. Similar to loc. cit, we really show that $p^{i-1}(pz + x^p y) = 0$ for some $z : S^{p(q,w)+(s,t)} \to \mathbf{E}$ with $y := (\tau_p)_*(\alpha)$. (Following the tradition of [M1], our notation here and below omits Σ_T^∞, and ocassionally does not distinguish between a stable map and its unstable representative.) Multiplying this equality with x, we deduce that $p^{i-1}x^{p+1}(\tau_p)_*(\alpha) = 0$ as claimed. To show the equality above, we need to compute $x^p \cdot y$ which is given by the following composition:

$$S^{p(q,w)+(s,t)} \overset{\alpha}{\to} D_p(S^{q,w}) \overset{\tau_p}{\to} S^{p(q,w)} \overset{\iota_p}{\to} D_p(S^{q,w}) \overset{D_p(x)}{\to} D_p(\mathbf{E}) \overset{\xi_p}{\to} \mathbf{E}.$$

Now we multiply this composition with p^{i-1} and apply Proposition 2.5 with $r = p^i$ (hence $\binom{r}{p}$ is divisible by p^{i-1}) to $p^{i-1} \cdot \iota_p \circ \tau_p$. Consequently, the latter equals (possibly up to a unit) $D_p(p^i) - p^i g$ for some g. Hence $p^{i-1} \cdot \iota_p \circ \tau_p$ can be written as the difference of two maps, one being already of the requested form and the other one given by

$$S^{p(q,w)+(s,t)} \overset{\alpha}{\to} D_p(S^{q,w}) \overset{D_p(p^i)}{\to} D_p(S^{q,w}) \overset{D_p(x)}{\to} D_p(\mathbf{E}) \overset{\xi_p}{\to} \mathbf{E}.$$

But that one is 0 as $D_p(p^i)D_p(x) = D_p(p^i x) = 0$ by assumption. $\square$

Theorem 2.7. *Assume that the unstable motivic Kahn-Priddy theorem holds for a field k at a prime p. Then for $s \geq 1$, any element $x \in \pi_{s,t}(S)$ over k annihilated by a power of p is nilpotent.*

Proof. We may argue as in [M2, Theorem II.2.9]. Namely, using the unstable motivic Kahn-Priddy theorem we see that $x = (\tau_p)_*(\alpha)$ for some

$$\alpha \in \pi_{p(q,w)+(s,t)}(D_p S^{q,w})$$

with q large enough and arbitrary w. More precisely, as in [M2, Theorem 2.9] we have a factorization $q = m \cdot s$ for a suitable integer m, and then we may take $w = m \cdot t$. Now thanks to our assumption $p^i x = 0$, we can apply Proposition 2.6 with $\mathbf{E} = S$ to α and x^m (rather than x). This yields the desired equality $p^{i-1} x^{1+m(p+1)} = 0$. We then may conclude via descending induction on i. (Note that for odd primes p, if $q - w$ is odd and w is even or $k = \mathbb{C}$, then we already have $x^2 = 0$ by the graded commutativity of [DI3, Proposition 2.5].) $\qquad\square$

As both η and μ_9 are 2-torsion over $\mathbb{C}$ (more generally η is if k is not formally real, as then $1 - \epsilon = 2$) and both are not nilpotent, the theorem above puts some restrictions on possible motivic Kahn-Priddy theorems. Let us look at what goes wrong for $\eta \in \pi_{1,1}(S)$.

Theorem 2.8. *The unstable motivic Kahn-Priddy theorem does not hold at $p = 2$. In particular:*

- *The map*

$$(\tau_2)_* : \pi_{9,2w+1}(D_2 S^{4,w}) \to \pi_{9,2w+1}(S^{8,2w})$$

  for $w = 1, 2, 3$ and k not formally real (e.g., $k = \mathbb{C}$) is not a 2-local epimorphism.
- *The map*

$$(\tau_2)_* : \pi_{1161,645}(D_2 S^{576,320}) \to \pi_{1161,645}(S^{1152,640})$$

  for $k = \mathbb{C}$ is not a 2-local epimorphism, provided μ_9 lifts to an unstable element in $\pi_{1161,645}(S^{1152,640})$.

