UNIFORM BOUNDS IN F-FINITE RINGS
AND LOWER SEMI-CONTINUITY OF THE F-SIGNATURE

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ABSTRACT. This paper establishes uniform bounds in characteristic p rings
which are either F-finite or essentially of finite type over an excellent local
ring. These uniform bounds are then used to show that the Hilbert-Kunz
length functions and the normalized Frobenius splitting numbers defined on the
spectrum of a ring converge uniformly to their limits, namely the Hilbert-Kunz
multiplicity function and the F-signature function. From this we establish that
the F-signature function is lower semi-continuous. Lower semi-continuity of the
F-signature of a pair is also established. We also give a new proof of the upper
semi-continuity of Hilbert-Kunz multiplicity, which was originally proven by
Ilya Smirnov.

1. INTRODUCTION

Throughout this paper all rings are assumed to be commutative, Noetherian,
with identity, and of prime characteristic p. We shall reserve q to denote a power
of p, i.e., $q = p^e$ for some nonnegative integer e, and $\lambda(-)$ denotes the standard
length function.

If $(R, \mathfrak{m})$ is a local ring of dimension d, M is a finitely generated R-module, and
I is an $\mathfrak{m}$-primary ideal of R, then *the qth Hilbert-Kunz length of M at I is given
by* $\frac{1}{q^{\dim(M)}}\lambda(M/I^{[q]}M)$. *The Hilbert-Kunz multiplicity of M at I is defined by*

$$e_{HK}(I, M) := \lim_{q \to \infty} \frac{1}{q^{\dim(M)}}\lambda\left(\frac{M}{I^{[q]}M}\right).$$

Paul Monsky showed this limit always exists in [14].

We say that a ring R is F-finite if the Frobenius endomorphism $F : R \to R$
which maps $r \mapsto r^p$ makes R a finite R-module. This is equivalent to R being
module finite over $F^e(R)$ for all $e \geq 1$, and when R is reduced this is equivalent to
$R^{1/q}$ being module finite over R for all q. If R is an F-finite ring, then so is any
finitely generated algebra and localization over R. If $(R, \mathfrak{m}, k)$ is local, then we let
$\alpha(R) = \log_p[k^{1/p} : k]$. If R is not necessarily local and $P \in \operatorname{Spec}(R)$, then we let
$\alpha(P) = \alpha(R_P)$.

In this paper we will be interested in uniform properties of Hilbert-Kunz func-
tions over F-finite rings and rings essentially of finite type over an excellent lo-
cal ring. For the sake of simplicity, assume that $I = \mathfrak{m}$ and $M = R$. Then
not only does $\frac{1}{q^d}\lambda(R/\mathfrak{m}^{[q]})$ converge to $e_{HK}(R) := e_{HK}(\mathfrak{m}, R)$, it is the case that
$\lambda(R/\mathfrak{m}^{[q]}R) = e_{HK}(R)q^d + O(q^{d-1})$; hence there exists a $C > 0$ such that for all q,
$\lambda(R/\mathfrak{m}^{[q]}R) \leq Cq^d$. Now suppose that R is a not necessarily local characteristic p

ring. Then for each $P \in \mathrm{Spec}(R)$ there exists a constant $C > 0$ such that for each q, $\lambda(R_P/P^{[q]}R_P) \leq Cq^{\mathrm{ht}(P)}$. This particular result is not very interesting since the constant C depends upon P and is easily obtained from well-known results about Hilbert-Kunz length functions. Recently, I. Smirnov showed that if R is an excellent ring of characteristic p, then for each $P \in \mathrm{Spec}(R)$ there exist a constant C and element $s \in R - P$ such that for all $Q \in D(s) \cap V(P)$, $\lambda_{R_Q}(R_Q/Q^{[q]}R_Q) \leq Cq^{\mathrm{ht}(Q)}$ [15, Lemma 15]. We significantly improve this result in Proposition 3.3 and Proposition 4.3 for rings which are either F-finite or are essentially of finite type over an excellent local ring, both of which are large classes of excellent rings. A consequence of Proposition 3.3 and Proposition 4.3 is that if R is F-finite or essentially of finite type over an excellent local ring, then there exists a constant C such that for all $P \in \mathrm{Spec}(R)$, $\lambda(R_P/P^{[q]}R_P) \leq Cq^{\mathrm{ht}(P)}$.

A *map of primary ideals* is a map $I(-) : \mathrm{Spec}(R) \to \{\text{Ideals of } R\}$ such that for each $P \in \mathrm{Spec}(R)$, $I(P)R_P$ is a PR_P-primary. If M is a finitely generated R-module, then we shall denote by $\lambda_{q_1}^M(I(-))$, or simply $\lambda_{q_1}(I(-))$ if M is understood, the function from the support of M, which we denote $\mathrm{Supp}(M)$, to the real numbers $\mathbb{R}$ which maps a prime $P \mapsto \frac{1}{q_1^{\dim(M_P)}}\lambda(M_P/I(P)^{[q_1]}M_P)$. We denote by $e_{HK}(I(-), M_-)$ the function which sends a prime $P \in \mathrm{Supp}(M)$ to $e_{HK}(I(P), M_P)$.

Let R, M, and $I(-)$ be as above. Then it is easy to see that $\lambda_{q_1}(I(-))$ converges pointwise to $e_{HK}(I(-), M_-)$ as $q_1 \to \infty$. Theorem 5.1 states that if R is an F-finite ring or essentially of finite type over an excellent local ring, M is a finitely generated R-module, and $I(-)$ is a map of primary ideals, then $\lambda_{q_1}^M(I(-))/\lambda_1^M(I(-))$ converges uniformly to $e_{HK}(I(-), M_-)/\lambda_1^M(I(-))$ as $q_1 \to \infty$. In particular, if $M = R$ and $I(P) = P$, then the qth Hilbert-Kunz function which sends a prime $P \mapsto \lambda(R_P/P^{[q]}R_P)$ converges uniformly to the Hilbert-Kunz multiplicity function which sends a prime $P \mapsto e_{HK}(R_P)$. In order to prove this we will need to establish some uniform bounds in F-finite rings and in rings essentially of finite type over an excellent local ring. Some of the uniform bounds established in Sections 3 and 4 of this paper are related to, but are often improvements on, uniform bounds in Section 3 of [15], which establishes the upper semi-continuity of Hilbert-Kunz multiplicity, and Section 3 of [16], which shows that the F-signature of a local ring exists.

Smirnov has recently shown in [15] that $e_{HK}(R_-)$ is upper semi-continuous on locally equidimensional rings which are F-finite or essentially of finite type over an excellent local ring. In showing that the Hilbert-Kunz multiplicity function is the uniform limit of upper semi-continuous functions on such rings, we easily recover Smirnov's result. It is still unknown if Hilbert-Kunz multiplicity is upper semi-continuous on an excellent locally equidimensional ring.

Another interesting invariant defined on a local ring $(R, \mathfrak{m})$ of characteristic p is the F-signature of R, defined originally in [11], by Huneke and Leuschke. For any module M and any $q = p^e$, we can view M as an R-module via restriction of scalars under F^e, which we denote by $F_*^q M$. In particular if R is F-finite and $M = R$, then $F_*^q R$ is a module finite R-module, and we can write $F_*^q R \simeq R^{a_q} \oplus M_q$ where M_q has no free R-summand. The number a_q is called the qth Frobenius splitting number of R. We denote by $b_q = \frac{a_q}{q^{\alpha(R)}}$. The number $s_q := \frac{a_q}{q^{\alpha(R)+\dim(R)}}$ is called the qth normalized Frobenius splitting number of R. Yao showed there is a way to measure b_q in way that is well defined even for rings which are not F-finite; hence one can define the qth normalized Frobenius splitting number for a ring which is not F-finite [17, Lemma 2.1]. Huneke and Leuschke defined the F-signature of a

local ring of dimension d to be the limit $\lim_{q\to} \frac{b_q}{q^d}$, provided the limit exists. Kevin Tucker showed in [16] that the F-signature of a local ring always exists.

If R is an F-finite ring which is not necessarily local, then define the qth Frobenius splitting number function $a_q : \mathrm{Spec}(R) \to \mathbb{R}$ by letting $a_q(P)$ be the qth Frobenius splitting number of the local ring R_P. If R is any characteristic p ring, then define the qth normalized Frobenius splitting number function $s_q : \mathrm{Spec}(R) \to \mathbb{R}$ by letting $s_q(P)$ be the qth normalized Frobenius splitting number of the local ring R_P. We let $b_q(P) = q^{\mathrm{ht}(P)} s_q(P)$ and we let $s : \mathrm{Spec}(R) \to \mathbb{R}$ be the F-signature function which sends a prime $P \mapsto s(R_P)$, the F-signature of the local ring R_P.

The problem of whether the F-signature function is a lower semi-continuous function with respect to the Zariski topology has been of interest for quite some time. Recall that a function $f : X \to \mathbb{R}$, where X is a topological space, is lower semi-continuous at $x \in X$ if for all $\epsilon > 0$, there is an open neighborhood U of x such that $f(x) - f(y) < \epsilon$ for all $y \in U$. In other words, a function f is lower semi-continuous at x if in a small enough open neighborhood of x the numbers $f(y)$ as y varies in the open neighborhood of x can only be slightly smaller than $f(x)$. We would like to briefly explain why it has been suspected that the F-signature function should satisfy this property.

The F-signature detects subtle information about the severity of the singularity of a local ring. Given a local ring $(R, \mathfrak{m})$, it is always the case that $0 \leq s(R) \leq 1$. Huneke and Leuschke showed in [11] that $s(R) = 1$ if and only if R is a regular local ring. Aberbach and Leuschke showed in [2] that $s(R) > 0$ if and only if R is strongly F-regular. Heuristically, the closer to 1 the F-signature of R is the "nicer" the singularity is, and the closer to 0 the "worse" the singularity is. One expects that given a ring or a scheme with decent geometric properties that the severity of a singularity of a point is controlled in an open neighborhood of that point. Given a prime P in the spectrum of such a ring, we expect that in a small enough open neighborhood of the prime that the singularities found in that open neighborhood are not too much worse than the singularity associated with P. Thus we should expect that in a small enough open neighborhood U of P, that $s(Q)$ is at most ϵ closer to 0, which is precisely what lower semi-continuity of the F-signature would say.

Another reason to expect the F-signature function to be lower semi-continuous is that Enescu and Yao showed in [6] that under mild conditions, the qth normalized Frobenius splitting number function is a lower semicontinuous function. For example, they showed that if R is a domain which is either F-finite or essentially of finite type over an excellent local ring, then the qth normalized Frobenius number function is lower semi-continuous. So after Kevin Tucker showed the F-signature to always exist, it has been known that the F-signature function naturally arises as the limit of lower semi-continuous functions.

Some light has been previously shed on the lower semi-continuity of the F-signature problem. Blickle, Schwede, and Tucker showed that if R is a regular and not necessarily local F-finite ring with $0 \neq f \in R$ and $t \geq 0$, then the function $\mathrm{Spec}(R) \to \mathbb{R}$ defined by $P \mapsto s(R_P, f^t)$ is lower semi-continuous. See [4] for more details. The results in Section 6 in this paper recapture Blickle, Schewede, and Tucker's result.

Theorem 5.6 proves that if R is either F-finite or is essentially of finite type over an excellent local ring, then the qth normalized Frobenius splitting number

functions converge uniformly to the F-signature function as $q \to \infty$. It will then follow by Enescu's and Yao's work, [6], that the F-signature function will be lower semi-continuous on all such rings. Kevin Tucker has independently found, and discussed with the author, an alternative proof of the lower semi-continuity of the F-signature.

The paper is organized as follows. In Section 2 we establish some preliminary results. In particular, Section 2 contains a generalized version of a lemma of Sankar Dutta which is crucial to the bounds given in Section 3. Section 3 establishes uniform bounds of Hilbert-Kunz length functions in F-finite rings. Section 3 is the most difficult section to work through, but the bounds that are established lead to a proof that the F-signature function is lower semi-continuous on F-finite rings and rings essentially of finite type over an excellent local ring. Section 4 establishes the bounds in Section 3 for rings which are essentially of finite type over an excellent local ring. In Section 5 we apply the results of Sections 3 and 4 to establish the uniform convergence of Hilbert-Kunz length functions and normalized Frobenius splitting numbers to their limits. In Section 6 lower semi-continuity of the F-signature of a pair $(R, \mathscr{D})$ is established for all Cartier subalgebras $\mathscr{D}$ on an F-finite ring R.

