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IMPROVED SUBCONVEXITY BOUNDS FOR $GL(2) \times GL(3)$ AND $GL(3)$ L-FUNCTIONS BY WEIGHTED STATIONARY PHASE

MARK MCKEE, HAIWEI SUN, AND YANGBO YE

ABSTRACT. Let f be a fixed self-contragradient Hecke–Maass form for $SL(3,\mathbb{Z})$, and let u be an even Hecke–Maass form for $SL(2,\mathbb{Z})$ with Laplace eigenvalue $1/4+k^2$, $k \geq 0$. A subconvexity bound $O\big((1+k)^{4/3+\varepsilon}\big)$ in the eigenvalue aspect is proved for the central value at $s = 1/2$ of the Rankin–Selberg L-function $L(s, f \times u)$. Meanwhile, a subconvexity bound $O\big((1+|t|)^{2/3+\varepsilon}\big)$ in the t aspect is proved for $L(1/2+it, f)$. These bounds improved corresponding subconvexity bounds proved by Xiaoqing Li (Annals of Mathematics, 2011). The main techniques in the proofs, other than those used by Li, are nth-order asymptotic expansions of exponential integrals in the cases of the explicit first derivative test, the weighted first derivative test, and the weighted stationary phase integral, for arbitrary $n \geq 1$. These asymptotic expansions sharpened the classical results for $n = 1$ by Huxley.

1. Introduction

Bounds for automorphic L-functions on the critical line $\mathrm{Re}(s) = 1/2$ are central questions in number theory and have far-reaching applications (cf. Iwaniec and Sarnak [13] and Michel [26]). The ultimate conjectured bounds are predicted by the Lindelöf Hypothesis, while trivial bounds include the convexity bounds as a consequence of the Phragmén-Lindelöf principle. Any bound which has a power saving over the corresponding convexity bound is highly nontrivial and called a subconvexity bound.

The strength of a subconvexity bound is crucial. There are important applications which depend on the strength of the subconvexity bounds. A notable example is the number of real zeros of a holomorphic Hecke cusp form f for $SL(2,\mathbb{Z})$ of weight k, i.e., zeros of f on $\{iy | y \geq 1\}$. By Ghosh and Sarnak [7], the number of such zeros is $\gg \log k$. Their proof uses a Weyl-like, i.e., a $1/3$ power-saving, subconvexity bound for $L(s, f)$ proved by Peng [28] and Jutila and Motohashi [15]. Note that a subconvexity bound for $L(s, f)$ with a power saving less than $1/3$ does not suffice in [7].

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These authors contributed equally to this work.

Yangbo Ye is the corresponding author.

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In this paper, we will prove subconvexity bounds for certain Rankin–Selberg L-functions for $GL(3)\times GL(2)$ and automorphic L-functions for $GL(3)$ over $\mathbb{Q}$ which improve bounds established by Xiaoqing Li [20].

Theorem 1.1. *Let f be a fixed self-contragradient Hecke–Maass form for $SL(3,\mathbb{Z})$ normalized by $A(1,1)=1$, and let $\{u_j\}$ be an orthonormal basis of even Hecke–Maass forms for $SL(2,\mathbb{Z})$. Denote by $1/4+t_j^2$, $t_j\geq 0$, the Laplace eigenvalue of u_j. Then for large T and $T^{1/3+\varepsilon}\leq M\leq T^{1/2}$ we have*

(1.1)
$$\sum_j e^{-(t_j-T)^2/M^2}L\Big(\frac{1}{2},f\times u_j\Big)+\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-(t-T)^2/M^2}\Big|L\Big(\frac{1}{2}-it,f\Big)\Big|^2dt\ll_{\varepsilon,f}T^{1+\varepsilon}M$$

for any $\varepsilon>0$.

Note that in [20] the same equation as (1.1) was proved for $T^{3/8+\varepsilon}\leq M\leq T^{1/2}$. As pointed out in [20],

$$L\Big(\frac{1}{2},f\times u_j\Big)\geq 0 \tag{1.2}$$

was proved by Lapid [17] because f is orthogonal and u_j is symplectic (Jacquet and Shalika [14]). The nonnegativity in (1.2) allows us to deduce a bound for individual terms from (1.1).

We remark that the normalization of u_j is different from the normalization $\lambda_{u_j}(1)=1$ as required in the definition of $L(s,f\times u_j)$, but the discrepancy is within t_j^{ε} as proved in Hoffstein and Lockhart [10]. The smaller allowable power of T for M in Theorem 1.1 gives us a smaller subconvexity bound.

Corollary 1.2. *Let f be a fixed self-contragradient Hecke–Maass form for $SL(3,\mathbb{Z})$ normalized by $A(1,1)=1$, and let u be an even Hecke–Maass form for $SL(2,\mathbb{Z})$ normalized by $\lambda_u(1)=1$. Denote by $1/4+k^2$, $k>0$, the Laplace eigenvalue of u. Then*

$$L\Big(\frac{1}{2},f\times u\Big)\ll_{\varepsilon,f}k^{4/3+\varepsilon}.$$

Note that Corollary 1.2 improved the bound $O\big(k^{11/8+\varepsilon}\big)$ proved in [20]. The convexity bound is $O\big(k^{3/2+\varepsilon}\big)$. Because of the nonnegativity in (1.2), the bound in (1.1) implies a square moment bound for $L(s,f)$ over a short interval.

Corollary 1.3. *Let f be a fixed self-contragradient Hecke–Maass form for $SL(3,\mathbb{Z})$ normalized by $A(1,1)=1$. Then for $T^{1/3+\varepsilon}\leq M\leq T^{1/2}$,*

$$\int_{-\infty}^{\infty}e^{-(t-T)^2/M^2}\Big|L\Big(\frac{1}{2}-it,f\Big)\Big|^2dt\ll_{\varepsilon,f}T^{1+\varepsilon}M. \tag{1.3}$$

Since f is a $GL(3)$ form, the square moment in (1.3) is comparable to a sixth power moment of the Riemann zeta function. Similar arguments were carried out for a $GL(2)$ form in Ye [32] and Lau, Liu, and Ye [18].

By a standard argument of analytic number theory (cf. Heath-Brown [9] or Ivić [12, p. 197]), we derived a subconvexity bound for $L(s,f)$ in the t aspect. Its improvement over [20]'s $O\big((1+|t|)^{11/16+\varepsilon}\big)$ is again based on the smaller allowable power of T for M. The convexity bound is $O\big((1+|t|)^{3/4+\varepsilon}\big)$.

Corollary 1.4. *Let f be a fixed self-contragradient Hecke–Maass form for $SL(3,\mathbb{Z})$ normalized by $A(1,1)=1$. Then*

$$L\Big(\frac{1}{2}+it,f\Big) \ll_{\varepsilon,f} (1+|t|)^{2/3+\varepsilon}.$$

Following Ye and Deyu Zhang [33], we can deduce the following result on zero density for $L(s,f)$ from (1.3). Let

$$N_f(\sigma,T,T+T^\delta) = \#\big\{\rho=\beta+i\gamma \big| L(\rho,f)=0, \sigma<\beta<1, T\le\gamma\le T+T^\delta\big\}$$

be the number of zeros of $L(s,f)$ in the box of $\sigma<\beta<1$ and $T\le\gamma\le T+T^\delta$.

Corollary 1.5. *Let f be a fixed self-contragradient Hecke–Maass form for $SL(3,\mathbb{Z})$. Then for $1/3<\delta\le 1$, we have*

$$\begin{aligned} N_f(\sigma,T,T+T^\delta) &\ll_{\varepsilon,f} T^{\frac{(2+4\delta)(1-\sigma)}{3-2\sigma}+\varepsilon} \text{ for } 1/2\le\sigma<\frac{2+\delta}{2+2\delta} \\ &\ll_{\varepsilon,f} T^{2(1+\delta)(1-\sigma)+\varepsilon} \text{ for } \frac{2+\delta}{2+2\delta}\le\sigma<1. \end{aligned} \tag{1.4}$$

We note that Corollary 1.5 shows that (1.4) is now valid on a shorter interval $[T,T+T^\delta]$ with $1/3<\delta\le 1$ than the interval with $3/8<\delta\le 1$ in [33] which uses Li [20].

As noted in [20], Theorem 1.1 can also be proved for f being the minimal Eisenstein series on $GL(3)$. This has been carried out in Lu [23]. Our proof and improvement can also be applied to that case.

P. Sarnak pointed out to us that for a holomorphic cusp form g for $SL(2,\mathbb{Z})$, the Dirichlet series for the L-functions $L(s,Sym^2 g)$ and $L(s,Sym^2 g\times u_j)$ have the same structure and properties as $L(s,f)$ and $L(s,f\times u_j)$, respectively, for f being a self-dual Maass form for $SL(3,\mathbb{Z})$ (cf. Bump [4, 5] and Luo and Sarnak [24]). Consequently our theorem and corollaries are also valid for such $L(s,Sym^2 g)$ and $L(s,Sym^2 g\times u_j)$.

The main techniques of our proof, other than those used in [20], include an asymptotic expansion of exponential integrals

$$\int_\alpha^\beta g(x)e(f(x))\,dx \tag{1.5}$$

when $f'(x)$ changes signs at a point $x=\gamma$ with $\alpha<\gamma<\beta$. Huxley [11] obtained the first-order asymptotic expansion of (1.5). His results [11] are widely used as standard techniques in analytic number theory and other branches of mathematics.

What we need in our proof, however, is an asymptotic expansion of (1.5) beyond the first order. Blomer, Khan and Young [3] proved such an asymptotic expansion for $f(x)$ being smooth and $g(x)$ being smooth of compact support. In [25] we proved a similar asymptotic expansion for $f(x)$ being continuously differentiable $2n+3$ times and $g(x)$ being continuously differentiable $2n+1$ times on a finite interval $[\alpha,\beta]$. Since the latter one is explicitly written, we will use it in the present paper:

$$\begin{aligned} \int_\alpha^\beta g(x)e(f(x))\,dx &= \frac{e(f(\gamma)\pm 1/8)}{\sqrt{|f''(\gamma)|}}\Big(g(\gamma)+\sum_{j=1}^n \varpi_{2j}\frac{(-1)^j(2j-1)!!}{(2\pi i f''(\gamma))^j}\Big) \\ &\quad + \text{Boundary terms} + \text{Error terms}. \end{aligned}$$

Here γ is the only zero of $f'(x)$ in (α, β), and ϖ_{2j} are given in (2.4). Note that the boundary terms do not appear in [3]. See Proposition 2.2 below for details. We will apply Voronoi's summation formula (Lemma 3.1) and its asymptotic expansion (Lemma 3.2) to the leading term of (2.4) for all ϖ_{2j} the second time.

In the following sections, ε is any arbitrarily small positive number. Its value may be different on each occurrence.

2. Oscillatory integrals

The following proposition is the weighted first derivative test, which strengthens Lemma 5.5.5 of [11, p. 113] with more boundary terms and smaller error terms. We can also use a similar formula proved in Jutila and Motohashi [15, Lemma 6].

Proposition 2.1 (McKee, Sun, and Ye [25]). *Let $f(x)$ be a real-valued function, $n+2$ times continuously differentiable for $\alpha \le x \le \beta$, and let $g(x)$ be a real-valued function, $n+1$ times continuously differentiable for $\alpha \le x \le \beta$. Suppose that there are positive parameters M, N, T, U, with $M \ge \beta - \alpha$, and positive constants C_r such that for $\alpha \le x \le \beta$,*

$$|f^{(r)}(x)| \le C_r \frac{T}{M^r}, \ |g^{(s)}(x)| \le C_s \frac{U}{N^s},$$

for $r = 2, \ldots, n+2$ and $s = 0, \ldots, n+1$. If $f'(x)$ and $f''(x)$ do not change signs on the interval $[\alpha, \beta]$, then we have

$$\begin{aligned}\int_\alpha^\beta g(x)e(f(x))dx \quad = \quad & \Big[e(f(x))\sum_{i=1}^n H_i(x)\Big]_\alpha^\beta \\ & +O\Big(\frac{M}{N}\sum_{j=1}^{[n/2]}\frac{UT^j}{\min|f'|^{n+j+1}M^{2j}}\sum_{t=j}^{n-j}\frac{1}{N^{n-j-t}M^t}\Big) \\ & +O\Big(\Big(\frac{M}{N}+1\Big)\frac{U}{N^n\min|f'|^{n+1}}\Big) \\ & +O\Big(\sum_{j=1}^{n}\frac{UT^j}{\min|f'|^{n+j+1}M^{2j}}\sum_{t=0}^{n-j}\frac{1}{N^{n-j-t}M^t}\Big),\end{aligned}$$

where

$$H_1(x) = \frac{g(x)}{2\pi i f'(x)}, \ H_i(x) = -\frac{H'_{i-1}(x)}{2\pi i f'(x)} \tag{2.1}$$

for $i = 2, \ldots, n$.

