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CONTINUITY OF THE SOLUTION MAP OF THE EULER EQUATIONS IN HÖLDER SPACES AND WEAK NORM INFLATION IN BESOV SPACES

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Abstract. We construct an example showing that the solution map of the Euler equations is not continuous in the Hölder space from $C^{1,\alpha}$ to $L^\infty_t C^{1,\alpha}_x$ for any $0 < \alpha < 1$. On the other hand we show that it is continuous when restricted to the little Hölder subspace $c^{1,\alpha}$. We apply the latter to prove an ill-posedness result for solutions of the vorticity equations in Besov spaces near the critical space $B^1_{2,1}$. As a consequence we show that a sequence of best constants of the Sobolev embedding theorem near the critical function space is not continuous.

1. Introduction

We study the Cauchy problem for the Euler equations of an incompressible and inviscid fluid

$$
\begin{aligned}
&u_t + u \cdot \nabla u = -\nabla p, \qquad t \geq 0,\ x \in \mathbb{R}^n,\\
&\operatorname{div} u = 0,\\
&u(0) = u_0,
\end{aligned}
\tag{1.1}
$$

where $u = u(t,x)$ and $p = p(t,x)$ denote the velocity field and the pressure function of the fluid, respectively. The first rigorous results for (1.1) were proved in the framework of Hölder spaces by Gyunter [20], Lichtenstein [29] and Wolibner [38]. More refined results using a similar functional setting were obtained subsequently by Kato [23], Swann [35], Bardos and Frisch [2], Ebin [16], Chemin [10], Constantin [12], and Majda and Bertozzi [30], among others. The main focus in these papers was on existence and uniqueness of $C^{1,\alpha}$ solutions, and the question of continuity with respect to initial conditions was not explicitly addressed. Recall that, according to the definition of Hadamard, a Cauchy problem is said to be locally well-posed in a Banach space X if for any initial data in X there exists a unique solution which persists for some $T > 0$ in the space $C([0,T), X)$ and which depends continuously on the data. Otherwise the problem is said to be ill-posed. It was pointed out by Kato [24] that this notion of well-posedness is rather strong and may not be suitable for certain problems studied in the literature. Instead, it is frequently required that the solution persist in a larger space such as $L^\infty([0,T), X)$ or $C_w([0,T), X)$ (the subscript w indicates weak continuity in the time variable).

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Systematic studies of ill-posedness of the Cauchy problem (1.1) are of a more recent date and concern a wide range of phenomena including gradual loss of regularity of the solution map, energy dissipation, and nonuniqueness of weak solutions. See, e.g. Yudovich [39], Koch [28], Morgulis, Shnirelman and Yudovich [33], Eyink [18], Constantin, E and Titi [13], or Shnirelman [34]. Recently, Bardos and Titi [3] found examples of solutions in Hölder spaces C^α and the Zygmund space $B^1_{\infty,\infty}$ which exhibit an instantaneous loss of smoothness in the spatial variable for any $0 < \alpha < 1$. Similar examples in logarithmic Lipschitz spaces logLip^α were given by the authors in [31]. In another direction Cheskidov and Shvydkoy [11] constructed periodic solutions that are discontinuous in time at $t = 0$ in the Besov spaces $B^s_{p,\infty}$ where $s > 0$ and $2 < p \leq \infty$. In particular, it follows that the Cauchy problem (1.1) is not well-posed in the sense of Hadamard in $C([0,T), B^s_{p,\infty})$, although it is known that the corresponding solution map is well defined in $L^\infty([0,T), B^s_{p,\infty})$; see for instance [1, Chap. 7]. More recently, in a series of papers Bourgain and Li [7, 8] constructed smooth solutions which exhibit instantaneous blowup in borderline spaces such as $W^{n/p+1,p}$ for any $1 \leq p < \infty$ and $B^{n/p+1}_{p,q}$ for any $1 \leq p < \infty$ and $1 < q \leq \infty$ as well as in the standard spaces C^k and $C^{k-1,1}$ for any integer $k \geq 1$; see also Elgindi and Masmoudi [17] and [32]. As observed in [8] the cases C^k and $C^{k-1,1}$ are particularly intriguing in view of the classical existence and uniqueness results mentioned above.