2. Preliminary results

We will need a global version of a lemma first proved by Sankar Dutta in order to establish the uniform bounds found in Sections 3 and 4. In [5], Dutta showed that if $(R, \mathfrak{m})$ is an F-finite local domain of dimension d, then there exist a finite set of nonzero primes $\mathcal{S}(R)$ and a constant C such that for all $q = p^e$ there is a containment of R-modules $R^{q^{\gamma(R)}} \subseteq R^{1/q}$ which has a prime filtration whose prime factors are isomorphic to R/P, where $P \in \mathcal{S}(R)$, and such a prime factor appears no more than $Cq^{\gamma(R)}$ times in the filtration. In particular, the length of the prime filtration of $R^{q^{\gamma(R)}} \subseteq R^{1/q}$ has length no more than $C|\mathcal{S}(R)|q^{\gamma(R)}$. This result, for local domains whose residue field is perfect, is Exercise 10.4 in [9], whose proof is given in the second appendix by Karen Smith, and this result is explicitly stated and proved in [10] as Lemma 4.

Remark 2.1. If R is an F-finite domain and $P \in \mathrm{Spec}(R)$ is a nonzero prime, then $\gamma(R/P) = \log_p[\frac{R_P}{PR_P}^{1/p} : \frac{R_P}{PR_P}] = \alpha(P) < \alpha(P) + \mathrm{ht}(P) = \gamma(R)$.

Lemma 2.2. *Let R be an F-finite domain. Then there exist a finite set of nonzero primes $\mathcal{S}(R)$ and a constant C such that for every $q = p^e$,*

  (1) *there is a containment of R-modules $R^{q^{\gamma(R)}} \subseteq R^{1/q}$,*
  (2) *which has a prime filtration whose prime factors are isomorphic to R/P, where $P \in \mathcal{S}(R)$,*
  (3) *and for each $P \in \mathcal{S}(R)$, the prime factor R/P appears no more than $Cq^{\gamma(R)}$ times in the prime filtration of the containment $R^{q^{\gamma(R)}} \subseteq R^{1/q}$.*

Proof. We shall prove the statement by induction on the Krull dimension of R. If the dimension R is 0, then R is a field and the lemma is trivial.

Now suppose that $\dim(R) > 0$. Then $R^{1/p}$ is a torsion-free R-module of rank $p^{\gamma(R)}$. Hence there is an injection of R-modules $R^{p^{\gamma(R)}} \subseteq R^{1/p}$ so that the support of the cokernel $R^{1/p}/R^{p^{\gamma(R)}}$ consists of nonzero primes. Therefore $R^{p^{\gamma(R)}} \subseteq R^{1/p}$

has a prime filtration of the following form with the quotients $M_i/M_{i-1} = R/P_i$, where P_i is a nonminimal prime of R,

$$R^{p^{\gamma(R)}} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_h = R^{1/p}.$$

The quotients R/P_i are F-finite domains of smaller Krull dimension than R, and so we may assume by induction that the result holds for each R/P_i, with finite collection of primes $\mathcal{S}(R/P_i)$ and constant C_i. Let $C' = \sum C_i$ and let

$$\mathcal{S}(R) = \bigcup (\mathcal{S}(R/P_i) \cup \{P_i\}).$$

Observe that the above filtration shows that $R^{p^{\gamma(R)}} \subseteq R^{1/p}$ has a prime filtration consisting of no more than C' quotients isomorphic to R/P for each $P \in \mathcal{S}(R)$ and all prime factors are isomorphic to R/P for some $P \in \mathcal{S}(R)$. We shall show by induction that $R^{q^{\gamma(R)}} \subseteq R^{1/q}$ has a prime filtration whose prime factors are isomorphic to R/P with $P \in \mathcal{S}(R)$ with no more than $C'q^{\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{q})$ quotients isomorphic to R/P for each $P \in \mathcal{S}(R)$.

Now suppose that $R^{q^{\gamma(R)}} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = R^{1/q}$ is a prime filtration of $R^{q^{\gamma(R)}} \subseteq R^{1/q}$ whose prime factors are isomorphic to R/P with $P \in \mathcal{S}(R)$ with no more than $C'q^{\gamma(R)}(1 + \frac{1}{p} + \cdots + \frac{1}{q})$ quotients isomorphic to R/P for each $P \in \mathcal{S}(R)$. Take qth roots of the modules in the filtration $R^{p^{\gamma(R)}} \subseteq R^{1/p}$ to get the new filtration

$$(R^{1/q})^{p^{\gamma(R)}} = M_0^{1/q} \subseteq M_1^{1/q} \subseteq \cdots \subseteq M_h^{1/q} = R^{1/pq}.$$

Each of the quotients $M_i^{1/q}/M_{i+1}^{1/q} = (R/P_i)^{1/q}$. By induction there exists a prime filtration of $M_{i-1}^{1/q} \subseteq M_i^{1/q}$ with precisely $q^{\gamma(R)}$ prime factors isomorphic to R/P_i, and each other prime factor is isomorphic to R/P for some $P \in \mathcal{S}(R/P_i)$, and such a prime factor appears no more than $C_i q^{\gamma(R/P_i)}$ times in the filtration. Furthermore, the prime filtration $R^{q^{\gamma(R)}} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = R^{1/q}$ gives the following filtration of $(R^{q^{\gamma(R)}})^{p^{\gamma(R)}} = R^{(pq)^{\gamma(R)}} \subseteq (R^{1/q})^{p^{\gamma(R)}}$:

$$R^{(pq)^{\gamma(R)}} = N_0^{p^{\gamma(R)}} \subseteq N_1^{p^{\gamma(R)}} \subseteq \cdots \subseteq N_m^{p^{\gamma(R)}} = (R^{1/q})^{p^{\gamma(R)}}.$$

Hence $R^{(pq)^{\gamma(R)}} \subseteq (R^{1/q})^{p^{\gamma(R)}}$ has a prime filtration with prime factors isomorphic to R/P with $P \in \mathcal{S}(R)$, and such a prime factor appears no more than

$$C'(pq)^{\gamma(R)} \left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right)$$

times in the filtration.

Putting the above information together we get that there is an embedding of $R^{(pq)^{\gamma(R)}} \subseteq R^{1/pq}$ with prime filtration whose prime factors are isomorphic to R/P with $P \in \mathcal{S}(R)$ and there is no more than the following number of quotients isomorphic to R/P for each $P \in \mathcal{S}(R)$:

$$C'(pq)^{\gamma(R)} \left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right) \quad + \quad \sum_{i=1}^{h} C_i q^{\gamma(R/P_i)}.$$

By Remark 2.1 we know that each $\gamma(R/P_i) \leq \gamma(R)-1$, and so we have the following estimates:

$$C'(pq)^{\gamma(R)}\left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right) \quad + \quad \sum_{i=1}^{h} C_i q^{\gamma(R/P_i)}$$

$$\leq \quad C'(pq)^{\gamma(R)}\left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right) + \sum_{i=1}^{h} C_i q^{\gamma(R)-1}$$

$$= \quad C'(pq)^{\gamma(R)}\left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right) + C' q^{\gamma(R)-1}$$

$$= \quad C'(pq)^{\gamma(R)}\left(1 + \frac{1}{p} + \cdots + \frac{1}{q} + \frac{1}{p^{\gamma(R)}q}\right)$$

$$\leq \quad C'(pq)^{\gamma(R)}\left(1 + \frac{1}{p} + \cdots + \frac{1}{q} + \frac{1}{pq}\right).$$

Each of the sums $1 + \frac{1}{p} + \cdots + \frac{1}{q}$ is no more than $1 + \frac{1}{2} + \cdots + \frac{1}{2^e}$ is no more than 2. It now follows by induction that for every q the containment $R^{q^{\gamma(R)}} \subseteq R^{1/q}$ will have a prime filtration whose factors are isomorphic to R/P for some $P \in \mathcal{S}(R)$ with no more than $C'(1 + \frac{1}{p} + \cdots + \frac{1}{q})q^{\gamma(R)} \leq 2C' q^{\gamma(R)}$ quotients isomorphic to R/P for each $P \in \mathcal{S}(R)$. $\qquad\square$

We will find it useful in Section 3 to have a version of Dutta's lemma with the inclusion of $R^{q^{\gamma(R)}} \subseteq R^{1/q}$ reversed.

Corollary 2.3. *Let R be an F-finite domain. Then there exist a finite set of nonzero primes $\mathcal{S}(R)$ and a constant C such that for every $q = p^e$,*

(1) *there is a containment of R-modules $R^{1/q} \subseteq R^{q^{\gamma(R)}}$,*
(2) *which has a prime filtration whose prime factors are isomorphic to R/P, where $P \in \mathcal{S}(R)$,*
(3) *and for each $P \in \mathcal{S}(R)$, the prime factor R/P appears no more than $Cq^{\gamma(R)}$ times in the prime filtration of the containment $R^{1/q} \subseteq R^{q^{\gamma(R)}}$.*

Proof. Since $R^{1/p}$ is torsion-free of rank $p^{\gamma(R)}$, there is an injection of R-modules $R^{1/p} \subseteq R^{p^{\gamma(R)}}$ so that the support of the cokernel $R^{p^{\gamma(R)}}/R^{1/p}$ consists of nonzero primes. Therefore there is prime filtration $R^{1/p} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_h = R^{p^{\gamma(R)}}$ with $M_i/M_{i-1} \simeq R/P_i$, with P_i a nonzero prime ideal. By Remark 2.1 $\gamma(R/P_i) < \gamma(R)$. We let $\mathcal{S}(R/P_i)$ and constant C_i be the collection of primes and constant as described in Lemma 2.2 for the F-finite domain R/P_i. As in the proof of Lemma 2.2 we let $C' = \sum C_i$ and $\mathcal{S}(M) = \bigcup(\mathcal{S}(R/P_i) \cup \{P_i\})$. Furthermore, once again as in Lemma 2.2, we can show by induction that $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ has a prime filtration whose prime factors are isomorphic to R/P with $P \in \mathcal{S}(M)$ with no more than $C' q^{\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{q})$ quotients isomorphic to R/P for each $P \in \mathcal{S}(R)$. The above filtration of $R^{1/p} \subseteq R^{p^{\gamma(R)}}$ shows the induction step when $q = p$.

Now suppose that $R^{1/q} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = R^{q^{\gamma(R)}}$ is a prime filtration of $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ such that each $N_j/N_{i-j} \simeq R/P_j$ for some $P_j \in \mathcal{S}(R)$ and such a prime factor appears no more than $C' q^{\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{q})$ times in the

filtration. Therefore $(R^{p^{\gamma(R)}})^{1/q} = (R^{1/q})^{p^{\gamma(R)}} \subseteq (R^{q^{\gamma(R)}})^{p^{\gamma(R)}} = R^{(qp)^{\gamma(R)}}$ has a prime filtration with prime factors R/P_j with $P_j \in \mathcal{S}(R)$ and such a prime factor appears no more than $C'(qp)^{\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{q})$ times in the filtration. Furthermore, the prime filtration $R^{1/p} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = R^{p^{\gamma(R)}}$ gives the following filtration of $R^{1/pq} = (R^{1/p})^{1/q} \subseteq (R^{p^{\gamma(R)}})^{1/q}$:

$$(R^{1/p})^{1/q} = M_0^{1/q} \subseteq M_1^{1/q} \subseteq \cdots \subseteq M_n^{1/q} = (R^{p^{\gamma(R)}})^{1/q}.$$

Since $M_i^{1/q}/M_{i-1}^{1/q} \simeq (R/P_i)^{1/q}$, we apply Lemma 2.2 to see that there is a prime filtration of each $M_{i-1}^{1/q} \subseteq M_i^{1/q}$ whose prime factors come from $\mathcal{S}(R/P_i)$ and such a prime factor appears no more than $C_i q^{\gamma(R/P_i)} \leq C' q^{\gamma(R)-1}$ times in the filtration. Putting all of this information together we get an embedding $R^{1/pq} \subseteq R^{(pq)^{\gamma(R)}}$ with a prime filtration whose prime factors come from $\mathcal{S}(R)$, and such a prime factor appears no more than the following number of times in the filtration:

$$p^{\gamma(R)} C' q^{\gamma(R)} \left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right) + \sum_{i=1}^{h} C_i q^{\gamma(R/P_i)} \leq C'(qp)^{\gamma(R)}(1 + \frac{1}{p} + \cdots + \frac{1}{pq})$$

$$\leq 2C'(qp)^{\gamma(R)}.$$

$\square$

We combine Lemma 2.2 and Corollary 2.3 into a single statement for convenience.

