

PREPERIODIC PORTRAITS FOR UNICRITICAL POLYNOMIALS OVER A RATIONAL FUNCTION FIELD

JOHN R. DOYLE

ABSTRACT. Let K be an algebraically closed field of characteristic zero, and let $\mathcal{K} := K(t)$ be the rational function field over K . For each $d \geq 2$, we consider the unicritical polynomial $f_d(z) := z^d + t \in \mathcal{K}[z]$, and we ask the following question: If we fix $\alpha \in \mathcal{K}$ and integers $M \geq 0$, $N \geq 1$, and $d \geq 2$, does there exist a place $\mathfrak{p} \in \text{Spec}K[t]$ such that, modulo \mathfrak{p} , the point α enters into an N -cycle after precisely M steps under iteration by f_d ? We answer this question completely, concluding that the answer is generally affirmative and explicitly giving all counterexamples. This extends previous work by the author in the case that α is a constant point.

1. INTRODUCTION

Let F be a field, and let $\varphi(z) \in F(z)$ be a rational function, thought of as a self-map of $\mathbb{P}^1(F)$. For an integer $n \geq 0$, we denote by φ^n the n -fold composition of φ ; that is, φ^0 is the identity map, and $\varphi^n = \varphi \circ \varphi^{n-1}$ for each $n \geq 1$. We say that $\alpha \in \mathbb{P}^1(F)$ is **periodic** for φ if there exists an integer $N \geq 1$ for which $\varphi^N(\alpha) = \alpha$; the minimal such N is called the **period** of α . More generally, we say that α is **preperiodic** if there exist integers $M \geq 0$ and $N \geq 1$ such that $f^M(\alpha)$ has period N ; if M is minimal, we say that (M, N) is the **preperiodic portrait** (or simply **portrait**) of α under φ . If $M \geq 1$, then we say that α is **strictly preperiodic**. The **orbit** of α under φ is the set

$$\mathcal{O}_\varphi(\alpha) := \{\varphi^n(\alpha) : n \in \mathbb{Z}_{\geq 0}\}.$$

Note that α is preperiodic for φ if and only if $\mathcal{O}_\varphi(\alpha)$ is finite. We say that a point is **wandering** if it is not preperiodic.

Let \mathcal{M}_F denote the set of places of F . (If F is a function field, we require the places to be trivial on the constant subfield.) For a place $\mathfrak{p} \in \mathcal{M}_F$, let $k_{\mathfrak{p}}$ denote the residue field at \mathfrak{p} . Given a rational map φ and a place \mathfrak{p} , one can consider the reduction of φ at \mathfrak{p} : Write $\varphi(z) = p(z)/q(z)$ with coprime $p, q \in F[z]$, normalized so that all coefficients are integral at \mathfrak{p} , and at least one coefficient is a unit at \mathfrak{p} . Then the **reduction** of φ at \mathfrak{p} is the map $\tilde{\varphi}(z) \in k_{\mathfrak{p}}(z)$ obtained by reducing the coefficients modulo \mathfrak{p} . We say that φ has **good reduction** modulo \mathfrak{p} if $\deg \tilde{\varphi} = \deg \varphi$. If \mathfrak{p} is a place of good reduction for φ , then we say that a point $\alpha \in \mathbb{P}^1(F)$ is **preperiodic for φ modulo \mathfrak{p}** if the reduction $\tilde{\alpha} \in \mathbb{P}^1(k_{\mathfrak{p}})$ is preperiodic for the map $\tilde{\varphi}$. We say that α has **(preperiodic) portrait (M, N) for φ modulo \mathfrak{p}** if $\tilde{\alpha}$ has preperiodic portrait (M, N) for $\tilde{\varphi}$.

Received by the editors April 7, 2016, and, in revised form, July 25, 2016.

2010 *Mathematics Subject Classification*. Primary 37P05; Secondary 37F10, 14H05.

Key words and phrases. Preperiodic points, *abc*-theorem, unicritical polynomials.

If α is not preperiodic for φ , it may still be true that α is preperiodic for φ modulo \mathfrak{p} at some place \mathfrak{p} of good reduction. For example, this will necessarily be true if F has finite residue fields. We now consider the following more specific question regarding preperiodicity modulo places of F :

Question 1.1. Let $\varphi \in F(z)$. Fix $\alpha \in F$ and integers $M \geq 0$, $N \geq 1$. Does there exist a place $\mathfrak{p} \in \mathcal{M}_F$ of good reduction for φ such that α has preperiodic portrait (M, N) for φ modulo \mathfrak{p} ?

If the answer to Question 1.1 is “yes”, we will say that α **realizes portrait** (M, N) **for** φ .

Question 1.1 has been studied by multiple authors in the case that F is a number field, dating back to related questions addressed by Bang [2] and Zsigmondy [24] in the late nineteenth century. Much more recently, Ingram and Silverman [12] conjectured that if F is a number field and $\alpha \in F$ is a wandering point for φ , then α realizes all but finitely many portraits for φ . Faber and Granville [8] later gave counterexamples to this conjecture, noting that if $\varphi(z) \in F(z)$ is totally ramified over all points of period N , then a given $\alpha \in F$ will *fail* to realize portrait (M, N) for all but finitely many M . Ghioca, Nguyen, and Tucker [10] subsequently pointed out that if φ is totally ramified over $\varphi^M(\alpha)$ for some $M \geq 1$, then α cannot realize portrait (M, N) for any $N \in \mathbb{N}$; their main result is that these are the only obstructions to the analogue of the Ingram-Silverman conjecture in the setting where F is the function field¹ of a curve over an algebraically closed field K of characteristic zero. For simplicity, we state a slightly weaker form of their result in the case that $\varphi(z)$ is *non-isotrivial*; that is, φ cannot, after a change of coordinates, be defined over K .

Theorem 1.2 ([10, Thm. 1.3]). *Let K be an algebraically closed field of characteristic zero, and let F be the function field of a curve defined over K . Let $\varphi(z) \in F(z)$ be non-isotrivial, let $X(\varphi)$ be the set of $N \in \mathbb{N}$ such that φ is totally ramified at all points of period N for φ , and for each $\alpha \in F$ let $Y(\varphi, \alpha)$ be the set of $M \in \mathbb{N}$ such that φ is totally ramified over $\varphi^M(\alpha)$. Then there exists an effectively computable finite subset $\mathcal{Z}_F(\varphi) \subset \mathbb{Z}_{\geq 0} \times \mathbb{N}$ such that the following holds: For every wandering point $\alpha \in F$, if $(M, N) \in (\mathbb{Z}_{\geq 0} \times \mathbb{N}) \setminus \mathcal{Z}_F(\varphi)$, $N \notin X(\varphi)$, and $M \notin Y(\varphi, \alpha)$, then α realizes portrait (M, N) for φ .*

The purpose of the present article is to explicitly describe all exceptions to this result of Ghioca-Nguyen-Tucker in a natural special case. For the remainder of the paper, K will be an algebraically closed field of characteristic zero, and $\mathcal{K} = K(t)$ will be the rational function field over K . Places of \mathcal{K} correspond naturally to points on $\mathbb{P}^1(K)$, and the residue field at each place is isomorphic to K . For a point $c \in \mathbb{P}^1(K)$, we denote by $\mathfrak{p}_c \in \mathcal{M}_{\mathcal{K}}$ the place corresponding to c .

We take our rational maps to be the *unicritical polynomials*

$$f_d(z) := z^d + t \in \mathcal{K}[z]$$

of degree $d \geq 2$, which have good reduction away from \mathfrak{p}_{∞} . The map f_d is non-isotrivial (see, for example, the proof of [1, Cor. 3.13]) and, using the notation of Theorem 1.2, we have $X(f_d) = \emptyset$, $Y(f_d, 0) = \{1\}$, and $Y(f_d, \alpha) = \emptyset$ for all

¹They also claim that the appropriate modification of the Ingram-Silverman conjecture over number fields may be proven, under the assumption of the *abc*-conjecture, by adapting the methods of [10].

$\alpha \in \mathcal{K}^\times$. For each $c \in K$, we denote by $f_{d,c}$ the specialization of f_d at \mathfrak{p}_c ; that is, $f_{d,c}(z) = z^d + c \in K[z]$.

Our main result fully answers Question 1.1 with $F = \mathcal{K}$ and $\varphi = f_d$. For what follows, let

$$\varphi_1(z) := -\frac{t(z+1)}{z-(t-1)} \quad \text{and} \quad \varphi_2(z) := \frac{(t+1)(z-1)}{z+t}.$$

Theorem 1.3. *Let K be an algebraically closed field of characteristic zero, let $\mathcal{K} := K(t)$ be the rational function field over K , and let $(\alpha, M, N, d) \in \mathcal{K} \times \mathbb{Z}^3$ with $M \geq 0$, $N \geq 1$, and $d \geq 2$. Then there exists a place $\mathfrak{p} \in \mathcal{M}_{\mathcal{K}} \setminus \{\mathfrak{p}_\infty\} = \text{Spec } K[t]$ such that α has preperiodic portrait (M, N) under f_d modulo \mathfrak{p} if and only if (α, M, N, d) does not satisfy one of the following conditions:*

- $M = 1$ and $\alpha = 0$;
- $(M, N, d) = (0, 2, 2)$ and $\alpha = -1/2$;
- $(M, N, d) = (1, 1, 2)$ and $\alpha \in \mathcal{O}_{\varphi_1}(0) \cup \mathcal{O}_{\varphi_1}(\infty)$;
- $(M, N, d) = (1, 2, 2)$ and $\alpha \in \mathcal{O}_{\varphi_2}(0) \cup \mathcal{O}_{\varphi_2}(1/2) \cup \mathcal{O}_{\varphi_2}(\infty)$; or
- $(M, N, d) = (2, 2, 2)$ and $\alpha = \pm 1$.

Remark 1.4. The families of counterexamples in the $(1, 1, 2)$ and $(1, 2, 2)$ cases were discovered experimentally, and it was unclear whether some dynamical properties of the maps φ_1 and φ_2 could explain their appearance. Tom Tucker later pointed out that φ_1 (resp., φ_2) fixes each of the points in $\overline{\mathcal{K}}$ of portrait $(1, 1)$ (resp., $(1, 2)$) for f_2 and preserves vanishing at \mathfrak{p}_0 (resp. \mathfrak{p}_{-1}), the unique place at which the totally ramified point 0 is fixed (resp., has period two) for $f_{2,c}$. Finally, we note that these exceptions have arbitrarily large height: for each $k \geq 0$, the points $\varphi_1^k(0)$ and $\varphi_1^k(\infty)$ have height k , as do the points $\varphi_2^k(0)$, $\varphi_2^k(1/2)$, and $\varphi_2^k(\infty)$ — see Propositions 5.17 and 5.18.

Using the notation from Theorem 1.2, our main result implies that

$$\mathcal{Z}_{\mathcal{K}}(f_d) = \begin{cases} \{(0, 2), (1, 1), (1, 2), (2, 2)\}, & \text{if } d = 2; \\ \emptyset, & \text{if } d \geq 3. \end{cases}$$

In fact, not only does Theorem 1.3 explicitly describe the exceptional set of portraits from Theorem 1.2 in this special case, but it also completely determines those points $\alpha \in \mathcal{K}$ that fail to realize each exceptional portrait in the $d = 2$ case.

Theorem 1.3 may also be viewed as a natural extension of a previous result of the author:

Theorem 1.5 ([7, Thm. 1.3]). *Let K be as before, and let $(\alpha, M, N, d) \in K \times \mathbb{Z}^3$ with $M \geq 0$, $N \geq 1$, and $d \geq 2$. There exists $c \in K$ for which α has portrait (M, N) under $f_{d,c}$ if and only if*

$$(\alpha, M) \neq (0, 1) \text{ and } (\alpha, M, N, d) \notin \left\{ \left(-\frac{1}{2}, 0, 2, 2 \right), \left(\frac{1}{2}, 1, 2, 2 \right), (\pm 1, 2, 2, 2) \right\}.$$

This is precisely the case of Theorem 1.3 in which α lies in the constant subfield K . The proof of Theorem 1.5 almost exclusively used the geometry of certain dynamical modular curves associated to the maps f_d , whereas the proof of Theorem 1.3 requires Diophantine methods much like those used in [10]. In particular, the argument for the $d \geq 3$ case of Theorem 1.3 provides a completely different proof of the $d \geq 3$ case of Theorem 1.5 — except for the case $M = 0$, for which

we simply refer to Theorem 1.5 for constant points. The same is true for $d = 2$, except for the cases where $M = 1$ and $N \leq 3$, where the Diophantine methods are insufficient for constant points.

We now give a brief overview of the article. In §2, we collect the main tools required for the proof of the main theorem. In §3, we prove the $M = 0$ case of Theorem 1.3, and we then show that the problem for $M \geq 1$ may essentially be reduced to $M = 1$.

We prove the general case ($d \geq 3$) of Theorem 1.3 in §4. Focusing on the situation with $M = 1$, we apply the *abc*-theorem for function fields due to Mason and Stothers [14, 22] to get a lower bound on the number of places at which $f(\alpha)$ and $f^{N+1}(\alpha)$ agree; we then show that this bound must be greater than the number of places at which either α is periodic or $f(\alpha)$ has period strictly less than N , so there must be some place at which α has portrait $(1, N)$. The arguments in this case are quite similar to those used in [10], though we make modifications based on the specific nature of our maps f_d in order to obtain sufficiently nice bounds.