One of our goals in this paper is to revisit the picture of local well-posedness in the sense of Hadamard for the Euler equations in Hölder spaces. We present a simple example based on a DiPerna–Majda type shear flow which shows that in general the data-to-solution map of (1.1) is not continuous into the space $L^\infty([0,T), C^{1,\alpha})$ for any $0 < \alpha < 1$. On the other hand, we show that continuity of this map is restored (in the strong sense) if the Cauchy problem is restricted to the so-called little Hölder space $c^{1,\alpha}$. The failure of continuity in our example does not seem to be related to the mechanism described in [8] which essentially relies on unboundedness of the double Riesz transform in L^∞. Rather, it can be explained by the fact that smooth functions are not dense in the standard $C^{1,\alpha}$ spaces. This phenomenon should be compared with the results of [11] where, however, as mentioned above, the ill-posedness mechanism is different and with those of [21] where it is shown that the solution map cannot be uniformly continuous in the Sobolev space H^s with $s > 0$. We point out that continuity of the solution map for the Euler equations in Sobolev spaces $W^{s,p}$ for $p \geq 2$ and $s > 2/p + 2$ is of course well known (see, e.g., Ebin and Marsden [15], Kato and Lai [25] or Kato and Ponce [26]); see also the Appendix. However, we could not find the corresponding result for $c^{1,\alpha}$ in the literature although it should be familiar to the experts in the field.

Theorem 1.1. *The solution map of the incompressible Euler equations* (1.1) *is not continuous as a map from* $C^{1,\alpha}(\mathbb{R}^3)$ *to* $L^\infty([0,T), C^{1,\alpha}(\mathbb{R}^3))$ *for any* $0 < \alpha < 1$.

Theorem 1.2. *The incompressible Euler equations* (1.1) *are locally well-posed in the sense of Hadamard in the little Hölder space* $c^{1,\alpha}(\mathbb{R}^n)$ *for any* $0 < \alpha < 1$ *and* $n = 2$ *or* 3.

In particular, Theorem 1.2 implies that the solution map is continuous from bounded subsets of $c^{1,\alpha}(\mathbb{R}^n)$ to $C([0,T), c^{1,\alpha}(\mathbb{R}^n))$.

Remark 1.3. Although in the present paper we are primarily concerned with the local-in-time problem, a few comments on global well-posedness are in order. In light of Theorem 1.2 it is natural to expect that Wolibner's global result can be reformulated as a global Hadamard well-posedness result in $c^{1,\alpha}$. In fact, the proof in Section 4 below shows that if u_0 is in $c^{1,\alpha}$, then the corresponding particle trajectories $\eta(t)$ will retain their $c^{1,\alpha}$ regularity for any finite interval of time (they will remain in the set $\mathscr{U}_\delta$ for some suitably chosen $\delta > 0$; cf. (2.7)–(2.8) below) provided that the flow can be continued in $C^{1,\alpha}$. The latter however is guaranteed by the Beale–Kato–Majda criterion (see, e.g., [30, Theorem 4.3]) because vorticity is conserved along particle trajectories in 2D.

Remark 1.4. The proof of Theorem 1.1 is based on a local property of $C^{1,\alpha}$ to construct a counterexample. It would be also interesting to find an explicit counterexample in the Besov space framework $B^{1+\alpha}_{\infty,\infty}$ such that $\inf_{\ell\in\mathbb{Z}_+} 2^{(1+\alpha)\ell}\|\hat{\psi}_\ell * u_0\|_{L^\infty} > 0$ (nondecaying property on the Fourier side).

As an application of Theorem 1.2 we prove an ill-posedness result for the vorticity equations that involves a family of Besov spaces. Although this result is weaker than instantaneous blowup described by Bourgain and Li, our methods can be applied in the borderline end-point spaces such as $B^2_{2,1}(\mathbb{R}^2)$ which lie just outside the range of the spaces considered in [7]. Recall that existence and uniqueness results for (1.1) in $B^2_{2,1}(\mathbb{R}^2)$ are already known; cf., e.g., Vishik [37] or Chae [9]. The proof uses continuity of the data-to-solution map in $c^{1,\alpha}$ as well as several technical lemmas proved in our earlier paper [32]. In this respect the present paper can be viewed as a continuation of [32].

