

RESTRICTIONS OF HÖLDER CONTINUOUS FUNCTIONS

OMER ANGEL, RICHÁRD BALKÁ, ANDRÁS MÁTHÉ, AND YUVAL PERES

ABSTRACT. For $0 < \alpha < 1$ let $V(\alpha)$ denote the supremum of the numbers v such that every α -Hölder continuous function is of bounded variation on a set of Hausdorff dimension v . Kahane and Katznelson (2009) proved the estimate $1/2 \leq V(\alpha) \leq 1/(2-\alpha)$ and asked whether the upper bound is sharp. We show that in fact $V(\alpha) = \max\{1/2, \alpha\}$. Let $\dim_{\mathcal{H}}$ and $\overline{\dim}_{\mathcal{M}}$ denote the Hausdorff and upper Minkowski dimension, respectively. The upper bound on $V(\alpha)$ is a consequence of the following theorem. Let $\{B(t) : t \in [0, 1]\}$ be a fractional Brownian motion of Hurst index α . Then, almost surely, there exists no set $A \subset [0, 1]$ such that $\overline{\dim}_{\mathcal{M}} A > \max\{1 - \alpha, \alpha\}$ and $B : A \rightarrow \mathbb{R}$ is of bounded variation. Furthermore, almost surely, there exists no set $A \subset [0, 1]$ such that $\overline{\dim}_{\mathcal{M}} A > 1 - \alpha$ and $B : A \rightarrow \mathbb{R}$ is β -Hölder continuous for some $\beta > \alpha$. The zero set and the set of record times of B witness that the above theorems give the optimal dimensions. We also prove similar restriction theorems for deterministic self-affine functions and generic α -Hölder continuous functions.

Finally, let $\{\mathbf{B}(t) : t \in [0, 1]\}$ be a two-dimensional Brownian motion. We prove that, almost surely, there is a compact set $D \subset [0, 1]$ such that $\dim_{\mathcal{H}} D \geq 1/3$ and $\mathbf{B} : D \rightarrow \mathbb{R}^2$ is non-decreasing in each coordinate. It remains open whether $1/3$ is best possible.

1. INTRODUCTION

Let $C^\alpha = C^\alpha[0, 1]$ denote the set of α -Hölder continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. In 2009, Kahane and Katznelson [11] proved the following result and asked whether it is sharp.

Theorem (Kahane–Katznelson). *For every $0 < \alpha < 1$ there exists a function $g_\alpha \in C^\alpha$ such that if $A \subset [0, 1]$ and $g_\alpha|_A$ is of bounded variation, then the Hausdorff dimension satisfies $\dim_{\mathcal{H}} A \leq 1/(2 - \alpha)$.*

Question (Kahane–Katznelson). *Is the above result the best possible?*

We answer this question negatively and determine the optimal bound. Let

$$V(\alpha) = \inf_{f \in C^\alpha} \sup_{A \subset [0, 1]} \{\dim_{\mathcal{H}} A : f|_A \text{ is of bounded variation}\},$$

so that the above theorem states $V(\alpha) \leq 1/(2 - \alpha)$; see Figure 1.

Received by the editors April 19, 2015 and, in revised form, November 11, 2016.

2010 *Mathematics Subject Classification*. Primary 26A16, 26A45, 28A78, 54E52, 60G17, 60G22, 60J65.

Key words and phrases. Fractional Brownian motion, Hölder continuous, restriction, bounded variation, Hausdorff dimension, box dimension, Minkowski dimension, self-affine function, generic, typical, Baire category.

The first author was supported in part by NSERC.

The second and third authors were supported by the National Research, Development and Innovation Office-NKFIH, 104178.

The third author was also supported by the Leverhulme Trust.

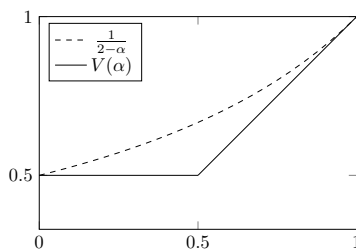


FIGURE 1. The Kahane–Katznelson bound $V(\alpha) \leq 1/(2-\alpha)$ compared to the actual value $V(\alpha) = \max\{1/2, \alpha\}$.

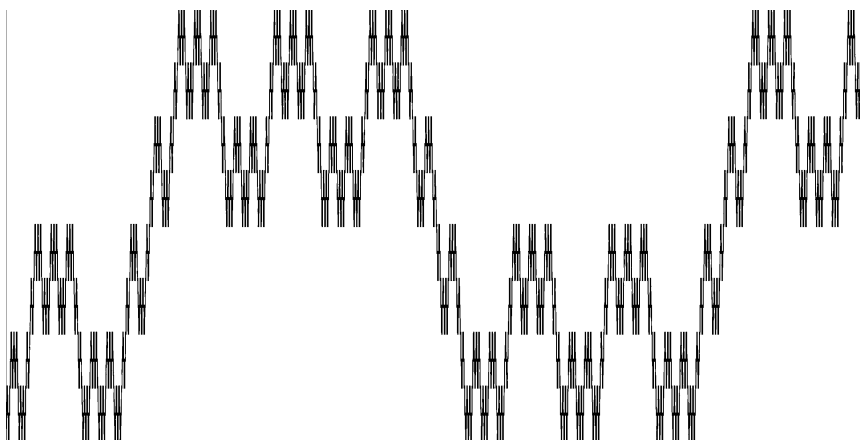


FIGURE 2. A self-affine function $f_\alpha: [0, 1] \rightarrow [0, 1]$ with Hölder exponent $\alpha = \log 2 / \log 6$. Its graph consists of 6 affine copies of itself.

Theorem 1.1. *For all $0 < \alpha < 1$ we have*

$$V(\alpha) = \max\{1/2, \alpha\}.$$

Kahane and Katznelson also asked about dimensions of sets A such that the restriction to A is Hölder continuous. (See the next section for related results.) We present two constructions, one deterministic and one stochastic, of functions that are not Hölder on any set of high enough dimension. First we consider self-affine functions. These are constructed in Definition 3.1 below; see Figure 2 for illustration.

Theorem 1.2. *There is a dense set $\Delta \subset (0, 1)$ with the following property. For each $\alpha \in \Delta$ there is a self-affine function $f_\alpha \in C^\alpha$ such that for all $A \subset [0, 1]$,*

- (1) *if $f_\alpha|_A$ is β -Hölder continuous for some $\beta > \alpha$, then $\overline{\dim}_{\mathcal{M}} A \leq 1 - \alpha$;*
- (2) *if $f_\alpha|_A$ is of bounded variation, then $\overline{\dim}_{\mathcal{M}} A \leq \max\{1 - \alpha, \alpha\}$.*

For a stochastically self-affine process, fractional Brownian motion (see Definition 3.2), we prove the following.

Theorem 1.3. *Let $0 < \alpha < 1$ and let $\{B(t) : t \in [0, 1]\}$ be a fractional Brownian motion of Hurst index α . Then, almost surely, for all $A \subset [0, 1]$,*

- (1) *if $B|_A$ is β -Hölder continuous for some $\beta > \alpha$, then $\overline{\dim}_{\mathcal{M}} A \leq 1 - \alpha$;*
- (2) *if $B|_A$ is of bounded variation, then $\overline{\dim}_{\mathcal{M}} A \leq \max\{1 - \alpha, \alpha\}$.*

Corollary 1.4. *Let $0 < \alpha < 1$ and let $\{B(t) : t \in [0, 1]\}$ be a fractional Brownian motion of Hurst index α . Then*

$$\mathbb{P}(\exists A : \overline{\dim}_{\mathcal{M}} A > \max\{1 - \alpha, \alpha\} \text{ and } B|_A \text{ is non-decreasing}) = 0.$$

Let \mathcal{Z} be the zero set of B and let $\mathcal{R} = \{t \in [0, 1] : B(t) = \max_{s \in [0, t]} B(s)\}$ be the set of record times of B . It is classical that, almost surely, $\dim_{\mathcal{H}} \mathcal{Z} = 1 - \alpha$; see [10, Chapter 18]. For the record, let us state the following more subtle fact.

Proposition 1.5. *Almost surely, $\dim_{\mathcal{H}} \mathcal{R} = \overline{\dim}_{\mathcal{M}} \mathcal{R} = \alpha$.*

We could not find a reference for this in the literature and include a proof in Section 6. Clearly \mathcal{Z} and \mathcal{R} witness that Theorem 1.3 and Corollary 1.4 are best possible.

Simon [20] proved that a standard linear Brownian motion is not monotone on any set of positive Lebesgue measure. Theorem 1.3 for $\alpha = 1/2$ with Hausdorff dimension in place of upper Minkowski dimension is due to Balka and Peres [4]. The methods used there do not extend to Minkowski dimension or to general exponents α . Related results in the discrete setting, concerning non-decreasing subsequences of random walks, can be found in [2].

Now we consider higher dimensional Brownian motion.

Definition 1.6. Let $d \geq 2$ and $f : [0, 1] \rightarrow \mathbb{R}^d$. We say that f is *non-decreasing* on a set $A \subset [0, 1]$ if all the coordinate functions of $f|_A$ are non-decreasing.

Theorem 1.7. *Let $\{B(t) : t \in [0, 1]\}$ be a standard two-dimensional Brownian motion. Then, almost surely, there exists a compact set $D \subset [0, 1]$ such that B is non-decreasing on D and $\dim_{\mathcal{H}} D \geq 1/3$.*

Corollary 1.4 (or [4, Theorem 1.2]) implies that, almost surely, the d -dimensional Brownian motion B cannot be non-decreasing on any set of Hausdorff dimension larger than $1/2$. The following problem remains open in all dimensions $d \geq 2$.

Question 1.8. *Let $d \geq 2$ and let $\{B(t) : 0 \leq t \leq 1\}$ be a standard d -dimensional Brownian motion. What is the supremum of the numbers γ such that, almost surely, B is non-decreasing on some set of Hausdorff dimension γ ?*

Finally, we prove restriction theorems for a generic α -Hölder continuous function (in the sense of Baire category); see the following section for the details.

2. RELATED WORK AND GENERAL STATEMENTS

Let $C[0, 1]$ denote the set of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ endowed with the maximum norm. Elekes [6, Theorems 1.4, 1.5] proved the following restriction theorem.

Theorem 2.1 (Elekes). *Assume that $0 < \beta < 1$. For a generic continuous function $f \in C[0, 1]$ (in the sense of Baire category) for all $A \subset [0, 1]$,*

- (1) *if $f|_A$ is β -Hölder continuous, then $\dim_{\mathcal{H}} A \leq 1 - \beta$;*
- (2) *if $f|_A$ is of bounded variation, then $\dim_{\mathcal{H}} A \leq 1/2$.*

Kahane and Katznelson [11, Theorems 2.1, 3.1], and independently Máthé [16, Theorems 1.4, 1.5] proved that the above result is sharp.

Theorem 2.2 (Kahane–Katznelson, Máthé). *Let $0 < \beta < 1$. For every $f \in C[0, 1]$ there are compact sets $A, C \subset [0, 1]$ such that*

- (1) $\dim_{\mathcal{H}} A = 1 - \beta$ and $f|_A$ is β -Hölder continuous;
- (2) $\dim_{\mathcal{H}} C = 1/2$ and $f|_C$ is of bounded variation.

In other words there is always a set A with the given properties and dimension, and for generic functions there is no A of larger dimension. Let us recall that the β -variation of a function $f : A \rightarrow \mathbb{R}$ is defined as

$$V^\beta(f) = \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})|^\beta : x_0 < \dots < x_n, x_i \in A, n \in \mathbb{N}^+ \right\}.$$

In the theorems above, bounded variation can be generalized to finite β -variation for all $\beta > 0$ by similar methods. For the following result see Máthé [16, Theorem 5.2].

Theorem 2.3 (Máthé). *Let $\beta > 0$ and $f \in C[0, 1]$. Then there is a compact set $A \subset [0, 1]$ such that $\dim_{\mathcal{H}} A = \beta/(\beta + 1)$ and $f|_A$ has finite β -variation.*

Our initial interest came from questions of Kahane and Katznelson [11] on restrictions of Hölder continuous functions. First we need the following definition.

