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# PRUNED HURWITZ NUMBERS

NORMAN DO AND PAUL NORBURY

ABSTRACT. Simple Hurwitz numbers count branched covers of the Riemann sphere and are well-studied in the literature. We define a new enumeration that restricts the count to branched covers satisfying an additional constraint. The resulting *pruned Hurwitz numbers* determine their simple counterparts, but have the advantage of satisfying simpler recursion relations and obeying simpler formulae. As an application of pruned Hurwitz numbers, we obtain a new proof of the Witten–Kontsevich theorem. Furthermore, we apply the idea of defining useful restricted enumerations to orbifold Hurwitz numbers and Belyi Hurwitz numbers.

## CONTENTS



## 1. INTRODUCTION

In 1891, Hurwitz introduced the problem of enumerating connected branched covers of $\mathbb{CP}^1$ with ramification over $\infty \in \mathbb{CP}^1$ given by a partition $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and simple ramification over $m$ fixed points [18]. The Riemann–Hurwitz formula asserts that the genus of the cover satisfies $m = m(g, \boldsymbol{\mu}) = 2g - 2 + n + |\boldsymbol{\mu}|$, where $|\boldsymbol{\mu}| = \mu_1 + \mu_2 + \cdots + \mu_n$. Hurwitz described the following equivalent factorisation problem in the symmetric group $S_{|\boldsymbol{\mu}|}$. We say that a product $\sigma_1\sigma_2 \cdots \sigma_m$ in $S_{|\boldsymbol{\mu}|}$ is *transitive* if the subgroup of $S_{|\boldsymbol{\mu}|}$ generated by $\sigma_1, \sigma_2, \ldots, \sigma_m$ acts transitively on the set $\{1, 2, \ldots, |\boldsymbol{\mu}|\}$. Given $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, choose $\tau \in S_{|\boldsymbol{\mu}|}$ of cycle type $\boldsymbol{\mu}$ and define $H_{g,n}(\boldsymbol{\mu})$ to be the number of transitive factorisations of $\tau$ into $m$ transpositions

$$\sigma_1\sigma_2 \cdots \sigma_m = \tau. \tag{1.1}$$

Note that the number $H_{g,n}(\boldsymbol{\mu})$ is independent of the choice of $\tau$.

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This problem was studied further by various people, including Hurwitz [19], Goulden–Jackson [15], and Ekedahl–Lando–Shapiro–Vainshtein [11], who showed that

$$\frac{H_{g,n}(\boldsymbol{\mu})}{m(g,\boldsymbol{\mu})!} = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i+1}}{\mu_i!}\, P_{g,n}(\mu_1,\ldots,\mu_n), \tag{1.2}$$

where $P_{g,n}$ is a polynomial of degree $3g-3+n$. The celebrated ELSV formula expresses the coefficients of these polynomials in terms of Hodge integrals over the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. See Section 2 for further details.

Each number in the set $\{1,2,\ldots,|\boldsymbol{\mu}|\}$ appears in at least one of the factors $\sigma_i$ on the left hand side of equation (1.1) due to the transitivity condition. In this paper, we introduce a new Hurwitz problem with one additional constraint.

**Definition 1.1.** Define the *pruned simple Hurwitz number* $K_{g,n}(\boldsymbol{\mu})$ to be the number of transitive factorisations (1.1) of a permutation $\tau$ of cycle type $\boldsymbol{\mu}$ into $m = 2g-2+n+|\boldsymbol{\mu}|$ transpositions such that each of the numbers $1,2,\ldots,|\boldsymbol{\mu}|$ appears in at least two of the factors $\sigma_i$.

The seemingly innocuous additional constraint that each number appears in at least two factors brings further deep structure to the problem. The pruned simple Hurwitz number enumeration is a restriction of the simple Hurwitz number enumeration, and it is rather surprising that it exhibits better behaviour. For example, pruned simple Hurwitz numbers obey a formula that is a vast simplification of the formula (1.2) for the usual simple Hurwitz numbers. The pruned condition can also be understood in terms of branched coverings, and the word *pruned* refers to the graphical description of simple Hurwitz numbers described in Section 2.

**Theorem 1.** *The pruned simple Hurwitz numbers satisfy:*

(i) $K_{g,n}(\mu_1,\ldots,\mu_n)$ *is equal to* $(2g-2+n+|\boldsymbol{\mu}|)!$ *multiplied by a polynomial in* $\mu_1,\ldots,\mu_n$ *of degree* $6g-6+3n$*;*
(ii) $K_{g,n}(\mu_1,\ldots,\mu_n)$ *satisfies an effective recursion; and*
(iii) *the numbers* $K_{g,n}(\mu_1,\ldots,\mu_n)$ *for fixed* $(g,n)$ *determine the numbers* $H_{g,n}(\mu_1,\ldots,\mu_n)$ *for fixed* $(g,n)$ *and vice versa.*

The recursion referred to in part (ii) of Theorem 1 is given explicitly by Proposition 3.3. It is not simply the restriction of the cut-and-join recursion for simple Hurwitz numbers, since the pruned condition — that each of the numbers $1,2,\ldots,|\boldsymbol{\mu}|$ appear in at least two of the factors $\sigma_i$ — is not preserved under the cut-and-join operations. In Section 3, we prove that the top degree coefficients of the polynomial $\frac{1}{m(g,\boldsymbol{\mu})!}K_{g,n}(\mu_1,\ldots,\mu_n)$ are intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Combining this result with the recursion leads to a new proof of the Witten–Kontsevich theorem. The relation referred to in part (iii) of Theorem 1 is given explicitly by Proposition 3.5.

The following example demonstrates how formulae for $K_{g,n}(\mu_1,\ldots,\mu_n)$ can be simpler than those for $H_{g,n}(\mu_1,\ldots,\mu_n)$.

**Example 1.2.** For a permutation $\tau \in S_{|\boldsymbol{\mu}|}$ of cycle type $\boldsymbol{\mu} = (\mu_1,\mu_2,\mu_3)$, consider transitive factorisations $\sigma_1\sigma_2\cdots\sigma_{|\boldsymbol{\mu}|+1} = \tau$ into transpositions. By the Riemann existence theorem, these correspond to genus 0 branched covers of $\mathbb{CP}^1$. We have the following two formulae for the number of such factorisations:

$$H_{0,3}(\mu_1,\mu_2,\mu_3) = (|\boldsymbol{\mu}|+1)!\prod_{i=1}^{3}\frac{\mu_i^{\mu_i+1}}{\mu_i!}, \qquad K_{0,3}(\mu_1,\mu_2,\mu_3) = (|\boldsymbol{\mu}|+1)!\,\mu_1\mu_2\mu_3.$$

Note that the second, which takes into account the pruned condition, is much simpler.

For any positive integer $a$, the *orbifold Hurwitz number* $H^{[a]}_{g,n}(\boldsymbol{\mu})$ is defined as follows. Given a tuple $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, choose $\tau \in S_{|\boldsymbol{\mu}|}$ of cycle type $\boldsymbol{\mu}$ and define $H^{[a]}_{g,n}(\boldsymbol{\mu})$ to be the number of transitive factorisations of $\tau$ into $m = 2g - 2 + n + \frac{|\boldsymbol{\mu}|}{a}$ transpositions and an element $\sigma_0$ of cycle type $(a, a, \ldots, a)$:

$$\sigma_0 \sigma_1 \sigma_2 \cdots \sigma_m = \tau. \tag{1.3}$$

In particular, $|\boldsymbol{\mu}|$ must be divisible by $a$ for the count to be non-zero. Such factorisations correspond to branched covers of $\mathbb{CP}^1$, and the Riemann–Hurwitz formula in this case asserts that the genus of the cover satisfies $m = 2g - 2 + n + \frac{|\boldsymbol{\mu}|}{a}$. The factor $\sigma_0$ in equation (1.3) defines a colouring of $\{1, 2, \ldots, |\boldsymbol{\mu}|\}$, where each of the $\frac{|\boldsymbol{\mu}|}{a}$ cycles of $\sigma_0$ is assigned a distinct colour. Each colour appears in at least one of the factors $\sigma_i$ for $i > 0$ of equation (1.3), due to the transitivity condition. We can generalise the pruning condition as follows.

**Definition 1.3.** Define the *pruned orbifold Hurwitz number* $K^{[a]}_{g,n}(\boldsymbol{\mu})$ to be the number of transitive factorisations (1.3) of a permutation $\tau$ of cycle type $\boldsymbol{\mu}$ into $m = 2g - 2 + n + \frac{|\boldsymbol{\mu}|}{a}$ transpositions and an element $\sigma_0$ of cycle type $(a, a, \ldots, a)$ such that each colour — determined by $\sigma_0$ — appears in at least two of the factors $\sigma_i$ for $i > 0$.

When $a = 1$, this reduces to the pruned simple Hurwitz numbers. Theorem 1 is a special case of the following result.

**Theorem 2.** *The pruned orbifold Hurwitz numbers satisfy:*

(i) $K^{[a]}_{g,n}(\mu_1, \ldots, \mu_n)$ *is equal to* $\big(2g - 2 + n + \frac{|\boldsymbol{\mu}|}{a}\big)!$ *multiplied by a polynomial in* $\mu_1, \ldots, \mu_n$ *of degree* $6g - 6 + 3n$*;*

(ii) $K^{[a]}_{g,n}(\mu_1, \ldots, \mu_n)$ *satisfies an effective recursion; and*

(iii) *the numbers* $K^{[a]}_{g,n}(\mu_1, \ldots, \mu_n)$ *for fixed* $(g, n)$ *determine the numbers* $H^{[a]}_{g,n}(\mu_1, \ldots, \mu_n)$ *for fixed* $(g, n)$ *and vice versa.*

The notion of pruning applies to a broader set of combinatorial problems arising from geometry. It is related to the rational behaviour of certain generating functions. For example, assemble the orbifold Hurwitz numbers into the following generating functions:

$$\mathcal{H}^{[a]}_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} \frac{H^{[a]}_{g,n}(\boldsymbol{\mu})}{m!} \frac{x_1^{\mu_1} \cdots x_n^{\mu_n}}{\mu_1 \cdots \mu_n}.$$

Then $\mathcal{H}^{[a]}_{g,n}(x_1, \ldots, x_n)$ is a convergent power series that extends to a rational function in $z_1, \ldots, z_n$, where $x_i = z_i \exp(-z_i^a)$. A local expansion of $\mathcal{H}^{[a]}_{g,n}(x_1, \ldots, x_n)$ in $z_1, \ldots, z_n$ yields a generating function for $K^{[a]}_{g,n}(\mu_1, \ldots, \mu_n)$. One of the main observations of this paper is that $K^{[a]}_{g,n}(\mu_1, \ldots, \mu_n)$ can be realised as the weighted count of an interesting combinatorial and geometric problem.

Another application of pruning — in which expansion of certain generating functions in a rational parameter gives rise to an interesting combinatorial and geometric problem — arises in the case of *Belyi Hurwitz numbers.* Consider connected genus $g$ branched covers $\pi : \Sigma \to \mathbb{CP}^1$ unramified over $\mathbb{CP}^1 - \{0, 1, \infty\}$, with ramification

$(\mu_1, \ldots, \mu_n)$ at labelled points $(p_1, \ldots, p_n)$ over $\infty$, ramification $(2, 2, \ldots, 2)$ over 1, and arbitrary ramification over 0. We call the weighted count of these branched covers a Belyi Hurwitz number because such branched covers are known as Belyi maps. Pruned Belyi Hurwitz covers are those with the additional constraint that all points over 0 have non-trivial ramification. There is an analogue of Theorem 1 for the case of pruned Belyi Hurwitz numbers that is discussed further in Section 5.

By pruning an enumerative problem, one aim is to produce a simpler problem that uncovers structure of the original version. Belyi Hurwitz numbers provide a good example of a case where the pruned and unpruned versions have independent interest. Unpruned Belyi Hurwitz numbers arise from discrete surfaces and matrix integral calculations, whereas pruned Belyi Hurwitz numbers can be interpreted as the enumeration of lattice points in the moduli space of curves $\mathcal{M}_{g,n}$. The latter viewpoint gives rise to deep geometric information, such as intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and the orbifold Euler characteristic of $\mathcal{M}_{g,n}$ [25].

The enumerative problems that are amenable to pruning discussed above have a further feature in common. They each satisfy the topological recursion of Chekhov, Eynard, and Orantin. Given a rational plane curve $C$ known as a *spectral curve*, the topological recursion defines a multidifferential $\omega_{g,n}(p_1, \ldots, p_n)$ — that is, a tensor product of meromorphic 1-forms — for integers $g \geq 0$ and $n \geq 1$ on the product $C^n$ [7, 13]. When $2g - 2 + n > 0$, the multidifferential $\omega_{g,n}(p_1, \ldots, p_n)$ is defined recursively in terms of local information around the poles of $\omega_{g',n'}(p_1, \ldots, p_{n'})$ for $2g' + 2 - n' < 2g - 2 + n$. The generating functions for each of the examples above have been shown to arise as expansions of these multidifferentials arising from particular rational plane curves. For simple Hurwitz numbers, this was known as the Bouchard–Mariño conjecture [6] and was subsequently proven in [4, 12]. A generalisation of this result to orbifold Hurwitz numbers was proven in [5, 8], and an analogue of this result for Belyi Hurwitz numbers was proven in [14, 25]. A natural question that arises is whether one can apply the idea of pruning to other enumerative problems that satisfy the topological recursion of Eynard and Orantin. A good candidate of geometric interest is the stationary Gromov–Witten theory of $\mathbb{CP}^1$, which is known to satisfy the topological recursion [10, 27]. We discuss this in Section 6.

## 2. Simple Hurwitz numbers

In this section, we consider the case of pruned simple Hurwitz numbers separately from pruned orbifold Hurwitz numbers. This is due to the independent interest of simple Hurwitz numbers and also because this easier case should help the reader understand the general case of orbifold Hurwitz numbers treated in Section 4.