Corollary 2.4. *Let R be an F-finite domain. There exist a finite set of nonzero primes $\mathcal{S}(R)$ and a constant C such that for every $q = p^e$, there are containments of R-modules $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ and $R^{q^{\gamma(R)}} \subseteq R^{1/q}$, each of which has a prime filtration whose prime factors are isomorphic to R/P, where $P \in \mathcal{S}(R)$, and such a prime factor appears no more than $Cq^{\gamma(R)}$ times in the filtration.*

We shall need the following two well-known lemmas, whose proofs are given for the sake of completion.

Lemma 2.5. *Let $(R, \mathfrak{m}, k)$ be an F-finite reduced local ring and let I be an $\mathfrak{m}$-primary ideal. Then $\lambda(R^{1/q}/IR^{1/q}) = q^{\alpha(R)}\lambda(R/I^{[q]}R)$.*

Proof. Consider a prime filtration of $I^{[q]}R \subseteq R$; say it is given by $I^{[q]}R = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = R$ with each $M_i/M_{i-1} \simeq k$. Then by taking qth roots we get a filtration $IR^{1/q} = M_0^{1/q} \subseteq M_1^{1/q} \subseteq \cdots \subseteq M_n^{1/q} = R^{1/q}$ with each quotient $M_i^{1/q}/M_{i-1}^{1/q} \simeq k^{1/q}$. It follows that $\lambda(R^{1/q}/IR^{1/q}) = q^{\alpha(R)}\lambda(R/I^{[q]}R)$. $\square$

Lemma 2.6. *Let $(R, \mathfrak{m}, k)$ be a local characteristic p ring, let I be an $\mathfrak{m}$-primary ideal, and let M be a finitely generated R-module. Then*

$$\lim_{q_2 \to \infty} \frac{1}{q_2^{\dim(M)}} \lambda(M/I^{[q_1 q_2]}M) = q_1^{\dim(M)} e_{HK}(I, M).$$

Proof. We only have to observe that

$$
\begin{aligned}
\lim_{q_2 \to \infty} \frac{1}{q_2^{\dim(M)}} \lambda(M/I^{[q_1 q_2]}M) &= \lim_{q_2 \to \infty} \frac{q_1^{\dim(M)}}{(q_1 q_2)^{\dim(M)}} \lambda(M/I^{[q_1 q_2]}M) \\
&= q_1^{\dim(M)} \lim_{q \to \infty} \frac{1}{q^{\dim(M)}} \lambda(M/I^{[q]}M) \\
&= q_1^{\dim(M)} e_{HK}(I, M).
\end{aligned}
$$

$\square$

3. Uniform bounds in F-finite rings

The goal of this section is to establish uniform bounds in not necessarily local F-finite rings. The purpose of establishing these uniform bounds is to better understand the global behavior of relative Hilbert-Kunz length functions, which can then be used to establish the lower semi-continuity of the F-signature.

Remark 3.1. If $(R, \mathfrak{m})$ is a local ring, an $\mathfrak{m}$-*primary pair of ideals* will be a containment of ideals of R, $I \subseteq J$, such that I is $\mathfrak{m}$-primary. Observe that J is either also $\mathfrak{m}$-primary or R itself. If $I \subseteq J$ is an $\mathfrak{m}$-primary pair of ideals, then there is an ascending chain of ideals $I \subseteq (I, u_1) \subseteq (I, u_1, u_2) \subseteq \cdots \subseteq (I, u_1, u_2, ..., u_{\lambda(J/I)}) = J$ where $u_{i+1} \in (I, u_1, ..., u_i) : \mathfrak{m}$. We shall let $I_0 = I$ and $I_i = (I, u_1, ..., u_i)$ for $1 \le i \le \lambda(J/I)$.

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a local ring of characteristic p and let M be a finitely generated R-module. If $I \subseteq J$ is an $\mathfrak{m}$-primary pair of ideals, then $\lambda(J^{[q]}M/I^{[q]}M) \le \lambda(M/\mathfrak{m}^{[q]}M)\lambda(J/I)$.*

Proof. Observe that $\lambda(J^{[q]}M/I^{[q]}M) = \sum_{i=1}^{\lambda(J/I)} \lambda(I_i^{[q]}M/I_{i-1}^{[q]}M)$; hence it is enough to show that if I is $\mathfrak{m}$-primary and $u \in (I : \mathfrak{m})$, then $\lambda((I, u)^{[q]}M/I^{[q]}M) \le \lambda(M/\mathfrak{m}^{[q]}M)$. Well, $(I, u)^{[q]}M/I^{[q]}M \simeq M/(I^{[q]}M :_M u^q)$. Since $u \in I : \mathfrak{m}$ we have that $\mathfrak{m}^{[q]}M \subseteq (I^{[q]}M :_M u^q)$; hence $\lambda(M/(I^{[q]}M :_M u^q)) \le \lambda(M/\mathfrak{m}^{[q]}M)$. $\square$

Proposition 3.3. *Let R be an F-finite ring and let M be a finitely generated R-module. There exists a constant $C > 0$ such that for all $P \in \operatorname{Spec}(R)$ and $q = p^e$, if $IR_P \subseteq JR_P$ is a PR_P-primary pair of ideals, then*

$$
\lambda\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right) \le Cq^{\dim(M_P)}\lambda\left(\frac{JR_P}{IR_P}\right).
$$

Proof. By Lemma 3.2 we only need to find a constant C such that for all $P \in \operatorname{Spec}(R)$ and all q, $\lambda(M_P/P^{[q]}M_P) \le Cq^{\dim(M_P)}$. If $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ is a prime filtration of M with $M_i/M_{i-1} \simeq R/P_i$, then $\lambda(M_P/P^{[q]}M_P) \le \sum_{i=1}^n \lambda(R_P/(P_i + P^{[q]})R_P)$. This reduces the proposition to showing that if R is an F-finite domain, then there is a constant C such that for all $P \in \operatorname{Spec}(R)$, $\lambda(R_P/P^{[q]}R_P) \le Cq^{\operatorname{ht}(P)}$.

Suppose that R is an F-finite domain. Let $\mathcal{S}(R)$ be the finite set of primes given by Proposition 2.2 for the R-module R, and suppose that $P \in \operatorname{Spec}(R)$. By Lemma 2.5, $\lambda(R_P/P^{[q]}R_P) = \frac{1}{q^{\alpha(R_P)}}\lambda(R_P^{1/q}/PR_P^{1/q})$, so it is equivalent to show

that $\lambda(R_P^{1/q}/PR_P^{1/q}) \le Cq^{\gamma(R)}$ for some C that does not depend on P. From the short exact sequence

$$0 \to R^{q^{\gamma(R)}} \to R^{1/q} \to R^{1/q}/R^{q^{\gamma(R)}} \to 0,$$

we have that

$$\lambda(R_P^{1/q}/PR_P^{1/q}) \le \lambda(R_P^{q^{\gamma(R)}}/PR_P^{q^{\gamma(R)}}) + \lambda(R^{1/q}/(R_P^{q^{\gamma(R)}} + PR_P^{1/q}))$$

$$= q^{\gamma(R)} + \lambda(R^{1/q}/(R_P^{q^{\gamma(R)}} + PR_P^{1/q})).$$

Therefore we only need to find a constant C, independent of P, such that

$$\lambda(R^{1/q}/(R_P^{q^{\gamma(R)}} + PR_P^{1/q})) \le Cq^{\gamma(R)}.$$

Before localizing at P we can apply Proposition 2.2 to know that there exists a filtration of $R^{q^{\gamma(R)}} \subseteq R^{1/q}$, say $R^{q^{\gamma(R)}} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = R^{1/q}$, such that $n \le C'|\mathcal{S}(R)|q^{\gamma(R)}$, where C' is completely independent of P, and each $N_i/N_{i-1} \simeq R/P_i$ for some $P_i \in \mathcal{S}(R)$. For convenience let $M_i = (N_i)_P$. Localizing at P and adding $PR_P^{1/q}$ to each module M_i give a filtration of $R_P^{q^{\gamma(R)}} + PR_P^{1/q} \subseteq R_P^{1/q}$ whose factors are $(M_i + PR_P^{1/q})/(M_{i-1} + PR_P^{1/q}) \simeq M_i/(M_{i-1} + (PR_P^{1/q} \cap M_i))$. Noticing that $PR_P^{1/q} \cap M_i \supseteq PM_i$ we get that

$$\lambda\left((M_i + PR_P^{1/q})/(M_{i-1} + PR_P^{1/q})\right) = \lambda\left(M_i/(M_{i-1} + (PR_P^{1/q} \cap M_i))\right)$$

$$\le \lambda(M_i/(M_{i-1} + PM_i))$$

$$= \lambda(R_P/(P_iR_P + PR_P))$$

$$\le \lambda(R_P/PR_P) = 1.$$

It now follows that $\lambda(R_P^{1/q}/PR_P^{1/q}) \le (1 + C'|\mathcal{S}(R)|)q^{\gamma(R)}$. $\qquad\square$

Corollary 3.4. *Let R be an F-finite ring, and let N, M be two finitely generated R-modules which are isomorphic at minimal primes of R. Then there is a constant C such that for all $P \in \operatorname{Spec}(R)$ and $q = p^e$, if $IR_P \subseteq JR_P$ is a PR_P-primary pair of ideals, then*

$$\left|\lambda\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right) - \lambda\left(\frac{J^{[q]}N_P}{I^{[q]}N_P}\right)\right| \le Cq^{\operatorname{ht}(P)-1}\lambda\left(\frac{JR_P}{IR_P}\right).$$

Proof. Using the notation in Remark 3.1 and applying the triangle inequality, we get

$$\left|\lambda\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right) - \lambda\left(\frac{J^{[q]}N_P}{I^{[q]}N_P}\right)\right| \le \sum_{i=1}^{\lambda(J/I)} \left|\lambda\left(\frac{I_i^{[q]}M_P}{I_{i-1}^{[q]}M_P}\right) - \lambda\left(\frac{I_i^{[q]}N_P}{I_{i-1}^{[q]}N_P}\right)\right|.$$

Thus we may reduce to the scenario that $J = (I, u)$ where $u \in (I : P)$. There are exact sequences $M \xrightarrow{\varphi} N \to T_1 \to 0$ and $N \xrightarrow{\psi} M \to T_2 \to 0$ for which T_1, T_2 are 0 when localized at minimal primes of R. Observe that

$$\varphi(I^{[q]}M :_M u^q) \subseteq (I^{[q]}N :_N u^q)$$

so that there is an induced map $\frac{M_P}{(I^{[q]}M_P:_{M_P}u^q)} \to \frac{N_P}{(I^{[q]}N_P:_{N_P}u^q)}$ whose cokernel, say $(T_1')_P$, is naturally the homomorphic image of $(T_1)_P$. Thus we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
M_P & \xrightarrow{\;\;\varphi\;\;} & N_P & \longrightarrow & (T_1)_P & \longrightarrow & 0 \\
\Big\downarrow & & \Big\downarrow{\scriptstyle\pi_1} & & \Big\downarrow{\scriptstyle\pi_2} & & \\
\frac{M_P}{(I^{[q]}M_P:_{M_P}u^q)} & \longrightarrow & \frac{N_P}{(I^{[q]}N_P:_{N_P}u^q)} & \longrightarrow & (T_1')_P & \longrightarrow & 0
\end{array}
$$

Therefore $\lambda\left(\frac{N_P}{(I^{[q]}N_P:_{N_P}u^q)}\right) - \lambda\left(\frac{M_P}{(I^{[q]}M_P:_{M_P}u^q)}\right) \le \lambda((T_1')_P)$. Observe that $P^{[q]}N_P \subseteq (I^{[q]}N_P :_{N_P} u^q)$ so that $\pi_1(P^{[q]}N_P) = 0$ and therefore $\pi_2(P^{[q]}(T_1)_P) = 0$. Hence $(T_1')_P$ is the homomorphic image of $\frac{(T_1)_P}{P^{[q]}(T_1)_P}$. Thus

$$
\lambda\left(\frac{(I,u)^{[q]}M_P}{I^{[q]}M_P}\right) - \lambda\left(\frac{(I,u)^{[q]}N_P}{I^{[q]}N_P}\right)
$$
$$
= \lambda\left(\frac{N_P}{(I^{[q]}N_P :_{N_P} u^q)}\right) - \lambda\left(\frac{M_P}{(I^{[q]}M_P :_{M_P} u^q)}\right)
$$
$$
\le \lambda\left(\frac{(T_1)_P}{P^{[q]}(T_1)_P}\right).
$$