Proof. By Proposition 2.6 above, we know the following: for any

$$\alpha \in \pi_{p(q,w)+(s,t)}(D_2 S^{q,w})$$

and any $x \in \pi_{q,w}(S)$ with $2 \cdot x = 0$, we have $(\tau_2)_*(\alpha) \cdot x^3 = 0$. Now we need to find some α with $(\tau_2)_*(\alpha) = \eta$. In the classical case (corresponding to $w = 0 = t$), the unstable Kahn-Priddy theorem tells us that there is some $\alpha \in \pi_{2 \cdot 4 + 1}(D_2 S^4)$ which is mapped to $\eta_{top} \in \pi_9(S^8)$ under $(\tau_2)_*$. Hence we may choose $q = m \cdot 1 = 4$ and apply Proposition 2.6 to $x = \eta_{top}^4$. The corresponding motivic map $(\tau_2)_*$ must not have a preimage α in certain bidegrees $(9, w)$ for η. Namely, the claim about the first map follows from Theorem 2.7, the above properties of η and the above discussion on $(\tau_2)_*$.

For the second claim, the fact that η lifts to $\pi_{3,2}(S^{2,1})$, 2 lifts to $\pi_{1,0}(S^{1,0})$ and σ lifts to $\pi_{15,8}(S^{8,4})$ [DI3, section 4] together with some explicit construction of Toda brackets makes it plausible that μ_9 lifts to an element in $\pi_{20+s,10+w}(S^{11+s,5+w})$ for small s and w similar to the classical case, although we still lack a proof of this. (The classical proof of [M2, Theorem II.2.9] seems to use Freudenthal's suspension theorem without mentioning it. So far we have a motivic version of this only with

respect to the simplicial sphere; see [Mo3, Theorem 6.61].) If this holds, the μ_9 is also an element in $\pi_{1161,645}(S^{1152,640})$. Now an argument similar to the one above for η leads to the claim about the second map, choosing $(q, w) = 64 \cdot (9, 5) = (576, 320)$. □

We leave it to the reader to deduce similar statements using other non-nilpotent elements mentioned earlier. Note that the element $d_1 \in \pi_{32,18}(S)$, see also Corollary 2.11 below, shows that the motivic Kahn-Priddy theorem also fails over $\mathbb{R}$, but we do not know in which precise bidegrees as we do not know of a specific unstable lift of d_1 yet. Still, it seems likely that the motivic Kahn-Priddy theorem holds in weight $w = 0$.

Over $\mathbb{R}$, there are further non-nilpotent elements in $\pi_{**}(S)$ which do not exist over $\mathbb{C}$, of course:

Lemma 2.9. *For $k = \mathbb{R}$, the elements $\epsilon \in \pi_{0,0}(S)$ and $\rho = \rho_{-1} = [-1] \in \pi_{-1,-1}(S)$ are not nilpotent.*

Proof. We have $\epsilon^2 = 1$. Concerning $[-1]$, one knows that there is a graded ring epimorphism $K_*^M(\mathbb{R}) \to \mathbb{Z}/2[t]$ given by $[-1] \mapsto t$. Hence $[-1]$ is not nilpotent in $K_*^M(\mathbb{R})$, and consequently not in $K_*^{MW}(\mathbb{R})$. □

Note that $[-1]$ is detected by $\mathbf{MGL}_{**}$ because $\mathbf{MGL}_{-1,-1} \cong K_1(\mathbb{R})$ [Mo1, Theorem 6.4.5]. In this sense, it behaves more classically [DHS] than η which is not detected.

