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AMBIENT OBSTRUCTION FLOW

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Abstract. We establish fundamental results for a parabolic flow of Riemannian metrics introduced by Bahuaud–Helliwell which is based on the Fefferman–Graham ambient obstruction tensor. First, we obtain local L2 smoothing estimates for the curvature tensor and use them to prove pointwise smoothing estimates for the curvature tensor. We use the pointwise smoothing estimates to show that the curvature must blow up for a finite time singular solution. We also use the pointwise smoothing estimates to prove a compactness theorem for a sequence of solutions with bounded C0 curvature norm and injectivity radius bounded from below at one point. Finally, we use the compactness theorem to obtain a singularity model from a finite time singular solution and to characterize the behavior at infinity of a nonsingular solution.

1. Introduction

1.1. **Introduction.** The uniformization theorem ensures that for a compact two-dimensional Riemannian manifold (M, g), there is a metric $\tilde{g}$ conformal to g for which $(M, \tilde{g})$ has constant sectional curvature equal to K. Moreover, the sign of K can be determined via the Gauss–Bonnet theorem. In higher dimensions, curvature functionals have been used with great success to define and locate optimal metrics in higher dimensions; see [29]. One conformally invariant curvature functional for a 4-dimensional Riemannian manifold (M, g) is given by

$$\mathcal{F}_W^4(g) = \int_M |W_g|^2 \, dV_g,$$

where W_{ijkl} is the Weyl tensor. The negative gradient of $\mathcal{F}_W^4$ is the Bach tensor B_{ij} defined as

$$B_{ij} = -\nabla^k \nabla^l W_{kijl} - \tfrac{1}{2} R^{kl} W_{kijl}.$$

The study of critical metrics for $\mathcal{F}_W^4$, i.e., Bach-flat metrics, has been fruitful. The class of Bach-flat metrics contains, as shown in [5], familiar metrics such as locally conformally Einstein metrics and scalar flat (anti) self-dual metrics.

Another conformally invariant functional for a 4-dimensional Riemannian manifold (M, g) is given by

$$\mathcal{F}_Q^4(g) = \int_M Q(g) \, dV_g,$$

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where $Q(g)$ is a scalar quantity introduced by Branson in [7] called the Q curvature. Via the Chern–Gauss–Bonnet theorem, this functional is related to $\mathcal{F}_W^4$ by $\mathcal{F}_Q^4 = 8\pi^2\chi(M) - \frac{1}{4}\mathcal{F}_W^4$. The Bach tensor is also the gradient of $\mathcal{F}_Q^4$. Unlike the Weyl tensor, the Q curvature is not pointwise conformally covariant.

One can generalize the Q curvature to a scalar quantity defined on n-dimensional Riemannian manifolds (M, g), where n is even. Consider the functionals defined for n even by

$$\mathcal{F}_Q^n(g) = \int_M Q(g)\, dV_g.$$

These functionals are conformally invariant. The gradient of $\mathcal{F}_Q^n$ is a symmetric 2-tensor $\mathcal{O}$, introduced by Fefferman and Graham in [16], called the ambient obstruction tensor. This tensor arises in physics: for example, Anderson and Chruściel use $\mathcal{O}$ in [1] to construct global solutions of the vacuum Einstein equation in even dimensions. In dimension 4, $\mathcal{O}$ is just the Bach tensor. The ambient obstruction tensor is conformally covariant in n dimensions. This is in contrast to the n-dimensional generalization of the Bach tensor, which is only conformally covariant in dimension 4. This fact follows from a result in Graham–Hirachi [19] stating that in even dimensions 6 and greater, the only conformally covariant tensors are essentially W and $\mathcal{O}$. Extending the 4-dimensional case, Fefferman and Graham showed in [17] that $\mathcal{O}$ vanishes for Einstein metrics for all even dimensions. However, there also exist nonconformally Einstein metrics for which $\mathcal{O} = 0$, as shown by Gover and Leitner in [18]. The conformal covariance of $\mathcal{O}$ and the fact that obstruction flat metrics generalize conformally Einstein metrics suggest that studying the critical points of $\mathcal{F}_Q^n$ via its gradient flow may aid in the study of optimal metrics on M. Our main goal is to establish fundamental results for this gradient flow.

1.2. **Main results.** We will continue the study of a variant of the gradient flow of $\mathcal{F}_Q^n$, that was introduced by Bahuaud and Helliwell in [3], establishing fundamental results. This flow, which we will refer to as the **ambient obstruction flow** (AOF), is defined for a family of metrics $g(t)$ on a smooth manifold M by

$$\begin{cases} \partial_t g = (-1)^{\frac{n}{2}}\mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g, \\ g(0) = h. \end{cases} \tag{1.1}$$

The conformal term involving the scalar curvature was added in order to counteract the invariance of $\mathcal{O}$ under the action of the conformal group on the space of metrics on M. In the papers [3, 4] they proved the short time existence and uniqueness, respectively, of solutions to AOF given by (1.1) when M is compact. Mantegazza and Martinazzi provided an existence proof for parabolic quasilinear PDE on compact manifolds in [32]. Kotschwar has given in [27] an alternate uniqueness proof via a classical energy argument without using the DeTurck trick.

Gradient flows have been studied extensively since Hamilton in [21–23] and Perelman in [34–36] (expositions are given in [9, 26, 33]) used the Ricci flow to study the geometry of 3-manifolds. In the past fifteen years, these have begun to include higher order flows. Mantegazza studied a family of higher order mean curvature flows in [31], Kuwert–Schätzle studied the gradient flow of the Willmore functional in [28], Streets studied the gradient flow of $\int_M |\mathrm{Rm}|^2$ in [39], Chen-He studied the Calabi flow in [11, 12], and Kişisel–Sarıoğlu–Tekin studied the Cotton flow in [25].

Bour studied the gradient flows of certain quadratic curvature functionals in [6], including some variants of $\int_M |W|^2$.

Our first result gives pointwise smoothing estimates for the C^0 norms of the derivatives of the curvature. Since the AOF PDE (1.1) is of order n, the maximum principle cannot be used to obtain these estimates. Instead, we first use interpolation inequalities derived by Kuwert and Schätzle in [28] in order to derive local integral Bernstein–Bando–Shi-type smoothing estimates. Then, we use a blowup argument adapted from Streets [40] in order to convert the integral smoothing estimates to pointwise smoothing estimates, as stated in the following theorem. During the proof, we use the local integral smoothing estimates to take a local subsequential limit of renormalized metrics.

Theorem 1.1. *Let $m \geq 0$ and let $n \geq 4$. There exists a constant $C = C(m,n)$ so that if $(M^n, g(t))$ is a complete solution to AOF on $[0,T]$ satisfying*

$$\max\left(1, \sup_{M\times[0,T]} |\mathrm{Rm}|\right) \leq K,$$

then for all $t \in (0,T]$,

$$\sup_M |\nabla^m \mathrm{Rm}|_{g(t)} \leq C\left(K + t^{-\frac{2}{n}}\right)^{1+\frac{m}{2}}.$$

We obtain from the pointwise smoothing estimates two additional theorems. The first theorem gives an obstruction to the long-time existence of the flow. Since the pointwise smoothing estimates do not require that the Sobolev constant be bounded on $[0,T)$, we rule out that the manifold collapses with bounded curvature.

Theorem 1.2. *Let $g(t)$ be a solution to the AOF on a compact manifold M that exists on a maximal time interval $[0,T)$ with $0 < T \leq \infty$. If $T < \infty$, then we must have*

$$\limsup_{t\uparrow T} \|\mathrm{Rm}\|_{C^0(g(t))} = \infty.$$

The second theorem allows us to extract convergent subsequences from a sequence of solutions to AOF with uniform C^0 curvature bound and uniform injectivity radius lower bound. We prove this in section 7 by using the Cheeger–Gromov compactness theorem to obtain subsequential convergence of solutions at one time. Then, after extending estimates on the covariant derivatives of the metrics from one time to the entire time interval, we obtain subsequential convergence over the entire time interval.

Theorem 1.3. *Let $\{(M_k^n, g_k(t), O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed solutions to AOF for $t \in (\alpha,\omega)$, with $t_0 \in (\alpha,\omega)$, such that*

(1) *$|\mathrm{Rm}(g_k)|_{g_k} \leq C_0$ on $M_k \times (\alpha,\omega)$ for some constant $C_0 < \infty$ independent of k,*
(2) *$\mathrm{inj}_{g_k(t_0)}(O_k) \geq \iota_0$ for some constant $\iota_0 > 0$.*

Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ such that $\{(M_{j_k}, g_{j_k}(t), O_{j_k})\}_{k\in\mathbb{N}}$ converges in the sense of families of pointed Riemannian manifolds to a complete pointed solution to AOF $(M_\infty^n, g_\infty(t), O_\infty)$ defined for $t \in (\alpha,\omega)$ as $k \to \infty$.

We use this compactness theorem to prove two corollaries. For a compact Riemannian manifold (M,g), let $C_S(M,g)$ denote the L^2 Sobolev constant of (M,g), defined as the smallest constant C_S such that

$$\|f\|^2_{L^{\frac{2n}{n-2}}} \le C_S\big(\|\nabla f\|^2_{L^2} + V^{-\frac{2}{n}}\|f\|^2_{L^2}\big),$$

where $V = \mathrm{vol}(M, g)$, for all $f \in C^1(M)$. The following result states that if the Sobolev constant and the integral of Q curvature are bounded along the flow, there exists a sequence of renormalized solutions to AOF that converge to a singularity model.

Theorem 1.4. *Let $(M^n, g(t))$, $n \ge 4$, be a compact solution to AOF that exists on a maximal time interval $[0,T)$ with $T < \infty$. Suppose that $\sup\{C_S(M, g(t)) : t \in [0,T)\} < \infty$. Let $\{(x_i, t_i)\}_{i\in\mathbb{N}} \subset M \times [0,T)$ be a sequence of points satisfying $t_i \to T$, $|\mathrm{Rm}(x_i,t_i)| = \sup\{|\mathrm{Rm}(x,t)| : (x,t) \in M \times [0,t_i]\}$, and $\lambda_i \to \infty$, where $\lambda_i = |\mathrm{Rm}(x_i,t_i)|$. Then the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i\in\mathbb{N}}$, with*

$$g_i(t) = \lambda_i g(t_i + \lambda_i^{-\frac{n}{2}} t), \quad t \in [-\lambda_i^{\frac{n}{2}} t_i, 0],$$

subsequentially converges in the sense of families of pointed Riemannian manifolds to a nonflat, noncompact complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF defined for $t \in (-\infty, 0]$. Moreover, if $n = 4$ or

$$\sup_{t\in[0,T)} \int_M Q(g(t))\, dV_{g(t)} < \infty,$$

then $\mathcal{O}(g_\infty(t)) \equiv 0$ for all $t \in (-\infty, 0]$.

The next result states that if a nonsingular solution to AOF does not collapse at time ∞ and the integral of Q curvature is bounded along the flow, there exists a sequence of times $t_i \to \infty$ for which $g(t_i)$ converges to an obstruction flat metric. We note that in cases (2) and (3), the boundedness of the integral of the Q curvature along the flow implies that $g_\infty(t)$ is obstruction flat. However, this does not imply that $\partial_t g_\infty = 0$. Rather, $\partial_t g_\infty = (-1)^{n/2} C(n)(\Delta^{\frac{n}{2}-1} R) g_\infty$, i.e., the metric is still flowing by the conformal term of AOF within the conformal class of $g_\infty(0)$.

Theorem 1.5. *Let $(M, g(t))$ be a compact solution to AOF on $[0,\infty)$ such that*

$$\sup_{t\in[0,\infty)} \|\mathrm{Rm}\|_{C^0(g(t))} < \infty.$$

Then exactly one of the following is true:

(1) *M collapses when $t = \infty$, i.e.,*

$$\lim_{t\to\infty} \inf_{x\in M} \mathrm{inj}_{g(t)}(x) = 0.$$

(2) *There exists a sequence $\{(x_i, t_i)\}_{i\in\mathbb{N}} \subset M \times [0,\infty)$ such that the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i\in\mathbb{N}}$, with*

$$g_i(t) = g(t_i + t), \quad t \in [-t_i, \infty),$$

subsequentially converges in the sense of pointed Riemannian manifolds to a complete noncompact finite volume pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF defined for $t \in (-\infty, \infty)$. If $n = 4$ or

$$\sup_{t\in[0,\infty)} \int_M Q(g(t))\, dV_{g(t)} < \infty,$$

then $g_\infty(t)$ is obstruction flat for all $t \in (-\infty, \infty)$.

(3) *There exists a sequence $\{(x_i, t_i)\}_{i\in\mathbb{N}} \subset M \times [0,\infty)$ such that the sequence of pointed solutions to AOF given by $\{(M, g_i(t), x_i)\}_{i\in\mathbb{N}}$, with*

$$g_i(t) = g(t_i + t), \quad t \in [-t_i, \infty),$$

subsequentially converges in the sense of pointed Riemannian manifolds to a compact pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF defined for $t \in (-\infty,\infty)$, where M_∞ is diffeomorphic to M. If $n = 4$ or

$$\sup_{t\in[0,\infty)} \int_M Q(g(t))\, dV_{g(t)} < \infty,$$

then $g_\infty(t)$ is obstruction flat for all $t \in (-\infty,\infty)$ and there exists a family of metrics $\hat{g}_\infty(t)$ conformal to $g_\infty(t)$ for all $t \in (-\infty,\infty)$, with $\hat{g}_\infty(t) = \hat{g}_\infty(0)$ for all $t \in (-\infty,\infty)$, such that $\hat{g}_\infty(0)$ is obstruction flat and has constant scalar curvature.

