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BOUNDED ORBITS OF CERTAIN DIAGONALIZABLE FLOWS ON $SL_n(R)/SL_n(Z)$

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Abstract. We prove that the set of points that have bounded orbits under certain diagonalizable flows is a hyperplane absolute winning subset of $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.

1. Introduction

1.1. **Statement of main results.** Let G be a connected Lie group, let Γ be a nonuniform lattice in G, and let $F = \{g_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of G with noncompact closure. We are interested in the dynamical properties of the action of F on the homogeneous space G/Γ by left translations. Specifically, we will focus on the study of the set

$$E(F) := \{\Lambda \in G/\Gamma : F\Lambda \text{ is bounded in } G/\Gamma\}.$$

In certain important cases, it turns out that $E(F)$ has zero Haar measure (for example, when G is semisimple without compact factors and Γ is irreducible, this follows from Moore's ergodicity theorem [18]). If F is Ad-unipotent, $E(F)$ is even smaller. In this case, by Ratner's Theorems [20], $E(F)$ is contained in a countable union of proper submanifolds, and hence has Hausdorff dimension $< \dim G$. When F is Ad-semisimple, the situation is quite different. Motivated by the work of Dani (cf. [9], [10]), Margulis proposed a conjecture in his 1990s ICM report [16], which was settled in a subsequent work of Kleinbock and Margulis [14]. In that work, they proved: if the flow $(G/\Gamma, F)$ has the so-called property (Q), then the set $E(F)$ is *thick*, i.e., for any nonempty open subset V of G/Γ the set $E(F) \cap V$ is of Hausdorff dimension equal to the dimension of the underlying space G/Γ. In particular, when F is Ad-semisimple, the flow $(G/\Gamma, F)$ always has property (Q).

Given countably many Ad-semisimple F_n, it is natural to ask whether the set of points Λ such that all the orbits $F_n\Lambda$ are bounded is still thick. This is natural from both the dynamical point of view and its relation to number theory. This is proved to be true for $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ in [15], and for $G = \mathrm{SL}_3(\mathbb{R})$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ in [3]. Note that this set is the intersection $\bigcap_n E(F_n)$. A powerful tool for studying intersection properties of different sets is a type of game introduced by Schmidt in [21], which is called Schmidt's (α, β)-game. The game can be played on any metric space, and it defines a class of so-called α-winning sets ($0 < \alpha < 1$). When the metric space is a Riemannian manifold, α-winning sets are thick and stable with respect to countable intersections. In this paper, we will use a variant of Schmidt's (α, β)-game, i.e., the hyperplane absolute game introduced in

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[7] and [15]. This game has the advantage that it can be naturally defined on a differential manifold without picking a Riemannian metric, while the hyperplane absolute winning (abbreviated as HAW) sets also enjoy the thickness and countable intersection properties. See Section 2 for details. Note that, in both [15] and [3], the authors prove their results by showing that $E(F)$ is HAW in the corresponding case. In fact, the following conjecture is proposed in [3].

Conjecture 1.1 ([3, Conjecture 7.1])**.** *Let G be a Lie group, let Γ be a lattice in G, and let F be a one-parameter Ad-diagonalizable subgroup of G. Then the set $E(F)$ is HAW on G/Γ.*

In this paper, we restrict ourselves to the cases

$$G = SL_n(\mathbb{R}), \quad \Gamma = SL_n(\mathbb{Z}).$$

Our main theorem is the following, verifying the above conjecture for a certain class of F.

Theorem 1.2. *Let F be a one-parameter subgroup of G satisfying the following property:*

(1.1)
it is diagonalizable and the eigenvalues of g_1 (denoted by $\lambda_1, \ldots, \lambda_n$) satisfy

$$\#\{i : |\lambda_i| < 1\} = 1 \text{ and } \#\{i : |\lambda_i| = \max_{1 \le j \le n} |\lambda_j|\} \ge n - 2.$$

Then the set $E(F)$ is HAW on G/Γ.

We also prove the following theorem verifying [3, Conjecture 7.2] for F satisfying (1.1).

Theorem 1.3. *Let F be a one-parameter subgroup of G satisfying* (1.1)*, and let $F^+ = \{g_t \in F : t \ge 0\}$. Let $H(F^+)$ denote the expanding horospherical subgroup of F^+ which is defined as*

$$H(F^+) = \left\{ h \in G : \lim_{t \to +\infty} g_t^{-1} h g_t = e \right\}. \tag{1.2}$$

Then for any $\Lambda \in G/\Gamma$, the set

$$\{h \in H(F^+) : h\Lambda \in E(F^+)\}$$

is HAW on $H(F^+)$.

1.2. **Connection to number theory.** To begin, let us define a *d-weight* $\mathbf{r}$ to be a d-tuple $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ such that each r_i is positive and their sum equals 1. Due to work of Dani [9] and Kleinbock [13], we know that for a d-weight $\mathbf{r}$ there is a close relation between the set of $\mathbf{r}$-badly approximable vectors (abbreviated as $\mathbf{Bad}(\mathbf{r})$) and bounded orbits of certain flow corresponding to $\mathbf{r}$ in $\mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$. We will not present the explicit definition of $\mathbf{Bad}(\mathbf{r})$ here, but we remark that they are natural generalizations of the classical badly approximable numbers. Recently, there has been rapid progress on the study of intersection properties of the sets $\mathbf{Bad}(\mathbf{r})$ for different weight $\mathbf{r}$; for example, see [1, 2, 5, 6, 12, 19]. Concerning the winning properties of such sets, Schmidt proved that $\mathbf{Bad}_d$ (abbreviation for $\mathbf{Bad}(\frac{1}{d}, \ldots, \frac{1}{d})$) is winning for his game for any $d \in \mathbb{N}$. They are also proved to be HAW in [7]. Recently, An [2] proved that $\mathbf{Bad}(\mathbf{r})$ are winning sets for Schmidt's game for any 2-weight $\mathbf{r}$. The HAW property is also established for such sets by Nesharim and Simmons [19]. To this end, we want to highlight the following theorem proved in [12], since it motivates the results of this paper.

Theorem 1.4 (cf. [12, Theorem 1.4]). *Let a d-weight $\mathbf{r} = (r_1, \cdots, r_d)$ satisfy*

$$\sum_{i=1}^{d} r_i = 1 \text{ and } r_1 = \ldots = r_{d-1} \geq r_d \geq 0. \tag{1.3}$$

Then $\mathbf{Bad}(\mathbf{r})$ *is HAW.*

Remark 1.5. Whether $\mathbf{Bad}(\mathbf{r})$ is winning (α-winning or HAW) for general weight $\mathbf{r}$ is a challenging open problem proposed by Kleinbock [13].

1.3. **Structure of the paper.** For the sake of convenience, from now on we will assume

$$G = \mathrm{SL}_{d+1}(\mathbb{R}), \Gamma = \mathrm{SL}_{d+1}(\mathbb{R}).$$

That is, the number n in the title of the paper is replaced by $d + 1$.

This paper is organized as follows. In Section 2 we recall some basics of certain Schmidt games, namely the hyperplane absolute game and the hyperplane potential game. In Section 3.1, we state Theorem 3.1 and then convert it to the Diophantine setting using Lemma 3.4. Note that Theorem 3.1, whose proof forms the most technical part of this paper, can be regarded as a special case of Theorem 1.3. In the rest of Section 3, we turn to the study of pairs (B, P), where B is a closed ball in $\mathbb{R}^{2d-1}$ and P is a rational vector in $\mathbb{Q}^d$. We manage to attach a rational hyperplane and a rational line in $\mathbb{R}^d$ to the pair (B, P). Section 5 is the core of this paper, in which Theorem 3.1 is proved using the information of the pairs (B, P) and some subdivisions prepared in Sections 3 and 4. In the last section, Theorem 1.2 and Theorem 1.3 are deduced from Theorem 3.1.

2. Schmidt games

In this section, we will recall definitions of certain Schmidt games, namely, the hyperplane absolute game and the hyperplane potential game. They are both variants of the (α, β)-game introduced by Schmidt in [21]. Since we don't make direct use of the (α, β)-game in this paper, we omit its definition here and refer the interested reader to [21, 22]. Instead, we list here some nice properties of the α-winning sets:

(1) If the game is played on a Riemannian manifold, then any α-winning set is thick.
(2) The intersection of countably many α-winning sets is α-winning.

2.1. **Hyperplane absolute game.** The hyperplane absolute game was introduced in [7]. It is played on a Euclidean space $\mathbb{R}^d$. Given a hyperplane $L \subset \mathbb{R}^d$ and a $\delta > 0$, we denote by $L^{(\delta)}$ the δ-neighborhood of L, i.e.,

$$L^{(\delta)} := \{\mathbf{x} \in \mathbb{R}^d : \mathrm{dist}(\mathbf{x}, L) < \delta\}.$$

For $\beta \in (0, \frac{1}{3})$, the *$\beta$-hyperplane absolute game* is defined as follows. Bob starts by choosing a closed ball $B_0 \subset \mathbb{R}^d$ of radius ρ_0. In the ith turn, Bob chooses a closed ball B_i with radius ρ_i, and then Alice chooses a hyperplane neighborhood $L_i^{(\delta_i)}$ with

$\delta_i \leq \beta\rho_i$. Then in the $(i+1)$th turn, Bob chooses a closed ball $B_{i+1} \subset B_i \smallsetminus L_i^{(\delta_i)}$ of radius $\rho_{i+1} \geq \beta\rho_i$. By this process there is a nested sequence of closed balls

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots .$$

We say that a subset $S \subset \mathbb{R}^d$ is *β-hyperplane absolute winning* (*β-HAW* for short) if no matter how Bob plays, Alice can ensure that

$$\bigcap_{i=0}^{\infty} B_i \cap S \neq \varnothing.$$

We say S is *hyperplane absolute winning* (*HAW* for short) if it is β-HAW for any $\beta \in (0, \frac{1}{3})$.

We have the following lemma collecting the basic properties of β-HAW subsets and HAW subsets of $\mathbb{R}^d$ ([7], [15], [12]).

