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THE SHAPE OF THE LEVEL SETS OF THE FIRST EIGENFUNCTION OF A CLASS OF TWO-DIMENSIONAL SCHRÖDINGER OPERATORS

THOMAS BECK

ABSTRACT. We study the first Dirichlet eigenfunction of a class of Schrödinger operators with a convex, non-negative, potential V on a convex, planar domain Ω. In the case where the diameter of Ω is large and the potential V varies on different length scales in orthogonal directions, we find two length scales L_1 and L_2 and an orientation of the domain Ω which determine the shape of the level sets of the eigenfunction. As an intermediate step, we also establish bounds on the first eigenvalue in terms of the first eigenvalue of an associated ordinary differential operator.

1. Introduction

We are interested in studying a class of Schrödinger operators

$$\mathcal{L} = -\Delta_{x,y} + V(x,y).$$

This operator acts on functions defined on the bounded, convex domain $\Omega \subset \mathbb{R}^2$, and $V(x,y)$ is a convex potential.

The operator $\mathcal{L}$ has an increasing sequence of Dirichlet eigenvalues

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \nearrow \infty,$$

with corresponding eigenfunctions $u_j(x,y)$ satisfying

$$\begin{cases} (-\Delta_{x,y} + V(x,y))u_j(x,y) &= \lambda_j u_j(x,y) \qquad \text{in } \Omega, \\ u_j(x,y) &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Our main focus will be to study the first eigenvalue $\lambda = \lambda_1$ and eigenfunction $u(x,y) = u_1(x,y)$. The first eigenfunction $u(x,y)$ does not change sign inside Ω, and so we normalise $u(x,y)$ so that it is positive inside Ω and attains a maximum of 1. In Definitions 1.1 and 1.3 below, we will define the class of convex domains Ω and potentials $V(x,y)$ that we are interested in. Our results will become non-trivial in the case that the diameter of Ω is large and the potential $V(x,y)$ varies on very different length-scales in directions along and perpendicular to a diameter. We will see that one consequence of the assumptions on Ω and $V(x,y)$ is that it ensures that the superlevel sets of $u(x,y)$,

$$W_c = \{(x,y) \in \Omega : u(x,y) \geq c\},$$

are convex subsets of Ω for all $0 \leq c \leq 1$.

A theorem of John, [18], therefore implies that for each c we can find an ellipse E_c contained within this superlevel set W_c, such that a dilate of E_c, with scaling factor

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bounded by an absolute constant contains W_c. We are interested in determining the shape of the level sets of $u(x, y)$, and to do this we will study the lengths and orientation of the axes of the ellipse E_c. One of the main steps in establishing the shape of the level sets of $u(x, y)$ will be to prove sufficiently precise bounds on the first eigenvalue λ.

We know that the level set $\{(x, y) \in \Omega : u(x, y) = 0\}$ is equal to the boundary, $\partial\Omega$, and so in particular the shape of this level set is determined solely by the geometry of Ω. However, we will see that, in general, for the intermediate level sets, for example $\{(x, y) \in \Omega : u(x, y) = \frac{1}{2}\}$, it is not solely the shape of $\partial\Omega$ that governs its shape, but instead the two length scales L_1 and L_2. These length scales L_1 and L_2 will be given in Definitions 1.5 and 1.8, but the key feature of their definitions is the following: The length scale L_1 will be defined purely in terms of the geometry of Ω and properties of the potential $V(x, y)$, but the length scale L_2 will also depend on a family of associated one-dimensional Schrödinger operators. Moreover, the definition of L_2 will also describe the orientation of these level sets of $u(x, y)$.

Our motivation for studying this problem is as follows: First, λ and $\Psi(t, x, y) = e^{-i\lambda t}u(x, y)$ are the lowest energy and ground state eigenfunction of the quantum system governed by the Schrödinger operator

$$i\partial_t\Psi(t, x, y) + \mathcal{L}\Psi(t, x, y) = 0.$$

The main motivation comes from the series of papers [17], [15], [16]. There, the authors study the first two Dirichlet eigenfunctions on two-dimensional convex domains Ω, normalised so that the inner radius is comparable to 1 and the diameter is equal to the large parameter N. We will describe their results and techniques in more detail below, but for now we will briefly describe one of the techniques used that is most relevant for us: Using their normalisation of the domain Ω, they write it as

$$\Omega = \{(x, y) : f_1(x) < y < f_2(x), a < x < b\},$$

for functions $f_1(x)$ and $f_2(x)$, which are convex and concave respectively, and they consider the concave *height function* $h(x)$,

$$h(x) = f_2(x) - f_1(x),$$

with $\max_{x\in[a,b]} h(x) = 1$. This allows us to define a large parameter L, purely in terms of the function $h(x)$ (and hence just depending on the geometry of the domain). This number L is the largest value such that

$$h(x) \geq 1 - L^{-2} \tag{1.1}$$

on an interval I of length at least L. Rather than the length of the diameter N, this parameter L is the relevant length scale to study the low energy eigenfunctions. Since the inner radius of their domain is comparable to 1, while the projection of the domain onto the x-axis is large compared to 1, it is natural to study the two-dimensional problem via an approximate separation of variables. For each fixed x, the domain Ω consists of the interval $[f_1(x), f_2(x)]$ of length $h(x)$, which has first eigenvalue $\pi^2 h(x)^{-2}$. Thus, the ordinary differential operator on the interval $[a, b]$, which is naturally associated with this separation of variables, is

$$-\frac{d^2}{dx^2} + \frac{\pi^2}{h(x)^2}, \tag{1.2}$$

with zero boundary conditions. In [17] the eigenvalues and eigenfunctions of this operator are used to generate appropriate test functions to provide bounds on the first eigenvalue in terms of L and to estimate the location and width of the nodal line of the second eigenfunction. In [15], they give a sharper estimate on the nodal line, and in [16] they study the location of the maximum of the first eigenfunction of Ω and its behaviour near this maximum, where they use this approximate separation of variables to relate it to the first eigenfunction of the one-dimensional operator. As a straightforward consequence of their work, it is this length scale L and orientation of the domain Ω given above which determine the shape of the level sets of the eigenfunction $u(x, y)$ in this special case.

The papers [17], [15], [16] also provide more motivation for studying the operators $\mathcal{L}$. In the same way that the one-dimensional Schrödinger operator in (1.2) is used in a crucial way to study the eigenfunctions of two-dimensional convex domains, it will be important to understand the properties of the eigenfunctions of $\mathcal{L}$ when considering the eigenfunctions of three- (and higher-)dimensional convex domains.

Before stating our results, let us define precisely the class of domains Ω and potentials $V(x, y)$ that we will be considering here.

Definition 1.1 (The domain Ω)**.** The domain Ω is a bounded, convex two-dimensional domain with inner radius N_1 and diameter N_2. We assume that the diameter is large compared to an absolute constant, while the inner radius is bounded below by an absolute constant.

Remark 1.2. Throughout, the constants that appear will depend on these absolute constants, but the dependence of any bounds on the diameter and inner radius themselves (and the other parameters introduced below) will be explicitly stated.

We now state the class of potentials of interest.

Definition 1.3 (The potential $V(x, y)$)**.** The potential $V(x, y)$ on the domain Ω satisfies

$$V(x, y) = \frac{1}{h(x, y)^2},$$

where $h(x, y)$ is a concave function with $0 \leq h(x, y) \leq 1$ and $\max_\Omega h(x, y) = 1$. In other words, $V(x, y)^{-1/2}$ is concave on Ω and

$$\min_\Omega V(x, y) = 1.$$

In particular, this also ensures that $V(x, y)$ is convex.

We see that this ensures that the first derivatives of V are bounded almost everywhere and that the second derivatives of V are positive measures. However, we do not impose any further regularity assumptions on the potential. Before continuing, let us briefly discuss the motivation behind Definition 1.3.

(1) One allowed potential is the constant potential $V(x, y) = 1$. In this case, our operator is analogous to the purely two-dimensional operator studied in [17]. In particular, we can renormalise our domain Ω to ensure that the inner radius is comparable to 1. Note in general, our potential $V(x, y)$ is not scale invariant, and so this is not as useful a normalisation for us.
(2) The assumption that $V(x, y)^{-1/2}$ is concave is a natural one when we recall the motivation for studying this class of Schrödinger operators. In the same way that the operator in (1.2) has been used to study the eigenfunctions

of two-dimensional domains, the potential $V(x,y)$ that we are considering is naturally related to the three-dimensional domain with height function proportional to $h(x,y)$. This assumption that $V(x,y)^{-1/2}$ is concave also appears in the work of Borell, [4], [5], when studying the concavity properties of the Green's functions associated to these Schrödinger operators.

(3) We do not claim that this is the only class of potentials for which the results below will be valid. In fact, many of the results can be restated to hold for a more general class of convex potentials (including those related to the harmonic oscillator). However, at times we will see that it is convenient to restrict to those potentials given in Definition 1.3, and so we will only state the results for this class of potentials.

Remark 1.4. The potential $V(x,y)$ may be unbounded, but can only tend to ∞ as (x,y) approaches the boundary of Ω. This means that our operator $\mathcal{L}$ can be written as a self-adjoint operator, with domain $D_{\mathcal{L}}$ containing all functions in $C_0^\infty(\Omega)$. The domain $D_{\mathcal{L}}$ is then equal to

$$\{\psi \in L^2(\Omega) : \int_\Omega (\mathcal{L}f).\psi = \int_\Omega f\phi_\psi, f \in C_0^\infty(\Omega), \text{ some } \phi_\psi \in L^2(\Omega)\}.$$

In particular, this allows us to consider the variational formulation of the first eigenvalue

$$\lambda = \inf_{\psi \in C_0^\infty(\Omega), \psi \neq 0} \frac{\int_\Omega |\nabla\psi|^2 + V\psi^2}{\int_\Omega \psi^2}.$$

We can now introduce the crucial parameters L_1 and L_2 that will appear as important length scales in our study of the first eigenfunction $u(x,y)$. For each $c \geq 0$, let us define the sublevel sets of $V(x,y)$ by

$$\Omega_c = \{(x,y) \in \Omega : V(x,y) \leq 1 + c\}.$$

Since $V(x,y)$ is convex, these sublevel sets Ω_c are convex subsets of Ω.

Definition 1.5 (The parameter L_1)**.** Let L_1 be the largest value such that the sublevel set $\Omega_{L_1^{-2}}$ has inner radius at least equal to L_1.

Remark 1.6. This definition is analogous to the definition of the parameter L from [17] described above and roughly speaking is equal to the largest length scale L_1 on which the potential increases by at most L_1^{-2} from its minimum.

With L_1 fixed, we let $\tilde{L}_1$ be the diameter of the set $\Omega_{L_1^{-2}}$. If L_1 and $\tilde{L}_1$ are comparable in size, then we define L_2 to be equal to L_1, but if

$$\tilde{L}_1 \gg L_1,$$

then we now describe how to find L_2.

Remark 1.7. Throughout, the notation $A \gg B$ denotes $A \geq \tilde{C}B$, for some large fixed absolute constant $\tilde{C} > 0$, and if this and the converse $B \gg A$ do not hold, then we say that A and B are comparable. In particular, we are not interested in the exact values of L_1 and L_2, but instead are interested in knowing whether any length scale is, or is not, comparable to L_1 and L_2. We will use the notation C to represent an absolute constant that is small compared to $\tilde{C}$, which may change from line to line.

To obtain a value for L_2, we first rotate our domain Ω, so that the projection of $\Omega_{L_1^{-2}}$ onto the y-axis is of the smallest length amongst the projections onto any line. In particular, this means that the projection of $\Omega_{L_1^{-2}}$ onto the x-axis is comparable to $\tilde{L}_1$, while the projection of $\Omega_{L_1^{-2}}$ onto the y-axis is comparable to L_1. This also fixes the orientation of Ω.

For each fixed x, let the interval $\Omega(x)$ be the cross-section of Ω at x, and consider the ordinary differential operator

$$\mathcal{L}(x) = -\frac{d^2}{dy^2} + V(x,y), \tag{1.3}$$

with zero boundary conditions on $\Omega(x)$. We let $\mu(x)$ be the first eigenvalue of $\mathcal{L}(x)$ and define the minimum of these eigenvalues,

$$\mu^* = \min_x \mu(x).$$

We can now define the parameter L_2.

Definition 1.8 (The parameter L_2)**.** We define L_2 to be the largest value such that

$$\mu^* \leq \mu(x) \leq \mu^* + L_2^{-2},$$

for all x in an interval I of length at least L_2.

Remark 1.9. Note that in this definition of L_2, we have used the orientation of $\Omega_{L_1^{-2}}$ fixed above. Therefore, from now on, whenever we consider any property of the eigenvalue or eigenfunction that depends on the value of L_2, we will have to use this orientation of $\Omega_{L_1^{-2}}$. In contrast, the definition of L_1 does not depend on the orientation of $\Omega_{L_1^{-2}}$.

Our main aim in the study of the first eigenfunction is to give precise information about the shape of the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ which are near the point where $u(x,y)$ attains its maximum of 1. Since the potential $V(x,y)$ is a convex function and Ω is a convex set, Theorem 6.1 in [8] tells us that $u(x,y)$ is log concave. Alternative proofs of this result have also been given in [9], [19], [20]. In particular, this tells us that the superlevel sets are all convex. Since $\{(x,y) \in \Omega : u(x,y) \geq 0\} = \Omega$, one way of viewing this result is that

$$\{(x,y) \in \Omega : u(x,y) \geq 0\} \text{ convex } \Rightarrow \{(x,y) \in \Omega : u(x,y) \geq c\} \text{ convex}$$

for all $0 \leq c \leq 1$.

We will use the convexity of the superlevel sets of $u(x,y)$ in a crucial way to describe their shape near its maximum.

Theorem 1.10. *Let Ω and $V(x,y)$ be a domain and potential from Definitions* 1.1 *and* 1.3*. Fix a small absolute constant $c_1 > 0$, and let L_1 and L_2 be as in Definitions* 1.5 *and* 1.8*. In particular, this means that we have fixed the orientation of the set $\Omega_{L_1^{-2}}$. Then, for any fixed absolute constant c, with $c_1 < c < 1 - c_1$, the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ has the following shape: There exists an ellipse E with minor axis in the y-direction of length comparable to L_1 and major axis in the x-direction of length comparable to L_2, such that E is contained inside this level set and a dilate of E, with a scaling factor bounded by an absolute constant, contains this level set.*

Remark 1.11. The level set $\{(x,y) \in \Omega : u(x,y) = 0\}$ is equal to $\partial\Omega$, the boundary of Ω. We will see that in general the parameters L_1 and L_2 are not comparable to the inner radius and diameter of the original domain Ω. Thus, the result of Theorem 1.10 does not remain valid when c becomes close to 0.

Corollary 1.12. *For a convex set W, we define the eccentricity of W, $\mathrm{ecc}(W)$, in the usual way:*

$$\mathrm{ecc}(W) = \frac{\mathrm{diam}(W)}{\mathrm{inradius}(W)}.$$

For $c = 0$, the eccentricity of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ is equal to the eccentricity of Ω, but as c increases (while bounded above by $1 - c_1$), the eccentricity of the superlevel set becomes comparable to L_2/L_1.

The log concavity of the eigenfunction and resulting convexity of its superlevel sets have been used previously in various situations. For example, in [2] moduli of convexity and concavity are introduced. Under certain conditions on the potential V, it is then possible to strengthen the log concavity of the first eigenfunction by finding an appropriate modulus of concavity. This allows the spectral gap for a class of Schrödinger operators to be compared to the case where the potential is identically zero and allows them to prove the Fundamental Gap Conjecture. In [14] the convexity of the superlevel sets of the Green's function are used in a crucial way to prove third derivative estimates on the eigenfunction which are valid up to the boundary of the convex domain.

As well as the convexity of the superlevel sets of $u(x,y)$, a very important part of the proof of Theorem 1.10 will be to obtain sufficiently precise eigenvalue bounds for the first eigenvalue λ. For $\mu(x)$ equal to the first eigenvalue of the operator $\mathcal{L}(x)$, we consider the ordinary differential operator

$$\mathcal{A} = -\frac{d^2}{dx^2} + \mu(x) \tag{1.4}$$

and let μ be the first eigenvalue of this operator. Our eigenvalue bounds relate the value of λ to this eigenvalue μ.

Theorem 1.13. *Let Ω and $V(x,y)$ be a domain and potential from Definitions 1.1 and 1.3. If L_2 is defined as in Definition 1.8 and μ is the first eigenvalue of the operator $\mathcal{A}$ in (1.4), then the first eigenvalue λ of the operator $\mathcal{L}$ satisfies*

$$\mu \leq \lambda \leq \mu + CL_2^{-2},$$

for an absolute constant C.

Remark 1.14. Theorems 1.10 and 1.13 are valid for all domains and potentials satisfying the assumptions of Definitions 1.1 and 1.3, and the bounds are uniform for domains Ω and potentials V leading to the same values for L_1 and L_2.

While it is much more straightforward to locate the eigenvalue λ to an interval of length comparable to L_1^{-2}, we will see that the more precise bound obtained in Theorem 1.13 is necessary to obtain sharp information about the length scale on which the eigenfunction $u(x,y)$ decays in the x-direction and hence prove Theorem 1.10.

Theorem 1.13 locates the first eigenvalue λ to an interval of length comparable to L_2^{-2}, provided we know the value of μ. However, μ is also an eigenvalue of a differential operator, and so it may seem like we have only been able to locate the unknown λ in terms of another unknown μ. Another reason why this theorem still has value is that whereas λ is the first eigenvalue of a two-dimensional partial differential operator (with a potential), μ is the first eigenvalue of an ordinary differential operator $\mathcal{A}$. Thus, from a computational standpoint, it is much easier to accurately approximate the value of μ compared to λ. Also, we notice that the parameter L_2 depends on the geometric properties of the domain Ω and potential $V(x, y)$, together with the eigenvalues of the differential operator $\mathcal{L}(x)$ given in (1.3). In other words, L_2 also only depends on knowledge of ordinary differential operators. Thus, the bound given in Theorem 1.13 gives information about the eigenvalue of a two-dimensional partial differential operator purely in terms of ordinary differential operators.