The case $d = 2$ must be handled separately; this case is discussed in §5. A technique similar to that used for $d \geq 3$ is used when $N \geq 4$. While this particular method is insufficient when $N = 3$, we are able to prove the result in this case by applying the *abc*-theorem together with properties of the period-three *dynatomic polynomial* associated to the map $z^2 + t$. Unfortunately, the *abc*-theorem can no longer be applied when $N = 1$ and $N = 2$, so we handle these cases with completely different techniques, again appealing to the appropriate dynatomic polynomials. Theorem 1.3 is then proven by combining Proposition 4.3 (for the case $d \geq 3$) with Propositions 5.2, 5.11, 5.17, and 5.18 (for $d = 2$ and $N \geq 4$, $N = 3$, $N = 2$, and $N = 1$, respectively).

We conclude this section by discussing the limitations of our methods in terms of possible generalizations of Theorem 1.3. One direction to generalize would be to replace $\mathcal{K} = \mathbb{C}(t)$ with an arbitrary finite extension \mathcal{L}/\mathcal{K} . At this level of generality, though, the statement of Theorem 1.2 is best possible, in the sense that the finite set $\mathcal{Z}_{\mathcal{L}}(f_d)$ does indeed depend on \mathcal{L} . More precisely, there does not exist a finite subset $\overline{\mathcal{Z}}(f_d) \subset \mathbb{Z}_{\geq 0} \times \mathbb{N}$ such that $\mathcal{Z}_{\mathcal{L}}(f_d) \subseteq \overline{\mathcal{Z}}(f_d)$ for all \mathcal{L}/\mathcal{K} . In fact, given any finite subset $\mathcal{S} \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$, there exists a finite extension \mathcal{L}/\mathcal{K} such that $\mathcal{S} \subseteq \mathcal{Z}_{\mathcal{L}}(f_d)$. Indeed, for each $(M, N) \in \mathcal{S}$, let $\alpha_{M,N} \in \overline{\mathcal{K}}$ be a root of $f_d^{M+N}(z) - f_d^M(z) + s$, where $s \in K^\times$ is chosen so that $\alpha_{M,N}$ is not preperiodic for f_d . That such s exists follows from the fact that K is infinite and that there are only finitely many preperiodic points for f_d of bounded degree; this, in turn, follows from the fact that the irreducible factors of the *generalized dynatomic polynomials* $\Phi_{m,n}(z)$, whose roots are precisely the points of portrait (m, n) for f_d , have degree exponential in $m + n$. Since the quantity $f_d^{M+N}(\alpha_{M,N}) - f_d^M(\alpha_{M,N})$ is a unit, it cannot vanish at any place in $\mathcal{K}(\alpha_{M,N})$, hence $\alpha_{M,N}$ does not realize portrait (M, N) for f_d . If we now let \mathcal{L} be the compositum $\prod_{(M,N) \in \mathcal{S}} \mathcal{K}(\alpha_{M,N})$, we have $\mathcal{S} \subseteq \mathcal{Z}_{\mathcal{L}}(f_d)$ by construction.

One might then ask for a statement like Theorem 1.3 over a *specific* extension \mathcal{L}/\mathcal{K} , but there are restrictions on the fields over which a similar classification could be proven using our methods. One major obstruction to carrying our proof over to larger function fields is the appearance of the genus in our bounds derived from the *abc*-theorem — see the proofs of Lemmas 4.2, 5.1, and 5.7, which rely on the fact that \mathcal{K} has genus 0. We are also implicitly using the fact that certain places of $K(t)$

are unramified² in \mathcal{K} , especially in the proofs of Lemma 2.1 and Proposition 3.2. We note, however, that an appropriate modification of Lemma 2.1 — hence also of Corollary 2.2 — holds over general function fields, and the height formula from Lemma 2.3 holds in full generality.

Another natural direction to generalize would be to replace $f_d(z) = z^d + t$ with an arbitrary polynomial in $K[z, t] \subset \mathcal{K}[z]$. However, our proof of Theorem 1.3 relies on the following property of the polynomials f_d with $d \geq 3$: given $\alpha \in \mathcal{K}^\times$, there are at least two distinct \mathcal{K} -rational preimages of $f_d(\alpha)$ different from α itself (namely $\zeta\alpha$ and $\zeta^2\alpha$, where ζ is a primitive d th root of unity). While this property is not exclusive to unicritical polynomials — it holds for any polynomial in z^d with $d \geq 3$ — it does put a significant restriction on the set of polynomials for which our method of proof could be applied. Moreover, in §3.1, where we prove the $M = 0$ case of Theorem 1.3, we rely on the smoothness of the affine curves defined by the dynatomic polynomials for the class of unicritical polynomials, a property not guaranteed to hold more generally. However, if one had a polynomial $f(z) \in \mathcal{K}[z]$ of the appropriate shape and with smooth dynatomic curves, one should expect a modification of our argument to give a similar result.

2. PRELIMINARIES

2.1. Valuations and heights. Let \mathcal{L} be a finite extension of \mathcal{K} , which corresponds to a finite morphism of curves $\mathcal{X}_{\mathcal{L}} \rightarrow \mathcal{X}_{\mathcal{K}} \cong \mathbb{P}^1_K$. For a place $\mathfrak{p} \in \mathcal{M}_{\mathcal{K}}$, we denote by $\mathcal{M}_{\mathcal{L}, \mathfrak{p}}$ the set of places of \mathcal{L} that restrict to \mathfrak{p} . Associated to each place $\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}$ is a valuation $v_{\mathfrak{q}}$ and its corresponding absolute value $|\cdot|_{\mathfrak{q}} = e^{-v_{\mathfrak{q}}(\cdot)}$. When $\mathcal{L} = \mathcal{K}$, so that places correspond to points on $\mathbb{P}^1(K)$, we abuse notation and write $v_c, |\cdot|_c$, and $\mathcal{M}_{\mathcal{L}, c}$ for $v_{\mathfrak{p}_c}, |\cdot|_{\mathfrak{p}_c}$, and $\mathcal{M}_{\mathcal{L}, \mathfrak{p}_c}$, respectively.

We normalize the valuations on \mathcal{L} so that $v_{\mathfrak{q}}(\mathcal{L}^\times) = \mathbb{Z}$; equivalently, if $\pi_{\mathfrak{q}}$ is a uniformizer at \mathfrak{q} , then $v_{\mathfrak{q}}(\pi_{\mathfrak{q}}) = 1$. Thus, if \mathfrak{p} is the restriction of \mathfrak{q} to \mathcal{K} , and if $\alpha \in \mathcal{K}$, then $v_{\mathfrak{q}}(\alpha) = e_{\mathfrak{q}/\mathfrak{p}} \cdot v_{\mathfrak{p}}(\alpha)$, where $e_{\mathfrak{q}/\mathfrak{p}}$ is the ramification degree of \mathfrak{q} over \mathfrak{p} . This normalization of the valuations also ensures that the product formula holds: For all $\alpha \in \mathcal{L}^\times$, we have

$$\prod_{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}} |\alpha|_{\mathfrak{q}} = 1, \text{ or equivalently, } \sum_{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}} v_{\mathfrak{q}}(\alpha) = 0.$$

For each $\alpha \in \mathcal{L}$, set

$$h_{\mathcal{L}}(\alpha) = - \sum_{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}} \min\{v_{\mathfrak{q}}(\alpha), 0\} = - \sum_{\substack{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}} \\ v_{\mathfrak{q}}(\alpha) < 0}} v_{\mathfrak{q}}(\alpha).$$

By the product formula, this is equivalent (when $\alpha \neq 0$) to defining

$$h_{\mathcal{L}}(\alpha) = \sum_{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}} \max\{v_{\mathfrak{q}}(\alpha), 0\} = \sum_{\substack{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}} \\ v_{\mathfrak{q}}(\alpha) > 0}} v_{\mathfrak{q}}(\alpha).$$

If we consider $\alpha \in \mathcal{L}$ as a rational map $\mathcal{X}_{\mathcal{L}} \rightarrow \mathbb{P}^1$, then $h_{\mathcal{L}}(\alpha)$ is simply the degree of the map. If \mathcal{L}' is a finite extension of \mathcal{L} , then $h_{\mathcal{L}'}(\alpha) = [\mathcal{L}' : \mathcal{L}]h_{\mathcal{L}}(\alpha)$ for all

²This is because, in our case, $\mathcal{K} = K(t)$.

$\alpha \in \mathcal{L}$. This allows us to give a well-defined **(absolute) height function** on all of $\overline{\mathcal{K}}$, given by

$$h(\alpha) := \frac{1}{[\mathcal{L} : \mathcal{K}]} \cdot h_{\mathcal{L}}(\alpha)$$

for any finite extension \mathcal{L}/\mathcal{K} containing α . Given a rational map $\varphi(z) \in \mathcal{K}(z)$ of degree $d \geq 2$, we also define the **canonical height** associated to φ :

$$\widehat{h}_{\varphi}(\alpha) := \lim_{n \rightarrow \infty} \frac{1}{d^n} h(\varphi^n(\alpha)).$$

That this is well-defined follows from the fact that $h(\varphi(\alpha)) = dh(\alpha) + O(1)$, where the implied constant depends only on φ ; see [21, §3.2]. Note that $\widehat{h}_{\varphi}(\varphi(\alpha)) = d\widehat{h}_{\varphi}(\alpha)$ for all $\alpha \in \overline{\mathcal{K}}$.

We now record a basic height identity for elements of the orbit of a point $\alpha \in \mathcal{K}$.

Lemma 2.1. *Let $\alpha \in \mathcal{K}$, and let $d \geq 2$. Then for each $n \geq 1$, the poles of $f_d^n(\alpha)$ are precisely \mathfrak{p}_{∞} and the poles of α . Moreover,*

- (A) *if \mathfrak{p} is a finite pole of α , then $v_{\mathfrak{p}}(f_d^n(\alpha)) = d^n v_{\mathfrak{p}}(\alpha)$;*
- (B) $v_{\infty}(f_d^n(\alpha)) = d^n \cdot \begin{cases} v_{\infty}(\alpha), & \text{if } v_{\infty}(\alpha) < 0; \\ -1/d, & \text{if } v_{\infty}(\alpha) \geq 0. \end{cases}$

Therefore $h(f_d^n(\alpha)) = d^n \cdot \begin{cases} h(\alpha), & \text{if } v_{\infty}(\alpha) < 0; \\ h(\alpha) + 1/d, & \text{if } v_{\infty}(\alpha) \geq 0. \end{cases}$

Proof. Since $f_d^n(z)$ is a polynomial in z and t , every pole of $f_d^n(\alpha)$ must be equal to \mathfrak{p}_{∞} or a pole of α . That the poles of $f_d^n(\alpha)$ are precisely \mathfrak{p}_{∞} and the poles of α then follows from parts (A) and (B), which we now prove by induction on n .

For $n = 1$, we have $f_d(\alpha) = \alpha^d + t$, so part (A) follows immediately from the ultrametric inequality. Furthermore, since $v_{\infty}(\alpha^d) \neq -1 = v_{\infty}(t)$, we have $v_{\infty}(f_d(\alpha)) = \min\{dv_{\infty}(\alpha), -1\}$.

Now suppose $n \geq 2$. First, let \mathfrak{p} be a finite pole of α . By the induction hypothesis, \mathfrak{p} is a pole of $f_d^{n-1}(\alpha)$ of order $d^{n-1}v_{\mathfrak{p}}(\alpha)$; applying the $n = 1$ case with α replaced by $f_d^{n-1}(\alpha)$ yields (A). We now consider $\mathfrak{p} = \mathfrak{p}_{\infty}$, in which case the induction hypothesis tells us that

$$v_{\infty}(f_d^{n-1}(\alpha)) = d^{n-1} \cdot \begin{cases} v_{\infty}(\alpha), & \text{if } v_{\infty}(\alpha) < 0 \\ -1/d, & \text{if } v_{\infty}(\alpha) \geq 0. \end{cases}$$

Since this quantity is necessarily negative, the $n = 1$ case implies $v_{\infty}(f_d^n(\alpha)) = dv_{\infty}(f_d^{n-1}(\alpha))$, which gives us (B).

Finally, we note that for all $n \geq 1$,

$$\begin{aligned} h(f_d^n(\alpha)) &= - \sum_{v_{\mathfrak{p}}(f_d^n(\alpha)) < 0} v_{\mathfrak{p}}(f_d^n(\alpha)) \\ &= - \sum_{\substack{v_{\mathfrak{p}}(\alpha) < 0 \\ \mathfrak{p} \neq \mathfrak{p}_{\infty}}} d^n v_{\mathfrak{p}}(\alpha) - d^n \cdot \begin{cases} v_{\infty}(\alpha), & \text{if } v_{\infty}(\alpha) < 0, \\ -1/d, & \text{if } v_{\infty}(\alpha) \geq 0 \end{cases} \\ &= d^n \cdot \left(- \sum_{v_{\mathfrak{p}}(\alpha) < 0} v_{\mathfrak{p}}(\alpha) + \begin{cases} 0, & \text{if } v_{\infty}(\alpha) < 0 \\ 1/d, & \text{if } v_{\infty}(\alpha) \geq 0 \end{cases} \right) \\ &= d^n \cdot \begin{cases} h(\alpha), & \text{if } v_{\infty}(\alpha) < 0 \\ h(\alpha) + 1/d, & \text{if } v_{\infty}(\alpha) \geq 0. \end{cases} \end{aligned}$$

□

The following description of the canonical height for points in \mathcal{K} now follows immediately from the definition.