Lemma 2.1.

(1) *A HAW subset is always $\frac{1}{2}$-winning.*
(2) *Given $\beta, \beta' \in (0, \frac{1}{3})$, if $\beta \geq \beta'$, then any β'-HAW set is β-HAW.*
(3) *A countable intersection of HAW sets is again HAW.*
(4) *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 diffeomorphism. If S is a HAW set, then so is $\varphi(S)$.*

The notion of HAW was extended to subsets of C^1 manifolds in [15]. This is done in two steps. First, one defines the hyperplane absolute game on an open subset $W \subset \mathbb{R}^d$. It is defined just as the hyperplane absolute game on $\mathbb{R}^d$, except for requiring that Bob's first move B_0 be contained in W. Now, let M be a d-dimensional C^1 manifold, and let $\{(U_\alpha, \phi_\alpha)\}$ be a C^1 atlas on M. A subset $S \subset M$ is said to be HAW on M if for each α, $\phi_\alpha(S \cap U_\alpha)$ is HAW on $\phi_\alpha(U_\alpha)$. The definition is independent of the choice of atlas by property (4) listed above. We have the following lemma that collects the basic properties of HAW subsets of a C^1 manifold (cf. [15]).

Lemma 2.2.

(1) *HAW subsets of a C^1 manifold are thick.*
(2) *A countable intersection of HAW subsets of a C^1 manifold is again HAW.*
(3) *Let $\phi : M \to N$ be a diffeomorphism between C^1 manifolds, and let $S \subset M$ be a HAW subset of M. Then $\phi(S)$ is a HAW subset of N.*
(4) *Let M be a C^1 manifold with an open cover $\{U_\alpha\}$. Then, a subset $S \subset M$ is HAW on M if and only if $S \cap U_\alpha$ is HAW on U_α for each α.*
(5) *Let M, N be C^1 manifolds, and let $S \subset M$ be a HAW subset of M. Then $S \times N$ is a HAW subset of $M \times N$.*

2.2. **Hyperplane potential game.** Being introduced in [11], the hyperplane potential game also defines a class of subsets of $\mathbb{R}^d$ called *hyperplane potential winning* (*HPW* for short) sets. The following lemma allows one to prove the HAW property of a set $S \subset \mathbb{R}^d$ by showing that it is winning for the hyperplane potential game. This is exactly the game we will use in this paper.

Lemma 2.3 (cf. [11, Theorem C.8])**.** *A subset S of $\mathbb{R}^d$ is HPW if and only if it is HAW.*

The hyperplane potential game involves two parameters, $\beta \in (0,1)$ and $\gamma > 0$. Bob starts the game by choosing a closed ball $B_0 \subset \mathbb{R}^d$ of radius ρ_0. In the ith turn, Bob chooses a closed ball B_i of radius ρ_i, and then Alice chooses a countable family of hyperplane neighborhoods $\{L_{i,k}^{(\delta_{i,k})} : k \in \mathbb{N}\}$ such that

$$\sum_{k=1}^{\infty} \delta_{i,k}^{\gamma} \leq (\beta\rho_i)^{\gamma}.$$

Then in the $(i+1)$th turn, Bob chooses a closed ball $B_{i+1} \subset B_i$ of radius $\rho_{i+1} \geq \beta\rho_i$. By this process there is a nested sequence of closed balls

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots .$$

We say a subset $S \subset \mathbb{R}^d$ is (β,γ)*-hyperplane potential winning* ((β,γ)*-HPW* for short) if no matter how Bob plays, Alice can ensure that

$$\bigcap_{i=0}^{\infty} B_i \cap \Big(S \cup \bigcup_{i=0}^{\infty}\bigcup_{k=1}^{\infty} L_{i,k}^{(\delta_{i,k})}\Big) \neq \varnothing.$$

We say S is *hyperplane potential winning* (*HPW* for short) if it is (β,γ)-HPW for any $\beta \in (0,1)$ and $\gamma > 0$.

3. Converting to the Diophantine setting

Fix $d \geq 2$. Recall that we have assumed

$$G = \mathrm{SL}_{d+1}(\mathbb{R}), \Gamma = \mathrm{SL}_{d+1}(\mathbb{R}).$$

Let

$$\pi : G \to G/\Gamma \text{ be the natural projection.}$$

We will fix a d-weight $\mathbf{r}$ satisfying (1.3) until the last section. For simplicity, sometimes we also write $\lambda = r_1 = \cdots = r_{d-1}$, $\mu = r_d$. Both Theorem 1.2 and Theorem 1.3 will be deduced from the following theorem.

Theorem 3.1. *Let* $\mathbf{r}$ *be a weight satisfying* (1.1)*. Denote*

$$F_{\mathbf{r}} := \{g_t = \mathrm{diag}(e^{r_1 t}, e^{r_2 t}, \cdots, e^{r_d t}, e^{-t}) : t \in \mathbb{R}\}, \quad F_{\mathbf{r}}^{+} := \{g_t \in F_{\mathbf{r}} : t \geq 0\},$$

and

$$U := \left\{u_{\mathbf{x},y,\mathbf{z}} : \mathbf{x},\mathbf{z} \in \mathbb{R}^{d-1}, y \in \mathbb{R}\right\}, \quad \text{where } u_{\mathbf{x},y,\mathbf{z}} := \begin{pmatrix} Id & \mathbf{z} & \mathbf{x} \\ & 1 & y \\ & & 1 \end{pmatrix} \in G. \tag{3.1}$$

Then the set $U \cap \pi^{-1}(E(F_{\mathbf{r}}^{+}))$ *is HAW on* U*.*

Remark 3.2. If $\mathbf{r}$ satisfies $r_1 > r_d$, then the expanding horospherical subgroup $H(F_{\mathbf{r}}^{+})$ defined as in (1.2) coincides with the group U given in (3.1). Thus in this case, Theorem 3.1 can be regarded as special case of Theorem 1.3 with $\Lambda = Id \cdot \Gamma$.

3.1. **Diophantine characterization.** For technical reasons, we will prove Theorem 3.1 by applying the diffeomorphism $\mathbb{R}^{2d-1} \to U$ defined as

$$(\mathbf{x}, y, \mathbf{z}) \mapsto u^{-1}_{\mathbf{x},y,\mathbf{z}}.$$

Remark 3.3. In view of Lemma 2.2(3), if we can prove that the set

$$\{(\mathbf{x}, y, \mathbf{z}) \in \mathbb{R}^{2d-1} : F^+_{\mathbf{r}} u^{-1}_{\mathbf{x},y,\mathbf{z}}\Gamma \text{ is bounded in } G/\Gamma\}$$

is HAW on $\mathbb{R}^{2d-1}$, then Theorem 3.1 will follow.

A rational vector $P \in \mathbb{Q}^d$ will always be written in the following reduced form: $P=\left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right)$, with $q>0$ and $\mathbf{p}=(p_1, \dots, p_{d-1})$ satisfying $\gcd(p_1, \dots, p_{d-1}, r, q) = 1$. Such a form is unique, thus we may write the denominator of P as a function $q(P)$.

We need the following Diophantine characterization of the boundedness of $F^+_{\mathbf{r}} u^{-1}_{\mathbf{x},y,\mathbf{z}}\Gamma$ in G/Γ. For $\epsilon > 0$ and a rational vector $P = (\frac{\mathbf{p}}{q}, \frac{r}{q}) \in \mathbb{Q}^d$ written in its reduced form, we denote

$$\Delta_\epsilon(P) := \left\{ (\mathbf{x}, y, \mathbf{z}) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1} : \left| y - \frac{r}{q} \right| < \frac{\epsilon}{q^{1+\mu}}, \right.$$
$$\left. \left\| \mathbf{x} - \frac{\mathbf{p}}{q} - \left(y - \frac{r}{q} \right) \mathbf{z} \right\|_\infty < \frac{\epsilon}{q^{1+\lambda}} \right\},$$

where $\|\cdot\|_\infty$ means the maximal norm on $\mathbb{R}^{d-1}$; that is, for $\mathbf{x} = (x_1, \dots, x_{d-1})$, $\|\mathbf{x}\|_\infty = \max\{|x_1|, \dots, |x_d|\}$. Then we set

$$S_\epsilon(\mathbf{r}) := \mathbb{R}^{2d-1} \setminus \bigcup_{P \in \mathbb{Q}^d} \Delta_\epsilon(P)$$

and

$$S(\mathbf{r}) := \bigcup_{\epsilon > 0} S_\epsilon(\mathbf{r}).$$

The following lemma allows us to convert our problem to the Diophantine setting. For the proof, one can refer to [13] (see also [3, Lemma 3.2]).

Lemma 3.4 (cf. [13, Theorem 2.5])**.** *The orbit $F^+_{\mathbf{r}} u^{-1}_{\mathbf{x},y,\mathbf{z}}\Gamma$ is bounded if and only if $(\mathbf{x}, y, \mathbf{z}) \in S(\mathbf{r})$; that is, there is $\epsilon = \epsilon(\mathbf{x}, y, \mathbf{z}) > 0$ such that*

$$\max\left\{q^\mu |qy - r|, q^\lambda \|q\mathbf{x} - \mathbf{p} - (qy - r)\mathbf{z}\|_\infty\right\} \geq \epsilon \quad \forall P = \left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right) \in \mathbb{Q}^d.$$

3.2. **Attaching hyperplanes.** Let $\mathcal{B}$ denote the set of closed balls in $\mathbb{R}^{2d-1}$ with radius smaller than $1/d$. For any $\mathbf{a}^+ : \mathscr{B} \times \mathbb{Q}^d \to \mathbb{Z}^d$ one can define a linear function on $\mathbb{R}^d$ that depends on the pair of a closed ball $B \in \mathscr{B}$ and $P = (\frac{\mathbf{p}}{q}, \frac{r}{q}) \in \mathbb{Q}^d$:

$$F_{B,P}(\mathbf{w}) = \mathbf{a}^+(B, P) \cdot \mathbf{w} - \mathbf{a}^+(B, P) \cdot \left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right), \quad \mathbf{w} \in \mathbb{R}^d. \tag{3.2}$$

We also write for simplicity

$$C(B, P) = \mathbf{a}^+(B, P) \cdot \left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right). \tag{3.3}$$

Then we can define a hyperplane attached to the pair (B, P) to be

$$\mathcal{H}_{B,P} := \{\mathbf{w} \in \mathbb{R}^d : F_{B,P}(\mathbf{w}) = 0\}.$$

Now we introduce a particular function $\mathbf{a}^+ : \mathscr{B} \times \mathbb{Q}^d \to \mathbb{Z}^d$ which we will use throughout the paper. We shall need the following lemma.