The following proposition is for a weighted stationary phase integral and sharpens Lemma 5.5.6 of [11, p. 114], with main terms up to the nth order, more boundary terms, and smaller error terms. In [3, Proposition 8.2], Blomer, Khan, and Young obtained the same main terms and the last big-O term as in (2.4), under the assymptions that $f(x)$ and let $g(x)$ be are smooth and $g(x)$ is compactly supported on $\mathbb{R}$. We may use their version in the present paper.

Proposition 2.2 (McKee, Sun, and Ye [25]). *Let $f(x)$ be a real-valued function, $2n+3$ times continuously differentiable for $\alpha \le x \le \beta$, and let $g(x)$ be a real-valued* *function, $2n+1$ times continuously differentiable for $\alpha \le x \le \beta$. Let $H_k(x)$ be defined as in (2.1). Assume that there are positive parameters M, N, T, U with*

$$M \ge \beta - \alpha, \tag{2.2}$$

and positive constants C_r such that for $\alpha \le x \le \beta$,

$$|f^{(r)}(x)| \le C_r \frac{T}{M^r}, \ |f^{(2)}(x)| \ge \frac{T}{C_2 M^2}, \ |g^{(s)}(x)| \le C_s \frac{U}{N^s}, \tag{2.3}$$

for $r = 2, \ldots, 2n+3$ and $s = 0, \ldots, 2n+1$. Suppose that $f'(x)$ changes signs only at $x = \gamma$, from negative to positive, with $\alpha < \gamma < \beta$. Let

$$\Delta = \min\Big\{ \frac{\log 2}{C_2}, \frac{1}{C_2^2 \max\limits_{2 \le k \le 2n+3} \{C_k\}} \Big\}.$$

If T is sufficiently large such that $T^{\frac{1}{2n+3}} \Delta > 1$, we have for $n \ge 2$ that

$$\begin{aligned} &\int_\alpha^\beta g(x) e(f(x)) dx \\ = \;& \frac{e\Big(f(\gamma) + \frac{1}{8}\Big)}{\sqrt{f''(\gamma)}} \Big(g(\gamma) + \sum_{j=1}^{n} \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \Big) + \Big[e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \Big]_\alpha^\beta \\ &+ O\Big(\frac{UM^{2n+5}}{T^{n+2} N^{n+2}} \Big(\frac{1}{(\gamma-\alpha)^{n+2}} + \frac{1}{(\beta-\gamma)^{n+2}} \Big)\Big) \\ &+ O\Big(\frac{UM^{2n+4}}{T^{n+2}} \Big(\frac{1}{(\gamma-\alpha)^{2n+3}} + \frac{1}{(\beta-\gamma)^{2n+3}} \Big)\Big) \\ &+ O\Big(\frac{UM^{2n+4}}{T^{n+2} N^{2n}} \Big(\frac{1}{(\gamma-\alpha)^{3}} + \frac{1}{(\beta-\gamma)^{3}} \Big)\Big) + O\Big(\frac{U}{T^{n+1}} \Big(\frac{M^{2n+2}}{N^{2n+1}} + M \Big)\Big), \end{aligned}$$

where

$$\lambda_j = \frac{f^{(j)}(\gamma)}{j!} \text{ for } j = 2, \ldots, 2n+2, \quad \eta_\ell = \frac{g^{(\ell)}(\gamma)}{\ell!} \text{ for } \ell = 0, \ldots, 2n$$

and

$$\varpi_k = \eta_k + \sum_{\ell=0}^{k-1} \eta_\ell \sum_{j=1}^{k-\ell} \frac{C_{k\ell j}}{\lambda_2^j} \sum_{\substack{3 \le n_1, \ldots, n_j \le 2n+3 \\ n_1 + \cdots + n_j = k - \ell + 2j}} \lambda_{n_1} \cdots \lambda_{n_j}, \tag{2.4}$$

with $C_{k\ell j}$ being some constant coefficients.

3. Background on automorphic forms

We will follow the setting and notation in Li [20]. Recall for $m, n \ge 1$ the Kuznetsov trace formula (Kuznetsov [16] and Conrey and Iwaniec [6])

$$\begin{aligned} &\sum_{j\ge1}{}' h(t_j)\omega_j \lambda_j(m)\lambda_j(n) + \frac{1}{4\pi}\int_{\mathbb{R}} h(t)\omega(t)\overline{\eta}\Big(m, \frac12 + it\Big)\eta\Big(n, \frac12 + it\Big)\, dt \\ = \;& \delta(m,n)\frac{H}{2} + \sum_{c\ge1}\frac{1}{2c}\Big\{ S(m,n;c)H^+\Big(\frac{4\pi\sqrt{mn}}{c}\Big) + S(-m,n;c)H^-\Big(\frac{4\pi\sqrt{mn}}{c}\Big)\Big\}. \end{aligned} \tag{3.1}$$

Here $\sum'$ in (3.1) means we are only summing over even Maass forms u_j, $\delta(m,n)$ is the Kronecker delta,

$$\omega_j = \frac{4\pi|\rho_j(1)|^2}{\cosh \pi t_j},\ \omega(t) = 4\pi\frac{|\phi(1,1/2+it)|^2}{\cosh \pi t}, \tag{3.2}$$

$$H = \frac{2}{\pi}\int_0^\infty h(t)\tanh(\pi t)t\ dt,\ H^+(x) = 2i\int_{\mathbb{R}} J_{2it}(x)\frac{h(t)t}{\cosh \pi t}\ dt,$$

$$H^-(x) = \frac{4}{\pi}\int_{\mathbb{R}} K_{2it}(x)\sinh(\pi t)h(t)t\ dt,\ \text{and}\ S(a,b;c) = \sum_{d\bar{d}\equiv 1(\text{mod } c)} e\Big(\frac{da+\bar{d}b}{c}\Big)$$

is the standard Kloosterman sum. Above, J_ν is the J-Bessel function.

We let f be a Maass form of type $\nu = (\nu_1,\nu_2)$ for $SL_3(\mathbb{Z})$ (cf. Goldfeld [8]). Then f has a Whittaker function expansion

$$f(z) = \sum_{\pm\Gamma^\infty\backslash SL_2(\mathbb{Z})}\sum_{m_1=1}^{\infty}\sum_{m_2\neq 0}\frac{A(m_1,m_2)}{m_1|m_2|}W_J\Big(M\big(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\big)z,\nu,\psi_{1,1}\Big),$$

where W_J is the Jacquet–Whittaker function, $M = \text{diag}(m_1|m_2|, m_1, 1)$, and $\psi_{1,1}$ is a fixed specific generic character on the abelianization of the standard unipotent upper triangular subgroup of $SL_3(\mathbb{Z})$. Put $\alpha = -\nu_1 - 2\nu_2 + 1$, $\beta = -\nu_1 + \nu_2$, $\gamma = 2\nu_1 + \nu_2 - 1$. These are the Langlands parameters of f at ∞. In the usual way, we put

$$\tilde{\psi}(s) = \int_0^\infty \psi(x)x^{s-1}\ dx$$

to be the Mellin transform of ψ which we assume is smooth and compactly supported on $(0,\infty)$.

For $k = 0, 1$ we define

$$\Psi_k(x) = \int_{\text{Re}s=\sigma}(\pi^3 x)^{-s}\frac{\Gamma(\frac{1+s+2k+\alpha}{2})\Gamma(\frac{1+s+2k+\beta}{2})\Gamma(\frac{1+s+2k+\gamma}{2})}{\Gamma(\frac{-s-\alpha}{2})\Gamma(\frac{-s-\beta}{2})\Gamma(\frac{-s-\gamma}{2})}\tilde{\psi}(-s-k)\ ds.$$

Here σ is taken sufficiently large depending on α,β,γ. We then define, for $k = 0, 1$,

$$\Psi_{0,1}^k(x) = \Psi_0(x) + (-1)^k\frac{1}{x\pi^3 i}\Psi_1(x). \tag{3.3}$$

Then the following is a crucial tool, the Voronoi formula for $GL(3)$.

Lemma 3.1 ([27]). *Let $\psi \in C_c^\infty(0,\infty)$. Let f be a $SL_3(\mathbb{Z})$ Maass form with corresponding Fourier coefficients $A(m,n)$ as in (3). Let $d,\bar{d},c \in \mathbb{Z}$ with $c \neq 0$, $(d,c) = 1$, and $d\bar{d} \equiv 1(\text{ mod } c)$. Then*

(3.4)

$$\sum_{n>0} A(m,n)e\Big(\frac{n\bar{d}}{c}\Big)\psi(n) = \frac{c}{4\pi^{5/2}i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1 n_2}S\Big(md,n_2;\frac{mc}{n_1}\Big)\Psi_{0,1}^0\Big(\frac{n_1^2 n_2}{c^3 m}\Big)$$
$$+\frac{c}{4\pi^{5/2}i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A(n_1,n_2)}{n_1 n_2}S\Big(md,-n_2;\frac{mc}{n_1}\Big)\Psi_{0,1}^1\Big(\frac{n_1^2 n_2}{c^3 m}\Big).$$

To use this formula, asymptotics of Ψ_0, Ψ_1 are needed which were proved in Li [19] and Ren and Ye [29] for $GL(3)$. (For $GL(m)$ see Ren and Ye [30].) Since $x^{-1}\Psi_1(x)$ has similar asymptotics to Ψ_0, following [20], we only deal with Ψ_0. We will use the following lemma ([19]).

Lemma 3.2. *Suppose $\psi \in C_c^\infty([X,2X])$. Then for any fixed integer $K \geq 1$ and $xX \gg 1$ we have*

$$\Psi_0(x) = 2\pi^3 xi \int_0^\infty \psi(y) \sum_{j=1}^{K} \frac{c_j \cos(6\pi(xy)^{1/3}) + d_j \sin(6\pi(xy)^{1/3})}{(xy)^{j/3}}\, dy + O((xX)^{\frac{2-K}{3}}).$$

Here c_j and d_j are constants depending on the Langlands parameters with $c_1 = 0$ and $d_1 = -2/\sqrt{3}\pi$.

We now assume f is a self-dual Hecke–Maass form for $SL_3(\mathbb{Z})$ of type (ν,ν), normalized so that $A(1,1) = 1$. The Rankin–Selberg L-function of f with itself is then defined by

$$L(s, f\times f) = \sum_{m\geq 1}\sum_{n\geq 1} \frac{|A(m,n)|^2}{(m^2n)^s}$$

for Res large. $L(s,f\times f)$ has meromorphic continuation to the complex plane, with a simple pole at $s=1$. By a standard analytic number theory argument using complex analysis, this gives

$$\sum_{m^2 n\leq N} |A(m,n)|^2 \ll_f N.$$

Applying Cauchy–Schwartz, this gives

$$\sum_{n\leq N} |A(m,n)| \ll_f |m|N. \tag{3.5}$$

We will use (3.5) and summation by parts in the estimates below. Here f being self-dual also means $A(m,n) = A(n,m)$ for all m,n.