Corollary 2.2. *Let $\alpha \in \mathcal{K}$ and $d \geq 2$. Then*

$$\widehat{h}_{f_d}(\alpha) = \begin{cases} h(\alpha), & \text{if } v_{\infty}(\alpha) < 0, \\ h(\alpha) + 1/d, & \text{if } v_{\infty}(\alpha) \geq 0. \end{cases}$$

Thus, for all $n \geq 1$, we have $h(f_d^n(\alpha)) = d^n \widehat{h}_{f_d}(\alpha)$.

2.2. Dynatonic polynomials for f_d . Throughout the article, we will require certain properties of the dynatonic polynomials for the maps $f_d(z) = z^d + t$. Suppose $x, c \in K$ are such that x has period N for $f_{d,c}(z) = z^d + c$. Then (x, c) is a solution to the equation $f_{d,c}^N(x) - x = 0$. However, this equation is also satisfied whenever x has period dividing N for $f_{d,c}$. We therefore define the N th **dynatonic polynomial** for f_d to be the polynomial

$$\Phi_N(z, t) := \prod_{n|N} (f_d^n(z) - z)^{\mu(N/n)} \in \mathbb{Z}[z, t],$$

where μ is the Möbius function. (To ease notation, we omit the dependence on d .) The dynatonic polynomials give a natural factorization $f_d^N(z) - z = \prod_{n|N} \Phi_n(z, t)$. If x has period N for $f_{d,c}$, then $\Phi_N(x, c) = 0$, and for each $N \geq 1$ the converse is true for all but finitely many pairs (x, c) . That $\Phi_N(z, t)$ is indeed a polynomial is shown in [18, Thm. 3.1]; see also [21, Thm. 4.5]. For each $N \geq 1$, we set

$$D(N) := \deg_z \Phi_N(z, t) = \sum_{n|N} \mu(N/n) d^n.$$

It is not difficult to verify that $\Phi_N(z, t)$ is monic in both z and t , that $\deg_t \Phi_N(z, t) = D(N)/d$, and that

$$\Phi_N(z, t) = z^{D(N)} + (\text{terms of lower total degree}).$$

In particular, this implies that if $\mathfrak{p} \in \mathcal{M}_{\mathcal{K}}$ is a pole of $\alpha \in \mathcal{K}$, or if $\mathfrak{p} = \mathfrak{p}_{\infty}$, then

$$(2.1) \quad v_{\mathfrak{p}}(\Phi_N(\alpha, t)) = \min \left\{ D(N) v_{\mathfrak{p}}(\alpha), \frac{D(N)}{d} v_{\mathfrak{p}}(t) \right\} < 0.$$

Lemma 2.3. *Let $\alpha \in \mathcal{K}$ and $N \geq 1$. Then $v_\infty(\Phi_N(\alpha, t)) < 0$ and*

$$h(\Phi_N(\alpha, t)) = D(N) \cdot \widehat{h}_{f_d}(\alpha).$$

In particular, $\Phi_N(\alpha, t)$ vanishes at precisely $D(N) \cdot \widehat{h}_{f_d}(\alpha)$ finite places, counted with multiplicity.

Proof. Since $\Phi_N(z, t)$ is a polynomial in z and t , if \mathfrak{p} is a pole of $\Phi_N(\alpha, t)$, then $\mathfrak{p} = \mathfrak{p}_\infty$ or \mathfrak{p} is a pole of α . It then follows from (2.1) that the poles of $\Phi_N(\alpha, t)$ are precisely \mathfrak{p}_∞ and the poles of α . Therefore

$$\begin{aligned} h(\Phi_N(\alpha, t)) &= - \sum_{\substack{v_{\mathfrak{p}}(\alpha) < 0 \\ \text{or } \mathfrak{p} = \mathfrak{p}_\infty}} \min \left\{ D(N)v_{\mathfrak{p}}(\alpha), \frac{D(N)}{d}v_{\mathfrak{p}}(t) \right\} \\ &= - \sum_{\substack{v_{\mathfrak{p}}(\alpha) < 0 \\ \mathfrak{p} \neq \mathfrak{p}_\infty}} D(N)v_{\mathfrak{p}}(\alpha) - \begin{cases} D(N)v_\infty(\alpha), & \text{if } v_\infty(\alpha) < 0 \\ -D(N)/d, & \text{if } v_\infty(\alpha) \geq 0 \end{cases} \\ &= D(N) \cdot \begin{cases} h(\alpha), & \text{if } v_\infty(\alpha) < 0 \\ h(\alpha) + 1/d, & \text{if } v_\infty(\alpha) \geq 0 \end{cases} \\ &= D(N) \cdot \widehat{h}_{f_d}(\alpha). \end{aligned}$$

□

Finally, we record the following geometric result:

Theorem 2.4. *For each integer $N \geq 1$ and $d \geq 2$, the affine plane curve defined by $\Phi_N(z_1, t) = 0$ is smooth and irreducible over K .*

Theorem 2.4 was originally proven in the $d = 2$ case by Douady and Hubbard (smoothness; [6, §XIV]) and Bousch (irreducibility; [4, Thm. 1 (§3)]), with a subsequent proof by Buff and Lei [5, Thm. 3.1]. For $d \geq 2$, irreducibility was proven by Lau and Schleicher [13, Thm. 4.1] using analytic methods and by Morton [16, Cor. 2] using algebraic methods, while both irreducibility and smoothness were later proven by Gao and Ou [9, Thms. 1.1, 1.2] using the methods of Buff-Lei. The theorem was originally proven over \mathbb{C} , but the Lefschetz principle allows us to extend the result to arbitrary fields of characteristic zero.

2.3. The *abc*-theorem for function fields. Our main tool for proving the general case of Theorem 1.3 is the *abc*-theorem for function fields due to Mason and Stothers [14, 22]; see also [20] and [11, Thm. F.3.6].

Theorem 2.5. *Let \mathcal{L}/\mathcal{K} be a finite extension, and let $g_{\mathcal{L}}$ be the genus of \mathcal{L} . Let $u \in \mathcal{L} \setminus K$, and define $S \subset \mathcal{M}_{\mathcal{L}}$ to be the set of places \mathfrak{q} for which $v_{\mathfrak{q}}(u) \neq 0$ or $v_{\mathfrak{q}}(1 - u) \neq 0$. Then*

$$h_{\mathcal{L}}(u) \leq 2g_{\mathcal{L}} - 2 + |S|.$$

3. AN ELEMENTARY REDUCTION

For the majority of this article, we focus on the case of Theorem 1.3 in which M is equal to 1. In this section, we justify this approach: First, we prove the theorem when $M = 0$, and then we show how the $M \geq 1$ case may essentially be reduced to $M = 1$.

3.1. Periodic points. In order to have α not realize portrait $(0, N)$ for f_d , it must be the case that whenever $\Phi_N(\alpha, t)$ vanishes, so too does $\Phi_n(\alpha, t)$ for some proper divisor n of N .

Lemma 3.1. *Fix integers $N \geq 1$ and $d \geq 2$.*

(A) *Let $x, c \in K$, and suppose $\Phi_N(x, c) = \Phi_n(x, c) = 0$ for some proper divisor n of N . Then*

$$\left. \frac{\partial \Phi_N(z, t)}{\partial z} \right|_{(x, c)} = 0.$$

(B) *There are strictly fewer than $D(N)$ elements $c \in K$ for which there exists $x \in K$ with $\Phi_N(x, c) = \Phi_n(x, c) = 0$ for some proper divisor n of N .*

Proof. For part (A), see [17, Thm. 2.4]; for part (B), see [19, Cor. 3.3]. □

We may now prove the $M = 0$ case of Theorem 1.3.

Proposition 3.2. *Let $\alpha \in \mathcal{K}$, and let $N \geq 1$ and $d \geq 2$ be integers. Then α realizes portrait $(0, N)$ for f_d if and only if $(\alpha, N, d) \neq (-1/2, 2, 2)$.*

Proof. For $\alpha \in K$, the result follows from Theorem 1.5, so we assume that $\alpha \in \mathcal{K} \setminus K$. In this case, we have $\widehat{h}_{f_d}(\alpha) \geq h(\alpha) \geq 1$, so it follows from Lemma 2.3 that the number of places, counted with multiplicity, at which $\Phi_N(\alpha, t)$ vanishes is at least $D(N)$. Now suppose \mathfrak{p}_c is a place at which $\Phi_N(\alpha, t)$ vanishes, but α has period $n < N$ modulo \mathfrak{p}_c . By Lemma 3.1(B), there are fewer than $D(N)$ such places, so to prove the proposition it suffices to show that $\Phi_N(\alpha, t)$ vanishes to order one at each such place. By Lemma 3.1(A), we have

$$\left. \frac{\partial \Phi_N(z, t)}{\partial z} \right|_{(\alpha(c), c)} = 0.$$

Here we write $\alpha(c)$ for the reduction of α modulo \mathfrak{p}_c , since this is the image of c under α if we consider α as a rational map. Since the affine curve $\{\Phi_N(z, t) = 0\}$ is smooth, we must also have

$$\left. \frac{\partial \Phi_N(z, t)}{\partial t} \right|_{(\alpha(c), c)} \neq 0.$$

This implies that

$$\left. \frac{\partial \Phi_N(\alpha, t)}{\partial t} \right|_{t=c} = \left. \frac{\partial \Phi_N(z, t)}{\partial z} \right|_{(\alpha(c), c)} \cdot \alpha'(c) + \left. \frac{\partial \Phi_N(z, t)}{\partial t} \right|_{(\alpha(c), c)} \neq 0,$$

so $t = c$ is a simple root of $\Phi_N(\alpha, t)$; i.e., $\Phi_N(\alpha, t)$ vanishes to order one at \mathfrak{p}_c , completing the proof. □

3.2. Strictly preperiodic points. As mentioned previously, we will generally restrict our attention to the case $M = 1$. We now justify this approach.

Lemma 3.3. *Fix $\alpha \in \mathcal{K}$ and integers $M \geq 2$, $N \geq 1$, and $d \geq 2$. Then α realizes portrait (M, N) for f_d if and only if $f_d^{M-1}(\alpha)$ realizes portrait $(1, N)$ for f_d .*

Proof. We simply note that both statements are equivalent to the statement that, at some place $\mathfrak{p} \in \mathcal{M}_{\mathcal{K}} \setminus \{\mathfrak{p}_{\infty}\}$, $f_d^M(\alpha)$ reduces to a point of period M while $f_d^{M-1}(\alpha)$ does not. □

We also record a useful characterization of points in K of portrait $(1, N)$.

Lemma 3.4. *Fix $x, c \in K$, $d \geq 2$, and $N \geq 1$. Then x has portrait $(1, N)$ for $f_{d,c}$ if and only if $x \neq 0$ and ζx has period N for $f_{d,c}$ for some d th root of unity $\zeta \neq 1$.*

Proof. We first note that two points x and y have the same image under $f_{d,c}$ if and only if $y = \zeta x$ for some d th root of unity ζ .

Suppose x has portrait $(1, N)$ for $f_{d,c}$. Then x is not periodic, but $f_{d,c}(x)$ has period N and therefore has exactly one preimage y with period N . We can write $y = \zeta x$ for some d th root of unity ζ , and since $x \neq y = \zeta x$, we must have $\zeta \neq 1$ and $x \neq 0$.

Now suppose $x \neq 0$ and ζx has period N for $f_{d,c}$ for some d th root of unity $\zeta \neq 1$. Since ζx has period N , so must $f_{d,c}(\zeta x) = f_{d,c}(x)$. However, since $x \neq 0$ and $\zeta \neq 1$, we have $x \neq \zeta x$. Thus x is not periodic, and therefore x has portrait $(1, N)$. □

Corollary 3.5. *Fix integers $d \geq 2$ and $N \geq 1$. Then 0 does not realize portrait $(1, N)$ for f_d .*

4. THE DEGREE $d \geq 3$ CASE

In this section, we show that if $d \geq 3$ and $\alpha \neq 0$, then α realizes portrait $(1, N)$ for every $N \geq 1$. We then use this to prove the $d \geq 3$ case of Theorem 1.3; see Proposition 4.3 below.

Fix $\alpha \in \mathcal{K}^\times$, integers $d \geq 3$ and $N \geq 1$, and a primitive d th root of unity ζ . Define the polynomial

$$\sigma(z) := \frac{1}{\zeta - 1} \left(\frac{1}{\zeta \alpha} z - 1 \right) \in \mathcal{K}[z],$$

which maps $\zeta \alpha$, $\zeta^2 \alpha$, and ∞ to 0, 1, and ∞ , respectively. Set $\gamma := f_d^N(\alpha)$, and define

$$\begin{aligned} \mathcal{A} &:= \{ \mathfrak{p} \in \mathcal{M}_{\mathcal{K}} : v_{\mathfrak{p}}(\sigma(\gamma)) \neq 0 \text{ or } v_{\mathfrak{p}}(\sigma(\gamma) - 1) > 0 \}; \\ \mathcal{B} &:= \{ \mathfrak{p}_{\infty} \} \cup \{ \mathfrak{p} \in \mathcal{M}_{\mathcal{K}} : v_{\mathfrak{p}}(\alpha) \neq 0 \\ &\quad \text{or } v_{\mathfrak{p}}(\Phi_n(\zeta^k \alpha)) > 0 \text{ for some } n < N \text{ and } k \in \{1, 2\} \}. \end{aligned}$$

Lemma 4.1. *If $\mathfrak{p} \in \mathcal{A} \setminus \mathcal{B}$, then α has portrait $(1, N)$ for f_d modulo \mathfrak{p} .*

Proof. Let $\mathfrak{p} \in \mathcal{A} \setminus \mathcal{B}$. Since $v_{\mathfrak{p}}(\alpha) = 0$, the map σ has good reduction — hence remains invertible — modulo \mathfrak{p} . Since $\mathfrak{p} \in \mathcal{A}$, $\sigma(\gamma)$ reduces to 0, 1, or ∞ modulo \mathfrak{p} , which implies that γ reduces to $\zeta \alpha$, $\zeta^2 \alpha$, or ∞ modulo \mathfrak{p} . Since the only poles of $\gamma = f_d^N(\alpha)$ are \mathfrak{p}_{∞} and the poles of α , and since such places lie in \mathcal{B} , it must be the case that $f_d^N(\alpha) \equiv \zeta^k \alpha \pmod{\mathfrak{p}}$ for some $k \in \{1, 2\}$.