Lemma 3.5. *Let $\mathbf{z} \in \mathbb{R}^{d-1}$. For any $P = (\frac{\mathbf{p}}{q}, \frac{r}{q}) \in \mathbb{Q}^d$, there exists $(\mathbf{a}, b) \in \mathbb{Z}^d$ with $(\mathbf{a}, b) \neq (\mathbf{0}, 0)$ such that $\mathbf{a} \cdot \mathbf{p} + br \in q\mathbb{Z}$ and $\|\mathbf{a}\|_\infty \leq q^\lambda$, $|b + \mathbf{z} \cdot \mathbf{a}| \leq q^\mu$.*

Proof. By Minkowski's linear forms theorem (cf. [8, Chapter III, Theorem III]), there exist $\mathbf{a} \in \mathbb{Z}^{d-1}, b, c \in \mathbb{Z}$ which are not all zero such that

$$|\mathbf{a} \cdot \mathbf{p} + br + cq| < 1, \quad \|\mathbf{a}\|_\infty \leq q^\lambda, \quad |b + \mathbf{z} \cdot \mathbf{a}| \leq q^\mu.$$

Since $\mathbf{a} \cdot \mathbf{p} + br + cq \in \mathbb{Z}$, it must be 0 by the first inequality above. Assume that $\mathbf{a} = \mathbf{0}$ and $b = 0$. Then it follows from $\mathbf{a} \cdot \mathbf{p} + br + cq = 0$ and $q \neq 0$ that $c = 0$, which is a contradiction. Thus $(\mathbf{a}, b) \neq (\mathbf{0}, 0)$. The lemma follows. $\square$

Now let us consider the following set:

$$\mathscr{A}_{B,P} := \big\{(\mathbf{a}, b) \in \mathbb{Z}^d : (\mathbf{a}, b) \neq (\mathbf{0}, 0), \mathbf{a} \cdot \mathbf{p} + br \in q\mathbb{Z}, \\ \|\mathbf{a}\|_\infty \leq q^\lambda, |b + \mathbf{z}_B \cdot \mathbf{a}| \leq q^\mu + \rho(B)^{\frac{1}{2}}\big\},$$

where $\mathbf{z}_B$ is the $\mathbf{z}$-coordinate of the center of B and $\rho(B)$ is the radius of B. It follows from Lemma 3.5 that $\mathscr{A}_{B,P} \neq \varnothing$. We choose and fix

$$\mathbf{a}^+(B, P) = \big(\mathbf{a}(B, P), b(B, P)\big) \in \mathscr{A}_{B,P}$$

such that

$$\begin{aligned} \xi(B, P) &:= \max\Big\{\|\mathbf{a}(B, P)\|_\infty, |b(B, P) + \mathbf{z}_B \cdot \mathbf{a}(B, P)|\Big\} \\ &= \min\Big\{\max\big\{\|\mathbf{a}\|_\infty, |b + \mathbf{z}_B \cdot \mathbf{a}|\big\} : (\mathbf{a}, b) \in \mathscr{A}_{B,P}\Big\}. \end{aligned} \tag{3.4}$$

This completes the definition of the function $\mathbf{a}^+$. Then we define the *height of P with respect to B*:

$$H_B(P) := q(P)\xi(B, P).$$

Remark 3.6. From its definition, one can see that the height function $H_B(P)$ is not canonically defined, i.e., it may depend on a choice. But we have the following lemma controlling the size of $H_B(P)$.

Lemma 3.7. *For any $(B, P) \in \mathscr{B} \times \mathbb{Q}^d$, we have*

$$q(P) \leq H_B(P) \leq q(P)^{1+\lambda}. \tag{3.5}$$

Proof. Write $q(P)$ simply as q; the first inequality is clear from the definition. By Lemma 3.5, $\mathscr{A}_{B,P}$ contains a vector $(\mathbf{a}_0, b_0)$ with $\|\mathbf{a}_0\|_\infty \leq q^\lambda$ and $|b_0 + \mathbf{z}_B \cdot \mathbf{a}_0| \leq q^\mu$. Thus, it follows from (3.4) that

$$\begin{aligned} \max\Big\{\|\mathbf{a}(B, P)\|_\infty, |b(B, P) + \mathbf{z}_B \cdot \mathbf{a}(B, P)|\Big\} &\leq \max\Big\{\|\mathbf{a}_0\|_\infty, |b_0 + \mathbf{z}_B \cdot \mathbf{a}_0|\Big\} \\ &\leq \max\{q^\lambda, q^\mu\} = q^\lambda. \end{aligned}$$

The second inequality follows. $\square$

Remark 3.8. It follows from the definition of $\mathbf{a}^+(B, P)$ that $C(B, P) \in \mathbb{Z}$; thus the coefficients of $F_{B,P}$ belong to $\mathbb{Z}$.

3.3. **Attaching lines.** We shall define another function,

$$\mathbf{v}^+ : \mathscr{B} \times \mathbb{Q}^d \to \mathbb{Q}^d,$$

in this subsection. The function $\mathbf{v}^+(*, P)$ takes values in the lattice Λ_P which is defined as follows:

$$\Lambda_P = \mathbb{Z}^d + \mathbb{Z}\left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right), \quad \text{where } P = \left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right).$$

The line attached to the pair (B, P) is defined to be

$$\mathcal{L}_{B,P} := \left\{\mathbf{w} \in \mathbb{R}^d : \mathbf{w} - \left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right) = t\mathbf{v}^+(B, P), \quad t \in \mathbb{R}\right\}.$$

The definition of the function $\mathbf{v}^+$ is given in the following lemma.

Lemma 3.9. *For any $(B, P) \in \mathscr{B} \times \mathbb{Q}^d$, there exists a nonzero vector*

$$\mathbf{v}^+(B, P) = (\mathbf{v}(B, P), u(B, P)) \in \Lambda_P$$

with $\mathbf{v}(B, P) \in \mathbb{R}^{d-1}, u(B, P) \in \mathbb{R}$ such that

$$\|\mathbf{v}(B, P) - u(B, P)\mathbf{z}_B\|_\infty \le 2dq(P)^{-\lambda}, \quad |u(B, P)| \le 2d\xi(B, P)q(P)^{-\lambda-\mu}. \tag{3.6}$$

Proof. Write $q(P)$ simply as q. It is easy to check that $d(\Lambda_P) = 1/q$, where $d(\Lambda_P)$ denotes the covolume of the lattice Λ_P. We will make use of the vector $\mathbf{a}^+(B, P)$ constructed in the previous subsection. For simplicity, write $\mathbf{a}^+(B, P)$, $\mathbf{a}(B, P), a_i(B, P), b(B, P), \xi(B, P), \mathbf{z}_B$ as $\mathbf{a}^+, \mathbf{a}, a_i, b, \xi, \mathbf{z}$, respectively.

We have the following two distinct cases:

(1) *Case* $|a_k|q^{-\lambda} = \max(|a_1|q^{-\lambda}, \dots, |a_{d-1}|q^{-\lambda}, |b + \mathbf{z}\cdot\mathbf{a}|\xi q^{-\lambda-\mu})$, where $1 \le k \le d-1$. Then it is obvious that $a_k \ne 0$. Consider the convex body

$$\begin{aligned}\Sigma_k := \big\{\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d : |w_i - z_i w_d| \le q^{-\lambda}, i \ne k, d;\\ |w_d| \le \xi q^{-\lambda-\mu}; \quad |\mathbf{a}^+ \cdot \mathbf{w}| < 1\big\}.\end{aligned}$$

A direct computation shows that

$$2^{-d}\mathrm{Vol}(\Sigma_k) = |a_k|^{-1}\xi q^{-\lambda-\mu} \prod_{i\ne k,d} q^{-\lambda} = |a_k|^{-1}\xi q^{-1} \ge q^{-1}.$$

Hence there is a nonzero Λ_P-lattice point $\tilde{\mathbf{w}} = (\tilde{w}_1, \dots, \tilde{w}_d)$ in Σ_k. By the definition of $\mathbf{a}^+$, we have $\mathbf{a}^+ \cdot \left(\frac{\mathbf{p}}{q}, \frac{r}{q}\right) = 0$. Consequently, $\mathbf{a}^+ \cdot \tilde{\mathbf{w}} \in \mathbb{Z}$ for all $\tilde{\mathbf{w}} \in \Lambda_P$. Hence $|\mathbf{a}^+ \cdot \tilde{\mathbf{w}}| < 1$ implies $\mathbf{a}^+ \cdot \tilde{\mathbf{w}} = 0$, and therefore

$$\begin{aligned}|\tilde{w}_k - z_k\tilde{w}_d| &= |a_k|^{-1}\Big|\sum_{i\ne k,d} a_i(\tilde{w}_i - z_i\tilde{w}_d) + (b + \mathbf{z}\cdot\mathbf{a})\tilde{w}_d\Big|\\ &\le |a_k|^{-1}\Big(\sum_{i\ne k,d} |a_i||\tilde{w}_i - z_i\tilde{w}_d| + |b + \mathbf{z}\cdot\mathbf{a}||\tilde{w}_d|\Big)\\ &\le |a_k|^{-1}\Big(\sum_{i\ne k,d} |a_i|q^{-\lambda} + |b + \mathbf{z}\cdot\mathbf{a}|\xi q^{-\lambda-\mu}\Big)\\ &\le (d-1)q^{-\lambda}.\end{aligned}$$