2. Background

2.1. **Q curvature.** Here we recall a description of Q curvature given by Chang et al. in [10]. The Q curvature was introduced in 4 dimensions by Riegert in [38] and Branson–Ørsted in [8] and in even dimensions by Branson in [7]. It is a scalar quantity defined on an even-dimensional Riemannian manifold (M^n, g). If $n = 2$, we define Q to be $Q = -\frac{1}{2}R = -K$, where K is the Gaussian curvature of M. The Gauss–Bonnet theorem gives $\int Q\, dV = -2\pi\chi(M)$. The Q curvature of a metric $\tilde{g} = e^{2f}g$ is given by $e^{2f}\widetilde{Q} = Q + \mathscr{P}f$, where the Paneitz operator $\mathscr{P}$ introduced by Graham–Jenne–Mason–Sparling in [20] is given by $\mathscr{P}f = \Delta f$. If $n = 4$, we define Q to be

$$Q = -\tfrac{1}{6}\Delta R - \tfrac{1}{2}R^{ab}R_{ab} + \tfrac{1}{6}R^2.$$

The Chern–Gauss–Bonnet theorem gives

$$\int Q\, dV = 8\pi^2\chi(M) - \tfrac{1}{4}\int |W|^2\, dV.$$

In particular, if M is conformally flat, then $\int Q\, dV = 8\pi^2\chi(M)$. The Q curvature of a metric $\tilde{g} = e^{2f}g$ is given by $e^{4f}\widetilde{Q} = Q + \mathscr{P}f$, where the Paneitz operator $\mathscr{P}$ is given by

$$\mathscr{P}f = \nabla_a[\nabla^a\nabla^b + 2R^{ab} - \tfrac{2}{3}Rg^{ab}]\nabla_b f.$$

In general when n is even, we are only able to write down the highest order terms of Q and $\mathscr{P}$:

$$Q = -\tfrac{1}{2(n-1)}\Delta^{\frac{n}{2}-1}R + \text{lots}, \quad \mathscr{P}f = \Delta^{\frac{n}{2}}f + \text{lots}.$$

Nonetheless, Q still has nice conformal properties. Under a conformal change of metric $\tilde{g} = e^{2f}g$, we have $e^{nf}\widetilde{Q} = Q + \mathscr{P}f$. The integral of Q is conformally invariant. In particular, if M is locally conformally flat, we have an analogue of the Gauss–Bonnet theorem:

$$\int Q\, dV = (-1)^{\frac{n}{2}}(\tfrac{n}{2}-1)!\, 2^{n-1}\pi^{\frac{n}{2}}\chi(M).$$

2.2. **Ambient obstruction tensor.** Fefferman and Graham proposed in [16] a method to determine the conformal invariants of a manifold from the pseudo-Riemannian invariants of an ambient space in which it is embedded. They introduced the **ambient obstruction tensor** $\mathcal{O}$ as an obstruction to such an embedding. They subsequently provided a detailed description of the properties of $\mathcal{O}$ in their monograph [17].

We define several tensors that we will use to express $\mathcal{O}$. The Schouten tensor A, Cotton tensor C, and Bach tensor B are defined as

$$\mathsf{A}_{ij} = \tfrac{1}{n-2}\big(R_{ij} - \tfrac{1}{2(n-1)}Rg_{ij}\big), \quad C_{ijk} = \nabla_k \mathsf{A}_{ij} - \nabla_j \mathsf{A}_{ik}, \quad B_{ij} = \nabla^k C_{ijk} - \mathsf{A}^{kl} W_{kijl}.$$

We obtain via the identity $\nabla^l \nabla^k W_{kijl} = (3-n)\nabla^k C_{ijk}$ that

$$B_{ij} = \tfrac{1}{3-n}\nabla^l \nabla^k W_{kijl} + \tfrac{1}{2-n} R^{kl} W_{kijl}.$$

We define the notation $P_k^m(A)$ for a tensor A by

$$P_k^m(A) = \sum_{i_1+\cdots+i_k=m} \nabla^{i_1} A * \cdots * \nabla^{i_k} A.$$

The following result describes $\mathcal{O}$. The form of the lower order terms is implied by the proofs.

Theorem 2.1 (Fefferman–Graham [17], Theorem 3.8; Graham–Hirachi [19], Theorem 2.1)**.** *Let $n \geq 4$ be even. The obstruction tensor $\mathcal{O}_{ij}$ of g is independent of the choice of ambient metric $\tilde{g}$ and has the following properties:*

(1) *$\mathcal{O}$ is a natural tensor invariant of the metric g; i.e., in local coordinates the components of $\mathcal{O}$ are given by universal polynomials in the components of g, g^{-1}, and the curvature tensor of g and its covariant derivatives, and can be written just in terms of the Ricci curvature and its covariant derivatives. The expression for $\mathcal{O}_{ij}$ takes the form*

$$\begin{aligned}\mathcal{O}_{ij} &= \Delta^{\frac{n}{2}-2}(\Delta \mathsf{A}_{ij} - \nabla_j \nabla_i \mathsf{A}_k{}^k) + \sum_{k=2}^{n/2} P_k^{n-2k}(\mathrm{Rm}) \\ &= \frac{1}{3-n}\Delta^{\frac{n}{2}-2}\nabla^l \nabla^k W_{kijl} + \sum_{k=2}^{n/2} P_k^{n-2k}(\mathrm{Rm}),\end{aligned}$$

where $\Delta = \nabla^i \nabla_i$.

(2) *One has $\mathcal{O}_i{}^i = 0$ and $\nabla^j \mathcal{O}_{ij} = 0$.*

(3) *$\mathcal{O}_{ij}$ is conformally invariant of weight $2-n$; i.e., if $0 < \Omega \in C^\infty(M)$ and $\hat{g}_{ij} = \Omega^2 g_{ij}$, then $\hat{\mathcal{O}}_{ij} = \Omega^{2-n}\mathcal{O}_{ij}$.*

(4) *If g_{ij} is conformal to an Einstein metric, then $\mathcal{O}_{ij} = 0$.*

C. R. Graham and K. Hirachi express the gradient of Q in terms of $\mathcal{O}$.

Theorem 2.2 ([19], Theorem 1.1)**.** *If $g(t)$ is a one-parameter family of metrics on a compact manifold M of even dimension $n \geq 4$ and $h = \partial_t|_{t=0}\, g(t)$, then*

$$\frac{\partial}{\partial t}\bigg|_{t=0} \int_M Q(g(t))\, dV_{g(t)} = (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M \big\langle \mathcal{O}(g(0)), h \big\rangle \, dV_{g(0)}.$$

Define the adjusted ambient obstruction tensor $\widehat{\mathcal{O}}$ to be

$$\widehat{\mathcal{O}} = (-1)^{\frac{n}{2}}\mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g. \tag{2.1}$$

We rewrite $\widehat{\mathcal{O}}$ in terms of the Ricci and scalar curvatures.

Proposition 2.3. *If (M, g) is a Riemannian manifold, then*

$$\mathcal{O} = \Delta^{\frac{n}{2}-1}\mathsf{A} - \frac{1}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \tag{2.2}$$

$$\widehat{\mathcal{O}} = \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).$$

Proof. First, we re-express $\mathcal{O}$:

$$\begin{aligned}
\mathsf{A}_k{}^k &= \tfrac{1}{n-2}\left[g^{jk}R_{kj} - \tfrac{1}{2(n-1)}Rg^{jk}g_{kj}\right] \\
&= \tfrac{1}{n-2}\left[R - \tfrac{n}{2(n-1)}R\right] \\
&= \tfrac{1}{2(n-1)}R
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{O}_{ij} &= \Delta^{\frac{n}{2}-2}(\Delta\mathsf{A}_{ij} - \nabla_j\nabla_i\mathsf{A}_k{}^k) + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\
&= \Delta^{\frac{n}{2}-1}\mathsf{A}_{ij} - \frac{1}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla_j\nabla_i R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).
\end{aligned}$$

Next, we re-express $\widehat{\mathcal{O}}$ using (2.2):

$$\begin{aligned}
\widehat{\mathcal{O}} &= (-1)^{\frac{n}{2}}\mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g \\
&= (-1)^{\frac{n}{2}}\Delta^{\frac{n}{2}-1}\mathsf{A} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\
&\quad + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g \\
&= \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R \\
&\quad + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\
&= \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).
\end{aligned}$$

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3. Short time existence and uniqueness

In this section, we derive the evolution equations for the covariant derivatives of the curvature tensor. We then give a theorem asserting the short time existence and uniqueness of solutions to AOF.

3.1. Preliminaries.

We collect some facts about Riemannian manifolds that will be used to derive the evolution equations.

Lemma 3.1 (Hamilton [21], Lemma 7.2)**.** *On any Riemannian manifold, the following identity holds:*

$$\Delta R_{jklm} = \nabla_j\nabla_m R_{lk} - \nabla_j\nabla_l R_{mk} + \nabla_k\nabla_l R_{mj} - \nabla_k\nabla_m R_{lj} + \mathrm{Rm}^{*2}.$$

The following proposition can be proved by adapting the proof of Proposition 13.22 in Chow et al. [14].

Proposition 3.2. *If A is a tensor on a Riemannian manifold and $k,l \geq 1$, then*

$$\nabla^k\Delta^l A = \Delta^l\nabla^k A + \sum_{i=0}^{2l+k-2} \nabla^{2l+k-2-i}\mathrm{Rm} * \nabla^i A.$$

The following proposition can be proved by adapting the proof of Proposition 13.26 in Chow et al. [14].

Proposition 3.3. *Let M be a manifold and let $g(t)$ be a 1-parameter family of metrics on M. If A is a tensor on M and $k \geq 1$, then*

$$\partial_t\nabla^k A = \nabla^k\partial_t A + \sum_{j=0}^{k-1}\nabla^j(\nabla\partial_t g * \nabla^{k-1-j}A).$$

3.2. Evolution equations.

We derive the equations for $\partial_t\nabla^k\mathrm{Rm}$ for every $k \geq 0$.

Proposition 3.4. *If $(M, g(t))$ is a solution to AOF, then*

$$\partial_t\mathrm{Rm} = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\mathrm{Rm} + \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(\mathrm{Rm}).$$

Proof. Let $\hat{g}(t)$ be a one-parameter family of metrics on M and let $h = \partial_t\hat{g}$. The evolution of Rm is given by ([21], Theorem 7.1)

$$\partial_t R_{ijkl} = \tfrac{1}{2}[\nabla_i\nabla_k h_{jl} + \nabla_j\nabla_l h_{ik} - \nabla_i\nabla_l h_{jk} - \nabla_j\nabla_k h_{il}] + \mathrm{Rm} * h.$$

If $h = \Delta^{\frac{n}{2}-1}\mathrm{Rc}$, then, using Proposition 3.2 in the second line and Lemma 3.1 in the third line,

$$\begin{aligned}
\partial_t R_{ijkl} &= \tfrac{1}{2}[\nabla_i\nabla_k\Delta^{\frac{n}{2}-1}R_{jl} + \nabla_j\nabla_l\Delta^{\frac{n}{2}-1}R_{ik} - \nabla_i\nabla_l\Delta^{\frac{n}{2}-1}R_{jk} - \nabla_j\nabla_k\Delta^{\frac{n}{2}-1}R_{il}] \\
&\quad + \mathrm{Rm} * \Delta^{\frac{n}{2}-1}\mathrm{Rc} \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-1}[\nabla_i\nabla_k R_{jl} + \nabla_j\nabla_l R_{ik} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il}] \\
&\quad + \sum_{i=0}^{n-2}\nabla^{n-2-i}\mathrm{Rm} * \nabla^i\mathrm{Rc} + P_2^{n-2}(\mathrm{Rm}) \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-1}[-\Delta R_{ijkl} + \mathrm{Rm}^{*2}] + P_2^{n-2}(\mathrm{Rm}) \\
&= -\tfrac{1}{2}\Delta^{\frac{n}{2}}R_{ijkl} + P_2^{n-2}(\mathrm{Rm}).
\end{aligned}$$

If $h = \Delta^{\frac{n}{2}-2}\nabla^2 R$, then, using Proposition 3.2 in the second and fourth lines,

$$\begin{aligned}
\partial_t R_{ijkl} &= \tfrac{1}{2}[\nabla_i\nabla_k\Delta^{\frac{n}{2}-2}\nabla_j\nabla_l R + \nabla_j\nabla_l\Delta^{\frac{n}{2}-2}\nabla_i\nabla_k R - \nabla_i\nabla_l\Delta^{\frac{n}{2}-2}\nabla_j\nabla_k R \\
&\quad - \nabla_j\nabla_k\Delta^{\frac{n}{2}-2}\nabla_i\nabla_l R] + \mathrm{Rm} * \Delta^{\frac{n}{2}-2}\nabla^2 R \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-2}[\nabla_i\nabla_k\nabla_j\nabla_l R + \nabla_j\nabla_l\nabla_i\nabla_k R - \nabla_i\nabla_l\nabla_j\nabla_k R - \nabla_j\nabla_k\nabla_i\nabla_l R] \\
&\quad + \sum_{i=0}^{n-2} \nabla^{n-2-i}\mathrm{Rm} * \nabla^i\nabla^2 R + P_2^{n-2}(\mathrm{Rm}) \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-2}[\nabla_i\nabla_k\nabla_j\nabla_l R + \nabla_j\nabla_l\nabla_i\nabla_k R - \nabla_i\nabla_l\nabla_j\nabla_k R - \nabla_j\nabla_k\nabla_i\nabla_l R] \\
&\quad + P_2^{n-2}(\mathrm{Rm}) \\
&= \tfrac{1}{2}\Delta^{\frac{n}{2}-2}[\nabla_i\nabla_k\nabla_j\nabla_l R + \nabla_j\nabla_l\nabla_i\nabla_k R - \nabla_i\nabla_k\nabla_j\nabla_l R - \nabla_j\nabla_l\nabla_i\nabla_k R \\
&\quad + \nabla\mathrm{Rm} * \nabla R + \mathrm{Rm} * \nabla^2 R] + P_2^{n-2}(\mathrm{Rm}) \\
&= P_2^{n-2}(\mathrm{Rm}).
\end{aligned}$$

If $h = \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm})$, then

$$\begin{aligned}
\partial_t \mathrm{Rm} &= \nabla^2 \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) + \mathrm{Rm} * \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}) \\
&= \sum_{j=2}^{n/2} P_j^{n-2j+2}(\mathrm{Rm}) + \sum_{j=2}^{n/2} P_{j+1}^{n-2j}(\mathrm{Rm}).
\end{aligned}$$

Combining these results, we conclude that if $h = \widehat{\mathcal{O}}$, then

$$\begin{aligned}
\partial_t \mathrm{Rm} &= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\mathrm{Rm} + P_2^{n-2}(\mathrm{Rm}) + P_2^{n-2}(\mathrm{Rm}) \\
&\quad + \sum_{j=2}^{n/2} P_j^{n-2j+2}(\mathrm{Rm}) + \sum_{j=2}^{n/2} P_{j+1}^{n-2j}(\mathrm{Rm}) \\
&= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\mathrm{Rm} + \sum_{j=2}^{n/2+1} P_j^{n-2j+2}(\mathrm{Rm}).
\end{aligned}$$

□

Proposition 3.5. *If $(M, g(t))$ is a solution to AOF, then*

$$\partial_t \nabla^k \mathrm{Rm} = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\nabla^k\mathrm{Rm} + \sum_{l=2}^{n/2+1} P_l^{n-2l+k+2}(\mathrm{Rm}).$$