Since $\zeta^k \alpha \equiv f_d^N(\alpha) = f_d^N(\zeta^k \alpha) \pmod{\mathfrak{p}}$ and \mathfrak{p} is not a pole of α , $\zeta^k \alpha$ reduces to a finite point of period dividing N for f_d modulo \mathfrak{p} , and the period must equal N since $\Phi_n(\zeta^k \alpha) \not\equiv 0 \pmod{\mathfrak{p}}$ for all $n < N$. Finally, since $\alpha \not\equiv 0 \pmod{\mathfrak{p}}$, α has portrait $(1, N)$ modulo \mathfrak{p} by Lemma 3.4. □

Lemma 4.2. *The set $\mathcal{A} \setminus \mathcal{B}$ is non-empty.*

Proof. We first get an upper bound for $|\mathcal{A}|$. In order to apply Theorem 2.5 with $u = \sigma(\gamma)$, we first verify that $\sigma(\gamma) \notin K$. Indeed, suppose $\sigma(\gamma) = \lambda \in K$. Then

$$f_d^N(\alpha) = \gamma = \sigma^{-1}(\lambda) = \zeta((\zeta - 1)\lambda + 1) \cdot \alpha,$$

so $f_d^N(\alpha)$ is a constant multiple of α . However, this implies that $h(f_d^N(\alpha)) \leq h(\alpha)$, contradicting Lemma 2.1. Therefore $\sigma(\gamma) \notin K$, so we may apply Theorem 2.5 to get

$$|\mathcal{A}| \geq h(\sigma(\gamma)) + 2.$$

Since $h(\sigma(\gamma)) = h(\gamma/\alpha) \geq h(\gamma) - h(\alpha)$ and $h(\gamma) = h(f_d^N(\alpha)) = d^N \widehat{h}_{f_d}(\alpha)$, we have

$$|\mathcal{A}| \geq h(\gamma) - h(\alpha) + 2 = d^N \widehat{h}_{f_d}(\alpha) - h(\alpha) + 2 \geq (d^N - 1) \widehat{h}_{f_d}(\alpha) + 2.$$

On the other hand, it is straightforward to verify that

$$\begin{aligned} |\mathcal{B}| &\leq 1 + 2h(\alpha) + \sum_{k=1}^2 \sum_{\substack{n|N \\ n < N}} h(\Phi_n(\zeta^k \alpha)) = 1 + 2h(\alpha) + \sum_{k=1}^2 \sum_{\substack{n|N \\ n < N}} \widehat{h}_{f_d}(\zeta^k \alpha) D(n) \\ &= 1 + 2h(\alpha) + 2\widehat{h}_{f_d}(\alpha) \sum_{\substack{n|N \\ n < N}} D(n) \\ &\leq 1 + \widehat{h}_{f_d}(\alpha) \left(2 + 2 \sum_{\substack{n|N \\ n < N}} D(n) \right). \end{aligned}$$

Combining these bounds on $|\mathcal{A}|$ and $|\mathcal{B}|$, we find that $|\mathcal{A} \setminus \mathcal{B}| \geq \kappa \widehat{h}_{f_d}(\alpha) + 1$, where

$$\kappa := d^N - 3 - 2 \sum_{\substack{n|N \\ n < N}} D(n).$$

To show that $\mathcal{A} \setminus \mathcal{B}$ is non-empty, it suffices to show that $\kappa \geq 0$, since $\widehat{h}_{f_d}(\alpha) > 0$ for all $\alpha \in \mathcal{K}$. Now observe that

$$\sum_{\substack{n|N \\ n < N}} D(n) \leq \sum_{\substack{n|N \\ n < N}} d^n \leq \sum_{n=1}^{\lfloor N/2 \rfloor} d^n \leq \frac{d}{d-1} (d^{N/2} - 1),$$

and therefore

$$\begin{aligned} \kappa &\geq d^N - 3 - \frac{2d}{d-1} (d^{N/2} - 1) \geq d^N - 3 - d(d^{N/2} - 1) \\ &= d^{N/2+1} (d^{N/2-1} - 1) + d - 3. \end{aligned}$$

This expression is non-negative for all $d \geq 3$ and $N \geq 2$; on the other hand, when $N = 1$, we have $\kappa = d - 3 \geq 0$. In either case, we conclude that $\mathcal{A} \setminus \mathcal{B}$ is non-empty. □

We may now prove Theorem 1.3 for the general case $d \geq 3$.

Proposition 4.3. *Let $(\alpha, M, N, d) \in \mathcal{K} \times \mathbb{Z}^3$ with $M \geq 0, N \geq 1$, and $d \geq 3$. Then α realizes portrait (M, N) for f_d if and only if $(\alpha, M) \neq (0, 1)$.*

Proof. For $M = 0$, this follows from Proposition 3.2. Corollary 3.5 says that 0 does not realize portrait $(1, N)$ for any $N \geq 1$ and $d \geq 2$, so suppose $\alpha \neq 0$. Letting \mathcal{A} and \mathcal{B} be as above, the set $\mathcal{A} \setminus \mathcal{B}$ is non-empty, thus α realizes portrait $(1, N)$ for f_d , proving the statement for $M = 1$.

Finally, let $M \geq 2$. Since $0 \notin f_d(\mathcal{K})$, the point $f_d^{M-1}(\alpha)$ is non-zero, so $f_d^{M-1}(\alpha)$ realizes portrait $(1, N)$ for f_d . By Lemma 3.3, we conclude that α realizes portrait (M, N) for f_d . \square

5. THE DEGREE $d = 2$ CASE

We henceforth drop the subscript and write $f = f_2$ and $f_c = f_{2,c}$. To prove Theorem 1.3 when $N \geq 4$, we proceed much like in §4; however, we require different methods when $N \leq 3$, so we consider these cases separately.

5.1. $N \geq 4$. The proof of Proposition 4.3 in the previous section relied on fixing two preimages of $f_d(\alpha)$ *different from α itself*, then counting the number of places at which $f_d^N(\alpha)$ had the same reduction as one of those two preimages. When $d = 2$, however, there is only *one* preimage of $f(\alpha)$ different from α (namely, $-\alpha$), so we require a minor modification of the technique from §4.

Fix $\alpha \in \mathcal{K}$ and $N \geq 4$. Let $\eta \in \overline{\mathcal{K}}$ satisfy $\eta^2 + t = -\alpha$, set $\mathcal{L} := \mathcal{K}(\eta)$, and set $\delta := [\mathcal{L} : \mathcal{K}] \in \{1, 2\}$. Define the polynomial

$$\sigma(z) := -\frac{1}{2} \left(\frac{z}{\eta} - 1 \right) \in \mathcal{K}[z],$$

which maps $\eta, -\eta$, and ∞ to 0, 1, and ∞ , respectively. Set $\gamma := f^{N-1}(\alpha)$, and define

$$\begin{aligned} \mathcal{A} &:= \{ \mathfrak{q} \in \mathcal{M}_{\mathcal{L}} : v_{\mathfrak{q}}(\sigma(\gamma)) \neq 0 \text{ or } v_{\mathfrak{q}}(\sigma(\gamma) - 1) > 0 \}; \\ \mathcal{B} &:= \mathcal{M}_{\mathcal{L}, \infty} \cup \{ \mathfrak{q} \in \mathcal{M}_{\mathcal{L}} : v_{\mathfrak{q}}(\eta) \neq 0, v_{\mathfrak{q}}(\alpha) > 0, \\ &\quad \text{or } v_{\mathfrak{q}}(\Phi_n(-\alpha)) > 0 \text{ for some } n < N \}. \end{aligned}$$

By a proof similar to that of Lemma 4.1, if $\mathfrak{q} \in \mathcal{A} \setminus \mathcal{B}$, then α has portrait $(1, N)$ for f modulo \mathfrak{q} . We now show that there exists at least one such place.

Lemma 5.1. *The set $\mathcal{A} \setminus \mathcal{B}$ is non-empty.*

Proof. We first get a lower bound for $|\mathcal{A}|$. By an argument similar to the beginning of the proof of Lemma 4.2, we have $\sigma(\gamma) \notin K$, so we apply Theorem 2.5 to get

$$|\mathcal{A}| \geq h_{\mathcal{L}}(\sigma(\gamma)) - (2g_{\mathcal{L}} - 2).$$

We get a lower bound on $h_{\mathcal{L}}(\sigma(\gamma))$ by noting that $h_{\mathcal{L}}(\sigma(\gamma)) = h_{\mathcal{L}}(\gamma/\eta) \geq h_{\mathcal{L}}(\gamma) - h_{\mathcal{L}}(\eta)$; since also $h_{\mathcal{L}}(\gamma) = \delta \cdot h(\gamma) = \delta \cdot 2^{N-1} \widehat{h}_f(\alpha)$, we have

$$h_{\mathcal{L}}(\sigma(\gamma)) \geq \delta(2^{N-1} \widehat{h}_f(\alpha) - h(\eta)).$$

We also require an upper bound on $2g_{\mathcal{L}} - 2$. Let $\mathcal{R}_{\mathcal{L}/\mathcal{K}}$ be the ramification divisor of the extension \mathcal{L}/\mathcal{K} . Since \mathcal{L}/\mathcal{K} is generated by $\eta = \sqrt{-(\alpha + t)}$, the only places in $\mathcal{M}_{\mathcal{K}}$ over which \mathcal{L} may ramify are zeros and poles of $\alpha + t$. Thus $\deg \mathcal{R}_{\mathcal{L}/\mathcal{K}} \leq (\delta - 1) \cdot 2h(\alpha + t) = 4(\delta - 1)h(\eta)$, so it follows from Riemann-Hurwitz that $2g_{\mathcal{L}} - 2 \leq -2\delta + 4(\delta - 1)h(\eta)$. Therefore

$$\begin{aligned} |\mathcal{A}| &\geq \delta(2^{N-1} \widehat{h}_f(\alpha) - h(\eta)) - (-2\delta + 4(\delta - 1)h(\eta)) \\ &= \delta \cdot \left(2^{N-1} \widehat{h}_f(\alpha) - \left(5 - \frac{4}{\delta} \right) h(\eta) + 2 \right). \end{aligned}$$

On the other hand, it is straightforward to verify that

$$\begin{aligned} |\mathcal{B}| &\leq \#\mathcal{M}_{\mathcal{L},\infty} + 2h_{\mathcal{L}}(\eta) + h_{\mathcal{L}}(\alpha) + \sum_{\substack{n|N \\ n < N}} h_{\mathcal{L}}(\Phi_n(-\alpha)) \\ &\leq \delta + 2\delta h(\eta) + \delta h(\alpha) + \sum_{\substack{n|N \\ n < N}} \delta h(\Phi_n(-\alpha)) \\ &= \delta \left(1 + 2h(\eta) + h(\alpha) + \widehat{h}_f(\alpha) \sum_{\substack{n|N \\ n < N}} D(n) \right). \end{aligned}$$

Combining these bounds on $|\mathcal{A}|$ and $|\mathcal{B}|$ yields

$$|\mathcal{A} \setminus \mathcal{B}| \geq \delta \cdot \left[\left(2^{N-1} - \sum_{\substack{n|N \\ n < N}} D(n) \right) \widehat{h}_f(\alpha) - h(\alpha) - \left(7 - \frac{4}{\delta} \right) h(\eta) + 1 \right].$$

It remains to show that this bound is positive for all $N \geq 4$ and $\alpha \in \mathcal{K}^\times$.