We begin by formally defining simple Hurwitz numbers via branched covers. For $g \geq 0$ and $n \geq 1$, define the set of simple Hurwitz covers as follows:

$$\mathcal{H}_{g,n}(\boldsymbol{\mu}) = \left\{ f : \Sigma \to \mathbb{CP}^1 \;\middle|\; \begin{array}{l} \Sigma \text{ a connected genus } g \text{ Riemann surface;} \\ \text{simple ramification over } m\text{th roots of unity;} \\ f^{-1}(\infty) = (p_1, \ldots, p_n) \text{ with respective} \\ \text{ramification } \boldsymbol{\mu} = (\mu_1, \ldots, \mu_n); \\ f \text{ unramified elsewhere} \end{array} \right\}.$$

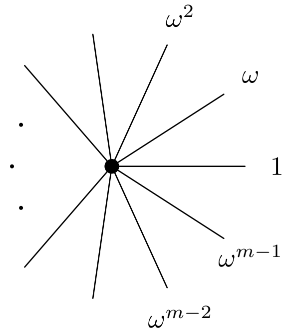


FIGURE 1. The star graph.

Here, $m = 2g-2+n+|\boldsymbol{\mu}|$ and we consider two branched covers $f_1 : \Sigma_1 \to \mathbb{CP}^1$ and $f_2 : \Sigma_2 \to \mathbb{CP}^1$ to be equivalent if there exists $h : \Sigma_1 \to \Sigma_2$ that satisfies $f_1 = f_2 \circ h$ while preserving the labels over $\infty$.

Define the simple Hurwitz numbers

$$H_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{f \in \mathcal{H}_{g,n}(\boldsymbol{\mu})} \frac{\mu_1 \cdots \mu_n}{|\mathrm{Aut}\ f|}. \tag{2.1}$$

The summands in equation (2.1) are integral essentially because the automorphism group is small. An automorphism of the branched cover $f : (\Sigma; p_1, p_2, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ is an automorphism $\phi$ of the marked Riemann surface $(\Sigma; p_1, p_2, \ldots, p_n)$ such that $f = f \circ \phi$. The automorphism group is only non-trivial for hyperelliptic covers of $\mathbb{CP}^1$ with one point at infinity. In such cases, the automorphism group has order 2, while the numerator of the summand in equation (2.1) is also 2.

*Remark* 2.1. By the Riemann existence theorem, such a branched cover is prescribed by the location of the ramification points and the monodromy around each. Therefore, equations (1.1) and (2.1) give equivalent definitions. Any cycle of $\tau$ acts by conjugation on factorisations (1.1) since it fixes $\tau$ and preserves the cycle type of transpositions. The orbits of this action have size equal to the summands of (2.1), which turns out to be $\mu_1 \cdots \mu_n$ generically, or 1 in the exceptional case $\boldsymbol{\mu} = (2)$.

*Remark* 2.2. Different normalisations of simple Hurwitz numbers are often defined in the literature. They may differ by factors of $\mu_1 \cdots \mu_n$ and $|\mathrm{Aut}\ \boldsymbol{\mu}|$ where $\mathrm{Aut}\ \boldsymbol{\mu}$ consists of the permutations of the tuple $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ that leave it fixed. Differing normalisations may arise depending on whether one labels the preimages of $\infty$ in the branched cover. For this reason, the form of the cut-and-join relation and the ELSV formula may appear different from other appearances in the literature.

Given $f \in \mathcal{H}_{g,n}(\boldsymbol{\mu})$, its *branching graph* is $f^{-1}(\Gamma_m) \subset \Sigma$, where $\Gamma_m \subset \mathbb{C}$ is the star graph given by the cone on the $m$th roots of unity [2].

Conversely, a branching graph gives rise to $f \in \mathcal{H}_{g,n}(\boldsymbol{\mu})$, so the simple Hurwitz number $H_{g,n}(\boldsymbol{\mu})$ can be interpreted as an enumeration of branching graphs. These are fatgraphs with certain edge labellings, where we allow fatgraphs to contain half-edges as well as full edges. For more information on fatgraphs — also referred to as ribbon graphs, embedded graphs, and graphs with rotation — see the book of Lando and Zvonkin [22].

**Definition 2.3.** We define a *branching graph of type* $(g; \boldsymbol{\mu})$ to be an edge-labelled fatgraph of type $(g, \ell(\boldsymbol{\mu}))$ such that for $m = 2g - 2 + \ell(\boldsymbol{\mu}) + |\boldsymbol{\mu}|$:

- there are $|\boldsymbol{\mu}|$ vertices and each of them is adjacent to $m$ half-edges that are cyclically labelled $1, 2, \ldots, m$;
- there are exactly $m$ full edges that are labelled $1, 2, \ldots, m$;
- the $n$ faces are labelled $1, 2, \ldots, n$ and have perimeters given by $(m\mu_1, m\mu_2, \ldots, m\mu_n)$; and
- each face has a marked $m$-label. (Note that the face labelled $k$ has $\mu_k$ appearances of an $m$-label.)

The set of all branching graphs of type $(g; \boldsymbol{\mu})$ is denoted $\mathrm{Br}_{g,n}(\boldsymbol{\mu})$.

The marked $m$-labels give locations for removing or attaching edges in the cut-and-join recursion below and remove non-trivial automorphisms. They also give rise to an unweighted count that produces simple Hurwitz numbers.

**Proposition 2.4.** *The simple Hurwitz number $H_{g,n}(\mu_1, \ldots, \mu_n)$ enumerates branching graphs in the following way:*

$$H_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\Gamma \in \mathrm{Br}_{g,n}(\boldsymbol{\mu})} 1.$$

We now assemble three fundamental results concerning simple Hurwitz numbers, using the normalisation

$$\widehat{H}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{H_{g,n}(\mu_1, \ldots, \mu_n)}{(2g - 2 + n + |\boldsymbol{\mu}|)!}.$$

The cut-and-join recursion for simple Hurwitz numbers can be obtained by considering the removal of an edge from a branching graph [16]:

(2.2)

$$m\, \widehat{H}_{g,n}(\boldsymbol{\mu}_S) = \sum_{i<j} \mu_i \mu_j\, \widehat{H}_{g,n-1}(\boldsymbol{\mu}_{S \setminus \{i,j\}}, \mu_i + \mu_j)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \mu_i \sum_{\alpha + \beta = \mu_i} \Bigg[ \widehat{H}_{g-1,n+1}(\boldsymbol{\mu}_{S \setminus \{i\}}, \alpha, \beta) \tag{2.3}$$

$$+ \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = S \setminus \{i\}}} \widehat{H}_{g_1, |I|+1}(\mu_I, \alpha)\, \widehat{H}_{g_2, |J|+1}(\mu_J, \beta) \Bigg].$$

Here, we have used the notation $S = \{1, 2, \ldots, n\}$ and $\boldsymbol{\mu}_I = (\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_k})$ for $I = \{i_1, i_2, \ldots, i_k\}$.

**Proposition 2.5** (ELSV formula [11])**.** *The simple Hurwitz number $\widehat{H}_{g,n}(\mu_1, \ldots, \mu_n)$ satisfies*

$$\widehat{H}_{g,n}(\mu_1, \ldots, \mu_n) = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i + 1}}{\mu_i!} \sum_{|\mathbf{d}| + \ell = 3g-3+n} (-1)^\ell \langle \tau_{d_1} \cdots \tau_{d_n} \lambda_\ell \rangle_g\, \mu_1^{d_1} \cdots \mu_n^{d_n},$$

*where $\langle \tau_{d_1} \cdots \tau_{d_n} \lambda_\ell \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_\ell$ is an intersection number of tautological classes on the moduli space of stable pointed curves.*

Let $\omega_{g,n}$ denote the multidifferentials output by the topological recursion applied to the spectral curve

$$x(z) = z\exp(-z) \qquad \text{and} \qquad y(z) = z.$$

The Bouchard–Mariño conjecture [6] was proven in [4,12] and states the following.

**Theorem 3.** *The expansion of $\omega_{g,n}$ at $x_1 = x_2 = \cdots = x_n = 0$ is given by*

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{H}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} x_i^{\mu_i-1}\,\mathrm{d}x_i. \tag{2.4}$$

## 3. Pruned Hurwitz numbers

3.1. **Pruned simple Hurwitz numbers.** In the previous section, we interpreted simple Hurwitz numbers as an enumeration of branching graphs. In this section, we define pruned simple Hurwitz numbers by restricting to the set of branching graphs that satisfy an additional constraint on the vertex degrees. We will show that simple Hurwitz numbers can be recovered from their pruned counterparts and vice versa. One advantage of studying pruned simple Hurwitz numbers is that they possess an inherent polynomial structure that allows geometric information to be easily extracted. We conclude the section with an application of this methodology to obtain a new proof of the Witten–Kontsevich theorem.

We define the *essential degree* of a vertex in a branching graph to be the number of incident full edges. The branching graph of $f \in \mathcal{H}_{g,n}(\boldsymbol{\mu})$ can be equivalently described as a triple $(X, \tau_0, \tau_1)$ where $X = f^{-1}(\{\zeta^0, \zeta^1, \ldots, \zeta^{m-1}\})$ for $\zeta$ a primitive $m$th root of unity, equipped with automorphisms $\tau_0 : X \to X$ given by the monodromy map around 0 and $\tau_1 : X \to X$ given by the monodromy maps around the roots of unity. The full edges, often simply called edges, correspond to orbits of $\tau_1$ of length 2, whereas half-edges correspond to fixed points of $\tau_1$.

For $g \geq 0$ and $n \geq 1$, define the set of pruned simple Hurwitz covers as follows.

$$\mathcal{K}_{g,n}(\boldsymbol{\mu}) = \big\{ f \in \mathcal{H}_{g,n}(\boldsymbol{\mu}) \mid \text{ all vertices of the branching graph } f^{-1}(\Gamma_m) \text{ have essential degree at least two} \big\}.$$

We call a branching graph *pruned* if all of its vertices have essential degree at least two and denote the set of pruned branching graphs of type $(g;\boldsymbol{\mu})$ by $\mathrm{PFat}_{g,n}(\boldsymbol{\mu})$. Define the pruned simple Hurwitz numbers by

$$K_{g,n}(\mu_1,\ldots,\mu_n) = \sum_{f \in \mathcal{K}_{g,n}(\boldsymbol{\mu})} \frac{\mu_1\cdots\mu_n}{|\mathrm{Aut}\ f|} = \sum_{\Gamma \in \mathrm{PFat}_{g,n}(\boldsymbol{\mu})} 1.$$

This definition agrees with the definition of pruned simple Hurwitz numbers given in the introduction via factorisations in the symmetric group. Furthermore, let $m(g,\boldsymbol{\mu}) = 2g-2+n+|\boldsymbol{\mu}|$ and define the normalisation

$$\widehat{K}_{g,n}(\boldsymbol{\mu}) = \frac{K_{g,n}(\boldsymbol{\mu})}{m(g,\boldsymbol{\mu})!}.$$

**Example 3.1.** The edges of a branching graph with $(g,n) = (0,1)$ necessarily form a tree. So in this case, there does not exist a pruned branching graph and we have $K_{0,1}(\mu_1) = 0$ for all positive integers $\mu_1$.

**Example 3.2.** The edges of a branching graph with $(g, n) = (0, 2)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ necessarily form a cycle with $\mu_1 + \mu_2$ edges. Remove the edge labelled $\mu_1 + \mu_2$ and record the labels of the remaining edges in an anticlockwise fashion around face 1 to obtain a permutation of the set $\{1, 2, \ldots, \mu_1 + \mu_2 - 1\}$. The contribution to the perimeter of face 1 is one more than the number of ascents of the resulting permutation. Therefore, we have

$$K_{0,2}(\mu_1, \mu_2) = \mu_1\mu_2\, A(\mu_1 + \mu_2 - 1, \mu_1 - 1)$$

for all positive integers $\mu_1$ and $\mu_2$. Here, $A(m, n)$ represents the Eulerian number that counts the number of permutations of the set $\{1, 2, \ldots, m\}$ with $n$ ascents.

The cut-and-join recursion provides a recursive method to calculate simple Hurwitz numbers [15]. The next result establishes an analogous recursion for pruned simple Hurwitz numbers.

**Proposition 3.3** (Cut-and-join recursion for pruned simple Hurwitz numbers)**.** *The following equation holds for all* $2g - 2 + n > 0$ *and* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ :

$$\begin{aligned} m(g, \boldsymbol{\mu})\, \widehat{K}_{g,n}(\boldsymbol{\mu}) &= \sum_{i<j} \mu_i\mu_j \sum_{\alpha+\beta=\mu_i+\mu_j+1} \beta\, \widehat{K}_{g,n-1}(\boldsymbol{\mu}_{S\setminus\{i,j\}}, \alpha) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \mu_i \sum_{\alpha+\beta+\gamma=\mu_i+1} \gamma \Bigg[ \widehat{K}_{g-1,n+1}(\boldsymbol{\mu}_{S\setminus\{i\}}, \alpha, \beta) \\ &\qquad + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} \widehat{K}_{g_1,|I|+1}(\boldsymbol{\mu}_I, \alpha)\, \widehat{K}_{g_2,|J|+1}(\boldsymbol{\mu}_J, \beta) \Bigg]. \end{aligned}$$

*We use the notation* $S = \{1, 2, \ldots, n\}$ *and* $\boldsymbol{\mu}_I = (\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_k})$ *for* $I = \{i_1, i_2, \ldots, i_k\}$*. The word* stable *over the final summation indicates that summands involving* $\widehat{K}_{0,1}$ *or* $\widehat{K}_{0,2}$ *are to be excluded.*

**Example 3.4.** As an example of the cut-and-join recursion for pruned simple Hurwitz numbers in action, consider the following calculation of $\widehat{K}_{0,4}(\mu_1, \mu_2, \mu_3, \mu_4)$, which uses $\widehat{K}_{0,3}(\mu_1, \mu_2, \mu_3) = \mu_1\mu_2\mu_3$:

$$\begin{aligned} (|\boldsymbol{\mu}| + 2)\, \widehat{K}_{0,4}(\mu_1, \mu_2, \mu_3, \mu_4) &= \mu_1\mu_2\mu_3\mu_4 \sum_{i<j} \sum_{\alpha+\beta=\mu_i+\mu_j+1} \alpha\beta \\ &= (|\boldsymbol{\mu}| + 2)\, \mu_1\mu_2\mu_3\mu_4\, \tfrac{1}{2}(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 + \mu_1 + \mu_2 + \mu_3 + \mu_4). \end{aligned}$$

Therefore, we conclude that

$$K_{0,4}(\mu_1, \mu_2, \mu_3, \mu_4) = \tfrac{1}{2}(|\boldsymbol{\mu}| + 2)!\, \mu_1\mu_2\mu_3\mu_4 \sum_{i=1}^{4} (\mu_i^2 + \mu_i).$$

In contrast, the calculation of $H_{0,4}(\mu_1, \mu_2, \mu_3, \mu_4)$ via the cut-and-join recursion (2.2) is far from straightforward since it involves combinatorial identities more difficult than sums of polynomials. In particular, note that $H_{0,4}$ terms would appear on both sides of the recursion.