Proof. We compute:

$$\begin{aligned}\sum_{j=0}^{k-1}\nabla^j(\nabla\partial_t g * \nabla^{k-1-j}\mathrm{Rm}) &= \sum_{j=0}^{k-1}\nabla^j\left(\sum_{l=2}^{n/2}P_l^{n-2l+1}(\mathrm{Rm}) * \nabla^{k-1-j}\mathrm{Rm}\right)\\ &= \sum_{j=0}^{k-1}\nabla^j\sum_{l=2}^{n/2}P_{l+1}^{n-2l+k-j}(\mathrm{Rm})\\ &= \sum_{j=0}^{k-1}\sum_{l=2}^{n/2}P_{l+1}^{n-2l+k}(\mathrm{Rm})\\ &= \sum_{l=2}^{n/2}P_{l+1}^{n-2l+k}(\mathrm{Rm})\\ &= \sum_{l=3}^{n/2+1}P_l^{n-2l+k+2}(\mathrm{Rm}).\end{aligned}$$

Then, using Proposition 3.3 in the first line, Proposition 3.4 in the second line, and Proposition 3.2 in the third line, we get

$$\begin{aligned}\partial_t\nabla^k\mathrm{Rm} &= \nabla^k\partial_t\mathrm{Rm} + \sum_{j=0}^{k-1}\nabla^j(\nabla\partial_t g * \nabla^{k-1-j}\mathrm{Rm})\\ &= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\nabla^k\Delta^{\frac{n}{2}}R_{ijkl} + \nabla^k\sum_{j=2}^{n/2+1}P_j^{n-2j+2}(\mathrm{Rm})\\ &\quad + \sum_{l=3}^{n/2+1}P_l^{n-2l+k+2}(\mathrm{Rm})\\ &= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\nabla^k R_{ijkl} + P_2^{n+k-2}(\mathrm{Rm}) + \sum_{j=2}^{n/2+1}P_j^{n-2j+k+2}(\mathrm{Rm})\\ &\quad + \sum_{l=3}^{n/2+1}P_l^{n-2l+k+2}(\mathrm{Rm})\\ &= \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\nabla^k R_{ijkl} + \sum_{l=2}^{n/2+1}P_l^{n-2l+k+2}(\mathrm{Rm}).\end{aligned}$$

□

3.3. **Short time existence and uniqueness.** The ambient obstruction flow is a quasilinear flow of order n in the metric g. E. Bahuaud and D. Helliwell have shown the following existence and uniqueness result for AOF.

Theorem 3.6 ([3], Theorem C; [4], Theorem C)**.** *Let h be a smooth metric on a compact manifold M of even dimension $n \geq 4$. Then there is a unique smooth short time solution to the following flow:*

$$\begin{cases}\partial_t g = \widehat{\mathcal{O}} = (-1)^{\frac{n}{2}}\mathcal{O} + \frac{(-1)^{\frac{n}{2}}}{2(n-1)(n-2)}(\Delta^{\frac{n}{2}-1}R)g,\\ g(0) = h,\end{cases} \tag{3.1}$$

where $\mathcal{O}$ is the ambient obstruction tensor on M and R is the scalar curvature of M.

We will only briefly illustrate that applying the DeTurck trick to the system (3.1) results in a strongly parabolic system. Due to the diffeomorphism invariance of M, the system (3.1) is not strongly parabolic. We define the following vector fields:

$$\begin{aligned} V^k &= g^{ij}(\Gamma_{ij}^k - \Gamma(h)_{ij}^k), \\ X &= \frac{(-1)^{\frac{n}{2}-1}}{2(n-2)}\Delta^{\frac{n}{2}-1}V, \\ Y &= \frac{(-1)^{\frac{n}{2}}}{4(n-1)}(\nabla\Delta^{\frac{n}{2}-2}R)^\sharp, \\ W &= X+Y. \end{aligned}$$

We show that the following system is strongly parabolic:

$$\begin{cases} \partial_t g = \widehat{\mathcal{O}} + \mathcal{L}_W g, \\ g(0) = h. \end{cases} \tag{3.2}$$

We show this by computing the principal symbol σ of the linearization of $\widehat{\mathcal{O}} + \mathcal{L}_W g$ at h. We know from Proposition 2.3 that

$$\widehat{\mathcal{O}} = \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}).$$

We then rewrite the system (3.2) as follows:

$$\partial_t g = \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc}+\mathcal{L}_X g+\frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R+\mathcal{L}_Y g+\sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}). \tag{3.3}$$

Let $\zeta \in T^*M$. The principal symbol of the first two terms of (3.3) is given by

$$\begin{aligned} &\sigma\left[D\left(\tfrac{(-1)^{n/2}}{n-2}\Delta^{n/2-1}\mathrm{Rc} + \mathcal{L}_X g\right)\right](\zeta)(h) \\ &= \tfrac{(-1)^{n/2-1}}{2(n-2)}\sigma[D(\Delta^{n/2-1})](\zeta)\cdot\sigma[D(-2\mathrm{Rc}+\mathcal{L}_V g)](\zeta)(h) \\ &= \tfrac{(-1)^{n/2-1}}{2(n-2)}|\zeta|^n h. \end{aligned}$$

We used the fact that the Ricci–DeTurck flow is strongly parabolic (Chow–Knopf [15], Theorem 3.13). The highest order terms of the next two terms of (3.3) cancel each other out, and the remaining terms are of lower order. Therefore the principal symbol of the system (3.2) is $\frac{(-1)^{n/2-1}}{2(n-2)}|\zeta|^n h$, implying that this system is strongly parabolic.

4. Local integral estimates

In this section, let (M^n, g) be a Riemannian manifold that is a solution to the AOF on a time interval $[0,T)$. We give local L^2 estimates for $\nabla^k\mathrm{Rm}$ for all $k \in \mathbb{N}$. We need to use local L^2 estimates since we can only locally convert L^2 estimates to pointwise estimates. These local pointwise estimates are used in the proof of the pointwise smoothing estimates given in Theorem 1.1. Specify the Laplace operator by $\Delta = -\nabla^*\nabla$. Let $\varphi \in C_c^\infty(M)$ be a cutoff function with constants $\Lambda, \Lambda_1 > 0$ such that

$$\sup_{t\in[0,T)} |\nabla\varphi| \le \Lambda_1, \qquad \max_{0\le i\le \frac{n}{2}} \sup_{t\in[0,T)} |\nabla^i\varphi| \le \Lambda.$$

Lemma 4.1. *Suppose M, φ satisfy the above hypotheses. Let A be any tensor and let $p \ge 1, q \ge 2$. Then*

$$\int_M \varphi^p \langle \Delta^q A, A\rangle = (-1)^q \int_{[\varphi>0]} \sum_{i=0}^{q} P_p^{q-i}(\varphi) * \nabla^i A * \nabla^q A + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A * A.$$

Proof. We first claim that if $q \ge 2$, then

$$\Delta^q A = (-1)^q (\nabla^*)^q \nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A.$$

If $q = 2$, we get, using Proposition 3.2, that

$$\begin{aligned}
\Delta^2 A &= -\nabla^*\nabla\Delta A \\
&= -\nabla^*\Delta\nabla A + \nabla^*[\nabla\mathrm{Rm} * A + \mathrm{Rm} * \nabla A] \\
&= (\nabla^*)^2\nabla^2 A + \nabla^2\mathrm{Rm} * A + \nabla\mathrm{Rm} * \nabla A + \mathrm{Rm} * \nabla^2 A,
\end{aligned}$$

which agrees with the claim. Suppose the claim is true for every integer less than q. First,

$$\begin{aligned}
\Delta^q A &= -\nabla^*\nabla\Delta^{q-1} A \\
&= -\nabla^*\left[\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-3} \nabla^{2q-3-i}\mathrm{Rm} * \nabla^i A\right] \\
&= -\nabla^*\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-3} \left[\nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A + \nabla^{2q-3-i}\mathrm{Rm} * \nabla^{i+1} A\right] \\
&= -\nabla^*\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-3} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A + \sum_{i=1}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A \\
&= -\nabla^*\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A.
\end{aligned}$$

Applying the last equation above and then the inductive hypothesis,

$$\begin{aligned}
\Delta^q A &= -\nabla^*\Delta^{q-1}\nabla A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A \\
&= -\nabla^* \left[(-1)^{q-1}(\nabla^*)^{q-1}\nabla^{q-1}\nabla A + \sum_{i=0}^{2q-4} \nabla^{2q-4-i}\mathrm{Rm} * \nabla^i \nabla A \right] \\
&\quad + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A \\
&= (-1)^q(\nabla^*)^q\nabla^q A + \sum_{i=0}^{2q-4} \nabla^{2q-3-i}\mathrm{Rm} * \nabla^{i+1} A + \sum_{i=0}^{2q-4} \nabla^{2q-4-i}\mathrm{Rm} * \nabla^{i+2} A \\
&\quad + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A \\
&= (-1)^q(\nabla^*)^q\nabla^q A + \sum_{i=1}^{2q-3} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^{i} A + \sum_{i=2}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^{i} A \\
&\quad + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A \\
&= (-1)^q(\nabla^*)^q\nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A.
\end{aligned}$$

This proves the claim. We compute

$$\begin{aligned}
(-1)^{q+1}\int_M \nabla^q A * \nabla^q(\varphi^p A) &= (-1)^{q+1}\int_M \nabla^q A * \sum_{i=0}^{q} \nabla^{q-i}(\varphi^p) * \nabla^i A \\
&= (-1)^{q+1}\int_{[\varphi>0]} \sum_{i=0}^{q} \sum_{|\alpha|=q-i} \nabla^i A * \nabla^q A * \prod_{j=1}^{p} \nabla^{\alpha_j}\varphi_j \\
&= (-1)^{q+1}\int_{[\varphi>0]} \sum_{i=0}^{q} P_p^{q-i}(\varphi) * \nabla^i A * \nabla^q A.
\end{aligned}$$

Finally, applying the claim,

$$\begin{aligned}
\int_M \varphi^p \langle \Delta^q A, A\rangle &= \int_M \varphi^p \left\langle (-1)^q(\nabla^*)^q\nabla^q A + \sum_{i=0}^{2q-2} \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A, A \right\rangle \\
&= (-1)^q \int_M \nabla^q A * \nabla^q(\varphi^p A) + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A * A \\
&= (-1)^q \int_{[\varphi>0]} \sum_{i=0}^{q} P_p^{q-i}(\varphi) * \nabla^i A * \nabla^q A \\
&\quad + \int_M \sum_{i=0}^{2q-2} \varphi^p \nabla^{2q-2-i}\mathrm{Rm} * \nabla^i A * A.
\end{aligned}$$

□

Proposition 4.2. *Suppose M, φ satisfy the above hypotheses. If $p \geq 1, k \geq 0$, then*

(4.1)
$$\frac{\partial}{\partial t}\int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 = -\frac{1}{n-2}\int_M \varphi^p |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 + \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) + \int_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i}\mathrm{Rm} * \nabla^{k+\frac{n}{2}}\mathrm{Rm}.$$

Proof. First, we have

$$\frac{\partial}{\partial t}\int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 \, dV_g = 2\int_M \varphi^p \left\langle \frac{\partial}{\partial t}\nabla^k \mathrm{Rm}, \nabla^k \mathrm{Rm}\right\rangle dV_g + \int_M \varphi^p |\nabla^k \mathrm{Rm}|^2 \frac{\partial g}{\partial t}\, dV_g.$$

We can expand the first integral by substituting Proposition 3.5, which states that for our flow,

$$\frac{\partial}{\partial t}\nabla^k \mathrm{Rm} = \frac{(-1)^{\frac{n}{2}+1}}{2(n-2)}\Delta^{\frac{n}{2}}\nabla^k \mathrm{Rm} + \sum_{i=2}^{\frac{n}{2}+1} P_i^{n-2i+k+2}(\mathrm{Rm}).$$

Applying Lemma 4.1 to the first term of $\frac{\partial}{\partial t}\nabla^k \mathrm{Rm}$ gives that

$$\begin{aligned}
&\tfrac{(-1)^{\frac{n}{2}+1}}{n-2}\int_M \varphi^p \langle \Delta^{\frac{n}{2}}\nabla^k \mathrm{Rm}, \nabla^k \mathrm{Rm}\rangle \\
&= \tfrac{(-1)^{n+1}}{n-2}\int_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}} [P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i}\mathrm{Rm} * \nabla^{k+\frac{n}{2}}\mathrm{Rm}] \\
&\quad + \int_M \sum_{i=0}^{n-2} \varphi^p \nabla^{n-2-i}\mathrm{Rm} * \nabla^{k+i}\mathrm{Rm} * \nabla^k \mathrm{Rm} \\
&= -\tfrac{1}{n-2}\int_M \varphi^p |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\
&\quad + \int_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i}\mathrm{Rm} * \nabla^{k+\frac{n}{2}}\mathrm{Rm} \\
&\quad + \int_M \varphi^p P_3^{n+2k-2}(\mathrm{Rm}).
\end{aligned}$$

Substituting the second term of $\frac{\partial}{\partial t}\nabla^k\mathrm{Rm}$ into the inner product gives that

$$\begin{aligned}\int\limits_M \varphi^p \left\langle \nabla^k\mathrm{Rm}, \sum_{i=2}^{\frac{n}{2}+1} P_i^{n-2i+k+2}(\mathrm{Rm})\right\rangle &= \int\limits_M \varphi^p \sum_{i=3}^{\frac{n}{2}+2} P_i^{n-2i+2k+4}(\mathrm{Rm})\\ &= \int\limits_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}).\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial g}{\partial t} &= \Delta^{\frac{n}{2}-1}\mathrm{Rc} + \Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{i=2}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm})\\ &= \nabla^{n-2}\mathrm{Rm} + \nabla^{n-4+2}\mathrm{Rm} + \sum_{i=2}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm})\\ &= \sum_{i=1}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm}),\end{aligned}$$

we have

$$\begin{aligned}\int_M \varphi^p|\nabla^k\mathrm{Rm}|^2\frac{\partial g}{\partial t} &= \int_M \varphi^p(\nabla^k\mathrm{Rm})^{*2}\sum_{i=1}^{\frac{n}{2}} P_i^{n-2i}(\mathrm{Rm})\\ &= \int_M \varphi^p \sum_{i=1}^{\frac{n}{2}} P_{i+2}^{n-2i+2k}(\mathrm{Rm})\\ &= \int_M \varphi^p \sum_{i=3}^{\frac{n}{2}+2} P_i^{n-2i+2k+4}(\mathrm{Rm})\\ &= \int_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}).\end{aligned}$$

Combining all of these results yields

$$\begin{aligned}\frac{\partial}{\partial t}\int\limits_M \varphi^p|\nabla^k\mathrm{Rm}|^2 &= -\frac{1}{n-2}\int\limits_M \varphi^p|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2\\ &\quad+\int\limits_{[\varphi>0]}\sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi)*\nabla^{k+i}\mathrm{Rm}*\nabla^{k+\frac{n}{2}}\mathrm{Rm}\\ &\quad+\int\limits_M \varphi^p P_3^{n+2k-2}(\mathrm{Rm}) + \int\limits_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm})\\ &\quad+\int\limits_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm})\\ &= -\frac{1}{n-2}\int\limits_M \varphi^p|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2\end{aligned}$$

$$+ \int\limits_{[\varphi>0]} \sum_{i=0}^{\frac{n}{2}-1} P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{k+i}\mathrm{Rm} * \nabla^{k+\frac{n}{2}}\mathrm{Rm}$$

$$+ \int\limits_M \varphi^p \sum_{l=k}^{\frac{n}{2}+k-1} P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}).$$

$\square$

We estimate the last two terms of (4.1). First, we recall two corollaries from the paper [28] of E. Kuwert and R. Schätzle.