The idea of relating the eigenfunctions and eigenvalues of a two-dimensional problem to an associated ordinary differential operator has also been used extensively by Friedlander and Solomyak in [11], [12], [13]. In these papers, they use this approximate separation of variables to obtain asymptotics for the eigenvalues and the resolvent of the Dirichlet Laplacian. They use a semiclassical method by sending a small parameter ϵ to 0 in order to give a one-parameter of 'narrow' domains, and then write asymptotics in terms of this small parameter. In Borisov-Freitas [6], they use similar techniques to study the asymptotics of eigenfunctions and eigenvalues for a class of planar, not necessarily convex, domains in the singular limit around a line segment. In Freitas and Krejčiřík [10] they also relate the study of eigenfunctions and eigenvalues of a class of 'thin' two-dimensional (not necessarily convex) domains to an associated ordinary differential operator, and in particular use this to deduce properties of the nodal line of the second eigenfunction.

Let us now describe how we will proceed in the sections below.

In Section 2 we study the parameters L_1 and L_2 from Definitions 1.5 and 1.8 in more detail. In particular, we will obtain bounds on L_1 and L_2 in terms of the diameter and inner radius of the domain and the potential and construct domains Ω and potentials $V(x, y)$ to show to what extent these estimates are sharp. We will also give a straightforward bound on λ in terms of L_1 by using the variational formulation for the first eigenvalue.

In Section 3 we will prove the eigenvalue bounds in Theorem 1.13. For each fixed x, $u(x, y)$ is an admissible test function for the operator $\mathcal{L}(x)$ from (1.3), and the lower bound on λ will follow straightforwardly from this. The proof of the upper bound on λ in Theorem 1.13 is more involved. The starting point of the proof is to use the first eigenfunction, $\psi^{(x)}(y)$, of the operator $\mathcal{L}(x)$ to construct a suitable test function in the variational formulation for the first eigenvalue. To obtain the required upper bound on λ it will be necessary to study the first variation of $\psi^{(x)}(y)$ in the cross-sectional variable x. To do this, we will derive the ordinary differential equation that this first variation satisfies for each fixed x. The bounds then follow from using the method of variation of parameters. It will be particularly important to have estimates on the relative size of the first derivative of the potential $V(x, y)$ and the size of $\psi^{(x)}(y)$.

Once we have established the bounds on λ in Theorem 1.13, in Section 4 we use them to study the first eigenfunction $u(x, y)$ itself. Our first aim is to prove an $L^2(\Omega)$-bound on $u(x, y)$ which is consistent with the shape of the level sets required in Theorem 1.10. We begin by using Theorem 1.13 to prove a Carleman-type estimate to show how the $L^2(\Omega(x))$-norm of the cross-sections of $u(x,y)$,

$$H(x) = \int_{\Omega(x)} u(x,y)^2 \,\mathrm{d}y,$$

decays from its maximum exponentially on a length scale comparable to L_2. To find the required bound on the $L^2(\Omega)$-norm of $u(x,y)$, we then need to estimate the size of the maximum of $H(x)$. We will do this by proving $L^2(\Omega)$-bounds on the first derivatives of $u(x,y)$, which are again consistent with Theorem 1.10. We finish Section 4 by proving an Agmon-type estimate to give an indication of the behaviour of $u(x,y)$ at points at a large distance from its maximum.

In Section 5 we study the shape of the level sets of $u(x,y)$ and complete the proof of Theorem 1.10. To do this we will use the results of Section 4 on the $L^2(\Omega)$-norms of $u(x,y)$ itself, and also its first derivatives. We will also use the log-concavity of the eigenfunction $u(x,y)$ in a crucial way, since it is this that ensures that the superlevel sets are convex.

Theorem 1.10 gives information about the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ whenever c is bounded away from 0 and 1. In Section 6, we briefly discuss what is known and what is conjectured about the behaviour of the eigenfunction $u(x,y)$ near its maximum. Studying this in more detail will be a subject of future work.

2. The parameters L_1 and L_2

Before proving Theorems 1.13 and 1.10, we first give some more properties of the parameters L_1 and L_2 defined in Definitions 1.5 and 1.8.

We first want to give upper and lower bounds for L_1, where we recall that L_1 is the largest value for which the sublevel set $\{(x,y) \in \Omega : V(x,y) \leq 1 + L_1^{-2}\}$ has inner radius at least L_1. We can think of this as being analogous to the parameter L from [17], which we described earlier in (1.1). In [17], it was shown that this parameter L satisfies

$$N^{1/3} \leq L \leq N,$$

where N is the diameter of the two-dimensional domain. The upper bound on L is attained by an exactly rectangular domain, $[0,N] \times [0,1]$, and the lower bound is attained by a right triangle of height 1 and length N. Moreover, any intermediate value for L can be attained by interpolating between these two extreme cases and forming the appropriate trapezoidal shape.

We now give an analogous description for the possible values of L_1. Rather than the potential $V(x,y)$, it will be more convenient to work with the *height function*

$$h(x,y) = V(x,y)^{-1/2}, \tag{2.1}$$

which, by the assumptions on the potential, is a concave function, satisfying

$$0 \leq h(x,y) \leq 1$$

and attaining its maximum of 1 at the minimum of $V(x,y)$.

Proposition 2.1. *Recalling that N_1 is the inner radius of the domain Ω, we have the bounds*

$$cN_1^{1/5} \leq L_1 \leq N_1,$$

for some absolute constant $c > 0$.

Remark 2.2. We will see in the proof of the proposition that we are using the stronger assumption that $h(x,y) = V(x,y)^{-1/2}$ is concave, instead of just the convexity of $V(x,y)$.

Proof of Proposition 2.1. The proposition follows easily when the inner radius N_1 is comparable to a constant, and so throughout we will assume that $N_1 \gg 1$.

The upper bound follows trivially from the definition of L_1 and is attained, for example, when $V(x,y)$ (and hence $h(x,y)$) is identically equal to 1.

Before proving the lower bound, we recall the following theorem of John, [18]:

Theorem 2.3. *Let $K \subset \mathbb{R}^m$ be a convex domain. Then, there exists an ellipsoid E such that if $c^* \in \mathbb{R}^m$ is the centre of E, then we have*

$$E \subset K \subset c^* + m(E - c^*).$$

That is, the ellipsoid E is contained within the convex set K, but if it is dilated by a constant depending only on the dimension, then it contains K.

We will also need the following simple property of concave functions:

Lemma 2.4. *Suppose $g(x)$ is a concave function on an interval of length M, with $0 \leq g(x) \leq 1$, and $g(0) = 1$. Let $0 < \beta < 1$ and suppose that $g(z) = 1 - \beta$ at some point $z \in (0, M)$. Then, we have the bound*

$$M \leq \beta^{-1} z.$$

Proof of Lemma 2.4. By the assumptions on the function $g(x)$, it decreases by at most 1 over an interval of length M. Thus, since it is a concave function, it must satisfy

$$g(x) \geq 1 - \frac{x}{M}.$$

Since $g(z) = 1 - \beta$, this gives

$$1 - \beta \geq 1 - \frac{z}{M}, \qquad \text{or equivalently} \qquad M \leq \beta^{-1} z,$$

as required. □

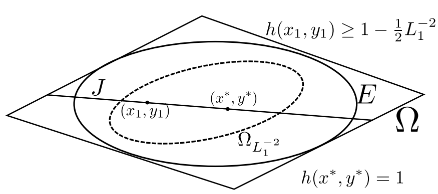

FIGURE 1. The domain Ω and other sets appearing in the proof of Proposition 2.1.

We can now prove Proposition 2.1. Let E be the ellipse coming from Theorem 2.3 for our two-dimensional domain Ω, and let (x^*, y^*) be a point where $h(x,y)$ attains its maximum of 1. Consider the ray J which is the intersection of our domain Ω, and the line containing the point (x^*, y^*) and the centre of the ellipse E (see Figure 1).

Since Ω has inner radius equal to N_1, by the properties of the ellipse E, we know that the ray J has length M with

$$M \geq c_1 N_1, \tag{2.2}$$

for some small absolute constant $c_1 > 0$. Now consider the intersection of J with the interior of the sublevel set

$$\Omega_{L_1^{-2}} = \{(x, y) \in \Omega : V(x, y) \leq 1 + L_1^{-2}\}.$$

Let J_1 be this interval. If $V(x, y) = 1 + L_1^{-2}$, then $1 - h(x, y) = 1 - V(x, y)^{-2}$ will be comparable to L_1^{-2}, and so applying Lemma 2.4 with $\beta = L_1^{-2}$, we see that J_1 will be of length A, where

$$M \leq C_1 L_1^2 A, \tag{2.3}$$

for a large absolute constant C_1.

Combining (2.2) and (2.3) gives us

$$c_1 N_1 \leq M \leq C_1 L_1^2 A. \tag{2.4}$$

Thus, the lower bound of the proposition is established unless

$$A \geq C_2 L_1^3, \tag{2.5}$$

for a large constant $C_2 > 0$.

Therefore, we will assume that (2.5) holds, and so in particular, A is large compared to L_1. Let $E_{L_1^{-2}}$ be the ellipse from Theorem 2.3 for the set $\Omega_{L_1^{-2}}$, and rotate so that the minor axis of $E_{L_1^{-2}}$ lies in the y-direction. Then, by the definition of L_1, the minor axis of $E_{L_1^{-2}}$ has length comparable to L_1.

This means that the ray of length A must approximately lie in the x-direction. Ω is a convex set with inner radius N_1, and the original ray, J, through Ω is of length M. Therefore, if we pick a point (x_1, y_1) in the interval J_1, which is at a distance of at least $A/4$ from the ends of J_1, then the height of Ω in the y-direction at $x = x_1$ must be at least

$$c_2 A N_1 / M, \tag{2.6}$$

for a constant $c_2 > 0$. In contrast, the height of $\Omega_{L_1^{-2}}$ at $x = x_1$ must be bounded above by $C_3 L_1$, since the minor axis of $E_{L_1^{-2}}$ lies in the y-direction and has length comparable to L_1.

Moreover, the concave function $h(x, y)$ varies from 1 to $1 - L_1^{-2}$ in the interval J_1 of length A. Thus, using Lemma 2.4 again, we have

$$h(x_1, y_1) \geq 1 - \frac{3}{4L_1^2} \tag{2.7}$$

at this point on the ray.

Thus, combining (2.6) and (2.7), we see that, for $x = x_1$ fixed, $h(x_1, y)$ is a concave function of y, which decreases by at most 1 on an interval of length comparable to AN_1/M and decreases by $\frac{1}{4}L_1^{-2}$ on an interval of length comparable to L_1. Thus, using Lemma 2.4 one more time, we see that

$$\frac{AN_1}{M} \leq C_4 L_1^2 L_1 = C_4 L_1^3, \tag{2.8}$$

for a constant C_4. Combining (2.4) and (2.8) we see that

$$M \leq C_1 L_1^2 A \leq C_1 L_1^2 C_4 L_1^3 \frac{M}{N_1} = C_5 L_1^5 \frac{M}{N_1},$$

for a constant $C_5 > 0$. Rearranging this inequality gives the desired lower bound on L_1. □

We noted in the proof of Proposition 2.1 that it is straightforward to give an example showing that the upper bound on L_1 is sharp. We now want to construct an example showing that the lower bound on L_1 is also optimal.

Lemma 2.5. *We can find a domain Ω and potential $V(x,y)$ satisfying the assumptions of Definitions* 1.1 *and* 1.3 *such that*

$$L_1 \leq \tilde{c} N_1^{1/5},$$

for some absolute constant $\tilde{c} > 0$.

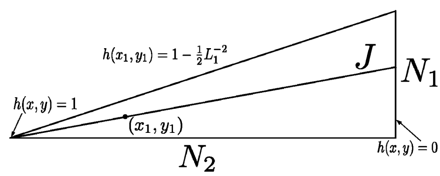

FIGURE 2. The domain in Lemma 2.5.

Proof of Lemma 2.5. We first construct the domain Ω. We have remarked earlier that for the two-dimensional domain case in [17], a right triangle gives the smallest possible value for L. Motivated by this, we let Ω be a right triangle of side lengths N_1 in the y-direction and side length N_2 in the x-direction (see Figure 2). We note that while the inner radius of this domain is not identical to N_1, it is comparable to N_1 (independently of the size of N_2), and this is all we need.

We now define the potential $V(x,y)$, via the function $h(x,y) = V(x,y)^{-2}$. We let $h(x,y) = 1$ at the point where the hypotenuse joins the side of length N_2, and set $h(x,y) = 0$ at the midpoint of the side of length N_1. We then require $h(x,y)$ to decay linearly on the interval connecting these two points. Finally, $h(x,y)$ decays linearly to 0 in the y-direction as we move away from this interval. This defines $h(x,y)$ everywhere on Ω and also ensures that $h(x,y)$ is a concave function. Thus the potential $V(x,y)$ satisfies the required properties.

We define L_1 as usual from Definition 1.5 for this domain Ω and potential $V(x,y)$. Consider the line segment J joining the vertex where $h(x,y) = 1$ to the midpoint of the opposite side, and let M be the length of the line segment $J_1 \subset J$ on which $h(x,y) \geq 1 - L_1^{-2}$. Then, since $h(x,y)$ decays linearly and the whole of J has length comparable to N_2, it is easy to see that

$$M = c_1 L_1^{-2} N_2, \tag{2.9}$$

for a constant $c_1 > 0$.

By the definition of L_1, the set $\{(x,y) \in \Omega : h(x,y) = 1 - L_1^{-2}\}$ has inner radius comparable to L_1. Thus, at the point (x_1, y_1) on the line segment J with

$$h(x_1, y_1) = 1 - \tfrac{1}{2} L_1^{-2}, \tag{2.10}$$

this set has height comparable to L_1 in the y-direction for $x = x_1$ fixed. Moreover, the point (x_1, y_1) is at a distance comparable to M from the vertex where $h(x, y) = 1$, and so the height of Ω at this point is equal to

$$c_2 M \frac{N_1}{N_2}, \tag{2.11}$$

for $c_2 > 0$. Thus, for $x = x_1$ fixed, $h(x_1, y)$ decays linearly to 0 on an interval of length comparable to $L_1 N_2/N_1$, and by (2.10) decreases linearly by $\frac{1}{2}L_1^{-2}$ on an interval of length comparable to L_1. This tells us that

$$L_1^3 = c_3 M \frac{N_1}{N_2}. \tag{2.12}$$

Combining (2.9) and (2.12) gives

$$L_1^3 = c_3 c_1 L_1^{-2} N_2 \frac{N_1}{N_2} = c_3 c_1 L_1^{-2} N_1,$$

and rearranging gives the desired estimate for L_1. □

Remark 2.6. By combining the two examples which show that the upper and lower bounds on L_1 from Proposition 2.1 are sharp, it is easy to construct examples where L_1 attains any intermediate length scale.

We now want to consider the parameter L_2 introduced in Definition 1.8. Before describing the bounds that L_2 must satisfy, we first give a simple bound on the eigenvalue λ.

Proposition 2.7. *The first eigenvalue λ satisfies*

$$1 \leq \lambda \leq 1 + C_1 L_1^{-2},$$

for an absolute constant $C_1 > 0$.

Remark 2.8. The proof of this proposition will in fact establish the lower bound $\lambda \geq 1 + \lambda(\Omega)$, where $\lambda(\Omega)$ is the first Dirichlet eigenvalue of Ω.

Proof of Proposition 2.7. We will establish these bounds by using the variational formulation of the first eigenvalue, λ. That is,

$$\lambda = \inf \left\{ \frac{\int_\Omega |\nabla \psi(x,y)|^2 + V(x,y)\psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y}{\int_\Omega \psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y} \middle| \psi \in W^{1,2}(\Omega), \psi|_{\partial\Omega} = 0, \psi \not\equiv 0 \right\}. \tag{2.13}$$

Since $V(x, y) \geq 1$ for all $(x, y) \in \Omega$, the lower bound, $\lambda \geq 1$, follows immediately.

To prove the upper bound, we need to construct a suitable test function $\psi(x, y)$ to use in (2.13). By the definition of L_1, we know that the sublevel set

$$\Omega_{L_1^{-2}} = \{(x, y) : V(x, y) \leq 1 + L_1^{-2}\}$$

has inner radius equal to L_1. Thus, we can choose a point (x_0, y_0) and a constant $c > 0$ such that the set

$$R = \{(x, y) : |x - x_0| \leq cL_1, |y - y_0| \leq cL_1\}$$

is contained in the interior of $\Omega_{L_1^{-2}}$. We then define $\psi(x, y)$ as

$$\psi(x, y) = \cos\left(\frac{\pi(x - x_0)}{2cL_1}\right) \cos\left(\frac{\pi(y - y_0)}{2cL_1}\right)$$

inside the square R, and set $\psi(x,y) = 0$ for all other $(x,y) \in \Omega$. It is then clear that

$$\frac{\int_\Omega |\nabla\psi(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \psi(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y} \leq C_2 L_1^{-2},$$

and since $V(x,y) \leq 1 + L_1^{-2}$ on the support of the test function $\psi(x,y)$, we also have

$$\frac{\int_\Omega V(x,y)\psi(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega \psi(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y} \leq 1 + C_3 L_1^{-2}.$$

Using these inequalities in (2.13) gives the desired upper bound on λ. □

We now consider the parameter L_2 from Definition 1.8. We recall that the sublevel set $\Omega_{L_1^{-2}}$ has inner radius L_1 and diameter $\tilde{L}_1$ and that we set L_2 to be equal to L_1 unless $\tilde{L}_1 \gg L_1$. The upper and lower bounds for L_2 from Definition 1.8 that we want to establish are the following.

Proposition 2.9. *The parameter L_2 satisfies*

$$c_1 \tilde{L}_1^{1/3} L_1^{2/3} \leq L_2 \leq \frac{1}{c_1}\tilde{L}_1,$$

for some absolute constant $c_1 > 0$.

Remark 2.10. In particular, the lower bound shows us that if we have $\tilde{L}_1 \gg L_1$, then also $L_2 \gg L_1$.