(2) *Case* $|b + \mathbf{z}\cdot\mathbf{a}|\xi q^{-\lambda-\mu} = \max(|a_1|q^{-\lambda}, \dots, |a_{d-1}|q^{-\lambda}, |b + \mathbf{z}\cdot\mathbf{a}|\xi q^{-\lambda-\mu})$.
Then we consider the convex body

$$\Sigma_d := \left\{\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d : |w_i - z_i w_d| \le 2q^{-\lambda}, i \ne d; \quad |\mathbf{a}^+ \cdot \mathbf{w}| < 1\right\}.$$

Since $B \in \mathscr{B}$, we have $\rho(B) \leq 1/d$. Thus $|b+\mathbf{z}\cdot\mathbf{a}| \leq q^{\mu} + \rho(B)^{\frac{1}{2}} < q^{\mu}+1$. A direct computation shows that

$$2^{-d}\mathrm{Vol}(\Sigma_d) = |b+\mathbf{z}\cdot\mathbf{a}|^{-1}\prod_{i\neq d} 2q^{-\lambda} \geq 2^{d-1}(q^{\mu}+1)^{-1}q^{\mu}q^{-1} \geq q^{-1}.$$

Thus there is a nonzero Λ_P-lattice point $\tilde{\mathbf{w}} = (\tilde{w}_1, \ldots, \tilde{w}_d)$ in Σ_k. Similarly we have

$$\begin{aligned}|\tilde{w}_d| &= |b+\mathbf{z}\cdot\mathbf{a}|^{-1}\Big|\sum_{i\neq d} a_i(\tilde{w}_i - z_i\tilde{w}_d)\Big| \\ &\leq |b+\mathbf{z}\cdot\mathbf{a}|^{-1}\Big(\sum_{i\neq d}|a_i||\tilde{w}_i - z_i\tilde{w}_d|\Big) \\ &\leq |b+\mathbf{z}\cdot\mathbf{a}|^{-1}\Big(\sum_{i\neq d}2|a_i|q^{-\lambda}\Big) \\ &\leq 2(d-1)\xi q^{-\lambda-\mu}.\end{aligned}$$

In each case above we set $\mathbf{v}^{+}(B,P) = \mathbf{w}$, and this completes the proof . $\square$

Remark 3.10. Let $\Pi_{B,P}$ denote the subset of $\mathbb{R}^d$ defined by the inequalities given in (3.6). Note that the volume of $\Pi_{B,P}$ may be smaller than $1/q$, so the above lemma does not follow directly from Minkowski's Theorem on linear forms.

4. Some subdivisions

As previously mentioned, we will use the hyperplane potential game in establishing Theorem 3.1. This section is devoted to some preparations for playing a hyperplane potential game on U defined in (3.1). Hence we will fix $\beta \in (0,1)$ and $\gamma > 0$, and a closed ball $B_0 \in \mathscr{B}$ in this section. We are going to define subfamilies $\mathscr{B}_n$ ($n \geq 0$) of $\mathscr{B}$ and decompositions of $\mathbb{Q}^d$ with respect to given β, γ, and B_0.

Firstly, denote

$$\kappa := \max_{(\mathbf{x},y,\mathbf{z})\in B_0} \max\{\|\mathbf{x}\|_\infty, |y|, \|\mathbf{z}\|_\infty\} + 1.$$

Then choose a positive number R satisfying

$$R \geq \max\{4\beta^{-1}, 10^4 d^6\kappa^4\}, \quad \text{and } (R^{\gamma}-1)^{-1} \leq \left(\frac{\beta^2}{3}\right)^{\gamma}, \tag{4.1}$$

and set

$$\epsilon = 10^{-3}d^{-6}\kappa^{-3}R^{-20d^2}\rho_0, \tag{4.2}$$

where $\rho_0 = \rho(B_0)$.

Let $\mathscr{B}_0 = \{B_0\}$. For $n \geq 1$, let $\mathscr{B}_n$ be the subfamily of $\mathscr{B}$ defined by

$$\mathscr{B}_n := \{B \subset B_0 : \beta R^{-n}\rho_0 < \rho(B) \leq R^{-n}\rho_0\}.$$

In view of (4.1), the families $\mathscr{B}_n$ are mutually disjoint.

Let $n \geq 0$, and fix a closed ball $B \in \mathscr{B}_n$ in this paragraph. We define

$$\mathcal{V}_B := \left\{P \in \mathbb{Q}^d : H_n \leq H_B(P) \leq 2H_{n+1}\right\},$$

where

$$H_n = 2d^2\epsilon\kappa\rho_0^{-1}R^{n+1}.$$

It follows from (3.5) that if $P \in \mathcal{V}_B$, then

$$H_n^{\frac{1}{1+\lambda}} \le q(P) \le 2H_{n+1}.$$

We shall also need the following subdivisions of $\mathcal{V}_B$:

$$\mathcal{V}_{B,1} := \left\{ P \in \mathcal{V}_B : H_n^{\frac{1}{1+\lambda}} \le q(P) \le H_n^{\frac{1}{1+\lambda}} R^{10d^2} \right\},$$

$$\mathcal{V}_{B,k} := \left\{ P \in \mathcal{V}_B : H_n^{\frac{1}{1+\lambda}} R^{10d^2+(2k-4)d} \le q(P) \le H_n^{\frac{1}{1+\lambda}} R^{10d^2+(2k-2)d} \right\}, \quad k \ge 2.$$

One can show an important inequality here: for $P \in \mathcal{V}_{B,k}, k \ge 2$,

$$\frac{\xi(B,P)}{q(P)^\lambda} = \frac{H_B(P)}{q(P)^{1+\lambda}} \le \frac{2H_{n+1}}{H_n R^{(1+\lambda)(10d^2+(2k-4)d)}} \le 2R^{-8d^2-2kd+1}. \tag{4.3}$$

Now we define a subfamily $\mathscr{B}'_n$ of $\mathscr{B}_n$ inductively as follows. Let $\mathscr{B}'_0 = \{B_0\}$. If $n \ge 1$ and $\mathscr{B}'_{n-1}$ has been defined, we let

$$\mathscr{B}'_n := \left\{ B \in \mathscr{B}_n : B \subset B' \text{ for some } B' \in \mathscr{B}'_{n-1}, \text{ and } B \cap \bigcup_{P \in \mathcal{V}_B} \Delta_\epsilon(P) = \varnothing \right\}.$$

The following lemma plays an important role in the proof of Theorem 3.1.

Lemma 4.1. *Let $n \ge 0$, and let $B \in \mathscr{B}'_n$. Then for any $P \in \mathbb{Q}^d$ with $q(P)^{1+\lambda} \le 2H_{n+1}$, we have $\Delta_\epsilon(P) \cap B = \varnothing$.*

Proof. Note that $2H_1 < 1$, and hence we may assume that $n \ge 1$. We denote $B_n = B$, and let $B_n \subset \cdots \subset B_0$ be such that $B_k \in \mathscr{B}'_k$. Assume to the contrary that the conclusion of the lemma is not true. Then there exists $P = (\frac{\mathbf{p}}{q}, \frac{r}{q}) \in \mathbb{Q}^d$ with $q^{1+\lambda} \le 2H_{n+1}$ such that $\Delta_\epsilon(P) \cap B_k \ne \varnothing$ for every $1 \le k \le n$. It then follows from the definition of $\mathscr{B}'_k$ that $P \notin \mathcal{V}_{B_k}$, that is,

$$H_{B_k}(P) \notin [H_k, 2H_{k+1}] \quad \forall 1 \le k \le n. \tag{4.4}$$

Let $1 \le n_0 \le n$ be such that

$$2H_{n_0} < q^{1+\lambda} \le 2H_{n_0+1}. \tag{4.5}$$

We claim that

$$H_{B_k}(P) < H_k, \ \forall 1 \le k \le n_0. \tag{4.6}$$

We prove the above claim inductively as follows. Since $H_{B_{n_0}}(P) \le q^{1+\lambda} \le 2H_{n_0+1}$, it follows from (4.4) that (4.6) holds for $k = n_0$. Suppose that $1 \le k \le n_0 - 1$ and (4.6) holds if k is replaced by $k+1$. We prove that

$$H_{B_k}(P) \le 2H_{B_{k+1}}(P). \tag{4.7}$$

Denote $\mathbf{a}^+(B_i, P) = (\mathbf{a}_i, b_i), \mathbf{z}_{B_i} = \mathbf{z}_i (i = k, k+1)$. We claim that

$$\mathbf{a}^+(B_{k+1}, P) \in \mathscr{A}(B_k, P). \tag{4.8}$$

Since $\mathbf{a}^+(B_{k+1}, P) \in \mathscr{A}(B_{k+1}, P)$, it is clear that

$$(\mathbf{a}_{k+1}, b_{k+1}) \ne (\mathbf{0}, 0), \ \mathbf{a}_{k+1} \cdot \mathbf{p} + b_{k+1} r + c_{k+1} q = 0, \ \|\mathbf{a}_{k+1}\|_\infty \le q^\lambda.$$

On the other hand, it follows from (4.5) and the induction hypothesis that

$$\begin{aligned}
(4.9) \qquad & |b_{k+1} + \mathbf{z}_k \cdot \mathbf{a}_{k+1}| \\
\leq & |b_{k+1} + \mathbf{z}_{k+1} \cdot \mathbf{a}_{k+1}| + |(\mathbf{z}_k - \mathbf{z}_{k+1}) \cdot \mathbf{a}_{k+1}| \\
\leq & |b_{k+1} + \mathbf{z}_{k+1} \cdot \mathbf{a}_{k+1}| + d\|\mathbf{a}_{k+1}\|_\infty \rho(B_k) \\
\leq & q^\mu + \rho(B_{k+1})^{\frac{1}{2}} + dq^{-1} H_{B_{k+1}}(P)\rho(B_k) \\
\leq & q^\mu + (\beta R)^{-\frac{1}{2}} \rho(B_k)^{\frac{1}{2}} + dH_{n_0}^{-\frac{1}{2}} H_{k+1}\rho(B_k) \\
\leq & q^\mu + \frac{1}{2}\rho(B_k)^{\frac{1}{2}} + d(2d^2\epsilon\kappa R^2)^{\frac{1}{2}} \rho(B_k)^{\frac{1}{2}} \qquad \text{(by (4.1) and } B_k \in \mathscr{B}_k) \\
\leq & q^\mu + \rho(B_k)^{\frac{1}{2}} \qquad \text{(by (4.2))}.
\end{aligned}$$