Proposition 4.3 ([28], Corollary 5.2)**.** *Suppose M, φ satisfy the above hypotheses. Let A be a tensor. If $2 \leq p < \infty$ and $s \geq p$, then for every $\epsilon > 0$,*

$$\left(\int_M |\nabla A|^p \varphi^s\right)^{\frac{1}{p}} \leq \epsilon \left(\int_M |\nabla^2 A|^p \varphi^{s+p}\right)^{\frac{1}{p}} + \frac{c}{\epsilon}\left(\int_{[\varphi>0]} |A|^p \varphi^{s-p}\right)^{\frac{1}{p}},$$

where $c = c(n, p, s, \Lambda_1)$.

Proposition 4.4 ([28], Corollary 5.5)**.** *Suppose M, φ satisfy the above hypotheses. Let A be a tensor. Let $0 \leq i_1, \dots, i_r \leq k$, $i_1 + \cdots + i_r = 2k$, and $s \geq 2k$. Then*

$$\left|\int_M \varphi^s \nabla^{i_1} A * \cdots * \nabla^{i_r} A\right| \leq c\|A\|_\infty^{r-2}\left(\int_M \varphi^s |\nabla^k A|^2\, dV + \|A\|^2_{2,[\varphi>0]}\right),$$

where $c = c(k, n, r, s, \Lambda_1)$.

We estimate the last term of (4.1).

Lemma 4.5. *Suppose M, φ satisfy the above hypotheses. If $l \geq 1, q \geq 0$, then for every $\epsilon > 0$,*

$$\int_M \varphi^{2l+q}|\nabla^l \mathrm{Rm}|^2 \leq \epsilon \int_M \varphi^{2l+q+2}|\nabla^{l+1}\mathrm{Rm}|^2 + \frac{C}{\epsilon^l}\int_{[\varphi>0]} \varphi^q |\mathrm{Rm}|^2, \tag{4.2}$$

where $C = C(n, l, \Lambda_1, q)$.

Proof. We prove the inequality (4.2) by induction on l. If $l = 1$, the inequality (4.2) follows immediately from Proposition 4.3. Assume that $l \geq 2$ and (4.2) is true for all integers at most l. Then, applying Proposition 4.3 in the first line and the inductive hypothesis in the second line,

$$\begin{aligned}
\int_M \varphi^{2l+2+q}|\nabla^{l+1}\mathrm{Rm}|^2 &\leq \frac{\epsilon}{2}\int_M \varphi^{2l+4+q}|\nabla^{l+2}\mathrm{Rm}|^2 + \frac{C}{\epsilon}\int_M \varphi^{2l+q}|\nabla^l \mathrm{Rm}|^2 \\
&\leq \frac{\epsilon}{2}\int_M \varphi^{2l+4+q}|\nabla^{l+2}\mathrm{Rm}|^2 + \frac{C}{\epsilon}\frac{\epsilon}{2C}\int_M \varphi^{2l+q+2}|\nabla^{l+1}\mathrm{Rm}|^2 \\
&\quad + \frac{C}{\epsilon}\frac{C}{\epsilon^l}\int_{[\varphi>0]} \varphi^q|\mathrm{Rm}|^2 \\
&= \frac{\epsilon}{2}\int_M \varphi^{2l+4+q}|\nabla^{l+2}\mathrm{Rm}|^2 + \frac{1}{2}\int_M \varphi^{2l+q+2}|\nabla^{l+1}\mathrm{Rm}|^2 \\
&\quad + \frac{C}{\epsilon^{l+1}}\int_{[\varphi>0]} \varphi^q|\mathrm{Rm}|^2.
\end{aligned}$$

Collecting terms, we see that (4.2) is also true for $l + 1$. $\square$

Lemma 4.6. *Suppose M, φ satisfy the above hypotheses. If $q \geq 0$ and $0 \leq l \leq q$, then for every $\epsilon > 0$,*

$$\int_M \varphi^{2l+r} |\nabla^l \mathrm{Rm}|^2 \leq \epsilon^{q-l} \int_M \varphi^{2q+r} |\nabla^q \mathrm{Rm}|^2 + C\epsilon^{-l} \int_{[\varphi>0]} \varphi^r |\mathrm{Rm}|^2,$$

where $C = C(n, l, \Lambda_1, r)$.

Proof. Let $m = q - l$. The desired inequality is equivalent to

$$\int_M \varphi^{2q-2m+r} |\nabla^{q-m} \mathrm{Rm}|^2 \leq \epsilon^m \int_M \varphi^{2q+r} |\nabla^q \mathrm{Rm}|^2 + C\epsilon^{m-q} \int_{[\varphi>0]} \varphi^r |\mathrm{Rm}|^2. \tag{4.3}$$

We prove this inequality by induction on m. If $m = 0$ the inequality is true:

$$\int_M \varphi^{2q+r} |\nabla^q \mathrm{Rm}|^2 \leq \int_M \varphi^{2q+r} |\nabla^q \mathrm{Rm}|^2 + C\epsilon^{-q} \int_{[\varphi>0]} \varphi^r |\mathrm{Rm}|^2.$$

Assume the inequality (4.3) is true for every integer less than m. Then

$$\begin{aligned} \int_M \varphi^{2q-2m+r} |\nabla^{q-m} \mathrm{Rm}|^2 &\leq \epsilon \int_M \varphi^{2q-2m+r+2} |\nabla^{q-m+1} \mathrm{Rm}|^2 \\ &\quad + C\epsilon^{m-q} \int_{[\varphi>0]} \varphi^r |\mathrm{Rm}|^2 \\ &\leq \epsilon\epsilon^{m-1} \int_M \varphi^{2q+r} |\nabla^q \mathrm{Rm}|^2 + \epsilon C\epsilon^{m-q-1} \int_{[\varphi>0]} \varphi^r |\mathrm{Rm}|^2 \\ &\quad + C\epsilon^{m-q} \int_{[\varphi>0]} \varphi^r |\mathrm{Rm}|^2 \\ &= \epsilon^m \int_M \varphi^{2q+r} |\nabla^q \mathrm{Rm}|^2 + C\epsilon^{m-q} \int_{[\varphi>0]} \varphi^r |\mathrm{Rm}|^2. \end{aligned}$$

We applied Lemma 4.5 in the first line and the inductive hypothesis in the second line. □

Lemma 4.7. *Suppose M, φ satisfy the above hypotheses. Let $0 \leq i \leq \frac{n}{2} - 1$ and $p \geq n + 2k$. Then for every $\delta > 0$,*

$$\begin{aligned} \int_M P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{i+k} \mathrm{Rm} * \nabla^{\frac{n}{2}+k} \mathrm{Rm} &\leq C\delta \int_M \varphi^p |\nabla^{\frac{n}{2}+k} \mathrm{Rm}|^2 \\ &\quad + C\delta^{\frac{-n-2i-4k}{n-2i}} \int_{[\varphi>0]} \varphi^{p-n-2k} |\mathrm{Rm}|^2, \end{aligned}$$

where $C = C(n, k, p, \Lambda, i)$.

Proof. We apply the Cauchy–Schwarz inequality:

$$\begin{aligned} \int_M P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{i+k} \mathrm{Rm} * \nabla^{\frac{n}{2}+k} \mathrm{Rm} &\leq C(\Lambda) \int_M |\varphi^{p-(\frac{n}{2}-i)} * \nabla^{i+k} \mathrm{Rm} * \nabla^{\frac{n}{2}+k} \mathrm{Rm}| \\ &\leq C\epsilon^\beta \int_M \varphi^p |\nabla^{\frac{n}{2}+k} \mathrm{Rm}|^2 \\ &\quad + C\epsilon^{-\beta} \int_{[\varphi>0]} \varphi^{p-n+2i} |\nabla^{i+k} \mathrm{Rm}|^2. \end{aligned}$$

The second term can be estimated using Lemma 4.6:

$$\begin{aligned}\int_{[\varphi>0]} \varphi^{p-n+2i}|\nabla^{i+k}\mathrm{Rm}|^2 &= \int_{[\varphi>0]} \varphi^{2(i+k)+(p-n-2k)}|\nabla^{i+k}\mathrm{Rm}|^2 \\ &\le \epsilon^{\frac{n}{2}-i}\int_M |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 + C\epsilon^{-i-k}\int_{[\varphi>0]} \varphi^{p-n-2k}|\mathrm{Rm}|^2.\end{aligned}$$

If $\beta = \frac{n}{2} - i - \beta$, then $\beta = \frac{n-2i}{4}$. If we set $\delta = \epsilon^{\frac{n-2i}{4}}$, then $\epsilon = \delta^{\frac{4}{n-2i}}$ and

$$\epsilon^{-\beta-i-k} = \delta^{\frac{4}{n-2i}\left(\frac{2i-n}{4}-i-k\right)} = \delta^{\frac{-n-2i-4k}{n-2i}}.$$

Therefore

$$\begin{aligned}\int_M P_p^{\frac{n}{2}-i}(\varphi) * \nabla^{i+k}\mathrm{Rm} * \nabla^{\frac{n}{2}+k}\mathrm{Rm} &\le C\epsilon^{\beta}\int_M \varphi^p|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + C\epsilon^{-\beta+\frac{n}{2}-i}\int_M |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + C\epsilon^{-\beta-i-k}\int_{[\varphi>0]} \varphi^{p-n-2k}|\mathrm{Rm}|^2 \\ &\le C\delta\int_M \varphi^p|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + C\delta^{\frac{-n-2i-4k}{n-2i}}\int_{[\varphi>0]} \varphi^{p-n-2k}|\mathrm{Rm}|^2.\end{aligned}$$

□

We estimate the penultimate term of (4.1).

Lemma 4.8. *Suppose M, φ satisfy the above hypotheses. Let $K = \max\{1, \|\mathrm{Rm}\|_\infty\}$. If $p \ge n + 2k$ and $k \le l \le \frac{n}{2} + k - l$, then for every δ satisfying $0 < \delta \le 1$,*

$$\begin{aligned}\int_M \varphi^p P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) &\le C\delta\int_M \varphi^{p+n+2k-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\|\mathrm{Rm}\|^2_{2,[\varphi>0]},\end{aligned}$$

where $C = C(n, k, p, \Lambda_1, l)$.

Proof. Since $p \ge n + 2k \ge n + 2k - 2 = 2(\frac{n}{2} + k - 1)$, Proposition 4.4 implies

$$\int_M \varphi^p P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) \le C\|\mathrm{Rm}\|_\infty^{\frac{n}{2}+k-l}\left(\int_M \varphi^p|\nabla^l\mathrm{Rm}|^2 + \|\mathrm{Rm}\|^2_{2,[\varphi>0]}\right).$$

Let $\epsilon = K^{-1}\delta^{\frac{2}{n+2k-2l}}$. We have $p - 2l \ge n + 2k - (n + 2k - 1) = 1$. Via Lemma 4.6,

$$\begin{aligned}C\|\mathrm{Rm}\|_\infty^{\frac{n}{2}+k-l}\int_M \varphi^p|\nabla^l\mathrm{Rm}|^2 &\le CK^{\frac{n}{2}+k-l}\epsilon^{\frac{n}{2}+k-l}\int_M \varphi^{n+2k+p-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k-l}\epsilon^{-l}\int_{[\varphi>0]} \varphi^{p-2l}|\mathrm{Rm}|^2 \\ &= C\delta\int_M \varphi^{n+2k+p-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\int_{[\varphi>0]} \varphi^{p-2l}|\mathrm{Rm}|^2.\end{aligned}$$

Since $k \leq l \leq \frac{n}{2} + k - l$ and $0 < \delta \leq 1$, we get $\delta^{\frac{2l}{2l-n-2k}} \geq \delta^{-\frac{2k}{n}} \geq 1$ and $K^{\frac{n}{2}+k-l} \leq K^{\frac{n}{2}}$. Therefore

$$\begin{aligned}\int_M \varphi^p P^{2l}_{\frac{n}{2}+k-l+2}(\mathrm{Rm}) &\leq C\delta \int_M \varphi^{n+2k+p-2l} |\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}} \int_{[\varphi>0]} \varphi^{p-2l}|\mathrm{Rm}|^2 \\ &\quad + K^{\frac{n}{2}+k-l}\|\mathrm{Rm}\|^2_{2,[\varphi>0]} \\ &\leq C\delta \int_M \varphi^{p+n+2k-2l}|\nabla^{\frac{n}{2}+k}\mathrm{Rm}|^2 \\ &\quad + CK^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\|\mathrm{Rm}\|^2_{2,[\varphi>0]}.\end{aligned}$$

□

Proposition 4.9. *Suppose M, φ satisfy the above hypotheses. Let*

$$K = \max\{1, \|\mathrm{Rm}\|_\infty\}.$$

If $p \geq n + 2k$, then for every δ satisfying $0 < \delta \leq 1$,

$$\partial_t \|\varphi^{\frac{p}{2}}\nabla^k \mathrm{Rm}\|_2^2 \leq -\tfrac{1}{2(n-2)}\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2 + CK^{\frac{n}{2}+k}\|\mathrm{Rm}\|^2_{2,[\varphi>0]},$$

where $C = C(n, k, p, \Lambda)$.