A similar argument applied to the exact sequence $N \to M \to T_2 \to 0$ implies that

$$
\left| \lambda\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right) - \lambda\left(\frac{J^{[q]}N_P}{I^{[q]}N_P}\right) \right| \le \max_{i=1,2}\left\{ \lambda\left(\frac{(T_i)_P}{P^{[q]}(T_i)_P}\right) \right\}.
$$

The corollary now follows by Proposition 3.3. $\qquad\square$

Corollary 3.5. *Let R be an F-finite ring and let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated R-modules. There exists a constant C such that for all $P \in \operatorname{Spec}(R)$ and $q = p^e$, if $IR_P \subseteq JR_P$ is a PR_P-primary pair of ideals, then*

$$
\left| \lambda\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right) - \lambda\left(\frac{J^{[q]}M_P'}{I^{[q]}M_P'}\right) - \lambda\left(\frac{J^{[q]}M_P''}{I^{[q]}M_P''}\right) \right| \le Cq^{\dim(M_P)-1}\lambda\left(\frac{JR_P}{IR_P}\right).
$$

Proof. Observe that $\lambda\left(\frac{(J+\operatorname{Ann}_R M)R_P}{(I+\operatorname{Ann}_R M)R_P}\right) \le \lambda\left(\frac{JR_P}{IR_P}\right)$. Therefore we can begin by replacing R with $R/\operatorname{Ann}_R M$ so that $\operatorname{ht}(P) = \dim M_P$ for all $P \in \operatorname{Spec}(R)$. If R is reduced, then M is isomorphic to $M' \bigoplus M''$ at minimal primes of R, and we can apply Corollary 3.4. Suppose R is not reduced. Using a standard argument, we can reduce to the scenario that R is reduced. See for example the proofs of Lemma 1.5 in [14] and Proposition 3.11 in [10]. Let e_0 be a large enough integer so that for $q_0 = p^{e_0}$, $\sqrt{0}^{[q_0]} = 0$. Let $F : R \to R$ be the Frobenius endomorphism. Then $F^{e_0}(R)$ is abstractly isomorphic to the reduced ring $R/\sqrt{0}$ and R is module finite

over $F^{e_0}(R)$. Then for all $P \in \mathrm{Spec}(R)$ and $IR_P \subseteq PR_P$ which is PR_P-primary,

$$\frac{1}{q_0^{\alpha(P)}} \lambda_{F^{e_0}(R_P)} \left(\frac{M_P}{(I^{[q_0]} \cap F^{e_0}(R))^{[q]} M_P} \right) = \lambda_{R_P} \left(\frac{M_P}{(I^{[q_0]} \cap F^{e_0}(R))^{[q]} M_P} \right)$$

$$= \lambda_{R_P} \left(\frac{M_P}{I^{[qq_0]} M_P} \right).$$

$\square$

Theorem 3.6. *Let R be an F-finite ring and let M be a finitely generated R-module. There exists a constant C such that for all $P \in \mathrm{Spec}(R)$, for all q_1, q_2, if $IR_P \subseteq JR_P$ is a PR_P-primary pair of ideals, then*

$$\left| \lambda \left(\frac{J^{[q_1]} M_P}{I^{[q_1]} M_P} \right) q_2^{\mathrm{ht}(P)} - \lambda \left(\frac{J^{[q_1 q_2]} M_P}{I^{[q_1 q_2]} M_P} \right) \right| \leq C q_2^{\dim(M_P)} q_1^{\dim(M_P)-1} \lambda \left(\frac{JR_P}{IR_P} \right).$$

Proof. As in the proof of Corollary 3.5, we may replace R by $R/\mathrm{Ann}_R M$ so that $\dim(M_P) = \mathrm{ht}\, P$ for all $P \in \mathrm{Spec}(R)$. If there is a short exact sequence $0 \to M' \to M \to M'' \to 0$, then

$$\left| \lambda \left(\frac{J^{[q_1]} M_P}{I^{[q_1]} M_P} \right) q_2^{\mathrm{ht}(P)} - \lambda \left(\frac{J^{[q_1 q_2]} M_P}{I^{[q_1 q_2]} M_P} \right) \right| \leq A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \left| \lambda \left(\frac{J^{[q_1]} M_P}{I^{[q_1]} M_P} \right) - \lambda \left(\frac{J^{[q_1]} (M'_P \oplus M''_P)}{I^{[q_1]} (M'_P \oplus M''_P)} \right) \right| q_2^{\mathrm{ht}(P)},$$

$$A_2 = \left| \lambda \left(\frac{J^{[q_1 q_2]} M_P}{I^{[q_1 q_2]} M_P} \right) - \lambda \left(\frac{J^{[q_1 q_2]} (M'_P \oplus M''_P)}{I^{[q_1 q_2]} (M'_P \oplus M''_P)} \right) \right|,$$

$$A_3 = \left| \lambda \left(\frac{J^{[q_1]} M'_P}{I^{[q_1]} M'_P} \right) q_2^{\mathrm{ht}(P)} - \lambda \left(\frac{J^{[q_1 q_2]} M'_P}{I^{[q_1 q_2]} M'_P} \right) \right|,$$

$$A_4 = \left| \lambda \left(\frac{J^{[q_1]} M''_P}{I^{[q_1]} M''_P} \right) q_2^{\mathrm{ht}(P)} - \lambda \left(\frac{J^{[q_1 q_2]} M''_P}{I^{[q_1 q_2]} M''_P} \right) \right|.$$

By Corollary 3.5 there is a constant C such that $A_1 \leq C q_1^{\mathrm{ht}(P)-1} \lambda \left(\frac{JR_P}{IR_P} \right) q_2^{\mathrm{ht}(P)}$ and $A_2 \leq C (q_2 q_1)^{\mathrm{ht}(P)-1} \lambda \left(\frac{JR_P}{IR_P} \right)$. Therefore by considering a prime filtration of the module M, we can reduce proving the theorem to the scenario that $M = R/P$ for some prime $P \in \mathrm{Spec}(R)$; i.e., we may assume that $M = R$ is an F-finite domain. Observe that by Lemma 2.5, $\lambda \left(\frac{J^{[q_1 q_2]} R_P}{I^{[q_1 q_2]} R_P} \right) = \frac{1}{q_2^{\alpha(P)}} \lambda \left(\frac{J^{[q_1]} R_P^{1/q_2}}{I^{[q_1]} R_P^{1/q_2}} \right)$. Therefore the theorem is now reduced to showing that there is a constant C independent of P, I, J, q_1, q_2 such that

$$\left| \lambda \left(\frac{J^{[q_1]} R_P}{I^{[q_1]} R_P} \right) q_2^{\gamma(R)} - \lambda \left(\frac{J^{[q_1]} R_P^{1/q_2}}{I^{[q_1]} R_P^{1/q_2}} \right) \right| \leq C q_2^{\gamma(R)} q_1^{\mathrm{ht}(P)-1} \lambda \left(\frac{JR_P}{IR_P} \right).$$

As in the proof of Corollary 3.4 we can further reduce to the scenario that $J = (I, u)$ where $u \in (I : P)$.

Let $C, \mathcal{S}(R)$ be as in Corollary 2.4 with corresponding inclusions of R-modules $R^{1/q} \to R^{q^{\gamma(R)}}$ and $R^{q^{\gamma(R)}} \to R^{1/q}$ whose cokernels are $T_1(q)$ and $T_2(q)$ respectively.

So there are exact sequences $0 \to R^{1/q} \to R^{q^{\gamma(R)}} \to T_1(q) \to 0$ and $0 \to R^{q^{\gamma(R)}} \to R^{1/q} \to T_2(q) \to 0$ so that both $T_1(q)$ and $T_2(q)$ have a prime filtration whose prime factors are isomorphic to R/Q where $Q \in \mathcal{S}(R)$ and such a prime factor appears no more than $Cq^{\gamma(R)}$ times in the filtration. As in the proof of Corollary 3.4 there will be the following commutative diagrams with all vertical maps being surjective:

$$
\begin{array}{ccccccc}
R_P^{1/q_2} & \longrightarrow & R_P^{q_2^{\gamma(R)}} & \longrightarrow & T_1(q_2)_P & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\dfrac{R_P^{1/q_2}}{(I^{[q]}R_P^{1/q_2} :_{R_P^{1/q_2}} u^q)} & \longrightarrow & \dfrac{R_P^{q^{\gamma(R)}}}{(I^{[q]} :_R u^q)R_P^{q^{\gamma(R)}}} & \longrightarrow & T_1'(q_2)_P & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{ccccccc}
R_P^{q_2^{\gamma(R)}} & \longrightarrow & R_P^{1/q_2} & \longrightarrow & T_2(q_2)_P & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\dfrac{R_P^{q^{\gamma(R)}}}{(I^{[q]} :_R u^q)R_P^{q^{\gamma(R)}}} & \longrightarrow & \dfrac{R_P^{1/q_2}}{(I^{[q]}R_P^{1/q_2} :_{R_P^{1/q_2}} u^q)} & \longrightarrow & T_2'(q_2)_P & \longrightarrow & 0
\end{array}
$$

Furthermore, $T_i'(q_2)_P$ will be the homomorphic image of $\frac{T_i(q_2)_P}{P^{[q]}T_i(q_2)_P}$ for $i = 1, 2$. It follows that

$$
\left| \lambda\left(\frac{J^{[q_1]}R_P}{I^{[q_1]}R_P}\right) q_2^{\gamma(R)} - \lambda\left(\frac{J^{[q_1]}R_P^{1/q_2}}{I^{[q_1]}R_P^{1/q_2}}\right) \right| \leq \max_{i=1,2}\left\{ \lambda\left(\frac{T_i(q_2)_P}{P^{[q]}T_i(q_2)_P}\right) \right\}.
$$

For each $i = 1, 2$, $\lambda\left(\frac{T_i(q_2)_P}{P^{[q]}T_i(q_2)_P}\right) \leq Cq^{\gamma(R)} \max_{Q \in \mathcal{S}(R)}\{\left(\frac{R_P}{(Q+P^{[q]})R_P}\right)\}$. We can now apply Proposition 3.3 to see that the desired bound exists. $\qquad\square$

4. UNIFORM BOUNDS IN RINGS ESSENTIALLY OF FINITE TYPE OVER AN EXCELLENT LOCAL RING

The purpose of this section is to establish Proposition 3.3 and Theorem 3.6 for rings which are essentially of finite type over an excellent local ring. The following well-known lemma will allow us to reduce our considerations to rings which are essentially of finite type over a complete local ring.

Lemma 4.1. *Let $R \to S$ be a faithfully flat homomorphism of characteristic p Noetherian rings with regular fibers. Let M be a finitely generated R-module, let $P \in \mathrm{Spec}(R)$, and let IR_P be an PR_P-primary ideal. Then*

$$
\frac{1}{q^{\mathrm{ht}(P)}}\lambda_{R_P}\left(\frac{M_P}{I^{[q]}M_P}\right) = \frac{1}{q^{\mathrm{ht}(Q)}}\lambda_{S_Q}\left(\frac{(S \otimes_R M)_Q}{(I, \underline{x})^{[q]}(S \otimes_R M)_Q}\right),
$$

where Q is a prime of S lying over P and $\underline{x}$ is a regular system of parameters for S_Q/PS_Q.