This proves our claim (4.8). It then follows from (4.8) and (3.4) that

$$\begin{aligned}
H_{B_k}(P) &= q \max\{\|\mathbf{a}_k\|_\infty, |b_k + \mathbf{z}_k \cdot \mathbf{a}_k|\} \\
&\leq q \max\{\|\mathbf{a}_{k+1}\|_\infty, |b_{k+1} + \mathbf{z}_k \cdot \mathbf{a}_{k+1}|\} \\
&\leq q \max\{\|\mathbf{a}_{k+1}\|_\infty, |b_{k+1} + \mathbf{z}_{k+1} \cdot \mathbf{a}_{k+1}| + d\|\mathbf{a}_{k+1}\|_\infty \rho(B_k)\} \\
&\leq 2q \max\{\|\mathbf{a}_{k+1}\|_\infty, |b_{k+1} + \mathbf{z}_{k+1} \cdot \mathbf{a}_{k+1}|\} \\
&= 2H_{B_{k+1}}(P).
\end{aligned}$$

Thus (4.7) holds. It follows from (4.7) and the induction hypothesis that $H_{B_k}(P) \leq 2H_{k+1}$. By (4.4), we have $H_{B_k}(P) < H_k$. Thus claim (4.6) follows. This means that $H_{B_1}(P) < H_1 < 1$, a contradiction. This completes the proof. ☐

5. Proof of Theorem 3.1

At first, we prove the following proposition which plays a key role in the proof of Theorem 3.1.

Proposition 5.1. *Fix $\beta \in (0,1)$ and $\gamma > 0$, and a closed ball $B_0 \in \mathscr{B}$ as in Subsection 4. Let R be a positive number satisfying (4.1) and ϵ given by (4.2). For $n \geq 0$, $B \in \mathscr{B}'_n$, and $k \geq 1$, consider the set*

$$\mathscr{C}_{B,k,\epsilon} = \Big\{(B', P) \in \mathscr{B} \times \mathbb{Q}^d : B' \in \mathscr{B}_{n+k},\ B' \subset B,\ P \in \mathcal{V}_{B',k},\ \text{and } \Delta_\epsilon(P) \cap B \neq \varnothing\Big\}.$$

Then there exists an affine hyperplane $E_k(B) \subset \mathbb{R}^{2d-1}$ such that for any $(B', P) \in \mathscr{C}_{B,k,\epsilon}$, we have

$$\Delta_\epsilon(P) \cap B' \subset E_k(B)^{(R^{-(n+k)}\rho_0)}.$$

We shall need the following two lemmas.

Lemma 5.2. *Let $(B_1, P_1), (B_2, P_2) \in \mathscr{C}_{B,k,\epsilon}$, and F_{B_2,P_2} be the function defined in (3.2). Then one has*

$$|F_{B_2,P_2}(P_1)| \leq 30 d^4 \kappa^2 \epsilon q_1^{-1} R^{e_k + k + 2} \tag{5.1}$$

with

$$e_k = \begin{cases} 10d^2, & k = 1; \\ 2d, & k > 1. \end{cases}$$

Proof. Write $P_j = (\frac{\mathbf{p}_j}{q_j}, \frac{r_j}{q_j})$ and let $(\mathbf{x}_j, y_j, \mathbf{z}_j) \in \Delta_\epsilon(P_j) \cap B, j = 1, 2$. Then

$$\left\| \mathbf{x}_j - \frac{\mathbf{p}_j}{q_j} - \left(y_j - \frac{r_j}{q_j} \right) \mathbf{z}_j \right\|_\infty < \frac{\epsilon}{q_j^{1+\lambda}}, \quad \left| y_j - \frac{r_j}{q_j} \right| < \frac{\epsilon}{q_j^{1+\mu}}.$$

The latter inequality implies that

$$\left| \frac{r_j}{q_j} \right| \leq |y_j| + \frac{\epsilon}{q_j^{1+\mu}} \leq \kappa.$$

One has

$$\begin{aligned}
&\left\| \frac{\mathbf{p}_1}{q_1} - \frac{\mathbf{p}_2}{q_2} - \left(\frac{r_1}{q_1} - \frac{r_2}{q_2} \right) \mathbf{z}_{B_2} \right\|_\infty \\
&= \Big\| - \left(\mathbf{x}_1 - \frac{\mathbf{p}_1}{q_1} - \left(y_1 - \frac{r_1}{q_1} \right) \mathbf{z}_1 \right) + \left(\mathbf{x}_2 - \frac{\mathbf{p}_2}{q_2} - \left(y_2 - \frac{r_2}{q_2} \right) \mathbf{z}_2 \right) + (\mathbf{x}_1 - \mathbf{x}_2) \\
&\quad + \frac{r_1}{q_1}(\mathbf{z}_1 - \mathbf{z}_{B_2}) + \frac{r_2}{q_2}(\mathbf{z}_{B_2} - \mathbf{z}_2) + (y_1 \mathbf{z}_1 - y_2 \mathbf{z}_2) \Big\| \\
&\leq \left\| \mathbf{x}_1 - \frac{\mathbf{p}_1}{q_1} - \left(y_1 - \frac{r_1}{q_1} \right) \mathbf{z}_1 \right\|_\infty + \left\| \mathbf{x}_2 - \frac{\mathbf{p}_2}{q_2} - \left(y_2 - \frac{r_2}{q_2} \right) \mathbf{z}_2 \right\|_\infty \\
&\quad + \|\mathbf{x}_1 - \mathbf{x}_2\|_\infty + \left\| \frac{r_1}{q_1}(\mathbf{z}_1 - \mathbf{z}_{B_2}) \right\|_\infty + \left\| \frac{r_2}{q_2}(\mathbf{z}_{B_2} - \mathbf{z}_2) \right\|_\infty + \|y_1 \mathbf{z}_1 - y_2 \mathbf{z}_2\|_\infty \\
&\leq \frac{\epsilon}{q_1^{1+\lambda}} + \frac{\epsilon}{q_2^{1+\lambda}} + 10\kappa\rho(B)
\end{aligned}$$

and

$$\left| \frac{r_1}{q_1} - \frac{r_2}{q_2} \right| = \left| - \left(y_1 - \frac{r_1}{q_1} \right) + \left(y_2 - \frac{r_2}{q_2} \right) + (y_1 - y_2) \right| \leq \frac{\epsilon}{q_1^{1+\mu}} + \frac{\epsilon}{q_2^{1+\mu}} + 2\rho(B).$$

As $F_{B_2,P_2}(P_2) = 0$, it follows that

$$\begin{aligned}
&|F_{B_2,P_2}(P_1)| \\
&= \left| \mathbf{a}_2 \cdot \left(\frac{\mathbf{p}_1}{q_1} - \frac{\mathbf{p}_2}{q_2} \right) + b_2 \left(\frac{r_1}{q_1} - \frac{r_2}{q_2} \right) \right| \\
&= \left| \mathbf{a}_2 \cdot \left(\frac{\mathbf{p}_1}{q_1} - \frac{\mathbf{p}_2}{q_2} - \left(\frac{r_1}{q_1} - \frac{r_2}{q_2} \right) \mathbf{z}_{B_2} \right) + (b_2 + \mathbf{z}_{B_2} \cdot \mathbf{a}_2) \left(\frac{r_1}{q_1} - \frac{r_2}{q_2} \right) \right| \\
&\leq d\|\mathbf{a}_2\|_\infty \left(\frac{\epsilon}{q_1^{1+\lambda}} + \frac{\epsilon}{q_2^{1+\lambda}} + 10\kappa\rho(B) \right) + |b_2 + \mathbf{z}_{B_2} \cdot \mathbf{a}_2| \left(\frac{\epsilon}{q_1^{1+\mu}} + \frac{\epsilon}{q_2^{1+\mu}} + 2\rho(B) \right) \\
&\leq d q_2^\lambda \left(\frac{\epsilon}{q_1^{1+\lambda}} + \frac{\epsilon}{q_2^{1+\lambda}} \right) + 2 q_2^\mu \left(\frac{\epsilon}{q_1^{1+\mu}} + \frac{\epsilon}{q_2^{1+\mu}} \right) \\
&\quad + 12 d\kappa\rho(B) \max\{\|\mathbf{a}_2\|_\infty, |b_2 + \mathbf{z}_{B_2} \cdot \mathbf{a}_2|\} \\
&\leq d\epsilon q_1^{-1} \left(\frac{q_2^\lambda}{q_1^\lambda} + \frac{q_1}{q_2} + 2\frac{q_2^\mu}{q_1^\mu} + 2\frac{q_1}{q_2} \right) + 12 d\kappa R^{-n} \rho_0 q_2^{-1} H_{B_2}(P_2) \\
&\leq 6 d\epsilon q_1^{-1} R^{e_k} + 48 d^3 \epsilon \kappa^2 q_2^{-1} R^{k+2} \qquad (\text{by } P_1 \in \mathcal{V}_{B_1,k} \text{ and } P_2 \in \mathcal{V}_{B_2,k}) \\
&\leq 30 d^4 \kappa^2 \epsilon q_1^{-1} R^{e_k+k+2}.
\end{aligned}$$

□

Lemma 5.3. *For any $(B_1, P_1), (B_2, P_2) \in \mathscr{C}_{B,k,\epsilon}$, we have $F_{B_2,P_2}(P_1) = 0$.*

Proof. For simplicity, we write the objects $\mathbf{a}^+(B_j,P_j)$, $\mathbf{v}^+(B_j,P_j)$, ξ_{B_j,P_j}, F_{B_j,P_j}, $\mathcal{L}_{B_j,P_j}$, $\mathcal{H}_{B_j,P_j}$ $(j=1,2)$ as $\mathbf{a}_j^+$, $\mathbf{v}_j^+$, ξ_j, F_j, $\mathcal{L}_j$, $\mathcal{H}_j$, respectively. There are three cases:

(1) *Case* $k=1$. Then by (5.1), we have

$$q_1|F_2(P_1)|\le 30d^4\kappa^2\epsilon R^{e_k+k+2}<1.$$

As $q_1|F_2(P_1)|\in\mathbb{Z}$, we have $F_2(P_1)=0$.