Finally, for $k = \mathbb{C}$ we have an element $\tau \in \pi_{0,1}(L_{S/2}S)$, sometimes also denoted by θ, where $L_{S/2}S$ denotes motivic Bousfield localization with respect to the mod-2 Moore spectrum $S/2$. The element τ can be explicitly constructed using inverse systems of roots of unity; see [HKO1, pp. 81–82]. It is not known if this element lifts to an element in "integral" $\pi_{0,-1}(S)$. This is a special case of the following more general problem:

For general base fields, the abutment of the motivic ASS or ANSS might be quite different from $\pi_{**}(S)$. Recall that for Morel's [Mo1] computation in simplicial degree 0 these spectral sequences are not necessary, and that the article [OO] on computations in simplicial degree 1 contains techniques to get rid of completions in some cases; see also Lemma 2.10 below. Even for $k = \mathbb{C}$, the situation is more complicated than in classical stable homotopy theory. Over $\mathbb{C}$, the motivic ASS converges to the nilpotent completion $\pi_{**}(S_{\mathbf{H}}^{\wedge})$ [DI2, Corollary 6.15], [HKO1], where $\mathbf{H}$ denotes the motivic Eilenberg-MacLane spectrum for $\mathbb{Z}/p$ with p a fixed prime. Furthermore, it is shown in [HKO1, Theorem 6] that the nilpotent completion $S_{\mathbf{H}}^{\wedge}$ has the same bigraded homotopy groups as the Bousfield localization $S_p^{\wedge} := L_{S/p}S \simeq holim\, S/p^n$ [HKO3, Lemma 18]. Assuming this, it still remains to understand the map

$$\pi_{**}(S) \to \pi_{**}(S_p^{\wedge}).$$

For this, one may use a motivic variant [HKO1] of the usual short exact sequence of Bousfield [Bo, Proposition 2.5]. This tells us that $\pi_{s,w}(S_p^{\wedge}) \cong \pi_{s,w}(S)_p^{\wedge}$ if $\pi_{s,w}(S)$ and $\pi_{s-1,w}(S)$ are finitely generated abelian groups, but in general the situation is more complicated. For the motivic ANSS, similar problems do occur. The most convenient approach is to apply a motivic variant of the chromatic square, as in the following example.

Lemma 2.10. *For $k = \mathbb{C}$, the canonical map*

$$\pi_{9,5}(S) \to \pi_{9,5}(L_{S/2}S)$$

is an isomorphism. In particular, the element μ_9 of [GI1]*,* [AM] *has a unique integral lift.*

Proof. We consider the restriction of the motivic arithmetic square of [OO, Appendix A] at the prime 2, that is, for the motivic localization functors $L_{S_{\mathbb{Q}}}$ and $L_{S/2}$. As we know by computations of [DI2] and [HKO1] that $\pi_{9,5}(L_{S/2}S)$ and $\pi_{8,5}(L_{S/2}S)$ are finite 2-torsion, the associated long exact sequence degenerates to the isomorphism

$$\pi_{9,5}(S) \overset{\cong}{\to} \pi_{9,5}(L_{S/2}S) \oplus \pi_{9,5}(L_{S_{\mathbb{Q}}}S).$$

By Morel's theorem [Mo1], see also [CD], together with the fact $H_{mot}^{-9,-5}(Spec(\mathbb{C})) = 0$, we know that $\pi_{9,5}(L_{S_{\mathbb{Q}}}S) = 0$. $\qquad\square$

Observe that similar arguments apply to many other elements over $k = \mathbb{C}$, e.g., also to the higher μ_{8k+1} and to d_1, but not to τ. Using the recent work of Ananyevskiy-Levine-Panin [ALP], we also get results over other base fields:

Corollary 2.11. *For $k = \mathbb{R}$, the element $d_1 \in \pi_{32,18}(L_{S/2}S)$ lifts to a unique non-nilpotent element in $\pi_{32,18}(S)$.*

Proof. This is similar to the previous lemma. One has to replace the computations of [DI2] by the recent unpublished ones of Isaksen in bidegrees $(32, 18)$ and $(31, 18)$, and Morel's theorem by its refinement provided in [ALP] which implies the rational vanishing in the required degrees. $\qquad\square$

Hence, also for real fields there are other examples than η for non-nilpotent elements! Finally, note that if we cannot establish an integral lift for an element in the completion, then Theorem 2.8 still holds with localization replaced by completion.