Definition 2.4. Let $C^\alpha(A)$ be the set of α -Hölder continuous functions $f : A \rightarrow \mathbb{R}$. For all $0 < \alpha < 1$ and $\beta > 0$ define

$$H(\alpha, \beta) = \inf_{f \in C^\alpha[0,1]} \sup_{A \subset [0,1]} \{ \dim_{\mathcal{H}} A : f|_A \in C^\beta(A) \},$$

$$V(\alpha, \beta) = \inf_{f \in C^\alpha[0,1]} \sup_{A \subset [0,1]} \{ \dim_{\mathcal{H}} A : V^\beta(f|_A) < \infty \}.$$

Replacing Hausdorff dimension by upper Minkowski dimension yields

$$\overline{H}(\alpha, \beta) = \inf_{f \in C^\alpha[0,1]} \sup_{A \subset [0,1]} \{ \overline{\dim}_{\mathcal{M}} A : f|_A \in C^\beta(A) \},$$

$$\overline{V}(\alpha, \beta) = \inf_{f \in C^\alpha[0,1]} \sup_{A \subset [0,1]} \{ \overline{\dim}_{\mathcal{M}} A : V^\beta(f|_A) < \infty \}.$$

Remark 2.5. As the Hausdorff dimension is smaller than or equal to the upper Minkowski dimension, $H(\alpha, \beta) \leq \overline{H}(\alpha, \beta)$ and $V(\alpha, \beta) \leq \overline{V}(\alpha, \beta)$. If $\beta \geq 1/\alpha$, then $V^\beta(f) < \infty$ for all $f \in C^\alpha[0, 1]$, so $V(\alpha, \beta) = \overline{V}(\alpha, \beta) = 1$.

For the following theorem see [11, Theorems 5.1, 5.2].

Theorem 2.6 (Kahane–Katznelson). *For all $0 < \alpha < \beta < 1$ we have*

$$H(\alpha, \beta) \leq \frac{1 - \beta}{1 - \alpha} \quad \text{and} \quad V(\alpha, 1) \leq \frac{1}{2 - \alpha}.$$

Question 2.7 (Kahane–Katznelson). *Are the above bounds optimal?*

We answer this question negatively and find the sharp bounds, which generalizes Theorem 1.1.

Theorem 2.8. *For all $0 < \alpha < 1$ we have*

$$H(\alpha, \beta) = \overline{H}(\alpha, \beta) = 1 - \beta \quad \text{for all } \alpha < \beta \leq 1,$$

$$V(\alpha, \beta) = \overline{V}(\alpha, \beta) = \max \{ \alpha\beta, \beta/(\beta + 1) \} \quad \text{for all } 0 < \beta < 1/\alpha.$$

In Section 4 we prove restriction theorems for functions which satisfy certain scaled local time estimates. This allows us to prove the following more general version of Theorems 1.2 and 1.3; see Sections 5 and 6, respectively.

Theorem 2.9. *There is a dense set $\Delta \subset (0, 1)$ with the following property. For each $\alpha \in \Delta$ there is a self-affine function $f_\alpha \in C^\alpha[0, 1]$ such that for all $A \subset [0, 1]$,*

- (1) *if $f_\alpha|_A \in C^\beta(A)$ for some $\beta > \alpha$, then $\overline{\dim}_{\mathcal{M}} A \leq 1 - \alpha$;*
- (2) *if $V^\beta(f_\alpha|_A) < \infty$ for some $\beta > 0$, then $\overline{\dim}_{\mathcal{M}} A \leq \max\{1 - \alpha, \alpha\beta\}$.*

Theorem 2.10. *Let $0 < \alpha < 1$ and let $\{B(t) : t \in [0, 1]\}$ be a fractional Brownian motion of Hurst index α . Then, almost surely, for all $A \subset [0, 1]$,*

- (1) *if $B|_A \in C^\beta(A)$ for some $\beta > \alpha$, then $\overline{\dim}_{\mathcal{M}} A \leq 1 - \alpha$;*
- (2) *if $V^\beta(B|_A) < \infty$ for some $\beta > 0$, then $\overline{\dim}_{\mathcal{M}} A \leq \max\{1 - \alpha, \alpha\beta\}$.*

The zero set of B and the following result (see [4, Theorem 4.3]) with Lemma 6.7 witness that Theorem 2.10(2) is sharp for all $\beta \leq 1/\alpha$.

Theorem 2.11. *Let $0 < \alpha < 1$ and $0 < \beta \leq 1/\alpha$ be fixed. Then there is a compact set $A \subset [0, 1]$ (which depends only on α and β) such that $\dim_{\mathcal{H}} A = \alpha\beta$, and if $f: [0, 1] \rightarrow \mathbb{R}$ is a function and $c \in \mathbb{R}^+$ such that for all $x, y \in [0, 1]$ we have*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \log(1/|x - y|),$$

then $f|_A$ has finite β -variation.

In Section 5 we prove Theorem 2.8 by using Theorem 2.9 to obtain the sharp upper bounds for $\overline{H}(\alpha, \beta)$ and $\overline{V}(\alpha, \beta)$. Theorem 2.10 may be used there instead of Theorem 2.9. Finally, Theorems 2.2, 2.3, and 2.11 provide the optimal lower bounds for $H(\alpha, \beta)$ and $V(\alpha, \beta)$.

In Section 7 we consider higher dimensional Brownian motion and prove Theorem 1.7. In order to do so, we establish a general limit theorem for random sequences with i.i.d. increments, which is of independent interest.

Finally, in Section 8 we study generic α -Hölder continuous functions in the sense of the Baire category.

Definition 2.12. For $0 < \alpha < 1$ let $C_1^\alpha[0, 1]$ be the set of functions $f: [0, 1] \rightarrow \mathbb{R}$ such that for all $x, y \in [0, 1]$ we have

$$|f(x) - f(y)| \leq |x - y|^\alpha.$$

Let us endow $C_1^\alpha[0, 1]$ with the maximum norm. Then it is a complete metric space, and hence we can use Baire category arguments.

We show that a generic $f \in C_1^\alpha[0, 1]$ witnesses $H(\alpha, \beta) = 1 - \beta$ and $V(\alpha, \beta) = \max\{\alpha\beta, \beta/(\beta + 1)\}$ for all β simultaneously.

Theorem 2.13. *Let $0 < \alpha < 1$. For a generic $f \in C_1^\alpha[0, 1]$ for all $A \subset [0, 1]$,*

- (1) *if $f|_A \in C^\beta(A)$ for some $\alpha < \beta \leq 1$, then $\dim_{\mathcal{H}} A \leq 1 - \beta$;*
- (2) *if $V^\beta(f|_A) < \infty$ for some $\beta > 0$, then $\dim_{\mathcal{H}} A \leq \max\{\alpha\beta, \beta/(\beta + 1)\}$.*

3. PRELIMINARIES

Let $A \subset [0, 1]$ be non-empty and let $\alpha > 0$. A function $f: A \rightarrow \mathbb{R}$ is called (uniformly) α -Hölder continuous if there exists a constant $c \in (0, \infty)$ such that $|f(x) - f(y)| \leq c|x - y|^\alpha$ for all $x, y \in A$. For the definitions of $C[0, 1]$, $C^\alpha[0, 1]$,

and $C_1^\alpha[0, 1]$ see Section 2. The diameter of A is denoted by $\text{diam } A$. For all $s \geq 0$ the s -Hausdorff content of A is

$$\mathcal{H}_\infty^s(A) = \inf \left\{ \sum_{i=1}^\infty (\text{diam } A_i)^s : A \subset \bigcup_{i=1}^\infty A_i \right\}.$$

The Hausdorff dimension of A is defined as

$$\dim_{\mathcal{H}} A = \inf \{s \geq 0 : \mathcal{H}_\infty^s(A) = 0\}.$$

Let $|F|$ denote the cardinality of the set F . Let $M \geq 2$ be an integer. For all $n \in \mathbb{N}$ and $A \subset [0, 1]$ define

$$\begin{aligned} \mathcal{D}_n(M) &= \{[pM^{-n}, (p+1)M^{-n}) : p \in \{0, \dots, M^n - 1\}\}, \\ (3.1) \quad \mathcal{D}_n(A, M) &= \{I \in \mathcal{D}_n(M) : I \cap A \neq \emptyset\}, \\ N_n(A, M) &= |\mathcal{D}_n(A, M)|. \end{aligned}$$

The upper Minkowski dimension of A is defined as

$$\overline{\dim}_{\mathcal{M}} A = \limsup_{n \rightarrow \infty} \frac{\log N_n(A, M)}{n \log M}.$$

It is easy to show that this definition is independent of the choice of M and we have $\dim_{\mathcal{H}} A \leq \overline{\dim}_{\mathcal{M}} A$ for all $A \subset [0, 1]$. For more on these concepts see [8].

Definition 3.1. A compact set $K \subset \mathbb{R}^2$ is called *self-affine* if for some $M \geq 2$ there are injective and contractive affine maps $F_1, \dots, F_M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$K = \bigcup_{i=1}^M F_i(K).$$

A continuous function $f \in C[0, 1]$ is *self-affine* if $\text{graph}(f) \subset \mathbb{R}^2$ is a self-affine set.

Definition 3.2. Let $0 < \alpha < 1$. The process $\{B(t) : t \geq 0\}$ is called a *fractional Brownian motion of Hurst index α* if

- B is a Gaussian process with stationary increments;
- $B(0) = 0$ and $t^{-\alpha}B(t)$ has standard normal distribution for every $t > 0$;
- almost surely, the function $t \mapsto B(t)$ is continuous.

The covariance function of B is $\mathbb{E}(B(t)B(s)) = (1/2)(|t|^{2\alpha} + |s|^{2\alpha} - |t - s|^{2\alpha})$. It is well known that almost surely B is γ -Hölder continuous for all $\gamma < \alpha$; see Lemma 6.7 below. For more information see [1, Chapter 8] and [10, Chapter 18].

Let X be a complete metric space. A set is *somewhere dense* if it is dense in a non-empty open set; otherwise it is *nowhere dense*. We say that $A \subset X$ is *meager* if it is a countable union of nowhere dense sets, and a set is called *co-meager* if its complement is meager. By Baire’s category theorem a set is co-meager iff it contains a dense G_δ set. We say that a *generic* element $x \in X$ has property \mathcal{P} if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager.

Let $(\mathcal{K}[0, 1], d_{\mathcal{H}})$ be the set of non-empty compact subsets of $[0, 1]$ endowed with the Hausdorff metric; that is, for each $K_1, K_2 \in \mathcal{K}[0, 1]$ we have

$$d_H(K_1, K_2) = \min \{r : K_1 \subset B(K_2, r) \text{ and } K_2 \subset B(K_1, r)\},$$

where $B(A, r) = \{x \in \mathbb{R} : \exists y \in A \text{ such that } |x - y| \leq r\}$. Then $(\mathcal{K}[0, 1], d_H)$ is a compact metric space; see [12] for more on this concept.

Let $\text{supp}(\mu)$ stand for the support of the measure μ . For $x \in \mathbb{R}$ let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the lower and upper integer part of x , respectively.

4. FUNCTIONS SATISFYING A SCALED LOCAL TIME ESTIMATE

In this section we prove restriction theorems for functions satisfying a scaled local time estimate. First we need some notation.

Definition 4.1. Let $\alpha \in [0, 1]$ and an integer $M \geq 2$ be fixed. Let $n \in \mathbb{N}$ and $0 \leq p \leq M^n - 1$. A *time interval of order n* is of the form

$$I_{n,p} = [pM^{-n}, (p + 1)M^{-n}).$$

Let $q \in \mathbb{Z}$. A *value interval of order n* is of the form

$$J_{n,q} = [qM^{-\alpha n}, (q + 1)M^{-\alpha n}).$$

For all $0 \leq m \leq n$ define

$$\mathcal{I}_{n,m,p} = \{I \in \mathcal{D}_n(M) : I \subset I_{m,p}\},$$

where $\mathcal{D}_n(M)$ is the set of time intervals of order n . Clearly, $|\mathcal{I}_{n,m,p}| = M^{n-m}$.