Theorem 1.5. *Let $M_j \nearrow \infty$ be an increasing sequence of positive numbers. There exist a sequence of smooth rapidly decaying initial data $\{\tilde{u}_{0,j}\}_{j=1}^\infty$ and two sequences of indices $\{r_j\}_{j=1}^\infty$ and $\{q_j\}_{j=1}^\infty$ with $r_j \to 2$ and $q_j \to 1$ such that*

$$\|\tilde{u}_{0,j}\|_{B^2_{r_j,q_j}} \lesssim 1 \quad \textit{and} \quad \|\tilde{u}_j(t)\|_{B^2_{r_j,q_j}} > M_j \quad \textit{for some } 0 < t < M_j^{-3}.$$

Remark 1.6. It is of interest to compare this result with the a priori estimates for local solutions in Besov norms. If we let $C_j > 0$ denote the best constant in the Besov version of the Sobolev embedding

$$\|\nabla\tilde{u}_j\|_\infty \le C_j\|\tilde{u}_j\|_{B^2_{r_j,q_j}} \tag{1.2}$$

(cf. Chae [9, Rem. 2.1]), then Theorem 1.5 implies that $C_j \nearrow \infty$ or else we get a contradiction with the standard bound $\sup_{0\le t\le T_j}\|\tilde{u}_j(t)\|_{B^2_{r_j,q_j}} \le C_{0,j}\|\tilde{u}_{0,j}\|_{B^2_{r_j,q_j}}$ where $C_{0,j}$ depends on C_j. On the other hand, for $1 \le p < \infty$ and $1 \le q \le \infty$, we also have the estimate

$$\|\nabla\tilde{u}_\infty\|_\infty \le C_\infty\|\tilde{u}_\infty\|_{B^{1+2/p}_{p,q}} \tag{1.3}$$

for some finite constant $C_\infty > 0$ if and only if $q = 1$ (see [36, Theorem 11.4, p. 170]). This suggests that the dependence of the constants in (1.2) and (1.3) on the Besov parameters is not continuous. To the best of our knowledge this dependence has not been investigated in the literature.

In the next section we recall the basic setup and notation. In Section 3 we prove Theorem 1.1 by constructing a shear flow counterexample in the $C^{1,\alpha}$ space. Local Hadamard well-posedness in $c^{1,\alpha}$ is shown in Section 4. The proof of Theorem 1.5 is given in Section 5.

2. The basic setup: Function spaces and diffeomorphisms

Let $\psi_0 \in \mathscr{S}(\mathbb{R}^n)$ be any function of Schwartz class satisfying $0 \le \psi_0 \le 1$ and $\operatorname{supp}\psi_0 \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$ and such that

$$\sum_{l\in\mathbb{Z}} \psi_l(\xi) = 1, \quad \text{for any } \xi \neq 0,$$

where $\psi_l(\xi) = \psi_0(2^{-l}\xi)$. For any $s > 0$ and $1 \le p, q \le \infty$ let $B^s_{p,q}(\mathbb{R}^n)$ denote the inhomogeneous Besov space equipped with the norm

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}, \tag{2.1}$$

where the homogeneous seminorm is given by

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{l\in\mathbb{Z}} 2^{slq}\|\widehat{\psi}_l * f\|^q_{L^p}\right)^{1/q} & \text{if} \quad 1 \le q < \infty, \\ \sup_{l\in\mathbb{Z}} 2^{sl}\|\widehat{\psi}_l * f\|_{L^p} & \text{if} \quad q = \infty \end{cases} \tag{2.2}$$

for any $f \in \mathscr{S}'(\mathbb{R}^n)$. In particular, if $s = k + \alpha$ is not an integer, then $B^s_{\infty,\infty}$ is the Hölder space $C^{k,\alpha}(\mathbb{R}^n)$ with the standard norm

$$\|\varphi\|_{k,\alpha} = \|\varphi\|_{C^k} + [D^k\varphi]_\alpha,$$

where

$$[D^k\varphi]_\alpha = \sum_{|\beta|=k} \sup_{x\neq y} \frac{|D^\beta\varphi(x) - D^\beta\varphi(y)|}{|x-y|^\alpha}, \qquad 0 < \alpha < 1,\ k \in \mathbb{N}.$$

Let $c^{k,\alpha}(\mathbb{R}^n)$ denote the closed subspace of $C^{k,\alpha}(\mathbb{R}^n)$ consisting of those functions whose derivatives satisfy the vanishing condition

$$\lim_{h\to 0} \sup_{0<|x-y|<h} \frac{|D^\beta\varphi(x) - D^\beta\varphi(y)|}{|x-y|^\alpha} = 0 \tag{2.3}$$

for any multi-index $|\beta| = k$. It is well known that $c^{k,\alpha}(\mathbb{R}^n)$ is an interpolation space containing the smooth functions as a dense subspace; cf., e.g., [5].