3. RELATING THE MOTIVIC SPHERE SPECTRUM TO THE WITT SPECTRUM

We now discuss an interesting relationship between the η-local motivic sphere spectrum and Balmer's 4-periodic Witt groups, represented by **KT** [Ho1, Theorem 5.8],[ST]. For this, consider the unit map $S \to \mathbf{KO}$ for hermitian K-theory, sometimes also denoted by **BO** or **KQ** rather than **KO**. Together with the equivalence $\mathbf{KO}[\eta^{-1}] \simeq \mathbf{KT}$, it induces a map $S \to \mathbf{KT}$, which is the unit map for the naive ring spectrum **KT**. Observe that the equivalence $\mathbf{KO}[\eta^{-1}] \simeq \mathbf{KT}$ can be deduced from [RO, Theorem 4.4] arguing as in [Ho1, sections 4 and 5]. The following question assumes that $4 \cdot (1 - \epsilon)^2 \cdot \sigma = 0$ for the base field k. As explained above, this fails for subfields $k \subset \mathbb{R}$. It seems reasonable to expect it is true if -1 is a sum of squares.

Question 3.1. For any field k with $char(k) \neq 2$ and $4 \cdot (1 - \epsilon)^2 \cdot \sigma = 0$, does the above map

$$u : S[\eta^{-1}] \to \mathbf{KT}$$

in $\mathcal{SH}(k)$ factor through a map

$$u : S[\eta^{-1}, \mu_9^{-1}] \to \mathbf{KT}$$

and if so, what can we say about this map?

One might think of this question as an integral refinement of the recent rational result of [ALP], which provides an answer with rational coefficients. Morel's [Mo1] computation of $\pi_{**}(S)$ in simplicial degree 0 is also related to this question, of course.

We have a complete answer to the question over the complex numbers.

Theorem 3.2. *If $k = \mathbb{C}$, the unit map u of Question 3.1 factors through a map*

$$u : S[\eta^{-1}, \mu_9^{-1}] \to \mathbf{KT}.$$

The latter map induces an epimorphism

$$u_{**} : \pi_{**}(S[\eta^{-1}, \mu_9^{-1}]) \to \pi_{**}\mathbf{KT}$$

with the kernel being isomorphic to $\pi_{**}\mathbf{KT}$ shifted by bidegree $(7, 4)$.

As the above maps are maps of $S[\eta^{-1}]$-modules, we really have isomorphisms of $\pi_{**}S[\eta^{-1}]$-modules.

Proof. We show that $u_{**}(\mu_9)$ is invertible in $\pi_{**}(\mathbf{KT})$. For this, we first recall that the groups $\mathbf{KT}^{*,*}$ are $(4, 0)$- and $(1, 1)$-periodic [Ho1]. We also know that $W(\mathbb{C}) = W^{0,0}(\mathbb{C}) = \pi_{0,0}\mathbf{KT}$, and that $\pi_{i,0}\mathbf{KT} = W^{-i,0}(\mathbb{C}) = 0$ for $i = 1, 2, 3$ (which is true more generally for any field). We then consider the following commutative diagram:

$$\begin{array}{ccccc}
\pi_{9,5}(S) & \xrightarrow{u_{**}} & KO_{9,5} & \xrightarrow{\simeq} & KT_{9,5} \simeq W(\mathbb{C}) \simeq \mathbb{Z}/2 \\
\downarrow{\scriptstyle R_\mathbb{C}} & & \downarrow{\scriptstyle R_\mathbb{C}} & & \\
\pi_9 & \xrightarrow{u_*^{top}} & KO_9^{top} \simeq \mathbb{Z}/2 & &
\end{array}$$

where u and u^{top} are unit maps of ring spectra and $R_\mathbb{C}$ denotes complex topological realization. (There still seems to be no published proof of the folklore theorem that $R_\mathbb{C}(\mathbf{KO}) = \mathbf{KO}^{top}$. The best way to prove this is probably to apply the geometric description of $\mathbf{KO}$ established in [ST].) The element μ_9 is mapped to the topological μ_9 in the topological stable stem π_9 and then to the non-zero element under u^{top} by [Ad, Theorem 1.2]. As the square commutes, it is also mapped to the non-zero element in $KT_{9,5}$, which is invertible by the periodicity isomorphisms above. Hence we obtain the desired factorization. We now prove that