Definition 4.2. For a function $f: [0, 1] \rightarrow \mathbb{R}$ the *scaled local time* $A_{n,m,p,q}(f)$ is the number of order n intervals in $\mathcal{I}_{n,m,p}$ in which f takes at least one value in $J_{n,q}$:

$$A_{n,m,p,q}(f) = |\{I \in \mathcal{I}_{n,m,p} : \exists x \in I, f(x) \in J_{n,q}\}|.$$

It is easy to see that if f is α -Hölder continuous, then for every n, m, p , for some q we have $A_{n,m,p,q}(f) \geq cM^{(1-\alpha)(n-m)}$, since the function cannot visit too many different value intervals in any given time interval. Finally, for each $n \in \mathbb{N}^+$ define

$$\mathcal{A}_n(\alpha, M) = \{f : A_{n,m,p,q}(f) \leq n^2 M^{(1-\alpha)(n-m)} \text{ for all } m \leq n, p < M^n, q \in \mathbb{Z}\}.$$

Thus the set $\mathcal{A}_n(\alpha, M)$ includes α -Hölder functions with scaled local times which are not much larger than the minimal values possible given their continuity. We shall see below that the self-affine functions we define, as well as fractional Brownian motion, belong (almost surely) to this class.

The main goal of this section is to prove Theorems 4.3 and 4.4.

4.1. Hölder restrictions. For the notation $\mathcal{A}_n(\alpha, M)$ and $N_n(A, M)$ see Definition 4.2 and (3.1), respectively.

Theorem 4.3. *Let $M \geq 2$ be an integer and let $\alpha \in [0, 1]$. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a function such that $f \in \mathcal{A}_n(\alpha, M)$ for all large enough n . Assume that $\beta > \alpha$ and $A \subset [0, 1]$ such that $f|_A$ is β -Hölder continuous. Then $\overline{\dim}_{\mathcal{M}} A \leq 1 - \alpha$. Moreover,*

$$(4.1) \quad N_n(A, M) \leq M^{(1-\alpha)n + O(\log^2 n)}.$$

Proof. Assume that $A \subset [0, 1]$ and $\alpha < \gamma < \beta$ are fixed such that $f|_A$ is β -Hölder continuous. Choose $N \in \mathbb{N}^+$ such that $f \in \mathcal{A}_n(\alpha, M)$ for all $n \geq N$. Clearly it is enough to prove (4.1). By decomposing A into finitely many pieces of small enough diameters, we may assume that $f|_A$ is γ -Hölder continuous with Hölder constant 1; that is, for all $x, y \in A$ we have

$$(4.2) \quad |f(x) - f(y)| \leq |x - y|^\gamma.$$

For all $n \in \mathbb{N}$ let

$$d_n = N_n(A, M).$$

Let $c = \gamma/\alpha > 1$. Assume that $s, t \in \mathbb{N}$ such that $s \leq t \leq [cs]$ and $t \geq N$. Now we will prove that

$$(4.3) \quad d_t \leq 2d_s t^2 M^{(1-\alpha)(t-s)}.$$

Let us fix an arbitrary $I_{s,p} \in \mathcal{D}_s(A, M)$ for some p . As $|\mathcal{D}_s(A, M)| = d_s$, in order to show (4.3) it is enough to prove that

$$(4.4) \quad |\{I \in \mathcal{D}_t(A, M) : I \subset I_{s,p}\}| \leq 2t^2 M^{(1-\alpha)(t-s)}.$$

Inequality (4.2) yields $\text{diam } f(I_{s,p} \cap A) \leq M^{-\gamma s} \leq M^{-\alpha t}$; therefore $f(I_{s,p} \cap A) \subset J_{t,q} \cup J_{t,q+1}$ for some $q \in \mathbb{Z}$. As $f \in \mathcal{A}_t(\alpha, M)$, we have $A_{t,s,p,q+j}(f) \leq t^2 M^{(1-\alpha)(t-s)}$ for $j \in \{0, 1\}$, which yields (4.4). Hence (4.3) follows.

Fix an integer $m_0 \geq \max\{N, c/(c-1)\}$ and let n be an arbitrary integer such that $n > m_0$. For all $i \in \mathbb{N}^+$ let $m_i = \min\{n, \lfloor cm_{i-1} \rfloor\}$. Let k be the minimal number such that $m_{k+1} = n$. Note that $cl > \ell + 1$ for every $\ell \geq m_0$; thus such a k exists. Then the recursion and $m_0 \geq c/(c-1)$ yield that

$$n \geq m_k \geq c^k m_0 - \sum_{i=0}^{k-1} c^i = c^k (m_0 - 1/(c-1)) \geq c^k;$$

therefore $k \leq \log n / \log c$. Applying (4.3) repeatedly and using that $d_{m_0} \leq M^{m_0}$ and $m_{i+1} \leq cm_i$ we obtain that

$$\begin{aligned} d_n &\leq d_{m_0} \prod_{i=1}^{k+1} 2m_i^2 M^{(1-\alpha)(m_i - m_{i-1})} \\ &\leq M^{(1-\alpha)n} M^{m_0+k+1} m_0^2 c^{2(1+\dots+(k+1))} \\ &\leq M^{(1-\alpha)n+O(k^2)} \leq M^{(1-\alpha)n+O(\log^2 n)}. \end{aligned}$$

Hence (4.1) follows, and the proof is complete. \square

4.2. Restrictions of finite β -variation. The notation $\mathcal{A}_n(\alpha, M)$ and $N_n(A, M)$ are given in Definition 4.2 and (3.1), respectively.

Theorem 4.4. *Let $M \geq 2$ be an integer, let $\alpha \in [0, 1]$, and let $\beta > 0$. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a function such that $f \in \mathcal{A}_n(\alpha, M)$ for all large enough n . Assume that $A \subset [0, 1]$ is such that $f|_A$ has finite β -variation. Then we have $\overline{\dim}_{\mathcal{M}} A \leq \max\{1 - \alpha, \alpha\beta\} =: \gamma$. Moreover,*

$$(4.5) \quad N_n(A, M) \leq M^{\gamma n + O(\sqrt{n \log n})}.$$

Proof. If the theorem holds for $\beta = (1-\alpha)/\alpha$, then it holds for every $\beta < (1-\alpha)/\alpha$. Thus we may assume that $\beta \geq (1-\alpha)/\alpha$, so $\gamma = \max\{1 - \alpha, \alpha\beta\} = \alpha\beta$. Suppose that $A \subset [0, 1]$ such that $f|_A$ has finite β -variation. Choose $N \in \mathbb{N}^+$ such that $f \in \mathcal{A}_n(\alpha, M)$ for all $n \geq N$. Clearly it is enough to prove (4.5). By decomposing A into finitely many pieces of small enough diameters, we may assume that the β -variation of $f|_A$ is at most 1, that is,

$$(4.6) \quad V^\beta(f|_A) \leq 1.$$

Let $s, t \in \mathbb{N}$ such that $s < t$ and $t \geq N$. Assume that $I = I_{s,p} \in \mathcal{D}_s(A, M)$ and I contains r sub-intervals in $\mathcal{D}_t(A, M)$. First we prove that

$$(4.7) \quad V^\beta(f|_{A \cap I}) \geq \left(\frac{r}{2t^2 M^{(1-\alpha)(t-s)}} - 1 \right) M^{-\alpha\beta t}.$$

Let $\{Q_1, Q_2, \dots, Q_m\}$ be the sub-intervals of I in $\mathcal{D}_t(A, M)$ such that for every $1 \leq i \leq m$ there is an even $q_i \in \mathbb{Z}$ such that $f(Q_i \cap A) \cap J_{t,q_i} \neq \emptyset$. We may assume that $m \geq r/2$; otherwise we switch to odd integers q_i and repeat the same proof. For all $i \in \{1, \dots, m\}$ choose an $x_i \in Q_i \cap A$ such that $f(x_i) \in J_{t,q_i}$. We may

assume that $x_i < x_j$ whenever $i < j$. Let $j_1 = 1$, and if $j_\ell \in \{1, \dots, m\}$ is defined, then let $j_{\ell+1} = \min\{u > j_\ell : q_u \neq q_{j_\ell}\}$ if the minimum exists. As $f \in \mathcal{A}_t(\alpha, M)$, for all $i, j \in \{1, \dots, m\}$ we have

$$|\{i : f(x_i) \in J_{t,q_j}\}| \leq A_{t,s,p,q_j}(f) \leq t^2 M^{(1-\alpha)(t-s)},$$

so if $j_\ell \leq m - t^2 M^{(1-\alpha)(t-s)}$, then $j_{\ell+1}$ is defined and $j_{\ell+1} \leq j_\ell + t^2 M^{(1-\alpha)(t-s)}$. Thus the length of the defined sequence $j_1 < \dots < j_k$ satisfies $k \geq r(2t^2 M^{(1-\alpha)(t-s)})^{-1}$. By construction $|f(x_{j_{\ell+1}}) - f(x_{j_\ell})| \geq M^{-\alpha t}$ for all $\ell < k$, which implies (4.7).

Index the elements $\mathcal{D}_s(A, M) = \{I_1, \dots, I_{N_s(A, M)}\}$, and assume that each I_i contains r_i intervals of $\mathcal{D}_t(A, M)$, so $\sum_{i=1}^{N_s(A, M)} r_i = N_t(A, M)$. Inequality (4.7) yields that

$$\begin{aligned} 1 &\geq V^\beta(f|_A) \geq \sum_{i=1}^{N_s(A, M)} V^\beta(f|_{A \cap I_i}) \\ &\geq \sum_{i=1}^{N_s(A, M)} \left(\frac{r_i}{2t^2 M^{(1-\alpha)(t-s)}} - 1 \right) M^{-\alpha \beta t} \\ &= \left(\frac{N_t(A, M)}{2t^2 M^{(1-\alpha)(t-s)}} - N_s(A, M) \right) M^{-\alpha \beta t}. \end{aligned}$$

Therefore

$$(4.8) \quad \frac{N_t(A, M)}{(2t)^2 M^{(1-\alpha)t}} - \frac{N_s(A, M)}{M^{(1-\alpha)s}} \leq \frac{M^{\alpha \beta t}}{M^{(1-\alpha)s}}.$$

Now assume that $m, k \in \mathbb{N}^+$ are fixed such that $m \geq N$, and we prove that

$$(4.9) \quad N_{km}(A, M) \leq M^{\alpha \beta km} (2km)^{2k} \left(1 + kM^{(1-\alpha)m} \right).$$

For all $0 \leq i \leq k$ let

$$d_i = \frac{N_{im}(A, M)}{(2km)^{2i} M^{(1-\alpha)im}}.$$

Applying inequality (4.8) for $t = im$ and $s = (i - 1)m$ and using that $t \leq km$ and $2km \geq 1$ imply that for every $1 \leq i \leq k$ we have

$$d_i - d_{i-1} \leq \frac{M^{\alpha \beta im}}{M^{(1-\alpha)(i-1)m} (2km)^{2i-2}} \leq M^{(1-\alpha)m} M^{(\alpha \beta - (1-\alpha))im}.$$

As $\alpha \beta \geq 1 - \alpha$, the above inequality implies that

$$(4.10) \quad d_k - d_0 = \sum_{i=1}^k (d_i - d_{i-1}) \leq kM^{(1-\alpha)m} M^{(\alpha \beta - (1-\alpha))km}.$$

Then $d_0 = 1$, $\alpha \beta \geq 1 - \alpha$, and (4.10) imply (4.9).

Finally, let n be an arbitrary integer with $n > N^2$. Let $m = \lceil \sqrt{n \log n} \rceil \geq N$ and $k = \lceil \sqrt{n / \log n} \rceil$. Then $km \geq n$, so $N_n(A, M) \leq N_{km}(A, M)$. Applying (4.9) for k, m easily yields that

$$N_n(A, M) \leq N_{km}(A, M) \leq M^{\alpha \beta n + O(\sqrt{n \log n})}.$$

As $\gamma = \alpha \beta$, inequality (4.5) follows. The proof is complete. □

5. SELF-AFFINE FUNCTIONS AND THE PROOF OF THEOREM 2.8

The main goal of this section is to prove Theorems 2.8 and 2.9. First we define a family of self-affine functions $f_{k,m}$, which will be used in Section 8 as well.