Proof. The first thing to observe is that $\frac{(S \otimes_R M)_Q}{(I,\underline{x})^{[q]}(S \otimes_R M)_Q} \simeq \frac{S_Q}{\underline{x}^{[q]}S_Q} \otimes_{R_P} \frac{M_P}{I^{[q]}M_P}$. Since $R_P \to S_Q$ is flat and S_Q/PS_Q is regular, we have that

$$
\begin{aligned}
\lambda_{S_Q}\left(\frac{(S \otimes_R M)_Q}{(I,\underline{x})^{[q]}(S \otimes_R M)_Q}\right) &= \lambda_{S_Q}\left(\frac{S_Q}{\underline{x}^{[q]}S_Q} \otimes_{R_P} \frac{M_P}{I^{[q]}M_P}\right) \\
&= \lambda_{S_Q}\left(\frac{S_Q}{(P+\underline{x}^{[q]})S_Q}\right)\lambda_{R_P}\left(\frac{M_P}{I^{[q]}M_P}\right) \\
&= q^{\operatorname{ht}(Q)-\operatorname{ht}(P)}\lambda_{R_P}\left(\frac{M_P}{I^{[q]}M_P}\right).
\end{aligned}
$$

Dividing both sides of the equation by $q^{\operatorname{ht}(Q)}$ gives the desired result. $\qquad\square$

Suppose that R is essentially of finite type over the excellent local ring A. Let $\hat{A}$ denote the completion of A with respect to its maximal ideal. Then $R \to \hat{A} \otimes_A R$ is a faithfully flat homomorphism with regular fibers [13, Section 33, Lemma 4]. This observation and Lemma 4.1 allow us to reduce proving statements about rings essentially of finite type over an excellent local ring to rings which are essentially of finite type over a complete local ring.

If R is essentially of finite type over a complete local ring A, then let Λ be a p-base of the residue field of A. We shall let Γ be a cofinite subset of Λ. For each such Γ there is an associated R-algebra, R^Γ, which satisfies the following.

Theorem 4.2 ([8, Section 6])**.** *Let R be a characteristic p ring essentially of finite type over a complete local ring. Then for each $\Gamma \leq \Lambda$, R^Γ is a faithfully flat, purely inseparable, F-finite R-algebra.*

To say that $R \to R^\Gamma$ is purely inseparable is to say that for each $s \in R^\Gamma$, there exists an $n \in \mathbb{N}$ such that $s^n \in R$. From this it follows that the induced map $\operatorname{Spec}(R^\Gamma) \to \operatorname{Spec}(R)$ is a homeomorphism. The inverse map sends a prime $P \in \operatorname{Spec}(R)$ to $\sqrt{PR^\Gamma}$. If $P \in \operatorname{Spec}(R)$ we shall let $P_\Gamma = \sqrt{PR^\Gamma}$.

If R is essentially of finite type over a complete local ring, then for each Γ we have that $PR_{P_\Gamma}^\Gamma$ is $P_\Gamma R_{P_\Gamma}^\Gamma$-primary. If R is essentially of finite type over an excellent local ring A, then Γ shall represent a cofinite subset of a p-base for a coefficient field of $\hat{A}$. If R is essentially of finite type over a complete local ring and M is a finitely generated R-module, then we let $M^\Gamma = R^\Gamma \otimes_R M$.

Proposition 4.3. *Let R be essentially of finite type over an excellent local ring and let M be a finitely generated R-module. There exists a constant $C > 0$ such that for all $P \in \operatorname{Spec}(R)$ and $q = p^e$, if $IR_P \subseteq JR_P$ is a PR_P-primary pair of ideals, then*

$$
\lambda\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right) \leq Cq^{\dim(M_P)}\lambda\left(\frac{JR_P}{IR_P}\right).
$$

Proof. By Lemma 4.1 and the remarks that follow, we may reduce to the scenario that R is essentially of finite type over a complete local ring. Choose any Γ. Then for each $P \in \operatorname{Spec}(R)$ one sees that by tensoring a prime filtration of $\frac{J^{[q]}M_P}{I^{[q]}M_P}$ with $R_{P_\Gamma}^\Gamma$ that

$$
\lambda_{R_{P_\Gamma}^\Gamma}\left(\frac{J^{[q]}M_{P_\Gamma}^\Gamma}{I^{[q]}M_{P_\Gamma}^\Gamma}\right) = \lambda_{R_P}\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right)\lambda_{R_{P_\Gamma}^\Gamma}(R_{P_\Gamma}^\Gamma/PR_{P_\Gamma}^\Gamma).
$$

We can now apply Proposition 3.3 to the F-finite ring R^Γ so that we know there exists a constant C such that for all $P \in \operatorname{Spec}(R)$ and for all q,

$$
\begin{aligned}
\lambda_{R_P}\left(\frac{J^{[q]}M_P}{I^{[q]}M_P}\right) &= \frac{\lambda_{R_{P_\Gamma}^\Gamma}\left(J^{[q]}M_{P_\Gamma}^\Gamma / I^{[q]}M_{P_\Gamma}^\Gamma\right)}{\lambda_{R_{P_\Gamma}^\Gamma}(R_{P_\Gamma}^\Gamma / PR_{P_\Gamma}^\Gamma)} \\[2ex]
&\leq \frac{Cq^{\operatorname{ht}(P_\Gamma)}\lambda_{R_{P_\Gamma}^\Gamma}\left(JR_{P_\Gamma}^\Gamma / IR_{P_\Gamma}^\Gamma\right)}{\lambda_{R_{P_\Gamma}^\Gamma}(R_{P_\Gamma}^\Gamma / PR_{P_\Gamma}^\Gamma)} \\[2ex]
&= Cq^{\operatorname{ht}(P)}\lambda_{R_P}\left(\frac{JR_P}{IR_P}\right).
\end{aligned}
$$

$\square$

Theorem 4.4. *Let R be essentially of finite type over an excellent local ring and let M be a finitely generated R-module. There exists a constant C such that for all $P \in \operatorname{Spec}(R)$, for all q_1, q_2, if $IR_P \subseteq JR_P$ is a PR_P-primary pair of ideals, then*

$$
\left|\lambda\left(\frac{J^{[q_1]}M_P}{I^{[q_1]}M_P}\right)q_2^{\operatorname{ht}(P)} - \lambda\left(\frac{J^{[q_1 q_2]}M_P}{I^{[q_1 q_2]}M_P}\right)\right| \leq Cq_2^{\operatorname{ht}(P)}q_1^{\operatorname{ht}(P)-1}\lambda\left(\frac{JR_P}{IR_P}\right).
$$

Proof. The proof of this theorem is identical to the proof of Proposition 4.3. Lemma 4.1 allows us to reduce to the scenario that R is essentially of finite type over an excellent local ring. Pick a Γ and let C be as in Theorem 3.6 for the F-finite ring R^Γ. Then

$$
\left|\lambda_{R_P}\left(\frac{J^{[q_1]}M_P}{I^{[q_1]}M_P}\right)q_2^{\operatorname{ht}(P)} - \lambda_{R_P}\left(\frac{J^{[q_1 q_2]}M_P}{I^{[q_1 q_2]}M_P}\right)\right|
$$

$$
= \left|\lambda_{R_{P_\Gamma}^\Gamma}\left(\frac{J^{[q_1]}M_{P_\Gamma}^\Gamma}{I^{[q_1]}(M_{P_\Gamma}^\Gamma)}\right)q_2^{\operatorname{ht}(P)} - \lambda_{R_{P_\Gamma}^\Gamma}\left(\frac{J^{[q_1 q_2]}M_{P_\Gamma}^\Gamma}{I^{[q_1 q_2]}M_{P_\Gamma}^\Gamma}\right)\right| \Bigg/ \lambda_{R_{P_\Gamma}^\Gamma}\left(\frac{R_{P_\Gamma}^\Gamma}{PR_{P_\Gamma}^\Gamma}\right)
$$

$$
\leq \frac{Cq_2^{\operatorname{ht}(P_\Gamma)}q_1^{\operatorname{ht}(P_\Gamma)-1}\lambda_{R_{P_\Gamma}}\left(\frac{JR_{P_\Gamma}}{IR_{P_\Gamma}}\right)}{\lambda_{R_{P_\Gamma}^\Gamma}\left(\frac{R_{P_\Gamma}^\Gamma}{PR_{P_\Gamma}^\Gamma}\right)} = Cq_2^{\operatorname{ht}(P)}q_1^{\operatorname{ht}(P)-1}\lambda_{R_P}\left(\frac{JR_P}{IR_P}\right).
$$

$\square$

5. UNIFORM CONVERGENCE AND CONTINUITY RESULTS

Theorem 5.1. *Let R be either F-finite or essentially of finite type over an excellent local ring and let M be a finitely generated R-module. Let $I(-)$ be a map of primary ideals. The sequence of functions $\frac{\lambda_{q_1}(I(-))}{\lambda_1(I(-))} : \operatorname{Supp}(M) \to \mathbb{R}$ which sends a prime $P \in \operatorname{Supp}(M)$ to $\frac{\lambda(M_P/I(P)^{[q_1]}M_P)}{q_1^{\dim(M_P)}\lambda(R_P/I(P)R_P)}$ converges uniformly to the scaled Hilbert-Kunz multiplicity function $\frac{e_{HK}(I(-),M_-)}{\lambda_1(I(-))}$, which sends a prime $P \in \operatorname{Supp}(M)$ to $\frac{e_{HK}(I(P),M_P)}{\lambda(R_P/I(P)R_P)}$ as $q_1 \to \infty$.*

Proof. Given $\epsilon > 0$, our goal is to show that there exists a q' such that for all $P \in \mathrm{Supp}(M)$ and for all $q_1 \geq q'$, $|\frac{1}{\lambda(R_P/I(P)R_P)}\lambda_{q_1}(I(P)) - \frac{1}{\lambda(R_P/I(P)R_P)}e_{HK}(I(P), M_P)|$ $< \epsilon$. After modding out R by $\mathrm{Ann}_R M$, it follows by Theorems 3.6 and 4.4 that there exists a constant $C > 0$ such that, for all $P \in \mathrm{Supp}(M)$ and for all q_1, q_2,

$$\left| \lambda\left(\frac{M_P}{I(P)^{[q_1]}M_P}\right) q_2^{\dim(M_P)} - \lambda\left(\frac{M_P}{I(P)^{[q_1 q_2]}M_P}\right) \right|$$

$$\leq C q_2^{\dim(M_P)} q_1^{\dim(M_P)-1} \lambda\left(\frac{R_P}{(I(P) + \mathrm{Ann}_R(M))R_P}\right)$$

$$\leq C q_2^{\dim(M_P)} q_1^{\dim(M_P)-1} \lambda\left(\frac{R_P}{I(P)R_P}\right).$$

Dividing both sides of the inequality by $q_2^{\dim(M_P)}$, letting $q_2 \to \infty$, and applying Lemma 2.6 we get that for all $P \in \mathrm{Supp}(M)$ and for all q_1,

$$\left| \lambda\left(\frac{M_P}{I(P)^{[q_1]}M_P}\right) - q_1^{\dim(M_P)}e_{HK}(I(P), M_P) \right| \leq C q_1^{\dim(M_P)-1} \lambda\left(\frac{R_P}{I(P)R_P}\right).$$

Choose q' large enough that $\frac{C}{q'} < \epsilon$ and let $q_1 \geq q'$. Dividing the above inequality by $q_1^{\dim(M_P)}\lambda(R_P/I(P)R_P)$ gives that for all $P \in \mathrm{Supp}(M)$ and all q_1,

$$\left| \frac{\lambda_{q_1}^M(I(P))}{\lambda(R_P/I(P)R_P)} - \frac{e_{HK}(I(P), M_P)}{\lambda(R_P/I(P)R_P)} \right| \leq \frac{C}{q_1} < \epsilon.$$

$\square$

Let $n \in \mathbb{N}$ and set $f_n(P) = \frac{1}{q_1^{\dim(M_P)}}\lambda(M_P/I(P)^{[q_1]}M_P)$ where $q_1 = p^n$ and let f be the limit function $f(P) = e_{HK}(I(P), M_P)$. What Theorem 5.1 is saying is that there exists a strictly positive function $g : \mathrm{Spec}(R) \to \mathbb{R}$, namely $g(P) = \frac{1}{\lambda(R_P/I(P)R_P)}$, which does not depend on n, such that gf_n converges uniformly to the function gf. If there exists a $\delta > 0$ such that for all $P \in \mathrm{Spec}(R)$, $g(P) \geq \delta$, then f_n converges uniformly to f. To see this we can choose n large enough such that $|gf_n - gf| < \epsilon\delta$. Then $|f_n - f| < \epsilon\delta/g \leq \epsilon\delta/\delta = \epsilon$. Using this observation we obtain the following corollary to Theorem 5.1.