Proof. Applying the estimates from Lemmas 4.8 and 4.7 to equation (4.1) in Proposition 4.2, we obtain

$$\begin{aligned}&\partial_t \|\varphi^{\frac{p}{2}}\nabla^k\mathrm{Rm}\|_2^2 \leq -\tfrac{1}{n-2}\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2 \\ &\quad + \sum_{l=k}^{\frac{n}{2}+k-1} \Big[C_1\delta\|\varphi^{\frac{p}{2}+\frac{n}{2}+k-l}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2 + C_1K^{\frac{n}{2}+k}\delta^{\frac{2l}{2l-n-2k}}\|\mathrm{Rm}\|^2_{2,[\varphi>0]}\Big] \\ &\qquad + \sum_{i=0}^{\frac{n}{2}-1}\Big[C_2\delta\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2 + C_2\delta^{\frac{-n-2i-4k}{n-2i}}\|\varphi^{\frac{p}{2}-\frac{n}{2}-k}\mathrm{Rm}\|^2_{2,[\varphi>0]}\Big],\end{aligned}$$

where $C_1 = C_1(n, k, p, \Lambda, l)$ and $C_2 = C_2(n, k, p, \Lambda_1, i)$. From the inequalities

$$1 - n - 2k \leq 1 - \frac{2n+4k}{n-2i} \leq -\frac{n+4k}{n}, \qquad \frac{2-n-2k}{2} \leq 1 + \frac{n+2k}{2l-n-2k} \leq -\frac{2k}{n}$$

we conclude

$$\max\Big(\{\delta^{\frac{2l}{2l-n-2k}} : k \leq l \leq \tfrac{n}{2}+k-1\} \cup \{\delta^{\frac{-n-2i-4k}{n-2i}} : 0 \leq i \leq \tfrac{n}{2}-1\}\Big) = \delta^{1-n-2k}.$$

Therefore

$$\begin{aligned}\partial_t\|\varphi^{\frac{p}{2}}\nabla^k\mathrm{Rm}\|_2^2 &\leq -\tfrac{1}{n-2}\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2 + \widetilde{C}\delta\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2 \\ &\quad + \widetilde{C}K^{\frac{n}{2}+k}\delta^{1-n-2k}\|\mathrm{Rm}\|^2_{2,[\varphi>0]} \\ &\leq -\tfrac{1}{2(n-2)}\|\varphi^{\frac{p}{2}}\nabla^{\frac{n}{2}+k}\mathrm{Rm}\|_2^2 + CK^{\frac{n}{2}+k}\|\mathrm{Rm}\|^2_{2,[\varphi>0]},\end{aligned}$$

where

$$\widetilde{C} \equiv \sum_{l=k}^{\frac{n}{2}+k-1} C_1 + \sum_{i=0}^{\frac{n}{2}-1} C_2, \qquad \delta \equiv \min\{\tfrac{1}{2(n-2)}\widetilde{C}^{-1}, 1\}.$$

□

Proposition 4.10. *Suppose M, φ satisfy the above hypotheses. Suppose*

$$\max\{\|\mathrm{Rm}\|_\infty, 1\} \leq K$$

for all $t \in [0, \alpha K^{-\frac{n}{2}}]$. Then

$$\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2 \leq Ct^{-\frac{m}{2}} \sup_{t\in[0,\alpha K^{-\frac{n}{2}}]} \|\mathrm{Rm}\|_{L^2(t),[\varphi>0]},$$

where $C = C(m, n, \alpha, \Lambda)$, for all $t \in (0, \alpha K^{-\frac{n}{2}}]$.

Proof. Let β_k for $0 \leq k \leq m$ denote constants given by $\beta_k = (2n-4)^{m-k}m!/k!$. Define

$$G(t) \equiv t^m \|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2^2 + \sum_{k=0}^{m-1} \beta_k t^k \|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}k}\mathrm{Rm}\|_{2,[\varphi>0]}^2.$$

Using Proposition 4.9,

$$\begin{aligned}
\frac{dG}{dt} \leq\ & mt^{m-1}\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2^2 \\
& + t^m\Big(-\tfrac{1}{2(n-2)}\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}(m+1)}\mathrm{Rm}\|_2^2 + C_{\frac{n}{2}m}K^{\frac{n}{2}(m+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big) \\
& + \sum_{k=1}^{m-1}\beta_k k t^{k-1}\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}k}\mathrm{Rm}\|_2^2 \\
& + \sum_{k=0}^{m-1}\beta_k t^k\Big(-\tfrac{1}{2(n-2)}\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}(k+1)}\mathrm{Rm}\|_2^2 + C_{\frac{n}{2}k}K^{\frac{n}{2}(k+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big) \\
\leq\ & mt^{m-1}\|\varphi^{\frac{n}{2}m}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2^2 + t^m\Big(C_{\frac{n}{2}m}K^{\frac{n}{2}(m+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big) \\
& + \sum_{k=0}^{m-2}\beta_{k+1}(k+1)t^k\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}(k+1)}\mathrm{Rm}\|_2^2 \\
& + \sum_{k=0}^{m-1}\beta_k t^k\Big(-\tfrac{1}{2(n-2)}\|\varphi^{\frac{n}{2}(k+1)}\nabla^{\frac{n}{2}(k+1)}\mathrm{Rm}\|_2^2 + C_{\frac{n}{2}k}K^{\frac{n}{2}(k+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2\Big).
\end{aligned}$$

Choose $t_0 \in [0, \alpha K^{-\frac{n}{2}}]$ such that

$$\|\mathrm{Rm}\|_{L^2(t_0),[\varphi>0]} = \sup_{t\in[0,\alpha K^{-\frac{n}{2}}]}\|\mathrm{Rm}\|_{L^2(t),[\varphi>0]}.$$

Our choice of the constants β_k yields

$$\begin{aligned}
\frac{dG}{dt} &\leq \alpha^m K^{-\frac{n}{2}m}C_{\frac{n}{2}m}K^{\frac{n}{2}(m+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2 \\
&\quad + \sum_{k=0}^{m-1}\beta_k\alpha^k K^{-\frac{n}{2}k}C_{\frac{n}{2}k}K^{\frac{n}{2}(k+1)}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2 \\
&= \sum_{k=0}^{m}\beta_k C_{\frac{n}{2}k}\alpha^k K^{\frac{n}{2}}\|\mathrm{Rm}\|_{2,[\varphi>0]}^2 \\
&= CK^{\frac{n}{2}}\|\mathrm{Rm}\|_{L^2(t_0),[\varphi>0]}^2.
\end{aligned}$$

Therefore

$$\begin{aligned} t^m\|\varphi^{\frac{n}{2}(m+1)}\nabla^{\frac{n}{2}m}\mathrm{Rm}\|_2^2 \leq G &\leq \beta_0\|\mathrm{Rm}\|^2_{L^2(0),[\varphi>0]} + CK^{\frac{n}{2}}\|\mathrm{Rm}\|^2_{L^2(t_0),[\varphi>0]}t \\ &\leq (\beta_0+\alpha C)\|\mathrm{Rm}\|^2_{L^2(t_0),[\varphi>0]} \\ &= C\|\mathrm{Rm}\|^2_{L^2(t_0),[\varphi>0]}, \end{aligned}$$

proving the proposition. □

Proposition 4.11. *Let $(M^n, g(t))$ be a solution to the AOF for $t \in [0,T)$. Let $\varphi \in C_c^\infty(M)$ be a cutoff function such that*

$$\max_{0\leq i\leq \frac{n}{2}} \sup_{t\in[0,T)} \|\nabla^i\varphi\|_{C^0(M,g(t))} \leq \Lambda.$$

Suppose $\max\{\|\mathrm{Rm}\|_{C^0(M,g(t))}, 1\} \leq K$ *for all $t \in [0, \alpha K^{-\frac{n}{2}}]$. Then, for every $l \geq 0$ and all $t \in (0, \alpha K^{-\frac{n}{2}}]$,*

$$\|\varphi^{l+\frac{n}{2}}\nabla^l\mathrm{Rm}\|_{L^2(M,g(t))} \leq C(1+t^{-\lceil 2l/n\rceil/2}) \sup_{t\in[0,\alpha K^{-\frac{n}{2}}]} \|\mathrm{Rm}\|_{L^2(\mathrm{supp}(\varphi),g(t))},$$

where $C = C(l,n,\alpha,\Lambda)$.

Proof. Let $l = \frac{n}{2}m + r, 1 \leq r \leq \frac{n}{2}$. Then, applying Lemma 4.6 and Proposition 4.10, we get

$$\begin{aligned} \int_M \varphi^{n(m+1)+2r}|\nabla^{\frac{n}{2}m+r}\mathrm{Rm}|^2 &\leq \int_M \varphi^{n(m+2)}|\nabla^{\frac{n}{2}(m+1)}\mathrm{Rm}|^2 + C'\int_{[\varphi>0]}\varphi^n|\mathrm{Rm}|^2 \\ &\leq t^{-(m+1)}C\Theta^2 + C'\Theta^2 \\ \|\varphi^{l+\frac{n}{2}}\nabla^l\mathrm{Rm}\|_{L^2(t)} &\leq \Theta(Ct^{-\frac{m+1}{2}} + C'), \end{aligned}$$

where

$$\Theta = \sup_{t\in[0,\alpha K^{-\frac{n}{2}}]} \|\mathrm{Rm}\|_{L^2(t),[\varphi>0]}.$$

□

5. Pointwise smoothing estimates

Let $(M, g(t))$ be a solution to AOF and let φ be a cutoff function on M. We give estimates of $|\nabla^i\varphi|_{g(t)}$ for $1 \leq i \leq \frac{n}{2}$ that depend on spacetime derivatives of the metric and $|\nabla^i\varphi|_{g(0)}$ for $0 \leq i \leq \frac{n}{2}$. We then give a proof of the pointwise smoothing estimates given in Theorem 1.1.

Lemma 5.1. *Let M be a manifold and let $g(t)$ be a one-parameter family of metrics on M. For a function $\varphi \in C^i(M)$ and $i \geq 2$,*

$$\partial_t\nabla^i\varphi = \sum_{j=1}^{i-1} \nabla^{i-j}\partial_t g * \nabla^j\varphi.$$

Proof. Apply Proposition 3.3 with $k = i-1$ and $A = \nabla\varphi$. □

Proposition 5.2. *Let M be a manifold and let $g(t)$ be a one-parameter family of metrics on M. For a function $\varphi \in C^i(M)$ and $i \geq 1$,*

$$\partial_t|\nabla^i\varphi|^2_{g(t)} = \sum_{j=1}^{i} \nabla^{i-j}\partial_t g * \nabla^j\varphi * \nabla^i\varphi.$$

Proof. We compute, using the preceding Lemma 5.1 in the second line:

$$\begin{aligned}\partial_t|\nabla^i\varphi|^2_{g(t)} &= \partial_t g * \nabla^i\varphi^{*2} + \partial_t\nabla^i\varphi * \nabla^i\varphi \\ &= \partial_t g * \nabla^i\varphi^{*2} + \sum_{j=1}^{i-1}\nabla^{i-j}\partial_t g * \nabla^j\varphi * \nabla^i\varphi \\ &= \sum_{j=1}^{i}\nabla^{i-j}\partial_t g * \nabla^j\varphi * \nabla^i\varphi.\end{aligned}$$

□

Proposition 5.3. *Let M be a Riemannian manifold with a one-parameter family of metrics $\{g(t)\}_{t\in[0,T]}$ and $\varphi \in C_c^\infty(M)$. Fix $i \geq 1$. Suppose that, for each j satisfying $0 \leq j \leq i-1$, there exists $K_j > 0$ such that $|\nabla^j\partial_t g(x,t)|_{g(t)} \leq K_j$ on $\operatorname{supp}\varphi \times [0,T]$ and, for each j satisfying $1 \leq j \leq i$, there exists $C'_j > 0$ such that $|\nabla^j\varphi|_{g(0)} \leq C'_j$ on $\operatorname{supp}\varphi$. Then there exists a constant C_i such that, for every $t \in [0,T]$,*

$$|\nabla^i\varphi|^2_{g(t)} \leq C_i = C_i(K_0,\dots,K_{i-1},C'_1,\dots,C'_i,T).$$

Proof. Let $i = 1$. Then Proposition 5.2 gives

$$\partial_t|\nabla\varphi|^2_{g(t)} = \partial_t g * \nabla\varphi^{*2} \leq CK_0|\nabla\varphi|^2_{g(t)}.$$

Solving the differential inequality, we get

$$|\nabla\varphi|^2_{g(t)} \leq |\nabla\varphi|^2_{g(0)}e^{CK_0T} \equiv C_1^2,$$

which proves the proposition for $i = 1$.

Fix $i \geq 2$ and suppose that the proposition is true for every j satisfying $1 \leq j \leq i-1$. Let $f(t) = |\nabla^i\varphi|^2_{g(t)}$. Then, via Proposition 5.2,

$$\begin{aligned}\frac{df}{dt} &\leq \sum_{j=1}^{i}\nabla^{i-j}\partial_t g * \nabla^j\varphi * \nabla^i\varphi \\ &\leq \sum_{j=1}^{i-1}|\nabla^{i-j}\partial_t g||\nabla^j\varphi||\nabla^i\varphi| + |\partial_t g||\nabla^i\varphi|^2 \\ &\leq \sum_{j=1}^{i-1}CK_{i-j}C_jf^{\frac{1}{2}} + CK_0f \\ &\leq \widetilde{C}(K_0,\dots,K_{i-1},C_1,\dots,C_{i-1})(1+f) \\ &= \widetilde{C}(K_0,\dots,K_{i-1},C'_1,\dots,C'_{i-1},T)(1+f).\end{aligned}$$

Solving the differential inequality, we get

$$\begin{aligned}1+f(t) &\leq (1+f(0))e^{\widetilde{C}T} \\ |\nabla^i\varphi|^2_{g(t)} &\leq (1+|\nabla^i\varphi|^2_{g(0)})e^{\widetilde{C}T} \\ &\leq (1+(C'_i)^2)e^{\widetilde{C}T} \equiv C_i^2.\end{aligned}$$

□

Proposition 5.4. *Let $(M^n, g(t))$ solve AOF on $[0,T]$, where $n \geq 4$. Fix $r > 0$. Suppose there exist $x \in M$, $r > 0$, and $K > 0$ such that*
(5.1)

$$\max\left[1, \sup_{[0,T]} \|\mathrm{Rm}\|_{C^0(B_{g(T)}(x,2r),g(t))}\right] + \sum_{j=1}^{3n/2-3} \sup_{[0,T]} \|\nabla^j \mathrm{Rm}\|^{\frac{2}{j+2}}_{C^0(B_{g(T)}(x,2r),g(t))} < K.$$

Then for all $l \geq 0$ and $t \in (0,T]$,

$$\|\nabla^l \mathrm{Rm}\|_{L^2(B_{g(T)}(x,r),g(t))} \leq C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t\in[0,T]} \|\mathrm{Rm}\|_{L^2(B_{g(T)}(x,2r),g(t))}, \tag{5.2}$$

where $C = C(n,l,K,T,r)$.