(2) *Case* $k\ge 2$ and $\mathcal{L}_1$ parallel to $\mathcal{H}_2$, that is,

$$\mathbf{a}_2^+\cdot\mathbf{v}_1^+=0. \tag{5.2}$$

Assume to the contrary that $F_2(P_1)\neq 0$. Write $\mathbf{v}_1^+=(\mathbf{v}_1,u_1)=(v_{1,1},\dots,v_{d-1,1},u_1)$. We claim that

$$q_1\big|F_2(P_1)v_{i,1}\big|,\ q_1\big|F_2(P_1)u_1\big|\in\mathbb{Z}\text{ for each }1\le i\le d-1. \tag{5.3}$$

Indeed, since $\mathbf{v}_1^+\in\Lambda_{P_1}\smallsetminus\{\mathbf{0}\}$, we can write

$$\mathbf{v}_1^+=c\left(\frac{\mathbf{p}_1}{q_1},\frac{r_1}{q_1}\right)+\mathbf{c}, \tag{5.4}$$

where $c\in\mathbb{Z}$, $\mathbf{c}\in\mathbb{Z}^d$. Combining (5.2) and (5.4), we get

$$cF_2(P_1)\in\mathbb{Z}. \tag{5.5}$$

According to (5.4), $q_1v_{i,1},q_1u_1\in c\mathbb{Z}+q_1\mathbb{Z}$. Then claim (5.3) follows directly from (5.5).

Note that $\mathbf{v}_1^+\neq\mathbf{0}$. It follows from (5.3) that

$$q_1\Big|F_2(P_1)\Big|\Big(\sum_{1\le i\le d-1}|v_{i,1}|+|u_1|\Big)\ge 1.$$

However, according to (3.6), (4.3), and (5.1), we have

$$\begin{aligned}
&q_1\Big|F_2(P_1)\Big|\Big(\sum_{1\le i\le d-1}|v_{i,1}|+|u_1|\Big)\\
&\le q_1\Big|F_2(P_1)\Big|\Big(\sum_{1\le i\le d-1}|v_{i,1}-\mathbf{z}_{B_1}u_1|+(1+(d-1)\|\mathbf{z}_{B_1}\|_\infty)|u_1|\Big)\\
&\le q_1\Big|F_2(P_1)\Big|\Big(2d(d-1)q_1^{-\lambda}+2d^2\kappa\xi_1q_1^{-\lambda-\mu}\Big)\qquad\text{(by (3.6))}\\
&\le 120d^6\kappa^3\epsilon R^{e_k+k+2}\xi_1q_1^{-\lambda}\qquad\text{(by (5.1))}\\
&\le 240d^6\kappa^3\epsilon R^{2d+k+2}R^{-8d^2-2kd+1}\qquad\text{(by (4.3))}\\
&<1,
\end{aligned}$$

which leads to a contradiction.

(3) *Case* $k\ge 2$ and $\mathcal{L}_1$ intersects $\mathcal{H}_2$.

Assume to the contrary that $F_2(P_1)\neq 0$. Let

$$P_0=\frac{\mathbf{p}_0^+}{q_0}=\left(\frac{\mathbf{p}_0}{q_0},\frac{r_0}{q_0}\right)$$

be the intersection of $\mathcal{L}_1$ and $\mathcal{H}_2$. Write

$$\frac{\mathbf{p}_0^+}{q_0}=\frac{\mathbf{p}_1^+}{q_1}+t_0\mathbf{v}_1^+.$$

Then $\left(\frac{\mathbf{p}_0^+}{q_0}, t_0\right)^T$ is the solution of the linear equations

$$\begin{pmatrix} q_1 I_d & -q_1 \mathbf{v}_1^{+T} \\ \mathbf{a}_2^+ & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ t \end{pmatrix} = \begin{pmatrix} \mathbf{p}_1^{+T} \\ C_2 \end{pmatrix},$$

where $\mathbf{v}_1^{+T}, \mathbf{p}_1^{+T}$ means the transpose of $\mathbf{v}_1^+, \mathbf{p}_1^+$ and where $C_2 = C(B_2, P_2)$ is defined in (3.3). Let M be the matrix

$$\begin{pmatrix} q_1 I_d & -q_1 \mathbf{v}_1^{+T} & \mathbf{p}_1^{+T} \\ \mathbf{a}_2^+ & 0 & C_2 \end{pmatrix}$$

and let M_i $(1 \leq i \leq d+2)$ be the matrix obtained by deleting the ith column of M. In view of the fact that $\mathbf{v}_1^+ \in \Lambda_{P_1}$, a simple computation immediately implies

$$\det(M_i) \in q_1^{d-1}\mathbb{Z} \quad (1 \leq i \leq d+2). \tag{5.6}$$

By Cramer's rule,

$$\left(\frac{\mathbf{p}_0^+}{q_0}, t_0\right) = \left(\frac{\det(M_1)}{\det(M_{d+2})}, \ldots, \frac{\det(M_{d+1})}{\det(M_{d+2})}\right). \tag{5.7}$$

Hence

$$|t_0| = \left|\frac{\det(M_{d+1})}{\det(M_{d+2})}\right| = \frac{|F_2(P_1)|}{|\mathbf{a}_2^+ \cdot \mathbf{v}_1^+|}. \tag{5.8}$$

In view of (5.6) and (5.7), we have

$$q_0 \leq q_1^{-d+1} |\det(M_{d+2})| = q_1 |\mathbf{a}_2^+ \cdot \mathbf{v}_1^+|. \tag{5.9}$$

According to (3.4), (3.6) and (4.3), we have

$$\begin{aligned}
&|\mathbf{a}_2^+ \cdot \mathbf{v}_1^+| \\
&\leq |\mathbf{a}_2 \cdot (\mathbf{v}_1 - u_1 \mathbf{z}_{B_1})| + |u_1(\mathbf{a}_2 \cdot \mathbf{z}_{B_2} + b_2)| + |u_1 \mathbf{a}_2 \cdot (\mathbf{z}_{B_1} - \mathbf{z}_{B_2})| \\
&\leq \sum_{1 \leq i \leq d-1} \left|\xi_2 \cdot 2d q_1^{-\lambda}\right| + \left|2d\xi_1 q_1^{-\lambda-\mu} \cdot (q_2^{\mu} + 1)\right| \\
&\quad + \left|2d\xi_1 q_1^{-\lambda-\mu} \cdot \xi_2 \cdot 2dR^{-n}\rho_0\right| \qquad \text{(by (3.4) and (3.6))} \\
&\leq 4d(d-1)R^{e_k+1}\xi_1 q_1^{-\lambda} + 4dR^{e_k}\xi_1 q_1^{-\lambda} + 4d^2\xi_1 q_1^{-\lambda}\xi_2 R^{-n} \\
&\leq 4d^2 R^{e_k+1}\xi_1 q_1^{-\lambda}(1 + 2q_2^{-1}R^{k+2}) \\
&\leq 12d^2 R^{e_k+1}\xi_1 q_1^{-\lambda} \qquad (\text{by } P_1 \in \mathcal{V}_{B_1,k}) && (5.10)\\
&\leq 24d^2 R^{-8d^2-(2k-2)d+2} \qquad \text{(by (4.3))}. && (5.11)
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{q_0}{q_1} &\leq |\mathbf{a}_2^+ \cdot \mathbf{v}_1^+| \\
&\leq 24d^2 R^{-8d^2-(2k-2)d+2} \\
&\leq \frac{1}{2}. && (5.12)
\end{aligned}$$

By combining the inequalities (4.3), (5.9), (5.10) and the obvious estimate $\lambda \geq 1/d$, we have

$$\begin{aligned}
q_0^{1+\lambda} &\leq q_1^{1+\lambda}|\mathbf{a}_2^+ \cdot \mathbf{v}_1^+|^{1+\lambda} \qquad \text{(by (5.9))} \\
&\leq q_1^{1+\lambda}(12d^2R^{e_k+1}\xi_1 q_1^{-\lambda})^{1+\lambda} \qquad \text{(by (5.10))} \\
&\leq 144d^4R^{6d}(\xi_1^\lambda q_1^{-\lambda^2})H_{B_1}(P_1) \\
&\leq 288d^4R^{6d}R^{-8d-2k+1}2R^{k+1}H_n \qquad \text{(by (4.3) and } \lambda \geq 1/d) \\
&\leq 600d^4R^{-2d-k+2}H_n \\
&\leq H_n. \qquad (5.13)
\end{aligned}$$

Note that

$$\left\| \frac{\mathbf{p}_1}{q_1} - \frac{\mathbf{p}_0}{q_0} - \left(\frac{r_1}{q_1} - \frac{r_0}{q_0}\right)\mathbf{z}_{B_1} \right\|_\infty = |t_0|\,\|\mathbf{v}_1 - \mathbf{z}_{B_1}u_1\|_\infty \leq \frac{2d|t_0|}{q_1^\lambda}$$

and

$$\left|\frac{r_1}{q_1} - \frac{r_0}{q_0}\right| = |t_0u_1| \leq \frac{2d|t_0|\xi_1}{q_1^{\lambda+\mu}}.$$