Let $k, m \geq 2$ be fixed integers such that m is odd. For every $i \in \{0, \dots, m - 1\}$ and $j \in \{0, \dots, k - 1\}$ define the one-to-one affine map $F_{ik+j} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as

$$F_{ik+j}(x, y) = \left(\frac{x + ik + j}{km}, (-1)^i \frac{y + j}{k} + (i \bmod 2) \right);$$

see Figure 3. As the F_{ik+j} are contractions, Hutchinson’s contraction mapping theorem [9, page 713 (1)] implies that there is a unique, non-empty compact set $K \subset \mathbb{R}^2$ such that

$$K = \bigcup_{\ell=0}^{km-1} F_{\ell}(K).$$

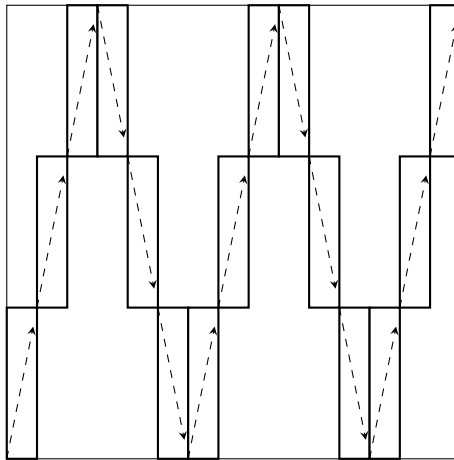


FIGURE 3. For $k = 3$ and $m = 5$ the illustration shows how the family $\{F_i\}_{i=0}^{14}$ maps $[0, 1]^2$ onto rectangles.

It is easy to see that K is a graph of a function $f_{k,m} : [0, 1] \rightarrow [0, 1]$, and $f_{k,m}$ can be approximated as follows. Let $D = \{(x, x) : x \in [0, 1]\}$ be the diagonal of $[0, 1]^2$ and define $\Gamma : \mathcal{P}(\mathbb{R}^2) \rightarrow \mathcal{P}(\mathbb{R}^2)$ as

$$\Gamma(A) = \bigcup_{\ell=0}^{km-1} F_{\ell}(A).$$

If Γ^i denotes the i^{th} iterate $\Gamma \circ \dots \circ \Gamma$, then $\Gamma^i(D)$ is the graph of a piecewise linear function $f_{k,m}^i$ which converges uniformly to $f_{k,m}$ as $i \rightarrow \infty$. Clearly $f_{k,m}$ is a self-affine function with $f_{k,m}(0) = 0$ and $f_{k,m}(1) = 1$, and the definition yields that $f_{k,m}$ is Hölder continuous with exponent $\log k / \log(km)$. Figure 4 shows the piecewise linear function $f_{3,5}^3$, which approximates $f_{3,5}$.

Define the set

$$\Delta = \left\{ \frac{\log k}{\log(km)} : k, m \geq 2 \text{ are integers and } m \text{ is odd} \right\}.$$

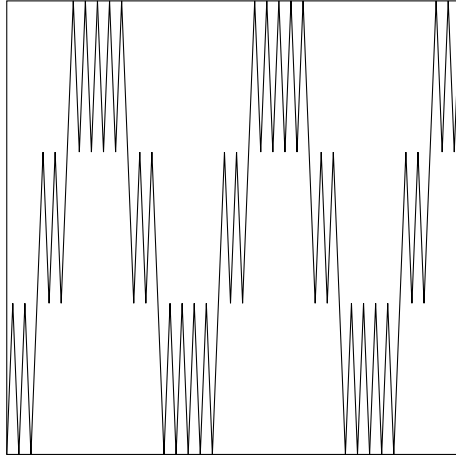


FIGURE 4. The third level approximation of the self-affine function $f_{3,5}$.

Then Δ is a countable dense subset of $(0, 1)$, since every rational $p/q \in (0, 1)$ is in Δ by $k = 3^p$ and $m = 3^{q-p}$. For all $\alpha \in \Delta$ fix k, m such that $\alpha = \log k / \log(km)$, and define $f_\alpha = f_{k,m} \in C^\alpha[0, 1]$.

Proof of Theorem 2.9. Fix $\alpha = \log k / \log(km) \in \Delta$ such that $f_\alpha = f_{k,m}$. We use the scaled local times with $M = km$. By Theorems 4.3 and 4.4 it is enough to show that $f_\alpha \in \mathcal{A}_n(\alpha, M)$ for all $n \geq 2$. Let us fix $n \geq 2$. Clearly $M^{1-\alpha} = m$, and the construction of f_α yields that for every $\ell \leq n, p < M^\ell$, and $0 \leq q < k^n$ we have

$$|I \in \mathcal{I}_{n,\ell,p} : \exists x \in I, f_\alpha(x) \in (qk^{-n}, (q+1)k^{-n})| = m^{n-\ell} = M^{(1-\alpha)(n-\ell)}.$$

Similarly,

$$|I \in \mathcal{I}_{n,\ell,p} : qk^{-n} \in f_\alpha(I)| \leq 2m^{n-\ell} = 2M^{(1-\alpha)(n-\ell)}.$$

The above and $M^{-\alpha} = 1/k$ yield that for all ℓ, p, q we have

$$A_{n,\ell,p,q}(f_\alpha) \leq 3M^{(1-\alpha)(n-\ell)}.$$

Thus $f_\alpha \in \mathcal{A}_n(\alpha, M)$, and the proof is complete. □

Remark 5.1. Note that the bound we get for $A_{n,\ell,p,q}(f_\alpha)$ does not use the n^2 factor. It is possible to go through the proof Theorem 4.3 with such a stronger assumption, which would slightly improve the bounds on $N_n(A, M)$ with $O(\log n)$ in place of an $O(\log^2 n)$ error term.

Proof of Theorem 2.8. Let $0 < \alpha < 1$. By Theorem 2.2, 2.3, and 2.11 it is enough to prove that

$$\begin{aligned} \overline{H}(\alpha, \beta) &\leq 1 - \beta \text{ for every } \alpha < \beta \leq 1; \\ \overline{V}(\alpha, \beta) &\leq \max \left\{ \alpha\beta, \frac{\beta}{\beta + 1} \right\} \text{ for all } \beta > 0. \end{aligned}$$

For the first inequality let $\gamma \in \Delta \cap (\alpha, \beta)$ be arbitrary; then $f_\gamma \in C^\alpha[0, 1]$. Suppose that f_γ is β -Hölder continuous on some set $A \subset [0, 1]$. Theorem 2.9(1) yields that $\overline{\dim}_{\mathcal{M}} A \leq 1 - \gamma$, thus $\overline{H}(\alpha, \beta) \leq 1 - \gamma$. Since Δ is dense in (α, β) , we obtain that $\overline{H}(\alpha, \beta) \leq 1 - \beta$.

For the second inequality define $\delta = \max\{\alpha, 1/(\beta + 1)\}$ and let $\gamma \in \Delta \cap (\delta, 1)$ be arbitrary. Then $f_\gamma \in C^\alpha[0, 1]$. Assume that f_γ has finite β -variation on some A . Theorem 2.9(2) and $\gamma > \delta$ imply that $\overline{\dim}_{\mathcal{M}} A \leq \max\{1 - \gamma, \gamma\beta\} = \gamma\beta$, so $\overline{V}(\alpha, \beta) \leq \gamma\beta$. As Δ is dense in $(\delta, 1)$, we have $\overline{V}(\alpha, \beta) \leq \delta\beta = \max\{\alpha\beta, \beta/(\beta + 1)\}$. \square

6. RESTRICTIONS OF FRACTIONAL BROWNIAN MOTION

Let $0 < \alpha < 1$ be fixed and let $B: [0, 1] \rightarrow \mathbb{R}$ be a fractional Brownian motion of Hurst index α . We think of B as a random function from $[0, 1]$ to \mathbb{R} .

The main goal of this section is to prove Theorem 2.10. By Theorems 4.3 and 4.4 it is enough to prove the following proposition. For the notation $\mathcal{A}_n(\alpha, 2)$ see Definition 4.2.

Proposition 6.1. *Let $0 < \alpha < 1$ and let $B: [0, 1] \rightarrow \mathbb{R}$ be a fractional Brownian motion of Hurst index α . Then, almost surely, $B \in \mathcal{A}_n(\alpha, 2)$ for all n large enough.*

First we define a discrete (truncated) scaled local time.

Definition 6.2. For all $n \in \mathbb{N}$ let

$$\mathcal{E}_n = \{i2^{-n} : 0 \leq i \leq 2^n - 1\}.$$

For all $0 \leq m \leq n$ let

$$\mathcal{L}_{n,m,p} = I_{m,p} \cap \mathcal{E}_n.$$

Clearly, $|\mathcal{L}_{n,m,p}| = 2^{n-m}$. For a function $f: [0, 1] \rightarrow \mathbb{R}$ the *discrete scaled local time* $S_{n,m,p,q}(f)$ is the number of points in $\mathcal{L}_{n,m,p}$ which are mapped to $J_{n,q}$ by f :

$$S_{n,m,p,q}(f) = |\{x \in \mathcal{L}_{n,m,p} : f(x) \in J_{n,q}\}|.$$

For every $n \in \mathbb{N}^+$ define

$$\mathcal{S}_n(\alpha) = \{f : S_{n,m,p,q}(f) \leq (n \log n)2^{(1-\alpha)(n-m)} \ \forall m \leq n, p < 2^n, |q| \leq n2^{\alpha n}\}.$$

First we need to prove some lemmas. To avoid technical difficulties we assume that the domain of B is extended to $[0, \infty)$ when necessary.

Definition 6.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which our fractional Brownian motion is defined, and let $\mathcal{F}_t = \sigma(B(s) : 0 \leq s \leq t)$ be the natural filtration. If $\tau: \Omega \rightarrow [0, \infty]$ is a stopping time, then define the σ -algebra

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

For all stopping times τ and integers $0 \leq m \leq n$ and q let

$$X_\tau^{n,m,q} = |\{k \in \{1, \dots, 2^{n-m}\} : B(\tau + k2^{-n}) \in J_{n,q}\}|.$$

Lemma 6.4. *There is a finite constant $c = c(\alpha)$ depending only on α such that for every bounded stopping time τ and integers $0 \leq m \leq n$ and q we have, almost surely,*

$$\mathbb{E}(X_\tau^{n,m,q} | \mathcal{F}_\tau) \leq c2^{(1-\alpha)(n-m)}.$$

Proof. Pitt [19, Lemma 7.1] showed that the property of *strong local non-determinism* holds for fractional Brownian motion; that is, there is a constant $c_1 = c_1(\alpha) > 0$ such that for all $t \geq 0$, almost surely,

$$(6.1) \quad \text{Var}(B(\tau + t) | \mathcal{F}_\tau) \geq c_1 t^{2\alpha}.$$

Let us fix $t > 0$. As B is Gaussian, almost surely the conditional distribution $B(\tau + t) | \mathcal{F}_\tau$ is normal, and (6.1) implies that its density function is bounded by $1/(\sqrt{c_1}t^\alpha)$. Therefore, almost surely,

$$(6.2) \quad \mathbb{P}(B(\tau + t) \in J_{n,q} | \mathcal{F}_\tau) \leq \int_{q2^{-\alpha n}}^{(q+1)2^{-\alpha n}} \frac{1}{\sqrt{c_1}t^\alpha} dx = c_2(t2^n)^{-\alpha},$$

where $c_2 = 1/\sqrt{c_1}$. Applying (6.2) for finitely many t implies that

$$\begin{aligned} \mathbb{E}(X_\tau^{n,m,q} | \mathcal{F}_\tau) &= \sum_{k=1}^{2^{n-m}} \mathbb{P}(B(\tau + k2^{-n}) \in J_{n,q} | \mathcal{F}_\tau) \\ &\leq \sum_{k=1}^{2^{n-m}} c_2 k^{-\alpha} \leq c_2 \int_0^{2^{n-m}} x^{-\alpha} dx = c2^{(1-\alpha)(n-m)}, \end{aligned}$$

where $c = c_2/(1 - \alpha)$. The proof is complete. □

Lemma 6.5. *There is a finite constant $C = C(\alpha)$ depending only on α such that for all m, n, p, q , and $\ell \in \mathbb{N}^+$ we have*

$$\mathbb{P}(S_{n,m,p,q}(B) \geq \ell C 2^{(1-\alpha)(n-m)}) \leq 2^{-\ell}.$$

Proof. Let c be the constant in Lemma 6.4; clearly we may assume that $c \geq 1$. We will show that $C = 3c$ satisfies the lemma. We define stopping times τ_0, \dots, τ_ℓ . Let $\tau_0 = 0$. If τ_k is defined for some $0 \leq k < \ell$, then let τ_{k+1} be the first time that the contribution of B to $S_{n,m,p,q}(B)$ on $(\tau_k, \tau_{k+1}] \cap \mathcal{L}_{n,m,p}$ is at least $2c2^{(1-\alpha)(n-m)}$ if such a time exists; otherwise let $\tau_{k+1} = 1$. Then $c \geq 1$ and the definition of stopping times yields that

$$\begin{aligned} \mathbb{P}(S_{n,m,p,q}(B) \geq 3\ell c 2^{(1-\alpha)(n-m)}) &\leq \mathbb{P}(S_{n,m,p,q}(B) \geq \ell(2c2^{(1-\alpha)(n-m)} + 1)) \\ &\leq \mathbb{P}(\tau_\ell < 1) = \prod_{k=1}^{\ell} \mathbb{P}(\tau_k < 1 | \tau_{k-1} < 1). \end{aligned}$$

Note that we may assume that $\mathbb{P}(\tau_\ell < 1) > 0$ and hence the above conditional probabilities are defined; otherwise we are done immediately. Therefore it is enough to prove that $\mathbb{P}(\tau_k < 1 | \tau_{k-1} < 1) \leq 1/2$ for all $1 \leq k \leq \ell$. The definition of $X_\tau^{n,m,q}$, Lemma 6.4, and the conditional Markov inequality imply that, almost surely,

$$\mathbb{P}(\tau_k < 1 | \mathcal{F}_{\tau_{k-1}}) \leq \mathbb{P}(X_{\tau_{k-1}}^{n,m,q} \geq 2c2^{(1-\alpha)(n-m)} | \mathcal{F}_{\tau_{k-1}}) \leq 1/2.$$

Therefore the tower property of conditional expectation yields that

$$\mathbb{P}(\tau_k < 1 | \tau_{k-1} < 1) \leq 1/2,$$

which completes the proof. □

The following lemma is a discrete version of Proposition 6.1.