Proof of Proposition 2.9. The value of L_2 depends on the function $\mu(x)$, where $\mu(x)$ is the first eigenvalue of the operator $\mathcal{L}(x)$ in (1.3). L_2 is the largest value such that $\mu(x)$ increases by L_2^{-2} from its minimum value, μ^*, on an interval of length at least L_2. Therefore, before proving the bounds on L_2, we first want to study the properties of the function $\mu(x)$.

We have rotated Ω so that the projection of the set $\Omega_{L_1^{-2}}$ onto the y-axis is of the smallest length amongst the projections onto any line. One immediate consequence of this is that if we set J to be the interval which is the projection of $\Omega_{L_1^{-2}}$ onto the x-axis, then the length of J is comparable to $\tilde{L}_1$, the diameter of $\Omega_{L_1^{-2}}$.

We now give a bound on the eigenvalues $\mu(x)$ for $x \in J$.

Lemma 2.11. *For x in the middle half of the interval J, there exists an absolute constant $C_1 > 0$ such that*

$$1 + \frac{1}{C_1 L_1^2} \leq \mu(x) \leq 1 + \frac{C_1}{L_1^2}.$$

Proof of Lemma 2.11. Since $\mu(x)$ is the first eigenvalue in the ordinary differential operator in (1.3), we want to apply Lemma 2.4(a) in [17]. This lemma implies that

$$1 + \frac{1}{C_1 L(x)^2} \leq \mu(x) \leq 1 + \frac{C_1}{L(x)^2}, \tag{2.14}$$

where $L(x)$ is the length scale associated to $V(x,y)$. In other words, for each fixed x, $L(x)$ is the largest value such that $V(x,y)$ varies from its minimum by $L(x)^{-2}$ on an interval of length at least $L(x)$. Thus, to prove the lemma it is enough to show that $L(x)$ is comparable to L_1 whenever x is in the middle half of the interval J.

The projections of $\Omega_{L_1^{-2}}$ onto the x- and y-axes have lengths comparable to $\tilde{L}_1$ and L_1 respectively. It follows from Theorem 2.3 that, for those x in the middle half of J, the height of $\Omega_{L_1^{-2}}$ in the y-direction is comparable to L_1. Since the potential $V(x,y)$ is convex, attains its minimum of 1, and is equal to $1+L_1^{-2}$ on the boundary of $\Omega_{L_1^{-2}}$, we know that for all x in the middle half of J, we must have $V(x,y) \leq 1+\frac{1}{2}L_1^{-2}$ for some y.

As a result, for all x fixed in the middle half of J, the potential $V(x,y)$ varies by an amount comparable to L_1^{-2} for y in an interval of length comparable to L_1. Therefore, for each x fixed the length scale $L(x)$ is comparable to L_1, and hence using (2.14) we have the required bound. □

Remark 2.12. Since Lemma 2.4(a) in [17] played a key role in the above, let us say a few words about its proof. The upper bound in (2.14) follows easily by choosing the appropriate test function, just as in the proof of Proposition 2.7. The proof of the lower bound is slightly more complicated and makes use of the convexity of the potential to ensure that it grows at a sufficiently fast rate once we move away from its minimum.

Before completing the proof of Proposition 2.9, we need one more property of the function $\mu(x)$.

Lemma 2.13. *The first eigenvalue $\mu(x)$ is a convex function of x.*

Proof of Lemma 2.13. This convexity property follows from Corollary 1.15 in [7]. The convexity of the eigenvalue is deduced from the log concavity of the fundamental solution of the associated diffusion operator. □

Remark 2.14. Although in the assumptions of Corollary 1.15 in [7], the potential does not depend on the x-variable, the proof of the log concavity of the fundamental solution (and hence the convexity of the first eigenvalue) follows in the same way if $V(x,y)$ is allowed to depend on x, provided it remains a convex function.

We can now combine Lemmas 2.11 and 2.13 to complete the proof of Proposition 2.9: Since the interval J is of length comparable to $\tilde{L}_1$, Lemma 2.11 tells us that $\mu(x)$ varies by an amount at most comparable to L_1^{-2} for x in an interval of length comparable to $\tilde{L}_1$. Thus, since $\mu(x)$ is a convex function, applying the same logic as in Lemma 2.4, we immediately obtain the lower bound

$$L_2 \geq c_1 \tilde{L}_1^{1/3} L_1^{2/3}. \tag{2.15}$$

By the convexity of $V(x,y)$, given $C_2>0$, we can find $C_3>0$ to ensure that

$$V(x,y) \geq 1 + C_2 L_1^{-2}$$

whenever the point (x,y) is at least $C_3\tilde{L}_1$ from $\Omega_{L_1^{-2}}$. This means that $\mu(x)$ certainly must increase by an amount comparable to L_1^{-2} when x is a distance comparable to $\tilde{L}_1$ from J, and this gives us the upper bound

$$L_2 \leq \frac{1}{c_1}\tilde{L}_1. \tag{2.16}$$

Combining the inequalities in (2.15) and (2.16) completes the proof of the proposition. □

3. The bound on the first eigenvalue λ

We recall from Proposition 2.7 that the first eigenvalue λ satisfies

$$1 \leq \lambda \leq 1 + C_1 L_1^{-2}.$$

In this section we will assume that we have $\tilde{L}_1 \gg L_1$ (and hence $L_2 \gg L_1$ also), and then prove the improved upper and lower bounds on the eigenvalue λ from Theorem 1.13. That is, we will show that λ satisfies

$$\mu \leq \lambda \leq \mu + C L_2^{-2}, \tag{3.1}$$

where μ is the first eigenvalue of the ordinary differential operator $\mathcal{A}$ in (1.4). The lower bound in (3.1) is more straightforward, and so we establish this bound first.

Proposition 3.1 (Lower bound on λ)**.** *The first eigenvalue λ satisfies*

$$\lambda \geq \mu.$$

Proof of Proposition 3.1. As before, for each x fixed, let $\Omega(x)$ be the cross-section of Ω at x. Then, the first Dirichlet eigenfunction $u(x,y)$ satisfies $u(x,y) = 0$ whenever y is at the endpoints of the interval $\Omega(x)$. In particular, for each fixed x, the function $u(x,\cdot)$ is an admissible test function for the variational formulation of the first eigenvalue of the operator $\mathcal{L}(x)$. Thus,

$$\int_{\Omega(x)} (\partial_y u(x,y))^2 + V(x,y)u(x,y)^2 \,\mathrm{d}y \geq \mu(x) \int_{\Omega(x)} u(x,y)^2 \,\mathrm{d}y.$$

Integrating this over x and using

$$\begin{cases} (-\Delta_{x,y} + V(x,y))u(x,y) &= \lambda u(x,y) \qquad \text{in } \Omega, \\ \qquad\qquad u(x,y) &= 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

we see that

$$\begin{aligned} \lambda \int_\Omega u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega (\partial_x u(x,y))^2 + (\partial_y u(x,y))^2 + V(x,y)u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\geq \int_\Omega (\partial_x u(x,y))^2 + \mu(x)u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\geq \mu \int_\Omega u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

To get the final inequality, we have defined $u(x,y) = 0$ outside Ω, used Fubini to calculate the interval in x first, and then used the variational formulation for the first eigenvalue μ of the operator $\mathcal{A}$ in (1.4). This gives us the bound $\lambda \geq \mu$, as required. $\square$

We now turn to the upper bound and prove the following.

Proposition 3.2 (Upper bound on λ)**.** *We have an upper bound on the first eigenvalue λ of the form*

$$\lambda \leq \mu + C L_2^{-2},$$

for an absolute constant $C > 0$.

Remark 3.3. From Lemma 4.2(e) in [17], the operator $\mathcal{A}$ has spectral gap bounded from below by a multiple of L_2^{-2}. Therefore, obtaining bounds on λ up to a precision of CL_2^{-2} is important if we want this separation of variables in the x and y variables to be of use to us.

Proof of Proposition 3.2. As in the proof of the simple bound on λ in Proposition 2.7, we will again make use of the variational formulation for λ given in (2.13). To do this we need to construct an appropriate test function, and our motivation will come from performing an approximate change of variables in the x- and y-directions. Before stating our test function, we need some definitions.

Definition 3.4. For each fixed x, we define $\psi_1^{(x)}(y)$ to be the L^2-normalised first eigenfunction of the ordinary differential operator $\mathcal{L}(x)$. That is, $\psi_1^{(x)}(y)$ is L^2-normalised on the cross-section $\Omega(x)$ and satisfies

$$\begin{cases} \left(-\frac{d^2}{dy^2} + V(x,y)\right)\psi_1^{(x)}(y) &= \mu(x)\psi_1^{(x)}(y) \qquad \text{in } \Omega(x), \\ \psi_1^{(x)}(y) &= 0 \qquad \text{on } \partial\Omega(x). \end{cases}$$

Definition 3.5. Let I be the interval of length L_2 from Definition 1.8. We define the cut-off function $\chi(x)$ to be a positive function which is comparable to its maximum in the middle half of the interval I and is supported in the middle three quarters of I, such that it decays smoothly to zero from its maximum. We also require that $\chi(x)$ be L^2-normalised on the interval I. In particular, this allows us to ensure that

$$|\chi'(x)| \leq C_1 L_2^{-3/2},$$

for some absolute constant C_1.

We will use the test function $f(x,y) = \chi(x)\psi_1^{(x)}(y)$ in (2.13) to prove the following intermediate step.

Proposition 3.6. *We have an upper bound for λ of the form*

$$\lambda \leq \mu + \int_\Omega \chi(x)^2(\partial_x\psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 L_2^{-2},$$

for a constant C_1.

Proof of Proposition 3.6. To obtain an upper bound on the first eigenvalue λ, we will calculate the quotient from (2.13) with $\psi(x,y) = f(x,y)$ as above. Since $\psi^{(x)}(y)$ is $L^2(\Omega(x))$-normalised in y for any fixed x, and $\chi(x)$ is $L^2(I)$-normalised in x, first computing the integral in y and then the integral in x, we see that the denominator in (2.13) is equal to 1. Thus, we have the bound

$$\lambda \leq \int_\Omega |\nabla_{x,y}\left(\chi(x)\psi_1^{(x)}(y)\right)|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega V(x,y)\chi(x)^2\psi_1^{(x)}(y)^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{3.2}$$

For each x, the function $\psi^{(x)}(y)$ satisfies

$$\int_{\Omega(x)} \psi_1^{(x)}(y)^2 \,\mathrm{d}y = 1, \tag{3.3}$$

and it is equal to 0 at the endpoints of the interval $\Omega(x)$. Therefore, differentiating (3.3) with respect to x, we obtain the orthogonality relation

$$\int_{\Omega(x)} \partial_x\psi_1^{(x)}(y)\psi_1^{(x)}(y) \,\mathrm{d}y = 0.$$

Thus, calculating the derivatives in the first integral in (3.2) and using this orthogonality relation, we see that (3.2) becomes

$$\lambda \leq \int_{\Omega} \chi'(x)^2 \psi_1^{(x)}(y)^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega} \chi(x)^2 (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y \\ + \int_{\Omega} \chi(x)^2 (\partial_y \psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega} V(x,y)\chi(x)^2 \psi_1^{(x)}(y)^2 \,\mathrm{d}x\,\mathrm{d}y.$$

The eigenfunction $\psi_1^{(x)}(y)$ of $\mathcal{L}(x)$ has eigenvalue $\mu(x)$, and so we have the inequality

$$\lambda \leq \int_I \chi'(x)^2 \,\mathrm{d}x + \int_{\Omega} \chi(x)^2 (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}x\,\mathrm{d}y + \int_I \chi(x)^2 \mu(x)\,\mathrm{d}x.$$

From Definition 1.8 we know that

$$|\mu(x) - \mu| \leq L_2^{-2}.$$

Therefore, combining this with the bound on $\chi'(x)$ given in Definition 3.5, we obtain the desired upper bound on λ. ☐

As a result of Proposition 3.6, to obtain an upper bound on λ, we need to consider the derivative with respect to x of the eigenfunction $\psi_1^{(x)}(y)$. We will prove the following proposition:

Proposition 3.7. *Let x be fixed in the support of the cut-off function $\chi(x)$. Then,*

$$\int_{\Omega(x)} (\partial_x \psi_1^{(x)}(y))^2 \,\mathrm{d}y \leq C_1 L_2^{-2},$$

with the constant C_1 independent of x.

Remark 3.8. Combining Proposition 3.6 with Proposition 3.7 establishes

$$\lambda \leq \mu + C_1 L_2^{-2}$$

and finishes the proof of Proposition 3.2.

Proof of Proposition 3.7. Throughout the proof of this proposition, $x \in I$ will be fixed in the support of the cut-off function x, and all bounds that appear will be uniform in x. We will also suppress the dependence of certain functions on x where this simplifies the notation.

Since

$$\left(-\frac{d^2}{dy^2} + V(x,y)\right) \psi_1^{(x)}(y) = \mu(x)\psi_1^{(x)}(y),$$

differentiating with respect to x we find that for $y \in \Omega(x)$, we have

$$\left(-\frac{d^2}{dy^2} + V(x,y) - \mu(x)\right) \partial_x \psi_1^{(x)}(y) = \mu'(x)\psi_1^{(x)}(y) - \partial_x V(x,y)\psi_1^{(x)}(y), \tag{3.4}$$

where the notation $'$ denotes differentiation with respect to x. Although, for each fixed x, $\psi_1^{(x)}(y)$ is equal to zero at the endpoints on $\Omega(x)$, the function $\partial_x \psi_1^{(x)}(y)$ will not in general be zero here.

Therefore, we will also need to take into account its boundary values. For those x in the support of the cut-off function $\chi(x)$, we can write the two parts of $\partial\Omega$ below and above in the y-direction as $\{y = g_1(x)\}$ and $\{y = g_2(x)\}$, where $g_1(x)$

and $g_2(x)$ are convex and concave functions respectively. We set $\alpha = \partial_x \psi_1^{(x)}(g_2(x))$ and define

$$g(y) = \partial_x \psi_1^{(x)}(y) - \alpha. \tag{3.5}$$

Our aim is to find an expression for the function $g(y)$ using (3.4). To do this we need to make the following definitions (again suppressing the dependence on x throughout).

Definition 3.9. We define the function $F(y)$ by

$$F(y) = V(x,y) - \mu(x).$$

We know that $\mu(x) \leq 1 + C_1 L_1^{-2}$ and that $\min_y V(x,y) \leq \mu(x)$ for all x in the support of $\chi(x)$. This allows us to define the three points y_1, y_2 and y_3.

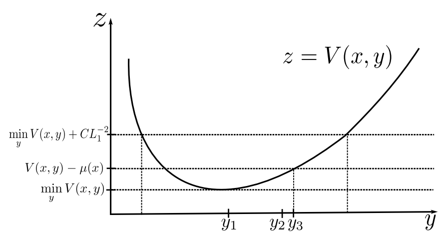

FIGURE 3. The points y_1, y_2 and y_3 from Definition 3.10.

Definition 3.10. We fix an absolute constant C. We define y_1 to be the middle point of the 'centre', where the centre is the interval on which $V(x,y) \leq \min_y V(x,y) + CL_1^{-2}$. We then choose $y_2 \geq y_1$ to be the largest value such that $[y_1, y_2]$ is contained in the middle half of the centre. Finally, we define $y_3 \geq y_2$ to be the value of y for which $F(y_3) = V(x, y_3) - \mu(x) = 0$. (See Figure 3.)

Definition 3.11. We set $\phi(y)$ to be the first eigenfunction of $\mathcal{L}(x)$, but this time normalised to be positive with a maximum of 1. Note that this function is equal to a multiple of $\psi_1^{(x)}(y)$ (where the multiple depends on the fixed value of x).

For $y \geq y_1$, we define the function $\tilde{\phi}(y)$ by

$$\tilde{\phi}(y) = \phi(y) \int_{y_1}^{y} \phi(t)^{-2}\,\mathrm{d}t.$$

We can now write an expression for the function $g(y)$.

Lemma 3.12. *Let $c_0(x)$ be the value such that*

$$g(y) - c_0(x)\psi_1^{(x)}(y) = 0$$

at $y = y_1$. Then, for $y \geq y_1$, the function $g(y)$ satisfies

$$g(y) - c_0(x)\psi_1^{(x)}(y) = \phi(y)\int_{y_1}^{y} \tilde{\phi}(t)G(x,t)\,\mathrm{d}t + \tilde{\phi}(y)\int_{y}^{g_2(x)} \phi(t)G(x,t)\,\mathrm{d}t, \tag{3.6}$$

where $G(x,y)$ is equal to

$$G(x,y) = \mu'(x)\psi_1^{(x)}(y) - \partial_x V(x,y)\psi_1^{(x)}(y) + (V(x,y) - \mu(x))\alpha. \tag{3.7}$$

Proof of Lemma 3.12. We see from the definition of $g(y)$ from (3.5) and the equation that $\partial_x \psi_1^{(x)}(y)$ satisfies in (3.4) that we have

$$\left(-\frac{d^2}{dy^2} + V(x,y) - \mu(x)\right) g(y) = \mu'(x)\psi_1^{(x)}(y) - \partial_x V(x,y)\psi_1^{(x)}(y) + (V(x,y) - \mu(x))\alpha.$$

The right-hand side of the above equation is equal to $G(x,y)$, so that

$$(\mathcal{L}(x) - \mu(x))(g(y) - c_0(x)\phi(y)) = G(x,y). \tag{3.8}$$

Since $\mathcal{L}(x)$ is a second order ordinary differential operator, to find an expression for $g(y)$ we will apply the method of variation of parameters to (3.8). From Definition 3.11, we know that

$$(\mathcal{L}(x) - \mu(x))\phi(y) = 0,$$

with $\phi(g_2(x)) = 0$. It is straightforward to check that the function $\tilde{\phi}(y)$ from Definition 3.11 also satisfies

$$(\mathcal{L}(x) - \mu(x))\tilde{\phi}(y) = 0,$$

for $y \geq y_1$, and is equal to 0 at $y = y_1$. Thus, since the function $g(y) - c_0(x)\phi(y)$ is equal to 0 at $y = y_1$ and $y = g_2(x)$, using (3.8) and variation of parameters, we have the desired expression. □

Looking at this expression for $g(y)$, we see that we will need to study how the magnitude of the functions $\phi(y)$ and $\tilde{\phi}(y)$ depends on the size of the potential $V(x,y)$ and its derivative with respect to x, $\partial_x V(x,y)$. Also, since $g(y) = \partial_x \psi_1^{(x)}(y) - \alpha$, where $\alpha = \partial_x \psi_1^{(x)}(g_2(x))$, we will also need to estimate the size of $\partial_x \psi_1^{(x)}(y)$ at the endpoints of the interval $\Omega(x)$.