We claim that

$$\Delta_\epsilon(P_1) \cap B \subset \Delta_\epsilon(P_0). \qquad (5.14)$$

In view of Lemma 4.1 and (5.13), (5.14) will contradict the assumption that $B \in \mathscr{B}_n'$. It remains to prove (5.14). Indeed, for $(\mathbf{x}, y, \mathbf{z}) \in \Delta_\epsilon(P_1) \cap B$, by (4.3), (5.1), (5.8), (5.9), and (5.12) we have

$$\begin{aligned}
q_0^{1+\mu}\left|y - \frac{r_0}{q_0}\right| &\leq q_0^{1+\mu}\left|y - \frac{r_1}{q_1}\right| + q_0^{1+\mu}\left|\frac{r_1}{q_1} - \frac{r_0}{q_0}\right| \\
&\leq q_0^{1+\mu}\frac{\epsilon}{q_1^{1+\mu}} + q_0^{1+\mu}\frac{2d|t_0|\xi_1}{q_1^{\lambda+\mu}} \\
&\leq \frac{\epsilon}{2} + 2dq_1|F_2(P_1)|\frac{\xi_1}{q_1^\lambda} \qquad \text{(by (5.8), (5.9), and (5.12))} \\
&\leq \frac{\epsilon}{2} + 120d^5\kappa^2R^{2d+k+3-8d^2-2kd}\epsilon \qquad \text{(by (4.3) and (5.1))} \\
&\leq \epsilon
\end{aligned}$$

and

$$\begin{aligned}
&q_0^{1+\lambda}\left\|\mathbf{x}-\frac{\mathbf{p}_0}{q_0}-\left(y-\frac{r_0}{q_0}\right)\mathbf{z}\right\|_\infty \\
&\le q_0^{1+\lambda}\left\|\mathbf{x}-\frac{\mathbf{p}_1}{q_1}-\left(y-\frac{r_1}{q_1}\right)\mathbf{z}\right\|_\infty + q_0^{1+\lambda}\left\|\frac{\mathbf{p}_1}{q_1}-\frac{\mathbf{p}_0}{q_0}-\left(\frac{r_1}{q_1}-\frac{r_0}{q_0}\right)\mathbf{z}\right\|_\infty \\
&\le q_0^{1+\lambda}\frac{\epsilon}{q_1^{1+\lambda}} + q_0^{1+\lambda}\frac{2d|t_0|}{q_1^{\lambda}} + q_0^{1+\lambda}\left|\frac{r_1}{q_1}-\frac{r_0}{q_0}\right|\|\mathbf{z}-\mathbf{z}_{B_1}\|_\infty \\
&\le q_0^{1+\lambda}\frac{\epsilon}{q_1^{1+\lambda}} + 2d|q_1F_2(P_1)|\cdot\frac{q_0^\lambda}{q_1^\lambda} \\
&\quad + 2d|q_1F_2(P_1)|\cdot\frac{q_0^\lambda}{q_1^\lambda}\cdot 2R^{-n}\rho_0\frac{q_1\xi_1}{q_1^{1+\mu}} \qquad \text{(by (5.8) and (5.9))} \\
&\le \frac{\epsilon}{2} + 2d|q_1F_2(P_1)|\cdot\frac{q_0^\lambda}{q_1^\lambda}\cdot\left(1+2R^{-n}q_1^{-1}H_{B_1}(P_1)\right) \qquad \text{(by (5.12))} \\
&\le \frac{\epsilon}{2} + 60d^5\kappa^2R^{2d+k+2}\epsilon\cdot\frac{q_0^\lambda}{q_1^\lambda}\cdot(1+4q_1^{-1}R^{k+2}) \qquad \text{(by (5.1) and } P_1\in\mathcal{V}_{B_1,k}) \\
&\le \frac{\epsilon}{2} + 300d^5\kappa^2R^{2d+k+2}\left(24d^2R^{-8d^2-(2k-2)d+2}\right)^{\frac{1}{d}} \qquad \text{(by (4.3) and } \lambda>1/d) \\
&\le \frac{\epsilon}{2} + 7200d^7\kappa^2R^{-6d+6-k}\epsilon \\
&\le \epsilon.
\end{aligned}$$

□

Proof of Proposition 5.1. Choose $(B_0', P_0)\in\mathscr{C}_{B,k,\epsilon}$ such that

$$q_0 = q(P_0) = \min\left\{q(P) : \exists \text{ closed ball } B' \text{ with } (B',P)\in\mathscr{C}_{B,k,\epsilon}\right\}.$$

Consider the attached hyperplane in $\mathbb{R}^{2d-1}$

$$\mathcal{H}_{B_0',P_0} = \left\{(\mathbf{x},y,\mathbf{z})\in\mathbb{R}^{2d-1} : \mathbf{a}_0\cdot\mathbf{x}+b_0y-C=0\right\},$$

where $\mathbf{a}_0^+=(\mathbf{a}_0,b_0)$ and $C=C(B_0',P_0)$ are given in Subsection 3.2. We claim that $\mathcal{H}_{B_0',P_0}$ is the $E_k(B)$ that we need. In other words, for any $(B',P)\in\mathscr{C}_{B,k,\epsilon}$,

$$\Delta_\epsilon(P)\cap B'\subset\mathcal{H}_{B_0',P_0}^{(R^{-(n+k)}\rho_0)}.$$

Indeed, we have proved in Lemma 5.3 that $P\in\mathcal{H}_{B_0',P_0}$ for $(B',P)\in\mathscr{C}_{B,k,\epsilon}$. Hence for any $(\mathbf{x},y,\mathbf{z})\in\Delta_\epsilon(P)\cap B'$, we have

$$\begin{aligned}
|\mathbf{a}_0\cdot\mathbf{x}+b_0y-C| &= \left|\mathbf{a}_0\cdot\left(\mathbf{x}-\frac{\mathbf{p}}{q}\right)+b_0\left(y-\frac{r}{q}\right)\right| \\
&\le (d-1)\|\mathbf{a}_0\|_\infty\left\|\mathbf{x}-\frac{\mathbf{p}}{q}-\left(y-\frac{r}{q}\right)\mathbf{z}\right\|_\infty + |b_0+\mathbf{z}\cdot\mathbf{a}_0|\left|y-\frac{r}{q}\right| \\
&\le (d-1)q_0^\lambda\frac{\epsilon}{q^{1+\lambda}} + 2q_0^\mu\frac{\epsilon}{q^{1+\mu}} \\
&\le (d+1)\frac{\epsilon}{q_0}.
\end{aligned}$$

Denote the width of this thickened hyperplane as ω. Then

$$\begin{aligned}\omega &\leq \frac{(d+1)\epsilon}{q_0 \max\{\|\mathbf{a}_0\|_\infty, |b_0|\}} \\ &\leq \frac{(d+1)\epsilon}{(1+(d-1)\kappa)^{-1} q_0 \max\{\|\mathbf{a}_0\|_\infty, |b_0 + \mathbf{z}\cdot\mathbf{a}_0|\}} \\ &\leq \frac{(d+1)(1+(d-1)\kappa)\epsilon}{H_{n+k}} \\ &\leq R^{-(n+k)}\rho_0,\end{aligned}$$

which finishes the proof. □

Proof of Theorem 3.1. In view of Remark 3.3, Lemma 2.3 and Lemma 3.4, to prove Theorem 3.1, it suffices to show that the set $S(\mathbf{r})$ is (β,γ)-HPW for any $\beta \in (0,1)$, $\gamma > 0$. Fix $\beta \in (0,1)$ and $\gamma > 0$ from now on. Bob starts the (β,γ)-hyperplane potential game on $\mathbb{R}^{2d-1}$ with target set $S(\mathbf{r})$ by choosing a closed ball $B_0 \subset \mathbb{R}^{2d-1}$ of radius ρ_0. As discussed in [3, Remark 2.4], without loss of generality we may assume that Bob will play so that $\rho_0 \leq 1/d$ and $\rho_i := \rho(B_i) \to 0$, where B_i is the ball chosen by Bob at the ith turn. Now we have that $\beta, \gamma > 0$, and B_0 satisfy the conditions of Proposition 5.1. Let R be a positive number satisfying (4.1) and let ϵ be the constant given by (4.2). Write i_n to be the smallest nonnegative integer with $B_{i_n} \in \mathscr{B}_n$. Let $\mathcal{N}$ denote the set of all $n \in \mathbb{N}$ with $B_{i_n} \in \mathscr{B}'_n$.

Let Alice play according to the strategy as follows. At the ith stage, if $i = i_n$ for some $n \in \mathcal{N}$, then Alice chooses the family of hyperplane neighborhoods $\{E_k(B_{i_n})^{(3R^{-(n+k)}\rho_0)} : k \in \mathbb{N}\}$, where the hyperplane $E_k(B_{i_n})$ is given by Proposition 5.1. Otherwise, Alice makes an empty move. Since $B_{i_n} \in \mathscr{B}_n$, it follows that $\rho_{i_n} > \beta R^{-n}\rho_0$. Hence Alice's move is legal as we have

$$\sum_{k=1}^{\infty} (3R^{-(n+k)}\rho_0)^\gamma = (3R^{-n}\rho_0)^\gamma (R^\gamma - 1)^{-1} \leq (\beta\rho_{i_n})^\gamma$$

by (4.1). We claim that this is a winning strategy for Alice, that is, the point $\mathbf{x}_\infty = \bigcap_{i=0}^{\infty} B_i$ lies in the set

$$S(\mathbf{r}) \cup \bigcup_{n\in\mathcal{N}} \bigcup_{k=1}^{\infty} E_k(B_{i_n})^{(3R^{-(n+k)}\rho_0)}.$$

There are two cases.

(1) *Case* $\mathcal{N} = \mathbb{N}\cup\{0\}$. For any $P \in \mathbb{Q}^d$, there is n such that $q^{1+\lambda} \leq 2H_{n+1}$. Since $n \in \mathcal{N}$, we have $B_{i_n} \in \mathscr{B}'_n$. Then we have $\Delta_\epsilon(P) \cap B_{i_n} = \varnothing$ by Lemma 4.1. Thus it follows from the definition of $S(\mathbf{r})$ that $\mathbf{x}_\infty \in S_\epsilon(\mathbf{r}) \subset S(\mathbf{r})$. Hence Alice wins.