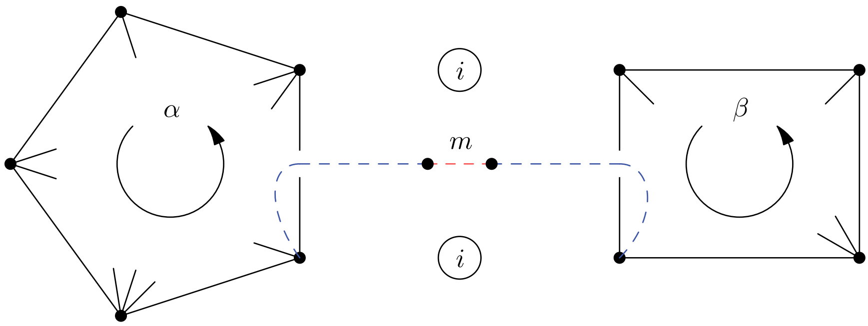


FIGURE 2. The edge labelled $m$ is adjacent to the face labelled $i$ on both sides.

*Proof of Proposition* 3.3. We begin by expressing the cut-and-join recursion without the normalisation:

$$K_{g,n}(\boldsymbol{\mu}) = \sum_{i<j} \mu_i\mu_j \sum_{\alpha+\beta=\mu_i+\mu_j+1} \beta \,\frac{(m-1)!}{(m-\beta)!}\, K_{g,n-1}(\boldsymbol{\mu}_{S\setminus\{i,j\}}, \alpha)$$
$$+ \frac{1}{2}\sum_{i=1}^{n} \mu_i \sum_{\alpha+\beta+\gamma=\mu_i+1} \gamma\,(m-1)! \Bigg[ \frac{K_{g-1,n+1}(\boldsymbol{\mu}_{S\setminus\{i\}}, \alpha, \beta)}{(m-\gamma)!}$$
$$+ \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} \frac{K_{g_1,|I|+1}(\boldsymbol{\mu}_I, \alpha)\; K_{g_2,|J|+1}(\boldsymbol{\mu}_J, \beta)}{m_1!\, m_2!} \Bigg].$$

Here, we use the notation $m_1 = 2g_1-1+|I|+|\boldsymbol{\mu}_I|+\alpha$ and $m_2 = 2g_2-1+|J|+|\boldsymbol{\mu}_J|+\beta$. The conditions $g_1 + g_2 = g$, $I \sqcup J = S \setminus \{i\}$, and $\alpha + \beta + \gamma = \mu_i + 1$ imply that $m_1 + m_2 = m - \gamma$.

Recall that $K_{g,n}(\boldsymbol{\mu})$ is the number of pruned branching graphs in $\mathrm{PFat}_{g,n}(\boldsymbol{\mu})$. Choose such a branching graph and remove the edge labelled $m$ from it. Repeatedly remove vertices with essential degree one and their incident edges until all of the vertices of the resulting branching graph have essential degree at least two. When removing an edge with a given label, we also remove all half-edges with the corresponding label. The removed edges necessarily form a path in the original branching graph. Observe that one of the following three cases must arise (see Figures 2 and 3).

- *The edge labelled $m$ is adjacent to the face labelled $i$ on both sides and its removal leaves a connected graph.*

  Suppose that $\gamma$ edges are removed in total, so that a branching graph in $\mathrm{PFat}_{g-1,n+1}(\boldsymbol{\mu}_{S\setminus\{i\}}, \alpha, \beta)$ remains, where $\alpha + \beta + \gamma = \mu_i + 1$.

  Conversely, there are $\frac{1}{2}\,\mu_i\gamma\,\frac{(m-1)!}{(m-\gamma)!}$ ways to reconstruct a branching graph in $\mathrm{PFat}_{g,n}(\boldsymbol{\mu})$ from a branching graph in $\mathrm{PFat}_{g-1,n+1}(\boldsymbol{\mu}_{S\setminus\{i\}}, \alpha, \beta)$ by adding a path of $\gamma$ edges. When adding an edge with a given label, we also add all possible half-edges with the corresponding label, while maintaining the correct cyclic ordering of the half-edges at every vertex. The factor $\mu_i$ accounts for the position of the new marked $m$-labelled edge. The factor $\gamma$ accounts for the position of the edge labelled $m$ along the path.

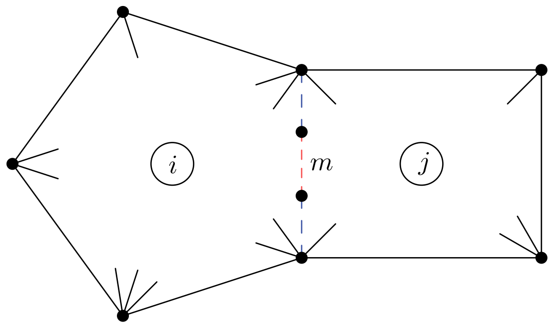


FIGURE 3. The edge labelled $m$ is adjacent to two distinct faces labelled $i$ and $j$.

The factor $\frac{(m-1)!}{(m-\gamma)!}$ accounts for the edge labels appearing on the remaining edges of the path. It is then necessary to adjust by the factor $\frac{1}{2}$ due to the overcounting caused by the symmetry in $\alpha$ and $\beta$.

- *The edge labelled $m$ is adjacent to the face labelled $i$ on both sides and its removal leaves the disjoint union of two connected graphs.*

  Suppose that $\gamma$ edges are removed in total, so that the disjoint union of two branching graphs in $\mathrm{PFat}_{g_1,|I|+1}(\boldsymbol{\mu}_I, \alpha)$ and $\mathrm{PFat}_{g_2,|J|+1}(\boldsymbol{\mu}_J, \beta)$ remain, where $\alpha + \beta + \gamma = \mu_i + 1$, $g_1 + g_2 = g$, and $I \sqcup J = S \setminus \{i\}$.

  Conversely, there are $\frac{1}{2}\, \mu_i \gamma \, \frac{(m-1)!}{m_1!\, m_2!}$ ways to reconstruct a branching graph in $\mathrm{PFat}_{g,n}(\boldsymbol{\mu})$ from a pair of branching graphs in $\mathrm{PFat}_{g_1,|I|+1}(\boldsymbol{\mu}_I, \alpha)$ and $\mathrm{PFat}_{g_2,|J|+1}(\boldsymbol{\mu}_J, \beta)$ by adding a path of $\gamma$ edges. When adding an edge with a given label, we also add all possible half-edges with the corresponding label, while maintaining the correct cyclic ordering of the half-edges at every vertex. The factor $\mu_i$ accounts for the position of the new marked $m$-labelled edge. The factor $\gamma$ accounts for the position of the edge labelled $m$ along the path. The factor $\frac{(m-1)!}{m_1!\, m_2!}$ accounts for the distribution of the edge labels $\{1, 2, \ldots, m-1\}$ between the two branching graphs. It is then necessary to adjust by the factor $\frac{1}{2}$ due to the overcounting caused by the symmetry in $(g_1, I, \alpha)$ and $(g_2, J, \beta)$.

- *The edge labelled $m$ is adjacent to two distinct faces labelled $i$ and $j$.*

  Suppose that $\beta$ edges are removed in total, so that a branching graph in $\mathrm{PFat}_{g,n-1}(\boldsymbol{\mu}_{S\setminus\{i,j\}}, \alpha)$ remains, where $\alpha + \beta = \mu_i + \mu_j + 1$.

  Conversely, there are $\mu_i \mu_j \beta \, \frac{(m-1)!}{(m-\beta)!}$ ways to reconstruct a branching graph in $\mathrm{PFat}_{g,n}(\boldsymbol{\mu})$ from a branching graph in $\mathrm{PFat}_{g,n-1}(\boldsymbol{\mu}_{S\setminus\{i,j\}}, \alpha)$ by adding a path of $\beta$ edges. When adding an edge with a given label, we also add all possible half-edges with the corresponding label, while maintaining the correct cyclic ordering of the half-edges at every vertex. The factor $\mu_i \mu_j$ accounts for the positions of the marked $m$-labelled edges on faces $i$ and $j$. The factor $\beta$ accounts for the position of the edge labelled $m$ along the path. The factor $\frac{(m-1)!}{(m-\beta)!}$ accounts for the edge labels appearing on the remaining edges of the path.

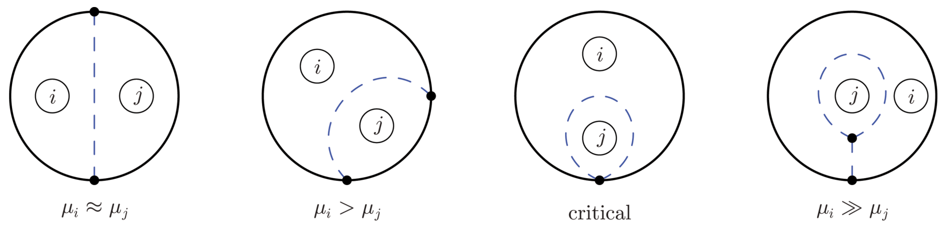


FIGURE 4. A subtlety arises when the edge labelled $m$ is adjacent to two distinct faces.

There is a crucial subtlety that arises in the third case, which we now address. One can discern the issue by considering the sequence of diagrams in Figure 4, in which $\mu_i$ increases from left to right, relative to $\mu_j$.

The factor $\mu_i \mu_j \beta \frac{(m-1)!}{(m-\beta)!}$ in the third case actually contributes to diagrams like the one on the far right, in which face $i$ completely surrounds face $j$, or vice versa. In fact, the edge labelled $m$ that we remove can lie anywhere along the dashed path in the schematic diagram. Note that this contributes to the second case, in which the edge labelled $m$ is adjacent to the face labelled $i$ on both sides and its removal leaves the disjoint union of two connected graphs. However, observe that this surplus contribution is precisely equal to the terms from the second case that involve $\widehat{K}_{0,2}$, so one can compensate simply by excluding such terms. Given that we have already witnessed in Example 3.1 that $\widehat{K}_{0,1} = 0$, we can restrict to the so-called *stable* terms in the second case, which are precisely those that do not involve $\widehat{K}_{0,1}$ or $\widehat{K}_{0,2}$.

Therefore, to obtain all fatgraphs in $\mathrm{PFat}_{g,n}(\boldsymbol{\mu})$ exactly once, it is necessary to perform the reconstruction process:

- in the first case for all values of $i$ and $\alpha + \beta + \gamma = \mu_i + 1$;
- in the second case for all *stable* values of $i$, $\alpha + \beta + \gamma = \mu_i + 1$, $g_1 + g_2 = g$, and $I \sqcup J = S \setminus \{i\}$; and
- in the third case for all values of $i$, $j$, and $\alpha + \beta = \mu_i + \mu_j + 1$.

We obtain the cut-and-join recursion for pruned simple Hurwitz numbers by summing up over all these contributions. □

3.2. **The pruning correspondence.** Despite the fact that $K_{g,n}(\boldsymbol{\mu})$ only counts a subset of the branching graphs enumerated by $H_{g,n}(\boldsymbol{\mu})$, simple Hurwitz numbers can be determined from their pruned counterparts, and vice versa. The crucial observation is the following result.

**Proposition 3.5.** *The following equation holds for all* $(g,n) \neq (0,1)$ *and* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ :

$$\widehat{H}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\nu_1, \ldots, \nu_n = 1}^{\mu_1, \ldots, \mu_n} \widehat{K}_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \frac{\mu_i^{\mu_i - \nu_i}}{(\mu_i - \nu_i)!}.$$

*Proof.* We begin by writing the proposition in the following way:

$$H_{g,n}(\boldsymbol{\mu}) = \sum_{\nu_1, \ldots, \nu_n = 1}^{\mu_1, \ldots, \mu_n} K_{g,n}(\boldsymbol{\nu}) \frac{(2g - 2 + n + |\boldsymbol{\mu}|)!}{(2g - 2 + n + |\boldsymbol{\nu}|)!\,(\mu_1 - \nu_1)! \cdots (\mu_n - \nu_n)!} \prod_{i=1}^{n} \mu_i^{\mu_i - \nu_i}.$$

This equation encapsulates the fact that, from a branching graph, one obtains a unique pruned branching graph by repeatedly removing vertices with essential degree one and their incident edges. The process continues until all of the vertices of the resulting branching graph have essential degree at least two. When removing an edge with a given label, we also remove all half-edges with the corresponding label. It is then necessary to relabel the edges and half-edges in the resulting branching graph so that the new labels come from a set of the form $\{1, 2, \ldots, m\}$, while maintaining the correct cyclic ordering of the half-edges at every vertex. We refer to the process described above as *pruning* and observe that it can be carried out one face at a time.