Corollary 5.2. *Let R be an F-finite ring of prime characteristic $p > 0$ and let M be a finitely generated R-module. Let $I(-)$ be a map of primary ideals. Suppose that there exists a q such that $P^{[q]} \subseteq I(P)$ for all $P \in \mathrm{Supp}(M)$, or more generally there exists a constant D such that $\lambda(R_P/I(P)R_P) \leq D$ for all $P \in \mathrm{Supp}(M)$. Then the sequence of functions $\lambda_{q_1}(I(-)) : \mathrm{Supp}(M) \to \mathbb{R}$ which sends a prime P to $\frac{1}{q_1^{\dim(M_P)}}\lambda(M_P/I(P)^{[q_1]}M_P)$ converges uniformly to the Hilbert-Kunz multiplicity function $e_{HK}(I(-), M_-)$, which sends a prime P to $e_{HK}(I(P), M_P)$.*

Proof. By the above remarks we only need to find $\delta > 0$ such that for all $P \in \mathrm{Supp}(M)$, $\frac{1}{\lambda(R_P/I(P)R_P)} \geq \delta$, or equivalently that there exists a D such that for all $P \in \mathrm{Supp}(M)$, $\lambda(R_P/I(P)R_P) \leq D$. We are assuming that for each $P \in \mathrm{Spec}(R)$ that $P^{[q]} \subseteq I(P)$. Hence by Proposition 3.3 there exists a constant C such that

for all $P \in \operatorname{Supp}(M)$, $\lambda(R_P/I(P)R_P) \leq \lambda(R_P/P^{[q]}R_P) \leq Cq^{\operatorname{ht}(P)} \leq Cq^{\dim(R)}$. Therefore $D = Cq^{\dim(R)}$ works. $\hfill\square$

Corollary 5.2 gives an alternative proof of Smirnov's result that if R is F-finite or essentially of finite type over an excellent local ring, then $e_{HK}(-)$ is upper semi-continuous at primes P such that R_P is equidimensional.

Corollary 5.3. *Let R be either F-finite or essentially of finite type over an excellent local ring. Then the Hilbert-Kunz function $e_{HK}(-) : \operatorname{Spec}(R) \to \mathbb{R}_{\geq 1}$ which sends a prime $P \mapsto e_{HK}(R_P)$ is upper semi-continuous at all $P \in \operatorname{Spec}(R)$ such that R_P is equidimensional.*

Proof. Consider the map of primary ideals $I(-)$ which sends a prime P to P. Then $P^{[1]} = P \subseteq I(P)$ for each $P \in \operatorname{Spec}(R)$. Corollary 5.2 says that $\lambda_{q_1}(-)$ converges uniformly to $e_{HK}(-)$. Kunz originally showed in [12] that for each q_1 the function $\lambda_{q_1}(-)$ which sends a prime $P \mapsto \frac{1}{q^{\operatorname{ht}(P)}}\lambda(R_P/P^{[q]}R_P)$ is upper semi-continuous on all rings which are locally equidimensional. If R_P is equidimensional, then R being catenary implies that there is an $s \in R - P$ such that R_s is locally equidimensional. The s which works is 1 if $\min(R_P) = \min(R)$. If $\min(R_P) \subsetneq \min(R)$, then just choose $s \in \bigcap_{Q \in Min(R) - \min(R_P)} Q \setminus P$. Therefore, if R_P is equidimensional, then in an open neighborhood of P, $e_{HK}(-)$ is the uniform limit of upper semi-continuous functions. Hence $e_{HK}(-)$ is upper semi-continuous as well. $\hfill\square$

Lemma 5.4. *Let $(R, \mathfrak{m}, k)$ be an excellent reduced local ring of dimension d. Let q_1, q_2 equal p^{e_1} and p^{e_2} respectively and let $b_{q_1} = q_1^d s_{q_1}, b_{q_1 q_2} = (q_1 q_2)^d s_{q_1 q_2}$, where s_{q_1} and $s_{q_1 q_2}$ are the q_1th and $q_1 q_2$th normalized Frobenius splitting numbers of R respectively. Then there is an irreducible $\mathfrak{m}$-primary ideal I and $u \in (I : \mathfrak{m})$ such that $b_{q_1} = \lambda((I, u)^{[q_1]}/I^{[q_1]})$ and $b_{q_1 q_2} = \lambda((I, u)^{[q_1 q_2]}/I^{[q_1 q_2]})$.*

Proof. Let $I_e = \{r \in R \mid F_*^e r \otimes u = 0 \text{ in } F_*^e R \otimes_R E_R(k)\}$ where u generates the socle of $E_R(k)$. Then $b_q = \lambda(R/I_e)$ [17, Remark 2.3]. Since R is reduced and excellent, R is approximately Gorenstein [7, Theorem 1.7]. So there exists a descending chain of irreducible $\mathfrak{m}$-primary ideals $\{I_t\}_{t \in \mathbb{N}}$ which is cofinal with $\{\mathfrak{m}^t\}_{t \in \mathbb{N}}$. Let u_t generate the socle mod I_t. Then $I_e = \bigcup_{t=1}^{\infty}(I_t^{[q]} : u_t^q)$. Therefore for each q there is a t_0 such that for all $t \geq t_0$, $b_q = \lambda(R/(I_t^{[q]} : u_t^q)) = \lambda((I_t, u_t)^{[q]}/I_t^{[q]})$. $\hfill\square$

Theorem 5.5. *Let R be either F-finite or essentially of finite type over an excellent local ring. There exists a constant C such that, for all $P \in \operatorname{Spec}(R)$ and for all q_1, q_2,*

$$|b_{q_1}(P)q_2^{\operatorname{ht}(P)} - b_{q_1 q_2}(P)| \leq Cq_2^{\operatorname{ht}(P)}q_1^{\operatorname{ht}(P)-1}.$$

Proof. It is well known that if $b_q(P) > 0$ for some, equivalently for all, q, then R_P is a reduced ring. Therefore $C = 0$ is a constant which works for all $P \in \operatorname{Spec}(R)$ such that R_P is not reduced. If R_P is reduced there exists an $s \in R - P$ such that R_s is reduced. Therefore by quasi-compactness of $\operatorname{Spec}(R)$, we may reduce our considerations to when R is a reduced ring. The theorem now follows by Lemma 5.4, Theorem 3.6, and Theorem 4.4. $\hfill\square$

Theorem 5.6. *Let R be either F-finite or essentially of finite type over an excellent local ring. The qth normalized Frobenius splitting number function which maps a prime $P \mapsto s_q(P)$ converges uniformly to the F-signature function which maps a prime $P \mapsto s(R_P)$ as $q \to \infty$.*

Proof. Let $\epsilon > 0$, let C be as in Theorem 5.5, and choose q so large that $\frac{C}{q} < \epsilon$. Then for all $P \in \mathrm{Spec}(R)$ we have that

$$|b_{q_1}(P)q_2^{\mathrm{ht}(P)} - b_{q_1 q_2}(P)| \leq Cq_2^{\mathrm{ht}(P)} q_1^{\mathrm{ht}(P)-1}.$$

Therefore

$$\left| b_{q_1}(P) - q_1^{\mathrm{ht}(P)} \frac{b_{q_1 q_2}(P)}{(q_1 q_2)^{\mathrm{ht}(P)}} \right| \leq Cq_1^{\mathrm{ht}(P)-1}.$$

Letting $q_2 \to \infty$ we have that for all $P \in \mathrm{Spec}(R)$,

$$|b_{q_1}(P) - q_1^{\mathrm{ht}(P)} s(R_P)| \leq Cq_1^{\mathrm{ht}(P)-1}.$$

Hence for all $q_1 \geq q$ and all $P \in \mathrm{Spec}(R)$,

$$\left| \frac{b_{q_1}(P)}{q_1^{\mathrm{ht}(P)}} - s(R_P) \right| \leq \frac{C}{q_1} \leq \frac{C}{q} < \epsilon.$$

This verifies that $\frac{b_q(P)}{q^{\mathrm{ht}(P)}} = s_q(P)$ converges uniformly to $s(R_P)$ as $q \to \infty$. $\qquad\square$

Theorem 5.7. *Let R be either F-finite or essentially of finite type over an excellent local ring. The F-signature function on $\mathrm{Spec}(R)$ is lower semi-continuous.*

Proof. Let $\epsilon > 0$ and let $P \in \mathrm{Spec}(R)$. If $s(R_P) = 0$, then it is the case that for all $Q \in \mathrm{Spec}(R)$, $s(R_P) - s(R_Q) \leq 0 < \epsilon$. Now suppose that $s(R_P) > 0$. Aberbach and Leuschke showed in [2], along with Tucker's proof of the existence of $s(R_P)$ in [16], that $s(R_P) > 0$ if and only if R_P is strongly F-regular. In particular we have that R_P is a domain. There then exists an $s \in R - P$ such that R_s is a domain. Enescu and Yao showed in [6] that if S is a locally equidimensional ring which is either F-finite or essentially of finite type over an excellent local ring, then the qth normalized Frobenius splitting number function is lower semi-continuous on $\mathrm{Spec}(S)$. By Lemma 5.4, we have that in a neighborhood of P, the F-signature function is the uniform limit of lower semi-continuous functions; hence is itself lower semi-continuous at P. $\qquad\square$

Suppose that $(R, \mathfrak{m}, k)$ is an F-finite domain of dimension d. Suppose that R is F-pure, i.e., that $a_q(R) \geq 1$ for all q. Recall that the main result of [2] is that the F-signature of R is positive if and only if R is strongly F-regular. Aberbach and Enescu introduced in [1] a way to study the asymptotic growth rate of the number of Frobenius splittings of R in the scenario that R is not strongly F-regular. They define the splitting dimension of R to be the largest integer k so that $\liminf_{q \to \infty} a_q(R)/q^{k+\alpha(R)} > 0$. The splitting dimension of R is denoted $\mathrm{sdim}(R)$. The F-splitting ratio of R is defined to be $r_F(R) = \lim_{q \to \infty} a_q(R)/q^{\mathrm{sdim}(R)+\alpha(R)}$, which was shown to always exist by Blickle, Schwede, and Tucker [3, Theorem 4.2]. In light of Theorem 5.7, we ask the following natural question.

Question 5.8. Suppose that R is an F-finite domain. What can be said about the global behavior of the functions $r_q : \mathrm{Spec}(R) \to \mathbb{R}$ and $r_F : \mathrm{Spec}(R) \to \mathbb{R}$ that send $P \mapsto a_q(R)/q^{\mathrm{sdim}(R)+\alpha(P)}$ and $P \mapsto r_F(R_P)$ respectively?

Given an F-pure domain $(R, \mathfrak{m}, k)$, Aberbach and Enescu show the existence of a prime ideal $\mathcal{P} \subseteq R$ so that $\dim(R/\mathcal{P}) \geq \mathrm{sdim}(R)$, $R/\mathcal{P}$ is strongly F-regular, and $\mathcal{P} = 0$ if and only if R is strongly F-regular; see [1, Theorem 1.1]. The prime $\mathcal{P}$ is referred to as the splitting prime of R. Blickle, Schwede, and Tucker show that $\dim(R/\mathcal{P}) = \mathrm{sdim}(R)$ [3, Corollary 4.3]. As the splitting prime of a local ring will

not necessarily localize to the splitting prime of a localization, it would be surprising if the functions r_q or r_F mentioned in Question 5.8 always enjoy semi-continuity properties.

6. Lower semi-continuity of F-signature of pairs

In this section all rings under consideration will be F-finite. We want to establish the lower semi-continuity of the F-signature of a pair $(R, \mathscr{D})$ where $\mathscr{D}$ is a Cartier algebra; see Section 2 of [3] for a more in-depth look at the basic notions of a Cartier subalgebra. Our main tool will be Proposition 3.3 in order to establish a uniform convergence result and the desired lower semi-continuity.