(2) *Case* $\mathcal{N} \neq \mathbb{N}\cup\{0\}$. Let n be the smallest integer with $n \notin \mathcal{N}$. Then we have $B_{i_n} \notin \mathscr{B}'_n$ and $B_{i_{n-1}} \in \mathscr{B}'_{n-1}$ as $n-1 \in \mathcal{N}$. By the definition of $\mathscr{B}'_n$, there exists $P \in \mathcal{V}_{B_{i_n},k}$ with $1 \leq k \leq n$ and $\Delta_\epsilon(P) \cap B_{i_n} \neq \varnothing$. By Proposition 5.1, we have $\Delta_\epsilon(P) \cap B_{i_n} \subset E_k(B_{i_{n-k}})^{(R^{-n}\rho_0)}$. In view of $\rho_{i_n} \leq R^{-n}\rho_0$, it follows that $\mathbf{x}_\infty \in B_{i_n} \subset E_k(B_{i_{n-k}})^{(3R^{-n}\rho_0)}$. Hence Alice wins.

This completes the proof of Theorem 3.1. □

6. Proof of the main theorems

In this section, we deduce Theorem 1.2 and Theorem 1.3 from Theorem 3.1. Indeed, the argument presented here is similar to the argument presented in [3, Section 6]. For the sake of completeness, we reproduce the proof in our setting here.

Proof of Theorem 1.2. The proof is divided into three steps:

Step 1. We show that it suffices to prove that the set $E(F^+)$ is HAW on G/Γ. Indeed, by applying the diffeomorphism

$$\tau : G/\Gamma \to G/\Gamma, \tau(g\Gamma) = (g^T)^{-1}\Gamma$$

to the set $E(F^+)$, we can see that the set $E(F^-)$ is also HAW if $E(F^+)$ is, where F^- denotes the subsemigroup $\{e\} \cup (F \setminus F^+)$. Hence, in view of the intersection stability of HAW sets, $E(F)$ will be HAW if $E(F^+)$ is as well.

Step 2. We show that it suffices to prove the theorem for $F^+ = F_{\mathbf{r}}^+$, which was defined in Theorem 3.1. Indeed, by the real Jordan decomposition (cf. [17, Proposition 4.3.3]), for any one-parameter diagonalizable subsemigroup F^+, there are one-parameter subsemigroups $F_i^+ = \{g_t^{(i)} : t > 0\}$ $(i = 1, 2)$ such that F_1^+ is $\mathbb{R}$-diagonalizable, F_2^+ has compact closure, and $g_t = g_t^{(1)} g_t^{(2)}$ with $g_t^{(1)}$ commuting with $g_t^{(2)}$. It is obvious that $E(F^+) = E(F_1^+)$, and as the eigenvalues of $g_t^{(2)}$ are of absolute 1, it follows that F_1^+ satisfies (1.1) if F^+ does. Hence we are reduced to consider the case where F satisfies (1.1) and is $\mathbb{R}$-diagonalizable, which is equivalent to saying that there exists $g' \in G$ and $\mathbf{r}$ satisfying (1.3) such that $F^+ = g' F_{\mathbf{r}}^+ g'^{-1}$. Note that in this case we have $E(F^+) = g' E(F_{\mathbf{r}}^+)$. Hence our statement follows from (3) of Lemma 2.2.

Step 3. We prove the theorem for $F_{\mathbf{r}}^+$. In view of Lemma 2.2, we have to prove that for any $\Lambda \in G/\Gamma$, there is an open neighborhood Ω of Λ in G/Γ such that $\Omega \cap E(F_{\mathbf{r}}^+)$ is HAW on Ω. Let

$$P = \Bigg\{ g \in G : \\ g = \begin{pmatrix} T & \mathbf{0} \\ N & T' \end{pmatrix}, T \in \mathrm{GL}_{d-1}(\mathbb{R}), N \in M_{2\times(d-1)}(\mathbb{R}), T' \text{ is lower triangular} \Bigg\}.$$

It is not hard to check that for any $g \in P$, the set $\{g_t g g_t^{-1} : t > 0\}$ is bounded in G. Consider the Bruhat decomposition of G viewed as the $\mathbb{R}$-point of an $\mathbb{R}$-split group [4, Theorem 21.15]. The set $G - PU$ is Zariski closed[1] by [4, Theorem 21.26]. Hence the set PU is nonempty and Zariski open in G. Moreover, the multiplication map $P \times U \to PU$ is a diffeomorphism since it is an algebraic isomorphism.

According to the Borel density theorem, the set $\pi^{-1}(\Lambda)$ is Zariski dense in G. Hence, $\pi^{-1}(\Lambda) \cap PU \neq \varnothing$; that is, there exists $p_0 \in P$ and $u_0 \in U$ such that $\Lambda = p_0 u_0 \Gamma$.

Let Ω_P and Ω_U be open neighborhoods of p_0 and u_0 in P and U, respectively, which are small enough such that the map $\phi : \Omega_P \times \Omega_U \to G/\Gamma, \phi(p, u) = pu\Gamma$ is a

[1] Here, the Zariski topology means the topology induced from the Zariski topology of $\mathrm{SL}_{d+1}(\mathbb{C})$.

diffeomorphism onto an open subset Ω in G/Γ. In view of Lemma 2.2(4), it suffices to prove that the set

$$\phi^{-1}(E(F_{\mathbf{r}}^+)\cap\Omega) = \{(p,u)\in\Omega_P\times\Omega_U : pu\Gamma\in E(F_{\mathbf{r}}^+)\} \tag{6.1}$$

is HAW on $\Omega_P\times\Omega_U$. By the definition of P, we have that $pu\Gamma\in E(F_{\mathbf{r}}^+)$ if and only if $u\Gamma\in E(F_{\mathbf{r}}^+)$. It follows that the set (6.1) is equal to

$$\Omega_P\times\{u\in\Omega_U : u\Gamma\in E(F_{\mathbf{r}}^+)\}.$$

Then it follows from Theorem 3.1 and (5) of Lemma 2.2 that the set $E(F_{\mathbf{r}}^+)$ is HAW. □

Proof of Theorem 1.3. We will prove the theorem only for $F^+ = F_{\mathbf{r}}^+$ with $\mathbf{r}$ satisfying (1.3) here, since the proof for general F^+ satisfying (1.1) follows along the same lines as Step 2 of the proof of Theorem 1.2 and will be omitted. There are two cases:

(1) *Case* $r_1 > r_d$. Then it is easy to check that $H(F^+)$ is equal to U. We need to prove that for any $\Lambda\in G/\Gamma$, the set $u\in U$ such that $u\Lambda\in E(F^+)$ is HAW on U. In view of Lemma 2.2, it suffices to prove that for any $u_0\in U$, there is an open neighborhood Ω of u_0 in U such that the set

$$\{u\in\Omega : u\Lambda\in E(F^+)\} \tag{6.2}$$

is HAW on Ω. Similar to the proof of Theorem 1.2, the Bruhat decompostion and the Borel density theorem imply that $\pi^{-1}(\Lambda)\cap u_0^{-1}PU\neq\varnothing$. Choose $g_0\in\pi^{-1}(\Lambda)\cap u_0^{-1}PU$. Then $\Lambda = g_0\Gamma$ and $u_0g_0\in PU$. Let Ω_1 be an open neighborhood of u_0 in U with $\Omega_1 g_0\subset PU$. Then there are smooth maps $\phi:\Omega_1\to P$ and $\psi:\Omega_1\to U$ such that

$$ug_0 = \phi(u)\psi(u) \quad \forall u\in\Omega_1. \tag{6.3}$$

We claim that

$$\text{the tangent map } (d\psi)_{u_0} \text{ is a linear isomorphism.} \tag{6.4}$$

It follows from claim (6.4) that the set (6.2) is HAW. Indeed, assuming (6.4), we can find a neighborhood $\Omega\subset\Omega_1$ such that ψ is a diffeomorphism when restricted on Ω. Note that for $u\in\Omega$, the set

$$F_{\mathbf{r}}^+u\Lambda = F_{\mathbf{r}}^+ug_0\Gamma = F_{\mathbf{r}}^+\phi(u)\psi(u)\Gamma$$

is bounded if and only if $F_{\mathbf{r}}^+\psi(u)\Lambda$ is bounded. Hence, in view of Theorem 3.1 and Lemma 2.2, we prove that the set (6.2) is HAW modulo claim (6.4).

Let's turn to the proof of claim (6.4). Write $p_1=\phi(u), u_1=\psi(u)$. Then it follows from (6.3) that

$$dr_{g_0}(Y) = dr_{u_1}\circ(d\phi)_{u_0}(Y) + dl_{p_1}\circ(d\psi)_{u_0}(Y) \quad \forall Y\in T_{u_0}U, \tag{6.5}$$

where r_u (resp., l_u) denotes the map defined by multiplying u on the right (resp., on the left) on G. If $(d\psi)_{u_0}(Y)=0$, then the left-hand side of (6.5) belongs to $T_{u_0g_0}(Uu_0g_0)$ and right-hand side belongs to $T_{u_0g_0}(Pu_0g_0)$; thus $Y=0$. This proves claim (6.4).

(2) *Case* $r_1 = r_d = \frac{1}{d}$. In this case, the expanding horospherical subgroup H coincides with the subgroup U_0 defined as

$$U_0 := \{u_{\mathbf{x}} : \mathbf{x}\in\mathbb{R}^d\}, \quad \text{where } u_{\mathbf{x}} = \begin{pmatrix} Id & \mathbf{x} \\ 0 & 1\end{pmatrix}\in G.$$

In view of the correspondence presented in [9, Theorem 2.20], the set $\{x \in \mathbb{R}^d : u_{\mathbf{x}}\Gamma \in E(F^+)\}$ coincides with the set of badly approximable vectors $\mathbf{Bad}_d$, which is proved to be HAW already in [7]. Then we omit the remaining part of the proof here, since it is similar to the proof of the above case $r_1 > r_d$. □

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