Conversely, every branching graph of type $(g; \boldsymbol{\mu})$ can be reconstructed from a pruned branching graph of type $(g; \boldsymbol{\nu})$ for $1 \leq \nu_i \leq \mu_i$ by adding $\mu_i - \nu_i$ edges to face $i$, for all $i = 1, 2, \ldots, n$. When adding an edge with a given label, we also add all possible half-edges with the corresponding label. It is then necessary to relabel the edges and half-edges in the resulting branching graph so that the new labels come from a set of the form $\{1, 2, \ldots, m\}$, while maintaining the correct cyclic ordering of the half-edges at every vertex.

There are $K_{g,n}(\boldsymbol{\nu})$ possibilities for the pruned branching graph and the factor

$$\frac{(2g - 2 + n + |\boldsymbol{\mu}|)!}{(2g - 2 + n + |\boldsymbol{\nu}|)!\,(\mu_1 - \nu_1)! \cdots (\mu_n - \nu_n)!}$$

accounts for the number of ways to choose the set of edge labels for the underlying pruned branching graph as well as the set of $\mu_i - \nu_i$ edge labels to be added to face $i$ for $i = 1, 2, \ldots, n$.

All that remains is to show that the factor $\mu^{\mu-\nu}$ is equal to the number of ways to add $\mu - \nu$ edges to a pruned face with perimeter $\nu$. To do this, we invoke the following generalisation of Cayley's formula:

> Let $N \subseteq M$ be sets of size $\nu \leq \mu$, respectively. Then the number of rooted forests on $\mu$ vertices labelled by $M$ with $\nu$ components whose roots are labelled by $N$ is precisely $T(\mu, \nu) = \nu\mu^{\mu-\nu-1}$.

See for example [1] for a proof of the formula for $T(\mu, \nu)$.

Consider a face of perimeter $\mu$ in a branching graph that has perimeter $\nu$ after pruning. By the definition of a branching graph, each edge label occurs precisely $\nu$ times in the pruned face, so we can divide its perimeter into $\nu$ disjoint *intervals*, each of which contains all of the edge labels. From the unpruned face of perimeter $\mu$, construct a rooted forest by contracting each of the intervals to a root vertex and reassign each edge label to the adjacent vertex that is further away from the root. We thus obtain a rooted forest with $\nu$ components, $\mu - \nu$ edges, and hence $\mu$ vertices. The $\nu$ roots are labelled by their corresponding intervals, while the remaining $\mu - \nu$ vertices are labelled by distinct positive integers derived from the original edge labels.

As an example, consider Figure 5, which shows a pruned face of perimeter $\nu = 3$ with $\mu - \nu = 8$ edges added to create a face of perimeter $\mu = 11$. The corresponding rooted forest is shown on the right.

So there are $\frac{\mu}{\nu} T(\mu, \nu) = \mu^{\mu-\nu}$ possibilities for the resulting rooted forest. Conversely, the process may be reversed to construct a face of perimeter $\mu$ from a

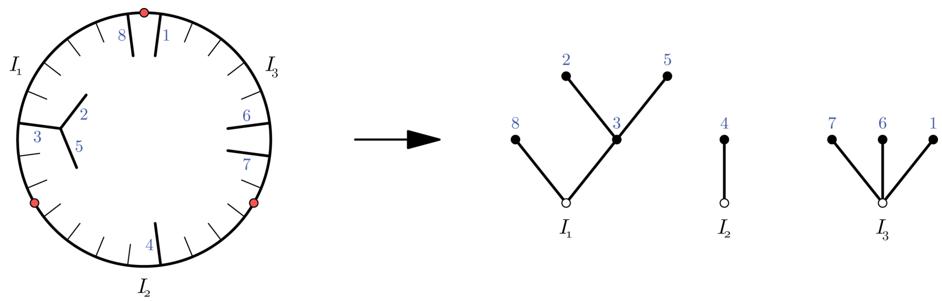


FIGURE 5. From a face in a branching graph to a pruned forest.

pruned face of perimeter $\nu$ together with a labelled rooted forest with $\nu$ components and $\mu$ vertices. The edge labels determine the cyclic orientations of the edges adjacent to a given vertex. □

Note that the system of linear equations in Proposition 3.5 relating the values of $\widehat{H}_{g,n}$ to those of $\widehat{K}_{g,n}$ is triangular in the sense that $\widehat{H}_{g,n}(\boldsymbol{\mu})$ depends only on values of $\widehat{K}_{g,n}(\boldsymbol{\nu})$ for which $\boldsymbol{\nu} \leq \boldsymbol{\mu}$ in the lexicographical order. Therefore, all of the information stored in the simple Hurwitz numbers is theoretically also stored in their pruned counterparts.

Theorem 3 states that the simple Hurwitz numbers comprise a natural enumerative problem in the context of the topological recursion. The following result demonstrates that the same is true of the pruned simple Hurwitz numbers and, furthermore, that they can be derived from the same spectral curve.

**Proposition 3.6.** *For $2g-2+n>0$, the expansions of the simple Hurwitz multidifferentials of equation* (2.4) *at the point $z_1 = z_2 = \cdots = z_n = 0$ satisfy*

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{K}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} z_i^{\mu_i-1}\,\mathrm{d}z_i.$$

*Proof.* Recall that the simple Hurwitz multidifferentials are defined in equation (2.4) by the formula

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{H}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} x_i^{\mu_i-1}\,\mathrm{d}x_i.$$

Furthermore, recall that $\omega_{g,n}$ is a meromorphic multidifferential on $C^n$, where $C$ is the rational spectral curve given parametrically by the equation $x(z) = z\exp(-z)$ and $y(z) = z$. We let $z_1, z_2, \ldots, z_n$ be the rational coordinates on $C^n$ and define $x_1 = x(z_1), x_2 = x(z_2), \ldots, x_n = x(z_n)$.

Now define another family of multidifferentials $\overline{\omega}_{g,n}$ on $C^n$ by the following local expansion at the point $z_1 = z_2 = \cdots = z_n = 0$:[1]

$$\overline{\omega}_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{K}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} z_i^{\mu_i-1}\,\mathrm{d}z_i.$$

[1]Proposition 3.8 below asserts that $\widehat{K}_{g,n}$ is a polynomial, so the equation does indeed define an analytic multidifferential.

Of course, we wish to prove that $\overline{\omega}_{g,n} = \omega_{g,n}$ and we proceed by calculating the following residue:

$$
\begin{aligned}
&\operatorname*{Res}_{x_1=0} \cdots \operatorname*{Res}_{x_n=0} \overline{\omega}_{g,n} \prod_{i=1}^{n} x_i^{-\mu_i} \\
&\quad = \operatorname*{Res}_{z_1=0} \cdots \operatorname*{Res}_{z_n=0} \sum_{\nu_1,\ldots,\nu_n=1}^{\infty} \widehat{K}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} z_i^{\nu_i-1}\,\mathrm{d}z_i\,[z_i \exp(-z_i)]^{-\mu_i} \\
&\quad = \operatorname*{Res}_{z_1=0} \cdots \operatorname*{Res}_{z_n=0} \sum_{\nu_1,\ldots,\nu_n=1}^{\infty} \widehat{K}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} z_i^{\nu_i-\mu_i-1}\,\mathrm{d}z_i \sum_{m_i=0}^{\infty} \frac{\mu_i^{m_i}}{m_i!} z_i^{m_i} \\
&\quad = \sum_{\nu_1,\ldots,\nu_n=1}^{\mu_1,\ldots,\mu_n} \widehat{K}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} \operatorname*{Res}_{z_i=0} z_i^{\nu_i-\mu_i-1}\,\mathrm{d}z_i \sum_{m_i=0}^{\infty} \frac{\mu_i^{m_i}}{m_i!} z_i^{m_i} \\
&\quad = \sum_{\nu_1,\ldots,\nu_n=1}^{\mu_1,\ldots,\mu_n} \widehat{K}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} \frac{\mu_i^{\mu_i-\nu_i}}{(\mu_i-\nu_i)!} \\
&\quad = \widehat{H}_{g,n}(\mu_1,\ldots,\mu_n).
\end{aligned}
$$

The last equality here is a direct consequence of Proposition 3.5. It now follows from the above residue calculation that

$$\overline{\omega}_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{H}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} x_i^{\mu_i-1}\,\mathrm{d}x_i = \omega_{g,n}. \qquad \square$$

In Example 3.8, we observed that the linear factor $m(g,\boldsymbol{\mu}) = 2g-2+n+|\boldsymbol{\mu}|$ on the left hand side of the cut-and-join recursion divides the right hand side in the case $(g,n) = (0,4)$, thereby establishing the fact that $\widehat{K}_{0,4}$ is a polynomial. In fact, we will see that this phenomenon is general.

**Lemma 3.7.** *For non-negative integers $d$, define the sequence $q_d(1), q_d(2), q_d(3), \ldots$ by the triangular system of linear equations*

$$\frac{\mu^{\mu+d+1}}{\mu!} = \sum_{\nu=1}^{\mu} q_d(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!} \qquad \textit{for } \mu = 1,2,3,\ldots.$$

*Then $q_d$ is a polynomial of degree $2d$.*

*Proof.* First, observe that $q_0(\nu) = 1$ for $\nu = 1,2,3,\ldots$, since this fact is equivalent to the identity

$$\mu^2\,T(\mu,1) = \sum_{\nu=1}^{\mu} \frac{\mu!}{(\mu-\nu)!}\,T(\mu,\nu) \qquad \text{for } \mu = 1,2,3,\ldots.$$

As in the proof of Proposition 3.5, $T(\mu,\nu) = \nu\mu^{\mu-\nu-1}$ denotes the number of rooted forests on $\mu$ labelled vertices with $\nu$ labelled roots. We interpret the left hand side of this equation as the number of trees with vertices labelled $1,2,\ldots,\mu$, along with a choice of an initial vertex and a terminal vertex, which are allowed to coincide. Given such a tree, suppose that there are $\nu$ vertices on the unique path from the initial vertex to the terminal vertex. Note that $1 \leq \nu \leq \mu$ and that there are $\frac{\mu!}{(\mu-\nu)!}$ possibilities for the labels of the vertices along the path. Removing the edges on the path yields a rooted forest, whose roots are precisely those vertices on the path. The number of such rooted forests is $T(\mu,\nu)$ by definition, which leads to the expression on the right hand side of this equation.

Second, consider the following sequence of equalities:

$$\begin{aligned}
\sum_{\nu=1}^{\mu} q_{d+1}(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\mu)!} &= \nu\sum_{\nu=1}^{\mu} q_d(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!}\\
&= \sum_{\nu=1}^{\mu} \nu q_d(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!} + \sum_{\nu=1}^{\mu}(\mu-\nu) q_d(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!}\\
&= \sum_{\nu=1}^{\mu} \nu q_d(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!} + \mu\sum_{\nu=1}^{\mu}(\nu-1) q_d(\nu-1)\,\frac{\mu^{\mu-\nu}}{(\mu-\nu)!}\\
&= \sum_{\nu=1}^{\mu} \nu q_d(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!} + \sum_{\nu=1}^{\mu}(\nu-1) q_d(\nu-1)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!}\\
&\quad + \mu\sum_{\nu=1}^{\mu}(\nu-2) q_d(\nu-2)\,\frac{\mu^{\mu-\nu}}{(\mu-\nu)!}\\
&= \sum_{\nu=1}^{\mu} \nu q_d(\nu)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!} + \sum_{\nu=1}^{\mu}(\nu-1) q_d(\nu-1)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!}\\
&\quad + \cdots + \sum_{\nu=1}^{\mu} 1q_d(1)\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!}\\
&= \sum_{\nu=1}^{\mu} \left[\nu q_d(\nu) + (\nu-1)q_d(\nu-1) + \cdots + 1q_d(1)\right]\,\frac{\nu\mu^{\mu-\nu}}{(\mu-\nu)!}.
\end{aligned}$$

Since the sequences $q_d(1), q_d(2), q_d(3), \ldots$ are defined by triangular systems of linear equations, we may deduce from the above sequence of equalities that

$$q_{d+1}(\nu) = \sum_{i=1}^{\nu} i q_d(i) \qquad \text{and} \qquad q_{d+1}(\nu) = q_{d+1}(\nu-1) + v q_d(\nu). \tag{3.1}$$

Using the base case $q_0(\nu) = 1$ and equation (3.1), it is now straightforward to prove by induction that $q_d$ is a polynomial of degree $2d$. □

**Proposition 3.8.** *For $2g-2+n > 0$, the normalised pruned simple Hurwitz number $\widehat{K}_{g,n}(\mu_1,\ldots,\mu_n)$ is a polynomial in $\mu_1,\ldots,\mu_n$ of degree $6g-6+3n$.*

*Proof.* Substitute the ELSV formula — see Proposition 2.5 — into the equation in the statement of Proposition 3.5 to obtain the following:

$$\begin{aligned}
&\sum_{|\mathbf{d}|+\ell=3g-3+n} (-1)^{\ell}\,\langle \tau_{d_1}\cdots\tau_{d_n}\lambda_\ell\rangle_g \prod_{i=1}^{n} \frac{\mu_i^{\mu_i+d_i+1}}{\mu_i!}\\
&\quad = \sum_{\nu_1,\ldots,\nu_n}^{\mu_1,\ldots,\mu_n} \widehat{K}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} \frac{\mu_i^{\mu_i-\nu_i}}{(\mu_i-\nu_i)!}.
\end{aligned}$$

From the definition of the polynomials $q_d$ for $d = 0, 1, 2, \ldots$, we may deduce from this equation that for all positive integers $\nu_1, \nu_2, \ldots, \nu_n$,

$$\widehat{K}_{g,n}(\nu_1,\ldots,\nu_n) = \sum_{|\mathbf{d}|+\ell=3g-3+n} (-1)^{\ell}\,\langle \tau_{d_1}\cdots\tau_{d_n}\lambda_\ell\rangle_g \prod_{i=1}^{n} \nu_i q_{d_i}(\nu_i). \tag{3.2}$$

Since we have already shown that $q_d$ is a polynomial of degree $2d$, the desired result follows. $\square$

The sequence of polynomials $q_0, q_1, q_2, \ldots$ plays a crucial part in the relation between simple Hurwitz numbers and their pruned counterparts. The numerators of the corresponding triangle of coefficients appear as sequence A202339 in the *On-Line Encyclopedia of Integer Sequences* [30]. We state without proof some simple facts about this sequence of polynomials, which follow from the base case $q_0 = 1$ and the recursive definition in equation (3.1).