First, suppose $v_\infty(\alpha) < 0$. Then $\widehat{h}_f(\alpha) = h(\alpha)$ and

$$h(\eta) = \frac{1}{2}h(\alpha + t) \geq \frac{1}{2}(h(\alpha) - 1).$$

Therefore

$$\begin{aligned} |\mathcal{A} \setminus \mathcal{B}| &\geq \delta \cdot \left[\left(2^{N-1} - \sum_{\substack{n|N \\ n < N}} D(n) \right) h(\alpha) - h(\alpha) - \left(7 - \frac{4}{\delta} \right) \frac{1}{2}(h(\alpha) - 1) + 1 \right] \\ &= \delta \cdot \left[\left(2^{N-1} - \frac{9}{2} + \frac{2}{\delta} - \sum_{\substack{n|N \\ n < N}} D(n) \right) h(\alpha) + \left(\frac{9}{2} - \frac{2}{\delta} \right) \right] \\ &\geq \delta \cdot \left[\left(2^{N-1} - \frac{7}{2} - \sum_{\substack{n|N \\ n < N}} D(n) \right) h(\alpha) + \frac{5}{2} \right], \end{aligned}$$

where the last inequality follows from the fact that $\delta \in \{1, 2\}$. Since $h(\alpha) \geq 0$ for all $\alpha \in \mathcal{K}$, it suffices to show that the quantity

$$(5.1) \quad 2^{N-1} - \frac{7}{2} - \sum_{\substack{n|N \\ n < N}} D(n)$$

is non-negative for all $N \geq 4$. We bound the sum just as in the proof of Lemma 4.2 to get

$$2^{N-1} - \frac{7}{2} - \sum_{\substack{n|N \\ n < N}} D(n) \geq 2^{N-1} - \frac{7}{2} - 2(2^{N/2} - 1) = 2^{N/2+1}(2^{N/2-2} - 1) - \frac{3}{2}.$$

The rightmost expression is positive for $N \geq 5$ but negative for $N = 4$; however, for $N = 4$ one can show directly that (5.1) is positive since $D(1) = D(2) = 2$. (Note that (5.1) is negative for $N \leq 3$.) Thus $|\mathcal{A} \setminus \mathcal{B}|$ is positive for all $N \geq 4$.

Now suppose that $v_\infty(\alpha) \geq 0$, in which case we have $\widehat{h}_f(\alpha) = h(\alpha) + \frac{1}{2}$ and $h(\eta) = \frac{1}{2}h(\alpha + t) = \frac{1}{2}(h(\alpha) + 1)$. By an estimate similar to the previous case, we find that

$$|\mathcal{A} \setminus \mathcal{B}| \geq \delta \cdot \left[\left(2^{N-1} - \frac{7}{2} - \sum_{\substack{n|N \\ n < N}} D(n) \right) h(\alpha) + \frac{1}{2} \left(2^{N-1} - 3 - \sum_{\substack{n|N \\ n < N}} D(n) \right) \right].$$

Since we have already shown that (5.1) is positive for all $N \geq 4$, it follows that this expression is positive for all $N \geq 4$ as well.

In both cases, our lower bound on $|\mathcal{A} \setminus \mathcal{B}|$ is positive, so $\mathcal{A} \setminus \mathcal{B}$ is non-empty. \square

The same argument as for Proposition 4.3 yields the $d = 2, N \geq 4$ case of Theorem 1.3.

Proposition 5.2. *Let $(\alpha, M, N) \in \mathcal{K} \times \mathbb{Z}^2$ with $M \geq 0$ and $N \geq 4$. Then α realizes portrait (M, N) for f_2 if and only if $(\alpha, M) \neq (0, 1)$.*

5.2. $N = 3$. Since the technique used for periods $N \geq 4$ gives a negative lower bound when $N \leq 3$, we again require a different method. For $N = 3$, we consider the third dynatomic polynomial

$$\begin{aligned} \Phi_3(z, t) &= \frac{f^3(z) - z}{f(z) - z} \\ &= z^6 + z^5 + (3t + 1)z^4 + (2t + 1)z^3 + (3t^2 + 3t + 1)z^2 \\ &\quad + (t + 1)^2z + t^3 + 2t^2 + t + 1. \end{aligned}$$

The roots of $\Phi_3(z, t)$ in $\overline{\mathcal{K}}$ are the points of period 3 for f , which fall naturally into two 3-cycles. Let η be one such root, and let $\mathcal{L} := \mathcal{K}(\eta)$; since $\Phi_3(z, t)$ is irreducible, we have $[\mathcal{L} : \mathcal{K}] = 6$. Set $\eta_1 := \eta, \eta_2 := f(\eta),$ and $\eta_3 := f^2(\eta)$, and note that each η_i is a root of $\Phi_3(z, t)$ that generates \mathcal{L}/\mathcal{K} . Denote by $\mathcal{X}_{\mathcal{L}}$ the normalization of the projective closure of the affine curve $\{\Phi_3(z, t) = 0\}$, and note that the extension \mathcal{L}/\mathcal{K} corresponds to the morphism $\mathcal{X}_{\mathcal{L}} \rightarrow \mathbb{P}^1$ mapping $(z, t) \mapsto t$. In his thesis, Bousch gave a general formula [4, §3, Thm. 2] for the genera of the curves $\{\Phi_N(z, t) = 0\}$, and in this case we have $g_{\mathcal{L}} = 0$. (For an explicit parametrization of the curve $\mathcal{X}_{\mathcal{L}}$; see [23].)

By a result of Morton [16, Prop. 10] for more general dynatomic curves, $\mathcal{M}_{\mathcal{L}, \infty}$ consists of three places, each of which has ramification degree two over \mathfrak{p}_∞ . To describe the ramification at finite places, Morton shows in [15, p. 358] that the discriminant of $\Phi_3(z, t) \in \mathcal{K}[z]$ is given by

$$(5.2) \quad \text{disc } \Phi_3(z, t) = \Delta_{3,1}^2 \Delta_{3,3}^3,$$

where $\Delta_{3,1} := \text{Res}_z(\Phi_3(z, t), \Phi_1(z, t)) = -(16t^2 + 4t + 7)$ and $\Delta_{3,3} := -(4t + 7)$. The roots c of $\Delta_{3,1}$ correspond to maps f_c for which one cycle of length three collapses to a fixed point, while the roots of $\Delta_{3,3}$ correspond to maps where the two 3-cycles collide to form a single 3-cycle.

Of particular relevance for us is the polynomial $\Delta_{3,1}$. Let $c_1, c_2 \in K$ be the two roots of $\Delta_{3,1}$. For each $i, \mathcal{M}_{\mathcal{L}, c_i}$ consists of four places: $\overline{\mathfrak{q}}_i$, which has ramification

degree three, and $\mathfrak{q}_{i,1}$, $\mathfrak{q}_{i,2}$, and $\mathfrak{q}_{i,3}$, each of which is unramified — see the proof of [16, Prop. 9]. The places $\overline{\mathfrak{q}}_1$ and $\overline{\mathfrak{q}}_2$ are precisely the places at which η has period-one; that is, the finite places at which η_1 , η_2 , and η_3 have the same reduction.

We briefly explain this geometrically: Let $c \in K$ be a root of $\Delta_{3,1}$. Instead of having two 3-cycles, f_c has one 3-cycle $\{x_1, x_2, x_3\}$ and a fixed point \overline{x} which satisfies $\Phi_3(\overline{x}, c) = 0$. Thus the only points on $\mathcal{X}_{\mathcal{L}}$ that map to $c \in \mathbb{P}^1$ are (\overline{x}, c) and (x_j, c) for $j \in \{1, 2, 3\}$. Since (\overline{x}, c) is fixed by the order three automorphism $(z, t) \mapsto (f(z), t)$, this point ramifies over c , while the three points (x_j, c) are unramified.

Lemma 5.3. *Let c be a root of $\Delta_{3,1}$, and let $\overline{\mathfrak{q}} \in \mathcal{M}_{\mathcal{L}}$ be the unique place ramified over \mathfrak{p}_c . For each $1 \leq i < j \leq 3$,*

- (A) $v_{\overline{\mathfrak{q}}}(\eta_i) = 0$;
- (B) $v_{\overline{\mathfrak{q}}}(\eta_i - \eta_j) = 1$;
- (C) $v_{\mathfrak{q}}(\eta_i) = -1$ for each place $\mathfrak{q} \in \mathcal{M}_{\mathcal{L},\infty}$;
- (D) there exists a place $\mathfrak{q}_{i,j} \in \mathcal{M}_{\mathcal{L},\infty}$ such that, for $\mathfrak{q} \in \mathcal{M}_{\mathcal{L},\infty}$,

$$v_{\mathfrak{q}}(\eta_i - \eta_j) = \begin{cases} 0, & \text{if } \mathfrak{q} = \mathfrak{q}_{i,j}; \\ -1, & \text{otherwise.} \end{cases}$$

Moreover, $\mathfrak{q}_{1,2}$, $\mathfrak{q}_{1,3}$, and $\mathfrak{q}_{2,3}$ are distinct.

Proof. Since $\Phi_3(z, t)$ is monic in z , the only poles of η_1 , η_2 , and η_3 must lie above \mathfrak{p}_{∞} . Therefore, to prove part (A), it suffices to show that none of the η_i vanish at $\overline{\mathfrak{q}}$. This follows by noting that $\Phi_3(0, c) \neq 0$.

For each i, j , the points η_i and η_j have the same reduction modulo $\overline{\mathfrak{q}}$, so $v_{\overline{\mathfrak{q}}}(\eta_i - \eta_j) \geq 1$. Now, we observe that the product

$$\prod_{1 \leq i < j \leq 3} (\eta_i - \eta_j)^2$$

divides $\text{disc } \Phi_3(z, t)$, which then implies that

$$\sum_{1 \leq i < j \leq 3} 2v_{\overline{\mathfrak{q}}}(\eta_i - \eta_j) \leq v_{\overline{\mathfrak{q}}}(\text{disc } \Phi_3(z, t)) = v_{\overline{\mathfrak{q}}}(\Delta_{3,1}^2 \overline{\Delta}_{3,3}^3).$$

This sum has three terms, and each term is at least 2, hence the sum is at least 6. Also, the polynomial $\Delta_{3,3}$ does not vanish at $\overline{\mathfrak{q}}$; moreover, if we factor $\Delta_{3,1}$ into linear factors, only the factor $(t - c)$ vanishes at $\overline{\mathfrak{q}}$. This implies that

$$6 \leq \sum_{1 \leq i < j \leq 3} 2v_{\overline{\mathfrak{q}}}(\eta_i - \eta_j) \leq 2v_{\overline{\mathfrak{q}}}(t - c) = 2e_{\overline{\mathfrak{q}}/\mathfrak{p}_c} = 6.$$

We therefore have equality throughout, so $v_{\overline{\mathfrak{q}}}(\eta_i - \eta_j) = 1$ for each $1 \leq i < j \leq 3$, proving (B).

Now fix $\mathfrak{q} \in \mathcal{M}_{\mathcal{L},\infty}$ and $1 \leq i \leq 3$. Note that $v_{\mathfrak{q}}(t) = -e_{\mathfrak{q}/\mathfrak{p}_{\infty}} = -2$. If $v_{\mathfrak{q}}(\eta_i) < -1$, then an induction argument shows that $v_{\mathfrak{q}}(f^n(\eta_i)) = 2^n v_{\mathfrak{q}}(\eta_i)$ for each $n \in \mathbb{N}$; in particular, we have $v_{\mathfrak{q}}(\eta_i) = v_{\mathfrak{q}}(f^3(\eta_i)) = 8v_{\mathfrak{q}}(\eta_i)$, a contradiction. If instead $v_{\mathfrak{q}}(\eta_i) > -1$, then $v_{\mathfrak{q}}(f(\eta_i)) = v_{\mathfrak{q}}(t) < -1$, so we reduce to the previous case to get a contradiction. Therefore $v_{\mathfrak{q}}(\eta_i) = -1$, proving (C).

The only finite zeros of $\eta_i - \eta_j$ are the two simple zeros at $\overline{\mathfrak{q}}_1$ and $\overline{\mathfrak{q}}_2$, so there must be at least two poles of $\eta_i - \eta_j$. By part (C), these must be simple poles lying above \mathfrak{p}_{∞} . Comparing the degrees of its zero and pole divisors, $\eta_i - \eta_j$ must have a simple pole at *precisely* two places above \mathfrak{p}_{∞} ; moreover, if we let $\mathfrak{q}_{i,j}$ be the remaining infinite place, then $v_{\mathfrak{q}_{i,j}}(\eta_i - \eta_j) = 0$.

Finally, suppose $(i, j) \neq (i', j')$ but $\mathfrak{q}_{i,j} = \mathfrak{q}_{i',j'}$. Reordering the indices if necessary, we may assume that $\mathfrak{q}_{1,2} = \mathfrak{q}_{1,3}$. Since $\eta_1 - \eta_2$ and $\eta_1 - \eta_3$ have exactly the same zeros and poles, with exactly the same orders, it must be that $\lambda := (\eta_1 - \eta_2)/(\eta_1 - \eta_3)$ is constant. This implies that

$$0 = (\eta_1 - \eta_2) - \lambda(\eta_1 - \eta_3) = \left(\eta - (\eta^2 + t)\right) - \lambda\left(\eta - ((\eta^2 + t)^2 + t)\right)$$

for some $\lambda \in K$, contradicting the fact that η has degree 6 over \mathcal{K} . We conclude that the places $\mathfrak{q}_{1,2}$, $\mathfrak{q}_{1,3}$, and $\mathfrak{q}_{2,3}$ are distinct, completing the proof. \square

Remark 5.4. It follows from Lemma 5.3 and its proof that $h_{\mathcal{L}}(\eta_i) = 3$ for each $1 \leq i \leq 3$, since the only poles of η_i are simple poles at the three places lying above \mathfrak{p}_{∞} . Similarly, we have $h_{\mathcal{L}}(\eta_i - \eta_j) = 2$ for each $1 \leq i < j \leq 3$.