Lemma 6.6. *Almost surely, $B \in \mathcal{S}_n(\alpha)$ for all large enough n .*

Proof. We give an upper bound for $\mathbb{P}(B \notin \mathcal{S}_n(\alpha))$ by applying Lemma 6.5 with $\ell = \lfloor (n/C) \log n \rfloor$ to each of the relevant m, p, q . Since $0 \leq m \leq n$, $0 \leq p \leq 2^m - 1$, and $|q| \leq n2^{\alpha n}$, there are at most $(n+1)2^n(2n2^{\alpha n} + 1)$ possibilities to choose m, p, q . Therefore Lemma 6.5 implies that

$$\mathbb{P}(B \notin \mathcal{S}_n(\alpha)) \leq (n+1)2^n(2n2^{\alpha n} + 1)2^{-\lfloor (n/C) \log n \rfloor} = 2^{-(n/C) \log n + O(n)}.$$

Therefore $\sum_{n=1}^\infty \mathbb{P}(B \notin \mathcal{S}_n(\alpha)) < \infty$, so the Borel–Cantelli lemma implies that

$$\mathbb{P}\left(B \in \liminf_n \mathcal{S}_n(\alpha)\right) = 1. \quad \square$$

The following lemma is well known; see the more general [14, Corollary 7.2.3].

Lemma 6.7. *Almost surely, we have*

$$\limsup_{h \rightarrow 0^+} \sup_{0 \leq t \leq 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h^{2\alpha} \log(1/h)}} \leq 1.$$

Now we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. By Lemmas 6.6 and 6.7 we can choose a random $N \in \mathbb{N}^+$ such that, almost surely, for every $n \geq N$ we have the following properties:

- (i) $\max_{t \in [0,1]} |B(t)| < N - 1$;
- (ii) $B \in \mathcal{S}_n(\alpha)$;
- (iii) $\text{diam} B(I_{n,p}) \leq 2\sqrt{n}2^{-\alpha n}$ for all $0 \leq p \leq 2^n - 1$;
- (iv) $(4\sqrt{n} + 3)(n \log n) \leq n^2$.

Fix a path of B for which the above properties hold. Let us fix an arbitrary $n \geq N$. It is enough to prove that $B \in \mathcal{A}_n(\alpha, 2)$. Let $0 \leq m \leq n$, $0 \leq p \leq 2^m - 1$, and $q \in \mathbb{Z}$ be given. We need to show that

$$(6.3) \quad A_{n,m,p,q}(B) \leq n^2 2^{(1-\alpha)(n-m)}.$$

Property (i) yields that if $q' \in \mathbb{Z}$ with $|q'| > n2^{\alpha n}$, then $S_{n,m,p,q'}(B) = 0$. Therefore (ii) implies that for all $q' \in \mathbb{Z}$ we have

$$(6.4) \quad S_{n,m,p,q'}(B) \leq (n \log n) 2^{(1-\alpha)(n-m)}.$$

Let $I_{n,p'}$ be a time interval of order n such that $I_{n,p'} \subset I_{m,p}$ and $B(I_{n,p'}) \cap J_{n,q} \neq \emptyset$. Then (iii) yields that

$$(6.5) \quad B(I_{n,p'}) \subset \bigcup_{q': |q'-q| \leq 2\sqrt{n}+1} J_{n,q'}.$$

Finally, (6.5), (6.4), and (iv) imply that

$$\begin{aligned} A_{n,m,p,q}(B) &\leq \sum_{q': |q'-q| \leq 2\sqrt{n}+1} S_{n,m,p,q'}(B) \\ &\leq (4\sqrt{n} + 3)(n \log n) 2^{(1-\alpha)(n-m)} \\ &\leq n^2 2^{(1-\alpha)(n-m)}. \end{aligned}$$

Hence (6.3) holds, and the proof of Proposition 6.1 is complete. □

6.1. Dimension of the record times. We include here a proof of Proposition 1.5, which we could not find in the literature. Recall that $\{B(t) : t \in [0, 1]\}$ is a fractional Brownian motion of Hurst index α . The lower bound is quite elementary, while the upper bound relies on a first moment computation and on a result of Molchan. He studied the distribution of the maximal value of fractional Brownian motion on $[0, t]$ and of the time $\tau_{\max}(t)$ when it is achieved. Specifically, Molchan [17, Theorem 2] proved the following.

Theorem 6.8. *We have $\mathbb{P}(\tau_{\max}(1) < x) = x^{1-\alpha+o(1)}$ as $x \rightarrow 0$.*

Proof of Proposition 1.5. With probability one B is γ -Hölder continuous for every $\gamma < \alpha$ and maps \mathcal{R} to the non-degenerate interval $I = [0, \max\{B(t) : t \in [0, 1]\}]$, thus $\dim_{\mathcal{H}} \mathcal{R} \geq \alpha \dim_{\mathcal{H}} I = \alpha$.

Therefore it is enough to prove that $\overline{\dim}_{\mathcal{M}} \mathcal{R} \leq \alpha$ almost surely. Using that $\tau_{\max}(t)$ and $t\tau_{\max}(1)$ have the same distribution and $\{B(1-t) - B(1) : t \in [0, 1]\}$ is also a fractional Brownian motion, for all $0 \leq \varepsilon < t \leq 1$ we obtain

$$\begin{aligned} \mathbb{P}(\mathcal{R} \cap [t - \varepsilon, t] \neq \emptyset) &= \mathbb{P}(\tau_{\max}(t) \geq t - \varepsilon) = \mathbb{P}(\tau_{\max}(1) \geq 1 - \varepsilon/t) \\ &= \mathbb{P}(\tau_{\max}(1) \leq \varepsilon/t) = (\varepsilon/t)^{1-\alpha+o(1)}, \end{aligned}$$

with the $o(1)$ term tending to 0 as $\varepsilon/t \rightarrow 0$.

Let $N(m)$ be the number of intervals $[(i-1)/m, i/m]$ which intersect \mathcal{R} . Let $\delta > 0$ be arbitrary and fix $s \in \mathbb{N}^+$ such that $\mathbb{P}(\mathcal{R} \cap [t - \varepsilon, t] \neq \emptyset) \leq (\varepsilon/t)^{1-\alpha-\delta}$ whenever $\varepsilon/t < 1/s$. Using the above for $i > s$ and the trivial bound 1 for $i \leq s$, for every large enough m we obtain

$$\mathbb{E}N(m) \leq s + \sum_{i=s+1}^m (1/i)^{1-\alpha-\delta} \leq s + Cm^{\alpha+\delta} \leq 2Cm^{\alpha+\delta},$$

where C is a finite constant depending only on $\alpha + \delta$. By Markov's inequality

$$\mathbb{P}(N(m) > m^{\alpha+2\delta}) \leq 2Cm^{-\delta}$$

for any m large enough. Applying this for $m = 2^n$ yields that

$$\sum_{n=1}^{\infty} \mathbb{P}(N(2^n) > 2^{n(\alpha+2\delta)}) < \infty.$$

Thus the Borel–Cantelli lemma implies that, almost surely, $N(2^n) \leq 2^{n(\alpha+2\delta)}$ for all large enough n . Therefore $\overline{\dim}_{\mathcal{M}} \mathcal{R} \leq \alpha + 2\delta$. As $\delta > 0$ was arbitrary, $\overline{\dim}_{\mathcal{M}} \mathcal{R} \leq \alpha$ almost surely. \square

7. HIGHER DIMENSIONAL BROWNIAN MOTION

The aim of this section is to prove Theorem 1.7. The idea is to find (in a greedy manner) large sets along which a simple random walk in \mathbb{Z}^2 is monotone. Since the scaled simple random walk converges to Brownian motion, this gives sets along which B is monotone. To control the dimension of the limit sets we estimate the energy of the discrete sets and apply a version of Frostman's lemma to bound the dimension.

Given a simple random walk $S: \mathbb{N} \rightarrow \mathbb{Z}^2$, define the *greedy increasing subset* by

$$a_0 = 0 \quad \text{and} \quad a_{i+1} = \min\{a > a_i : S(a) - S(a_i) \in \mathbb{Z}_+^2\}.$$

Our first task is to prove tightness for the number and structure of record times in $[0, n)$. Since our argument may apply in similar situations, we state some of our arguments in the more general context of sums of i.i.d. variables with power law tails.

Before focusing on the case of random walks, we prove Theorem 7.6, a limit theorem, which will allow us to transfer estimates from the random walk setting to Brownian motion. As Theorem 7.6 below is a quite general result about random sequences with i.i.d. increments, we hope that it will find further applications.

7.1. Energy of renewal processes. Fix $0 < \alpha < 1$. Let $\tau \in \mathbb{N}^+$ be some random variable, and assume that there are $c_1, c_2 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}^+$ we have

$$(7.1) \quad c_1 n^{-\alpha} \leq \mathbb{P}(\tau > n) \leq c_2 n^{-\alpha}.$$

Let $\{\tau_i\}_{i \geq 1}$ be an i.i.d. sequence with the law of τ . Define

$$T_k = \sum_{i=1}^k \tau_i \quad \text{and} \quad \mathcal{T} = \{T_k : k \geq 1\}.$$

The number of steps before reaching n is denoted by $m_n = |\mathcal{T} \cap [0, n)|$.

The following lemma is fairly standard.

Lemma 7.1. *There are constants c_3, c_4 such that for all $t, n > 0$ we have*

- (i) $\mathbb{P}(m_n < tn^\alpha) \leq c_3 t$.
- (ii) $\mathbb{P}(m_n > tn^\alpha) \leq e^{-c_1 t}$.
- (iii) $\mathbb{E}m_n \leq c_4 n^\alpha$, and more generally, for all integers $i < j < k$ we have

$$\mathbb{E}(|\mathcal{T} \cap [j, k)| | i \in \mathcal{T}) \leq c_4(k - j)^\alpha.$$

Proof. Claim (i) is given by [2, Lemma 4.2]. Inequality (7.1) and $(1 - u) \leq e^{-u}$ imply that

$$\begin{aligned} \mathbb{P}(m_n > tn^\alpha) &\leq \mathbb{P}(\tau_i \leq n \text{ for all } i \leq \lceil tn^\alpha \rceil) \leq (\mathbb{P}(\tau \leq n))^{tn^\alpha} \\ &\leq (1 - c_1 n^{-\alpha})^{tn^\alpha} \leq e^{-c_1 t}, \end{aligned}$$

so (ii) holds. The first bound of (iii) follows easily from (ii). The general bound holds since the first ℓ with $T_\ell \in [j, k)$ (if there is such ℓ) is a stopping time. Applying (ii) to the sequence starting at time ℓ completes the proof. \square

Definition 7.2. Let μ be a non-atomic mass distribution on a metric space (E, ρ) (that is, a Borel measure on E with $0 < \mu(E) < \infty$). For $\gamma > 0$, define the γ -energy of μ by

$$\mathcal{E}_\gamma(E, \mu) = \iint_{E^2} \frac{d\mu(x) d\mu(y)}{\rho(x, y)^\gamma}.$$

For the following theorem see [18, Theorem 4.27].