$$u_{**} : \pi_{**}(S[\eta^{-1}, \mu_9^{-1}]) \to \mathbf{KT}^{-*,-*}$$

is an epimorphism. By the Theorem of [AM], see Theorem 1.1 above, the groups $\pi_{**}(S[\eta^{-1}\mu_9^{-1}])$ are $(4, 0)$ and $(1, 1)$-periodic, and so are the groups $\mathbf{KT}^{*,*}$. As u_{**} is induced by the unit, it maps the generator of $\pi_{0,0}(S[\eta^{-1}\mu_9^{-1}])$ to the generator of $W^{0,0}(\mathbb{C}) \simeq \mathbb{Z}/2$. Moreover, we already saw that u_{**} maps the invertible elements η and μ_9 to invertible elements. The claim now follows from the above explicit structure of the two bigraded rings. $\qquad\square$

Remark 3.3. In a preliminary version of this article, it was incorrectly stated that the map on π_{**} is an isomorphism when replacing $S[\eta^{-1}, \mu_9^{-1}]$ by $S/(\sigma)[\eta^{-1}, \mu_9^{-1}]$. This is obviously wrong, as Bachmann pointed out. Note that in general the behaviour of motivic homotopy groups when coning out elements is at least as interesting as in the classical case; see, e.g., [Jo, Proposition 9.3.2]

What can we say about the question for other base fields? Assume that we have $4 \cdot (1 - \epsilon)^2 \cdot \sigma = 0$, or $2^t \cdot (1 - \epsilon)^{4-t} \cdot \sigma = 0$ for some other t. Then the

element μ_9 exists, and it is non-nilpotent – in particular non-zero – for any subfield $k \subset \mathbb{C}$ as the base change $\mathcal{SH}(k) \to \mathcal{SH}(\mathbb{C})$ preserves multiplication, Toda brackets and motivic Hopf elements. Hence the factorization in the question follows if we can show that μ_9 is mapped to a unit (this might follow for subfields $k \subset \mathbb{C}$ as above). Now is $u_* : \pi_{**}(S[\eta^{-1}, \mu_9^{-1}]) \to \mathbf{KT}^{-*,-*}$ an epimorphism in all bidegrees? By periodicity, this reduces to show the following. First, u_* is an isomorphism in bidegree $(0, 0)$. This corresponds to Morel's theorem [Mo1] before inverting μ_9. Second, the groups $\pi_{s,0}(S[\eta^{-1}, \mu_9^{-1}])$ have to be studied for $s = 1, 2, 3$. This will be the hard part, of course. The computations of $\pi_{1,*}(S)$ of [OO] for "low dimensional fields" and very recently by Röndigs-Spitzweck-Ostvaer [RSO] and Röndigs [R] for general base fields might be useful here, as well as their study of the behaviour of the unit map $S \to \mathbf{KT}$ on slices.

Observe that looking at the real realization functor does not help us, as it maps μ_9 to zero. We have not used the main theorem of [ALP] when we proved the conjecture for $k = \mathbb{C}$. However, looking at other fields it tells us at least that the odd torsion before inverting μ_9 and killing σ is not too large. As the conjecture does not apply to $k = \mathbb{R}$, the computations of [DI4] do not apply here, either. The recent article [GI1] contains computations for $\pi_{*,*}(S)[\eta^{-1}]$ over $\mathbb{R}$, and even more recently we have the theorem of Bachmann [B] at odd primes. Also, there is work in progress by Röndigs (now available, see [R]) on a related cell structure of $\mathbf{KT}$ over $\mathbb{C}$ which also relies on [AM] and implies that $\pi_{**}(S[\eta^{-1}])$ vanishes in simplicial degree 0 and 1. Finally, it remains to state a precise conjecture relating S and $\mathbf{KT}$ (or some other spectrum?) for those fields not covered by Question 3.1. Part (2) of [DI4, Theorem 1.1] gives a first hint on what kind of phenomena might occur.

Appendix: Nishida nilpotence in Milnor-Witt K-theory

As in the main part of this paper, we say that Nishida nilpotence holds in a certain degree of a graded ring if all torsion elements in that degree are nilpotent. Question 2.1 above asks over which fields and in which bidegrees Nishida nilpotence holds for $\pi_{**}(S)$, where S is the motivic sphere spectrum. In this short appendix, we answer this question for the zero-line $\bigoplus_d \pi_{d,d}(S)$ as follows:

> Let S denote the motivic sphere spectrum over a perfect field F of characteristic not two. Nishida nilpotence holds in $\pi_{d,d}(S)$ for all non-positive d. It holds for all d if and only if F is formally real.