The Rankin–Selberg L-function of f with u_j is (for Res sufficiently large)

$$L(s, f\times u_j) = \sum_{m\geq 1}\sum_{n\geq 1} \frac{\lambda_j(n)A(m,n)}{(m^2n)^s}.$$

$L(s,f\times u_j)$ can be completed to $\Lambda(s,f\times u_j)$ with six Γ factors at ∞ (involving the Langlands parameters of f, and t_j).

We now need to define the Rankin–Selberg L-function of f with the Eisenstein series. (See Li [20] for the definition of $E(z,s)$ and $\eta(n,s)$.)

$$L(s, f\times E) = \sum_{m\geq 1}\sum_{n\geq 1} \frac{\bar\eta(n,1/2+it)A(m,n)}{(m^2n)^s}.$$

Following Goldfeld [8], comparing Euler products, we have

$$L\Big(\frac{1}{2}, f\times E\Big) = \Big|L\Big(\frac{1}{2}-it, f\Big)\Big|^2.$$

We need to set up the approximate functional equation. We define

$$\begin{aligned}\gamma(s,t) &= \pi^{-3s}\Gamma\Big(\frac{s-it-\alpha}{2}\Big)\Gamma\Big(\frac{s-it-\beta}{2}\Big)\Gamma\Big(\frac{s-it-\gamma}{2}\Big)\\ &\quad\times\Gamma\Big(\frac{s+it-\alpha}{2}\Big)\Gamma\Big(\frac{s+it-\beta}{2}\Big)\Gamma\Big(\frac{s+it-\gamma}{2}\Big).\end{aligned}$$

Here $\alpha = -3\nu+1$, $\beta = 0$, and $\gamma = 3\nu - 1$ are the Langlands parameters of f at ∞. We define $F(u) = \big(\cos(\pi u/A)\big)^{-3A}$ for A a positive integer. For $|\mathrm{Im} t| \leq 1000$

we now define

$$V(y,t) = \frac{1}{2\pi i}\int_{(1000)} y^{-u} F(u) \frac{\gamma(1/2+u,t)}{\gamma(1/2,t)}\ \frac{du}{u}. \tag{3.6}$$

By known bounds for the Langlands parameters, this integral converges. We have the following important approximate functional equation (cf. [20]).

Lemma 3.3. *For f a self-dual Maass form of type (ν,ν) for $SL_3(\mathbb{Z})$ and u_j a Hecke–Maass form for $SL_2(\mathbb{Z})$ corresponding to the eigenvalue $1/4+t_j^2$ in an orthonormal basis, as above,*

$$L\Big(\frac{1}{2}, f\times u_j\Big) = 2\sum_{m\ge1}\sum_{n\ge1}\frac{\lambda_j(n)A(m,n)}{\sqrt{m^2n}} V(m^2n, t_j). \tag{3.7}$$

The point of using V in the expansion (3.7) is that V decays rapidly for $m^2 n \gg |t_j|^{3+\varepsilon}$, and so in an effective way, we can take both sums above to be finite. For the precise decay rate, see Lemma 2.3 of Li [20]. We also have the approximate functional equation for $L(s, f\times E)$:

$$L\Big(\frac{1}{2}, f\times E\Big) = 2\sum_{m\ge1}\sum_{n\ge1}\frac{\eta(n,1/2+it)A(m,n)}{\sqrt{m^2n}} V(m^2n, t). \tag{3.8}$$

Following Li [20] we now define

$$W = \sum_j{}' e^{-(\frac{t_j-T}{M})^2}\omega_j L\Big(\frac{1}{2}, f\times u_j\Big) + \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(\frac{t-T}{M})^2}\omega(t)\Big|L\Big(\frac{1}{2}-it, f\Big)\Big|^2\ dt.$$

Here ω_j and $\omega(t)$ are defined in (3.2). It is known that $\omega_j \gg t_j^{-\varepsilon}$ and $\omega(t)\gg t^{-\varepsilon}$. See the references in Li [20]. It follows that

$$\sum_j{}' e^{-(\frac{t_j-T}{M})^2} L\Big(\frac{1}{2}, f\times u_j\Big) + \frac{1}{4\pi}\int_{\mathbb{R}} e^{-(\frac{t-T}{M})^2}\Big|L\Big(\frac{1}{2}-it, f\Big)\Big|^2\ dt \ll WT^{\varepsilon}.$$

Consequently, Theorem 1.1 will be proved if we show $W \ll_{\varepsilon,f} T^{1+\varepsilon}M$. As Li [20] points out, the function $e^{-(\frac{t-T}{M})^2}$ cannot be used as a test function in the Kuznetsov trace formula simply because it is not even. Following Li [20] we will use the modified function

$$k(t) = e^{-(\frac{t-T}{M})^2} + e^{-(\frac{t+T}{M})^2} \tag{3.9}$$

which essentially captures the size of $e^{-(\frac{t-T}{M})^2}$ for t near T. Thus, we define

$$\mathcal{W} = \sum_j{}' k(t_j)\omega_j L\Big(\frac{1}{2}, f\times u_j\Big) + \frac{1}{4\pi}\int_{\mathbb{R}} k(t)\omega(t)\Big|L\Big(\frac{1}{2}-it, f\Big)\Big|^2\ dt. \tag{3.10}$$

By plugging (3.7) and (3.8) into $\mathcal{W}$ in (3.10) we see that we need to analyze $\mathcal{R}$, which we define by the equation

$$\begin{aligned}\mathcal{R} = 2\sum_j{}' k(t_j)\omega_j \sum_{m\ge1}\sum_{n\ge1}\frac{\lambda_j(n)A(m,n)}{\sqrt{m^2n}} V(m^2n,t_j) g\Big(\frac{m^2n}{N}\Big)\\ + \frac{1}{2\pi}\int_{\mathbb{R}} k(t)\omega(t)\sum_{m\ge1}\sum_{n\ge1}\frac{\eta(n,1/2+it)A(m,n)}{\sqrt{m^2n}} V(m^2n,t) g\Big(\frac{m^2n}{N}\Big)\ dt.\end{aligned} \tag{3.11}$$

Here and for the rest of this article we take $N = T^{3+\varepsilon}$ and g is a fixed nonnegative function with compact support in $[1, 2]$. This is the trick of using a dyadic partition of unity which is best outlined in Lau, Liu, and Ye [18].

Now, we apply the Kuznetsov trace formula (3.1) to $\mathcal{R}$ (3.11). Consequently, we write

$$\mathcal{R} = \mathcal{D} + \mathcal{R}^+ + \mathcal{R}^-; \tag{3.12}$$

$$\mathcal{D} = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g\Big(\frac{m^2 n}{N}\Big)\delta(n,1)H_{m,n}; \tag{3.13}$$

$$H_{m,n} = \frac{2}{\pi}\int_{\mathbb{R}} k(t)V(m^2 n, t)\tanh(\pi t)t\ dt;$$

$$\mathcal{R}^+ = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g\Big(\frac{m^2 n}{N}\Big)\sum_{c>0}\frac{S(n,1;c)}{c}H^+_{m,n}\Big(\frac{4\pi\sqrt{n}}{c}\Big); \tag{3.14}$$

$$H^+_{m,n}(x) = 2i\int_{\mathbb{R}} J_{2it}(x)\frac{k(t)V(m^2 n,t)t}{\cosh(\pi t)}\ dt; \tag{3.15}$$

$$\mathcal{R}^- = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g\Big(\frac{m^2 n}{N}\Big)\sum_{c>0}\frac{S(n,-1;c)}{c}H^-_{m,n}\Big(\frac{4\pi\sqrt{n}}{c}\Big); \tag{3.16}$$

$$H^-_{m,n}(x) = \frac{4}{\pi}\int_{\mathbb{R}} K_{2it}(x)\sinh(\pi t)k(t)V(m^2 n,t)t\ dt. \tag{3.17}$$

By the estimates in Section 3 of Li [20], we see easily that $\mathcal{D}$ in (3.13) is negligible for any M with $T^{\varepsilon} \leq M \leq T^{1-\varepsilon}$, and we leave the details for the reader. In the next two sections we will estimate $\mathcal{R}^+$ in (3.14) and $\mathcal{R}^-$ in (3.16).

4. Estimates for the J-Bessel function terms

In this section we provide estimates for $\mathcal{R}^+$ in (3.14). In this section and the next, we show estimates under the assumption $T^{1/3+2\varepsilon} \leq M \leq T^{1/2}$. Following Li [20] we define the parameters

$$C_1 = T^{100}, \text{ and } C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon}M}, \tag{4.1}$$

and we split $\mathcal{R}^+ = \mathcal{R}^+_1 + \mathcal{R}^+_2 + \mathcal{R}^+_3$ with

$$\mathcal{R}^+_1 = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g\Big(\frac{m^2 n}{N}\Big)\sum_{c\geq C_1/m}\frac{S(n,1;c)}{c}H^+_{m,n}\Big(\frac{4\pi\sqrt{n}}{c}\Big), \tag{4.2}$$

$$\mathcal{R}^+_2 = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g\Big(\frac{m^2 n}{N}\Big)\sum_{C_2/m\leq c\leq C_1/m}\frac{S(n,1;c)}{c}H^+_{m,n}\Big(\frac{4\pi\sqrt{n}}{c}\Big), \tag{4.3}$$

$$\mathcal{R}^+_3 = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)}{\sqrt{m^2 n}} g\Big(\frac{m^2 n}{N}\Big)\sum_{c\leq C_2/m}\frac{S(n,1;c)}{c}H^+_{m,n}\Big(\frac{4\pi\sqrt{n}}{c}\Big). \tag{4.4}$$

For $\mathcal{R}_1^+$ in (4.2), Li [20] shifts the integral defining $H_{m,n}^+$ (see (3.15)) and uses an integral representation of the J-Bessel function and Stirling's formula to conclude

$$H_{m,n}^+(x) \ll x^{\frac{3}{4}} T^{\frac{3}{8}} (m^2 n)^{-\frac{3}{8}} T^{1+\varepsilon} M. \tag{4.5}$$

Consequently (4.2) is bounded

$$\mathcal{R}_1^+ \ll T^{\frac{11}{8}+\varepsilon} M \sum_{m \le \sqrt{2N}} \sum_{n \le 2N/m^2} \frac{|A(m,n)|}{m\sqrt{n}} \sum_{c \ge C_1/m} \frac{|S(n,1;c)|}{c} \Big(\frac{\sqrt{n}}{c}\Big)^{\frac{3}{4}} \cdot (m^2 n)^{-\frac{3}{8}}. \tag{4.6}$$

Using Weil's bound for $S(n,1;c)$, we see

$$\sum_{c \ge C_1/m} \frac{|S(n,1;c)|}{c^{\frac{7}{4}}} \ll \sum_{c \ge C_1/m} \frac{c^{\frac{1}{2}+\varepsilon}}{c^{\frac{7}{4}}} \ll \Big(\frac{C_1}{m}\Big)^{-\frac{1}{4}+\varepsilon}. \tag{4.7}$$

By (3.5) and summation by parts, we have

$$\sum_{n \le 2N/m^2} \frac{|A(m,n)|}{\sqrt{n}} \ll m\Big(\frac{N}{m^2}\Big)^{\frac{1}{2}}. \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6) we get

$$\mathcal{R}_1^+ \ll T^{\frac{11}{8}+\varepsilon} M N^{\frac{1}{2}} C_1^{-\frac{1}{4}} \sum_{m \le \sqrt{2N}} \frac{1}{m^{\frac{3}{2}}}. \tag{4.9}$$

Plugging in $C_1 = T^{100}$ from (4.1), $N = T^{3+\varepsilon}$, and noticing that the sum on m in (4.9) converges, we have $\mathcal{R}_1^+ \ll 1$ for any M with $T^\varepsilon \le M \le T^{1-\varepsilon}$.