**Proposition 3.9.** *The function $q_d$ is in fact a polynomial of degree $2d$ with leading coefficient $a_d = \frac{1}{(2d)!!}$. For all non-negative integers $d$ and positive integers $\nu$, $q_d(\nu) = S(\nu + d, \nu)$, the Stirling number of the second kind that represents the number of ways to partition a set with $\nu + d$ objects into $\nu$ non-empty subsets.*

The combinatorial significance of the Stirling numbers of the second kind for pruned simple Hurwitz numbers is presently unclear.

| $d$ | $q_d(\nu)$ |
|---|---|
| 0 | $1$ |
| 1 | $\frac{1}{2}(\nu^2 + \nu)$ |
| 2 | $\frac{1}{24}(3\nu^4 + 10\nu^3 + 9\nu^2 + 2\nu)$ |
| 3 | $\frac{1}{48}(\nu^6 + 7\nu^5 + 17\nu^4 + 17\nu^3 + 6\nu^2)$ |
| 4 | $\frac{1}{5760}(15\nu^8 + 180\nu^7 + 830\nu^6 + 1848\nu^5 + 2015\nu^4 + 900\nu^3 + 20\nu^2 - 48\nu)$ |
| 5 | $\frac{1}{11520}(3\nu^{10}+55\nu^9+410\nu^8+1598\nu^7+3467\nu^6+4055\nu^5+2120\nu^4+ 52\nu^3-240\nu^2)$ |

3.3. **Witten–Kontsevich theorem.** We apply the earlier results of this section to give a direct proof of the Witten–Kontsevich theorem, which governs intersection numbers of psi-classes on Deligne–Mumford moduli spaces of stable pointed curves $\overline{\mathcal{M}}_{g,n}$. We adopt the following notation of Witten for such intersection numbers, which are defined to be zero unless the condition $|\boldsymbol{d}| = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$ is satisfied:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

The psi-classes $\psi_1, \psi_2, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ are the first Chern classes of the cotangent line bundles at the marked points. For more information on Deligne–Mumford moduli spaces of curves, psi-classes, and the Witten–Kontsevich theorem, see the book of Harris and Morrison [17].

One of the virtues of the cut-and-join recursion for pruned simple Hurwitz numbers is that, although it is primarily an equality of numbers, it can be interpreted as an equality of polynomials in light of Proposition 3.8. In order to do this, we define the following functions for non-negative integers $i$ and $j$:

$$P_i(x, y) = \sum_{\alpha+\beta=x+y+1} \alpha\beta\, q_i(\alpha) \qquad \text{and} \qquad P_{i,j}(x) = \sum_{\alpha+\beta+\gamma=x+1} \alpha\beta\gamma\, q_i(\alpha)\, q_j(\beta).$$

The following lemma will be useful to determine the leading order behaviour of $P_i$ and $P_{i,j}$.

**Lemma 3.10.** *The expression*

$$\sum_{\alpha_1+\cdots+\alpha_m=n} \alpha_1^{k_1}\cdots\alpha_m^{k_m}$$

*is a polynomial in $n$ of degree $|\boldsymbol{k}|+m-1$ with leading coefficient $\frac{k_1!\cdots k_m!}{(|\boldsymbol{k}|+m-1)!}$.*

One proof of this fact expresses monomials $\alpha^k$ as linear combinations of binomial coefficients $\alpha^k = k!\binom{\alpha}{k}+\cdots$ and uses the combinatorial fact

$$\sum_{\alpha_1+\cdots+\alpha_m=n} \binom{\alpha_1}{k_1}\cdots\binom{\alpha_m}{k_m} = \binom{n+m-1}{|\boldsymbol{k}|+m-1}.$$

As a direct consequence of this lemma and Proposition 3.9, we have the following result.

**Corollary 3.11.** *For non-negative integers $i$ and $j$, $P_i$ is a polynomial of degree $2i+3$ and $P_{i,j}$ is a polynomial of degree $2i+2j+5$. Their leading coefficients are given by the formulae*

$$\left[x^{2a+1}y^{2b}\right] P_{a+b-1}(x,y) = \frac{(2a+2b-1)!!}{(2a+1)!\,(2b)!}$$

*and*

$$\left[x^{2a+2b+5}\right] P_{a,b}(x) = \frac{(2a+1)!!\,(2b+1)!!}{(2a+2b+5)!}.$$

**Example 3.12.** The summations over $\alpha,\beta,\gamma$ in the cut-and-join recursion for pruned simple Hurwitz numbers can be replaced by expressions involving $P_i$ and $P_{i,j}$. For example, consider the case $(g,n)=(1,2)$:

$$\begin{aligned}
&(\mu_1+\mu_2+2)\,\widehat{K}_{1,2}(\mu_1,\mu_2)\\
&= \sum_{\alpha+\beta=\mu_1+\mu_2+1} \alpha\beta\,\widehat{K}_{1,1}(\alpha) + \frac{1}{2}\sum_{\alpha+\beta+\gamma=\mu_1+1} \alpha\beta\gamma\,\widehat{K}_{0,3}(\mu_2,\alpha,\beta)\\
&\quad + \frac{1}{2}\sum_{\alpha+\beta+\gamma=\mu_2+1} \alpha\beta\gamma\,\widehat{K}_{0,3}(\mu_1,\alpha,\beta)\\
&= \sum_{\alpha+\beta=\mu_1+\mu_2+1} \alpha\beta\,\frac{q_1(\alpha)-q_0(\alpha)}{24} + \sum_{\alpha+\beta+\gamma=\mu_1+1} \alpha\beta\gamma\,\frac{q_0(\mu_2)q_0(\alpha)q_0(\beta)}{2}\\
&\quad + \sum_{\alpha+\beta+\gamma=\mu_2+1} \alpha\beta\gamma\,\frac{q_0(\mu_1)q_0(\alpha)q_0(\beta)}{2}\\
&= \frac{1}{24}P_1(\mu_1,\mu_2) - \frac{1}{24}P_0(\mu_1,\mu_2) + \frac{1}{2}P_{0,0}(\mu_1)\,q_0(\mu_2) + \frac{1}{2}P_{0,0}(\mu_2)\,q_0(\mu_1).
\end{aligned}$$

We are now in a position to deduce the Witten–Kontsevich theorem from equation (3.2) and Proposition 3.3, the cut-and-join recursion for pruned simple Hurwitz numbers.

**Theorem 4** (Witten–Kontsevich theorem)**.** *The intersection numbers of psi-classes on the Deligne–Mumford moduli spaces of stable pointed curves $\overline{\mathcal{M}}_{g,n}$ satisfy the following equation for all $d_1, d_2, \ldots, d_n$:*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{j=2}^{n} \frac{(2d_1+2d_j-1)!!}{(2d_1+1)!!\,(2d_j-1)!!} \langle \tau_{\boldsymbol{d}_{S\setminus\{1,j\}}} \tau_{d_1+d_j-1} \rangle$$
$$+ \frac{1}{2} \sum_{i+j=d_1-2} \frac{(2i+1)!!\,(2j+1)!!}{(2d_1+1)!!} \Big[ \langle \tau_i \tau_j \tau_{\boldsymbol{d}_{S\setminus\{1\}}} \rangle + \sum_{I \sqcup J = S\setminus\{1\}} \langle \tau_i \tau_{\boldsymbol{d}_I} \rangle \langle \tau_j \tau_{\boldsymbol{d}_J} \rangle \Big].$$

*Remark* 3.13. In actual fact, the original formulation of Witten posited that a certain natural generating function for intersection numbers of psi-classes — the Gromov–Witten potential of a point — is a solution to the KdV integrable hierarchy [32]. This is equivalent to the fact that the generating function is annihilated by the Virasoro differential operators $L_{-1}, L_0, L_1, \ldots$, which satisfy the Virasoro relation $[L_m, L_n] = (m-n)L_{m+n}$. The annihilation by $L_{-1}$ and $L_0$ is equivalent to the dilaton and string equations, which have straightforward geometric interpretations that already appear in the original paper of Witten. It is straightforward to prove that Theorem 4 is equivalent to the fact that $L_{d_1-1}$ annihilates the Gromov–Witten potential of a point.

*Proof of Theorem* 4. Take the cut-and-join recursion for pruned simple Hurwitz numbers and consider the coefficient of $\mu_1 \boldsymbol{\mu}^{2\boldsymbol{d}} = \mu_1^{2d_1+1} \mu_2^{2d_2} \cdots \mu_n^{2d_n}$ for $|\boldsymbol{d}| = 3g-3+n$. This condition ensures that no terms involving non-trivial Hodge classes $\lambda_\ell$ appear.

The desired coefficient of the left hand side of the cut-and-join recursion can be expressed as follows:

$$\begin{aligned}
&\left[\mu_1 \boldsymbol{\mu}^{2\boldsymbol{d}}\right] (2g-2+n+|\boldsymbol{\mu}|)\, \widehat{K}_{g,n}(\boldsymbol{\mu}_S) \\
&= \left[\mu_1 \boldsymbol{\mu}^{2\boldsymbol{d}}\right] (2g-2+n+|\boldsymbol{\mu}|) \sum_{|\mathbf{k}|+\ell=3g-3+n} (-1)^\ell \langle \tau_{k_1} \cdots \tau_{k_n} \lambda_\ell \rangle_g \prod_{i=1}^{n} q_{k_i}(\mu_i) \\
&= \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} a_{d_i}.
\end{aligned}$$

The first equality uses equation (3.2), while the second makes use of the fact that $q_d$ is a polynomial of degree $2d$ with leading coefficient $a_d = \frac{1}{(2d)!!}$, as stated in Proposition 3.9.

The desired coefficient of the first term on the right hand side of the cut-and-join recursion can be expressed as follows:

$$\begin{aligned}
&\big[\mu_1\boldsymbol{\mu}^{\boldsymbol{2d}}\big]\sum_{i<j}\sum_{\alpha+\beta=\mu_i+\mu_j+1}\alpha\beta\,\widehat{K}_{g,n-1}(\boldsymbol{\mu}_{S\setminus\{i,j\}},\alpha)\\
&=\big[\mu_1\boldsymbol{\mu}^{\boldsymbol{2d}}\big]\sum_{i<j}\sum_{|\boldsymbol{k}_{S\setminus\{i,j\}}|+s+\ell=3g-4+n}(-1)^{\ell}\langle\tau_{\boldsymbol{k}_{S\setminus\{i,j\}}}\tau_s\lambda_\ell\rangle_g\,P_s(\mu_i,\mu_j)\prod_{m\in S\setminus\{i,j\}}q_{k_m}(\mu_m)\\
&=\sum_{j=2}^{n}\big\langle\tau_{\boldsymbol{d}_{S\setminus\{i,j\}}}\tau_{d_1+d_j-1}\big\rangle_g\left[\mu_1^{2d_1+1}\mu_j^{2d_j}\right]P_{d_1+d_j-1}(\mu_1,\mu_j)\prod_{m\in S\setminus\{1,j\}}a_{d_m}\\
&=\sum_{j=2}^{n}\big\langle\tau_{\boldsymbol{d}_{S\setminus\{i,j\}}}\tau_{d_1+d_j-1}\big\rangle_g\,\frac{(2d_1+2d_j-1)!!}{(2d_1+1)!(2d_j)!}\prod_{m\in S\setminus\{1,j\}}a_{d_m}.
\end{aligned}$$

The first equality uses equation (3.2), the second takes into account the fact that $|\boldsymbol{d}|=3g-3+n$, while the third follows from Corollary 3.11.

In an analogous fashion, the desired coefficients of the second and third terms on the right hand side of the cut-and-join recursion can be expressed as follows:

$$\frac{1}{2}\sum_{s+t=d_1-2}\big\langle\tau_{\boldsymbol{d}_{S\setminus\{1\}}}\tau_s\tau_t\big\rangle_{g-1}\left[\mu_1^{2d_1+1}\right]P_{s,t}(\mu_1)\prod_{m\in S\setminus\{1\}}a_{d_m},$$

$$\frac{1}{2}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S\setminus\{1\}}}^{\text{stable}}\sum_{s+t=d_1-2}\big\langle\tau_{\boldsymbol{d}_I}\tau_s\big\rangle_{g_1}\big\langle\tau_{\boldsymbol{d}_J}\tau_t\big\rangle_{g_2}\frac{(2s+1)!!(2t+1)!!}{(2d_1+1)!}\prod_{m\in S\setminus\{1\}}a_{d_m}.$$

Now substitute these expressions into the cut-and-join recursion and divide both sides by $a_{d_1}a_{d_2}\cdots a_{d_n}$ to obtain the desired result. □

It is worth remarking that Okounkov and Pandharipande also deduce the Witten–Kontsevich theorem using the ELSV formula as a starting point [29]. Their approach expresses the asymptotics of simple Hurwitz numbers as a sum over trivalent fatgraphs, thereby obtaining Kontsevich's combinatorial formula [21]. The Witten–Kontsevich theorem is then derived as a consequence of this formula using the theory of matrix models. In contrast, the notion of pruning reduces the enumeration of simple Hurwitz numbers to an equivalent problem that is inherently polynomial. The asymptotic analysis of pruned simple Hurwitz numbers is then stored in the top degree terms of the cut-and-join recursion. As shown in the proof of Theorem 4 above, the Witten–Kontsevich theorem emerges directly from this analysis without necessitating the use of a matrix model.