Let $\mathscr{C}_q := \operatorname{Hom}_R(F_*^q R, R)$ and $\mathscr{C}^R = \bigoplus_{q=p^e, e \geq 0} \mathscr{C}_q$. If $\varphi \in \mathscr{C}_q$ and $\psi \in \mathscr{C}_{q'}$, then $\varphi \cdot \psi := \varphi \circ F_*^q \psi \in \mathscr{C}_{qq'}$, where $F_*^q \psi(F_*^{qq'} r) := F_*^q \psi(F_*^{q'} r)$. We call $\mathscr{C}^R$ the (*total*) *Cartier algebra* of R. Note that the multiplication defined on homogenous elements of $\mathscr{C}^R$ makes $\mathscr{C}^R$ a noncommutative $\mathbb{F}_p$-algebra. Even though the 0th graded piece of $\mathscr{C}^R$ is $\mathscr{C}_{p^0} = \mathscr{C}_1 = \operatorname{Hom}_R(R, R) \simeq R$, R is not central in $\mathscr{C}^R$; hence $\mathscr{C}^R$ is not an R-algebra. We say that $\mathscr{D} \subseteq \mathscr{C}^R$ is a *Cartier subalgebra* of R if $\mathscr{D}$ is an $\mathbb{F}_p$-subalgebra of $\mathscr{C}^R$ and $\mathscr{D}_1 = \mathscr{C}_1 \simeq R$.

Suppose that $(R, \mathfrak{m}, k)$ is a local ring and $\mathscr{D}$ is a Cartier subalgebra of R. Suppose that $F_*^q R \simeq \bigoplus M_i$ as an R-module. The summand M_i is called a $\mathscr{D}$-*summand* if $M_i \simeq R$ and the projection $F_*^q R \to M_j \simeq R$ is an element of $\mathscr{D}_q$. The *qth F-splitting number* of $(R, \mathscr{D})$ is the maximal number $a_q^{\mathscr{D}}$ of $\mathscr{D}$-summands appearing in the various direct sum decompositions of $F_*^q R$. Observe that $a_q^{\mathscr{C}} = a_q$ for all q, the usual qth F-splitting number of R. For each $q = p^e$ let $I_q^{\mathscr{D}} = \{r \in R \mid \varphi(F_*^q r) \in \mathfrak{m} \text{ for all } \varphi \in \mathscr{D}_q\}$. The following lemma is a list of basic properties about the sets $I_q^{\mathscr{D}}$ which can all be found in Section 3 of [3].

Lemma 6.1. *Let $(R, \mathfrak{m}, k)$ be a local F-finite ring, let $\mathscr{D}$ be a Cartier subalgebra, and let q, q_1, q_2 be various powers of p and $\varphi \in \mathscr{D}_{q_1}$. Then:*

(1) $I_q^{\mathscr{D}} \subseteq R$ *is an ideal,*
(2) $\mathfrak{m}^{[q]} \subseteq I_q^{\mathscr{D}}$,
(3) $\varphi(F_*^{q_1} I_{q_1 q_2}^{\mathscr{D}}) \subseteq I_{q_2}^{\mathscr{D}}$,
(4) $\lambda(R/I_q^{\mathscr{D}}) = \dfrac{a_q^{\mathscr{D}}}{q^{\alpha(R)}}$.

Let $(R, \mathfrak{m}, k)$ be local and let $\mathscr{D}$ be a Cartier subalgebra. Set $\Gamma_{\mathscr{D}}$ to be the semigroup $\{q \mid a_q^{\mathscr{D}} \neq 0\}$. The main result of Blickle, Schwede, and Tucker in [3] is that if $(R, \mathfrak{m}, k)$ is local and $\mathscr{D}$ is a Cartier subalgebra, then the limit $\lim_{q \in \Gamma_{\mathscr{D}} \to \infty} \dfrac{a_q^{\mathscr{D}}}{q^{\alpha(R)+\dim(R)}}$ $= \lim_{q \in \Gamma_{\mathscr{D}} \to \infty} \dfrac{1}{q^{\dim(R)}} \lambda(R/I_q^{\mathscr{D}})$ exists, is called the *F-signature of the pair* $(R, \mathscr{D})$, and is denoted $s(R, \mathscr{D})$.

Suppose that R is F-finite and not necessarily local. Let $\mathscr{D}$ be a Cartier algebra of R. Suppose that $S \subseteq R$ is a multiplicatively closed set. Since R is F-finite, $S^{-1} \operatorname{Hom}_R(F_*^q R, R) \simeq \operatorname{Hom}_{S^{-1}R}(F_*^q S^{-1}R, S^{-1}R)$. Therefore there is a naturally induced Cartier subalgebra $S^{-1}\mathscr{D}$ of $S^{-1}R$ such that $(S^{-1}\mathscr{D})_q = S^{-1}(\mathscr{D}_q)$. If $P \in \operatorname{Spec}(R)$ and $s \in R$ we write $\mathscr{D}_P$ and $\mathscr{D}_s$ for the induced Cartier subalgebras of R_P and R_s respectively. For each $P \in \operatorname{Spec}(R)$ let $a_q(P, \mathscr{D})$ be the qth F-splitting number of $(R_P, \mathscr{D}_P)$, $s_q(P, \mathscr{D}) = a_q(P, \mathscr{D})/q^{\alpha(P)+\operatorname{ht}(P)}$, and let $s(P, \mathscr{D}) = s(R_P, \mathscr{D}_P)$. Then $s_q(P, \mathscr{D}) : \operatorname{Spec}(R) \to \mathbb{R}$ converges to $s(P, \mathscr{D}) : \operatorname{Spec}(R) \to \mathbb{R}$ as

$q \in \Gamma_{\mathscr{D}} \to \infty$ as a limit of functions. For each q and $P \in \mathrm{Spec}(R)$, let

$$I_q^{\mathscr{D}}(P) = \{r \in R_P \mid \forall \varphi \in (\mathscr{D}_P)_q, \varphi(F_*^q r) \in PR_P\}.$$

Remark 6.2. If R is an F-finite ring, $\mathscr{D}$ is a Cartier subalgebra, $N, M \in \mathbb{N}$, $f \in \mathscr{D}_{q_1}^N$, $g \in \mathscr{D}_{q_2}^M$, then the natural map $\underbrace{(g, g, ..., g)}_{N \text{ times}} \circ F_*^{q_2} f : F_*^{q_1 q_2} R \to R^{NM}$ is an element of

$\mathscr{D}_{q_1 q_2}^{NM}$.

The remark follows from the assumption that if $\varphi \in \mathscr{D}_{q_1}$ and $\psi \in \mathscr{D}_{q_2}$, then $\psi \circ F_*^{q_2} \varphi \in \mathscr{D}_{q_1 q_2}$.

Our first goal will be to establish a version of Corollary 2.3 for a pair $(R, \mathscr{D})$ when R is an F-finite domain. If R is a not necessarily local F-finite domain and $\mathscr{D}$ is a Cartier subalgebra, we let $\Gamma_{\mathscr{D}} = \Gamma_{\mathscr{D}_0}$. We will now be interested only in containments of R-modules $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ when $q \in \Gamma_{\mathscr{D}}$. Not only will we need to know that for each $q \in \Gamma_{\mathscr{D}}$ that a prime filtration of $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ has only finitely many prime factors up to isomorphism and such prime factors only appear a controlled number of times, but we will need that each of the following $q^{\gamma(R)}$ maps is an element of $\mathscr{D}_q$, $R^{1/q} \subseteq R^{q^{\gamma(R)}} \xrightarrow{\pi_i} R$, where π_i is the projection onto the ith factor.

Lemma 6.3. *Let R be an F-finite domain and let $\mathscr{D}$ be a Cartier subalgebra. There exist a finite set of nonzero primes $\mathcal{S}(R, \mathscr{D})$ and a constant C such that for every $q \in \Gamma_{\mathscr{D}}$,*

(1) *there is a containment of R-modules $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ which is an element of $\mathscr{D}_q^{q^{\gamma(R)}}$,*

(2) *which has a prime filtration whose prime factors are isomorphic to R/P, where $P \in \mathcal{S}(R, \mathscr{D})$,*

(3) *and for each $P \in \mathcal{S}(R, \mathscr{D})$, the prime factor R/P appears no more than $Cq^{\gamma(R)}$ times in the prime filtration of the containment $R^{1/q} \subseteq R^{q^{\gamma(R)}}$.*

Proof. Let $\Gamma_{\mathscr{D}}$ be generated by $\Lambda_{\mathscr{D}} = \{q_1, ..., q_m\}$ as a semigroup. For each $q_i \in \Lambda_{\mathscr{D}}$ we can fix an embedding $R^{1/q_i} \subseteq R^{q_i^{\gamma(R)}}$ which is an element of $\mathscr{D}_{q_i}^{q_i^{\gamma(R)}}$, which is an isomorphism when localized at 0. To see this, let $W = R - 0$ and let $q \in \Lambda_{\mathscr{D}}$ so that $R_W^{1/q} \simeq R_W^{q^{\gamma(R)}}$. As R_W is a field, $\mathrm{Hom}_{R_W}(R_W^{1/q}, R_W) \simeq R_W^{1/q}$ is an $R_W^{1/q}$-module. Suppose that $0 \neq \varphi \in \mathscr{D}_q$; then φ_W generates $\mathrm{Hom}_{R_W}(R_W^{1/q}, R_W) \simeq R_W^{1/q}$ as an $R_W^{1/q}$-module. As $\mathscr{D}_0 = \mathrm{Hom}_R(R, R)$, we have that an $R_W^{1/q}$-multiple of φ_W is still an element of $(\mathscr{D}_W)_q$. Therefore the isomorphism $R_W^{1/q} \simeq R_W^{q^{\gamma(R)}}$ is an element of $(\mathscr{D}_W)_q^{q^{\gamma(R)}}$. As R is an F-finite domain, the isomorphism $R_W^{1/q} \simeq R_W^{q^{\gamma(R)}}$ is the localization of an embedding $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ which is an element of $\mathscr{D}_q^{q^{\gamma(R)}}$.

For each $q \in \Lambda_{\mathscr{D}}$ we consider a prime filtration of $R^{1/q} \subseteq R^{q^{\gamma(R)}}$, say $R^{1/q} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = R^{q^{\gamma(R)}}$, say $N_i/N_{i-1} \simeq R/P_i$. Let $\mathcal{S}(R/P_{q,i})$ and $C_{q,i}$ be as in Lemma 2.2 and let $\mathcal{S}_q(R, \mathscr{D}) = \bigcup_{i=1}^n \mathcal{S}(R/P_{q,i}) \cup \{P_{q,i}\}$ and $\mathcal{S}(R, \mathscr{D}) = \bigcup_{q \in \Lambda_{\mathscr{D}}} \mathcal{S}_q(R, \mathscr{D})$. We can now set $C' = \sum C_{q,i}$. Every $q \in \Gamma_{\mathscr{D}}$ can be expressed as $\prod_{q_i \in \Lambda_{\mathscr{D}}} q_i^{e_i}$, where $e_i \in \mathbb{N}$. We show by induction on $\sum e_i$ that for each $q \in \Gamma_{\mathscr{D}}$ there is a containment of R-modules $R^{1/q} \subseteq R^{q^{\gamma(R)}}$ which is an element of $\mathscr{D}_q^{q^{\gamma(R)}}$, which has a prime filtration whose prime factors are isomorphic to R/P, where

$P \in \mathcal{S}(R, \mathscr{D})$, and such a prime factor appears no more than $C'q^{\gamma(R)}\left(1 + \frac{1}{p} + \cdots + \frac{1}{q}\right)$ times in the filtration. This trivially holds for $\sum e_i = 1$.