We now deal with $\mathcal{R}_2^+$ in (4.3). We do not wish to reproduce all the estimates in Li [20], so we will summarize. As used in Liu and Ye [21], [22] and Li [20] we need an integral representation for

$$\frac{J_{2it}(x) - J_{-2it}(x)}{\cosh(\pi t)}$$

from 1.13(69) of [2, vol. 1, p. 59]. Using integration by parts, a change of variables, and the fact that $k(t)$ (recall (3.9)) is a Schwartz function, we define

$$W_{m,n}(x) = T \int_{\mathbb{R}} \widehat{k^*}(\zeta) \cos\Big(x \cosh\Big(\frac{\zeta\pi}{M}\Big)\Big) e\Big(-\frac{T\zeta}{M}\Big)\, d\zeta.$$

Here

$$k^*(t) = e^{-t^2} V(m^2 n, tM + T)$$

is a Schwartz function and $\widehat{k^*}$ is its Fourier transform. We remark that derivatives of $k^*(t)$ are $\ll 1$. In fact, by (3.6) $\frac{\partial^\ell}{\partial t^\ell} V(y, tM+T)$ can be expressed in terms of derivatives of $\gamma(s, tM+T)$ and hence in terms of $\frac{d}{dz}\log\Gamma(z) =: \psi(z)$ and $\psi^{(\ell)}(z)$ (Bateman [1, p. 15, 1.7(1), and p. 45, 1.16(9)]). By their asymptotic expansions in [1, p. 47, 1.18(7), and p. 48, 1.18(9)], we can see

$$\frac{\partial^\ell}{\partial t^\ell} V(y, tM+T) \ll \Big(\frac{M}{T}\Big)^{\ell}.$$

We define

$$W_{m,n}^*(x) = T \int_{\mathbb{R}} \widehat{k^*}(\zeta) e\Big(-\frac{T\zeta}{M} - \frac{x}{2\pi}\cosh\Big(\frac{\zeta\pi}{M}\Big)\Big)\, d\zeta \tag{4.10}$$

so that

$$W_{m,n}(x) = \frac{W^*_{m,n}(x) + W^*_{m,n}(-x)}{2}.$$

The upshot here is that up to a lower-order term (which can be handled in a similar way) and a negligible amount, we have $H^+_{m,n}(x) = 4W_{m,n}(x)$.

The contribution to the integral in (4.10) from $|\zeta| \geq T^\varepsilon$ is a negligible amount, so in what follows we can assume $|\zeta| \leq T^\varepsilon$. The phase $\phi(\zeta)$ in the exponential (4.10) is

$$2\pi\phi(\zeta) = -\frac{T\zeta}{M} - \frac{x}{2\pi}\cosh\Big(\frac{\zeta\pi}{M}\Big).$$

Looking at $\phi'(\zeta)$, we see $W^*_{m,n}(x)$ is negligible for $|x| \leq T^{1-\varepsilon}M$. So in what follows we assume $T^{1-\varepsilon}M \leq |x| \leq T^2$. Using a Taylor expansion in ζ (within the exponential) of

$$e\Big(-\frac{T\zeta}{M} - \frac{x}{2\pi}\cosh\Big(\frac{\zeta\pi}{M}\Big)\Big)$$

in (4.10), using the Fourier transform of a Gaussian, using Parseval's Theorem, completing the square, and working out many estimates, Lau, Liu, and Ye (Lemma 5.1 of [18]) and Li (Proposition 4.1 of [20]) proved similar propositions, estimating $W^*_{m,n}(x)$ by a finite series involving derivatives of $\widehat{k^*}$, based on ideas in Sarnak [31]. For our purposes we can modify the proof of Proposition 4.1 of [20].

Lemma 4.1. 1) *For $|x| \leq T^{1-\varepsilon}M$ we have $W^*_{m,n}(x) \ll_{\varepsilon,A} T^{-A}$.*

2) *For $T^{1-\varepsilon}M \leq |x| \leq T^2$, with $T^{1/3+2\varepsilon} \leq M \leq T^{1/2}$ and $L_1, L_2 \geq 1$,*

(4.11)

$$\begin{aligned}W^*_{m,n}(x) = {} & \frac{TM}{\sqrt{|x|}} e\Big(-\frac{x}{2\pi} + \frac{T^2}{\pi x}\Big) \sum_{l=0}^{L_1} \sum_{0 \leq l_1 \leq 2l} \sum_{\frac{l_1}{4} \leq l_2 \leq L_2} c_{l,l_1,l_2} \frac{M^{2l-l_1}T^{4l_2-l_1}}{x^{l+3l_2-l_1}} \\ & \times \Big[\widehat{k^*}^{(2l-l_1)}\Big(-\frac{2MT}{\pi x}\Big) - \frac{\pi^6 ix}{6!M^6}(y^6\widehat{k^*}(y))^{(2l-l_1)} \\ & + \frac{\pi^{12}i^2x^2}{2!(6!)^2M^{12}}(y^{12}\widehat{k^*}(y))^{(2l-l_1)}\Big(-\frac{2MT}{\pi x}\Big)\Big] \\ & + O\Big(\frac{TM}{\sqrt{|x|}}\Big(\frac{T^4}{|x|^3}\Big)^{L_2+1} + T\Big(\frac{M}{\sqrt{|x|}}\Big)^{2L_1+3} + \frac{T|x|^3}{M^{18}}\Big),\end{aligned}$$

where c_{l,l_1,l_2} are constants depending only on the indices.

Note that part 1) is valid for $T^\varepsilon \leq M \leq T^{1-\varepsilon}$ and part 2) is valid for $T^{1/3+\varepsilon} \leq M \leq \sqrt{T}$ with the assumption of $T^{1-\varepsilon}M \leq |x| \leq T^2$. With our assumption $T^{1/3+2\varepsilon} \leq M \leq \sqrt{T}$ on M, to acquire the desired decay rate of the

$$O\Big(\frac{TM}{\sqrt{|x|}}\Big(\frac{T^4}{|x|^3}\Big)^{L_2+1}\Big) \tag{4.12}$$

term, L_2 could depend on ε. From 1) of Lemma 4.1 and (4.5) we see $\mathcal{R}^+_2$ is negligible. The extra term in the brackets in (4.11), as compared to [20], comes from a degree 2 Taylor expansion in x (with remainder) of $e(-\pi^6 ix\zeta^6/(2\cdot 6!M^6))$.

In the rest of this section, we estimate $\mathcal{R}^+_3$ as in (4.4). By choosing L_1, L_2 large enough (possibly depending on ε) in (4.11) the contribution to $\mathcal{R}^+_3$ from the first two error terms in (4.11) can be made as small as desired. We need to estimate the contribution from the last error term in (4.11). By the support of g we may assume $x^2 = 16\pi^2 n/c^2 \ll N = T^{3+\varepsilon}$. By our assumptions on M and T we then have $T|x|^3/M^{18} \ll T|x|/M^9$. Plugging in $x = 4\pi\sqrt{n}/c$ into $T|x|/M^9$, we estimate this error term contribution to $\mathcal{R}_3^+$ in (4.4), using (4.5), Weil's bound for the Kloosterman sum, and the compact support of g. This error can be seen to be bounded by $O(TN/M^9)$, which is smaller than $O(T^{1+\varepsilon}M)$ by a power of T with our assumption $T^{1/3+2\varepsilon} \le M \le \sqrt{T}$. In the finite series (4.11) with our assumptions we also have $M^{2l-l_1}T^{4l_2-l_1}x^{l_1-l-3l_3} \ll 1$. All the terms in (4.11) are similar and can be estimated in a similar way, so we will only work with the first term. Following Li [20] we define

$$\begin{aligned}(4.13)\qquad \widetilde{\mathcal{R}}_3^+ \;=\;& \frac{i(i+1)MT}{\sqrt{2}\pi}\sum_{m\ge 1}\sum_{n\ge 1}\frac{A(m,n)}{mn^{3/4}}g\Big(\frac{m^2n}{N}\Big)\\ &\times\sum_{c\le C_2/m}\frac{S(n,1;c)}{\sqrt{c}}e\Big(\frac{2\sqrt{n}}{c}-\frac{T^2c}{4\pi^2\sqrt{n}}\Big)\widehat{k^*}\Big(\frac{MTc}{2\pi^2\sqrt{n}}\Big).\end{aligned}$$

Li [20] points out here that even with Weil's bound for $S(n,1;c)$ simple estimates for $\widetilde{\mathcal{R}}_3^+$ are too large. So we expand the Kloosterman sum $S(n,1;c)$ and use the Voronoi formula (Lemma 3.1) with

$$(4.14)\qquad \psi(y) = y^{-\frac{3}{4}}g\Big(\frac{m^2y}{N}\Big)e\Big(\frac{2\sqrt{y}}{c}-\frac{T^2c}{4\pi^2\sqrt{y}}\Big)\widehat{k^*}\Big(\frac{MTc}{2\pi^2\sqrt{y}}\Big).$$

We get

$$(4.15)\quad \widetilde{\mathcal{R}}_3^+ = \frac{(i-1)MT}{\sqrt{2}\pi}\sum_{m\ge 1}\frac{1}{m}\sum_{c\le C_2/m}\frac{1}{\sqrt{c}}\sideset{}{^*}\sum_{d\,(\bmod\, c)}e\Big(\frac{d}{c}\Big)\sum_{n\ge 1}A(m,n)e\Big(\frac{n\bar{d}}{c}\Big)\psi(n),$$

where the innermost sum in (4.15) will be replaced by the right hand side of (3.4).

From the function $g(m^2y/N)$ in (4.14) we can see that $X = N/m^2$. Recall $x = n_2n_1^2/(c^3m)$ from Lemma 3.1. Then by $c \le C_2/m$

$$xX = \frac{n_2n_1^2N}{c^3m^3} \ge \frac{n_2n_1^2N}{C_2^3} = \frac{n_2n_1^2T^{3-3\varepsilon}M^3}{\sqrt{N}} \ge n_2n_1^2T^{3/2-3\varepsilon}M^3 \gg 1.$$

Consequently we can apply Lemma 3.2 to (4.15) with (3.4) to get

$$(4.16)\qquad \Psi_0(x) = \pi^3d_1x^{2/3}\int_0^\infty e(u_1(y))a(y)\,dy - \pi^3d_1x^{2/3}\int_0^\infty e(u_2(y))a(y)\,dy$$

with

$$(4.17)\qquad u_1(y) = \frac{2\sqrt{y}}{c}+3(xy)^{1/3},\ u_2(y) = \frac{2\sqrt{y}}{c}-3(xy)^{1/3},$$

and

$$(4.18)\qquad a(y) = g\Big(\frac{m^2y}{N}\Big)\widehat{k^*}\Big(\frac{MTc}{2\pi^2\sqrt{y}}\Big)e\Big(\frac{-T^2c}{4\pi^2\sqrt{y}}\Big)y^{-13/12}.$$

Note that u_1 has no stationary points; indeed, simple calculus estimates give the first integral in (4.16) a negligible contribution to $\widetilde{\mathcal{R}}_3^+$.

The second integral in (4.16) requires more analysis. As in [20, p. 319], if $x \ge 2\sqrt{N}/(c^3m)$ or $x \le 2\sqrt{N}/(3c^3m)$, then $u_2'(y)$ will be effectively bounded away from zero, making the integral negligible by multiple integration by parts. Thus we assume the contrary in what follows, namely

$$\frac{2\sqrt{N}}{3n_1^2} \le n_2 \le \frac{\sqrt{N}}{n_1^2}. \tag{4.19}$$

We have

$$\int_0^\infty e(u_2(y))a(y)\, dy = \int_{\frac{1}{4}x^2c^6}^{\frac{9}{2}x^2c^6} e(u_2(y))a(y)\, dy. \tag{4.20}$$

We explain the limits of integration. The compact support of the integral on the right side of equation (4.20) follows from the compact support of g, and so that of a. Further, recall $x = n_2n_1^2/(c^3m)$. As Li [20] points out, the stationary phase point of the integral in (4.20) is at $y_0 = x^2c^6$. The constants $1/4$ and $9/2$ in the limits of this integral give a segment that the support of a is contained in, since $g \in C_c^\infty([1,2])$. In (4.18), from the support of g, and since $\widehat{k^*}$ is a Schwartz function, we can assume

$$\frac{N}{m^2} \le y \le \frac{2N}{m^2} \text{ and } \frac{MTc}{2\pi^2\sqrt{y}} \ll T^\varepsilon.$$

Using this information, simple calculus estimates give us

$$u_2^{(r)}(y) \ll T_1M_1^{-r} \text{ for } r = 1, 2, \ldots, 2n_0 + 3 \tag{4.21}$$

and

$$a^{(r)}(y) \ll U_1N_1^{-r} \text{ for } r = 0, 1, 2, \ldots, 2n_0 + 1 \tag{4.22}$$

for y in the segment. Here $n_0 \in \mathbb{N}$ will be chosen in terms of ε_0 later, and

$$M_1 = \frac{N}{m^2},\ T_1 = \frac{\sqrt{N}}{cm},\ N_1 = \frac{N^{3/2}}{T^2cm^3},\ U_1 = \Big(\frac{N}{m^2}\Big)^{-13/12}. \tag{4.23}$$

Further, $u_2^{(2)}(y) \gg T_1M_1^{-2}$ for $y \in [\frac{1}{4}x^2c^6, \frac{9}{2}x^2c^6]$. The condition $N_1 \ge M_1/\sqrt{T_1}$ is then consistent with our assumption $c \le C_2/m$ when $M \ge T^{1/3+2\varepsilon}$.