Now consider the affine rational map

$$\sigma(z) := \frac{\eta_2 - \eta_3}{\eta_2 - \eta_1} \cdot \frac{z - \eta_1}{z - \eta_3},$$

which is constructed to map η_1 , η_2 , and η_3 to 0, 1, and ∞ respectively. Note that

$$\sigma^{-1}(z) = \frac{\eta_3(\eta_1 - \eta_2)z + \eta_1(\eta_2 - \eta_3)}{(\eta_1 - \eta_3)z + (\eta_2 - \eta_3)}.$$

Lemma 5.5. *For all $x \in \mathcal{L}$, $h_{\mathcal{L}}(\sigma(x)) \geq h_{\mathcal{L}}(x) - 4$.*

Proof. By considering $y = \sigma(x)$, it suffices to show that $h_{\mathcal{L}}(\sigma^{-1}(y)) \leq h_{\mathcal{L}}(y) + 4$ for all $y \in \mathcal{L}$. A standard height argument (see [21, pp. 90–92]) shows that for any $y \in \mathcal{L}$ we have

$$\begin{aligned} h_{\mathcal{L}}(\sigma^{-1}(y)) &\leq h_{\mathcal{L}}(y) - \sum_{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}} \min\{v_{\mathfrak{q}}(\eta_3(\eta_1 - \eta_2)), v_{\mathfrak{q}}(\eta_1(\eta_2 - \eta_3)), v_{\mathfrak{q}}(\eta_1 - \eta_3), v_{\mathfrak{q}}(\eta_2 - \eta_3)\}, \end{aligned}$$

so it suffices to show that the quantity

$$(5.3) \quad \sum_{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}} \min\{v_{\mathfrak{q}}(\eta_3) + v_{\mathfrak{q}}(\eta_1 - \eta_2), v_{\mathfrak{q}}(\eta_1) + v_{\mathfrak{q}}(\eta_2 - \eta_3), v_{\mathfrak{q}}(\eta_1 - \eta_3), v_{\mathfrak{q}}(\eta_2 - \eta_3)\}$$

is equal to -4 . We now determine the places \mathfrak{q} at which the ‘min’ term is non-zero.

If the minimum is positive at \mathfrak{q} , then $\mathfrak{q} \in \{\overline{\mathfrak{q}}_1, \overline{\mathfrak{q}}_2\}$. By Lemma 5.3, at each such \mathfrak{q} and for each $1 \leq i < j \leq 3$, we have $v_{\mathfrak{q}}(\eta_i - \eta_j) = 1$ and $v_{\mathfrak{q}}(\eta_i) = 0$. Thus the contribution to (5.3) at each such place is equal to 1, so the combined contribution at both places is equal to 2.

If the minimum is negative at \mathfrak{q} , then \mathfrak{q} necessarily lies above \mathfrak{p}_{∞} . Again applying Lemma 5.3, $v_{\mathfrak{q}}(\eta_i) = -1$ for each $1 \leq i \leq 3$, and exactly one of $v_{\mathfrak{q}}(\eta_1 - \eta_2)$, $v_{\mathfrak{q}}(\eta_1 - \eta_3)$, and $v_{\mathfrak{q}}(\eta_2 - \eta_3)$ is non-negative. In particular, this means that $\min\{v_{\mathfrak{q}}(\eta_1 - \eta_2), v_{\mathfrak{q}}(\eta_2 - \eta_3)\} = -1$, so the combined contribution to (5.3) at the three infinite places is $3 \cdot (-2) = -6$. Since we have accounted for all non-zero terms in the sum, we conclude that the expression (5.3) is equal to -4 , as claimed. \square

Now let $\alpha \in \mathcal{K}^{\times}$ be arbitrary. We will show that α realizes portrait $(1, 3)$ for f . Define

$$\begin{aligned} \mathcal{A} &:= \{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}} : v_{\mathfrak{q}}(\sigma(-\alpha)) \neq 0 \text{ or } v_{\mathfrak{q}}(\sigma(-\alpha) - 1) > 0\}; \\ \mathcal{B} &:= \mathcal{M}_{\mathcal{L}, \infty} \cup \{\overline{\mathfrak{q}}_1, \overline{\mathfrak{q}}_2\} \cup \{\mathfrak{q} \in \mathcal{M}_{\mathcal{L}} : v_{\mathfrak{q}}(\eta_i) > 0 \text{ for some } i = 1, 2, 3\}. \end{aligned}$$

Lemma 5.6. *If $\mathfrak{q} \in \mathcal{A} \setminus \mathcal{B}$, then α has portrait $(1, 3)$ for f modulo \mathfrak{q} .*

Proof. Let $\mathfrak{q} \in \mathcal{A} \setminus \mathcal{B}$. Since $\mathfrak{q} \notin \mathcal{M}_{\mathcal{L}, \infty}$, f has good reduction at \mathfrak{q} . Since η has period 3 for f , it follows that η has period 1 or 3 for f modulo \mathfrak{q} . The only poles of η are places at infinity, so $\eta \not\equiv \infty \pmod{\mathfrak{q}}$; moreover, since $\mathfrak{q} \notin \{\overline{\mathfrak{q}}_1, \overline{\mathfrak{q}}_2\}$ we have

$$f(\eta) - \eta = \eta_2 - \eta_1 \not\equiv 0 \pmod{\mathfrak{q}},$$

so η cannot have period 1. Thus η has period 3 for f modulo \mathfrak{q} .

The zeros and poles of the coefficients of σ all lie in \mathcal{B} , so each coefficient is a unit in the residue field $k_{\mathfrak{q}}$. Since also $\eta_1 \not\equiv \eta_3 \pmod{\mathfrak{q}}$, the reduction of σ has degree one over $k_{\mathfrak{q}}$; i.e., σ remains invertible modulo \mathfrak{q} . Thus, since $\sigma(-\alpha)$ reduces to 0, 1, or ∞ modulo \mathfrak{q} , $-\alpha$ must reduce to η_i for some $i \in \{1, 2, 3\}$. We have already shown that each η_i has period 3 modulo \mathfrak{q} , so the same is true for $-\alpha$. Finally, since $-\alpha \equiv \eta_i \not\equiv 0 \pmod{\mathfrak{q}}$, α must reduce to a point of portrait $(1, 3)$. \square

Lemma 5.7. *Suppose $h(\alpha) \geq 3$. Then $\mathcal{A} \setminus \mathcal{B}$ is non-empty.*

Proof. We have by Lemma 5.5 that $h_{\mathcal{L}}(\sigma(-\alpha)) \geq 14$, so $\sigma(-\alpha) \notin K$. Applying Theorem 2.5, it follows that the set \mathcal{A} has size at least $h_{\mathcal{L}}(\sigma(-\alpha)) - (2g_{\mathcal{L}} - 2) = h_{\mathcal{L}}(\sigma(-\alpha)) + 2$. Again applying the bound in Lemma 5.5 gives

$$|\mathcal{A}| \geq (h_{\mathcal{L}}(\alpha) - 4) + 2 = 6h(\alpha) - 2.$$

We also have $|\mathcal{B}| \leq |\mathcal{M}_{\mathcal{L}, \infty}| + 2 + \sum_{i=1}^3 h_{\mathcal{L}}(\eta_i) = 14$, which implies that $|\mathcal{A} \setminus \mathcal{B}| \geq 6h(\alpha) - 16$. Therefore $\mathcal{A} \setminus \mathcal{B}$ is non-empty when $h(\alpha) \geq 3$. \square

We have just shown that if $h(\alpha) \geq 3$, then there exists a place $\mathfrak{q} \in \mathcal{M}_{\mathcal{L}}$ for which α has portrait $(1, 3)$ modulo \mathfrak{q} ; choosing $\mathfrak{p} \in \mathcal{M}_{\mathcal{K}}$ below \mathfrak{q} , the same holds modulo \mathfrak{p} . It remains to prove this in the case $h(\alpha) \leq 2$. The constant point case $h(\alpha) = 0$ is covered by Theorem 1.5, so we henceforth assume $h(\alpha) \in \{1, 2\}$. To handle these remaining cases — as well as the $N = 2$ and $N = 1$ cases in § 5.3 — we use the following consequence of Lemma 3.4:

Corollary 5.8. *Let $\alpha \in \mathcal{K}$ and $N \in \mathbb{N}$. The following are equivalent:*

- (A) *The point α does not realize portrait $(1, N)$ for f .*
- (B) *For every place $\mathfrak{p} \in \mathcal{M}_{\mathcal{K}}$ at which $\Phi_N(-\alpha, t)$ vanishes, either $\Phi_n(-\alpha, t)$ also vanishes at \mathfrak{p} for some proper divisor n of N , or α also vanishes at \mathfrak{p} .*

Remark 5.9. If $\Phi_N(\beta, t)$ and $\Phi_n(\beta, t)$ both vanish at \mathfrak{p} for some proper divisor n of N , then it follows from the proof of Proposition 3.2 that $v_{\mathfrak{p}}(\Phi_N(\beta, t)) = 1$.

If α does not realize portrait $(1, 3)$ for f , then wherever $\Phi_3(-\alpha, t)$ vanishes, either α or $\Phi_1(-\alpha, t)$ must vanish as well. We will show that such behavior is impossible when $h(\alpha) \in \{1, 2\}$, having already handled all other cases.

Lemma 5.10. *Let $\beta \in \mathcal{K}$, and suppose $h(\beta) = 1$ or $h(\beta) = 2$. There exists a place $\mathfrak{p} \in \mathcal{M}_{\mathcal{K}} \setminus \{\mathfrak{p}_{\infty}\}$ such that $\Phi_3(\beta, t)$ vanishes at \mathfrak{p} but $\Phi_1(\beta, t)$ and β do not.*

Proof. Suppose to the contrary that there does not exist such a place. Then, if we fix a place $\mathfrak{p} = \mathfrak{p}_c$ for which $v_c(\Phi_3(\beta, t)) > 0$, we must also have $v_c(\Phi_1(\beta, t)) > 0$ or $v_c(\beta) > 0$.

If $v_c(\Phi_1(\beta, t)) > 0$, then it must be that $c \in \{c_1, c_2\}$, where c_1 and c_2 are as above in Lemma 5.3. Moreover, in this case the order of vanishing of $\Phi_3(\beta, t)$ at \mathfrak{p}_c must equal one by Remark 5.9.

By Lemma 2.3, $\Phi_3(\beta, t)$ vanishes at precisely $6\widehat{h}_f(\beta) \geq 6$ places, counted with multiplicity. Since $\Phi_3(\beta, t)$ can have at most simple roots at c_1 and c_2 , there must be a place \mathfrak{p}_c at which both $\Phi_3(\beta, t)$ and β vanish. In this case, we must have

$$\Phi_3(0, c) = c^3 + 2c^2 + c + 1 = 0.$$

Let C_1, C_2 , and C_3 be the three roots of $\Phi_3(0, t)$. Since $h(\beta) \leq 2$, β can have at most two roots; reordering the roots if necessary, we assume that $\Phi_3(\beta, t)$ and β both vanish at C_1 and possibly C_2 .

We have put certain restrictions on the places at which $\Phi_3(\beta, t)$ may vanish as well as the order of vanishing at each such place. Let $\rho(\beta) \in K[t]$ denote the numerator of $\Phi_3(\beta, t)$; scaling if necessary, we assume that $\rho(\beta)$ is monic. Set $R := \deg \rho(\beta) = 6\widehat{h}_f(\beta)$.

If $\Phi_3(\beta, t)$ vanishes at \mathfrak{p}_{C_1} but not \mathfrak{p}_{C_2} , then $\beta = (t - C_1)p/q$ for some $p, q \in K[t]$ with $\deg p \leq 1$ and $\deg q \leq 2$, and

$$(5.4) \quad \rho(\beta) = (t - c_1)^{\varepsilon_1}(t - c_2)^{\varepsilon_2}(t - C_1)^{R - \varepsilon_1 - \varepsilon_2}$$

for some $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. If $\Phi_3(\beta, t)$ vanishes at both \mathfrak{p}_{C_1} and \mathfrak{p}_{C_2} , then $\beta = (t - C_1)(t - C_2)/q$ for some $q \in K[t]$ with $\deg q \leq 2$, and

$$(5.5) \quad \rho(\beta) = (t - c_1)^{\varepsilon_1}(t - c_2)^{\varepsilon_2}(t - C_1)^k(t - C_2)^{R - \varepsilon_1 - \varepsilon_2 - k}$$

for some $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ and $1 \leq k \leq R - \varepsilon_1 - \varepsilon_2 - 1$.

The idea is to write p and q as polynomials with indeterminate coefficients; then compare the coefficients of both sides of (5.4) (resp., (5.5)). This will determine an affine scheme over K ; if this scheme is empty, or if the only points on the scheme yield a constant map β , then we will have completed the proof of the lemma.

We illustrate this computation in one case. Suppose that $\Phi_3(\beta, t)$ vanishes at \mathfrak{p}_{C_1} but not \mathfrak{p}_{C_2} , and let us suppose that $\deg q = 2$, in which case $R = 15$. Write $\beta = (t - C_1)(p_1t + p_0)/(t^2 + q_1t + q_0)$. Then $\rho(\beta)$ is a polynomial in t with coefficients in $K[p_0, p_1, q_0, q_1]$, and comparing the coefficients of $\rho(\beta)$ with the coefficients of $(t - c_1)^{\varepsilon_1}(t - c_2)^{\varepsilon_2}(t - C_1)^{15 - \varepsilon_1 - \varepsilon_2}$ for each pair $(\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2$ yields four different subschemes of $\mathbb{A}_K^4 = \text{Spec } K[p_0, p_1, q_0, q_1]$. A computation in Magma [3] verifies that each of these schemes is empty. The proof of the lemma is then completed by a number of similar computations; for the interested reader, the Magma code and output have been included as an ancillary file with this article’s arXiv submission. □

Applying Lemma 5.10 with $\beta = -\alpha$ shows that if $h(\alpha) \in \{1, 2\}$, then α realizes portrait (1, 3) for f ; as mentioned above, we may now conclude that this holds for all $\alpha \in \mathcal{K}^\times$. Finally, arguing as in the proof of Proposition 4.3, we have the $d = 2, N = 3$ case of Theorem 1.3:

Proposition 5.11. *Let $\alpha \in \mathcal{K}$, and let $M \geq 0$. Then α realizes portrait $(M, 3)$ for f_2 if and only if $(\alpha, M) \neq (0, 1)$.*

5.3. $N \leq 2$. In this section, we prove Theorem 1.3 in the $d = 2, N = 2$ case. The proof for $N = 1$ uses essentially the same technique, so we omit the proof in that case.