Our answer rests on Morel's concrete description of the zero-line as the Milnor-Witt K-ring of the base field: $\pi_{d,d}(S) \cong K_{-d}^{MW}(F)$ for any field F as above [Mo1, Thm 6.2.1]. Indeed, given this identification, the answer is immediately implied by the following result:

Proposition A.1. *Let F be a field of characteristic not two. Nishida nilpotence holds in all non-negative degrees of the Milnor-Witt K-ring $K_*^{MW}(F)$. It holds in all degrees of $K_*^{MW}(F)$ if and only if F is formally real.*

The remainder of this appendix constitutes a proof of this proposition.

Throughout, F will denote a field of characterstic not two. Recall that F is either formally real (i.e., there exists at least one ordering on F) or non-real (i.e., -1 is a sum of squares in F), and that these two possibilities are mutually exclusive [La, Thm VIII.1.10]. For non-real fields, Nishida nilpotence fails because the non-nilpotent element $\eta \in K_{-1}^{MW}(F)$ is torsion in this case. Before we describe the

situation in non-positive degrees in more detail, let us recall the structure of the
Milnor-Witt K-ring in these degrees:

$$K_n^{MW}(F) \cong \begin{cases} GW(F) & \text{for } n = 0, \\ W(F) & \text{for } n < 0. \end{cases}$$

The ring structure on $K_{\leq 0}^{MW}(F)$ is determined by the ring structure on $GW(F)$ and
the fact that multiplication with the element $\eta \in K_{-1}^{MW}(F)$ corresponding to the
unit $\langle 1 \rangle \in W(F)$ induces the canonical projection $GW(F) \twoheadrightarrow W(F)$ from degree 0
to degree -1 and the identity in lower degrees. In particular, any homogeneous
element of $K_{\leq 0}^{MW}$ can be written as $\phi \eta^n$ for some $\phi \in W(F)$ and some $\eta \geq 0$.

Proposition A.2. *All torsion in $K_{\leq 0}^{MW}(F)$ is 2-primary torsion. In degree zero,
the nilpotent elements are precisely the torsion elements. In negative degrees, the
homogeneous nilpotent elements are precisely the elements of the form $\phi \eta^n$ with
$\phi \in W(F)$ a torsion element of even rank.*

The assertions of Proposition A.1 concerning non-positive degrees easily follow
from this proposition in view of facts (W_3) and (W_4) below. Proposition A.2 itself
is just a reformulation of the remaining following well-known facts:

(GW_1) All torsion in $GW(F)$ is 2-primary.
(GW_2) The nilpotent elements in $GW(F)$ are precisely the torsion elements.
(W_1) All torsion in $W(F)$ is two-primary.
(W_2) The nilpotent elements in $W(F)$ are precisely the torsion elements of even
rank.
(W_3) When F is formally real, all torsion elements in $W(F)$ are of even rank.
(W_4) When F is non-real, $W(F)$ is torsion. In particular, the unit $\langle 1 \rangle \in W(F)$
is a torsion element of odd rank.

Facts (W_3) and (W_4) are immediate consequences of Pfister's description of the
torsion subgroup of $W(F)$ as the kernel of the total signature homomorphism [La,
Thm VIII.3.2]: observe that the rank homomorphism $W(F) \to \mathbb{Z}/2$ factors through
any signature. Fact (W_1) is likewise stated in [La, Thm VIII.3.2]. For (W_2), see
eq. (8.16) at the end of section VIII.8 in [La] and observe that nilpotent elements
in $W(F)$ necessarily have even rank. The corresponding statements (GW_1) and
(GW_2) easily follow by considering the following cartesian square of rings:

$$\begin{array}{ccc} GW(F) & \longrightarrow\!\!\!\!\!\longrightarrow & W(F) \\ {\scriptstyle \text{rank}}\downarrow & & \downarrow{\scriptstyle \text{rank}} \\ \mathbb{Z} & \longrightarrow\!\!\!\!\!\longrightarrow & \mathbb{Z}/2 \end{array}$$