Theorem 7.3. *Let μ be a non-atomic mass distribution on a metric space E with $\mathcal{E}_\gamma(E, \mu) < \infty$. Then $\dim_{\mathcal{H}} E \geq \gamma$.*

Consider the set $S_n = (\mathcal{T} + [0, 1)) \cap [0, n)$, endowed with Lebesgue measure λ . We next estimate the γ -energy of λ .

Lemma 7.4. *Let $0 < \gamma < \alpha$. There is a finite constant $c(\gamma)$ such that for all n we have*

$$\mathbb{E}\mathcal{E}_\gamma(S_n, \lambda) \leq c(\gamma)n^{2\alpha-\gamma}.$$

Proof. The argument is to consider the contribution to the energy from pairs x, y with distance at various scales, and the largest scale will dominate the rest.

Up to a factor of 2 we may restrict the integral to $x < y$. We split the integral on $S_n \times S_n$ into several parts. Note that S_n is a disjoint union of unit intervals. Let P_0 be the contribution to $\mathcal{E}_\gamma(S_n, \lambda)$ from pairs $x \in [i, i + 1)$ and $y \in [j, j + 1)$

where $i, j \in \mathcal{T}$ and $0 \leq j - i \leq 1$. The number of such pairs $\{i, j\}$ is at most m_n , so Lemma 7.1(iii) yields that

$$\mathbb{E}P_0 \leq (\mathbb{E}m_n) \int_0^1 \int_x^2 |x - y|^{-\gamma} dy dx = O_\gamma(n^\alpha).$$

For $k \geq 1$ let P_k be the contribution to $\mathcal{E}_\gamma(S_n, \lambda)$ from pairs $x \in [i, i + 1)$ and $y \in [j, j + 1)$ where $i, j \in \mathcal{T}$ and $i + 2^{k-1} < j \leq i + 2^k$. Let M_k denote the number of such pairs $\{i, j\}$. For such $\{i, j\}$ the contribution from $x \in [i, i + 1)$ and $y \in [j, j + 1)$ to P_k is at most

$$\int_i^{i+1} \int_j^{j+1} |x - y|^{-\gamma} dy dx \leq 2^{-(k-1)\gamma},$$

where we used $y - x \geq 2^{k-1}$. Thus $P_k \leq 2^{-(k-1)\gamma} M_k$. Lemma 7.1(iii) yields that for every i , conditioned on $i \in \mathcal{T}$, the expected number of j in $\mathcal{T} \cap (i + 2^{k-1}, i + 2^k]$ is at most $c_4 2^{(k-1)\alpha}$. Lemma 7.1(iii) also implies that the expected number of $i < n$ in \mathcal{T} equals $\mathbb{E}m_n \leq c_4 n^\alpha$. Therefore $\mathbb{E}M_k \leq c_4^2 n^\alpha 2^{(k-1)\alpha}$, and we obtain that

$$\mathbb{E}P_k \leq c_4^2 n^\alpha 2^{(k-1)(\alpha-\gamma)}.$$

This partition gives the identity $\mathcal{E}_\gamma(S_n, \lambda) = 2 \sum_{k=0}^\infty P_k$. As $P_k = 0$ whenever $2^{k-1} > n$, we have $\mathbb{E}\mathcal{E}_\gamma(S_n, \lambda) = 2 \sum_{2^k \leq 2n} \mathbb{E}P_k$. With the bounds above, the largest k dominates the sum and we arrive at the inequality $\mathbb{E}\mathcal{E}_\gamma(S_n, \lambda) \leq c(\gamma)n^{2\alpha-\gamma}$. \square

We will wish to work with rescaled sets. For all $n \in \mathbb{N}^+$ let

$$D_n = \left(\frac{1}{n}\mathcal{T}\right) \cap [0, 1) \quad \text{and} \quad C_n = \frac{1}{n}S_n = D_n + [0, 1/n).$$

Define the measure $\mu_n = n^{1-\alpha}\lambda|_{C_n}$, that is, $n^{1-\alpha}$ times the Lebesgue measure restricted to C_n .

Lemma 7.5. *Let $0 < \gamma < \alpha$ and $\varepsilon > 0$. Then there exist $N, N_\gamma \in \mathbb{N}^+$ depending on ε so that with probability at least $1 - \varepsilon$ we have*

$$N^{-1} \leq \mu_n([0, 1]) \leq N \quad \text{and} \quad \mathcal{E}_\gamma(C_n, \mu_n) \leq N_\gamma.$$

Proof. Let $I = [0, 1]$ and let $N \in \mathbb{N}^+$ be arbitrary. For all $n \in \mathbb{N}^+$ the following three inequalities hold. Markov’s inequality and $\mathbb{E}\mu_n(I) = 1$ yield that

$$\mathbb{P}(\mu_n(I) > N) \leq \frac{\mathbb{E}\mu_n(I)}{N} = \frac{1}{N}.$$

Lemma 7.1(iii) and (i) yield that

$$\mathbb{P}(\mu_n(I) < N^{-1}) = \mathbb{P}(m_n < N^{-1}\mathbb{E}m_n) \leq \mathbb{P}(m_n < N^{-1}c_4 n^\alpha) \leq \frac{c_3 c_4}{N}.$$

Lemma 7.4 yields that $\mathbb{E}\mathcal{E}_\gamma(C_n, \mu_n) \leq c(\gamma)$. Indeed, $\mathcal{E}_\gamma(C_n, \lambda) = n^{\gamma-2}\mathcal{E}_\gamma(S_n, \lambda)$ and since $\mu_n = n^{1-\alpha}\lambda$ on C_n , changing to μ_n gives a further factor of $n^{2-2\alpha}$. Now Markov’s inequality implies that

$$\mathbb{P}(\mathcal{E}_\gamma(C_n, \mu_n) > N_\gamma) \leq \frac{c(\gamma)}{N_\gamma}.$$

The above three inequalities with large enough N, N_γ complete the proof. \square

7.2. A limit theorem. Theorem 7.6 is concerned with a sequence of sequences of i.i.d. variables $\{\tau_i^{(n)}\}_{i \geq 1}$ satisfying (7.1). That is, for each fixed n the variables $\{\tau_i^{(n)}\}_{i \geq 1}$ are i.i.d., but there could be arbitrary dependencies between variables with different numbers n . The superscript n is also the parameter used for scaling sums of the n^{th} sequence. Thus we denote

$$T_k^{(n)} = \sum_{i=1}^k \tau_i^{(n)} \quad \text{and} \quad \mathcal{T}^{(n)} = \{T_k^{(n)} : k \geq 1\}.$$

For all $n \in \mathbb{N}^+$ define

$$D_n = \left(\frac{1}{n} \mathcal{T}^{(n)} \right) \cap [0, 1) \quad \text{and} \quad C_n = D_n + [0, 1/n).$$

Theorem 7.6. *With the notation above, almost surely, $\{D_n\}_{n \geq 1}$ has an accumulation point D in the Hausdorff metric such that $\dim_{\mathcal{H}} D \geq \alpha$.*

Proof. Let μ_n be $n^{1-\alpha}$ times the Lebesgue measure on C_n . Fix $\varepsilon > 0$. For some finite constants $\{N_\gamma\}_{0 < \gamma < \alpha}$ for all n define the event

$$\mathcal{B}_n = \{N_0^{-1} \leq \mu_n([0, 1]) \leq N_0 \text{ and } \mathcal{E}_\gamma(C_n, \mu_n) \leq N_\gamma \text{ for all } 0 < \gamma < \alpha\}.$$

Since the map $\gamma \mapsto \mathcal{E}_\gamma(C_n, \mu_n)$ is non-decreasing, applying Lemma 7.5 for a sequence of parameters $\gamma_k \nearrow \alpha$ with $\varepsilon_k = 2^{-k}\varepsilon$ implies that there are constants N_γ such that $\mathbb{P}(\mathcal{B}_n) \geq 1 - \varepsilon$ for all n . Let $\mathcal{B} = \limsup_n \mathcal{B}_n$; then $\mathbb{P}(\mathcal{B}) \geq 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it is enough to prove that the theorem is satisfied whenever \mathcal{B} holds.

Assume that \mathcal{B} holds. Then there is a random subsequence $\{n_i\}_{i \geq 1}$ such that the events \mathcal{B}_{n_i} hold for all $i \in \mathbb{N}^+$. Since $N_0^{-1} \leq \mu_{n_i}([0, 1]) \leq N_0$ for all i , by passing to a subsequence we may assume that $\mu_{n_i} \rightarrow \mu$ weakly, where μ is a measure on $[0, 1]$ satisfying $N_0^{-1} \leq \mu([0, 1]) \leq N_0$. Similarly, we may assume that $D_{n_i} \rightarrow D$ in the Hausdorff metric for some compact set $D \subset [0, 1]$. As $C_n = D_n + [0, 1/n]$ is close in the Hausdorff metric to D_n , we get $\text{supp}(\mu) \subset D$. For all $0 < \gamma < \alpha$ we obtain

$$\mathcal{E}_\gamma(D, \mu) \leq \liminf_i \mathcal{E}_\gamma(C_{n_i}, \mu_{n_i}) \leq N_\gamma < \infty;$$

for the first inequality see e.g. [15, Lemma 2.2]. Theorem 7.3 now implies that $\dim_{\mathcal{H}} D \geq \alpha$, and the proof is complete. \square

7.3. Application to random walks. We now apply Theorem 7.6 to random walks on \mathbb{Z}^2 and thus prove Theorem 1.7.

For each n , let $S^{(n)}$ be a simple random walk on \mathbb{Z}^2 , and define the rescaled random walks by $W_n(t) = \sqrt{2}n^{-1/2}S^{(n)}(\lfloor nt \rfloor)$. It is well known that it is possible to construct the walks $S^{(n)}$ and two-dimensional Brownian motion $\{B(t) : t \in [0, 1]\}$ on the same probability space so that W_n converges uniformly to B on the interval $[0, 1]$; see e.g. [13, Theorem 3.5.1] or [18]. We henceforth assume such a coupling.

Recall that for each walk we construct the *greedy increasing subset* by

$$a_0^{(n)} = 0 \quad \text{and} \quad a_{i+1}^{(n)} = \min\{a > a_i^{(n)} : S^{(n)}(a) - S^{(n)}(a_i^{(n)}) \in \mathbb{Z}_+^2\}.$$

For every n , this sequence has i.i.d. increments with the law of

$$\tau = \inf\{k > 0 : S(k) \in \mathbb{Z}_+^2\}.$$

We use the notation $a_n \sim b_n$ if $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$. We need the following known estimate.

Theorem 7.7. *Let $S: \mathbb{N} \rightarrow \mathbb{Z}^2$ be a two-dimensional simple random walk. Let τ be the hitting time of the positive quadrant: $\tau = \inf\{k > 0 : S(k) \in \mathbb{Z}_+^2\}$. Then there is a $c \in \mathbb{R}^+$ so that*

$$\mathbb{P}(\tau > n) \sim cn^{-1/3}.$$

For the above theorem see the general result of Denisov and Wachtel [5, Theorem 1] or a bit weaker one due to Varopoulos [21, (0.3.3) and (0.4.1)]. On exit times of planar Brownian motion from cones see Evans [7, Corollary 5(i)] or the somewhat weaker [18, Lemma 10.40]. In the Brownian case the exponent $1/3$ can be calculated by mapping the complement of \mathbb{R}_+^2 onto a half plane by the conformal map $z \mapsto z^{2/3}$ and using the conformal invariance of planar Brownian motion. The continuous case can be transformed to the discrete one by coupling. For the history of similar estimates and for further references see Denisov and Wachtel [5].