Then, all assumptions (2.2) and (2.3) are satisfied for parameters in (4.23), and we apply Proposition 2.2 (where we take $n = n_0$). Or, one may use Blomer, Khan, and Young's version in [3]. The main term of the integral in (4.20) is

$$\frac{e(u_2(y_0) \pm 1/8)}{\sqrt{|u_2''(y_0)|}}\Big(a(y_0) + \sum_{j=1}^{n_0} \varpi_{2j}\frac{(-1)^j(2j-1)!!}{(4\pi i\lambda_2)^j}\Big), \tag{4.24}$$

where ϖ_{2j} are defined above and $\lambda_2 = u_2''(y_0)/2$. Notice we have used $\gamma - \alpha \asymp \beta - \gamma \asymp M_1$, with $\alpha = \frac{1}{4}x^2c^6$, $\beta = \frac{9}{2}x^2c^6$, and $\gamma = y_0 = x^2c^6$. To save time in estimates, notice there are no boundary terms here. This is due to the compact support of a, with itself and all of its derivatives zero at $\frac{1}{4}x^2c^6$ and $\frac{9}{2}x^2c^6$. The sum of the four error terms in Proposition 2.2 can be simplified to

$$O\Big(\frac{U_1M_1^{2n_0+2}}{T_1^{n_0+1}N_1^{2n_0+1}}\Big). \tag{4.25}$$

This estimate uses the current assumptions on c and m, and the size of N compared to T. Note that $M_1 \gg N_1$.

We need to estimate this error term, as well as error terms coming from the ϖ_{2j} terms which will be very similar. First we need a nifty estimate from Li [20]. Using the basic definitions, as Li points out (equation (4.22) of [20])

$$\sum_{0\le d\le c}^{*} e\Big(\frac{d}{c}\Big) S(md, n_2; mcn_1^{-1}) = \sum_{u\,(\mathrm{mod}\, mcn_1^{-1})}^{*} S(0, 1+un_1; c) e\Big(\frac{n_2\bar{u}}{mcn_1^{-1}}\Big). \tag{4.26}$$

Here $u\bar{u} \equiv 1(\bmod mcn_1^{-1})$ and

$$S(0,a;c) = \sum_{v(\bmod c)}^{*} e\Big(\frac{av}{c}\Big)$$

is the Ramanujan sum, which is $\ll (a,c)$. Then (4.26) is bounded by

$$\ll \sum_{\substack{u\,(\mathrm{mod}\, mcn_1^{-1})\\ (1+un_1,c)}}^{*} = \sum_{d|c} d \sum_{\substack{u\,(\mathrm{mod}\, mcn_1^{-1})\\ (1+un_1,c)=d}} 1 \ll \sum_{d|c} d \sum_{\substack{u\,(\mathrm{mod}\, mcn_1^{-1})\\ un_1\equiv -1(\bmod d)}} 1. \tag{4.27}$$

Now $(n_1, d) = 1$ and so $\bar{n}_1$ exists (mod d). Thus the last inner sum in (4.27) is over all u with $0 \le u < mcn_1^{-1}$ and $u \equiv -\bar{n}_1(\bmod d)$. The number of such terms is clearly $\asymp mc/(dn_1)$. Plugging this into (4.27) we see that (4.26) is bounded by

$$\sum_{0\le d\le c}^{*} e\Big(\frac{d}{c}\Big) S(md, n_2; mcn_1^{-1}) \ll \frac{mc}{n_1}\sum_{d|c} 1 \ll \frac{mc^{1+\varepsilon}}{n_1}. \tag{4.28}$$

Now let us turn back to (4.15) with (3.4) and (3.3). As we pointed out before, we will only consider the contribution from $\Psi_0(x)$ for $x = n_2n_1^2/(c^3m)$. In other words,

$$\begin{aligned}\widetilde{\mathcal{R}}_3^{+} \ll MT \sum_{m\le C_2}\frac{1}{m}\sum_{c\le C_2/m} c^{1/2}\sum_{n_1|cm}\sum_{n_2>0}\frac{|A(n_2,n_1)|}{n_1n_2}\\ \times \Big|\Psi_0\Big(\frac{n_2n_1^2}{c^3m}\Big)\Big|\,\Big|\sum_{0\le d\le c}^{*} e\Big(\frac{d}{c}\Big)S(md,n_2;mcn_1^{-1})\Big|.\end{aligned} \tag{4.29}$$

We know that we need to actually consider the contribution from the second term in (4.16). Using (4.28), (4.29) can be reduced to

$$\begin{aligned}\widetilde{\mathcal{R}}_3^{+} \ll MT \sum_{m\le C_2} m^{-2/3}\sum_{c\le C_2/m} c^{-1/2+\varepsilon}\sum_{n_1|cm} n_1^{-2/3}\sum_{n_2\asymp\sqrt{N}/n_1^2}\\ \times \frac{|A(n_2,n_1)|}{n_2^{1/3}}\int_0^\infty e(u_2(y))a(y)dy.\end{aligned} \tag{4.30}$$

The following lemma is specific to the estimation of (4.30).

Lemma 4.2. *Assume $\alpha \ge -1/2$ and $\delta - \alpha \ge 1/6$. Suppose we have a term bounded by $O(c^\alpha T^\beta N^\gamma m^\delta)$ with specific numbers α, β, δ, and γ for the integral in (4.30). Then the contribution of this term to $\widetilde{\mathcal{R}}_3^{+}$ is*

$$\ll M^{2/3-\delta-2\varepsilon}T^{13/6+\beta+3\gamma+\delta/2+\varepsilon_1},$$

where ε is arbitrarily small from (4.28) and $\varepsilon_1 = \varepsilon(11/6 + 3\delta/2 + \gamma) + 3\varepsilon^2$.

Proof. By (4.30) the contribution of $O(c^\alpha T^\beta N^\gamma m^\delta)$ to $\widetilde{\mathcal{R}}_3^+$ is

(4.31)
$$\ll MT\sum_{m\le C_2} m^{-2/3}\sum_{c\le C_2/m} c^{-1/2+\varepsilon}\sum_{n_1|cm} n_1^{-2/3}\sum_{n_2\asymp\sqrt{N}/n_1^2}\frac{|A(n_1,n_2)|}{n_2^{1/3}}c^\alpha T^\beta N^\gamma m^\delta.$$

Note that the innermost sum in (4.31) is over (4.19). Also note Li [20] seems to have used the estimate $(mc)^{1+\varepsilon}$ instead of the estimate $mc^{1+\varepsilon}/n_1$ from (4.28). Since the sum on n_1 is a divisor sum, this is not an issue here. Using the estimates for $|A(n_1,n_2)|$ (see (3.5)) and partial summation one has

$$\sum_{n_2\asymp\sqrt{N}/n_1^2}\frac{|A(n_1,n_2)|}{n_2^{1/3}}\ll n_1\Big(\frac{\sqrt{N}}{n_1^2}\Big)^{2/3}.$$

Since the number of divisors of cm is $\ll (cm)^\varepsilon$ this simplifies the contribution to (4.31) to

$$\ll\ MT^{1+\beta}N^{1/3+\gamma}\sum_{m\le C_2} m^{-2/3+\varepsilon+\delta}\sum_{c\le C_2/m} c^{-1/2+2\varepsilon+\alpha}. \tag{4.32}$$

From a calculus estimate, we have

$$\sum_{c\le C_2/m} c^{-1/2+2\varepsilon+\alpha}\ll\Big(\frac{C_2}{m}\Big)^{1/2+2\varepsilon+\alpha},$$

because $\alpha\ge -1/2$ and $m\le C_2$. Plugging this into (4.32) and using $C_2 = \sqrt{N}/(T^{1-\varepsilon}M)$ we have

$$\ll MT^{1+\beta}N^{1/3+\gamma}\Big(\frac{\sqrt{N}}{T^{1-\varepsilon}M}\Big)^{1/2+2\varepsilon+\alpha}\sum_{m\le C_2} m^{-7/6+\delta-\alpha-\varepsilon}. \tag{4.33}$$

Now, since $\delta-\alpha\ge 1/6$, we have

$$\sum_{m\le C_2} m^{-7/6+\delta-\alpha-\varepsilon}\ll C_2^{-1/6+\delta-\alpha-\varepsilon}+1\ll C_2^{-1/6+\delta-\alpha}, \tag{4.34}$$

because $C_2=\sqrt{N}/(T^{1-\varepsilon}M) = T^{1/2+\varepsilon}/M\ge T^\varepsilon$. Inserting (4.34) into (4.33), we see (4.31) is bounded by

$$\ll M^{2/3-\delta-2\varepsilon}T^{2/3+\beta-\delta+\varepsilon(\delta-5/3+2\varepsilon)}N^{1/2+\gamma+\delta/2+\varepsilon}.$$

Now plugging in $N=T^{3+\varepsilon}$ gives our lemma. □

Now let us turn back to the error term (4.25). By (4.23), (4.25) can be written as

$$O\Big(c^{3n_0+2}T^{4n_0+2}N^{-\frac{3}{2}n_0-\frac{13}{12}}m^{3n_0+\frac{13}{6}}\Big). \tag{4.35}$$

Since $(3n_0+13/6)-(3n_0+2)=1/6$, we may apply Lemma 4.2 to (4.35) and get its contribution to $\widetilde{\mathcal{R}}_3^+$ as

$$O(M^{-3n_0-3/2-2\varepsilon}T^{n_0+2+\varepsilon_1}), \tag{4.36}$$

where $\varepsilon>0$ is arbitrarily small as in (4.28) and $\varepsilon_1=\varepsilon(3n_0+4)+3\varepsilon^2$. For any $\varepsilon_0>0$ arbitrarily small, we want to make (4.36) $\ll T^{1+\varepsilon_0}M$. This can be done if

$$M\ge T^{\frac{n_0+1+\varepsilon_1-\varepsilon_0}{3n_0+5/2+\varepsilon}}. \tag{4.37}$$

We will choose n_0 later depending on ε_0. Notice that if $n_0 = 1/2$, we pick up the 3/8 constant of Li [20] from (4.37). This concludes the estimation of contribution of error terms (4.25) in Proposition 2.2 to $\widetilde{R}_3^+$.