There are now myriad proofs of the Witten–Kontsevich theorem, most of which involve the theory of matrix models in one way or another. Exceptional in this respect is the proof by Mirzakhani, who analyses the volume $V_{g,n}(L_1,\ldots,L_n)$ of the moduli space of genus $g$ hyperbolic surfaces with $n$ geodesic boundary components of lengths $L_1,\ldots,L_n$ [23]. Her proof consists of two parts: a theorem that relates $V_{g,n}(L_1,\ldots,L_n)$ to the intersection theory of moduli spaces of curves and a recursion that can be used to compute $V_{g,n}(L_1,\ldots,L_n)$. It is natural to consider these as analogous to the ELSV formula and the cut-and-join recursion, respectively. Our proof of the Witten–Kontsevich theorem bears strong resemblance to that of Mirzakhani, but uses a combinatorial argument rather than hyperbolic geometry to obtain the recursion.

We finish the section with a table of the polynomials $\widehat{K}_{g,n}(\mu_1,\ldots,\mu_n)$, which give pruned simple Hurwitz numbers.

| $g$ | $n$ | $\widehat{K}_{g,n}(\mu_1,\mu_2,\ldots,\mu_n)$ |
|---|---|---|
| 0 | 3 | 1 |
| 0 | 4 | $\frac{1}{2}\sum\mu_i^2+\frac{1}{2}\sum\mu_i$ |
| 0 | 5 | $\frac{1}{8}\sum\mu_i^4+\frac{1}{2}\sum\mu_i^2\mu_j^2+\frac{5}{12}\sum\mu_i^3+\frac{1}{2}\sum\mu_i^2\mu_j+\frac{3}{8}\sum\mu_i^2+\frac{1}{2}\sum\mu_i\mu_j+\frac{1}{12}\sum\mu_i$ |
| 1 | 1 | $\frac{1}{48}\mu_1^2+\frac{1}{48}\mu_1-\frac{1}{24}$ |
| 1 | 2 | $\frac{1}{192}(\mu_1^4+\mu_2^4)+\frac{1}{96}\mu_1^2\mu_2^2+\frac{5}{288}(\mu_1^3+\mu_2^3)+\frac{1}{96}(\mu_1^2\mu_2+\mu_1\mu_2^2)$ $-\frac{1}{192}(\mu_1^2+\mu_2^2)+\frac{1}{96}\mu_1\mu_2-\frac{5}{288}(\mu_1+\mu_2)$ |
| 2 | 1 | $\frac{1}{442368}\mu_1^8+\frac{1}{36864}\mu_1^7+\frac{271}{3317760}\mu_1^6-\frac{7}{276480}\mu_1^5-\frac{1873}{6635520}\mu_1^4$ $-\frac{53}{552960}\mu_1^3+\frac{329}{1658880}\mu_1^2+\frac{13}{138240}\mu_1$ |

## 4. Orbifold Hurwitz numbers

4.1. **Orbifold Hurwitz numbers.** In this section, we generalise the results for simple Hurwitz numbers in the previous section to the case of orbifold Hurwitz numbers.

**Definition 4.1.** For a fixed positive integer $a$, the *orbifold Hurwitz number* $H^{[a]}_{g,n}(\mu_1,\ldots,\mu_n)$ is the weighted enumeration of connected genus $g$ branched covers $f:(\Sigma;p_1,\ldots,p_n)\to(\mathbb{CP}^1;\infty)$ such that

- the preimage of $\infty$ is given by the divisor $\mu_1p_1+\mu_2p_2+\cdots+\mu_np_n$;
- the ramification profile over 0 is $(a,a,\ldots,a)$; and
- the only other ramification is simple and occurs over $m=2g-2+n+\frac{|\boldsymbol{\mu}|}{a}$ fixed points.

Note that we recover the definition of simple Hurwitz numbers in the case $a=1$. Justification for the terminology *orbifold Hurwitz number* stems from the following generalisation of the ELSV formula due to Johnson, Pandharipande, and Tseng [20].

**Theorem 5** (Orbifold ELSV formula [20])**.** *The orbifold Hurwitz number $H^{[a]}_{g,n}(\mu_1,\ldots,\mu_n)$ satisfies*

$$\frac{H^{[a]}_{g,n}(\mu_1,\ldots,\mu_n)}{(2g-2+n+\frac{|\boldsymbol{\mu}|}{a})!}=a^{1-g+\sum\{\mu_i/a\}}\prod_{i=1}^{n}\frac{\mu_i^{\lfloor\mu_i/a\rfloor+1}}{\lfloor\mu_i/a\rfloor!}\int_{\overline{\mathcal{M}}_{g,[-\mu]}(\mathcal{B}\mathbb{Z}_a)}\frac{\sum_{i=0}^{\infty}(-a)^i\lambda_i^U}{\prod_{i=1}^{n}(1-\mu_i\overline{\psi}_i)},$$

*where $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a)$ is the moduli space of stable maps to $\mathcal{B}\mathbb{Z}_a$, the classifying stack of $\mathbb{Z}_a$ given by a point with trivial $\mathbb{Z}_a$-action, and $\lambda_i^U$ are generalisations of the Hodge class.*

The following result is the orbifold analogue of Theorem 3 [5, 8].

**Theorem 6.** *For a fixed positive integer $a$, consider the topological recursion applied to the rational spectral curve $C$ given by*

$$x(z)=z\exp(-z^a)\qquad\textit{and}\qquad y(z)=z^a.$$

*The expansion of the resulting multidifferential* $\omega_{g,n}$ *at* $x_1 = x_2 = \cdots = x_n = 0$ *is given by*

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \frac{H_{g,n}^{[a]}(\mu_1,\ldots,\mu_n)}{(2g-2+n+\frac{|\boldsymbol{\mu}|}{a})!} \prod_{i=1}^{n} x_i^{\mu_i-1}\mathrm{d}x_i. \tag{4.1}$$

**Definition 4.2.** For a fixed positive integer $a$, we define an *$a$-fold branching graph of type* $(g;\boldsymbol{\mu})$ to be an edge-labelled fatgraph of type $(g,\ell(\boldsymbol{\mu}))$ such that for $m = 2g-2+\ell(\boldsymbol{\mu})+\frac{|\boldsymbol{\mu}|}{a}$:

- there are $\frac{|\boldsymbol{\mu}|}{a}$ vertices and at each of them there are $am$ adjacent half-edges that are cyclically labelled
$$1,2,3,\ldots,m,1,2,3,\ldots,m,\ldots,1,2,3,\ldots,m;$$
- there are exactly $m$ full edges that are labelled $1,2,3,\ldots,m$;
- the $n$ faces are labelled $1,2,\ldots,n$ and have perimeters given by $(m\mu_1, m\mu_2,\ldots,m\mu_n)$; and
- each face has a marked $m$-label. (Note that the face labelled $k$ has $\mu_k$ appearances of an $m$-label.)

Of course, this definition anticipates the following result [8].

**Proposition 4.3.** *The orbifold Hurwitz number* $H_{g,n}^{[a]}(\mu_1,\ldots,\mu_n)$ *enumerates $a$-fold branching graphs of type* $(g;\boldsymbol{\mu})$.

4.2. **Pruned orbifold Hurwitz numbers.** One obtains pruned orbifold Hurwitz numbers by restricting the enumeration to the set of pruned orbifold branching graphs, which are obtained by introducing a simple condition on vertex degrees. We call an orbifold branching graph *pruned* if all of its vertices have essential degree at least two. Let $K_{g,n}^{[a]}(\mu_1,\ldots,\mu_n)$ be the number of pruned $a$-fold branching graphs of type $(g;\boldsymbol{\mu})$, where $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_n)$. Furthermore, let $m = m(g,\boldsymbol{\mu}) = 2g-2+n+\frac{|\boldsymbol{\mu}|}{a}$ and define the normalisation

$$\widehat{K}_{g,n}^{[a]}(\mu_1,\ldots,\mu_n) = \frac{K_{g,n}(\mu_1,\ldots,\mu_n)}{m!}.$$

**Proposition 4.4** (Cut-and-join recursion for pruned orbifold Hurwitz numbers)**.** *For* $2g-2+n>0$,

$$\begin{aligned} m(g,\boldsymbol{\mu})\,\widehat{K}_{g,n}^{[a]}(\boldsymbol{\mu}) &= \sum_{i<j}\mu_i\mu_j \sum_{\alpha+a\beta=\mu_i+\mu_j+a} \beta\widehat{K}_{g,n-1}^{[a]}(\boldsymbol{\mu}_{S\setminus\{i,j\}},\alpha) \\ &+ \frac{1}{2}\sum_{i=1}^{n}\mu_i \sum_{\alpha+\beta+\gamma=\mu_i+1}\gamma\Bigg[\widehat{K}_{g-1,n+1}^{[a]}(\boldsymbol{\mu}_{S\setminus\{i\}},\alpha,\beta) \\ &\qquad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} \widehat{K}_{g_1,|I|+1}^{[a]}(\boldsymbol{\mu}_I,\alpha)\,\widehat{K}_{g_2,|J|+1}^{[a]}(\boldsymbol{\mu}_J,\beta)\Bigg]. \end{aligned}$$

The proof follows from considering the behaviour of removing an edge from a branching graph and pruning the result. It is essentially the same as the proof of Proposition 3.3.

### 4.3. The pruning correspondence.

**Proposition 4.5.** *For $(g,n) \neq (0,1)$,*

$$\widehat{H}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n) = \sum_{\nu_1,\ldots,\nu_n=1}^{\mu_1,\ldots,\mu_n} \widehat{K}^{[a]}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} \frac{\mu_i^{\frac{\mu_i-\nu_i}{a}}}{(\frac{\mu_i-\nu_i}{a})!}.$$

*Proof.* As in the proof of Proposition 3.5, the factor

$$\frac{(2g-2+n+\frac{|\boldsymbol{\mu}|}{a})!}{(2g-2+n+\frac{|\boldsymbol{\nu}|}{a})!\,(\mu_1-\nu_1)!\,\cdots\,(\mu_n-\nu_n)!}$$

accounts for the number of ways to choose the set of edge labels for the underlying pruned branching graph as well as the set of $\frac{\mu_i-\nu_i}{a}$ edge labels to be added to face $i$ for $i=1,2,\ldots,n$. The only difference from the proof of Proposition 3.5 is that we now require $T^{[a]}_{k,e}$, which we define to be the number of rooted forests with $k$ labelled components and $e$ labelled edges, counted with weight $a^{\#\text{internal edges}}$. We have the following generalisation of Cayley's formula:

$$T^{[a]}_{k,e} = k(ae+k)^{e-1}.$$

One uses $\mu^{\frac{\mu-\nu}{a}} = \frac{\mu}{\nu} T^{[a]}_{\nu,\frac{\mu-\nu}{a}}$ in the proof of Proposition 3.5 to obtain the desired result. $\square$

**Proposition 4.6.** *For $2g-2+n>0$, the expansions of the orbifold Hurwitz multidifferentials of equation* (4.1) *at the point $z_1 = z_2 = \cdots = z_n = 0$ satisfy*

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{K}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} z_i^{\mu_i-1}\mathrm{d}z_i.$$

*Proof.* Recall that the orbifold Hurwitz multidifferentials are defined in equation (4.1) by the formula

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{H}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} x_i^{\mu_i-1}\,\mathrm{d}x_i.$$

Furthermore, recall that $\omega_{g,n}$ is a meromorphic multidifferential on $C^n$, where $C$ is the rational spectral curve given parametrically by the equation $x(z) = z\exp(-z^a)$ and $y(z) = z^a$. We let $z_1, z_2, \ldots, z_n$ be the rational coordinates on $C^n$ and define $x_1 = x(z_1), x_2 = x(z_2), \ldots, x_n = x(z_n)$.

Now define another family of multidifferentials $\overline{\omega}_{g,n}$ on $C^n$ by the following local expansion at the point $z_1 = z_2 = \cdots = z_n = 0$:

$$\overline{\omega}_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{K}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} z_i^{\mu_i-1}\,\mathrm{d}z_i.$$

We will prove that $\overline{\omega}_{g,n} = \omega_{g,n}$ by calculating the following residue:

$$\begin{aligned}
&\operatorname*{Res}_{x_1=0} \cdots \operatorname*{Res}_{x_n=0} \overline{\omega}_{g,n} \prod_{i=1}^{n} x_i^{-\mu_i} \\
&\quad = \operatorname*{Res}_{z_1=0} \cdots \operatorname*{Res}_{z_n=0} \sum_{\nu_1,\ldots,\nu_n=1}^{\infty} \widehat{K}^{[a]}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} z_i^{\nu_i-1} \mathrm{d}z_i \, [z_i \exp(-z_i^a)]^{-\mu_i} \\
&\quad = \operatorname*{Res}_{z_1=0} \cdots \operatorname*{Res}_{z_n=0} \sum_{\nu_1,\ldots,\nu_n=1}^{\infty} \widehat{K}^{[a]}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} z_i^{\nu_i-1} \mathrm{d}z_i \, z_i^{-\mu_i} \sum_{m_i=0}^{\infty} \frac{\mu_i^{am_i}}{m_i!} \, z_i^{am_i} \\
&\quad = \sum_{\nu_1,\ldots,\nu_n=1}^{\mu_1,\ldots,\mu_n} \widehat{K}^{[a]}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} \operatorname*{Res}_{z_i=0} z_i^{\nu_i-1} \mathrm{d}z_i \, z_i^{-\mu_i} \sum_{m_i=0}^{\infty} \frac{\mu_i^{am_i}}{m_i!} \, z_i^{am_i} \\
&\quad = \sum_{\nu_1,\ldots,\nu_n=1}^{\mu_1,\ldots,\mu_n} \widehat{K}^{[a]}_{g,n}(\nu_1,\ldots,\nu_n) \prod_{i=1}^{n} \frac{\mu_i^{\frac{\mu_i-\nu_i}{a}}}{(\frac{\mu_i-\nu_i}{a})!} \\
&\quad = \widehat{H}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n).
\end{aligned}$$

It follows that

$$\overline{\omega}_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{H}_{g,n}(\mu_1,\mu_2,\ldots,\mu_n) \prod_{i=1}^{n} \mu_i x_i^{\mu_i-1} \mathrm{d}x_i = \omega_{g,n}. \qquad \square$$

A consequence of Theorem 5 is that we may express the orbifold Hurwitz numbers as

$$\widehat{H}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n) = a^{1-g+\sum\{\mu_i/a\}} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \mu_i/a \rfloor}+1}{\lfloor \mu_i/a \rfloor !} \times Q^{[a]}_{g,n}(\mu_1,\ldots,\mu_n),$$

where $Q^{[a]}_{g,n}$ is a quasi-polynomial modulo $a$ of degree $3g-3+n$. The following result is the orbifold analogue of Proposition 3.8.