Proof of Theorem 1.7. Recall that $S^{(n)}$ are two-dimensional simple random walks so that the rescaled walks W_n converge uniformly to a Brownian motion B . Let τ be the hitting time of the positive quadrant by S , that is, $\tau = \inf\{k > 0 : S(k) \in \mathbb{Z}_+^2\}$. For every $n \in \mathbb{N}^+$, the greedy increasing subsequence of $S^{(n)}$ has i.i.d. increments, distributed as τ . By Theorem 7.7 we have

$$\mathbb{P}(\tau > n) \sim cn^{-1/3}$$

with some $c \in \mathbb{R}^+$. Thus we can apply Theorem 7.6 with $\alpha = 1/3$. This yields that, almost surely, there is an accumulation point D of $\{D_n\}_{n \geq 1}$ in the Hausdorff metric such that $\dim_{\mathcal{H}} D \geq 1/3$. As $W_n \rightarrow B$ uniformly, B is non-decreasing on D . This completes the proof. \square

Remark 7.8. For a higher dimensional simple random walk $S: \mathbb{N} \rightarrow \mathbb{Z}^d$ define the hitting time

$$\tau = \inf\{k > 0 : S(k) \in \mathbb{Z}_+^d\}.$$

Then $\mathbb{P}(\tau > n) \sim cn^{-\alpha}$ for some $c, \alpha \in (0, \infty)$; see [5, Theorem 1]. Our argument proves an analogue of Theorem 1.7 in higher dimensions with this α instead of $1/3$.

8. RESTRICTIONS OF GENERIC α -HÖLDER CONTINUOUS FUNCTIONS

The goal of this section is to prove Theorem 2.13. First we need some preparation. The following lemma is probably well known. However, we could not find an explicit reference for its second claim, so we outline the proof.

Lemma 8.1. *Let $0 < \alpha \leq 1$ and $c > 0$. Assume that $A \subset \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$ is a function such that for all $x, y \in A$ we have*

$$(8.1) \quad |f(x) - f(y)| \leq c|x - y|^\alpha.$$

Then f extends to $F: \mathbb{R} \rightarrow \mathbb{R}$ satisfying the above inequality for all $x, y \in \mathbb{R}$. If A is closed, then F can be chosen to be linear on the components of $\mathbb{R} \setminus A$.

Proof. As f admits a unique continuous extension to the closure of A which clearly satisfies (8.1), we may assume that A is closed. Let I be any component of $\mathbb{R} \setminus A$. It is enough to prove that f extends to $A \cup I$ such that (8.1) holds. If $I = (-\infty, a)$ or $I = (a, \infty)$ for some $a \in A$, then $F|_I \equiv f(a)$ works. Now let $I = (a, b)$ for some $a, b \in A$ and let F be the linear extension of f to I . The concavity of the function $x \mapsto x^\alpha$ implies that $|F(x) - F(y)| \leq c|x - y|^\alpha$ for all $x, y \in A \cup I$. The straightforward calculation is left to the reader. \square

Let $\|\cdot\|$ denote the maximum norm of $C[0, 1]$.

Lemma 8.2. *Let $f \in C_1^\alpha[0, 1]$ and $\varepsilon > 0$ be arbitrary. Then there is a piecewise linear function g with non-zero slopes and $c < 1$ such that $\|g - f\| \leq \varepsilon$ and for all $x, y \in [0, 1]$ we have*

$$|g(x) - g(y)| \leq c|x - y|^\alpha.$$

Proof. Let $0 = x_0 < x_1 < \dots < x_\ell = 1$ such that the oscillation of f on $[x_{i-1}, x_i]$ is at most $\varepsilon/3$ for all $i \in \{1, \dots, \ell\}$. Let g_0 be the piecewise linear function passing through the points $(x_i, f(x_i))$; then clearly $\|g_0 - f\| \leq \varepsilon/3$. Applying Lemma 8.1 ℓ -times we obtain that $g_0 \in C_1^\alpha[0, 1]$. We can choose $c_0 < 1$ such that $g_1 = c_0 g_0$ satisfies $\|g_1 - g_0\| \leq \varepsilon/3$. Hence for all $x, y \in [0, 1]$ we have

$$|g_1(x) - g_1(y)| \leq c_0|x - y|^\alpha.$$

Let $c \in (c_0, 1)$. Then it is easy to see that every horizontal line segment of the graph of g_1 (if there are any) can be replaced by two line segments of non-zero slopes such that the resulting function g satisfies $\|g - g_1\| \leq \varepsilon/3$ and for all $x, y \in [0, 1]$ we have

$$|g(x) - g(y)| \leq c|x - y|^\alpha.$$

Clearly $\|g - f\| \leq \varepsilon$, and the proof is complete. □

Now we are ready to prove the first part of Theorem 2.13. The concept of the proof is similar to that of [6, Theorem 1.4], but the technical details are much more difficult, and in order to create appropriate Hölder continuous functions some new ideas are needed as well.

Proof of Theorem 2.13(1). Let $\beta \in (\alpha, 1)$ be arbitrarily fixed, and define

$$\mathcal{F}_\beta = \{f \in C_1^\alpha[0, 1] : \dim_{\mathcal{H}}\{f = g\} \leq 1 - \beta \text{ for all } g \in C_1^\beta[0, 1]\},$$

where we use the notation $\{f = g\} = \{x \in [0, 1] : f(x) = g(x)\}$. First we show that it is enough to prove that \mathcal{F}_β is co-meager in $C_1^\alpha[0, 1]$. Indeed, since co-meager sets are closed under countable intersection, this implies that for a countable dense set $\Gamma \subset (\alpha, 1)$ the set $\mathcal{F} := \bigcap_{\gamma \in \Gamma} \mathcal{F}_\gamma$ is co-meager in $C_1^\alpha[0, 1]$. Now assume that $f \in \mathcal{F}$ and $A \subset [0, 1]$ such that $f|_A$ is β -Hölder for some $\beta > \alpha$. We need to prove that $\dim_{\mathcal{H}} A \leq 1 - \beta$. Choose a sequence $\gamma_n \in \Gamma$ such that $\gamma_n \nearrow \beta$ and fix an $n \in \mathbb{N}^+$. As $f|_A$ is β -Hölder, there is an $\varepsilon > 0$ such that for all $E \subset A$ with $\text{diam } E \leq \varepsilon$ the function $f|_E$ is γ_n -Hölder with Hölder constant 1. Let $A = \bigcup_{i=1}^k A_i$ such that $\text{diam } A_i \leq \varepsilon$ for all $i \in \{1, \dots, k\}$. Then $f|_{A_i}$ are γ_n -Hölder with Hölder constant 1, so by Lemma 8.1 there are functions $g_i \in C_1^{\gamma_n}[0, 1]$ such that $A_i \subset \{f = g_i\}$ for all i . Therefore $f \in \mathcal{F}_{\gamma_n}$ implies that $\dim_{\mathcal{H}} A_i \leq \dim_{\mathcal{H}}\{f = g_i\} \leq 1 - \gamma_n$ for all i ; thus the countable stability of Hausdorff dimension yields $\dim_{\mathcal{H}} A \leq 1 - \gamma_n$. This holds for all $n \in \mathbb{N}^+$, so $\dim_{\mathcal{H}} A \leq 1 - \beta$.

Now let $\beta \in (\alpha, 1)$ be fixed, and for all $N, M \in \mathbb{N}^+$ define

$$\mathcal{F}(N, M) = \left\{f \in C_1^\alpha[0, 1] : \mathcal{H}_\infty^{1-\beta+1/N}(\{f = g\}) \leq 1/M \text{ for all } g \in C_1^\beta[0, 1]\right\}.$$

Clearly $\mathcal{F}_\beta = \bigcap_{N=1}^\infty \bigcap_{M=1}^\infty \mathcal{F}(N, M)$; thus it is enough to show that each $\mathcal{F}(N, M)$ contains a dense open subset of $C_1^\alpha[0, 1]$. Assume that $M, N \in \mathbb{N}^+$, $r_0 > 0$, $c_0 < 1$, and a piecewise linear function $f_0 \in C_1^\alpha[0, 1]$ with non-zero slopes are given such that for all $x, y \in [0, 1]$ we have

$$|f_0(x) - f_0(y)| \leq c_0|x - y|^\alpha.$$

By Lemma 8.2 it is enough to find a function $f \in C_1^\alpha[0, 1]$ and $r > 0$ such that

$$(8.2) \quad B(f, r) \subset B(f_0, r_0) \cap \mathcal{F}(N, M),$$

where $B(f, r)$ denotes the closed ball in $C_1^\alpha[0, 1]$ centered at f with radius r . Now we define f . We can fix integers $k_0, m_0 \geq 2$ such that m_0 is odd and

$$\max \left\{ \alpha, \beta - \frac{1}{N} \right\} < \frac{\log(k_0/2)}{\log(k_0 m_0)} < \frac{\log k_0}{\log(k_0 m_0)} < \beta.$$

Let $\gamma = \log k_0 / \log(k_0 m_0)$ and let $f_1 = f_{k_0, m_0} \in C^\gamma[0, 1]$ be the self-affine function defined in Section 5. We will approximate f_0 by re-scaled copies of f_1 . As f_1 is γ -Hölder continuous, there exists $c_1 \in \mathbb{R}^+$ such that for all $x, y \in [0, 1]$ we have

$$(8.3) \quad |f_1(x) - f_1(y)| \leq c_1 |x - y|^\gamma.$$

Assume that $0 = x_1 < \dots < x_{\ell+1} = 1$ such that f_0 is linear on each interval $[x_i, x_{i+1}]$ with non-zero slopes. Let $y_i = f_0(x_i)$ for all $1 \leq i \leq \ell+1$; then $y_{i+1} - y_i \neq 0$ for all $i \leq \ell$. Let us define $\theta, \xi > 0$ and $n_0 \in \mathbb{N}^+$ such that

$$\begin{aligned} \theta &= \min_{1 \leq i \leq \ell} (x_{i+1} - x_i), \\ \xi &= \max_{1 \leq i \leq \ell} |y_{i+1} - y_i|, \\ n_0 &\geq \max \left\{ \frac{2\xi}{r_0}, \left(\frac{2\xi}{(1 - c_0)\theta^\alpha} \right)^{1/(1-\alpha)}, \left(\frac{2\xi c_1}{\theta^\gamma} \right)^{1/(1-\gamma)} \right\}. \end{aligned}$$

For all $i \in \{1, \dots, \ell\}$ and $j \in \{0, \dots, n_0\}$ let

$$x_{i,j} = x_i + \frac{j}{n_0}(x_{i+1} - x_i) \quad \text{and} \quad y_{i,j} = f_0(x_{i,j}).$$

Now we are ready to define f . If for some $i \in \{1, \dots, \ell\}$, $j \in \{0, \dots, n_0 - 1\}$, and $a \in [0, 1]$ we have

$$(8.4) \quad x = x_{i,j} + \frac{a}{n_0}(x_{i+1} - x_i),$$

then let

$$f(x) = y_{i,j} + \frac{f_1(a)}{n_0}(y_{i+1} - y_i).$$

Note that the linearity of f_0 implies that if x satisfies (8.4), then

$$f_0(x) = y_{i,j} + \frac{a}{n_0}(y_{i+1} - y_i).$$

Now we prove that

$$(8.5) \quad f \in C_1^\alpha[0, 1] \quad \text{and} \quad f \in B(f_0, r_0/2).$$

As the range of f_0 is $[0, 1]$, the definition of f and n_0 imply that for all $x \in [0, 1]$ we have

$$(8.6) \quad |f(x) - f_0(x)| \leq \frac{\xi}{n_0} \leq \frac{r_0}{2}.$$

Thus it is enough to prove for (8.5) that $f \in C_1^\alpha[0, 1]$. Assuming that $x, y \in [0, 1]$ and $x < y$, we need to prove that $|f(x) - f(y)| \leq |x - y|^\alpha$. We consider three cases.

First case: Suppose that $y - x \geq \theta/n_0$. Then (8.6), $f_0 \in C_1^\alpha[0, 1]$, and the definition of n_0 imply that

$$|f(x) - f(y)| \leq |f_0(x) - f_0(y)| + \frac{2\xi}{n_0} \leq c_0|x - y|^\alpha + \frac{2\xi}{n_0} \leq |x - y|^\alpha.$$

Second case: Assume that x, y are adjacent points; that is, $x, y \in [x_{i,j}, x_{i,j+1}]$ for some $1 \leq i \leq \ell$ and $0 \leq j \leq n_0 - 1$. Then the definitions of f and ξ , inequality (8.3), the definition of n_0 , and $\alpha < \gamma$ yield that

$$\begin{aligned} |f(x) - f(y)| &= \frac{|y_{i+1} - y_i|}{n_0} \left| f_1 \left(n_0 \frac{x - x_{i,j}}{x_{i+1} - x_i} \right) - f_1 \left(n_0 \frac{y - x_{i,j}}{x_{i+1} - x_i} \right) \right| \\ &\leq \frac{\xi c_1}{n_0} \left(\frac{n_0|x - y|}{\theta} \right)^\gamma \leq |x - y|^\gamma \leq |x - y|^\alpha. \end{aligned}$$

Third case: Suppose that there exists $z \in (x, y)$ such that x, z and z, y are adjacent points. The triangle inequality, the above inequality, the definition of n_0 , and $\alpha < \gamma$ imply that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(z)| + |f(z) - f(y)| \\ &\leq \frac{\xi c_1}{n_0} \left(\left(\frac{n_0|x - z|}{\theta} \right)^\gamma + \left(\frac{n_0|z - y|}{\theta} \right)^\gamma \right) \\ &\leq \frac{2\xi c_1}{n_0} \left(\frac{n_0|x - y|}{\theta} \right)^\gamma \leq |x - y|^\gamma \leq |x - y|^\alpha. \end{aligned}$$

By the definition of θ for all x, y at least one of the above three cases holds, which concludes the proof of (8.5).