We now need to deal with the ϖ_{2j} terms in (4.24) and their contribution to $\widetilde{R}_3^+$. Recall the expression for ϖ_{2j} in (2.4). Here we take $2 \le 2j \le 2n_0$. One can see from (2.4) that the main term from ϖ_{2j} is $a^{(2j)}(y_0)$. (Here $a(y)$ given in (4.18) and $u_2(y)$ in (4.17) take the place of g and f in Proposition 2.2. Further y_0 takes the place of γ.) Using the estimates in (4.21) and (4.22) along with $|u_2''(y_0)| \gg T_1/M_1^2$ and along with our current assumptions on c and m in (4.13), we have

$$\varpi_{2j} - a^{(2j)}(y_0) = O\Big(\frac{U_1}{M_1 N_1^{2j-1}}\Big). \tag{4.38}$$

The constant ultimately depends on n_0, and we have used $M_1 \gg N_1$. To estimate the contribution of this error term (4.38) to $\widetilde{\mathcal{R}}_3^+$, we must divide by $\lambda_2^{j+\frac{1}{2}}$ and sum over j. (See (4.24).) Since $y_0 \asymp N/m^2$, we have $\lambda_2 \asymp m^3 N^{-3/2}/c$. We then have that this contribution is

$$\ll \Big(\frac{N}{m^2}\Big)^{-\frac{25}{12}} \Big(\frac{T^2 c m^3}{N^{\frac{3}{2}}}\Big)^{2j-1} \Big(\frac{cN^{\frac{3}{2}}}{m^3}\Big)^{j+\frac{1}{2}} = O\Big(c^{3j-\frac{1}{2}} T^{4j-2} N^{-\frac{3}{2}j+\frac{1}{6}} m^{3j-\frac{1}{3}}\Big).$$

Since $(3j - 1/3) - (3j - 1/2) = 1/6$, by Lemma 4.2 the nonleading terms (4.38) of ϖ_{2j} contribute the following to $\widetilde{R}_3^+$:

$$O\big(M^{1-3j-2\varepsilon} T^{j+1/2+\varepsilon_1}\big) \text{ with } \varepsilon_1 = \varepsilon(3j + 3/2) + 3\varepsilon^2, \tag{4.39}$$

which is

$$\ll T^{1+\varepsilon_0} M \text{ if } M \ge T^{\frac{j-1/2+\varepsilon_1-\varepsilon_0}{3j+2\varepsilon}}. \tag{4.40}$$

So we have

$$\frac{j - 1/2 + \varepsilon_1 - \varepsilon_0}{3j + 2\varepsilon} \le \frac{1}{3} - \frac{1}{6j} + 3\varepsilon$$

for $j \ge 1$. Thus the condition on M in (4.40) is always true for $M \ge T^{1/3}$.

We must now estimate the $a^{(2j)}(y_0)$ term in ϖ_{2j} in (4.24). Recall that $a(y)$ is given in (4.18). Then $a^{(2j)}(y)$ will consist of a sum of terms of the following form. Let i_1 be the number of times $g(m^2y/N)$ is differentiated (with respect to y) plus the number of times a power of y is differentiated. So at every differentiation either the factor m^2/N comes out or, up to a constant, the factor $1/y$ comes out. Notice that $1/y \asymp m^2/N$. Let i_2 be the number of times $\widehat{k^*}\Big(\frac{MTc}{2\pi^2\sqrt{y}}\Big)$ is differentiated, and put i_3 to be the number of times $e\Big(\frac{-T^2c}{4\pi^2\sqrt{y}}\Big)$ is differentiated. (Note that we have no restriction on the order of differentiation and that $a^{(2j)}(y)$ will be a sum of these terms over different possible orders of differentiation with various coefficients.) Then $i_1 + i_2 + i_3 = 2j$, and neglecting coefficients (which ultimately depend on n_0), $a^{(2j)}(y_0)$ is bounded by the sum over all combinatorial possibilities of

$$\Big(\frac{N}{m^2}\Big)^{-\frac{13}{12}-i_1} \Big(\frac{MTcm^3}{N^{\frac{3}{2}}}\Big)^{i_2} \Big(\frac{T^2cm^3}{N^{\frac{3}{2}}}\Big)^{i_3}. \tag{4.41}$$

The main term is (4.41) when $i_3 = 2j$, and we will estimate this separately, below. So we can assume in all terms (4.41), now, that $i_1 + i_2 \ge 1$. To estimate this error term, which is all but one term in $a^{(2j)}(y_0)$, as before, in (4.24), we must divide by $\lambda_2^{j+\frac{1}{2}}$, where $\lambda_2 \asymp m^3N^{-3/2}/c$ with our assumption on y_0. We have then a sum of error terms which are all

$$O\Big(M^{i_2}c^{j+i_2+i_3+\frac{1}{2}}T^{i_2+2i_3}N^{\frac{3}{2}j-i_1-\frac{3}{2}i_2-\frac{3}{2}i_3-\frac{1}{3}}m^{-3j+2i_1+3i_2+3i_3+\frac{2}{3}}\Big). \tag{4.42}$$

Using $i_3 = 2j - i_1 - i_2$, by Lemma 4.2 this error term (4.42) can be seen to be

$$\ll M^{-3j+i_1+i_2-\varepsilon}T^{j-i_1-i_2+\frac{3}{2}+\varepsilon} \le T^{1+\varepsilon_0}M \text{ if } M \ge T^{\frac{j-i_1-i_2+\frac{1}{2}+\varepsilon_1-\varepsilon_0}{3j-i_1-i_2+1+\varepsilon}}. \tag{4.43}$$

Here $\varepsilon_1 = \varepsilon(3j - i_1 + 9/2) + 3\varepsilon^2$. Now

$$\frac{j-i_1-i_2+\frac{1}{2}+\varepsilon_1-\varepsilon_0}{3j-i_1-i_2+1+\varepsilon} \le \frac{j-i_1-i_2+\frac{1}{2}}{3j-i_1-i_2+1} + 10\varepsilon.$$

We are assuming $1 \le i_1 + i_2 \le 2j$ with $j \ge 1$, and so

$$\frac{j-i_1-i_2+\frac{1}{2}}{3j-i_1-i_2+1} + 10\varepsilon \le \frac{1}{3} - \frac{1}{6j} + 10\varepsilon.$$

Consequently, the latter condition on M in (4.43) is always true for $M \ge T^{1/3}$.

This leaves the main term of $a^{(2j)}(y_0)$ (where $i_3 = 2j$ and $i_1 = i_2 = 0$), which is

$$\alpha_j \left(\frac{T^2c}{y^{\frac{3}{2}}}\right)^{2j} g\Big(\frac{m^2y}{N}\Big)\widehat{k^*}\Big(\frac{MTc}{2\pi^2\sqrt{y}}\Big)e\Big(\frac{-T^2c}{4\pi^2\sqrt{y}}\Big)y^{-13/12}\Big|_{y_0} =: a_{2j}(y_0). \tag{4.44}$$

Here, the constant α_j depends on j which ultimately can be bounded in terms of n_0. If we estimate this similarly, we will get an estimate similar to (4.37) with $2j$ replacing n_0. Instead, we will apply the Voronoi formula to (4.44). This is very similar to Li [20], in applying the Voronoi formula a second time, but only to the main term

$$\frac{e(u_2(y_0)+1/8)}{\sqrt{|u_2''(y_0)|}}a(y_0)$$

in (4.24). It appears that the term $(T^2cy^{-\frac{3}{2}})^{2j}$ in (4.44) for $1 \le j \le n_0$ is on average $\asymp 1$ in summing over m and c, and so we do not improve upon the second application of Voronoi to the term for just $j = 0$.

Recall that in (4.16) we have

$$x = \frac{n_2n_1^2}{c^3m}, \quad y_0 = x^2c^6 = \frac{n_2^2n_1^4}{m^2}.$$

Further, $\lambda_2 = \frac{1}{12}c^{-1}y_0^{-\frac{3}{2}}$. The contribution to $\widetilde{\mathcal{R}}_3^+$ of $a_{2j}(y_0)$ in (4.13) is then $\asymp \widetilde{\mathcal{R}}_{3,j}^+$, where

$$\begin{aligned}\widetilde{\mathcal{R}}_{3,j}^+ = MT\sum_{m\le C_2}\frac{1}{m}\sum_{c\le C_2/m}\frac{1}{c^{\frac{1}{2}}}\sum_{n_1|cm}\sum_{n_2>0}c\frac{A(n_1,n_2)}{n_1n_2} \\ \times \sideset{}{^*}\sum_{u\,(\mathrm{mod}\, mcn_1^{-1})} S(0,1+un_1;c)e\Big(\frac{n_2\bar{u}}{mcn_1^{-1}}\Big)e(-xc^2)x^{\frac{2}{3}}\frac{a_{2j}(y_0)}{\lambda_2^{j+\frac{1}{2}}}.\end{aligned} \tag{4.45}$$

Inserting what x, y_0, and λ_2 are in terms of n_1, n_2, c, and m into (4.45) we have

$$\widetilde{\mathcal{R}}^+_{3,j} = MT^{4j+1}\sum_{m\le C_2} m^{3j-1}\sum_{c\le C_2/m} c^{3j-1}\sum_{n_1|cm}\frac{1}{n_1^{6j+1}}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_2^{3j+1}} \tag{4.46}$$
$$\times \sideset{}{^*}\sum_{u\,(\mathrm{mod}\, mcn_1^{-1})} S(0,1+un_1;c)e\Big(\frac{n_2\bar{u}}{mcn_1^{-1}}\Big)e\Big(-\frac{n_2n_1^2}{cm}\Big)$$
$$\times g\Big(\frac{n_2^2n_1^4}{N}\Big)\widehat{k^*}\Big(\frac{MTcm}{2\pi^2n_2n_1^2}\Big)e\Big(-\frac{T^2cm}{4\pi^2n_2n_1^2}\Big).$$

In (4.46) we can switch the sums over n_2 and u, and pull out $S(0,1+un_1;c)$ which does not depend on n_2. Then the inner sum on n_2 is

$$\sum_{n_2>0}A(n_2,n_1)e\Big(\frac{n_2u'}{c'}\Big)b_j(n_2), \tag{4.47}$$

where

$$b_j(y)=\frac{1}{y^{3j+1}}g\Big(\frac{y^2n_1^4}{N}\Big)\widehat{k^*}\Big(\frac{MTcm}{2\pi^2yn_1^2}\Big)e\Big(-\frac{T^2cm}{4\pi^2yn_1^2}\Big) \tag{4.48}$$

and

$$\frac{u'}{c'}=\frac{\bar{u}-n_1}{mcn_1^{-1}},\ \text{with } (u'c')=1 \text{ and } c'|mcn_1^{-1}. \tag{4.49}$$

We now apply the Voronoi formula for $GL(3)$ (Lemma 3.1) a second time to (4.47). (See (4.25) of Li [20].) We have

$$\sum_{n_2\ge1}A(n_1,n_2)e\Big(\frac{n_2u'}{c'}\Big)b(n_2) \tag{4.50}$$
$$= \frac{c'}{4\pi^{5/2}i}\sum_{l_1|c'n_1}\sum_{l_2>0}\frac{A(l_2,l_1)}{l_1l_2}S(n_1\bar{u}',l_2;n_1c'l_1^{-1})B^0_{0,1}\Big(\frac{l_1^2l_2}{c'^3n_1}\Big)$$
$$+\frac{c'}{4\pi^{5/2}i}\sum_{l_1|c'n_1}\sum_{l_2>0}\frac{A(l_2,l_1)}{l_1l_2}S(n_1\bar{u}',-l_2;n_1c'l_1^{-1})B^1_{0,1}\Big(\frac{l_1^2l_2}{c'^3n_1}\Big).$$

(We followed Li [20] in using the notation B rather than Ψ.) From (4.50) we have $x=l_2l_1^2/(c'^3n_1)$. From the function $g(y^2n_1^4/N)$ in (4.48) we have $X=\sqrt{N}/n_1^2$. Then

$$xX=\frac{l_2l_1^2\sqrt{N}}{c'^3n_1^3}\ge\frac{l_2l_1^2\sqrt{N}}{c^3m^3}\ge\frac{l_2l_1^2\sqrt{N}}{C_2^3}\ge l_2l_1^2T^{3/2-3\varepsilon}M^3\gg1$$

by (4.49). Consequently we can apply Lemma 3.2 to $B_0(x)$ in (4.50) which is, up to a negligible amount and lower order terms (up to a constant),

$$x^{2/3}\int_0^\infty e(v_2(y))q_j(y)\,dy, \tag{4.51}$$

where

$$v_2(y)=-3(xy)^{1/3}-\frac{T^2cm}{4\pi^2yn_1^2} \tag{4.52}$$

and

$$q_j(y)=y^{-3j-\frac{4}{3}}g\Big(\frac{y^2n_1^4}{N}\Big)\widehat{k^*}\Big(\frac{MTcm}{2\pi^2yn_1^2}\Big). \tag{4.53}$$