Proof. Let φ be a cutoff function that is equal to 1 on $B_{g(T)}(x,r)$ and supported on $B_{g(T)}(x,2r)$. The inequality (5.1) provides C^0 bounds for the first $\frac{n}{2} - 1$ covariant derivatives of Rm so that

$$\max_{0 \leq j \leq \frac{n}{2}} \|\nabla^j \varphi\|_{C^0(M,g(T))} \leq C'(n,K,r). \tag{5.3}$$

The inequality (5.3) provides bounds for the first $\frac{n}{2}$ covariant derivatives of φ at time T, and the inequality (5.1) induces bounds on the first $\frac{n}{2} - 1$ covariant derivatives of $\widehat{\mathcal{O}}$. We therefore are able to, for each $t \in [0,T]$ and j satisfying $0 \leq j \leq \frac{n}{2}$, to obtain via Proposition 5.3 bounds given by

$$\|\nabla^j \varphi\|_{C^0(M,g(t))} \leq \widetilde{C}_j(n,K,r,T).$$

Therefore, via Proposition 4.11,

$$\begin{aligned}
\|\nabla^l \mathrm{Rm}\|_{L^2(B_{g(T)}(x,r),g(t))} &\leq \|\varphi^{l+\frac{n}{2}} \nabla^l \mathrm{Rm}\|_{L^2(M,g(t))} \\
&\leq C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t\in[0,T]} \|\mathrm{Rm}\|_{L^2(\mathrm{supp}(\varphi),g(t))} \\
&= C(1 + t^{-\lceil 2l/n \rceil/2}) \sup_{t\in[0,T]} \|\mathrm{Rm}\|_{L^2(B_{g(T)}(x,2r),g(t))},
\end{aligned}$$

where $C = C(n,l,K,T,r)$. $\square$

We are now able to prove the pointwise smoothing estimates given in Theorem 1.1.

Proof of Theorem 1.1. We adapt the proof of Theorem 1.3 in Streets [40]. We will show that if this inequality fails, we can construct a blowup limit that is flat and has nonzero curvature. Consider the function given by

$$f_m(x,t,g) = \sum_{j=1}^{m} |\nabla^j \mathrm{Rm}(g(x,t))|^{\frac{2}{j+2}}_{g(t)}.$$

It suffices to show that

$$f_m(x,t,g) \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right) \tag{5.4}$$

since for every l satisfying $1 \leq l \leq m$,

$$|\nabla^l \mathrm{Rm}(g(x,t))|^{\frac{2}{l+2}}_{g(t)} \leq \sum_{j=1}^{m} |\nabla^j \mathrm{Rm}(g(x,t))|^{\frac{2}{j+2}}_{g(t)} = f_m(x,t,g) \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right)$$

and

$$|\nabla^l \mathrm{Rm}(g(x,t))|_{g(t)} \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right)^{\frac{l+2}{2}} \leq C\left(K + \frac{1}{t^{\frac{2}{n}}}\right)^{\frac{m+2}{2}}.$$

Suppose that the inequality (5.4) fails. It suffices to take $m \geq \frac{3n}{2} - 3$. Without loss of generality, for each $i \in \mathbb{N}$ there exists a solution to AOF $(M_i^n, g_i(t))$ and $(x_i, t_i) \in M_i \times (0, T]$ such that

$$i < \frac{f_m(x_i, t_i, g_i)}{K + t_i^{-\frac{2}{n}}} = \sup_{M_i \times (0,T]} \frac{f_m(x, t, g_i)}{K + t^{-\frac{2}{n}}} < \infty,$$

and define a new sequence of blown up metrics by

$$\widetilde{g}_i(t) = \lambda_i g_i(t_i + \lambda_i^{-\frac{n}{2}} t),$$

where $\lambda_i = f_m(x_i, t_i, g_i)$. We will show in the proof of Theorem 1.4 that these metrics also solve AOF. These metrics, which are defined for $t \in [-\lambda_i^{\frac{n}{2}} t_i, 0]$, are eventually defined on $[-1, 0]$ since as $i \to \infty$,

$$t_i^{\frac{2}{n}} \lambda_i = \frac{f_m(x_i, t_i, g_i)}{t_i^{-\frac{2}{n}}} \geq \frac{f_m(x_i, t_i, g_i)}{K + t_i^{-\frac{2}{n}}} \to \infty.$$

Replace the sequence of AOF solutions $\{(M_i, \widetilde{g}_i(t))\}_{i \in \mathbb{N}}$ with the tail subsequence for which $\lambda_i^{\frac{n}{2}} t_i > 1$. The curvatures of these manifolds converge to 0 since as $i \to \infty$,

$$|\mathrm{Rm}(\widetilde{g}_i)|_{\widetilde{g}_i} \leq \frac{K}{\lambda_i} = \frac{K}{f_m(x_i, t_i, g_i)} \leq \frac{K + t_i^{-\frac{2}{n}}}{f_m(x_i, t_i, g_i)} \to 0. \tag{5.5}$$

Furthermore, there is a uniform C^m estimate on the curvature given by

$$\begin{aligned} f_m(x, t, \widetilde{g}_i) &= \frac{f_m(x, t_i + t\lambda_i^{-\frac{n}{2}}, g_i)}{\lambda_i} \\ &= \frac{f_m(x, t_i + t\lambda_i^{-\frac{n}{2}}, g_i)}{f_m(x_i, t_i, g_i)} \\ &\leq \frac{K + (t_i + t\lambda_i^{-\frac{n}{2}})^{-\frac{2}{n}}}{K + t_i^{-\frac{2}{n}}} \\ &\leq \frac{K + t_i^{-\frac{2}{n}}(1 + \frac{t}{2})^{-\frac{2}{n}}}{K + t_i^{-\frac{2}{n}}} \\ &\leq 2^{\frac{2}{n}} \end{aligned} \tag{5.6}$$

for all $i \in \mathbb{N}$ and $(x, t) \in M_i \times [-1, 0]$.

Let $B(0,1)$ be the open Euclidean ball in $\mathbb{R}^n$ centered at 0 with radius 1, let $\varphi_i : B(0,1) \to M_i$ be given by $\exp_{x_i}$ with respect to $g_i(0)$ for each $i \in \mathbb{N}$, and let $h_i(t) \equiv \varphi_i^* g_i(t)$. The uniform C^0 bound on $\mathrm{Rm}(\widetilde{g}_i(t))$ given by (5.6) induces a uniform bound on $(\varphi_i)_*$ (see Petersen [37]) which permits the uniform C^m estimate (5.6) on $\mathrm{Rm}(\widetilde{g}_i(t))$ to lift to a uniform C^m estimate on $\mathrm{Rm}(h_i(t))$. Furthermore, $h_i(t)$ solves AOF for all i since φ_i does not depend on t.

Since $m \geq \frac{3n}{2} - 3$, we have uniform C^0 bounds on $\nabla^j \widehat{\mathcal{O}}(h(t))$ for $0 \leq j \leq \frac{n}{2} - 1$. Via Proposition 5.4, we obtain uniform bounds on the $L^2(B_{h_i(0)}(0, \frac{1}{2}))$-norms of all covariant derivatives of $\mathrm{Rm}(h_i(0))$. Since the metrics $h_i(0)$ are uniformly equivalent to the Euclidean metric, the Sobolev constant of $B_{h_i(0)}(0, \frac{1}{2})$ is uniformly bounded for all i. Via the Kondrakov compactness theorem, we thus obtain uniform bounds on the $C^0(B_{h_i(0)}(0,\frac{1}{2}))$-norms of all covariant derivatives of $\mathrm{Rm}(h_i(0))$. The Taylor expansion for h_i in terms of geodesic coordinates about 0 with curvature coefficients can then be used to obtain uniform bounds on the $C^0(B_{h_i(0)}(0,\frac{1}{2}))$-norms of all partial derivatives of $h_i(0)$. Finally, by the Arzelà–Ascoli theorem, after taking a subsequence, still named $\{h_i(0)\}_{i\in\mathbb{N}}$, we get $h_i(0)\to h_\infty$ in $C^\infty(B(0,\frac{1}{2}))$ for some Riemannian metric h_∞. We have already shown with inequality (5.5) that $(B(0,\frac{1}{2}),h_\infty)$ is flat. However, for all $i\in\mathbb{N}$,

$$\begin{aligned} f_m(x_i,0,\tilde{g}_i) &= \sum_{j=1}^{m} |\nabla^j_{\tilde{g}_i}\mathrm{Rm}(\tilde{g}_i)(x_i,0)|^{\frac{2}{2+j}}_{\tilde{g}_i(0)} \\ &= \sum_{j=1}^{m}\left(\lambda_i^{-\frac{j+2}{2}}|\nabla^j\mathrm{Rm}(x_i,t_i)|_{g(t_i)}\right)^{\frac{2}{2+j}} \\ &= \sum_{j=1}^{m}\lambda_i^{-1}|\nabla^j\mathrm{Rm}(x_i,t_i)|^{\frac{2}{2+j}}_{g(t_i)} \\ &= \lambda_i^{-1}\lambda_i = 1. \end{aligned}$$

Also, $f_m(0,0,h_i)=1$ for all i since $(\varphi_i)_*$ is the identity map at $0=\varphi_i^{-1}(x_i)$. Therefore $f_m(0,0,h_\infty)=1$. This is a contradiction, thereby proving the inequality (5.4). $\square$

6. Long time existence

In this section, we prove that if a solution $(M,g(t))$ to the AOF only exists for a finite time T, then $\|\mathrm{Rm}\|_{C^0(g(t))}$ becomes unbounded along a sequence $\{(x_n,t_n)\}_{n=1}^\infty\subset M\times[0,T)$ with $t_n\uparrow T$. We will prove this theorem by showing that if actually

$$\sup_{t\in[0,T)}\|\mathrm{Rm}\|_{C^0(g(t))}=K<\infty, \tag{6.1}$$

then the solution $g(t)$ exists past the time T. In order to show this, we show that (6.1) and the pointwise smoothing estimates on $|\nabla^k\mathrm{Rm}|_{g(t)}$ induce bounds on $|\bar{\nabla}^k g(t)|_{\bar{g}}$ with respect to some fixed background metric $\bar{g}$ and connection $\bar{\nabla}$. We also show that (6.1) implies uniform convergence of $g(t)$ to some continuous metric $g(T)$. The bounds on $|\bar{\nabla}^k g(t)|_{\bar{g}}$ imply that $g(T)$ is smooth so that we can extend the solution $g(t)$ past the time T via the short time existence Theorem 3.6.

We first show that if (6.1) holds, the metrics $g(t)$ converge uniformly as $t\uparrow T$ to a continuous metric $g(T)$ equivalent to each $g(t)$. The following lemma is from Chow–Knopf [15].

Lemma 6.1. *Let M be a closed manifold. For $0\le t<T\le\infty$, let $g(t)$ be a one-parameter family of metrics on M depending smoothly on both space and time. If there exists a constant $C<\infty$ such that*

$$\int_0^T\left|\frac{\partial}{\partial t}g(x,t)\right|_{g(t)}dt\le C$$

for all $x\in M$, then

$$e^{-C}g(x,0)\le g(x,t)\le e^{C}g(x,0)$$

for all $x \in M$ and $t \in [0,T)$. Furthermore, as $t \uparrow T$, the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ such that for all $x \in M$,

$$e^{-C}g(x,0) \le g(x,T) \le e^{C}g(x,0).$$

Lemma 6.2. *Let M be a compact manifold and let $(M, g(t))$ be a solution to AOF on $[0,T)$ such that*

$$\sup_{t\in[0,T)} \|\mathrm{Rm}\|_{C^0(g(t))} = K < \infty.$$

Then $g(t)$ converges uniformly as $t \uparrow T$ to a continuous metric $g(T)$ that is uniformly equivalent to $g(t)$ for every $t \in [0,T]$.

Proof. Since Proposition 2.3 states that

$$\frac{\partial g}{\partial t} = \frac{(-1)^{\frac{n}{2}}}{n-2}\Delta^{\frac{n}{2}-1}\mathrm{Rc} + \frac{(-1)^{\frac{n}{2}-1}}{2(n-1)}\Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}),$$

in order to apply the preceding Lemma 6.1 it suffices to show that $|\nabla^k \mathrm{Rm}|_{g(t)}$ is bounded on $M \times [0,T)$ for all k satisfying $0 \le k \le n-2$. Using the smoothing estimate provided in Theorem 1.1, we get

$$\max_{0\le k\le n-2}\sup_{M\times[0,T)} |\nabla^k \mathrm{Rm}|_{g(t)} \le \max_{0\le k\le n-2}\sup_{M\times[0,\frac{T}{2}]} |\nabla^k \mathrm{Rm}|_{g(t)} + C\big(\widetilde{K} + (\tfrac{T}{2})^{-\frac{2}{n}}\big)^{\frac{n}{2}},$$

where $C = C(n)$ and $\widetilde{K} = \max\{K, 1\}$.

So $\frac{\partial g}{\partial t}$ is bounded on $M \times [0,T)$ and the metrics $g(t)$ converge uniformly as $t \uparrow T$ to a continuous metric $g(T)$ uniformly equivalent to each $g(t)$. $\square$

Since M is a compact manifold, we can obtain bounds on $|\bar{\nabla}^k g(t)|_{\bar{g}}$ by taking the maximum of bounds taken on finitely many coordinate patches. On such a coordinate patch, we can assume that the fixed metric is just the Euclidean one. Thus we will only need to bound the partial derivatives of g and $\widehat{\mathcal{O}}$.