3.1. **Properties of $\phi(y)$.** We first study the function $\phi(y)$. For x fixed in the support of I, let us set $L(x)$ to be the largest value such that $V(x,y)$ varies from its minimum value by $L(x)^{-2}$ on an interval in y of length at least $L(x)$. Then, as we remarked in the proof of Lemma 2.11, $L(x)$ is comparable to L_1. Thus, from Lemma 2.4(b), (d) in [17], we immediately get the following estimates on $\phi(y)$ (uniformly in x).

Lemma 3.13. *There exists an absolute constant C_1 such that the eigenfunction $\phi(y)$ (which we recall will depend on x) satisfies*

$$|\phi'(y)| \leq C_1/L_1 \text{ for all } y \in \Omega(x)$$

and

$$\phi(y) \leq C_1 e^{-c|y-y_1|/L_1},$$

where y_1 is the point in the 'centre' given in Definition 3.10.

This second inequality gives an L^∞ exponential decay estimate for $\phi(y)$ as we move away from the minimum of $V(x,y)$ on a length scale comparable to L_1. In particular, this means that the $L^2(\Omega(x))$ norm of $\phi(y)$ is bounded above by a multiple of $L_1^{1/2}$. (In fact, it follows from Lemma 2.4 in [17] that the $L^2(\Omega(x))$-norm also has a lower bound that is comparable to $L_1^{1/2}$.)

We now want to sharpen this L^∞ exponential decay estimate for $\phi(y)$ as $V(x,y)$ increases from its minimum.

Proposition 3.14. *Define the interval* J_k *by*

$$J_k = [t_k, t_{k+1}] = \{t \geq y_3 : \partial_t V(x,t) \in [2^{-k}, 2^{-k+1}]\}. \tag{3.9}$$

Then, for all $t_k \leq t \leq g_2(x)$,

$$\phi(t) \leq \phi(t_k) \exp(-(t-t_k)2^{-k/3}/10),$$

for all $y_3 \leq t \leq t_{k+1}$,

$$\phi(t_{k+1}) \leq \phi(t) \exp(-(t_{k+1}-t)2^{-k/3}/10),$$

and, for all $t \in J_k$,

$$\phi(t) \leq |\phi'(t)| 2^{k/3}.$$

For the interval $\tilde{J}_k$ *defined by*

$$\tilde{J}_k = [\tilde{t}_k, \tilde{t}_{k+1}] = \{t \geq y_3 : V(x,t) - \min_t V(x,t) \in [2^{-2k/3}, 2^{-2(k-1)/3}]\}, \tag{3.10}$$

we have the analogous bounds on $\phi(t)$.

Remark 3.15. We have the analogous decay estimates for $\phi(y)$ as we move away from the region where $V(x,y) \leq \min_y V(x,y) + L_1^{-2}$ in the other direction.

Remark 3.16. We recall that $y = y_3$ is the point where $V(x,y) - \mu(x) = 0$. Since $\min_y V(x,y) - \mu(x) \leq -cL_1^{-2}$, by convexity, J_k and $\tilde{J}_k$ are only non-empty for those k satisfying $2^k \leq CL_1^3$, for some absolute constant $C > 0$.

Proof of Proposition 3.14. The proposition follows from the key inequality given in the proof of Theorem A in [17],

$$\left|(\log \phi(t))'\right| = |\phi'(t)|/\phi(t) \geq 2^{-k/3}/10 \qquad \text{for all } t \in J_k.$$

Integrating this inequality from both $t = t_k$ and $t = t_{k+1}$ gives all of the desired estimates involving the intervals J_k.

By the definition of the intervals $\tilde{J}_k$, we have $V(x,t) - \mu(x) \geq 2^{-2k/3}$ for $t \in \tilde{J}_k$. Therefore, it is straightforward to obtain the same bounds for $(\log \phi(t))'$, and hence $\phi(t)$ itself on $\tilde{J}_k$, as for the intervals J_k. □

We now show to what extent $\phi'(y)$ inherits this exponential decay as we move away from the centre.

Proposition 3.17. *Let the intervals* J_k *be defined as in Proposition* 3.14. *Then, for all* $t \geq t_k$,

$$|\phi'(t)| \leq C|\phi'(t_k)| \exp(-c|t-t_k|2^{-k/3}),$$

for some absolute constants c *and* $C > 0$.

Proof of Proposition 3.17. The function $\phi(t)$ satisfies the equation

$$\phi''(t) = F(t)\phi(t),$$

with the function $F(t) = V(x,t) - \mu(x)$ as before. On the intervals J_k, we know that $t \geq y_3$, and so certainly $F(t) \geq 0$. Also, $\phi'(t) \leq 0$, and so this means that $|\phi'(t)|$ is decreasing. Thus, for $t \geq t_k$, we have

$$|\phi'(t)| \leq |\phi'(t_k)|.$$

If $|t - t_k| \leq 2^{k/3}$, then this is enough to establish the required bound.

Now suppose that $|t-t_k| \in [N2^{k/3},(N+1)2^{k/3}]$ for some $N \geq 1$. Then, by Proposition 3.14, we know that $\phi(t)$ satisfies

$$\phi(t) \leq C2^{k/3}|\phi'(t_k)|\exp(-cN2^{-k/3}).$$

In particular, $\phi(t)$ changes by at most $C2^{k/3}|\phi'(t_k)|\exp(-cN2^{-k/3})$, as t ranges over this interval of length $2^{k/3}$. Since $\phi'(t)$ is negative here, this gives us a bound on the integral of $|\phi'(t)|$ over this interval.

Moreover, as we noted above, by convexity, $|\phi'(t)|$ decreases as t increases. In particular, since the interval $[N2^{k/3},(N+1)2^{k/3}]$ has length $2^{k/3}$, this means that

$$|\phi'(t)| \leq C2^{k/3}|\phi'(t_k)|\exp(-cN2^{-k/3}).2^{-k/3} = C|\phi'(t_k)|\exp(-cN2^{-k/3}),$$

for t at the right endpoint of the interval. This concludes the proof of the proposition. □

It will often be important to measure the distance of a point (x,y) from the level sets $\{(x,y)\in\Omega : V(x,y) = 1+L_1^{-2}\}$.

Definition 3.18. Fix a large absolute constant C^*. Then, suppressing the dependence on x, let $y^* \geq y_1$ be the first point where $V(x,y) \geq 1 + C^*L_1^{-2}$.

We can now write an immediate corollary of Proposition 3.17.

Corollary 3.19. *For any $t \geq t_k$, we have the first derivative estimate*

$$|\phi'(t)| \leq CL_1^{-1}\exp(-c|t-t_k|2^{-k/3})\exp(-c|t_k-y^*|/L_1).$$

Proof of Corollary 3.17. We can apply Proposition 3.17 with t replaced by t_k and t_k replaced by y^* to obtain a bound on $|\phi'(t_k)|$ of the form

$$|\phi'(t_k)| \leq CL_1^{-1}\exp(-c|t_k-y^*|/L_1).$$

We then use this bound in the right-hand side of the estimate for $|\phi'(t)|$ in Proposition 3.17 to get the desired result. □

3.2. **Properties of $\tilde{\phi}(y)$.** From Lemma 3.12, we see that as well as $\phi(y)$, it will also be important to study the properties of $\tilde{\phi}(y)$ from Definition 3.11. We recall from Definition 3.10 that $y_2 \geq y_1$ is the largest value of y_2 such that $[y_1,y_2]$ is contained in the middle half of the 'centre', where $V(x,y) \leq \min_t V(x,t) + CL_1^{-2}$ and that $y_3 \geq y_2$ is the value of y for which $F(y_3) = V(x,y_3) - \mu(x) = 0$. We now prove:

Lemma 3.20. *The function $\tilde{\phi}(y)$ satisfies*

$$\tilde{\phi}(y) \leq C_1L_1,$$

for $y_1 \leq y \leq y_3$, and

$$\tilde{\phi}(y) \leq C_1L_1 + C_1|\phi'(y)|^{-1},$$

for $y_3 \leq y \leq g_2(x)$.

Proof of Lemma 3.20. We first consider the interval $[y_1,y_2]$. By the definition of the point y_2, Lemma 2.4 in [17] implies that we have an absolute lower bound on $\phi(t)$ for $t \in [y_1,y_2]$, and we know that this interval is of length comparable to L_1. Thus, for $y \in [y_1,y_2]$, we have

$$\tilde{\phi}(y) \leq C_1L_1\phi(y).$$

Before considering $y \in [y_2, y_3]$, we first assume that $y \geq y_3$. Here $F(y) \geq 0$, and so $|\phi'(y)|$ is decreasing ($\phi'(y)$ is becoming less negative). Therefore, for $t \in [y_3, y]$, we have the lower bound

$$\phi(t) \geq \phi(y) + |\phi'(y)|(y - t).$$

This gives us the bound

$$\int_{y_3}^{y} \phi(t)^{-2}\,\mathrm{d}t \leq C_1 \phi(y)^{-1} |\phi'(y)|^{-1}. \tag{3.11}$$

We now want to bound

$$\int_{y_2}^{y_3} \phi(t)^{-2}\,\mathrm{d}t.$$

Since $\phi''(y) = F(y)\phi(y)$, we have

$$\phi'(y) = \int_{\tilde{y}}^{y} F(t)\phi(t)\,\mathrm{d}t,$$

where $\phi(y)$ attains its maximum of 1 at $y = \tilde{y}$. For $t \in [y_2, y_3]$, $F(t) \leq 0$, and so $|\phi'(t)|$ is increasing from 0, $\phi(t)$ is decreasing from 1, and $|y_3 - y_2| \leq C_1 L_1$. Therefore, either $\phi(t)$ is bounded below by an absolute constant or else $|\phi'(y)| \geq C_1 L_1^{-1}$. This gives us the bound

$$\int_{y_2}^{y_3} \phi(t)^{-2}\,\mathrm{d}t \leq C_1 L_1. \tag{3.12}$$

Combining the bounds in (3.11) and (3.12) shows that

$$\tilde{\phi}(y) \leq C_1 L_1,$$

for $y \in [y_2, y_3]$, and

$$\tilde{\phi}(y) \leq C_1 L_1 + C|\phi'(y)|^{-1},$$

for $y \geq y_3$, as required. ☐

3.3. **An estimate for $\partial_x \psi_1^{(x)}(y)$ at the boundary.** We can now bound $\partial_x \psi_1^{(x)}(y)$ at the endpoints of the interval $\Omega(x)$. For each fixed x, $\psi_1^{(x)}(y)$ has zero boundary conditions on $\Omega(x)$. However, since the interval $\Omega(x)$ will in general depend on x, $\partial_x \psi_1^{(x)}(y)$ will not necessarily be zero when y is at the endpoints of $\Omega(x)$.

We recall from Definition 3.18 that $y^* \geq y_1$ is the first point where $V(x, y) \geq 1 + C^* L_1^{-2}$, for a fixed large constant C^*. The upper endpoint of the interval $\Omega(x)$ is equal to $g_2(x)$, and we set

$$M = g_2(x) - y^*, \tag{3.13}$$

which is the distance between the endpoint of $\Omega(x)$ and the region where the potential $V(x, y)$ is less than $1 + C^* L_1^{-2}$. We can prove a bound on $\partial_x \psi_1^{(x)}(g_2(x))$ in terms of M.

Proposition 3.21. *For $y = g_2(x)$ equal to the upper endpoint of the interval $\Omega(x)$, we have the bound*

$$|\alpha| = \left|\partial_x \psi_1^{(x)}(g_2(x))\right| \leq C L_2^{-1} L_1^{-3/2} (L_1 + M) \exp(-cML_1^{-1}).$$

We also have an analogous bound for y equal to the lower endpoint of $\Omega(x)$.

Proof of Proposition 3.21. We can view $\psi_1^{(x)}(y)$ as a function of two variables on the domain Ω, with $\psi_1^{(x)}(y)$ identically equal to 0 on $\partial\Omega$. In particular, for those x in the support of the cut-off function $\chi(x)$, we have written the upper boundary of Ω as the graph of the function $y = g_2(x)$, and so $\psi_1^{(x)}(g_2(x))$ is identically zero as a function of x. Differentiating this with respect to x gives

$$\partial_x\psi_1^{(x)}(g_2(x)) = -g_2'(x)\partial_y\psi_1^{(x)}(g_2(x)). \tag{3.14}$$

Thus, to obtain a bound on $\partial_x\psi_1^{(x)}(g_2(x))$, it is enough to consider $\partial_y\psi_1^{(x)}(y)$ and the slope of $\partial\Omega$ at $(x, g_2(x))$.

We remarked in the definition of $\phi(y)$ in Definition 3.11 that the eigenfunction $\psi_1^{(x)}(y)$ is equal to a multiple of $\phi(y)$. Since $\phi(y)$ has $L^2(\Omega(x))$-norm comparable to $L_1^{1/2}$, whereas $\psi_1^{(x)}(y)$ is $L^2(\Omega(x))$-normalised, this multiple is comparable to $L_1^{-1/2}$. Thus, by the bound on $\phi'(y)$ from Proposition 3.17, with 2^k comparable to L_1^3, we have the bound

$$\left|\partial_y\psi_1^{(x)}(g_2(x))\right| \leq CL_1^{-3/2}\exp\left(-cML_1^{-1}\right).$$

Therefore, by (3.14), to conclude the proof of the proposition it is enough to show that

$$|g_2'(x)| \leq C(L_1 + M)L_2^{-1}, \tag{3.15}$$

for an absolute constant $C > 0$. Recall the set $\Omega_{L_1^{-2}} = \{(x, y) \in \Omega : V(x, y) \leq 1 + L_1^{-2}\}$. This is a convex subset of Ω with height comparable to L_1 in the y-direction and length comparable to $\tilde{L}_1$ in the x-direction. Moreover, for x fixed in the support of $\chi(x)$, we are at a distance at least comparable to L_2 from the left and right ends of $\Omega_{L_1^{-2}}$. Therefore, if we write the upper boundary of $\Omega_{L_1^{-2}}$ of this set as the graph of a function $y = v(x)$, then certainly we have the derivative bound

$$|v'(x)| \leq CL_1L_2^{-1}.$$

In particular, if the distance M is bounded above by a multiple of L_1, then by convexity, the part of $\partial\Omega$ for x contained in the support of $\chi(x)$ has slope bounded by a multiple of $L_1L_2^{-1}$. This gives the desired bound for $g_2'(x)$ in (3.15),

$$|g_2'(x)| \leq CL_1L_2^{-1}.$$

If the distance M is large compared to L_1, then the domain Ω is convex and contains an ellipse of height comparable to M in the y-direction and length comparable to L_2 in the x-direction. Thus, the part of $\partial\Omega$ with x in the support of $\chi(x)$ has slope bounded by a multiple of ML_2^{-1}. Again we get a bound for $g_2'(x)$,

$$|g_2'(x)| \leq CML_2^{-1},$$

which implies the bound in (3.15).

This establishes the estimate in (3.15) in all cases and completes the proof of the proposition. □

We have now established the properties of the functions $\phi(y)$ and $\tilde{\phi}(y)$ together with the bound required on $\alpha = \partial_x\psi_1^{(x)}(g_2(x))$. Thus, we return to the expression for $g(y) = \partial_x\psi_1^{(x)}(y) - \alpha$ that we derived in Lemma 3.12 to obtain the desired bound on $\partial_x\psi_1^{(x)}(y)$.

Proposition 3.22. *We have the pointwise bound*

$$\left|\partial_x \psi_1^{(x)}(y) - c_0(x)\psi_1^{(x)}(y)\right| \leq F_1(y) + F_2(y)$$

for all $y \in \Omega(x)$ with $y \geq y_1$. Here $F_1(y)$ is a positive function on $\Omega(x)$, with a maximum comparable to $L_2^{-1}L_1^{-1/2}$ and decaying exponentially from this maximum on a length scale comparable to L_1 as y moves away from the interval where $V(x,y) \leq 1 + L_1^{-2}$. The function $F_2(y)$ is also a positive function on $\Omega(x)$, with a maximum comparable to $L_2^{-1}L_1^{-1/2}$, but it decays exponentially from this maximum within each interval J_k from (3.9) *on a length scale comparable to $2^{k/3}$. We also have the analogous exponential decay estimate on the corresponding intervals as we move away from the 'centre' region where $V(x,y) \leq 1 + L_1^{-2}$ in the opposite direction with $y \leq y_1$.*

Before we prove this proposition, let us show how it implies the $L^2(\Omega(x))$-bound on $\partial_x \psi_1^{(x)}(y)$ given in Proposition 3.7: Since $\psi_1^{(x)}(y)$ is $L^2(\Omega(x))$-normalised, the orthogonality of $\psi_1^{(x)}(y)$ and its derivative in x implies

$$c_0(x) = \int_{\Omega(x)} \left(c_0(x)\psi_1^{(x)}(y) - \partial_x \psi_1^{(x)}(y)\right) \psi_1^{(x)}(y)\,\mathrm{d}y.$$

Using the bound on $c_0(x)\psi_1^{(x)}(y) - \partial_x \psi_1^{(x)}(y)$ in Proposition 3.22 we obtain

$$|c_0(x)| \leq C_1 L_2^{-1}L_1^{-1/2} \int_{\Omega(x)} \psi_1^{(x)}(y)\,\mathrm{d}y \leq C_1 L_2^{-1}, \tag{3.16}$$

where the final inequality holds since $\psi_1^{(x)}(y)$ has $L^2(\Omega(x))$-norm equal to 1, and decays exponentially away from its maximum on a length scale comparable to L_1.