**Proposition 4.7.** *For a fixed positive integer $a$ and $2g-2+n>0$, the normalised pruned orbifold Hurwitz number $\widehat{K}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n)$ is a quasi-polynomial modulo $a$ in $\mu_1,\ldots,\mu_n$ of degree $6g-6+3n$.*

*Proof.* One can prove this in an analogous way to Proposition 3.8. However, we will adopt here an approach that instead uses the spectral curve. Recall from Proposition 4.6 that

$$\omega_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \widehat{K}^{[a]}_{g,n}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} z_i^{\mu_i-1} \mathrm{d}z_i.$$

By the general theory of topological recursion, $\omega_{g,n}$ is a meromorphic multidifferential on the spectral curve

$$x(z) = z\exp(-z^a) \qquad \text{and} \qquad y(z) = z^a.$$

Therefore, it is rational in $z_1,\ldots,z_n$. Furthermore, it has poles only at the zeros of $\mathrm{d}x_i$, which are the $a$th roots of unity in this case. A rational function in $z$ with poles only at $\{z \mid z^a = 1\}$ has an expansion $\sum p(n)\, z^n$ at $z=0$, where $p(n)$ is a quasi-polynomial modulo $a$. The degree of $K^{[a]}_{g,n}$ follows from the order of the poles of $\omega_{g,n}$, which is $6g-4+2n$ by the general theory of topological recursion [13]. $\square$

## 5. Belyi Hurwitz numbers

For $g \geq 0$ and $n \geq 1$, define the set of Belyi Hurwitz covers as follows:

$$\mathcal{Z}_{g,n}(\boldsymbol{\mu}) = \left\{ f : \Sigma \to \mathbb{CP}^1 \;\middle|\; \begin{array}{l} \Sigma \text{ a connected genus } g \text{ Riemann surface;} \\ f^{-1}(\infty) = (p_1, \ldots, p_n) \text{ with} \\ \text{respective ramification } \boldsymbol{\mu} = (\mu_1, \ldots, \mu_n); \\ \text{ramification } (2, 2, \ldots, 2) \text{ over } 1; \\ \text{arbitrary ramification over } 0;\ f \text{ unramified} \\ \text{elsewhere} \end{array} \right\}.$$

Here, we consider two branched covers $f_1 : \Sigma_1 \to \mathbb{CP}^1$ and $f_2 : \Sigma_2 \to \mathbb{CP}^1$ to be equivalent if there exists $h : \Sigma_1 \to \Sigma_2$ that satisfies $f_1 = f_2 \circ h$ while preserving the labels over $\infty$.

Define the Belyi Hurwitz numbers

$$M_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{f \in \mathcal{Z}_{g,n}(\boldsymbol{\mu})} \frac{1}{|\text{Aut } f|}.$$

Now define the set of pruned Belyi Hurwitz covers

$$\mathcal{Z}^0_{g,n}(\boldsymbol{\mu}) = \left\{ f \in \mathcal{Z}_{g,n}(\boldsymbol{\mu}) \mid \text{all points in } f^{-1}(0) \text{ have non-trivial ramification} \right\},$$

as well as the corresponding pruned Belyi Hurwitz numbers

$$N_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{f \in \mathcal{Z}^0_{g,n}(\boldsymbol{\mu})} \frac{1}{|\text{Aut } f|}.$$

There is a recursion for $M_{g,n}$ that uses a cut-and-join argument and is known as the Tutte recursion in the planar case [31]. More generally, such a recursion arises from expansions of matrix integrals [3, 14]. See also [9], where $M_{g,n}(\boldsymbol{\mu})$ is treated as a generalised Catalan number. An analogous recursion for $N_{g,n}$ appears in [24].

To any $f \in \mathcal{Z}_{g,n}(\boldsymbol{\mu})$, one can associate a fatgraph $\Gamma_f = f^{-1}([0,1]) \subset \Sigma$. Note that $\Sigma - f^{-1}([0,1])$ is a union of disks, so that $\Gamma_f$ can be considered as a discrete surface of genus $g$ obtained by gluing together $n$ polygonal faces of perimeters $\mu_1, \ldots, \mu_n$. Equivalently, a fatgraph is defined by the set $X$ of its oriented edges, equipped with automorphisms $\tau_0, \tau_1 : X \to X$. Then $\Gamma_f$ is the fatgraph given by $(X_f, \tau_0, \tau_1)$, where $X_f = f^{-1}(0,1)$, $\tau_0 : X_f \to X_f$ is the monodromy map around 0, and $\tau_1 : X_f \to X_f$ is the monodromy map around 1. The vertices of the fatgraph correspond to $V_f = f^{-1}(0) \cong X_f/\tau_0$, while the edges correspond to $E_f = X_f/\tau_1 \cong f^{-1}(1)$. The boundary components correspond to $X_f/\tau_2 \cong f^{-1}(\infty)$ for $\tau_2 = \tau_0\tau_1$, and its perimeter is simply the size of the corresponding orbit of $\tau_2$. An automorphism of a fatgraph $\Gamma = (X, \tau_0, \tau_1)$ is a map $g : X \to X$ that commutes with $\tau_0$ and $\tau_1$. From the fatgraph, one can reconstruct the map $f$. Hence the Belyi Hurwitz numbers can be equivalently defined as

$$M_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\Gamma \in \text{Fat}_{g,n}(\boldsymbol{\mu})} \frac{1}{|\text{Aut } \Gamma|},$$

where $\mathrm{Fat}_{g,n}(\boldsymbol{\mu})$ is the set of all genus $g$ fatgraphs with $n$ labelled boundary components of respective lengths $(\mu_1, \ldots, \mu_n)$. Similarly,

$$N_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\Gamma \in \mathrm{Fat}^2_{g,n}(\boldsymbol{\mu})} \frac{1}{|\mathrm{Aut}\,\Gamma|},$$

where $\mathrm{Fat}^2_{g,n}(\boldsymbol{\mu}) \subset \mathrm{Fat}_{g,n}(\boldsymbol{\mu})$ consists of those fatgraphs without degree one vertices, namely, pruned fatgraphs. It is this graph representation that justifies the term *pruned Belyi Hurwitz number*.

Let $\mathcal{M}_{g,n}$ be the moduli space of genus $g$ curves with $n$ marked points. For each $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, there is the Penner–Harer–Mumford–Thurston cell decomposition

$$\mathcal{M}_{g,n} \cong \bigcup_{\Gamma \in \mathrm{Fat}^3_{g,n}} P_\Gamma(\mu_1, \ldots, \mu_n), \tag{5.1}$$

where the indexing set $\mathrm{Fat}^3_{g,n}$ is the finite set of genus $g$ fatgraphs with all vertices of degree at least three and $n$ labelled boundary components.

The cell decomposition (5.1) arises from the existence and uniqueness of Strebel differentials on a compact Riemann surface $\Sigma$ with $n$ marked points $(p_1, \ldots, p_n)$ and $n$ positive reals $(\mu_1, \ldots, \mu_n)$. A Strebel differential is a meromorphic quadratic differential $\omega$ that is holomorphic on $\Sigma - \{\mu_1, \ldots, \mu_n\}$ that satisfies further conditions. Any quadratic differential gives rise to vertical and horizontal foliations along which $\omega$ is positive and negative, respectively. In terms of a local coordinate $z$ away from zeros and poles, one can write $\omega = (\mathrm{d}z)^2 = (\mathrm{d}x)^2 - (\mathrm{d}y)^2 + 2i\,\mathrm{d}x\,\mathrm{d}y$. This is positive when $y$ is constant and negative when $x$ is constant. A Strebel differential is distinguished by the fact that its horizontal foliation has compact leaves and its poles occur at the $p_k$ with principal part $\frac{\mu_k}{z^2}(\mathrm{d}z)^2$. It has a unique singular compact leaf that is a fatgraph endowed with a positive length on each edge. The important point is that this singular compact leaf has no degree one vertices, since such a vertex would correspond to a singularity of the form $\frac{1}{z}(\mathrm{d}z)^2$. In summary, Strebel differentials give rise to pruned fatgraphs without degree two vertices, endowed with a positive length on each edge.

Consider the natural map $\mathcal{Z}_{g,n}(\boldsymbol{\mu}) \to \mathcal{M}_{g,n}$ that sends $f : \Sigma \to \mathbb{CP}^1$ to its domain curve $(\Sigma; p_1, \ldots, p_n)$, where $f^{-1}(\infty) = \{p_1, \ldots, p_n\}$. It can be combined with the cell decomposition of equation (5.1) with the same $\boldsymbol{\mu}$ to assign to $f$ a fatgraph $\Gamma^f$ without degree two vertices. Note that in general, $\Gamma^f \neq \Gamma_f$.

Underlying $\Gamma_f$ is a fatgraph $\widetilde{\Gamma}_f$ without degree two vertices, obtained by ignoring the degree two vertices of $\Gamma_f$. On the level of the sets $X_f$ and $\widetilde{X}_f$ of oriented edges, there exist maps $\pi : X_f \to \widetilde{X}_f$ and $\iota : \widetilde{X}_f \to X_f$ satisfying $\pi \circ \iota = \mathrm{id}$, $\pi \circ \tau_1 = \tau_1 \circ \pi$, and $\iota \circ \tau_0 = \tau_0 \circ \iota$. The induced map $\pi_* : E_f \to \widetilde{E}_f$ is surjective and one-to-one except on edges adjacent to degree two vertices. The induced map $\iota_* : \widetilde{V}_f \to V_f$ is injective with image all of $V_f$ except for degree two vertices. For general $f \in \mathcal{Z}_{g,n}(\boldsymbol{\mu})$, we have $\widetilde{\Gamma}_f \neq \Gamma^f$ since $\Gamma_f$ usually has degree one vertices. However,

$$f \in \mathcal{Z}^0_{g,n}(\boldsymbol{\mu}) \quad \Rightarrow \quad \widetilde{\Gamma}_f = \Gamma^f.$$

In other words $\mathcal{Z}^0_{g,n}(\boldsymbol{\mu})$ sits naturally inside $\mathcal{M}_{g,n}$ and this gives rise to a third description of $N_{g,n}(\mu_1, \ldots, \mu_n)$ as the number of integral points inside rational polytopes that form a cell decomposition of $\mathcal{M}_{g,n}$. The cells described by (5.1) are convex polytopes

$$P_\Gamma(\mu_1, \ldots, \mu_n) = \{\mathbf{x} \in \mathbb{R}_+^{E(\Gamma)} \mid A_\Gamma \mathbf{x} = \boldsymbol{\mu}\},$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and $A_\Gamma : \mathbb{R}^{E(\Gamma)} \to \mathbb{R}^n$ is the incidence matrix that maps an edge to the sum of its two incident boundary components. If we define $N_\Gamma(\mu_1, \ldots, \mu_n) = \#\{\mathbb{Z}_+^{E(\Gamma)} \cap P_\Gamma(\mu_1, \ldots, \mu_n)\}$, then

$$N_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\Gamma \in \mathrm{Fat}_{g,n}^3} \frac{1}{|\mathrm{Aut}\ \Gamma|} N_\Gamma(\mu_1, \ldots, \mu_n).$$

An important consequence of this interpretation of $N_{g,n}(\boldsymbol{\mu})$ as the enumeration of lattice points in $\mathcal{M}_{g,n}$ is the identity

$$N_{g,n}(0, \ldots, 0) = \chi(\mathcal{M}_{g,n}).$$

One makes sense of evaluation of $N_{g,n}$ at $(0, \ldots, 0)$ by using the fact that $N_{g,n}(\mu_1, \ldots, \mu_n)$ is quasi-polynomial in $\mu_1, \ldots, \mu_n$ [25].