Lemma 6.3. *Let M be a compact manifold and let $(M, g(t))$ be a solution to AOF on $[0,T)$. Let U be a coordinate patch on M. Fix $m \ge 0$. Suppose that for $0 \le i \le m+n-1$, there exist constants C_i such that $|\nabla^i_{g(t)}\mathrm{Rm}(g(t))|_{g(t)} < C_i$ on $M \times [0,T)$. Then for all $(x,t) \in U \times [0,T)$,*

$$|\partial^m g(x,t)|_{g(t)} < \widetilde{C}_1(g(0), C_0, \ldots, C_{m+n-1}),$$
$$|\partial^m \widehat{\mathcal{O}}(x,t)|_{g(t)} < \widetilde{C}_2(g(0), C_0, \ldots, C_{m+n-1}).$$

Proof. We adapt the proof of Proposition 6.48 in Chow–Knopf [15]. The given estimates for $\nabla^k\mathrm{Rm}$ imply $C^0(M \times [0,T))$ estimates on $\nabla^k\widehat{\mathcal{O}}$ for all k. We first estimate ∂g. We bound Γ by integrating $\partial_t\Gamma = \nabla\widehat{\mathcal{O}}$ and estimate ∂g by integrating $\partial_t \partial g = \nabla\widehat{\mathcal{O}} + \Gamma * \widehat{\mathcal{O}}$. Next, we estimate $\partial^m g$ and $\partial^{m-1}\Gamma$ for $m \ge 2$ using induction. It suffices to bound $\partial^m\widehat{\mathcal{O}}$ since $\partial_t\partial^m g = \partial^m\widehat{\mathcal{O}}$. We define $\mathsf{P}^m(\Gamma)$ to be a polynomial in $\Gamma, \ldots, \partial^{m-1}\Gamma$, in which each term contains m partial derivatives of g:

$$\mathsf{P}^m(\Gamma) = \sum_{k+i_1+\cdots+i_k=m} \partial^{i_1}\Gamma * \cdots * \partial^{i_k}\Gamma.$$

We first bound $\partial^{m-1}\Gamma$ in terms of $\nabla^k \widehat{\mathcal{O}}$ for $k \geq 0$, $\partial^k g$ for $0 \leq k \leq m-1$, and $\partial^k \Gamma$ for $0 \leq k \leq m-2$ by using the inequality

$$|\partial_t \partial^{m-1}\Gamma| \leq \sum_{i=0}^{m-1} |\partial^i \nabla \widehat{\mathcal{O}} * \partial^{m-1-i} g|$$

and the equation

$$\partial^i \nabla \widehat{\mathcal{O}} = \nabla^{i+1}\widehat{\mathcal{O}} + \sum_{j=1}^{i} \nabla^j \widehat{\mathcal{O}} * \mathsf{P}^{i-j+1}(\Gamma).$$

It then follows from the equation

$$\partial^m \widehat{\mathcal{O}} = \nabla^m \widehat{\mathcal{O}} + \sum_{i=1}^{m} \partial^{m-i}\widehat{\mathcal{O}} * \mathsf{P}^i(\Gamma)$$

that we can bound $\partial^m g$ in terms of $\nabla^k \widehat{\mathcal{O}}$ for $k \geq 0$, $\partial^k g$ for $0 \leq k \leq m-1$, and $\partial^k \Gamma$ for $0 \leq k \leq m-2$. This completes the induction. $\square$

Proof of Theorem 1.2. Suppose that equation (6.1) holds. By Lemma 6.2, the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ as $t \uparrow T$. We show that $g(T)$ is C^∞ on M. It suffices to show for each $k \in \mathbb{N}$ that $g(T)$ is C^k on any coordinate patch since we can take a maximum over finitely many of them to show that $g(T)$ is C^k on M. We have

$$g(t) = g(0) + \int_0^t \widehat{\mathcal{O}}(\tau)\, d\tau.$$

Taking limits as $t \uparrow T$, we get

$$g(T) = g(0) + \int_0^T \widehat{\mathcal{O}}(\tau)\, d\tau.$$

This permits us to take the kth partial derivative:

$$\partial^k g(T) = \partial^k g(0) + \int_0^T \partial^k \widehat{\mathcal{O}}(\tau)\, d\tau.$$

The bounds on $\partial^k g$ and $\partial^k \widehat{\mathcal{O}}$ from Lemma 6.3 therefore imply a bound on $\partial^k g(T)$. So $g(T)$ is C^∞ on M. Furthermore, since

$$|\partial^k g(T) - \partial^k g(t)| \leq \int_t^T |\partial^k \widehat{\mathcal{O}}(\tau)|\, d\tau \leq C_k(T-t),$$

the metrics $g(t)$ converge in C^∞ to $g(T)$. So $g(t)$ is a C^∞ solution to AOF on $[0,T]$. Then the short time existence Theorem 3.6 applied to $g(t)$ with initial metric $g(T)$ allows us to extend $g(t)$ past T. This contradicts the assumption that T was the maximal time for the solution $(M, g(t))$. $\square$

7. Compactness of solutions

In this section, we give compactness results for an AOF similar to Hamilton's compactness theorem for solutions of the Ricci flow. We first prove a proposition that states that for a sequence of metrics, uniform bounds on the spacetime derivatives of curvature and the derivatives of the metric at one time extend to uniform bounds on the spacetime derivatives of the metric. This is used to prove the compactness Theorem 1.3 for a sequence of complete pointed solutions of AOF. We then give the proofs of Theorem 1.4, which allow us to obtain a singularity model from a singular solution, and Theorem 1.5, which describe the behavior at time ∞ of a nonsingular solution.

The type of convergence of manifolds we will consider is the pointed C^∞ Cheeger–Gromov convergence.

Definition 7.1 (C^∞ Cheeger–Gromov convergence ([13], Definition 3.5))**.** A sequence $\{(M_k^n, g_k, O_k)\}_{k\in\mathbb{N}}$ of complete pointed Riemannian manifolds **converges** (in the Cheeger–Gromov topology) to a complete pointed Riemannian manifold $(M_\infty^n, g_\infty, O_\infty)$ if there exist

(1) an exhaustion $\{U_k\}_{k\in\mathbb{N}}$ of M_∞ by open sets with $O_\infty \in U_k$,
(2) a sequence of diffeomorphisms $\Phi_k : U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(O_\infty) = O_k$ such that $(U_k, \Phi_k^*[g_k|_{V_k}])$ converges in C^∞ to (M_∞, g_∞) uniformly on compact sets in M_∞.

The following compactness result of Hamilton allows us to extract a convergent subsequence of manifolds at a fixed time.

Theorem 7.2 (Cheeger–Gromov compactness theorem ([23], Theorem 2.3))**.** *Let $\{(M_k^n, g_k, O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds that satisfy*

$$|\nabla_k^p \mathrm{Rm}_k|_k \le C_p \text{ on } M_k$$

for all $p \ge 0$ and k, where $C_p < \infty$ is a sequence of constants independent of k and

$$\mathrm{inj}_{g_k}(O_k) \ge \iota_0$$

for some constant $\iota_0 > 0$. Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ such that $\{M_{j_k}, g_{j_k}, O_{j_k})\}_{k\in\mathbb{N}}$ converges to a complete pointed Riemannian manifold

$$(M_\infty^n, g_\infty, O_\infty)$$

as $k \to \infty$.

The following proposition allows us to extend bounds on the derivatives of a sequence of metrics at one time to bounds that are uniform over an interval.

Proposition 7.3. *Let (M, g) be a Riemannian manifold and let L be a compact subset of M. Let $\{g_i\}_{i\in\mathbb{N}}$ be a collection of Riemannian metrics that are solutions of AOF on neighborhoods containing $L \times [\beta, \psi]$. Let $t_0 \in [\beta, \psi]$ and fix $k \ge n-2$. Let unmarked objects such as ∇ and $|\cdot|$ be taken with respect to g, and let objects such as ∇_k and $|\cdot|_k$ be taken with respect to g_k. Suppose that:*

(1) *The metrics $g_i(t_0)$ are uniformly equivalent to g for every $i \in \mathbb{N}$: for some $B_0 > 0$, $B_0^{-1} g \le g_i(t_0) \le B_0 g$.*
(2) *For each $1 \le p \le k$, there exists a uniform bound C_p on L independent of i such that $|\nabla^p g_i(t_0)| \le C_p$.*
(3) *For each $0 \le p+q \le k+n-2$, there exists a uniform bound $C'_{p,q}$ on $L \times [\beta, \psi]$ independent of i such that $|\partial_t^q \nabla_{g_i}^p \mathrm{Rm}(g_i)|_{g_i} \le C'_{p,q}$.*

Then:

(1) *The metrics $g_k(t)$ are uniformly equivalent to g for every $i \in \mathbb{N}$ and $t \in [\beta, \varphi]$: for some $B = B(t, t_0) > 0$, $B^{-1} g \le g_i(t) \le Bg$.*
(2) *For every p, q satisfying $0 \le p+q \le k$, there is a uniform bound $\widetilde{C}_{p,q}$ on $L \times [\beta, \psi]$ independent of i such that $|\partial_t^q \nabla^p g_i(t)| \le \widetilde{C}_{p,q}$.*

Proof. We adapt the proof of Lemma 3.11 in Chow et al. [13]. The uniform equivalence of the g_k and g on $L \times [\beta, \phi]$ follow from the given bounds for $|\nabla^p_{g_i} \mathrm{Rm}(g_i)|_{g_i}$ on $L \times [\beta, \psi]$. Define the bounds $\overline{C}_j$ for j satisfying $0 \leq j \leq j - n + 2$ by

$$|\nabla^j_k \widehat{\mathcal{O}}_k| \leq \sum_{p=j}^{n-2+j} a_p C C'_{p,0} \equiv \overline{C}_j.$$

Suppose that $(p, q) = (1, 0)$. Hamilton showed in Theorem 7.1 of [21] that $\partial_t \Gamma = g^{-1} * \nabla \partial_t g$. Then

$$|\partial_t(\Gamma_k - \Gamma)|_k \leq C |\nabla_k \widehat{\mathcal{O}}_k|_k \leq C \overline{C}_1.$$

It follows that

$$\begin{aligned}
|\nabla g_k(t)| &\leq B(t, t_0)^{3/2} |\nabla g_k(t)|_k \\
&\leq B(\psi, \beta)^{3/2} 2 |\Gamma_k(t) - \Gamma|_k \\
&\leq B(\psi, \beta)^{3/2} (C \overline{C}_1 |\psi - \beta| + 3 B_0^{3/2} C_1) \equiv \tilde{C}_{1,0}.
\end{aligned}$$

Next, we prove the lemma for p satisfying $p \leq k$ when $q = 0$. We will show that for $p \geq 1$,

$$|\nabla^p \partial_t g_k| \leq C''_p |\nabla^p g_k| + C'''_p, \quad |\nabla^p g_k| \leq \tilde{C}_{p,0}. \tag{7.1}$$

If $p = 1$, then

$$\begin{aligned}
|\nabla \partial_t g_k(t)| &\leq B(t, t_0)^{3/2} |(\nabla - \nabla_k) \partial_t g_k + \nabla_k \partial_t g_k|_k \\
&\leq B(t, t_0)^{3/2} C |\Gamma - \Gamma_k|_k |\partial_t g_k|_k + |\nabla_k \partial_t g_k|_k \\
&\leq B(t, t_0)^{3/2} C |\nabla g_k| \overline{C}_0 + \overline{C}_1,
\end{aligned}$$

and we have already shown that $|\nabla g_k| \leq \tilde{C}_{1,0}$.

Let $N \geq 2$ and assume that (7.1) is true for $0 \leq p \leq N - 1$. The telescoping identity

$$\nabla^N A - \nabla^N_k A = \sum_{i=1}^{N} \nabla^{N-i} (\nabla - \nabla_k) \nabla^{i-1}_k A$$

results in the following inequality:

$$|\nabla^N \partial_t g_k| \leq |\nabla^{N-1} (\nabla - \nabla_k) \partial_t g_k| + \sum_{i=2}^{N} |\nabla^{N-i} (\nabla - \nabla_k) \nabla^{i-1}_k \partial_t g_k| + |\nabla^N_k \partial_t g_k|.$$

Using the induction hypothesis and the given estimates for $|\nabla^p_{g_i} \mathrm{Rm}(g_i)|_{g_i}$, we estimate the terms of the preceding inequality. Collecting terms yields

$$|\nabla^N \partial_t g_k| \leq C''_N |\nabla^N g_k| + C'''_N.$$

Applying the preceding inequality, we get

$$\begin{aligned}
\partial_t |\nabla^N g_k|^2 &= 2 \langle \partial_t \nabla^N g_k, \nabla^N g_k \rangle \\
&\leq |\partial_t \nabla^N g_k|^2 + |\nabla^N g_k|^2 \\
&\leq (1 + 2(C''_N)^2) |\nabla^N g_k|^2 + 2(C'''_N)^2.
\end{aligned}$$

After solving an ODE, we obtain

$$|\nabla^N g_k|^2(t) \leq e^{(1+2(C''_N)^2)(\psi - t_0)} \left[C_N + \frac{2(C'''_N)^2}{1 + 2(C''_N)^2} \left(1 - e^{(1+2(C''_N)^2)(t_0 - \beta)}\right) \right] \equiv \tilde{C}^2_{N,0}.$$

This completes the inductive proof of (7.1) and the proof of the proposition when $q = 0$. Since $\partial_t^q \nabla^p g_k = \nabla^p \partial_t^q g_k$, a similar procedure may be used to prove the proposition when $q > 0$. $\square$

We are now able to prove the compactness Theorem 1.3 for solutions of the AOF via a modification of the proof given by Hamilton in [23] of the compactness theorem for Ricci flow.

Proof of Theorem 1.3. Since we are given a uniform bound on $|\mathrm{Rm}(g_k)|_{g_k}$, the pointwise smoothing estimates given by Theorem 1.1 furnish uniform bounds on $\|\nabla^m_{g_k(t_0)} \mathrm{Rm}(g_k(t_0))\|_{C^0(g_k(t_0))}$ for all $m \in \mathbb{N}$. Therefore, since the (M_k, g_k) are complete, the Cheeger–Gromov compactness Theorem 7.2 yields a subsequence of $\{(M_k, g_k(t), O_k)\}_{k\in\mathbb{N}}$ that converges to a complete pointed Riemannian manifold $(M^n_\infty, h, O_\infty)$.

Theorem 1.1 provides uniform C^0 estimates for the covariant derivatives of curvature on a closed interval. This enables us to apply Proposition 7.3 in order to obtain uniform estimates on closed time intervals of the spacetime derivatives of the metrics in the subsequence that converges at time t_0. Finally, an application of the Arzelà–Ascoli theorem provides the desired subsequence that converges on the time interval (α, ω). $\square$

As our first corollary of the compactness Theorem 1.3, we show that under suitable conditions, we can obtain a singularity model for the ambient obstruction flow.