It remains to prove Proposition A.1 in positive degrees. The idea is to use
Morel's description of the Milnor-Witt K-ring as a fibre product of graded rings,
generalizing the cartesian square above. Let $K_*^M(F)$ denote the Milnor K-ring of
F, and let $k_*^M(F)$ denote the Milnor K-ring modulo two. Write $I(F) \subset W(F)$ for
the fundamental ideal, consisting of all elements of even rank, and let $I_*(F)$ denote
the graded ring given by $W(F)$ in negative degrees and by $I(F)^n$ in degrees $n > 0$.
By the positive answer to one of the famous questions in [Mi], the graded Witt
ring associated with the powers of the fundamental ideal is naturally isomorphic
to $k_*^M(F)$ [OVV]. In particular, we have a graded ring homomorphism $I_*(F) \to$

$k_*^M(F)$. Morel shows in [Mo2, Thm 5.3] that the Milnor-Witt K-ring fits into the following cartesian square of graded rings:

$$\begin{array}{ccc} K_*^{MW}(F) & \longrightarrow & I_*(F) \\ \downarrow & & \downarrow \\ K_*^M(F) & \longrightarrow & k_*^M(F) \end{array}$$

We briefly dwell on the lower left corner.

Lemma A.3. *Every element α of positive degree in $K_*^M(F)$ has a power of the form $\alpha^m = \{-1\}\gamma$, for some $m > 0$ and some $\gamma \in K_*^M(F)$.*

(Both m and γ depend on α, of course. The braces $\{-\}$ indicate the canonical isomorphism from the units of F to $K_1^M(F)$, translating multiplicative into additive notation.)

Proof. Suppose first that α is a generator of $K_n^M(F)$ ($n > 0$). Then [Mi, Lemma 1.2] implies $\alpha^2 = \pm\{-1\}^n\alpha$. In general, we can write α as $\alpha = \alpha_1 + \cdots + \alpha_k$ for certain generators α_i. Then α^{k+1} is a sum of products of the α_is, and in each summand at least one of the α_is appears at least twice. Thus, $\alpha^{k+1} = \{-1\}^n\gamma$ for some γ. $\square$

Lemma A.4. *Every element of positive degree in $K_*^M(F)$ that becomes nilpotent in $k_*^M(F)$ is already nilpotent in $K_*^M(F)$.*

Proof. Let α be such an element of positive degree. By assumption, α has a power of the form $\alpha^k = 2\beta$ for some β. By Lemma A.3, α also has some power of the form $\alpha^m = \{-1\}\gamma$ for some γ. So $\alpha^{m+k} = 2\beta\{-1\}\gamma$. This vanishes as $2\{-1\} = \{(-1)^2\} = 0$ in $K_1^M(F)$. $\square$

We now prove Proposition A.1 in degrees $n > 0$. Using the above cartesian square, we can write any element of $K_n^{MW}(F)$ as a pair (α, ϕ) with $\alpha \in K_n^M(F)$ and $\phi \in I^n(F)$ such that the image of ϕ in $k_n^M(F)$ agrees with the reduction of α modulo two. Let (α, ϕ) be such an element, and assume it is torsion. Then in particular, ϕ is an even-rank torsion element of $W(F)$, hence nilpotent in $I_*(F)$ by (W_2). A fortiori, its image in $k_*^M(F)$ is nilpotent. So by Lemma A.4, α is nilpotent in $K_*^M(F)$. Hence (α, ϕ) is nilpotent in $K_*^{MW}(F)$, as claimed.

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FACHGRUPPE MATHEMATIK UND INFORMATIK, BERGISCHE UNIVERSITÄT WUPPERTAL, GAUSS-STRASSE 20, 42119 WUPPERTAL, GERMANY

E-mail address: `hornbostel@math.uni-wuppertal.de`

MATHEMATISCHES INSTITUT, HEINRICH-HEINE-UNIVERSITÄT DÜSSELDORF, UNIVERSITÄTS-STRASSE 1, 40225 DÜSSELDORF, GERMANY

E-mail address: `marcus.zibrowius@uni-duesseldorf.de`