See equation (4.26) of Li [20]. We need only consider the case

$$\frac{T^6c^3m^3n_1^2}{10^3\pi^6N^2} \le x \le \frac{T^6c^3m^3n_1^2}{10\pi^6N^2}.$$

Thus

$$x = \frac{l_2l_1^2}{c'^3n_1} \asymp \frac{T^6c^3m^3n_1^2}{\pi^6N^2}. \tag{4.54}$$

By the compact support of g, we may assume the integral (4.51) is taken over a compact segment in y so that $1 \le y^2n_1^4/N \le 2$. With these assumptions, differentiating (4.52) we have

$$|v_2''(y)| \gg \frac{T^2cmn_1^4}{N^{3/2}}.$$

By (4.53) the variation of q_j over this interval can be seen to be $\ll y_0^{-3j-\frac{4}{3}}T^\varepsilon$. This computation uses basic estimates with simple calculus. Also needed is that

$$y \asymp \frac{\sqrt{N}}{n_1^2}, \quad n_1 \le cm \le C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon}M}, \quad \text{and} \quad M \ge T^{1/3+2\varepsilon}.$$

Then, by the second derivative test (see Huxley [11]), we have by (4.54) that

$$\begin{aligned} B_0(x) &\ll \Big(\frac{l_2l_1^2}{c'^3n_1}\Big)^{\frac{2}{3}}\Big(\frac{T^2cmn_1^4}{N^{-3/2}}\Big)^{-1/2}\Big(\frac{\sqrt{N}}{n_1^2}\Big)^{-3j-\frac{4}{3}}T^\varepsilon \\ &\ll T^{3+\varepsilon}c^{3/2}N^{-\frac{3}{2}j-\frac{5}{4}}n_1^{6j+2}m^{3/2}. \end{aligned} \tag{4.55}$$

Put

$$L_2 = \frac{T^6c^3m^3n_1^3c'^3}{\pi^6N^2l_1^2}.$$

Combining (4.55), (4.46), and (4.50) we see

(4.56)

$$\begin{aligned} \widetilde{\mathcal{R}}_{3,j}^+ \ll MT^{4j+1} &\sum_{m\le C_2} m^{3j-1} \sum_{c\le C_2/m} c^{3j-1} \sum_{n_1|cm} \frac{1}{n_1^{6j+1}} \sum_{u(\bmod mcn_1^{-1})} (1+un_1,c)c' \\ &\times \sum_{l_1|c'n_1}\sum_{l_2\asymp L_2} \frac{|A(l_1,l_2)|}{l_1l_2} \times \Big(\frac{n_1c'}{l_1}\Big)(T^{3+\varepsilon}c^{\frac{3}{2}}N^{-\frac{3}{2}j-\frac{5}{4}}n_1^{6j+2}m^{\frac{3}{2}}). \end{aligned}$$

Here $l_2 \asymp L_2$ means $L_2/10^3 \le l_2 \le L_2/10$. Also, we have used the trivial bound for the Kloosterman sum:

$$\Big|S\Big(n_1\bar{u}, l_2; \frac{n_1c'}{l_1}\Big)\Big| \le \frac{n_1c'}{l_1}.$$

Using the estimate (4.28) and that $c' \le mc/n_1$, we deduce from (4.56) that

$$\begin{aligned} \widetilde{\mathcal{R}}_{3,j}^+ \ll N^{-\frac{3}{2}j-\frac{5}{4}}MT^{4+\varepsilon} &\sum_{m\le C_2} m^{3j+\frac{7}{2}} \sum_{c\le C_2/m} c^{3j+\frac{7}{2}+\varepsilon} \\ &\times \sum_{n_1|cm}\frac{1}{n_1}\sum_{l_1|c'n_1}\frac{1}{l_1^2}\sum_{l_2\asymp L_2}\frac{|A(l_1,l_2)|}{l_2}. \end{aligned} \tag{4.57}$$

Now

$$\sum_{l_2\asymp L_2}\frac{|A(l_1,l_2)|}{l_2} \ll l_1L_2^\varepsilon \ll l_1^{1-2\varepsilon}\frac{T^{6\varepsilon}c^{6\varepsilon}m^{6\varepsilon}}{N^{2\varepsilon}}, \tag{4.58}$$

$$(4.59)\qquad \sum_{l_1|c'n_1}\frac{1}{l_1^{1+2\varepsilon}}=O(\varepsilon^{-1}),\quad \sum_{n_1|cm}\frac{1}{n_1}\le\sum_{n_1\le cm}\frac{1}{n_1}\ll c^\varepsilon m^\varepsilon.$$

Consequently by (4.58) and (4.59), (4.57) is bounded by

$$\widetilde{\mathcal{R}}_{3,j}^{+}\ll N^{-\frac32 j-\frac54}MT^{4j+4+7\varepsilon}\sum_{m\le C_2}m^{3j+\frac72+7\varepsilon}\sum_{c\le C_2/m}c^{3j+\frac72+8\varepsilon}.$$

Simple calculus and similar estimates then give us

$$(4.60)\qquad \widetilde{\mathcal{R}}_{3,j}^{+}\ll N^{-\frac32 j-\frac54}MT^{4j+4+7\varepsilon}C_2^{3j+\frac92+8\varepsilon}.$$

Plugging in $N=T^{3+\varepsilon}$ and $C_2=\sqrt{N}/(T^{1-\varepsilon}M)$ into (4.60), we see

$$(4.61)\qquad \widetilde{\mathcal{R}}_{3,j}^{+}\ll M^{-3j-\frac72-8\varepsilon}T^{j+\frac52+\varepsilon_2}.$$

Here $\varepsilon_2=\varepsilon(3j+33/2)+12\varepsilon^2$. This final term (4.61) is $\le MT^{1+\varepsilon_0}$ if

$$(4.62)\qquad M\ge T^{\frac{j+\frac32+\varepsilon_2-\varepsilon_0}{3j+\frac92+8\varepsilon}}.$$

Now $0\le j\le n_0$ and (with $0<\varepsilon\le 1/2$)

$$\frac{j+\frac32+\varepsilon_2-\varepsilon_0}{3j+\frac92+8\varepsilon}\le\frac13+\frac{3j}{3j+\frac92}\varepsilon+\frac{33/2}{3j+\frac92}\varepsilon+\frac{12}{3j+\frac92}\varepsilon^2\le\frac13+6\varepsilon.$$

Thus (4.62) is always true for $M\ge T^{\frac13+6\varepsilon}$.

Now we have showed that $\mathcal{R}_1^+\ll 1$ after (4.9) and that $\mathcal{R}_2^+$ is negligible after (4.12). For $\mathcal{R}_3^+$, other than negligible terms, if we take arbitrarily small $\varepsilon_0>0$, we have proved the bound $O(T^{1+\varepsilon_0}M)$ for $M\ge T^{1/3}$ in (4.40) and (4.43), and for $M\ge T^{1/3+6\varepsilon}$ in (4.61) and (4.62), where $\varepsilon>0$ is arbitrarily small independently. The only bound left is (4.36) which is $O(T^{1+\varepsilon_0}M)$ when (4.37) holds, where $\varepsilon>0$ is arbitrarily small as in (4.28) and $\varepsilon_1=\varepsilon(3n_0+4)+3\varepsilon^2$. To have $O(T^{1+\varepsilon_0}M)$ for any $M\ge T^{1/3+\varepsilon_0}$ we require

$$(4.63)\qquad \frac{n_0+1+\varepsilon_1-\varepsilon_0}{3n_0+5/2+\varepsilon}\le\frac13+\varepsilon_0.$$

Solving (4.63) for n_0 we conclude that (4.36) is $\ll T^{1+\varepsilon_0}M$ for $M\ge T^{1/3+\varepsilon_0}$ provided we take n_0 sufficiently large, i.e., if we take sufficiently many main terms in (4.24) when we apply Proposition 2.2:

$$(4.64)\qquad n_0\ge\frac{1}{\varepsilon_0-\varepsilon}\Big(\frac{1}{18}+\frac{11\varepsilon}{9}-\frac{7\varepsilon_0}{6}+\varepsilon^2-\frac{\varepsilon\varepsilon_0}{3}\Big).$$

Here we may simply take $\varepsilon=\varepsilon_0/6$.

Therefore, we have proved that $\mathcal{R}^+$ in (3.14) is bounded by $T^{1+\varepsilon_0}M$ for $M\ge T^{1/3+\varepsilon_0}$ by choosing n_0 satisfying (4.64) and setting the ε in (4.61) equal to $\varepsilon_0/6$.

5. K-Bessel function terms

Following Li [20] we split $\mathcal{R}^-$ as in (3.16) into $\mathcal{R}_1^-+\mathcal{R}_2^-$ with

$$(5.1)\qquad \mathcal{R}_1^- = \sum_{m\ge1}\sum_{n\ge1}\frac{A(m,n)}{(m^2n)^{\frac12}}g\Big(\frac{m^2n}{N}\Big)\sum_{c\ge C/m}c^{-1}S(n,-1;c)H_{m,n}^-\Big(\frac{4\pi\sqrt{n}}{c}\Big),$$

$$(5.2)\qquad \mathcal{R}_2^- = \sum_{m\ge1}\sum_{n\ge1}\frac{A(m,n)}{(m^2n)^{\frac12}}g\Big(\frac{m^2n}{N}\Big)\sum_{c\le C/m}c^{-1}S(n,-1;c)H_{m,n}^-\Big(\frac{4\pi\sqrt{n}}{c}\Big),$$

where $H^-_{m,n}$ is defined in (3.17) and $C = \sqrt{N} + T$. In estimating $\mathcal{R}_1^-$, one can express the K-Bessel function in terms of the I-Bessel function. Set $\sigma = 100$. Then the estimates for the I-Bessel function, along with Li's previous estimates of V (see (4.7) and (5.6) of Li [20]), give a bound for (5.1) (using the trivial bound for the Kloosterman sum):

$$\mathcal{R}_1^- \ll MT^{\sigma+1+\varepsilon} \sum_{m\le\sqrt{2N}} \frac{1}{m^{1+2\sigma}} \sum_{n\le\frac{2N}{m^2}} \frac{A(m,n)}{n^{\frac{1}{2}}} \sum_{c\ge C/m} \frac{1}{c^{2\sigma}} e^{4\pi\frac{\sqrt{n}}{c}}. \tag{5.3}$$

Using $n \le 2N/m^2$ and $c \ge C/m$ we see that $e^{4\pi\sqrt{n}/c} \ll 1$. Further,

$$\sum_{c\ge C/m} \frac{1}{c^{2\sigma}} \ll \Big(\frac{C}{m}\Big)^{1-2\sigma} \quad\text{and}\quad \sum_{n\le\frac{2N}{m^2}} \frac{A(m,n)}{n^{\frac{1}{2}}} \ll m\Big(\frac{2N}{m^2}\Big)^{\frac{1}{2}}.$$

Plugging this into (5.3) and noting that the sum over m converges, we have

$$\mathcal{R}_1^- \ll \sqrt{N}MT^{\sigma+1+\varepsilon}C^{1-2\sigma} \ll 1 \tag{5.4}$$

for ε sufficiently small. Notice this bound holds for $T^\varepsilon \le M \le T^{1-\varepsilon}$.