Proof of Theorem 1.4. We first show that the g_i are also solutions to AOF by showing that if $\tilde{g} = \lambda g$ and g satisfies AOF, given up to constants by

$$\partial_t g = \Delta^{\frac{n}{2}-1}\mathrm{Rc} + \Delta^{\frac{n}{2}-2}\nabla^2 R + \sum_{j=2}^{n/2} P_j^{n-2j}(\mathrm{Rm}),$$

then $\tilde{g}$ satisfies

$$\partial_t \tilde{g} = \widetilde{\Delta}^{\frac{n}{2}-1}\widetilde{\mathrm{Rc}} + \widetilde{\Delta}^{\frac{n}{2}-2}\widetilde{\nabla}^2 \tilde{R} + \sum_{j=2}^{n/2} P_j^{n-2j}(\widetilde{\mathrm{Rm}}). \tag{7.2}$$

We evaluate the first term of the right side of (7.2):

$$\widetilde{\Delta}^{\frac{n}{2}-1}\widetilde{\mathrm{Rc}} = \left(\lambda^{-1} g^{-1}\nabla^2\right)^{\frac{n}{2}-1}\mathrm{Rc} = \lambda^{1-\frac{n}{2}}\Delta^{\frac{n}{2}-1}\mathrm{Rc}.$$

Similarly, the second term is equal to $\lambda^{1-\frac{n}{2}}\Delta^{\frac{n}{2}-1}\mathrm{Rc}$. The remaining terms are contractions of terms of the form

$$\widetilde{\nabla}^{i_1}\widetilde{\mathrm{Rm}} \otimes \cdots \otimes \widetilde{\nabla}^{i_j}\widetilde{\mathrm{Rm}}$$

with $2 \le j \le \frac{n}{2}$ and $i_1 + \cdots + i_j = n - 2j$. In order to contract on all but two indices of the above term, we need to contract $\frac{1}{2}(i_1 + \cdots + i_j + 3j - j - 2) = \frac{n}{2} - 1$ pairs of indices. This implies that $P_j^{n-2j}(\widetilde{\mathrm{Rm}}) = \lambda^{1-\frac{n}{2}} P_j^{n-2j}(\mathrm{Rm})$. The left side of (7.2) is equal to $\lambda^{1-\frac{n}{2}}\partial_t g$. So $\tilde{g}$ satisfies (7.2).

We have $|\mathrm{Rm}(g_i)|_{g_i} \le 1$ on $M \times [-\lambda_i^{n/2} t_i, 0]$ for each i since the definition of λ_i implies

$$|\mathrm{Rm}(g_i)|^2_{g_i} = \lambda_i^{-2}|\mathrm{Rm}|^2 \le \lambda_i^{-2}\lambda_i^2 = 1.$$

Let $k \in \mathbb{N}$. There exists i_k such that if $i \geq i_k$, then $\lambda_i^{n/2} t_i > k$. Then $\{g_i\}_{i \geq i_k}$ is a sequence of complete pointed solutions to AOF on $(-k, 0]$. Since the Sobolev constant is scaling invariant, the uniform bound of $C_S(M, g)$ on $[0, T)$ implies a uniform bound independent of i of $C_S(M, g_i)$ on $[0, T)$. We conclude from Lemma 3.2 of Hebey [24] that there exists a uniform lower bound independent of i for $\inf_{x \in M} \operatorname{vol}(B_{g_i}(x, 1))$. This and the bound $|\mathrm{Rm}(g_i)|_{g_i} \leq 1$ on $M \times [-\lambda_i^{n/2} t_i, 0]$ for all i give a uniform lower bound independent of i for $\mathrm{inj}_{g_i(0)}(x_i)$ via the Cheeger–Gromov–Taylor theorem.

The proof of the compactness Theorem 1.3 is unchanged if we replace (α, ω) with $(-k, 0]$. Thus, by Theorem 1.3, we obtain subsequential convergence of

$$\{(M, g_i(t), x_i)\}_{i \geq i_k}$$

to a complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF for $t \in (-k, 0]$. By taking a further diagonal subsequence over k, we get that $\{(M, g_i(t), x_i)\}_{i \geq 1}$ subsequentially converges to a complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF for $t \in (-\infty, 0]$. The limit $(M_\infty, g_\infty(t))$ is not flat since

$$|\mathrm{Rm}(g_\infty(0))(x_\infty)|_{g_\infty(0)} = 1$$

by the definition of $g_i(t)$.

We show that M_∞ is noncompact. Lemma 3.9 of Chow–Knopf [15] states that for a one-parameter family of Riemannian manifolds $(M, g(t))$, the volume element evolves by $\partial_t dV_g = \frac{1}{2} g^{ij} \partial_t g_{ij}$. By applying the fact that $\mathcal{O}$ is traceless and the divergence theorem,

$$\begin{aligned}
\frac{\partial}{\partial t} \operatorname{vol}(M, g(t)) &= \frac{1}{2} \int_M g^{ij} \frac{\partial g_{ij}}{\partial t} \, dV_{g(t)} \\
&= \frac{1}{2} \int_M [(-1)^{\frac{n}{2}} g^{ij} \mathcal{O}_{ij} + C(n)(\Delta^{\frac{n}{2}-1} R) g^{ij} g_{ij}] \, dV_{g(t)} \\
&= C(n) \int_M \Delta^{\frac{n}{2}-1} R \, dV_{g(t)} \\
&= 0.
\end{aligned}$$

Therefore the volume of $(M, g(t))$ is preserved along the flow. Since $\lambda_i \to \infty$,

$$\operatorname{vol}(M_\infty, g_\infty(t)) = \lim_{i \to \infty} \operatorname{vol}(M, g_i(t)) = \lim_{i \to \infty} \lambda_i^{n/2} \operatorname{vol}(M, g(t_i + \lambda_i^{\frac{n}{2}} t)) = \infty$$

for all $t \in (-\infty, 0]$. So the volume of $(M, g_\infty(t))$ is infinite for all $t \in (-\infty, 0]$. The uniform volume lower bound for (M, g_i) passes in the limit to a uniform volume lower bound for (M, g_∞). Therefore M_∞ is noncompact by Lemma 8.1 of Bour [6].

Next, we show that the integral of the Q curvature is nondecreasing along the flow on M. Along the flow, the derivative of $\int_M Q$ is given by

$$\begin{aligned}
\frac{\partial}{\partial t} \int_M Q &= (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M \langle \mathcal{O}, \partial_t g \rangle \\
&= (-1)^{\frac{n}{2}} \frac{n-2}{2} \int_M (-1)^{\frac{n}{2}} |\mathcal{O}|^2 + C(n) \int_M \langle \mathcal{O}, (\Delta^{\frac{n}{2}-1} R) g \rangle \\
&= \frac{n-2}{2} \int_M |\mathcal{O}|^2,
\end{aligned}$$

where the third line holds since $\mathcal{O}$ is traceless. So the integral of the Q curvature does not decrease along the flow.

Suppose that

$$\sup_{t\in[0,T)} \int_M Q(g(t))\, dV_{g(t)} < \infty.$$

This is always true when $n = 4$ since the Chern–Gauss–Bonnet theorem gives that for all $t \in [0, T)$,

$$\int_M Q = 8\pi^2\chi(M) - \tfrac{1}{4}\int_M |W|^2 \leq 8\pi^2\chi(M).$$

So if the integral of the Q curvature is bounded along the flow,

$$\begin{aligned}\int_0^T \int_M |\mathcal{O}|^2 &= \int_0^T \frac{\partial}{\partial t}\int_M Q \\ &= \lim_{t\uparrow T}\int_M Q(g(t)) - \int_M Q(g(0)) \\ &< \infty.\end{aligned}$$

Let $\{(M, g_i(t), x_i)\}_{i\geq 1}$ be the convergent subsequence previously found in the proof. Fix $k \in \mathbb{N}$. Since $t_i \to T$ and $\lambda_i \to \infty$, we can choose a subsequence of times $\{t_{i_j}\}_{j\in\mathbb{N}}$ as follows:

$$i_1 = \inf\left\{i : t_i \geq \tfrac{T}{2}, \lambda_i \geq \left(\tfrac{2k}{T}\right)^{\frac{2}{n}}\right\}, \quad i_j = \inf\left\{i : t_i \geq \tfrac{1}{2}(T + t_{i_{j-1}}), \lambda_i \geq \left(\tfrac{2k}{T - t_{i_{j-1}}}\right)^{\frac{2}{n}}\right\}$$

for $j \geq 2$. We relabel $\{t_{i_j}\}_{j\in\mathbb{N}}$ as $\{t_i\}_{i\in\mathbb{N}}$. Then

$$\sum_{i=1}^{\infty}\int_{t_i - k\lambda_i^{-\frac{n}{2}}}^{t_i}\int_M |\mathcal{O}|^2 < \int_0^T\int_M |\mathcal{O}|^2 < \infty,$$

implying that, using the scaling law $\mathcal{O}(\lambda g) = \lambda^{\frac{2-n}{2}}\mathcal{O}(g)$,

$$\begin{aligned}0 &= \lim_{i\to\infty}\int_{t_i - k\lambda_i^{-\frac{n}{2}}}^{t_i}\int_M |\mathcal{O}(g)|_g^2\, dV_g\, dt \\ &= \lim_{i\to\infty}\int_{-k}^0\int_M \lambda_i^n |\mathcal{O}(g_i)|_{g_i}^2 \lambda_i^{-\frac{n}{2}}\lambda_i^{-\frac{n}{2}}\, dV_{g_i}\, dt \\ &= \lim_{i\to\infty}\int_{-k}^0\int_M |\mathcal{O}(g_i)|_{g_i}^2\, dV_{g_i}\, dt.\end{aligned}$$

Since $\mathcal{O}(g_i) \to \mathcal{O}(g_\infty)$ in C^∞ on compact subsets, this implies that $\mathcal{O}(g_\infty) \equiv 0$ on $[-k, 0]$. So for each $k \in \mathbb{N}$, there exists a sequence of pointed solutions to AOF that converge to an obstruction flat pointed solution to AOF on $[-k, 0]$. By taking a further diagonal subsequence over k, we obtain a sequence of pointed solutions to AOF that converge to an obstruction flat complete pointed solution to AOF on $(-\infty, 0]$. □

Finally, we provide a corollary of the compactness Theorem 1.3 characterizing limits of nonsingular solutions to AOF.

Proof of Theorem 1.5. Suppose M does not collapse at ∞. Then there exists a sequence $\{(x_i, t_i)\}_{i\in\mathbb{N}} \subset M \times [0, \infty)$ such that $\inf_i \operatorname{inj}_{g(t_i)}(x_i) > 0$. Let $g_i(t) = g(t + t_i)$ for $t \in [-t_i, \infty)$. Let $k \in \mathbb{N}$. Then there exists $i_k \in \mathbb{N}$ such that $t_i > k$ for all $i \geq i_k$. Since $\sup_{t\in[0,\infty)} \|\mathrm{Rm}\|_\infty < \infty$ and $\inf_i \operatorname{inj}_{g_i(0)}(x_i) > 0$, we apply Theorem 1.3 to obtain subsequential convergence in the sense of families of pointed Riemannian manifolds of $\{(M, g_i(t), x_i)\}_{i\geq i_k}$ to a complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$

to AOF on $(-k,\infty)$. By taking a further diagonal subsequence over k, we get that $\{(M, g_i(t), x_i)\}_{i\geq 1}$ subsequentially converges to a complete pointed solution $(M_\infty, g_\infty(t), x_\infty)$ to AOF on $(-\infty,\infty)$.

If M_∞ is compact, then by the definition of convergence of complete pointed Riemannian manifolds, M_∞ is diffeomorphic to M. Just as in the proof of Theorem 1.4, the volume of $(M, g(t))$ is preserved along the flow. So for all $t \in (-\infty,\infty)$,

$$\mathrm{vol}(M_\infty, g_\infty(t)) = \lim_{i\to\infty} \mathrm{vol}(M, g_i(t)) = \lim_{i\to\infty} \mathrm{vol}(M, g(t_i + t)) < \infty.$$

Suppose that

$$\sup_{t\in[0,\infty)} \int_M Q(g(t))\, dV_{g(t)} < \infty.$$

This is always true when $n = 4$ by the Chern–Gauss–Bonnet theorem. Using the same argument as in the proof of Theorem 1.4, we obtain

$$\int_0^\infty \int_M |\mathcal{O}|^2 < \infty.$$

Let $\{(M, g_i(t), x_i)\}_{i\geq 1}$ be the convergent subsequence previously found in the proof. Since $t_i \to \infty$, we can choose a subsequence of times $\{t_{i_j}\}_{j\in\mathbb{N}}$ as follows:

$$i_1 = \inf\{i : t_i \geq k\}, \quad i_j = \inf\{i : t_i \geq t_{i_{j-1}} + 2k\}$$

for $j \geq 2$. We relabel $\{t_{i_j}\}_{j\in\mathbb{N}}$ as$\{t_i\}_{i\in\mathbb{N}}$. Then

$$\sum_{i=1}^\infty \int_{t_i-k}^{t_i+k} \int_M |\mathcal{O}|^2 < \int_0^\infty \int_M |\mathcal{O}|^2 < \infty$$

implies that

$$0 = \lim_{i\to\infty} \int_{t_i-k}^{t_i+k} \int_M |\mathcal{O}(g)|_g^2 \, dV_g \, dt = \lim_{i\to\infty} \int_{-k}^{k} \int_M |\mathcal{O}(g_i)|_{g_i}^2 \, dV_{g_i} \, dt.$$

Since $\mathcal{O}(g_i) \to \mathcal{O}(g_\infty)$ in C^∞ on compact subsets, this implies that $\mathcal{O}(g_\infty) \equiv 0$ on $[-k,k]$. So for each $k \in \mathbb{N}$, there exists a sequence of pointed solutions to AOF that converge to an obstruction flat pointed solution to AOF on $[-k,k]$. By taking a further diagonal subsequence over k, we obtain a sequence of pointed solutions to AOF that converge to an obstruction flat complete pointed solution to AOF on $(-\infty,\infty)$. Since g_∞ solves the conformal flow $\partial_t g_\infty = (-1)^{n/2} C(n)(\Delta^{\frac{n}{2}-1} R) g$, we see that $g_\infty(t)$ is in the conformal class of $g_\infty(0)$ for all $t \in (-\infty,\infty)$. If M_∞ is compact, we can solve the Yamabe problem for $(M_\infty, [g_\infty(0)])$; the Yamabe problem was solved by Aubin, Trudinger, and Schoen (see [2, 30]). Due to the conformal covariance of $\mathcal{O}$, we obtain an obstruction flat, constant scalar curvature complete pointed solution $(M_\infty, \hat{g}_\infty(t))$ to AOF with $\hat{g}_\infty(t) = \hat{g}_\infty(0)$ for all $t \in (-\infty,\infty)$. □

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