Now suppose that $q = \prod_{q_i \in \Lambda_{\mathscr{D}}} q_i^{e_i}$ with $\sum e_i > 1$. Without loss of generality we may suppose that $e_1 \geq 1$ so that $q' = \frac{q}{q_1} \in \Gamma_{\mathscr{D}}$. By induction, we can find $R^{1/q'} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = R^{q'^{\gamma(R)}}$ is a prime filtration of an embedding $R^{1/q'} \subseteq R^{q'^{\gamma(R)}}$ in $\mathscr{D}_{q'}^{q'^{\gamma(R)}}$, each $N_j/N_{i-j} \simeq R/P_j$ for some $P_j \in \mathcal{S}(R, \mathscr{D})$, and such a prime factor appears no more than $C'q'^{\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{q'})$ times in the filtration. Therefore $(R^{q_1^{\gamma(R)}})^{1/q'} = (R^{1/q'})^{q_1^{\gamma(R)}} \subseteq (R^{q'^{\gamma(R)}})^{q_1^{\gamma(R)}} = R^{q^{\gamma(R)}}$ has a prime filtration with prime factors R/P_j with $P_j \in \mathcal{S}(R, \mathscr{D})$, and such a prime factor appears no more than $C'q^{\gamma(R)}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{q})$ times in the filtration. Furthermore, the prime filtration $R^{1/q_1} = N_{q_1,0} \subseteq N_{q_1,1} \subseteq \cdots \subseteq N_{q_1,n} = R^{q_1^{\gamma(R)}}$ gives the following filtration of $R^{1/q} = (R^{1/q_1})^{1/q'} \subseteq (R^{q_1^{\gamma(R)}})^{1/q'}$:

$$(R^{1/q_1})^{1/q'} = N_{q_1,0}^{1/q'} \subseteq N_{q_1,1}^{1/q'} \subseteq \cdots \subseteq R_{q_1,n}^{1/q'} = (R^{q_1^{\gamma(R)}})^{1/q'}.$$

Since $N_{q_1,i}^{1/q'}/N_{q_1,i-1}^{1/q'} \simeq (R/P_{q_1,i})^{1/q'}$, we apply Lemma 2.2 to see that there is a prime filtration of each $N_{q_1,i-1}^{1/q'} \subseteq N_{q_1,i}^{1/q'}$ whose prime factors come from $\mathcal{S}(R/P_{q_1,i})$ and such a prime factor appears no more than $C_i q^{\gamma(R/P_{q_1,i})} \leq C'q^{\gamma(R)-1}$ times in the filtration. Putting all of this information together we get an embedding $R^{1/q} \subseteq R^{q^{\gamma(R)}}$, which is an element of $\mathscr{D}_q^{q^{\gamma(R)}}$ by Remark 6.2, with a prime filtration whose prime factors come from $\mathcal{S}(R, \mathscr{D})$, and such a prime factor appears no more than the following number in the filtration:

$$q_1^{\gamma(R)} C' q'^{\gamma(R)} \left(1 + \frac{1}{p} + \cdots + \frac{1}{q'}\right) + \sum_{i=1}^{h} C_i q^{\gamma(R/P_{q_1,i})} \leq C'q^{\gamma(R)}(1 + \frac{1}{p} + \cdots + \frac{1}{q'} + \frac{1}{q})$$
$$\leq 2C'q^{\gamma(R)}.$$

$\square$

Enescu and Yao showed that $a_q(P, \mathscr{C}) : \operatorname{Spec}(R) \to \mathbb{R}$, hence $s_q(P, \mathscr{C})$, is lower semi-continuous on an F-finite ring which is locally equidimensional [6, Corollary 2.5]. We provide a very similar proof that shows $a_q(P, \mathscr{D})$, hence $s_q(P, \mathscr{D})$, is a lower semi-continuous function for any Cartier subalgebra $\mathscr{D}$ whenever R is locally equidimensional. It is well known that a function $f : X \to \mathbb{R}$, X a topological space, is lower semi-continuous if and only if for each $r \in \mathbb{R}$ the sets $f^{-1}((r, \infty)) = \{x \in X \mid f(x) > r\}$ is open in X.

Lower semi-continuity is a local condition. We may assume R_P, hence R is reduced, else $a_q(P, \mathscr{D}) = 0$ and $a_q(-, \mathscr{D})$ is trivially lower semi-continuous at P. Suppose that $q \in \Gamma_{\mathscr{D}}$, $r \in \mathbb{R}$, and $P \in \{Q \in \operatorname{Spec}(R) \mid a_q(Q, \mathscr{D}) > r\}$. Then $R_P^{1/q} \simeq R_P^{a_q(P, \mathscr{D})} \oplus M_P$ is such that each of the $a_q(P, \mathscr{D})$ projections $R_P^{1/q} \to R_P$ is an element of $(\mathscr{D}_P)_q$. It follows that there is an $s \in R - P$ such that $R_s^{1/q} \simeq R_s^{a_q(P, \mathscr{D})} \oplus M_s$ and each of the $a_q^{\mathscr{D}}(P)$-projections $R_s^{1/q} \to R_s$ is an element of $(\mathscr{D}_s)_q$. Hence for all $P' \in D(s)$, $a_q(P', \mathscr{D}) \geq a_q(P, \mathscr{D}) > r$ and $\{Q \in \operatorname{Spec}(R) \mid a_q^{\mathscr{D}}(Q) > r\}$ is indeed an open set. This shows that $a_q^{\mathscr{D}}(P)$ is lower semi-continuous and so is $s_q(P, \mathscr{D})$, since $a_q(P, \mathscr{D})$ and $s_q(P, \mathscr{D})$ differ only by a constant on connected components of $\operatorname{Spec}(R)$.

Consider the following condition we could impose on a Cartier subalgebra $\mathscr{D}$:

$$(1) \qquad\qquad (I_q^{\mathscr{D}}(P))^{[p]} \subseteq I_{qp}^{\mathscr{D}}(P).$$

Suppose R is an F-finite domain and $\mathscr{D}$ is a Cartier subalgebra. Then $r \in I_q^{\mathscr{D}}(P)$ if and only if $\varphi(r^{1/q}) \in PR_P$ for all $\varphi \in \mathscr{D}_q \subseteq \operatorname{Hom}_R(R^{1/q}, R)$. Thus to impose condition (1) is to impose for each $r \in I_q^{\mathscr{D}}(P)$ that $\psi(r^{1/q}) \in PR_P$ for all $\psi \in \mathscr{D}_{qp} \subseteq \operatorname{Hom}_R(R^{1/qp}, R)$. This condition is seen to be satisfied if for each $\psi \in \mathscr{D}_{qp}$ we require $\varphi \circ i \in \mathscr{D}_q$, where i is the natural inclusion $R^{1/q} \subseteq R^{1/qp}$.

Theorem 6.4. *Let R be an F-finite domain and let $\mathscr{D}$ be a Cartier subalgebra of R. Then the F-signature function which sends $P \in \operatorname{Spec}(R)$ to $s(P, \mathscr{D})$ is lower semi-continuous. Moreover, if the Cartier subalgebra satisfies (1), then the function $s_q(P, \mathscr{D})$ converges uniformly to the F-signature function $s(P, \mathscr{D})$ as $q \in \Gamma_{\mathscr{D}} \to \infty$.*

Proof. Let $C, \mathcal{S}(R)$ be as in Lemma 6.3. Let $q_1 \in \Gamma_{\mathscr{D}}$ so that $\mathscr{D}_{q_1} \neq 0$. Let $S(q_1)$ be the cokernel of $R^{1/q_1} \to R^{q_1^{\gamma(R)}}$. Therefore we have the short exact sequences

$$0 \to R^{1/q_1} \to R^{q_1^{\gamma(R)}} \to S(q_1) \to 0.$$

By Remark 6.2 we have exact sequences

$$\frac{R_P^{1/q_1}}{I_{q_1 q_2}^{\mathscr{D}}(P)^{1/q_1}} \to \frac{R_P^{q_1^{\gamma(R)}}}{I_{q_2}^{\mathscr{D}}(P) R_P^{q_1^{\gamma(R)}}} \to \tilde{S}(q_1) \to 0,$$

where $\tilde{S}(q_1)$ is the homomorphic image of $S(q_1)_P / I_{q_1}^{\mathscr{D}}(P) S(q_1)_P$. Therefore by parts (2) and (4) of Lemma 6.1,

$$\frac{a_{q_2}(P, \mathscr{D})}{q_2^{\alpha(P)}} q_1^{\gamma(R)} - \frac{a_{q_1 q_2}(P, \mathscr{D})}{(q_1 q_2)^{\alpha(P)}} q_1^{\alpha(P)} \leq \lambda(\tilde{S}(q_1)_P) \leq \lambda\left(\frac{S(q_1)_P}{I_{q_2}^{\mathscr{D}}(P) S(q_1)_P}\right)$$

$$\leq \lambda\left(\frac{S(q_1)_P}{P^{[q_2]} S(q_1)_P}\right).$$

By Proposition 3.3 there is a constant C_1, independent of P, q_2, such that

$$\max_{Q \in \mathcal{S}(R)} \lambda\left(R_P / (Q + P^{[q_2]}) R_P\right) \leq C_1 q_2^{\operatorname{ht}(P) - 1}.$$

It follows that

$$\frac{a_{q_2}(P, \mathscr{D})}{q_2^{\alpha(P)}} q_1^{\gamma(R)} - \frac{a_{q_1 q_2}(P, \mathscr{D})}{q_2^{\alpha(P)}} \leq \lambda\left(\frac{S(q_1)_P}{P^{[q_2]} S(q_1)_P}\right) \leq C C_1 |\mathcal{S}(R)| q_1^{\gamma(R)} q_2^{\operatorname{ht}(P) - 1}.$$

Dividing both sides of the inequality by $q_1^{\gamma(R)} q_2^{\operatorname{ht}(P) - 1}$ shows that

$$s_{q_2}(P, \mathscr{D}) - s_{q_1 q_2}(P, \mathscr{D}) \leq \frac{C C_1 |\mathcal{S}(R)|}{q_2}.$$

Letting $q_1 \to \infty$ and relabeling constants we show that there is a constant C, independent of P, q, such that

$$s_q(P, \mathscr{D}) - s(P, \mathscr{D}) < \frac{C}{q}.$$

To see that $s(-, \mathscr{D})$ is lower semi-continuous at $P \in \operatorname{Spec}(R)$, we may assume that $s(P, \mathscr{D}) > 0$, else $s(-, \mathscr{D})$ is trivially lower semi-continuous. Thus we may assume that R_P is a strongly F-regular domain. In particular, $s(-, \mathscr{D})$ is the limit of lower semi-continuous functions in an open neighborhood of P. Thus

the lower semi-continuity of the $s_q(-, \mathscr{D})$ will now imply the lower semi-continuity of $s(-, \mathscr{D})$, since there is a constant C independent of $Q \in \mathrm{Spec}(R)$ such that $s_q(Q, \mathscr{D}) - s(Q, \mathscr{D}) < \frac{C}{q}$. Observe that up to this point in the proof we have not used the assumption that the Cartier subalgebra $\mathscr{D}$ satisfies condition (1).

Now assume that the Cartier subalgebra $\mathscr{D}$ satisfies condition (1). Let C and $\mathcal{S}(R)$ be as in Lemma 2.2 applied to the F-finite domain R. Let $S(q_1)$ be the cokernels of the inclusions $R^{q_1^{\gamma(R)}} \xrightarrow{f_{q_1}} R^{1/q_1}$. Then there are short exact sequences

$$0 \to R^{q_1^{\gamma(R)}} \xrightarrow{f_{q_1}} R^{1/q_1} \to S(q_1) \to 0.$$

We claim that $f_{q_1}(I_{q_2}^{\mathscr{D}}(P)R^{q_1^{\gamma(R)}}) \subseteq I_{q_1 q_2}^{\mathscr{D}}(P)^{1/q_1}$. Let $x \in R^{q_1^{\gamma(R)}}$ and $r \in I_{q_2}^{\mathscr{D}}(P)$. Then $f_{q_1}(rx)^{q_1} = r^{q_1} f_{q_1}(x)^{q_1} \in I_{q_2}^{\mathscr{D}}(P)^{[q_1]} \subseteq I_{q_1 q_2}^{\mathscr{D}}(P)$ by part (3) of Lemma 6.1 and the assumption (1). Therefore there are induced exact sequences

$$\frac{R_P^{q_1^{\gamma(R)}}}{I_{q_2}^{\mathscr{D}}(P)R_P^{q_1^{\gamma(R)}}} \xrightarrow{f_{q_1}} \frac{R_P^{1/q_1}}{I_{q_1 q_2}^{\mathscr{D}}(P)^{1/q_1}} \to \tilde{S}(q_1) \to 0.$$

Observe that by part (2) of Lemma 6.1 that $P^{[q_2]}R_P$ kills $R_P^{1/q_2}/I_{q_1 q_2}^{\mathscr{D}}(P)^{1/q_1}$; hence $P^{[q_1]}R_P$ kills $\tilde{S}(q_1)$. Thus $\tilde{S}(q_1)$ is the homomorphic image of $S(q_1)/P^{[q_2]}S(q_1)_P$. We can now proceed as before to get a constant C independent of P and q such that

$$s(P, \mathscr{D}) - s_q(P, \mathscr{D}) < \frac{C}{q}.$$

Hence there is a constant C independent of $P \in \mathrm{Spec}(R)$ such that $|s(P, \mathscr{D}) - s_q(P, \mathscr{D})| < \frac{C}{q}$, which implies that $s_q(-, \mathscr{D})$ converges uniformly to $s(-, \mathscr{D})$. $\square$

Acknowledgments

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