Consider the rational map

$$\varphi_2(z) := \frac{(t + 1)(z - 1)}{z + t},$$

with inverse

$$\varphi_2^{-1}(z) = -\frac{tz + (t + 1)}{z - (t + 1)}.$$

It follows from Theorem 1.5 that $0, 1/2,$ and ∞ (which is a fixed point for f) are the only points in $\mathbb{P}^1(K)$ that fail to realize portrait $(1, 2)$ for f . We will show that the points in \mathcal{K} that fail to realize portrait $(1, 2)$ are precisely the points in the orbits of $0, 1/2,$ and ∞ under φ_2 :

Proposition 5.12. *Let $\alpha \in \mathcal{K}$. Then α does not realize portrait $(1, 2)$ for f if and only if*

$$\alpha \in \mathcal{O}_{\varphi_2}(0) \cup \mathcal{O}_{\varphi_2}(1/2) \cup \mathcal{O}_{\varphi_2}(\infty).$$

Moreover, for each $k \geq 0$ we have

$$h(\varphi^k(0)) = h(\varphi^k(1/2)) = h(\varphi^k(\infty)) = k.$$

We begin by giving an alternative description of those points that do not realize portrait $(1, 2)$:

Lemma 5.13. *Let $\alpha \in \mathcal{K}$. Then α does not realize portrait $(1, 2)$ for f if and only if $\alpha = 1/2$ or α satisfies the following conditions:*

$$(*) \quad \begin{cases} \alpha \text{ vanishes at } \mathfrak{p}_{-1}; \\ \Phi_2(-\alpha, t) \text{ vanishes at } \mathfrak{p}_{-1} \text{ and possibly at } \mathfrak{p}_{-3/4}, \text{ but nowhere else; and} \\ \text{if } \Phi_2(-\alpha, t) \text{ vanishes at } \mathfrak{p}_{-3/4}, \text{ then } \alpha - 1/2 \text{ also vanishes at } \mathfrak{p}_{-3/4}. \end{cases}$$

Moreover, if in this case $\Phi_2(-\alpha, t)$ vanishes at $\mathfrak{p}_{-3/4}$, then it does so to order 1.

Proof. By Theorem 1.5, the only constant points $\alpha \in K \subset \mathcal{K}$ which fail to realize portrait $(1, 2)$ are $\alpha = 0$, which satisfies $(*)$, and $\alpha = 1/2$. We henceforth assume $\alpha \in \mathcal{K} \setminus K$, so that $\widehat{h}_f(\alpha) \geq h(\alpha) \geq 1$.

First, suppose that α does not realize portrait $(1, 2)$ for f . Let $\mathfrak{p}_c \in \mathcal{M}_{\mathcal{K}}$ be a place at which $\Phi_2(-\alpha, t)$ vanishes. Then $\Phi_1(-\alpha, t)$ or α also vanishes at \mathfrak{p}_c by Corollary 5.8. If $\Phi_1(-\alpha, t)$ vanishes at \mathfrak{p}_c , then $\Phi_2(-\alpha(c), c) = \Phi_1(-\alpha(c), c) = 0$. Thus $z = -\alpha(c)$ satisfies both

$$\Phi_2(z, c) = z^2 + z + c + 1 = 0 \quad \text{and} \quad \Phi_1(z, c) = z^2 - z + c = 0,$$

which implies that $c = -3/4$ and $\alpha(c) = \alpha(-3/4) = 1/2$. That $t = -3/4$ is a simple root of $\Phi_2(-\alpha, t)$ follows from Remark 5.9. Since $h(\Phi_2(-\alpha, t)) = 2\widehat{h}_f(\alpha) \geq 2$, and since $\Phi_2(-\alpha, t)$ has at most a simple root at $\mathfrak{p}_{-3/4}$, $\Phi_2(-\alpha, t)$ must vanish at some place $\mathfrak{p}_c \neq \mathfrak{p}_{-3/4}$. In this case, $\Phi_2(-\alpha, t)$ and α must both vanish at \mathfrak{p}_c ; hence $0 = \Phi_2(0, c) = c + 1$, so $c = -1$. Therefore α satisfies $(*)$.

Now suppose α satisfies $(*)$. By Corollary 5.8, it suffices to show that wherever $\Phi_2(-\alpha, t)$ vanishes, so too must $\Phi_1(-\alpha, t)$ or α . Since α satisfies $(*)$, $\Phi_2(-\alpha, t)$ vanishes only at \mathfrak{p}_{-1} and possibly $\mathfrak{p}_{-3/4}$. The result follows by noting that condition $(*)$ forces α to vanish at \mathfrak{p}_{-1} , and if $\Phi_2(-\alpha, t)$ vanishes at $\mathfrak{p}_{-3/4}$, then condition $(*)$ says that α reduces to $1/2$ — and therefore $\Phi_1(-\alpha, t)$ reduces to $\Phi_1(-1/2, -3/4) = 0$ — modulo $\mathfrak{p}_{-3/4}$. □

We now verify that φ_2 and φ_2^{-1} preserve property $(*)$, and that φ_2 behaves nicely with respect to the heights of points satisfying $(*)$.

Lemma 5.14. *Let $\alpha \in \mathcal{K}$ satisfy property (*). Then*

- (A) $\varphi_2(\alpha)$ satisfies (*) as well, and
- (B) $h(\varphi_2(\alpha)) = h(\alpha) + 1$.

Proof. Since α vanishes at \mathfrak{p}_{-1} , it is clear that $\varphi_2(\alpha)$ also vanishes at \mathfrak{p}_{-1} . Also, we have

$$\begin{aligned} \Phi_2(-\varphi_2(\alpha), t) &= \varphi_2(\alpha)^2 - \varphi_2(\alpha) + (t + 1) = \frac{(t + 1)^2(\alpha^2 - \alpha + (t + 1))}{(\alpha + t)^2} \\ &= \frac{(t + 1)^2\Phi_2(-\alpha, t)}{(\alpha + t)^2}, \end{aligned}$$

so $\Phi_2(-\varphi_2(\alpha), t)$ vanishes at \mathfrak{p}_{-1} ; moreover, since $\Phi_2(-\alpha, t)$ only vanishes at \mathfrak{p}_{-1} and possibly to order one at $\mathfrak{p}_{-3/4}$, the same holds for $\Phi_2(-\varphi_2(\alpha), t)$. (Any pole of $(\alpha + t)^2$ is a pole of at least the same order for $(t + 1)^2\Phi_2(-\alpha, t)$, so the above expression may only vanish at the zeros of its numerator.) Finally, if $\Phi_2(-\varphi_2(\alpha), t)$ vanishes at $\mathfrak{p}_{-3/4}$, then so must $\Phi_2(-\alpha, t)$, in which case $v_{-3/4}(\alpha - 1/2) > 0$. Hence

$$\varphi_2(\alpha) - \frac{1}{2} = \frac{(t + 1/2)(\alpha - 1/2) - (t + 3/4)}{(\alpha - 1/2) + (t + 1/2)}$$

vanishes at $\mathfrak{p}_{-3/4}$. Therefore $\varphi_2(\alpha)$ satisfies (*).

We now prove (B). Letting

$$\gamma := \frac{1}{\varphi_2(\alpha)} = \frac{1}{t + 1} + \frac{1}{\alpha - 1},$$

it suffices to show that $h(\gamma) = h(\alpha) + 1$. Suppose that \mathfrak{p} is a pole of γ . Then either $\mathfrak{p} = \mathfrak{p}_{-1}$, in which case property (*) implies that $v_{\mathfrak{p}}(\alpha - 1) = 0$, hence $v_{\mathfrak{p}}(\gamma) = -v_{\mathfrak{p}}(t + 1) = -1$; or else $\mathfrak{p} \neq \mathfrak{p}_{-1}$ is a zero of $\alpha - 1$, in which case $v_{\mathfrak{p}}(\gamma) = -v_{\mathfrak{p}}(\alpha - 1)$. Therefore

$$h(\gamma) = - \sum_{v_{\mathfrak{p}}(\gamma) < 0} v_{\mathfrak{p}}(\gamma) = -v_{-1}(\gamma) + \sum_{v_{\mathfrak{p}}(\alpha - 1) > 0} v_{\mathfrak{p}}(\alpha - 1) = 1 + h(\alpha - 1).$$

Since $h(\alpha - 1) = h(\alpha)$, we are done. □

Lemma 5.15. *Let $\alpha \in \mathcal{K}$ satisfy (*), and assume $\alpha \notin \{0, t + 1, -\frac{t+1}{2t+1}\}$.*

- (A) *We have*

$$\begin{aligned} v_{-1}(\alpha) &= 1, \\ v_{-1}(\alpha - (t + 1)) &= 2, \text{ and} \\ v_{-1}(t\alpha + (t + 1)) &\geq 3. \end{aligned}$$

- (B) *The point $\varphi_2^{-1}(\alpha)$ also satisfies (*).*

Proof. We first show that $v_{-1}(\Phi_2(-\alpha, t)) \geq 3$. Since $\Phi_2(-\alpha, t)$ can only vanish at \mathfrak{p}_{-1} and possibly, to order one, at $\mathfrak{p}_{-3/4}$, we have

$$\begin{aligned} v_{-1}(\Phi_2(-\alpha, t)) &= h(\Phi_2(-\alpha, t)) - v_{-3/4}(\Phi_2(-\alpha, t)) \\ &= 2\widehat{h}_f(\alpha) - \begin{cases} 0, & \text{if } \Phi_2(-\alpha, t) \text{ does not vanish at } \mathfrak{p}_{-3/4}, \\ 1, & \text{if } \Phi_2(-\alpha, t) \text{ vanishes at } \mathfrak{p}_{-3/4}. \end{cases} \end{aligned}$$

Thus $v_{-1}(\Phi_2(-\alpha, t)) \leq 2$ if and only if $\widehat{h}_f(\alpha) \leq 1$ or $\widehat{h}_f(\alpha) = 3/2$ and $\Phi_2(-\alpha, t)$ vanishes at $\mathfrak{p}_{-3/4}$. A simple calculation then verifies that the only such $\alpha \in \mathcal{K}$

satisfying (*) are the three values of α excluded from the statement of the lemma. Therefore $v_{-1}(\Phi_2(-\alpha, t)) \geq 3$.

Now, write

$$\alpha = (t + 1) + \alpha^2 - (\alpha^2 - \alpha + (t + 1)) = (t + 1) + \alpha^2 - \Phi_2(-\alpha, t).$$

Since $v_{-1}(\alpha) \geq 1$ by assumption, we have $v_{-1}(\alpha^2) \geq 2$, thus $v_{-1}(\alpha) = v_{-1}(t + 1) = 1$. This implies that $v_{-1}(\alpha^2) = 2$, and therefore

$$v_{-1}(\alpha - (t + 1)) = v_{-1}(\alpha^2 - \Phi_2(-\alpha, t)) = 2$$

as well. Finally, we write

$$t\alpha + (t + 1) = \Phi_2(-\alpha, t) - \alpha(\alpha - (t + 1)),$$

and by what we have already shown, this has valuation at least 3 at \mathfrak{p}_{-1} , proving (A).

We now show that $\varphi_2^{-1}(\alpha)$ satisfies (*). By part (A), we have

$$v_{-1}(\varphi_2^{-1}(\alpha)) = v_{-1}(t\alpha + (t + 1)) - v_{-1}(\alpha - (t + 1)) \geq 1,$$

so $\varphi_2^{-1}(\alpha)$ vanishes at \mathfrak{p}_{-1} . Now consider

$$\Phi_2(-\varphi_2^{-1}(\alpha), t) = (-\varphi_2^{-1}(\alpha))^2 - \varphi_2^{-1}(\alpha) + t + 1 = \frac{(t + 1)^2 \Phi_2(-\alpha, t)}{(\alpha - (t + 1))^2}.$$

Since $\Phi_2(-\alpha, t)$ vanishes to order at least three and $(\alpha - (t + 1))^2$ vanishes to order four at \mathfrak{p}_{-1} , it follows that $\Phi_2(-\varphi_2^{-1}(\alpha), t)$ vanishes at \mathfrak{p}_{-1} . Moreover, $\Phi_2(-\varphi_2^{-1}(\alpha), t)$ can only vanish at \mathfrak{p}_{-1} and the places at which $\Phi_2(-\alpha, t)$ vanishes, which are only \mathfrak{p}_{-1} and possibly $\mathfrak{p}_{-3/4}$ by assumption. (As before, we are using the fact that the above expression cannot vanish at poles of its denominator.)