Finally, we prove that (8.2) holds for some $r > 0$. Let us define $\delta > 0$ as

$$\delta = \min_{1 \leq i \leq \ell} |y_{i+1} - y_i|.$$

Since $\gamma < \beta$, we have $k_0 < (k_0 m_0)^\beta$. Thus by $\beta < 1$ we can fix an $n_1 \in \mathbb{N}^+$ such that for all $i \in \mathbb{N}^+$ and $n \geq n_1$ we have

$$(8.7) \quad \left(\frac{i + 2}{(k_0 m_0)^n} \right)^\beta < \frac{i \delta}{3n_0 k_0^n}.$$

Since $\log(k_0/2)/\log(k_0 m_0) > \beta - 1/N$, we can define

$$\sigma = \frac{2m_0}{(k_0 m_0)^{1-\beta+1/N}} < 1.$$

Let us fix an integer $n_2 > n_1$ such that

$$(8.8) \quad \sigma^{n_2} < \frac{1}{k_0^{n_1} M \ell n_0}.$$

Let us define $r > 0$ as

$$r = \min \left\{ \frac{r_0}{2}, \frac{\delta}{3n_0 k_0^{n_2}} \right\}.$$

Then clearly $B(f, r) \subset B(f_0, r_0)$. Let us fix $g \in B(f, r)$. It is enough to show that $g \in \mathcal{F}(N, M)$. Fix $i \in \{1, \dots, \ell\}$ and $j \in \{0, \dots, n_0 - 1\}$ arbitrarily, and let $I_0 = [x_{i,j}, x_{i,j+1}]$. Let $h \in C_1^\beta[0, 1]$. By the subadditivity of \mathcal{H}_∞^s it is enough to show that

$$(8.9) \quad \mathcal{H}_\infty^{1-\beta+1/N}(\{g = h\} \cap I_0) \leq \frac{1}{M \ell n_0}.$$

Assume that $n \in \mathbb{N}$ and $I \subset I_0$ is a closed interval. We divide I into $(k_0 m_0)^n$ non-overlapping closed intervals of equal length; the resulting intervals are called the *elementary intervals of I of level n* . Assume that $n_1 \leq n \leq n_2$ and let I_1 be an elementary interval of I_0 of level $n - 1$. Now we show that $\{g = h\}$ intersects at most $2m_0$ many first level elementary intervals of I_1 . Let us decompose I_1 into m_0 non-overlapping intervals of equal length, letting I_2 be one of them. Let $J_1, J_2 \subset I_2$ be two non-consecutive first level elementary intervals of I_1 . It is enough to show that $\{g = h\}$ cannot intersect both J_1 and J_2 . Assume to the contrary that there are $z_1 \in J_1$ and $z_2 \in J_2$ such that $g(z_1) = h(z_1)$ and $g(z_2) = h(z_2)$. If there are $i \in \{1, \dots, k_0 - 2\}$ first level elementary intervals of I_1 between J_1 and J_2 , then

$$|z_1 - z_2| \leq (i + 2) \text{diam } J_1 \leq (i + 2)(k_0 m_0)^{-n}.$$

Therefore $h \in C_1^\beta[0, 1]$ yields that

$$|h(z_1) - h(z_2)| \leq \left(\frac{i + 2}{(k_0 m_0)^n} \right)^\beta.$$

On the other hand, the definitions of g, f , and r imply that

$$|g(z_1) - g(z_2)| \geq |f(z_1) - f(z_2)| - 2r \geq i \frac{\delta}{n_0 k_0^n} - 2r \geq \frac{i\delta}{3n_0 k_0^n}.$$

The above inequalities and (8.7) yield that $|h(z_1) - h(z_2)| < |g(z_1) - g(z_2)|$, which is a contradiction.

Therefore $\{g = h\} \cap I_0$ intersects at most $(k_0 m_0)^{n_1} (2m_0)^{n_2 - n_1}$ many elementary intervals of I_0 of level n_2 . Since the length of these intervals is less than $(k_0 m_0)^{-n_2}$, the definition of σ and inequality (8.8) yield that

$$\begin{aligned} \mathcal{H}_\infty^{1-\beta+1/N}(\{g = h\} \cap I_0) &\leq (k_0 m_0)^{n_1} (2m_0)^{n_2 - n_1} (k_0 m_0)^{-n_2(1-\beta+1/N)} \\ &\leq k_0^{n_1} \sigma^{n_2} \leq \frac{1}{M \ell n_0}. \end{aligned}$$

Hence (8.9) holds, and the proof is complete. □

In order to prove the second part of Theorem 2.13 we need a bridge between the notions of α -Hölder continuity and β -variation. The following lemma is [4, Lemma 4.1]; see also [3, Lemma 4.1].

Lemma 8.3. *Let $\beta, \gamma > 0$ and let $A \subset [0, 1]$. If the function $f: A \rightarrow \mathbb{R}$ has finite β -variation, then there are sets $A_n \subset A$ such that*

- (1) $f|_{A_n}$ is γ -Hölder continuous for all $n \in \mathbb{N}^+$,
- (2) $\dim_{\mathcal{H}}(A \setminus \bigcup_{n=1}^\infty A_n) \leq \gamma\beta$.

Proof of Theorem 2.13 (2). Clearly it is enough to prove the theorem for a countable dense set of parameters β . Since co-meager sets are closed under countable intersection, it is enough to show the statement for an arbitrary fixed $\beta > 0$. For all $f \in C_1^\alpha[0, 1]$ let $A_f \subset [0, 1]$ be given such that $V^\beta(f|_{A_f}) < \infty$. Fix an arbitrary $\delta > \max\{\alpha\beta, \beta/(\beta + 1)\}$ and let $\gamma = \delta/\beta > \max\{\alpha, 1/(\beta + 1)\}$. It is enough to prove that $\dim_{\mathcal{H}} A_f \leq \delta$ for a generic $f \in C_1^\alpha[0, 1]$. Applying Lemma 8.3 we obtain that for all $f \in C_1^\alpha[0, 1]$ there are sets $A_{f,n} \subset A_f$ such that

- (1) $f|_{A_{f,n}}$ is γ -Hölder continuous for all $n \in \mathbb{N}^+$,
- (2) $\dim_{\mathcal{H}}(A_f \setminus \bigcup_{n=1}^\infty A_{f,n}) \leq \gamma\beta = \delta$.

As $\gamma > \alpha$ and $f|_{A_{f,n}}$ are γ -Hölder continuous, Theorem 2.13 (1) and the definition of γ imply that for a generic $f \in C_1^\alpha[0, 1]$ for all $n \in \mathbb{N}^+$ we have

$$(8.10) \quad \dim_{\mathcal{H}} A_{f,n} \leq 1 - \gamma < 1 - \frac{1}{\beta + 1} = \frac{\beta}{\beta + 1} < \delta.$$

Inequalities (2), (8.10), and the countable stability of Hausdorff dimension yield that $\dim_{\mathcal{H}} A_f \leq \delta$ for a generic $f \in C_1^\alpha[0, 1]$. The proof is complete. \square

REFERENCES

- [1] Robert J. Adler, *The geometry of random fields*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Ltd., Chichester, 1981. MR611857
- [2] Omer Angel, Richárd Balka, and Yuval Peres, *Increasing subsequences of random walks*, Math. Proc. Cambridge Philos. Soc. **163** (2017), no. 1, 173–185, DOI 10.1017/S0305004116000797. MR3656355
- [3] Tonći Antunović, Krzysztof Burdzy, Yuval Peres, and Julia Ruscher, *Isolated zeros for Brownian motion with variable drift*, Electron. J. Probab. **16** (2011), no. 65, 1793–1814, DOI 10.1214/EJP.v16-927. MR2842087
- [4] Richárd Balka and Yuval Peres, *Restrictions of Brownian motion* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **352** (2014), no. 12, 1057–1061, DOI 10.1016/j.crma.2014.09.023. MR3276819
- [5] Denis Denisov and Vitali Wachtel, *Random walks in cones*, Ann. Probab. **43** (2015), no. 3, 992–1044, DOI 10.1214/13-AOP867. MR3342657
- [6] Márton Elekes, *Hausdorff measures of different dimensions are isomorphic under the continuum hypothesis*, Real Anal. Exchange **30** (2004/05), no. 2, 605–616. MR2177422
- [7] Steven N. Evans, *On the Hausdorff dimension of Brownian cone points*, Math. Proc. Cambridge Philos. Soc. **98** (1985), no. 2, 343–353, DOI 10.1017/S0305004100063519. MR795899
- [8] Kenneth Falconer, *Fractal geometry*, Mathematical foundations and applications, 2nd ed., John Wiley & Sons, Inc., Hoboken, NJ, 2003. Mathematical foundations and applications. MR2118797
- [9] John E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747, DOI 10.1512/iumj.1981.30.30055. MR625600
- [10] Jean-Pierre Kahane, *Some random series of functions*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 5, Cambridge University Press, Cambridge, 1985. MR833073
- [11] Jean-Pierre Kahane and Yitzhak Katznelson, *Restrictions of continuous functions*, Israel J. Math. **174** (2009), 269–284, DOI 10.1007/s11856-009-0114-x. MR2581219
- [12] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597
- [13] Gregory F. Lawler and Vlada Limic, *Random walk: a modern introduction*, Cambridge Studies in Advanced Mathematics, vol. 123, Cambridge University Press, Cambridge, 2010. MR2677157
- [14] Michael B. Marcus and Jay Rosen, *Markov processes, Gaussian processes, and local times*, Cambridge Studies in Advanced Mathematics, vol. 100, Cambridge University Press, Cambridge, 2006. MR2250510
- [15] András Máthé, *Hausdorff measures of different dimensions are not Borel isomorphic*, Israel J. Math. **164** (2008), 285–302, DOI 10.1007/s11856-008-0030-5. MR2391150
- [16] András Máthé, *Measurable functions are of bounded variation on a set of dimension $\frac{1}{2}$* , Bull. Lond. Math. Soc. **45** (2013), no. 3, 580–594, DOI 10.1112/blms/bds106. MR3065027
- [17] G. M. Molchan, *Maximum of a fractional Brownian motion: probabilities of small values*, Comm. Math. Phys. **205** (1999), no. 1, 97–111, DOI 10.1007/s002200050669. MR1706900
- [18] Peter Mörters and Yuval Peres, *Brownian motion*, with an appendix by Oded Schramm and Wendelin Werner, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 30, Cambridge University Press, Cambridge, 2010. MR2604525
- [19] Loren D. Pitt, *Local times for Gaussian vector fields*, Indiana Univ. Math. J. **27** (1978), no. 2, 309–330, DOI 10.1512/iumj.1978.27.27024. MR0471055
- [20] K. Simon, *Some dual statements concerning Wiener measure and Baire category*, Proc. Amer. Math. Soc. **106** (1989), no. 2, 455–463, DOI 10.2307/2048827. MR961409

- [21] N. Th. Varopoulos, *Potential theory in conical domains*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 2, 335–384, DOI 10.1017/S0305004198002771. MR1643806

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA V6T 1Z2, CANADA

Email address: `angel@math.ubc.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, AND PACIFIC INSTITUTE FOR THE MATHEMATICAL SCIENCES, VANCOUVER, BRITISH COLUMBIA V6T 1Z2, CANADA

Email address: `balka@math.ubc.ca`

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UNITED KINGDOM

Email address: `a.mathe@warwick.ac.uk`

MICROSOFT RESEARCH, 1 MICROSOFT WAY, REDMOND, WASHINGTON 98052

Email address: `peres@microsoft.com`