Combining this bound on $c_0(x)$ in (3.16) with Proposition 3.22, we see that $\partial_x \psi_1^{(x)}(y)$ can be bounded by functions $F_1(y) + F_2(y)$ with the same properties as in the statement of Proposition 3.22. This gives us an $L^2(\Omega(x))$-bound on $\partial_x \psi_1^{(x)}(y)$ of the form

$$\int_{\Omega(x)} \left(\partial_x \psi_1^{(x)}(y)\right)^2 \mathrm{d}y \leq C_1 L_2^{-2} + \sum_{2^k \leq CL_1^3} 2^{k/3} L_2^{-2} L_1^{-1} \leq C_1 L_2^{-2}.$$

This completes the proof of Proposition 3.7. □

Since Proposition 3.7 implies the desired upper bound on the eigenvalue λ in Proposition 3.2, we just need to prove Proposition 3.22.

Proof of Proposition 3.22. From Proposition 3.21 we know that

$$|\partial_x \psi_1^{(x)}(y) - g(y)| = |\partial_x \psi_1^{(x)}(g_2(x))| = |\alpha| \leq C L_2^{-1} L_1^{-3/2}(L_1 + M)\exp\left(-cML_1^{-1}\right),$$

and this bound has the same properties as the function $F_1(y)$ in the statement of the proposition. Therefore, to prove Proposition 3.22, it is enough to show that

$$g(y) - c_0(x)\psi_1^{(x)}(y)$$

has the desired bounds.

To do this, we want to bound the right-hand side of (3.6), which contains the functions $\phi(y)$, $\tilde{\phi}(y)$ together with $G(x,y)$. The two remaining functions which we have not discussed above are the functions $\mu'(x)$ and $\partial_x V(x,y)$ appearing in

$G(x,y)$. Therefore, let us prove two simple lemmas concerning these functions, and then we will be in a position to bound (3.6).

Lemma 3.23. *Let x be in the support of the cut-off function $\chi(x)$. Then, we have the bound*

$$|\mu'(x)| \leq C_1 L_2^{-3},$$

for an absolute constant $C_1 > 0$.

Proof of Lemma 3.23. We recall from Lemma 2.13 that the function $\mu(x)$ is a convex function of x. Moreover, by the definition of the parameter L_2, we know that $\mu(x)$ varies by L_2^{-2} for x in an interval of length at least L_2. Since the support of $\chi(x)$ is contained within the middle half of this interval, we immediately obtain the required bound by convexity. □

Lemma 3.24. *Let x be in the support of $\chi(x)$, and as in Definition* 3.18 *let $y = y^*$ be the first point where $V(x,y) \geq 1 + C^* L_1^{-2}$. Then, for $y \geq y^*$,*

$$|\partial_x V(x,y)| \leq C_1(|y - y^*| + L_1) L_2^{-1} |\partial_y V(x,y)|,$$

and for $y_1 \leq y \leq y^*$,

$$|\partial_x V(x,y)| \leq C_1 L_1^{-2} L_2^{-1} + C_1 L_1 L_2^{-1} |\partial_y V(x,y)|.$$

Proof of Lemma 3.24. Given c, let $y = f(x)$ be a parameterisation of the upper part of the level set $\{(x,y) \in \Omega : V(x,y) = c\}$. Differentiating this with respect to x, we see that

$$\partial_x V(x, f(x)) = -f'(x) \partial_y V(x, f(x)). \tag{3.17}$$

Assume first that $y = f(x) \geq y^*$. The sublevel set $\{(x,y) \in \Omega : V(x,y) \leq 1 + C^* L_1^{-2}\}$ is convex with height comparable to L_1 in the y-direction and length comparable to $\tilde{L}_1$ in the x-direction, and x is at distance comparable to L_2 from the ends of this set. Thus, by the convexity of the sublevel sets, we certainly have a bound on the slope of

$$|f'(x)| \leq C_1(|y - y^*| + L_1) L_2^{-1}.$$

Using this bound in the right-hand side of (3.17) gives the desired bound for $y \geq y^*$.

We now suppose that $y_1 \leq y = f(x) \leq y^*$. If y is in the middle half of the interval $\{t : V(x,t) \leq 1 + L_1^{-2}\}$, then we certainly have the bound

$$|\partial_x V(x, f(x))| \leq C_1 L_1^{-2} L_2^{-1},$$

by the convexity of the potential $V(x,y)$. For the remaining points $(x, f(x))$ of interest, we can again use the shape of the level set to obtain the desired bound

$$|\partial_x V(x, f(x))| \leq C_1 L_1^{-2} L_2^{-1} + C_1 L_1 L_2^{-1} |\partial_y V(x, f(x))|.$$

This is because for these points we can find a direction $\mathbf{e}$ such that the directional derivative of V at $(x, f(x))$ is bounded by $L_1^{-2} L_2^{-1}$, and this direction makes an angle comparable to $L_1 L_2^{-1}$ with the x-axis. □

Combining Lemmas 3.23 and 3.24, we see from (3.7) that

$$\begin{aligned}|G(x,t)| &\leq C_1 L_1^{-2} L_2^{-1} \psi_1^{(x)}(t) + C_1(|t-y^*| + L_1) L_2^{-1} |\partial_t V(x,t)| \psi_1^{(x)}(t) \\ &\quad + |V(x,t) - \mu(x)|\alpha \\ &\leq C_1 L_1^{-5/2} L_2^{-1} \phi(t) + C_1(|t-y^*| + L_1) L_1^{-1/2} L_2^{-1} |\partial_t V(x,t)| \phi(t) \\ &\quad + |V(x,t) - \mu(x)|\alpha. \end{aligned} \tag{3.18}$$

The final inequality comes from

$$\psi_1^{(x)}(t) \leq C_1 L_1^{-1/2} \phi(t),$$

which holds since $\psi_1^{(x)}(t)$ is $L^2(\Omega(x))$-normalised, whereas $\phi(t)$ has $L^2(\Omega(x))$-norm comparable to $L_1^{1/2}$.

Everything is now set up to show that the two integrals in (3.6) have the bounds required in the statement of Proposition 3.22.

3.4. A bound on $\phi(y) \int_{y_1}^{y} \tilde{\phi}(t) G(x,t)\,\mathrm{d}t$.

We start by considering the first integral in (3.6). Using (3.18), it is enough to bound

$$\begin{aligned}\phi(y) \int_{y_1}^{y} \tilde{\phi}(t) \Big(& C_1 L_1^{-5/2} L_2^{-1} \phi(t) + C_1(|t-y^*| + L_1) L_1^{-1/2} L_2^{-1} |\partial_t V(x,t)| \phi(t) \\ & + |V(x,t) - \mu(x)|\alpha \Big)\,\mathrm{d}t. \end{aligned} \tag{3.19}$$

We now bound the three terms in equation (3.19).

Lemma 3.25. *We have a bound on the first term in* (3.19),

$$\phi(y) \int_{y_1}^{y} \tilde{\phi}(t) L_1^{-5/2} L_2^{-1} \phi(t)\,\mathrm{d}t \leq C_1 L_1^{-1/2} L_2^{-1}.$$

Remark 3.26. We will see in the proof of the lemma that the function decays exponentially from its maximum away from the region where $V(x,y) \leq 1 + L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 3.22.

Proof of Lemma 3.25. By Lemma 3.20, we can bound the left-hand side by

$$\phi(y) \int_{y_1}^{y_3} C_1 L_1 L_1^{-5/2} L_2^{-1} \phi(t)\,\mathrm{d}t + \phi(y) \int_{y_3}^{y} (C_1 L_1 + C_1 |\phi'(t)|^{-1}) L_1^{-5/2} L_2^{-1} \phi(t)\,\mathrm{d}t.$$

Using Proposition 3.14 we have the bound $\phi(t) \leq 2^{k/3} |\phi'(t)| \leq C_1 L_1 |\phi'(t)|$ for $t \in J_k$, and so these integrals can be bounded by

$$C_1 \phi(y) \int_{y_1}^{y} L_2^{-1} L_1^{-3/2}\,\mathrm{d}t. \tag{3.20}$$

The eigenfunction $\phi(y)$ has a maximum of 1 and decays exponentially away from this maximum on a length scale comparable to L_1. Thus, we can bound (3.20) by $C_1 L_1^{-1/2} L_2^{-1}$ as required, and it also has the decay properties of the function $F_1(y)$. □

Lemma 3.27. *We have a bound on the second term in* (3.19),

$$\phi(y) \int_{y_1}^{y} \tilde{\phi}(t)(|t-y^*| + L_1) L_1^{-1/2} L_2^{-1} |\partial_t V(x,t)| \phi(t)\,\mathrm{d}t \leq C_1 L_1^{-1/2} L_2^{-1}.$$

Remark 3.28. We will again see in the proof of the lemma that the function decays exponentially from its maximum on a length scale comparable to $2^{k/3}$ within each interval J_k. Therefore we can include this term in the function $F_2(y)$ in the statement of Proposition 3.22.

Proof of Lemma 3.27. We first consider the part of this integral over $[y_1, y_3]$. Here, $\tilde{\phi}(t) \leq C_1 L_1$, and by the convexity of the potential

$$\int_{y_1}^{y_3} |\partial_t V(x,t)| \, \mathrm{d}t \leq 2C_1 L_1^{-2}.$$

Therefore, we immediately obtain a bound of $C_1 L_1^{-1/2} L_2^{-1} \phi(y)$. This is certainly at most $C_1 L_1^{-1/2} L_2^{-1}$, and by the properties of $\phi(y)$ it also has the decay properties of the function $F_2(y)$.

We now consider the part of the integral over $[y_3, y]$. Let y be in the interval J_{k^*} for some k^*, where as usual the intervals J_k are as in (3.9). We decompose the integral between y_3 and y as an integral over the relevant intervals J_k where $k \geq k^*$.

By Proposition 3.14,

$$\phi(t) \leq |\phi'(t)| 2^{k/3},$$

and so using the bound on $\tilde{\phi}(t)$ from Lemma 3.20, to estimate the contribution to the integral from J_k, we have to bound

$$\begin{aligned}(3.21)\quad & \phi(y) \int_{J_k} \phi(t)(L_1 + |\phi'(t)|^{-1})(|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} |\partial_t V(x,t)| \, \mathrm{d}t \\ & \leq C_1 \phi(y) \int_{J_k} 2^{k/3} |\phi'(t)| (L_1 + |\phi'(t)|^{-1})(|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} 2^{-k} \, \mathrm{d}t \\ & \leq C_1 2^{-2k/3} \phi(y) \int_{J_k} (|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} \, \mathrm{d}t.\end{aligned}$$

Using Proposition 3.14 again, we find that for any $k \geq k^*$,

$$\phi(y) \leq \phi(t_{k^*}) \exp(-(y - t_{k^*}) 2^{-k^*/3}/10) \leq 2^{k^*/3} |\phi'(t_{k^*})| \exp(-(y - t_{k^*}) 2^{-k^*/3}/10).$$

By Corollary 3.19, we can bound the factor of $|\phi'(t_{k^*})|$ as

$$|\phi'(t_{k^*})| \leq C L_1^{-1} \exp(-c|t - t_k|/2^{k/3}) \exp(-c|t_k - y^*|/L_1).$$

Inserting these estimates into the integral in (3.21) and integrating over the interval J_k, we have the bound

$$C \exp(-(y - t_{k^*}) 2^{-k^*/3}/10) 2^{k^*/3} 2^{-k/3} L_1^{-1/2} L_2^{-1}.$$

Summing over $k \geq k^*$ gives a bound for the integral over $y_3 \leq t \leq y$ of the form

$$C_1 L_1^{-1/2} L_2^{-1} \exp(-(y - t_{k^*}) 2^{-k^*/3}/10).$$

Note that this quantity is bounded by a multiple of $L_1^{-1/2} L_2^{-1}$ and has the required decay properties that we can include it in the function $F_2(y)$. □

Lemma 3.29. *We have a bound on the final term in* (3.19),

$$\phi(y) \int_{y_1}^{y} \tilde{\phi}(t) |V(x,t) - \mu(x)| |\alpha| \, \mathrm{d}t \leq C_1 L_1^{-1/2} L_2^{-1}. \tag{3.22}$$

Remark 3.30. We will see that the function decays exponentially from its maximum away from the region where $V(x,y) \le 1 + L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 3.22.

Proof of Lemma 3.29. For the part of the integral in (3.22) over $[y_1, y_3]$, we know that $|V(x,t) - \mu(x)| \le C_1 L_1^{-2}$, $|y_3 - y_1| \le C_1 L_1$ and $\tilde{\phi}(t) \le C_1 L_1$. Combining this with the bound on α from Proposition 3.21 immediately gives us the desired bound of $L_2^{-1} L_1^{-1/2} \exp(-cML_1^{-1})$ for this part of the integral in (3.22).

For $t \ge y_3$, we decompose $[y_3, y]$ into the intervals $\tilde{J}_k$ given in (3.10). Since $\mu(x) \ge \min_t V(x,t)$ on $\tilde{J}_k$ we know that

$$|V(x,t) - \mu(x)| \le 2^{-2k/3}.$$

So, for the part of the integral in (3.22) over $\tilde{J}_k$, combining this with the bound on α and the usual bound on $\tilde{\phi}(t)$ from Lemma 3.20, we have

$$\begin{aligned} &\phi(y) \int_{\tilde{J}_k} \tilde{\phi}(t) |V(x,t) - \mu(x)| |\alpha| \, \mathrm{d}t \\ &\le C_1 L_2^{-1} L_1^{-3/2} \phi(y) \int_{\tilde{J}_k} (L_1 + |\phi'(t)|^{-1}) 2^{-2k/3} (L_1 + M) \exp(-cML_1^{-1}) \, \mathrm{d}t. \end{aligned} \tag{3.23}$$

Similarly to the proof of Lemma 3.27, let us assume that $y \in \tilde{J}_{k^*}$ for some k^*. Then, using Proposition 3.14 and then Proposition 3.17 twice, we obtain

$$\begin{aligned} \phi(y) &\le C_1 2^{k^*/3} |\phi'(y)| \le C_1 2^{k^*/3} |\phi'(t_{k^*})| \exp\left(-c|y - t_{k^*}| 2^{-k^*/3}\right) \\ &\le C_1 2^{k^*/3} |\phi'(t)| \exp\left(-c|y - t_{k^*}| 2^{-k^*/3}\right) \exp\left(-c|t_{k^*} - t| 2^{-k/3}\right). \end{aligned}$$

Inserting this bound for $\phi(y)$ into the right-hand side of (3.23) and integrating over $\tilde{J}_k$ gives us the bound for the part of the integral over $\tilde{J}_k$ of

$$C_1 2^{k^*/3} 2^{-k/3} L_2^{-1} L_1^{-1/2} \exp(-cML_1^{-1}/2).$$

We finally sum over those k with $k \ge k^*$ to get the desired bound on the part of the integral (3.22) with $y_3 \le t \le y$. □

Combining Lemmas 3.25, 3.27 and 3.29, we see that the part of $g(y) - c_0(x)\psi_1^{(x)}(y)$ coming from the first term in (3.6) has the bounds required in Proposition 3.22.

Therefore to finish the proof of Proposition 3.22 we need to establish the analogous estimates for the second integral in (3.6).

3.5. **A bound on $\tilde{\phi}(y) \int_y^{g_2(x)} \phi(t) G(x,t) \, \mathrm{d}t$.** The estimates for the various parts of this integral will be similar to the estimates we used above. However, there will be places where we have to use different methods to obtain the desired bounds. We again use (3.18) to bound the integral by

$$\begin{aligned} \tilde{\phi}(y) \int_y^{g_2(x)} \phi(t) \Big(&C_1 L_1^{-5/2} L_2^{-1} \phi(t) + C_1 (|t - y^*| + L_1) L_1^{-1/2} L_2^{-1} |\partial_t V(x,t)| \phi(t) \\ &+ |V(x,t) - \mu(x)| \alpha \Big) \, \mathrm{d}t, \end{aligned} \tag{3.24}$$

and we split this into three terms that we need to estimate.

Lemma 3.31. *We have a bound on the first term in* (3.24),

$$\tilde{\phi}(y)\int_y^{g_2(x)}\phi(t)L_1^{-5/2}L_2^{-1}\phi(t)\,\mathrm{d}t \le C_1 L_1^{-1/2}L_2^{-1}.$$

Remark 3.32. The function also decays exponentially from its maximum away from the region where $V(x,y)\le 1+L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 3.22.

Proof of Lemma 3.31. We know that $\tilde{\phi}(y)\le\tilde{\phi}(t)$ and $\phi(t)$ decays exponentially on a length scale comparable to L_1 as we move away from y^*. Therefore, this bound follows in a very straightforward manner. □

Before bounding the second term in (3.24), we first want to establish the following lemma.