Following the derivation in Li [20], up to a negligible term, we can write

$$H^-_{m,n}(x) = H^{-;1}_{m,n}(x) + H^{-;2}_{m,n}(x), \tag{5.5}$$

where

$$\begin{aligned} H^{-;j}_{m,n}(x) &= \frac{4M^jT^{2-j}}{\pi} \int_{\mathbb{R}} \int_{|\zeta|\le T^\varepsilon} t^{j-1}e^{-t^2}V(m^2n, tM+T) \\ &\quad \times \cos(x\sinh\zeta) e\Big(-\frac{(tM+T)\zeta}{\pi}\Big)\, dtd\zeta, \end{aligned}$$

for $j = 1, 2$. In (5.5) $H^{-;2}_{m,n}(x)$ is a lower-order term. We only work with $H^{-;1}_{m,n}(x)$, since the analysis with $H^{-;2}_{m,n}(x)$ is similar. Up to a negligible amount, we can write $H^{-;1}_{m,n}(x) = 4Y_{m,n}(x)$, where

$$Y_{m,n}(x) = \frac{Y^*_{m,n}(x) + Y^*_{m,n}(-x)}{2},$$

with

$$Y^*_{m,n}(x) = T\int_{\mathbb{R}} \widehat{k^*}(\zeta) e\Big(-\frac{T\zeta}{M} + \frac{x}{2\pi}\sinh\frac{\zeta\pi}{M}\Big)\, d\zeta. \tag{5.6}$$

The part of the integral over $|\zeta| \ge M^{\varepsilon/2}$ in (5.6) is negligible. Further, with this assumption, it can be shown by integration by parts that $Y^*_{m,n}(x)$ is negligible unless

$$\frac{T}{100} \le |x| \le 100T \text{ and } \frac{x}{M^3} \ll T^{-\varepsilon}, \tag{5.7}$$

which we now assume. Recall $M \ge T^{\frac{1}{3}+2\varepsilon}$. Thus, the sum over c in (5.2) for which

$$c \ge \frac{400\pi\sqrt{N}}{Tm} \quad\text{or}\quad c \le \frac{\sqrt{2}\pi\sqrt{N}}{25Tm}$$

is negligible. We thus may assume

$$\frac{\sqrt{2}\pi\sqrt{N}}{25Tm} \le c \le \frac{400\pi\sqrt{N}}{Tm}$$

and we will denote this by $c \asymp \sqrt{N}/(Tm)$.

Using one more nonzero term in the Taylor expansion than Li [20], estimating, we have

$$Y^*_{m,n}(x) = T\int_{\mathbb{R}} \widehat{k^*}(\zeta) e\Big(-\frac{T\zeta}{M} + \frac{x\zeta}{2M} + \frac{\pi^2 x\zeta^3}{12M^3} + \frac{\pi^4 x\zeta^5}{240M^5} + \frac{\pi^6 x\zeta^7}{2\cdot 7!M^7}\Big)\, d\zeta + O\Big(T\int_{\mathbb{R}} |\widehat{k*}(\zeta)| \frac{|\zeta|^9|x|}{M^9}\Big). \tag{5.8}$$

Now, expanding

$$e\Big(\frac{\pi^2 x\zeta^3}{12M^3} + \frac{\pi^4 x\zeta^5}{240M^5} + \frac{\pi^6 x\zeta^7}{2\cdot 7!M^7}\Big)$$

in (5.8) into a Taylor series of order L_2 (which could depend on ε) we have

$$\begin{aligned} Y^*_{m,n}(x) = T\int_{\mathbb{R}} \widehat{k^*}(\zeta) e\Big(-\frac{(2T-x)\zeta}{2M}\Big) \\ \times \sum_{j_1+j_2+j_3\le L_2} d_{j_1,j_2,j_3}\Big(\frac{x\zeta^3}{M^3}\Big)^{j_1}\Big(\frac{x\zeta^5}{M^5}\Big)^{j_2}\Big(\frac{x\zeta^7}{M^7}\Big)^{j_3}\, d\zeta \\ + O\Big(\frac{T|x|^{L_2+1}}{M^{3L_2+3}} + \frac{T|x|}{M^9}\Big), \end{aligned}$$

where d_{j_1,j_2,j_3} are constants with $d_{0,0,0} = 1$ with the sum taken over $j_1 \ge 0$, $j_2 \ge 0$, and $j_3 \ge 0$. It follows that

$$\begin{aligned} Y^*_{m,n}(x) &= T\sum_{j_1+j_2+j_3\le L_2} \frac{d_{j_1,j_2,j_3}\cdot x^{j_1+j_2+j_3}}{(2\pi i M)^{3j_1+5j_2+7j_3}} k^{*(3j_1+5j_2+7j_3)}\Big(\frac{x-2T}{2M}\Big) \\ &+ O\Big(\frac{T|x|^{L_2+1}}{M^{3L_2+3}}\Big) + O\Big(\frac{T|x|}{M^9}\Big). \end{aligned} \tag{5.9}$$

We take L_2 large enough (possibly depending on ε) so that the first error term in (5.9) is negligible, or rather has as fast an inverse polynomial decay as desired. (Recall (5.7).) The contribution to $\mathcal{R}_2^-$ coming from the error term $O(T|x|/M^9)$ can be seen to be bounded by

$$\frac{T^2}{M^9}\sum_{m\le\sqrt{2N}} \frac{1}{m}\sum_{n\le 2N/m^2} \frac{|A(m,n)|}{n^{\frac12}} \sum_{c\le C/m} \frac{|S(n,-1;c)|}{c}. \tag{5.10}$$

Using Weil's bound for $S(n,-1;c)$ we see

$$\sum_{c\le C/m} \frac{|S(n,-1;c)|}{c} \ll \Big(\frac{C}{m}\Big)^{\frac12+\varepsilon}.$$

Estimating similarly to the above, we see that (5.10) is bounded by

$$\ll \frac{T^2}{M^9} C^{\frac12+\varepsilon}\sqrt{N} = \frac{T^{2+\frac32+\frac34+\varepsilon}}{M^9}.$$

The above is $\ll T^{1+\varepsilon}M$ by a power of T for $M \ge T^{\frac13+2\varepsilon}$.

We take the leading term in the finite series for $Y^*_{m,n}(x)$ in (5.9), as the terms with higher derivatives of k^* can be handled in the same way. It follows that we need to bound

$$\widetilde{\mathcal{R}}_2^- = T\sum_{m\ge1}\sum_{n\ge1} \frac{A(m,n)}{(m^2n)^{\frac12}} g\Big(\frac{m^2n}{N}\Big) \sum_{c\asymp\frac{\sqrt{N}}{Tm}} \frac{S(n,-1;c)}{c} k^*\Big(\frac{4\pi\sqrt{n}/c-2T}{2M}\Big). \tag{5.11}$$

Denote

$$r(y) = g\Big(\frac{m^2y}{N}\Big)k^*\Big(\frac{4\pi\sqrt{y}/c - 2T}{2M}\Big)y^{-\frac{1}{2}},$$

which is a smooth function of compact support. From $x = n_2n_1^2/(c^3m)$ and $X = N/m^2$ we know

$$xX = \frac{n_2n_1^2N}{c^3m^3} \geq \frac{n_2n_1^2N}{C^3} \geq T^{\frac{3}{2}-\varepsilon} \gg 1.$$

Consequently we may apply the Voronoi formula (Lemma 3.1) and its asymptotic expansion (Lemma 3.2) to the sum over n in (5.11). As in Li [20] we only consider $R_0(x)$ (see (5.11) of [20]), which is (up to lower order terms)

$$R_0(x) = 2\pi^4 xi\int_0^\infty r(y)\frac{d_1\sin(6\pi(xy)^{\frac{1}{3}})}{\pi(xy)^{\frac{1}{3}}}\,dy.$$

Li [20] states that (in an equivalent form) if $n_2 \gg \frac{N^{\frac{1}{2}}T^\varepsilon}{M^3n_1^2}$, then $r'(y)x^{-\frac{1}{3}}y^{\frac{2}{3}} \ll T^{-\varepsilon}$. For this assumption on n_2, the integral term in R_0 as well as the contribution to $\widetilde{\mathcal{R}}_2^-$ is found to be negligible.

Thus, we may assume $n_2 \ll \frac{N^{\frac{1}{2}}T^\varepsilon}{M^3n_1^2}$. Now, $r(y)$ is negligible unless

$$\Big|\frac{2\pi\sqrt{y}/c - T}{M}\Big| \leq T^\varepsilon.$$

This gives us an interval of width $\ll T^{1+\varepsilon}Mc^2$ where $y \asymp N/m^2$, and so

$$R_0(x) \ll \Big(\frac{n_2n_1^2}{c^3m}\Big)^{\frac{2}{3}}\Big(\frac{N}{m^2}\Big)^{-\frac{5}{6}}T^{1+\varepsilon}Mc^2.$$

Using this estimate along with (4.28) it follows from (5.11) that

(5.12)

$$\begin{aligned}\widetilde{\mathcal{R}}_2^- &\ll T\sum_{m\leq\sqrt{2N}}\sum_{c\asymp\frac{\sqrt{N}}{Tm}}\sum_{n_1|cm}\sum_{n_2\ll\sqrt{N}T^\varepsilon/(M^3n_1^2)}\frac{|A(n_1,n_2)|}{n_1n_2}\frac{mc^{1+\varepsilon}}{n_1}\\ &\quad\times\Big(\frac{n_2n_1^2}{c^3m}\Big)^{\frac{2}{3}}\Big(\frac{N}{m^2}\Big)^{-\frac{5}{6}}T^{1+\varepsilon}Mc^2\\ &= T^{2+\varepsilon}MN^{-\frac{5}{6}}\sum_{m\leq\sqrt{2N}}m\sum_{c\asymp C/m}c^{1+\varepsilon}\sum_{n_1|cm}n_1^{-\frac{2}{3}}\sum_{n_2\ll\sqrt{N}T^\varepsilon/(M^3n_1^2)}\frac{|A(n_1,n_2)|}{n_2^{\frac{1}{3}}}.\end{aligned}$$

Estimating similarly to the last section, the inner sum in (5.12) is

$$\sum_{n_2\ll\sqrt{N}T^\varepsilon/(M^3n_1^2)}\frac{|A(n_1,n_2)|}{n_2^{1/3}} \ll n_1\Big(\frac{\sqrt{N}T^\varepsilon}{M^3n_1^2}\Big)^{2/3}.$$

Plugging this and

$$\sum_{n_1|cm}\frac{1}{n_1} \ll (cm)^\varepsilon$$

into (5.12) we see

$$\widetilde{\mathcal{R}}_2^- \ll T^{2+5\varepsilon/3}M^{-1}N^{-1/2}\sum_{m\leq\sqrt{2N}}m^{1+\varepsilon}\sum_{c\asymp\frac{\sqrt{N}}{Tm}}c^{1+2\varepsilon}.$$

Now

$$\sum_{c \asymp \frac{\sqrt{N}}{Tm}} c^{1+2\varepsilon} \ll \Big(\frac{\sqrt{N}}{Tm}\Big)^{2+2\varepsilon} \quad \text{and} \quad \sum_{m \leq \sqrt{2N}} \frac{1}{m^{1+\varepsilon}} \ll \frac{1}{\varepsilon}.$$

Consequently, $\widetilde{\mathcal{R}}_2^- \ll T^{\frac{3}{2}+\frac{13}{6}\varepsilon} M^{-1}$. This is clearly smaller than $T^{1+\varepsilon_0}M$ if $M \geq T^{1/4+13\varepsilon/12-\varepsilon_0/2}$.

Together with (5.4) for $T^{\varepsilon} \leq M \leq T^{1-\varepsilon}$, we conclude that $\mathcal{R}^- \ll T^{1+\varepsilon_0}M$ if $M \geq T^{1/3}$. Recall that $\mathcal{D}$ in (3.13) is negligible for $T^{\varepsilon} \leq M \leq T^{1-\varepsilon}$ as we pointed at the end of Section 3. Together with our conclusion at the end of Section 4 for $\mathcal{R}^+$, we have proved that $\mathcal{R}$ in (3.12) is bounded by $O(T^{1+\varepsilon_0}M)$ for $T^{1/3+\varepsilon_0} \leq M \leq T^{1/2}$. This implies Theorem 1.1. □

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Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242-1419
Email address: `mark.mckee.zoso@gmail.com`

School of Mathematics and Statistics, Shandong University, Weihai, Shandong 264209, People's Republic of China
Email address: `hwsun@sdu.edu.cn`

Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242-1419
Email address: `yangbo-ye@uiowa.edu`