Finally, suppose $\Phi_2(-\varphi_2^{-1}(\alpha), t)$ vanishes at $\mathfrak{p}_{-3/4}$. This is equivalent to the vanishing of $\Phi_2(-\alpha, t)$, necessarily to order one, at $\mathfrak{p}_{-3/4}$, in which case $\alpha - 1/2$ also vanishes at $\mathfrak{p}_{-3/4}$. Thus

$$v_{-3/4}(\Phi_2(-\varphi_2^{-1}(\alpha), t)) = v_{-3/4}(\Phi_2(-\alpha, t)) = 1.$$

Moreover, we have

$$\varphi_2^{-1}(\alpha) - \frac{1}{2} = -\frac{(t + 1/2)(\alpha - 1/2) + (t + 3/4)}{\alpha - (t + 1)},$$

which vanishes at $\mathfrak{p}_{-3/4}$ by our assumptions on α . Therefore $\varphi_2^{-1}(\alpha)$ satisfies (*) as well. □

Proof of Proposition 5.12. Suppose first that α fails to realize portrait (1, 2) for f . Since clearly $1/2 \in \mathcal{O}_{\varphi_2}(1/2)$, we will assume that $\alpha \neq 1/2$, so α satisfies (*). We proceed by induction on $h(\alpha)$.

By Theorem 1.5, the only constant points $\alpha \in K \subset \mathcal{K}$ that do not realize portrait (1, 2) for f are $\alpha \in \{0, 1/2\}$; thus the $h(\alpha) = 0$ case holds. Now suppose $h(\alpha) \geq 1$. Since $t + 1 = \varphi_2(\infty)$ and $-(t + 1)/(2t + 1) = \varphi_2(1/2)$ we will assume $\alpha \notin \{t + 1, -(t + 1)/(2t + 1)\}$. Then Lemma 5.15 says that $\varphi_2^{-1}(\alpha)$ satisfies (*), hence $h(\varphi_2^{-1}(\alpha)) = h(\alpha) - 1$ by Lemma 5.14. By induction, we have $\varphi_2^{-1}(\alpha) \in \mathcal{O}_{\varphi_2}(\delta)$ for some $\delta \in \{0, 1/2, \infty\}$, and therefore $\alpha \in \mathcal{O}_{\varphi_2}(\delta)$ as well.

Now suppose $\alpha \in \mathcal{O}_{\varphi_2}(\delta)$ for some $\delta \in \{0, 1/2, \infty\}$. Write $\alpha = \varphi_2^k(\delta)$ for some $k \geq 0$. We show by induction on k that $h(\alpha) = k$ and that α satisfies (*), which is equivalent (for $\alpha \neq 1/2$) to the assertion that α fails to realize portrait (1, 2) for f .

Since 0 and 1/2 fail to realize portrait (1, 2) for f , the statement holds for $k = 0$. The points $\varphi_2(0) = -(t + 1)/t$, $\varphi_2(1/2) = -(t + 1)/(2t + 1)$, and $\varphi_2(\infty) = t + 1$ all satisfy (*), and certainly all three have height equal to one, establishing the $k = 1$ case. Now suppose $k \geq 2$. By induction, $\varphi_2^{k-1}(\delta)$ satisfies property (*) and has height $k - 1$. By Lemma 5.14, we conclude that $\alpha = \varphi_2^k(\delta)$ satisfies (*) and has height k , completing the proof. \square

In order to prove the more general statement involving points of portrait $(M, 2)$ with $M \geq 2$, we require the following:

Lemma 5.16. *Let $k \geq 1$. Then*

$$v_\infty(\varphi_2^k(0)) = v_\infty(\varphi_2^k(1/2)) = 0; \text{ and}$$

$$v_\infty(k\varphi_2^k(\infty) - t) \geq 0.$$

Proof. The map φ_2 reduces to $z - 1$ modulo \mathfrak{p}_∞ , so $\varphi_2^k(\delta) \equiv \delta - k \pmod{\mathfrak{p}_\infty}$ for all $\delta \in K$ and $k \in \mathbb{N}$. If we take $\delta \in \{0, 1/2\}$, then $\delta - k$ is a non-zero constant for all $k \geq 1$, thus $v_\infty(\varphi_2^k(\delta)) = 0$.

Now, for each $k \geq 1$ we set $u_k := k\varphi_2^k(\infty) - t$. We show that $v_\infty(u_k) \geq 0$ by induction on k . The result clearly holds for $k = 1$, since $u_1 = \varphi_2(\infty) - t = 1$, so we assume $k \geq 2$. Then

$$u_k = k\varphi_2(\varphi_2^{k-1}(\infty)) - t = k\varphi_2\left(\frac{u_{k-1} + t}{k - 1}\right) - t$$

$$= \frac{t((k - 1)u_{k-1} - k^2 + 2k) + (ku_{k-1} - k^2 + k)}{u_{k-1} + kt}.$$

By induction, we have $v_\infty(u_{k-1}) \geq 0$, and it therefore follows that $v_\infty(u_k) \geq 0$. \square

We now prove Theorem 1.3 in the case $d = N = 2$.

Proposition 5.17. *Let $\alpha \in \mathcal{K}$, and let $M \geq 0$. Then α realizes portrait $(M, 2)$ for f_2 if and only if (α, M) does not satisfy one of the following conditions:*

- $M = 0$ and $\alpha = -1/2$;
- $M = 1$ and $\alpha \in \mathcal{O}_{\varphi_2}(0) \cup \mathcal{O}_{\varphi_2}(1/2) \cup \mathcal{O}_{\varphi_2}(\infty)$; or
- $M = 2$ and $\alpha = \pm 1$.

Moreover, for each $k \geq 0$, we have $h(\varphi_2^k(0)) = h(\varphi_2^k(1/2)) = h(\varphi_2^k(\infty)) = k$.

Proof. The $M = 0$ and $M = 1$ cases follow from Propositions 3.2 and 5.12, respectively. We therefore assume $M \geq 2$.

That ± 1 do not realize portrait (2, 2) for f follows from Theorem 1.5. Now suppose that α does not realize portrait $(M, 2)$ for f ; equivalently, suppose that $f^{M-1}(\alpha)$ does not realize portrait (1, 2) for f . By Proposition 5.12, we have $f^{M-1}(\alpha) = \varphi_2^k(\delta)$ for some $k \geq 0$ and $\delta \in \{0, 1/2, \infty\}$. Lemma 2.1 asserts that any point in the image of f must have a pole at \mathfrak{p}_∞ ; since points in the orbits of 0 and 1/2 under φ_2 do not have poles at \mathfrak{p}_∞ by Lemma 5.16, we must have $f^{M-1}(\alpha) = \varphi_2^k(\infty)$ for some $k \geq 0$. Since the only preimage of ∞ is ∞ itself, we must have $k \geq 1$.

Set $\beta := f^{M-2}(\alpha)$. Then $f(\beta) = \varphi_2^k(\infty)$, so by Lemma 5.16 we have that

$$k\varphi_2^k(\infty) - t = kf(\beta) - t = k\beta^2 + (k - 1)t$$

is regular at \mathfrak{p}_∞ . This implies that $k = 1$, so $f^{M-1}(\alpha) = \varphi_2(\infty) = t + 1$. The preimages of $t + 1$ under f are ± 1 , and the preimages of ± 1 lie outside of \mathcal{K} .

Therefore, if $f^{M-1}(\alpha) = t + 1$ for some $M \geq 2$ and $\alpha \in \mathcal{K}$, then $M = 2$ and $\alpha = \pm 1$, as claimed. \square

As mentioned at the beginning of this section, the proof of the following statement — the $d = 2$, $N = 1$ case of the main theorem — uses the same ideas as for Proposition 5.17, and we therefore omit the proof.

Proposition 5.18. *Let $\alpha \in \mathcal{K}$, and let $M \geq 0$. Then α fails to realize portrait $(M, 1)$ for f_2 if and only if $M = 1$ and $\alpha \in \mathcal{O}_{\varphi_1}(0) \cup \mathcal{O}_{\varphi_1}(\infty)$, where*

$$\varphi_1(z) = -\frac{t(z+1)}{z-(t-1)}.$$

Moreover, for each $k \geq 0$, we have $h(\varphi_1^k(0)) = h(\varphi_1^k(\infty)) = k$.

Proposition 5.18 is the final case of Theorem 1.3. Therefore, by combining Propositions 5.17 and 5.18 with the results of the previous sections, we have proven the main theorem.

ACKNOWLEDGMENTS

The author would like to thank Tom Tucker for introducing him to this problem, as well as for a number of very helpful discussions over the course of writing this article. The author would also like to thank the anonymous referee for a careful reading and for suggestions to improve the exposition.

REFERENCES

- [1] Matthew Baker and Laura DeMarco, *Preperiodic points and unlikely intersections*, Duke Math. J. **159** (2011), no. 1, 1–29, DOI 10.1215/00127094-1384773. MR2817647
- [2] A. S. Bang, *Taltheoretiske undersøgelser*, Tidskrift Math. **5** (1886), 70–80 and 130–137.
- [3] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, DOI 10.1006/jsco.1996.0125. Computational algebra and number theory (London, 1993). MR1484478
- [4] Thierry Bousch, *Sur quelques problèmes de dynamique holomorphe*, Ph.D. thesis, Université de Paris-Sud, Centre d’Orsay, 1992.
- [5] Xavier Buff and Tan Lei, *The quadratic dynatomic curves are smooth and irreducible*, Frontiers in complex dynamics, Princeton Math. Ser., vol. 51, Princeton Univ. Press, Princeton, NJ, 2014, pp. 49–72. MR3289906
- [6] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes. Partie II* (French), Publications Mathématiques d’Orsay [Mathematical Publications of Orsay], vol. 85, Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac. MR812271
- [7] John R. Doyle, *Preperiodic portraits for unicritical polynomials*, Proc. Amer. Math. Soc. **144** (2016), no. 7, 2885–2899, DOI 10.1090/proc/13075. MR3487222
- [8] Xander Faber and Andrew Granville, *Prime factors of dynamical sequences*, J. Reine Angew. Math. **661** (2011), 189–214, DOI 10.1515/CRELLE.2011.081. MR2863906
- [9] Yan Gao and YaFei Ou, *The dynatomic periodic curves for polynomial $z \mapsto z^d + c$ are smooth and irreducible*, Sci. China Math. **57** (2014), no. 6, 1175–1192, DOI 10.1007/s11425-014-4807-1. MR3202010
- [10] Dragos Ghioca, Khoa Nguyen, and Thomas J. Tucker, *Portraits of preperiodic points for rational maps*, Math. Proc. Cambridge Philos. Soc. **159** (2015), no. 1, 165–186, DOI 10.1017/S0305004115000274. MR3349337
- [11] Marc Hindry and Joseph H. Silverman, *Diophantine geometry: An introduction*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000. MR1745599
- [12] Patrick Ingram and Joseph H. Silverman, *Primitive divisors in arithmetic dynamics*, Math. Proc. Cambridge Philos. Soc. **146** (2009), no. 2, 289–302, DOI 10.1017/S0305004108001795. MR2475968

- [13] Dierk Schleicher, *Internal addresses in the Mandelbrot set and irreducibility of polynomials*, ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)—Cornell University. MR2691195
- [14] R. C. Mason, *Diophantine equations over function fields*, London Mathematical Society Lecture Note Series, vol. 96, Cambridge University Press, Cambridge, 1984. MR754559
- [15] Patrick Morton, *Arithmetic properties of periodic points of quadratic maps*, Acta Arith. **62** (1992), no. 4, 343–372. MR1199627
- [16] Patrick Morton, *On certain algebraic curves related to polynomial maps*, Compositio Math. **103** (1996), no. 3, 319–350. MR1414593
- [17] Patrick Morton and Pratiksha Patel, *The Galois theory of periodic points of polynomial maps*, Proc. London Math. Soc. (3) **68** (1994), no. 2, 225–263, DOI 10.1112/plms/s3-68.2.225. MR1253503
- [18] Patrick Morton and Joseph H. Silverman, *Periodic points, multiplicities, and dynamical units*, J. Reine Angew. Math. **461** (1995), 81–122, DOI 10.1515/crll.1995.461.81. MR1324210
- [19] Patrick Morton and Franco Vivaldi, *Bifurcations and discriminants for polynomial maps*, Nonlinearity **8** (1995), no. 4, 571–584. MR1342504
- [20] Joseph H. Silverman, *The S -unit equation over function fields*, Math. Proc. Cambridge Philos. Soc. **95** (1984), no. 1, 3–4, DOI 10.1017/S0305004100061235. MR727073
- [21] Joseph H. Silverman, *The arithmetic of dynamical systems*, Graduate Texts in Mathematics, vol. 241, Springer, New York, 2007. MR2316407
- [22] W. W. Stothers, *Polynomial identities and Hauptmoduln*, Quart. J. Math. Oxford Ser. (2) **32** (1981), no. 127, 349–370, DOI 10.1093/qmath/32.3.349. MR625647
- [23] Ralph Walde and Paula Russo, *Rational periodic points of the quadratic function $Q_c(x) = x^2 + c$* , Amer. Math. Monthly **101** (1994), no. 4, 318–331, DOI 10.2307/2975624. MR1270956
- [24] K. Zsigmondy, *Zur Theorie der Potenzreste* (German), Monatsh. Math. Phys. **3** (1892), no. 1, 265–284, DOI 10.1007/BF01692444. MR1546236

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627
Current address: Department of Mathematics and Statistics, GTMH 330, Louisiana Tech University, Ruston, Louisiana 71272
E-mail address: jdoyle@latech.edu