Lemma 3.33. *For any $\tilde{y}\ge y_3$, we have the bound*

$$\int_{\tilde{y}}^{g_2(x)}\phi(t)^2\partial_t V(x,t)\,\mathrm{d}t \le (\phi'(\tilde{y}))^{1/2}.$$

Proof of Lemma 3.33. To prove this lemma, we will consider the 'energy'

$$\mathcal{E}(t) = (\phi'(t))^2 - F(x,t)\phi(t)^2. \tag{3.25}$$

Differentiating $\mathcal{E}(t)$ we find that

$$\mathcal{E}'(t) = 2\phi'(t)(\phi''(t)-F(x,t)\phi(t)) - \partial_t F(x,t)\phi(t)^2 = -\partial_t F(x,t)\phi(t)^2,$$

where the final equality holds because $\phi''(t)=F(x,t)\phi(t)$. Since $F(x,t)=V(x,t)-\mu(x)$, we have

$$\partial_t F(x,t) = \partial_t V(x,t),$$

and so

$$\int_{\tilde{y}}^{g_2(x)}\partial_t V(x,t)\phi(t)^2\,\mathrm{d}t = -\int_{\tilde{y}}^{g_2(x)}\mathcal{E}'(t)\,\mathrm{d}t = \mathcal{E}(\tilde{y})-\mathcal{E}(g_2(x)).$$

Since $F(x,t)\ge 0$ for $t\ge y_3$, we know that

$$\mathcal{E}(\tilde{y}) = (\phi'(\tilde{y}))^2 - F(x,\tilde{y})\phi(\tilde{y})^2 \le (\phi'(\tilde{y}))^2.$$

Thus, to finish the proof of the lemma we need to show that $\mathcal{E}(g_2(x))\ge 0$. We know that

$$\mathcal{E}(g_2(x)) \ge -F(x,g_2(x))\phi(g_2(x))^2$$

and that $\phi(g_2(x))=0$. However, we are not assuming that the potential $V(x,y)$ remains bounded as y approaches $g_2(x)$, and so we cannot immediately deduce that

$$F(x,g_2(x))\phi(g_2(x))^2 = 0.$$

Instead we argue as follows. The function $\phi'(t)$ is in $L^\infty(\Omega(x))$, and this has two consequences. First, the eigenfunction $\phi(y)$ decays at least linearly to 0 at $y=g_2(x)$. It also means that $\phi''(y)$ is in $L^1(\Omega(x))$, and hence $F(x,y)\phi(y)$ is in $L^1(\Omega(x))$. This means that $F(x,y)\phi(y)$ cannot grow as fast as $(y-g_2(x))^{-1}$ as we approach the boundary and so

$$\liminf_{y\to g_2(x)}\phi(y)F(x,y)\phi(y) = 0.$$

This implies that $\mathcal{E}(g_2(x))\ge 0$ and concludes the proof of the lemma. □

Remark 3.34. This energy $\mathcal{E}(t)$ has also been used in [16] in their proof of Theorem 2.1(B). There they obtain a pointwise estimate comparing the first eigenfunction of the two-dimensional domain with the first eigenfunction of the associated ordinary differential operator.

We can now bound the contribution from the second term in $G(x,t)$.

Lemma 3.35. *We have a bound on the second term in* (3.24),

$$\tilde{\phi}(y)\int_y^{g_2(x)} \phi(t)^2(|t-y^*|+L_1)L_1^{-1/2}L_2^{-1}|\partial_t V(x,t)|\,\mathrm{d}t \leq C_1 L_1^{-1/2}L_2^{-1}.$$

Remark 3.36. We will see in the proof that the function also decays exponentially from its maximum away from the region where $V(x,y)\leq 1+L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 3.22.

Proof of Lemma 3.35. If $y\leq y_3$, we first consider the part of the integral where t lies in the interval $[y,y_3]$ of length at most C_1L_1. In this case, we know that $\tilde{\phi}(y)\leq C_1L_1$, and the estimates follow easily.

For $t\geq y_3$, we first consider the integral between $\tilde{y}$ and $\tilde{y}+L_1$, where $\tilde{y}$ is some point with $\tilde{y}\geq y_3$ and $\tilde{y}\geq y$. Since $\partial_t V(x,t)\geq 0$ here, we have

$$\tilde{\phi}(y)\int_{\tilde{y}}^{\tilde{y}+L_1} \phi(t)^2(|t-y^*|+L_1)L_2^{-1}|\partial_t V(x,t)|L_1^{-1/2}\,\mathrm{d}t$$

$$\leq C_1\tilde{\phi}(y)(|\tilde{y}-y^*|+L_1)L_2^{-1}L_1^{-1/2}\int_{\tilde{y}}^{g_2(x)}\phi(t)^2\partial_t V(x,t)\,\mathrm{d}t. \tag{3.26}$$

Applying Lemma 3.33, we can bound the right-hand side of (3.26) by

$$C_1\tilde{\phi}(y)(|\tilde{y}-y^*|+L_1)L_2^{-1}L_1^{-1/2}(\phi'(\tilde{y}))^2.$$

Lemma 3.20 shows that

$$\tilde{\phi}(\tilde{y})|\phi'(\tilde{y})|\leq C_1,$$

and using Proposition 3.17 with 2^k comparable to L_1^3, we have the derivative bound

$$|\phi'(\tilde{y})|\leq C_1L_1^{-1}\exp\left(-c|\tilde{y}-y^*|/L_1\right).$$

Thus the right-hand side of (3.26) has the bound

$$C_1L_2^{-1}L_1^{-1/2}\exp\left(-c|\tilde{y}-y^*|/L_1\right).$$

Summing over $\tilde{y}$ between y and $g_2(x)$ at intervals of length comparable to L_1 then gives the desired bound. □

We finally have to bound the contribution from the third term in $G(x,t)$.

Lemma 3.37. *We have a bound on the third term in* (3.24),

$$\tilde{\phi}(y)\int_y^{g_2(x)}\phi(t)|V(x,t)-\mu(x)||\alpha|\,\mathrm{d}t\leq C_1L_1^{-1/2}L_2^{-1}. \tag{3.27}$$

Remark 3.38. As for the previous two lemmas, we will see in the proof that the function also decays exponentially from its maximum away from the region where $V(x,y)\leq 1+L_1^{-2}$ on a length scale comparable to L_1. Therefore we can include this term in the function $F_1(y)$ in the statement of Proposition 3.22.

Proof of Lemma 3.37. From Proposition 3.21 we have the estimate on α of the form

$$|\alpha| = |\partial_x \psi_1^{(x)}(g_2(x))| \leq CL_2^{-1}L_1^{-3/2}(L_1 + M)\exp(-cML_1^{-1}). \tag{3.28}$$

For the part of the integral in (3.19) for t between y and y_3, we know that $|V(x,t) - \mu(x)|$ is at most $C_1L_1^{-2}$ and $\tilde{\phi}(y) \leq C_1L_1$. Thus, we immediately get a bound of

$$C_1L_1^{-1/2}L_2^{-1}\exp(-cML_1^{-1})$$

for this part.

For $t \geq y_3$, we know that $F(x,t) = V(x,t) - \mu(x) \geq 0$. Thus, we can bound this part of the integral in (3.19) by

(3.29)

$$C_1\tilde{\phi}(y)L_2^{-1}L_1^{-3/2}(L_1 + M)\exp(-cML_1^{-1})\int_{\max\{y_3,y\}}^{g_2(x)} \phi(t)(V(x,t) - \mu(x))\,\mathrm{d}t.$$

Since

$$\phi''(t) = (V(x,t) - \mu(x))\phi(t)$$

and $|\phi'(t)|$ is decreasing for $t \geq y_3$, we find that (3.29) can be bounded by

$$C_1\tilde{\phi}(y)L_2^{-1}L_1^{-3/2}(L_1 + M)\exp(-cML_1^{-1})|\phi'(y)| \leq C_1L_1^{-1/2}L_2^{-1}\exp(-cML_1^{-1}/2),$$

where the last inequality comes from Lemma 3.20 as usual. This concludes the proof of the lemma. □

By the bounds on the right-hand side in Lemmas 3.25, 3.27, 3.29 and Lemmas 3.31, 3.35, 3.37, we have shown that

$$\left|g(y) - c_0(x)\psi_1^{(x)}(y)\right| \leq F_1(y) + F_2(y). \tag{3.30}$$

Here the functions $F_1(y)$ and $F_2(y)$ have the desired properties from the statement of Proposition 3.22. As we remarked at the beginning of the proof, by the bound on α that we obtained in Proposition 3.21, the estimate in (3.30) is sufficient to conclude the proof of Proposition 3.22. □

After the statement of Proposition 3.22, we showed that this implied Proposition 3.7 and the bound

$$\int_{\Omega(x)} \left(\partial_x\psi_1^{(x)}(y)\right)^2 \,\mathrm{d}y \leq C_1L_2^{-2}.$$

Combining this with the estimate on the first eigenvalue λ from Proposition 3.6,

$$\lambda \leq \mu + \int_\Omega \chi(x)^2\left(\partial_x\psi_1^{(x)}(y)\right)^2 \,\mathrm{d}x\,\mathrm{d}y + C_1L_2^{-2},$$

gives

$$\lambda \leq \mu + CL_2^{-2}.$$

This completes the proof of the upper bound on λ in Proposition 3.2. □

By Propositions 3.1 and 3.2, we see that the first eigenvalue λ satisfies

$$\mu \leq \lambda \leq \mu + CL_2^{-2},$$

and so we have established Theorem 1.13.

4. $L^2(\Omega)$ BOUNDS FOR THE FIRST EIGENFUNCTION $u(x, y)$

Now that we have established the improved eigenvalue bound on λ in Theorem 1.13, we want to use it to study the corresponding eigenfunction $u(x, y)$. We recall that $u(x, y)$ is normalised to be positive inside Ω with a maximum of 1. Our main aim is to prove Theorem 1.10 and show that the level sets $\{(x, y) \in \Omega : u(x, y) = c\}$ have lengths comparable to L_2 and L_1 in the x- and y-directions, respectively, whenever c is bounded away from 0 and 1.

Before we prove this theorem, in this section we will first establish an $L^2(\Omega)$-bound for $u(x, y)$. More precisely, we will prove the following proposition.

Proposition 4.1. *There exists an absolute constant $C > 0$ such that*

$$\int_{\Omega} u(x, y)^2 \,\mathrm{d}x \,\mathrm{d}y \leq CL_1L_2.$$

Remark 4.2. Note that this $L^2(\Omega)$ bound is consistent with the shape of the level sets described in Theorem 1.10. We will use the eigenvalue bound on λ from Theorem 1.13 in a critical way in the proof.

Proof of Proposition 4.1. The function $H(x)$ is given by $H(x) = \int_{\Omega(x)} u(x, y)^2 \,\mathrm{d}y$. We first study the rate at which the function $H(x)$ decays from its maximum, and we will then prove an estimate for the maximum of $H(x)$.

To study the decay of $H(x)$, we prove a Carleman-type inequality. For the convex function $\mu(x)$ let x^* be a point where it achieves its minimum of μ^*. We now prove:

Proposition 4.3. *For any x we have the differential inequality*

$$H''(x) \geq 2(\mu(x) - \lambda)H(x).$$

In particular, for $|x - x^*| \geq CL_2$, with C a sufficiently large absolute constant, we have

$$H''(x) \geq \frac{1}{L_2^2}H(x).$$

Remark 4.4. This type of Carleman inequality has been used frequently in the study of the ground state Dirichlet eigenfunction of Schrödinger operators. For example, in Lemma 3.9 [16] it has been used to establish the exponential decay of the first Fourier mode of the ground state eigenfunction of the two-dimensional convex domain. This first Fourier mode comes from a Fourier decomposition of the cross-section of the domain at each fixed x. A similar argument has also been used in Section 3 of [11] to study the decay of the L^2-norm of the cross-section at x of the eigenfunction for a two-dimensional domain which is periodic in the x-direction and with height in the y-direction depending on a small parameter $\epsilon > 0$.

Proof of Proposition 4.3. The eigenfunction $u(x, y)$ is equal to 0 when y is at the endpoints of the interval $\Omega(x)$. This allows us to differentiate $H(x)$ twice and pass the derivative inside the integral to obtain

$$\begin{aligned} H''(x) &= 2\int_{\Omega(x)} u(x, y)\partial_x^2 u(x, y) + (\partial_x u(x, y))^2 \,\mathrm{d}y \\ &= 2\int_{\Omega(x)} (V(x, y) - \lambda)u(x, y)^2 - u(x, y)\partial_y^2 u(x, y) + (\partial_x u(x, y))^2 \,\mathrm{d}y. \end{aligned}$$

Integrating by parts one time in y in the term containing a factor of $\partial_y^2 u(x,y)$, we can rewrite this as

$$
\begin{aligned}
H''(x) &= 2\int_{\Omega(x)} (V(x,y)-\lambda)u(x,y)^2 + (\partial_y u(x,y))^2 + (\partial_x u(x,y))^2 \,\mathrm{d}y \\
&\geq 2\int_{\Omega(x)} (V(x,y)-\lambda)u(x,y)^2 + (\partial_y u(x,y))^2 \,\mathrm{d}y. \qquad (4.1)
\end{aligned}
$$

Since $\mu(x)$ is the first eigenvalue of the operator $\mathcal{L}(x) = -\frac{d^2}{dy^2} + V(x,y)$ and $u(x,\cdot)$ vanishes at the endpoints of $\Omega(x)$, (4.1) gives us the lower bound

$$
H''(x) \geq 2(\mu(x)-\lambda)\int_{\Omega(x)} u(x,y)^2 \,\mathrm{d}y = 2(\mu(x)-\lambda)H(x). \tag{4.2}
$$

Since $\mu(x^*) = \mu^*$ is the minimum value of the function $\mu(x)$, by the definition of the length scale L_2, we know that $|\mu(x^*) - \mu| \leq C_1 L_2^{-2}$. Thus, applying Theorem 1.13, we have the bound

$$
|\lambda - \mu(x^*)| \leq C_1 L_2^{-2}. \tag{4.3}
$$

The function $\mu(x)$ increases from its minimum by L_2^{-2} as x varies in an interval of length comparable to L_2 from x^*. Moreover, $\mu(x)$ is a convex function. Therefore, provided we choose $C > 0$ sufficiently large, we have

$$
\mu(x) - \mu(x^*) \geq (C_1+1)L_2^{-2} \tag{4.4}
$$

whenever x satisfies $|x - x^*| \geq CL_2$. Combining the inequalities in (4.3) and (4.4) shows that $\mu(x) - \lambda \geq L_2^{-2}$, and using this bound in (4.2) gives $H''(x) \geq 2L_2^{-2}H(x)$ as required. □

Before giving a corollary of this proposition, we recall the generalised maximum principle.

Proposition 4.5. *Suppose that the functions v_1 and v_2 satisfy*

$$
\Delta v_1 + c(x)v_1 = 0, \qquad \Delta v_2 + c(x)v_2 \leq 0
$$

in a bounded domain D, where $c(x)$ is a continuous function. If in addition v_1 and v_2 are continuous in $\bar{D}$, $v_1 > 0$ in D and $v_2 > 0$ in $\bar{D}$, then

$$
\max_{\bar{D}} v_1/v_2 \leq \max_{\partial D} v_1/v_2.
$$

This is proven in [21], Theorem 10, page 73, and follows from applying the usual maximum principle to the function v_1/v_2. We now prove a corollary of Proposition 4.3.

Corollary 4.6. *Let $A = \max_x H(x)$. Then, the function $H(x)$ satisfies the upper bound*

$$
H(x) \leq C_1 A \exp\left(-c|x-x^*|/L_2\right).
$$

Proof of Corollary 4.6. With $C > 0$ as in the statement of Proposition 4.3, let $x_1 = x^* + CL_2$. We also define the function $R(x)$ for $x > x_1$ by

$$
R(x) = Ae^{-(x-x_1)/L_2}.
$$

Then, $R(x)$ satisfies $R''(x) = L_2^{-2} R(x)$, and $H(x_1) \leq A = R(x_1)$. By Proposition 4.3 we know that

$$H''(x) \geq L_2^{-2} H(x)$$

for all $x \geq x_1$. Therefore, setting D to be the interval $\{x \geq x_1\}$, the conditions of the generalised maximum principle are satisfied and hence

$$H(x) \leq R(x)$$

for all $x \geq x_1$. There is also an analogous bound for $x \leq x^* - CL_2$, and this completes the proof. □

Remark 4.7. In fact, we see from the proof that we can replace A by $H(x_1)$ and conclude that for any $x_1 \geq x^* + CL_2$ we have the bound

$$H(x) \leq H(x_1) e^{-(x-x_1)/L_2} \tag{4.5}$$

for all $x > x_1$.

In particular, as a result of this corollary, we see that $H(x)$ decays exponentially from its value at $x = x^*$ at least at a length scale comparable to L_2.

Our next aim is to obtain an upper bound for

$$A = \max_x H(x) = \max_x \int_{\Omega(x)} u(x,y)^2 \, \mathrm{d}y. \tag{4.6}$$

If we can show that A satisfies $A \leq C_1 L_1$, then by Proposition 4.3 we have

$$H(x) \leq C_1 L_1 e^{-c|x-x^*|/L_2},$$

and so integrating over x gives

$$\int_\Omega u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y = \int H(x) \, \mathrm{d}x \leq CL_1 L_2.$$

Therefore to complete the proof of Proposition 4.1, it is sufficient to prove this upper bound on A. To do this we first define a cut-off function $\chi_1(x)$ as follows.

Definition 4.8. We define $\chi_1(x)$ to be a smooth cut-off function which satisfies

$$0 \leq \chi_1(x) \leq 1$$

and is equal to 1 on the interval $[x^* - 2CL_2, x^* + 2CL_2]$ of length $4CL_2$, with C as in the statement of Proposition 4.3. Moreover, the function $\chi_1(x)$ is supported on the interval $[x^* - 3CL_2, x^* + 3CL_2]$ and has the derivative estimate

$$\left|\partial^k \chi_1(x)\right| \leq (CL_2)^{-k},$$

for $k = 1, 2$.

We now prove the following.

Proposition 4.9. *Let $\chi_1(x)$ be the cut-off function above in Definition* 4.8. *Then,*

$$\int_\Omega \chi_1(x) u(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \leq C_1 L_1 L_2,$$

for an absolute constant $C_1 > 0$. Note that this is consistent with $u(x,y)$ decaying on a length scale comparable to L_1 in the y-direction.