5.1. **The pruning correspondence.** Both the Belyi Hurwitz numbers and their pruned counterparts arise from topological recursion on the spectral curve

$$x(z) = z + \frac{1}{z} \qquad \text{and} \qquad y(z) = z.$$

More explicitly, the results of [26] and [25] show that for $2g - 2 + n > 0$, the multi-differentials arising from the topological recursion have the following expansions on the spectral curve:

$$\begin{aligned} \omega_{g,n} &= \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} M_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \mu_i x_i^{-\mu_i - 1} \\ &= \sum_{\nu_1, \ldots, \nu_n = 1}^{\infty} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \nu_i z_i^{\nu_i - 1}. \end{aligned} \tag{5.2}$$

**Proposition 5.1.** *We have*

$$M_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \mu_i = \sum_{\nu_1, \ldots, \nu_n = 1}^{\mu_1, \ldots, \mu_n} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \nu_i \binom{\mu_i}{\frac{\mu_i - \nu_i}{2}}. \tag{5.3}$$

*Proof.* In the context of Belyi Hurwitz numbers, the pruning correspondence follows immediately from equation (5.2):

$$\begin{aligned}
&M_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \mu_i \\
&\quad = \operatorname*{Res}_{x_1=\infty} \cdots \operatorname*{Res}_{x_n=\infty} \omega_{g,n} \prod_{i=1}^{n} x_i^{\mu_i} \\
&\quad = \operatorname*{Res}_{z_1=0} \cdots \operatorname*{Res}_{z_n=0} \sum_{\nu_1,\ldots,\nu_n=1}^{\infty} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \nu_i z_i^{\nu_i - 1} \mathrm{d}z_i x_i^{\mu_i} \\
&\quad = \operatorname*{Res}_{z_1=0} \cdots \operatorname*{Res}_{z_n=0} \sum_{\nu_1,\ldots,\nu_n=1}^{\infty} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \nu_i z_i^{\nu_i - 1} \mathrm{d}z_i \left( z_i + \frac{1}{z_i} \right)^{\mu_i} \\
&\quad = \operatorname*{Res}_{z_1=0} \cdots \operatorname*{Res}_{z_n=0} \sum_{\nu_1,\ldots,\nu_n=1}^{\infty} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \nu_i z_i^{\nu_i - 1} \mathrm{d}z_i \sum_{k_i=0}^{\mu_i} \binom{\mu_i}{k_i} z_i^{\mu_i - 2k_i} \\
&\quad = \sum_{\nu_1,\ldots,\nu_n=1}^{\mu_1,\ldots,\mu_n} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \nu_i \binom{\mu_i}{\frac{\mu_i+\nu_i}{2}} \\
&\quad = \sum_{\nu_1,\ldots,\nu_n=1}^{\mu_1,\ldots,\mu_n} N_{g,n}(\nu_1, \ldots, \nu_n) \prod_{i=1}^{n} \nu_i \binom{\mu_i}{\frac{\mu_i-\nu_i}{2}}.
\end{aligned}$$

Hence, we have deduced equation (5.3). □

One can also give a combinatorial proof that simply formalises the fact that repeatedly removing vertices of degree one and their incident edges from a fatgraph yields a pruned fatgraph in a unique way.

*Remark* 5.2. We could have naturally defined $M_{g,n}(\mu_1, \ldots, \mu_n)$ and $N_{g,n}(\mu_1, \ldots, \mu_n)$ to include an extra factor of $\mu_1 \cdots \mu_n$, in which case (5.3) would further resemble the pruning correspondence for simple Hurwitz numbers in Proposition 3.5.

## 6. Gromov–Witten invariants of $\mathbb{CP}^1$

In this section, we apply the idea of pruning to the Gromov–Witten invariants of $\mathbb{CP}^1$. Unlike the previous sections, the picture is not yet complete and the aim here is to predict interesting structure.

The Gromov–Witten invariants of $\mathbb{CP}^1$ are known to arise from the spectral curve $C$ given parametrically by

$$x(z) = z + \frac{1}{z} \qquad \text{and} \qquad y(z) = \log z \sim \sum_{k=1}^{\infty} -\frac{(1-z^2)^k}{2k}. \tag{6.1}$$

In particular, one can apply the topological recursion to the rational spectral curve $C_N$ that uses the $N$th partial sum for the above expansion of $y$. In that case, the multidifferentials $\omega_{g,n}$ stabilise for $N \geq 6g - 6 + 2n$.

Assemble the Gromov–Witten invariants of $\mathbb{CP}^1$ into the generating function

$$\Omega_n^g(x_1, \ldots, x_n) = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \Big\langle \prod_{i=1}^{n} \tau_{\mu_i}(\omega) \Big\rangle_d^g \prod_{i=1}^{n} \frac{(\mu_i+1)!}{x_i^{\mu_i+2}} \, \mathrm{d}x_i. \tag{6.2}$$

The following was predicted and proven in [27] for the case $g = 0, 1$ and for all $g$ in [10].

**Theorem 7.** *For $2g - 2 + n > 0$, the multidifferentials obtained from topological recursion on the spectral curve defined by* (6.1) *agree with the generating function for Gromov–Witten invariants of $\mathbb{CP}^1$ defined by* (6.2):

$$\omega_n^g \sim \Omega_n^g(x_1, \ldots, x_n).$$

*More precisely, $\Omega_n^g(x_1, \ldots, x_n)$ is an analytic expansion of $\omega_n^g$ around a branch of $x_1 = x_2 = \cdots = x_n = \infty$.*

By analogy with the pruning correspondence of the previous sections, it is natural to consider the expansion of $\omega_n^g$ at $z_1 = z_2 = \cdots = z_n = 0$. The equation

$$\omega_n^g \sim \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} G_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \mu_i z_i^{\mu_i - 1} \, \mathrm{d}z_i$$

defines the coefficients $G_{g,n}(\mu_1, \ldots, \mu_n)$, which are quasi-polynomials modulo 2 in $\mu_1, \ldots, \mu_n$. These quasi-polynomials are shown for small values of $g$ and $n$ in the table below.

| **g** | **n** | # odd $\mu_i$ | $G_{g,n}(\mu_1, \ldots, \mu_n)$ |
|---|---|---|---|
| 0 | 3 | $0, 2$ | 0 |
| 0 | 3 | $1, 3$ | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | $\frac{1}{48}(\mu_1^2 - 3)$ |
| 0 | 4 | $0, 4$ | $\frac{1}{4}(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2)$ |
| 0 | 4 | $1, 3$ | 0 |
| 0 | 4 | 2 | $\frac{1}{4}(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 - 2)$ |
| 1 | 2 | 0 | $\frac{1}{384}\,(\mu_1^2 + \mu_2^2 - 8)\,(\mu_1^2 + \mu_2^2)$ |
| 1 | 2 | 1 | 0 |
| 1 | 2 | 2 | $\frac{1}{384}\,(\mu_1^2 + \mu_2^2 - 6)\,(\mu_1^2 + \mu_2^2 - 2)$ |
| 2 | 1 | 0 | 0 |
| 2 | 1 | 1 | $\frac{1}{2^{16}3^{3}5}\,(\mu_1^2 - 1)^2\,(5\mu_1^4 - 186\mu_1^2 + 1605)$ |

One can show that the quasi-polynomials satisfy the following relations:

$$G_{g,n+1}(0, \mu_1, \ldots \mu_n) = \sum_{j=1}^{n} \sum_{k=1}^{\mu_j} k\, G_{g,n}(\mu_1, \ldots, \mu_n)|_{\mu_j = k},$$

$$G_{g,n+1}(1, \mu_1, \ldots, \mu_n) = \sum_{j=1}^{n} \sum_{k=1}^{\mu_j} k\, G_{g,n}(\mu_1, \ldots, \mu_n)|_{\mu_j = k} - \frac{2g - 2 + n + |\boldsymbol{\mu}|}{2}\, G_{g,n}(\mu_1, \ldots, \mu_n).$$

A natural question is whether these quasi-polynomials $G_{g,n}$ obtained from the expansion of $\omega_n^g$ around $z_1 = z_2 = \cdots = z_n = 0$ yield an interesting and useful enumerative problem. In all calculated cases, the values of $G_{0,n}$ take on integral values, which lends credence to the idea that there is an underlying enumerative problem.

6.1. **Cycle Hurwitz problem.** The following Hurwitz problem was introduced and studied by Okounkov and Pandharipande [28]. Given $\{x_1, \ldots, x_n\} \subset \mathbb{CP}^1$, define

$$\mathcal{C}_{g,n}(\boldsymbol{\mu}) = \left\{ f : \Sigma \to \mathbb{CP}^1 \;\middle|\; \begin{array}{l} \Sigma \text{ a connected genus } g \text{ Riemann surface; ramification} \\ (\mu_k, 1, \ldots, 1) \text{ over } x_k;\ f \text{ unramified elsewhere} \end{array} \right\}.$$

Now define the cycle Hurwitz numbers by the equation

$$P_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{f \in \mathcal{C}_{g,n}(\boldsymbol{\mu})} \frac{1}{|\mathrm{Aut}\ f|}.$$

The Gromov–Witten invariants of $\mathbb{CP}^1$ compactify the cycle Hurwitz enumeration by allowing stable domains, in the following precise sense.

**Theorem 8** (Okounkov–Pandharipande [28])**.** *The cycle Hurwitz number $P_{g,n}(\mu_1, \ldots, \mu_n)$ is precisely the contribution to the Gromov–Witten invariant $\prod_{i=1}^{n} (\mu_i - 1)! \left\langle \prod_{i=1}^{n} \tau_{\mu_i - 1}(\omega) \right\rangle_d^g$ by stable maps with smooth domain curves.*

**Lemma 6.1.** *Let $\mu_1, \mu_2, \mu_3$ be positive integers that sum to an odd integer and satisfy the triangle inequalities. Then $P_{0,3}(\mu_1, \mu_2, \mu_3) = G_{0,3}(\mu_1, \mu_2, \mu_3) = 1$.*

*Proof.* We already know that $G_{0,3}(\mu_1, \mu_2, \mu_3) = 1$ if and only if $\mu_1, \mu_2, \mu_3$ are positive integers that sum to an odd integer. So the point is to calculate $P_{0,3}(\mu_1, \mu_2, \mu_3)$ with the added constraint that $\mu_1, \mu_2, \mu_3$ satisfy the triangle inequality.

One only obtains a non-trivial cycle Hurwitz problem if $\mu_1 + \mu_2 + \mu_3$ is odd, since we require $\mu_1 + \mu_2 + \mu_3 = 2d + 1$, where $d$ is the degree of the branched cover. Furthermore, we require $\mu_1, \mu_2, \mu_3 \leq d$, from which one can deduce the triangle inequalities. So let us assume henceforth that $\mu_1, \mu_2, \mu_3$ are positive integers that sum to an odd integer and satisfy the triangle inequalities.

Denote by $C_\mu \subset S_d$ the conjugacy class in the symmetric group consisting of all permutations with cycle structure $(\mu, 1, \ldots, 1)$. The lemma is equivalent to the statement

(6.3)
$$\#\{(\sigma_1, \sigma_2, \sigma_3) \mid \sigma_i \in C_{\mu_i} \text{ and } \sigma_1\sigma_2\sigma_3 = \mathrm{id} \text{ is a transitive factorisation}\} = d!.$$

To obtain the cycle Hurwitz number, we divide (6.3) by $d!$, which corresponds to identifying equivalent products

$$(\sigma_1, \sigma_2, \sigma_3) \sim (g\sigma_1 g^{-1}, g\sigma_2 g^{-1}, g\sigma_3 g^{-1}), \qquad \text{for } g \in S_d.$$

These factorisations in turn correspond to isomorphic branched covers. Furthermore, if a product is fixed by conjugation, then this defines an automorphism of the branched cover.

So it remains to prove equation (6.3). By the Riemann–Hurwitz formula, the degree $d$ of the cover satisfies $\mu_1 + \mu_2 + \mu_3 = 2d + 1$. As a preliminary example, suppose that $(\mu_1, \mu_2, \mu_3) = (d, d, 1)$, which describes the unique cover of $\mathbb{CP}^1$ with two totally ramified points. It has automorphism group of size $d$, leading to a contribution of $\frac{1}{d}$. Equivalently, the number of transitive factorisations $\sigma_1\sigma_2 = \mathrm{id}$ is $(d-1)!$. However, an extra factor of $d$ comes from the choice of the point corresponding to $\sigma_3 = \mathrm{id}$. Equivalently, one can think of the third point making the automorphism group trivial.

More generally, identify $\sigma_i$ with its cycle of length $\mu_i$, and if $\mu_i = 1$, identify $\sigma_i$ with one of the $d$ 1-cycles $(i)$. For the factorisation to be transitive and have product equal to the identity, there must be exactly one number common to all three cycles, which we suppose to be 1. Also suppose that $\sigma_1$ is given by the cycle $(1, \ldots, a) \in S_d$. The numbers $2, 3, \ldots, a$ appear in exactly one of $\sigma_2$ and $\sigma_3$ and their location is uniquely determined. Also, $\sigma_2$ and $\sigma_3$ both contain the numbers $1, a+1, a+2, \ldots, d$, and the order of these numbers in $\sigma_2$ determines their order in $\sigma_3$. Hence, the number of transitive factorisations that we are interested in is

$$\binom{d}{a} \cdot (a-1)! \cdot a \cdot (d-a)! = d!.$$

Here, the factor $\binom{d}{a}$ chooses the elements of $\sigma_1$, the factor $(a-1)!$ chooses the cycle $\sigma_1$, the factor $a$ chooses the number common to all three factors, and the factor $(d-a)!$ chooses the order of $a+1, a+2, \ldots, d$ in $\sigma_2$. □

In general $G_{g,n}(\mu_1, \ldots, \mu_n) \neq P_{g,n}(\mu_1, \ldots, \mu_n)$, which can be seen from the calculations $G_{0,3}(2d-1, 1, 1) = 1$ and $P_{0,3}(2d-1, 1, 1) = 0$ for $d > 1$. Nevertheless, Lemma 6.1 suggests that $P_{g,n}(\mu_1, \ldots, \mu_n)$ may equal $G_{g,n}(\mu_1, \ldots, \mu_n)$ under certain conditions on $(\mu_1, \ldots, \mu_n)$. More generally, $G_{g,n}(\mu_1, \ldots, \mu_n)$ might be realised as the solution to a generalised cycle Hurwitz problem that corresponds to allowing certain stable curves for the domains of branched covers.

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School of Mathematical Sciences, Monash University, Victoria 3800, Australia
*E-mail address*: norm.do@monash.edu

School of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia
*E-mail address*: pnorbury@ms.unimelb.edu.au