Proof of Proposition 4.9. Integrating the eigenfunction equation against the function $\chi_1(x)u(x,y)$ we obtain

$$\int_\Omega -\chi_1(x)u(x,y)\Delta_{x,y}u(x,y) + \chi_1(x)(V(x,y)-\lambda)u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y = 0,$$

and integrating by parts one time in x and y gives

$$\begin{aligned}\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1'(x)\partial_x u(x,y)u(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ + \int_\Omega \chi_1(x)(V(x,y)-\lambda)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y = 0.\end{aligned} \tag{4.7}$$

In the second integral in (4.7) we can write

$$\chi_1'(x)\partial_x u(x,y)u(x,y) = \tfrac{1}{2}\chi_1'(x)\partial_x(u(x,y)^2)$$

and integrate by parts in x again to rewrite this integral as

$$-\frac{1}{2}\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y.$$

Thus, from (4.7) we have

$$\begin{aligned}&\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(V(x,y)-\lambda)_+u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(V(x,y)-\lambda)_-u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{4.8}$$

where we have decomposed $V(x,y)-\lambda$ into its positive and negative parts via

$$V(x,y)-\lambda = (V(x,y)-\lambda)_+ - (V(x,y)-\lambda)_-.$$

By the simple eigenvalue bound for λ from Proposition 2.7, we know that

$$(V(x,y)-\lambda)_- \le C_1L_1^{-2}.$$

This also means that for any fixed x, we can only have $V(x,y)-\lambda \le 0$ for y in an interval of length at most comparable to L_1. Since the eigenfunction is normalised to have a maximum of 1, and $\chi_1(x)$ is only non-zero in an interval of length comparable to L_2, this gives us a bound on the final term in the right-hand side of (4.8) of

$$\int_\Omega \chi_1(x)(V(x,y)-\lambda)_-u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \le C_1L_1^{-2}L_1L_2 = C_1L_1^{-1}L_2. \tag{4.9}$$

We now turn to the second integral on the left-hand side of (4.8),

$$\int_\Omega \chi_1(x)(V(x,y)-\lambda)_+u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y.$$

Fix a large constant $C_2>0$. For each fixed x, $V(x,y)-\lambda$ is only bounded above by $C_2L_1^{-2}$ on an interval in y of length comparable to L_1. Therefore, again combining this with the bound $u(x,y)\le 1$, we can write

(4.10)

$$C_2L_1^{-2}\int_\Omega \chi_1(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y - C_1L_1^{-1}L_2 \le \int_\Omega \chi_1(x)(V(x,y)-\lambda)_+u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y.$$

Inserting the estimates in (4.9) and (4.10) back into (4.8) we see that

$$\begin{aligned}&\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + C_2L_1^{-2}\int_\Omega \chi_1(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y\\ &\le \frac{1}{2}\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y + C_1L_1^{-1}L_2.\end{aligned}\tag{4.11}$$

The first integral in (4.11) is positive, and so we can drop it from the estimate. Therefore, dividing by $C_2L_1^{-2}$ gives us

$$\int_\Omega \chi_1(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \le \frac{1}{2}C_2^{-1}L_1^2\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y + C_1L_1L_2.\tag{4.12}$$

To conclude the proof of the proposition, we will use Corollary 4.6 and the remark following it. By (4.5), for any $x_1 \ge x^* + CL_2$ and any $x \ge x_1$, we have

$$\int_{\Omega(x)} u(x,y)^2\,\mathrm{d}y \le e^{-(x-x_1)/L_2}\int_{\Omega(x_1)} u(x_1,y)^2\,\mathrm{d}y.$$

Therefore, we certainly have the estimate

$$\int_{x^*+2CL_2}^{x^*+3CL_2}\int_{\Omega(x)} u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \le \int_{x^*+CL_2}^{x^*+2CL_2}\int_{\Omega(x)} u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y,\tag{4.13}$$

and an analogous estimate for $x_1 \le x^* - CL_2$ and $x \le x_1$. By the definition of the cut-off function $\chi_1(x)$, the second derivative $\chi_1''(x)$ is supported on the intervals $[x^*-3CL_2, x^*-2CL_2]$ and $[x^*+2CL_2, x^*+3CL_2]$ and is of order L_2^{-2} here. Also, $\chi_1(x)$ is equal to 1 on the intervals $[x^*-2CL_2, x^*-CL_2]$ and $[x^*+CL_2, x^*+2CL_2]$. Therefore, using the estimate in (4.13) the integral on the right-hand side of (4.12) is certainly at most $\frac{1}{2}$ the size of the integral on the left-hand side. This means that in (4.12) we can bring over the integral to the left-hand side and get the bound

$$\int_\Omega \chi_1(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \le C_1L_1L_2,$$

as required. $\square$

Corollary 4.10. *We have the derivative bound*

$$\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \le C_1L_1^{-1}L_2.$$

Proof of Corollary 4.10. In the proof of Proposition 4.9 in (4.11) we established the estimate

$$\begin{aligned}&\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + C_2L_1^{-2}\int_\Omega \chi_1(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y\\ &\le \frac{1}{2}\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y + C_1L_1^{-1}L_2.\end{aligned}\tag{4.14}$$

We also showed that

$$L_1^2\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y$$

is bounded by $\int_\Omega \chi_1(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y$, and hence by Proposition 4.9 is bounded by $C_1L_1^{-1}L_2$. Using this estimate in (4.14) gives the desired result. $\square$

The derivative bound

$$\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \le C_1 L_1^{-1} L_2$$

is of order L_1^{-2} smaller than the bound we obtained for the eigenfunction $u(x,y)$ itself in Proposition 4.9. For the y-derivative $\partial_y u(x,y)$, this bound is consistent with our eventual aim to show that $u(x,y)$ decays away from its maximum on a length scale comparable to L_1. However, in the x-direction, our aim is to show that $u(x,y)$ decays away from its maximum on a length scale comparable to L_2. Therefore, we want to improve the bound on $\partial_x u(x,y)$ given in Corollary 4.10.

Proposition 4.11. *Let $\chi_1(x)$ be as in Definition* 4.8. *Then, there exists an absolute constant $C_1 > 0$ such that*

$$\int_\Omega \chi_1(x)(\partial_x u(x,y))^2\,\mathrm{d}x\,\mathrm{d}y \le C L_1 L_2^{-1}.$$

Note that for $L_2 \gg L_1$ this is an improvement on the bound in Corollary 4.10.

Proof of Proposition 4.11. We begin by proceeding as in the proof of Proposition 4.9 to obtain the equality in (4.8):

$$\begin{aligned}\int_\Omega \chi_1(x)|\nabla_{x,y}u(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(V(x,y)-\lambda)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \\ -\frac{1}{2}\int_\Omega \chi_1''(x)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y = 0.\end{aligned} \tag{4.15}$$

We know that the integral of $\chi_1(x)u(x,y)^2$ is at most $C_1 L_1 L_2$. Since $|\chi_1''(x)| \le C_1 L_2^{-2}$ this means that

$$\frac{1}{2}\int_\Omega |\chi_1''(x)|\,u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \le C_1 L_1 L_2^{-1},$$

and so from (4.15) we have

$$\begin{aligned}\int_\Omega \chi_1(x)(\partial_x u(x,y))^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(\partial_y u(x,y))^2\,\mathrm{d}x\,\mathrm{d}y \\ + \int_\Omega \chi_1(x)(V(x,y)-\lambda)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \le C_1 L_1 L_2^{-1}.\end{aligned} \tag{4.16}$$

For each fixed x, the eigenfunction $u(x,y)$ is an admissible test function for our usual ordinary differential operator $\mathcal{L}(x)$. Since this operator has first eigenvalue equal to $\mu(x)$, we obtain the lower bound

$$\int_{\Omega(x)} (\partial_y u(x,y))^2 + (V(x,y)-\lambda)u(x,y)^2\,\mathrm{d}y \ge (\mu(x)-\lambda)\int_{\Omega(x)} u(x,y)^2\,\mathrm{d}y. \tag{4.17}$$

Multiplying the inequality in (4.17) by $\chi_1(x)$ and integrating over x, (4.16) becomes

$$\int_\Omega \chi_1(x)(\partial_x u(x,y))^2\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \chi_1(x)(\mu(x)-\lambda)u(x,y)^2\,\mathrm{d}x\,\mathrm{d}y \le C_1 L_1 L_2^{-1}. \tag{4.18}$$

By the definition of L_2, we have $\mu(x) - \mu \geq -C_1 L_2^{-2}$, and by the eigenvalue bounds in Theorem 1.13, we know that $\mu - \lambda \geq -C_1 L_2^{-2}$. Therefore, (4.18) tells us that

$$\int_\Omega \chi_1(x)(\partial_x u(x,y))^2 \,\mathrm{d}x\,\mathrm{d}y \leq C_1 L_2^{-2} \int_\Omega \chi_1(x) u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y + C_1 L_1 L_2^{-1}.$$

Applying Proposition 4.9 then gives the desired bound. □

Now that we have established L^2-bounds for the first derivative $\nabla_{x,y} u(x,y)$ in Propositions 4.9 and 4.11, we can return to establishing the required upper bound for

$$A = \max_x H(x) = \max_x \int_{\Omega(x)} u(x,y)^2 \,\mathrm{d}y.$$

Proposition 4.12. *A is bounded by L_1 multiplied by an absolute constant.*

Proof of Proposition 4.12. Suppose that we have

$$\max_x H(x) = H(x^*) = \int_{\Omega(x^*)} u(x^*,y)^2 \,\mathrm{d}y \geq C^* L_1, \tag{4.19}$$

where $C^* > 0$ is a large absolute constant that we will specify later. Then, for any (x,y), extending $u(x,y)$ to be 0 outside Ω, we can write

$$u(x,y) = u(x^*,y) + \int_{x^*}^{x} \partial_t u(t,y) \,\mathrm{d}t,$$

and so

$$u(x,y)^2 \geq \tfrac{1}{2} u(x^*,y)^2 - C_1 |x - x^*| \int_{x^*}^{x} (\partial_t u(t,y))^2 \,\mathrm{d}t, \tag{4.20}$$

for a fixed constant C_1. Integrating the inequality in (4.20) over y we find that

$$H(x) \geq \tfrac{1}{2} H(x^*) - C_1 |x - x^*| \int_{x^*}^{x} \int_{\Omega(t)} (\partial_t u(t,y))^2 \,\mathrm{d}y\,\mathrm{d}t,$$

and so by the assumption on $H(x^*)$ in (4.19), this gives

$$H(x) \geq \tfrac{1}{2} C^* L_1 - C_1 |x - x^*| \int_{x^*}^{x} \int_{\Omega(t)} (\partial_t u(t,y))^2 \,\mathrm{d}y\,\mathrm{d}t. \tag{4.21}$$

Let us restrict to those values of x with $|x - x^*| \leq c_1 L_2$ for a small constant $c_1 > 0$. Then by the derivative bound on $\partial_t u(t,y)$ in Proposition 4.11, we can ensure that the second term in (4.21) is small compared to $\frac{1}{4} C^* L_1$. Moreover, this constant c_1 can be chosen to be independent of C^*. Therefore, this tells us that for all x in an interval of length $2c_1 L_2$, we have the lower bound $H(x) \geq \frac{1}{4} C^* L_1$. In particular, this shows that

$$\int_\Omega \chi_1(x) u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \geq \int_{x^* - c_1 L_2}^{x^* + c_1 L_2} H(x) \,\mathrm{d}x \geq \tfrac{1}{2} c_1 C^* L_1 L_2.$$

Since c_1 is independent of C^*, we can contradict the $L^2(\Omega)$-bound from Proposition 4.9 by choosing C^* sufficiently large. □

By the discussion after the proof of Corollary 4.6, this upper bound on A from Proposition 4.12 implies the $L^2(\Omega)$-bound $\int_\Omega u(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y \leq C L_1 L_2$. This completes the proof of Proposition 4.1. □

In Proposition 4.1 we derived an $L^2(\Omega)$-bound for the first eigenfunction $u(x,y)$. For our purposes of studying the shape of the level sets of $u(x,y)$ near its maximum this will be sufficient. However, another interesting question is to study the rate at which $u(x,y)$ decays from its maximum. Therefore, before continuing with our study of the level sets, let us give some indication about the decay of $u(x,y)$ as we move away from its maximum.

We will do this by using an Agmon-type estimate, but first we need some definitions.

Definition 4.13. Fix a large absolute constant $C > 0$, and let Ω_1 be the subset of Ω given by

$$\Omega_1 = \{(x,y) \in \Omega : V(x,y) \geq 1 + CL_1^{-2}\}.$$

Note that the boundary of Ω_1 consists of parts of the two convex curves coming from $\partial\Omega$ and the level set $\{(x,y) \in \Omega : V(x,y) = 1 + CL_1^{-2}\}$.

Definition 4.14. With $\Omega_1 \subset \Omega$ as above, we also define the distance function

$$h^* : \Omega_1 \to [0,\infty)$$

as follows. We first define the function $\nu^*(x,y)$ to be equal to $V(x,y) - \lambda$. For (x,y) in Ω_1 we then define $h^*(x,y)$ by

$$h^*(x,y) = \inf_{\gamma} \frac{1}{2} \int_0^1 \nu^*(\gamma(t))^{1/2} |\gamma'(t)| \, \mathrm{d}t,$$

where the infimum is taken over all paths $\gamma : [0,1] \to \Omega_1$ between the inner boundary of Ω_1 and (x,y).

We are now in a position to state our Agmon-type estimate.

Proposition 4.15. *For Ω_1 and $h^*(x,y)$ defined as above, we have*

$$\int_{\Omega_1} u(x,y)^2 e^{2h^*(x,y)} \, \mathrm{d}x \, \mathrm{d}y \leq C_2 L_1 L_2,$$

for some absolute constant $C_2 > 0$.

Remark 4.16. Since we certainly have the lower bound $V(x,y) - \lambda \geq C_1 L_1^{-2}$ on Ω_1, roughly speaking this proposition shows that, in an $L^2(\Omega)$-sense, the function $u(x,y)$ decays at least on a length scale comparable to L_1 as we move away from the region where $V(x,y) \leq 1 + CL_1^{-2}$. However, as $V(x,y) - \lambda$ grows, this rate of exponential decay also increases.

Proof of Proposition 4.15. This proposition will follow from a classical Agmon estimate in [1]. Let us restate Theorem 1.5 from [1] (using slightly different notation).

Theorem 4.17 (Theorem 1.5 in [1])**.** *Let D be a bounded connected open set in $\mathbb{R}^2$. Let $q(x,y)$ be a real valued function on D, and suppose that $\nu(x,y)$ is a positive continuous function on D such that*

$$\int_D |\nabla_{x,y}\psi(x,y)|^2 + q(x,y)\psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \geq \int_D \nu(x,y)\psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \tag{4.22}$$

for all $\psi \in C_0^\infty(D)$.

Fix a point $(x_0, y_0) \in D$, and define the distance $\rho_\nu(x,y)$ by

$$\rho_\nu(x,y) = \inf_{\gamma} \int_0^1 \nu(\gamma(t))^{1/2} |\gamma'(t)| \, \mathrm{d}t, \tag{4.23}$$

where the infimum is taken over all continuous paths $\gamma : [0,1] \to D$ *in* D *between* (x_0, y_0) *and* (x, y)*. We also define* $\rho_\nu((x,y),\{\infty\})$ *to be the distance from the point* (x,y) *to* ∂D *under the distance function* $\rho_\nu(x,y)$ *and define* D_s *by*

$$D_s = \{(x,y) \in D : \rho_\nu((x,y),\{\infty\}) > s\}.$$

Finally, suppose that

$$-\Delta_{x,y} W(x,y) + q(x,y)W(x,y) = 0$$

and that the function $g(x,y)$ *satisfies*

$$|\nabla_{x,y} g(x,y)|^2 < \nu(x,y) \tag{4.24}$$

in D*. Then, we have the estimate*

$$\begin{aligned} &\int_{D_s} W(x,y)^2(\nu(x,y) - |\nabla_{x,y} g(x,y)|^2) e^{2g(x,y)} \, \mathrm{d}x \, \mathrm{d}y \\ &\leq \frac{2(1+2s)}{s^2} \int_{D \backslash D_s} W(x,y)^2 \nu(x,y) e^{2g(x,y)} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{4.25}$$

We will now apply this theorem with $W(x,y) = u(x,y)$ and $q(x,y) = V(x,y) - \lambda$. We will choose the set D as follows: We recall that Ω_1 consists of those points (x,y) with $V(x,y) - \lambda \geq 1 + CL_1^{-2}$. We then define D to be all points in $\mathbb{R}^2$ outside the inner boundary of Ω_1.

Since $u(x,y) = 0$ on $\partial\Omega$, we can extend $u(x,y)$ to D by setting it to be 0 for $D \backslash \Omega_1$, and we extend the potential $V(x,y)$ to D arbitrarily.

We clearly have the estimate

$$\int_D |\nabla_{x,y}\psi(x,y)|^2 + (V(x,y) - \lambda)\psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y \geq \int_D (V(x,y) - \lambda)\psi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y$$

for all $\psi \in C_0^\infty(D)$. Also, $V(x,y) - \lambda \geq C_1 L_1^{-2}$ for $(x,y) \in D$. As a result of this, from (4.22) we see that we can set $\nu^*(x,y)$ to be equal to the function described in the definition of $h^*(x,y)$ in Definition 4.14.

In Theorem 4.17 we are free to choose the value for s, and we will choose $s = 1$. Then, we see that

$$D \backslash D_1 = \{(x,y) \in D : \rho_\nu((x,y),\{\infty\}) \leq 1\}$$

consists of the region near the inner boundary of D with width comparable to at most L_1. This is because we have ensured that $\nu(x,y) \geq cL_1^{-2}$ when the point (x,y) is within a distance L_1 of the boundary of D.

We finally need to choose $g(x,y)$ to ensure that (4.24) holds, and so we need

$$|\nabla_{x,y} g(x,y)|^2 < \nu^*(x,y) = V(x,y) - \lambda.$$

We can achieve this by setting $g(x,y)$ to be equal to the function $h^*(x,y)$ as in Definition 4.14. This certainly satisfies the required derivative bound.

Thus, we can apply Theorem 4.17 to get

$$\begin{aligned} &\int_{D_1} u(x,y)^2(\nu(x,y) - |\nabla_{x,y} h^*(x,y)|^2) e^{2h^*(x,y)} \, \mathrm{d}x \, \mathrm{d}y \\ &\leq 6 \int_{D \backslash D_1} u(x,y)^2 \nu(x,y) e^{2h^*(x,y)} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{4.26}$$

On $D\backslash D_1$, we know that $\nu(x,y) \le L_1^{-2}$, and $h^*(x,y) \le 1$. Therefore, by the L^2 bound on $u(x,y)$ from Proposition 4.1, the right-hand side of (4.26) is bounded by

$$C_1 L_1^{-2} L_1 L_2 = C_1 L_1^{-1} L_2.$$

Since for $(x,y) \in D$, we have $\nu(x,y) - |\nabla_{x,y} h^*(x,y)|^2 \ge c_1 L_1^{-2}$, we can therefore conclude from (4.26) that

$$\int_{\Omega_2} u(x,y)^2 e^{2h^*(x,y)}\,\mathrm{d}x\,\mathrm{d}y \le C_1 L_1 L_2$$

as required. □

5. The shape of the level sets of $u(x,y)$

We now return to the problem of studying the shape of the level sets of the first eigenfunction $u(x,y)$. As we mentioned earlier, since the potential $V(x,y)$ is convex, a theorem of Brascamp and Lieb [8] tells us that $u(x,y)$ is log concave. In particular, this means that the superlevel sets of $u(x,y)$ are convex subsets of Ω.

We will use the results of the previous section to estimate the lengths of the projections of these level sets onto the x- and y-axis. In particular, in this section we will establish Theorem 1.10 about the shape of the level sets. Throughout this section we let $c_1 > 0$ be a small absolute constant as in the statement of Theorem 1.10. The constant $c > 0$ which appears in the propositions below is bounded away from 0 and 1 by satisfying

$$c_1 < c < 1 - c_1,$$

and all other constants will depend on the choice of c_1.

We first use the bound on A from Proposition 4.12 to find an upper bound on the behaviour of the level sets of $u(x,y)$ in the y-direction.

Proposition 5.1. *Let $0 < c < 1$ be a fixed absolute constant. Then, for any fixed x, the cross-section of the superlevel set $\{(x,y) \in \Omega : u(x,y) \ge c\}$ at x consists of an interval of length at most L_1 multiplied by an absolute constant.*

Proof of Proposition 5.1. By Proposition 4.12 we know that $A = \max_x H(x) \le C_1 L_1$. If $u(x,y) \ge c$ for y in an interval of length CL_1 for C sufficiently large, this immediately gives a contradiction. □

We can also prove an upper bound on the length of the projection of the level sets of $u(x,y)$ in the y-direction.

Proposition 5.2. *For sufficiently small $\delta > 0$ fixed, there exists an $\eta > 0$ such that if the point (x,y) is within a distance ηL_1 of the level set $\{(x,y) \in \Omega : V(x,y) = 1 + \eta^{-1} L_1^{-2}\}$, then*

$$u(x,y) \le \delta.$$

In particular, the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ are at a distance comparable to L_1 away from the level set $\{(x,y) \in \Omega : V(x,y) = 1 + CL_1^{-2}\}$, for some absolute constant $C > 0$.

Remark 5.3. The proof of this proposition follows closely the proof of Lemma 3.17 in [16], where an analogous property has been established for the first eigenfunction of a two-dimensional convex domain.

Before proving this proposition, let us show the following corollary.

Corollary 5.4. *Let $0 < c < 1$ be a fixed absolute constant. Then, the projection of the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y-axis has length bounded from above by an absolute constant multiplied by L_1.*

Proof of Corollary 5.4. By the definition of the length scale L_1 and the orientation of the level set $\Omega_{L_1^{-2}} = \{(x,y) \in \Omega : V(x,y) = 1 + L_1^{-2}\}$ we used when defining L_2, we know that the projection of the level set $\Omega_{L_1^{-2}}$ onto the y-axis has length comparable to L_1. Moreover, by the convexity of the potential $V(x,y)$, this is true for any level set

$$\{(x,y) \in \Omega : V(x,y) = 1 + CL_1^{-2}\},$$

for any absolute constant $C > 0$. Therefore, the upper bound on the length of the projection of the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y-axis follows from Proposition 5.2. □

Proof of Proposition 5.2. Let (x', y') be a point which is within a distance ηL_1 of the level set $\{(x,y) \in \Omega : V(x,y) = 1 + \eta^{-1}L_1^{-2}\}$. After a rotation, we may assume that the nearest point of $\{(x,y) \in \Omega : V(x,y) = 1 + \eta^{-1}L_1^{-2}\}$ to (x', y') is equal to (x', y_1), with $y_1 < y'$ and $y' - y_1 < \eta L_1$.

We will need to use two properties of the potential $V(x,y)$. Firstly, by the simple eigenvalue bounds on λ in Proposition 2.7 we have seen before that

$$\Delta_{x,y}u(x,y) = (V(x,y) - \lambda)u(x,y) \geq -\frac{C_1^2}{L_1^2}u(x,y) \tag{5.1}$$

for all values of (x,y), for some absolute constant C_1. Moreover, $V(x,y)$ has convex sublevel sets, and by the rotation we made above we have $V(x,y_1) = 1 + \eta^{-1}L_1^{-2}$. Therefore,

$$\Delta_{x,y}u(x,y) = (V(x,y) - \lambda)u(x,y) \geq \frac{1}{2\eta L_1^2}u(x,y) \tag{5.2}$$

whenever $y \leq y' - \eta L_1 < y_1$.

We define the comparison function $v_1(x,y)$ by

$$v_1(x,y) = \sin\left(\frac{C_1(y-y')}{2L_1} + \frac{C_1\eta}{2} + C_1\delta\right) \tag{5.3}$$

for $y \geq y' - \eta L_1$, and by

$$v_1(x,y) = (\sin(C_1\delta))\exp\left(\frac{\delta}{2} + \frac{(y-y')}{2\delta L_1}\right) \tag{5.4}$$

for $y < y' - \eta L_1$. We make the choice $\eta = \delta^2$, and this ensures that $v_1(x,y)$ is continuous at $y = y' - \eta L_1$ for all values of x.

For $\delta > 0$ sufficiently small, using $\sin(C_1\delta) > C_1\delta\cos(C_1\delta)$, we find that $\partial_y^2 v_1(x,y)$ has a negative delta function along $y = y' - \eta L_1$. Everywhere else, calculating $\Delta_{x,y}v_1(x,y)$ from its definition in (5.3) and (5.4) and using the inequalities for $\Delta_{x,y}u(x,y)$ in (5.1) and (5.2), we see that

$$\frac{\Delta_{x,y}v_1(x,y)}{v_1(x,y)} \leq \frac{\Delta_{x,y}u(x,y)}{u(x,y)}.$$

Moreover, for those $(x,y) \in \partial\Omega$, with $y \leq y' + \left(\frac{\pi}{C_1} - \eta - 2\delta\right) L_1$, we have

$$v_1(x,y) > 0 = u(x,y)|_{\partial\Omega},$$

and for $(x,y) \in \Omega$ with $y = y' + \left(\frac{\pi}{C_1} - \eta - 2\delta\right) L_1$, we have

$$v_1\left(x, y' + (\pi/C_1 - \eta - 2\delta)L_1\right) = 1 \geq u\left(x, y' + (\pi/C_1 - \eta - 2\delta)L_1\right).$$

Thus, applying the generalised maximum principle in Proposition 4.5 to those (x,y) in Ω with $y \leq y' + \left(\frac{\pi}{C_1} - \eta - 2\delta\right) L_1$, $v_1(x,y)$ is a positive supersolution, and in particular

$$u(x',y') \leq v_1(x',y') = \sin\left(\frac{C_1\eta}{2} + C_1\delta\right) \leq C_2\delta.$$

Thus, repeating the argument with a suitable multiple of δ gives the desired result. □

We now want to obtain a lower bound on the height of the level sets in the y-direction.

Proposition 5.5. *Let $0 < c < 1$ be a fixed absolute constant. Then, the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ has inner radius bounded below by an absolute constant multiplied by L_1. In particular, the projection of the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y-axis has length bounded from below by an absolute constant multiplied by L_1.*

Remark 5.6. The proof of this proposition only considers the parameter L_1 and does not use any properties of the eigenvalue or eigenfunction that depend on L_2. In particular, this means that we do not need to fix the orientation of the level set

$$\Omega_{L_1^{-2}} = \{(x,y) \in \Omega : V(x,y) = 1 + L_1^{-2}\},$$

and we are free to rotate Ω in the course of the proof.

Proof of Proposition 5.5. Let us consider the case $c = 1/4$ and study the level set

$$\{(x,y) \in \Omega : u(x,y) = \tfrac{1}{4}\}.$$

Suppose that the shortest projection of the set onto any direction is of length α. By the convexity of the superlevel sets of $u(x,y)$, after a rotation and a translation, we may then assume that this level set lies between the two lines $y = 0$ and $y = \alpha$.

We will use the comparison function

$$W(x,y) = \frac{1}{2}\sin\left(\frac{\pi}{6} + \frac{2\pi}{3\alpha}y\right).$$

This function is equal to $1/4$ when $y = 0$ or $y = \alpha$ and satisfies

$$(5.5) \quad (\Delta_{x,y} - V(x,y) + \lambda)W(x,y) = -\left(\frac{2\pi}{3\alpha}\right)^2 W(x,y) + (\lambda - V(x,y))W(x,y).$$

Since $V(x,y) \geq 1$, by the straightforward eigenvalue bound on λ from Proposition 2.7 we have

$$\lambda - V(x,y) \leq C^2 L_1^{-2},$$

for an absolute constant $C > 0$. Therefore, from (5.5) we obtain

$$(\Delta_{x,y} - V(x,y) + \lambda)W(x,y) \leq \left(- \left(\frac{2\pi}{3\alpha} \right)^2 + C^2 L_1^{-2} \right) W(x,y). \tag{5.6}$$

Let us assume that

$$\alpha < \frac{2\pi L_1}{3C}. \tag{5.7}$$

Then, from (5.6) we see that

$$(\Delta_{x,y} - V(x,y) + \lambda)W(x,y) < 0,$$

while $(\Delta_{x,y} - V(x,y) + \lambda)u(x,y) = 0$ in Ω. Also, for all points (x,y) with $y = 0, \alpha$ we have $u(x,y) \leq W(x,y) = \frac{1}{4}$, and $u(x,y) = 0 < W(x,y)$ for $(x,y) \in \partial\Omega$, with $0 \leq y \leq \alpha$. Therefore, by the generalised maximum principle in Proposition 4.5 we find that

$$u(x,y) \leq W(x,y) \qquad \text{for } (x,y) \in D \text{ with } 0 \leq y \leq \alpha.$$

However, $W(x,y) \leq \frac{1}{2}$, while $u(x,y)$ attains its maximum of 1 at some point (x,y) with $0 \leq y \leq \alpha$. This gives a contradiction, and so from (5.7) we must have

$$\alpha > \frac{2\pi L_1}{3C}.$$

Therefore the projection of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq \frac{1}{4}\}$ onto any direction has length at least comparable to L_1, and this gives us the required lower bound on the inner radius of this superlevel set. We can also repeat the argument above for the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ for any fixed absolute constant c with $c_1 < c < 1 - c_1$ to obtain the same result. $\square$

Corollary 5.7. *As an immediate consequence of Proposition* 5.5, *we see that*

$$A = \max_x \int_{\Omega(x)} u(x,y)^2 \,\mathrm{d}y \geq \tilde{c} L_1,$$

for an absolute constant $\tilde{c} > 0$.

Combining Propositions 5.1 and 5.5, the height of the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ in the y-direction is comparable to L_1. We now turn to studying the length of the level sets of $u(x,y)$ in the x-direction. We first use Corollary 4.6 to obtain an upper bound on the length of the level sets.

Proposition 5.8. *Let* $0 < c < 1$ *be a fixed absolute constant. Then, the projection of the level set* $\{(x,y) \in \Omega : u(x,y) = c\}$ *onto the* x*-axis has length bounded by an absolute constant multiplied by* L_2.

Proof of Proposition 5.8. Suppose that the length of the projection of $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the x-axis is bounded below by $2CL_2$, where $C > 0$ is a large absolute constant that we will specify later in the proof. For each fixed x, the cross-section of the superlevel set $\{(x,y) \in \Omega : u(x,y) \geq c\}$ at x consists of an interval. Since the superlevel set is convex, the length of this interval is greater than half of its maximum length for x lying in an interval of length CL_2.

By Proposition 5.5, this maximum length is bounded below by $2C_1L_1$ for an absolute constant $C_1 > 0$. In other words, $u(x,y) \geq c$ for all (x,y) in a rectangle of height C_1L_1 and width CL_2.

As a result of this, we have

$$H(x) = \int_{\Omega(x)} u(x,y)^2 \,\mathrm{d}y \geq c^2 C_1 L_1, \tag{5.8}$$

for all x in an interval of length CL_2. By Proposition 4.12, A is bounded by an absolute constant multiplied by L_1, and by Corollary 4.6 we have the bound

$$H(x) \leq Ae^{-c|x-x^*|/L_2}. \tag{5.9}$$

Therefore, combining (5.8) and (5.9), we obtain a contradiction if we choose C to be sufficiently large. This completes the proof of the proposition. $\square$

To complete the proof of Theorem 1.10 we finally want to obtain a comparable lower bound on the length of the level set of $u(x,y)$ in the x-direction. To do this we will use the L^2-bound on the first derivative $\partial_x u(x,y)$ from Proposition 4.11.

Proposition 5.9. *Let $0 < c < 1$ be a fixed absolute constant. Then, the projection of the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the x-axis has length bounded from below by an absolute constant multiplied by L_2.*

Proof of Proposition 5.9. We first prove the proposition for $c = 1/4$. By applying Proposition 5.5 with $c = \frac{1}{2}$, there exists a point $x = x_0$ and an interval J of length equal to $2c^* L_1$ for a constant $c^* > 0$, such that $u(x_0,y) \geq \frac{1}{2}$ for all y in J. Therefore,

$$\int_J u(x_0,y)^2 \,\mathrm{d}y \geq \tfrac{1}{4} c^* L_1. \tag{5.10}$$

Extending $u(x,y)$ to be zero outside Ω, for any other x, we can write

$$u(x,y) = u(x_0,y) + \int_{x_0}^{x} \partial_t u(t,y) \,\mathrm{d}t,$$

and so

$$u(x,y)^2 \geq \tfrac{3}{4} u(x_0,y)^2 - C_1|x-x_0| \int_{I(x)} (\partial_t u(t,y))^2 \,\mathrm{d}t,$$

where $I(x)$ consists of those points between x_0 and x. Integrating this over $y \in J$, we find that

$$\int_J u(x,y)^2 \,\mathrm{d}y \geq \tfrac{3}{4} \int_J u(x_0,y)^2 \,\mathrm{d}y - C_1|x-x_0| \int_J \int_{I(x)} (\partial_t u(t,y))^2 \,\mathrm{d}t \,\mathrm{d}y. \tag{5.11}$$

By (5.10), the first term on the right-hand side of (5.11) is bounded from below by $\frac{3}{8} c^* L_1$. Provided $|x - x_0| \leq c_2 L_2$ for $c_2 > 0$ sufficiently small, we can use Proposition 4.11 to show that the second term on the right-hand side of (5.11) is bounded above by

$$C_1|x-x_0| L_1 L_2^{-1},$$

for an absolute constant $C_1 > 0$. Thus, if $|x - x_0| \leq c_3 L_2$ for $c_3 > 0$ sufficiently small, we can ensure that

$$\int_J u(x,y)^2 \,\mathrm{d}y \geq \tfrac{3}{16} c^* L_1 - \tfrac{1}{16} c^* L_1 = \tfrac{1}{8} c^* L_1.$$

Since the interval J has length equal to c^*L_1, this means that for each x with $|x - x_0| \leq c_3L_2$, $u(x,y)$ must be at least $\frac{1}{4}$ at some point $y \in J$. In particular, the level set

$$\{(x,y) \in \Omega : u(x,y) = 1/4\}$$

must have length in the x-direction of at least c_3L_2 as required. A lower bound on the length in the x-direction of the other level sets of $u(x,y)$ follows in an analogous way. $\square$

Combining Corollary 5.4 and Proposition 5.5 concerning the height of the level sets in the y-direction with Propositions 5.8 and 5.9 concerning the length of the level sets in the x-direction we have established the following: For any c with $c_1 < c < 1 - c_1$, the projections of the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$ onto the y- and x-axes are of lengths comparable to L_1 and L_2, respectively, and, moreover, the inner radius of the corresponding superlevel set is comparable to L_1 while the diameter is comparable to L_2. This implies that the level sets have the desired shape and completes the proof of Theorem 1.10.

6. The behaviour of $u(x,y)$ near its maximum

In Theorem 1.10 we described the shape of the level sets $\{(x,y) \in \Omega : u(x,y) = c\}$, where c is bounded away from 0 and 1 by $c_1 < c < 1 - c_1$. In Proposition 4.15, we gave an indication of the behaviour of $u(x,y)$ as c becomes small. It is natural to ask what happens when c becomes close to 1 and we approach the maximum of the eigenfunction. Two interesting questions that one can ask in this case are the following:

Where is the maximum of $u(x,y)$ located relative
to the minimum of the potential $V(x,y)$?

What happens to the shape of the level sets of $u(x,y)$ as c approaches 1?

In the author's thesis [3], the first question has been partially studied, and this proposition has been established.

Proposition 6.1. *Suppose that the eigenfunction $u(x,y)$ attains its maximum at the point (x^*, y^*). Then, there exists an absolute constant $c^* > 0$ such that*

$$V(x^*, y^*) - \lambda \leq -c^* L_1^{-2}.$$

Regarding the second question, there is the following conjecture.

Conjecture 6.2. *For $c = 1 - \epsilon$, with $0 < \epsilon \leq \frac{1}{2}$, the level set $\{(x,y) \in \Omega : u(x,y) = c\}$ has the following shape: There exists an ellipse E with minor axis in the y-direction of length comparable to $\sqrt{\epsilon}L_1$ and major axis in the x-direction of length comparable to $\sqrt{\epsilon}L_2$, such that E is contained inside this level set and a dilate of E, with a scaling factor bounded by an absolute constant, contains this level set. Here all constants are independent of ϵ.*

A way of viewing this conjecture is that near the maximum of $u(x,y)$ at $(0,0)$, say, the eigenfunction resembles the polynomial

$$P(x,y) = 1 - \frac{x^2}{L_2^2} - \frac{y^2}{L_1^2}$$

in a suitable sense. This conjecture holds for all cases where the eigenfunction is explicitly known, but is still open even for the case of the Dirichlet Laplacian on a two-dimensional convex domain with constant potential V. In the author's thesis [3], a first step towards this conjecture is established by showing that the inner radius of the level sets is indeed comparable to $\sqrt{\epsilon}L_1$. Studying the behaviour of the eigenfunction near its maximum in more detail is a subject of future work.

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Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139-4307

E-mail address: tdbeck@mit.edu