In what follows we will use an alternative formulation of the fluid equations in terms of particle trajectories and vorticity. Any sufficiently smooth velocity field u solving (1.1) has a flow which traces out a curve $t \to \eta(t,x)$ of diffeomorphisms starting at the identity configuration $e(x) = x$ with initial velocity u_0. Using the incompressibility constraint $\det D\eta(t,x) = 1$ and the Biot–Savart law the equations satisfied by the flow can be written in the form

$$\begin{aligned} &\frac{d\eta}{dt}(t,x) = \int_{\mathbb{R}^n} K_n\big(\eta(t,x) - \eta(t,y)\big)\omega(t,\eta(t,y))\,dy, \qquad t \ge 0,\ x \in \mathbb{R}^n, \\ &\eta(0,x) = x, \end{aligned} \tag{2.4}$$

where $\omega = \operatorname{curl} u$ is the vorticity[1] and the kernel K_n is given by

$$K_2(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right), \qquad x \in \mathbb{R}^2, \tag{2.5}$$

[1]If $n = 2$ we can identify the vorticity of u with the function $\omega = \nabla^\perp \cdot u$, and if $n = 3$ with the vector field $\omega = \nabla \times u$.

and

$$K_3(x)y = \frac{1}{4\pi}\frac{x \times y}{|x|^3}, \qquad x, y \in \mathbb{R}^3. \tag{2.6}$$

For our purposes it will be sufficient to take as a configuration space of the fluid the set of those diffeomorphisms of $\mathbb{R}^n$ which differ from the identity by a function of class $c^{1,\alpha}$. Let

$$\mathscr{U}_\delta = \Big\{\eta : \mathbb{R}^n \to \mathbb{R}^n : \eta = e + \varphi_\eta,\ \varphi_\eta \in c^{1,\alpha}(\mathbb{R}^n) \text{ and } \|\varphi_\eta\|_{1,\alpha} < \delta\Big\}, \tag{2.7}$$

where $\delta > 0$ is chosen small enough so that

$$\inf_{x\in\mathbb{R}^n} \det D\eta(x) > \frac{1}{2}. \tag{2.8}$$

Clearly, $\mathscr{U}_\delta$ can be identified with an open ball centered at the origin in $c^{1,\alpha}(\mathbb{R}^n)$. The next two lemmas collect some elementary properties of compositions and inversions of diffeomorphisms in $\mathscr{U}_\delta$ that will be used in Section 4.

Lemma 2.1. *Let $0 < \alpha < 1$. Suppose that η and ξ are in $\mathscr{U}_\delta$ and $\psi \in c^{1,\alpha}(\mathbb{R}^n)$. Then $\psi \circ \eta$ and $\psi \circ \eta^{-1}$ are also of class $c^{1,\alpha}$, and we have*

$$\|\psi \circ \eta\|_{1,\alpha} \lesssim C\|\psi\|_{1,\alpha} \quad \textit{and} \quad \|\psi \circ \eta^{-1}\|_{1,\alpha} \lesssim C\|\psi\|_{1,\alpha}, \tag{2.9}$$

where $C > 0$ depends only on δ and α. Furthermore, we have that $\xi \circ \eta - e$ and $\eta^{-1} - e$ are also of class $c^{1,\alpha}$.

Proof. First, observe that $\|\psi \circ \eta\|_\infty = \|\psi\|_\infty$ and $\|D(\psi \circ \eta)\|_\infty = \|D\psi\|_\infty\|D\eta\|_\infty$, and therefore the first of the inequalities in (2.9) follows at once from

$$\begin{aligned} [D(\psi \circ \eta)]_\alpha &\le [D\psi \circ \eta]_\alpha\|D\eta\|_\infty + \|D\psi \circ \eta\|_\infty[D\eta]_\alpha \\ &\le [D\psi]_\alpha\|D\eta\|_\infty^{1+\alpha} + \|D\psi\|_\infty[D\eta]_\alpha \end{aligned} \tag{2.10}$$

and $[D\eta]_\alpha = [D\varphi_\eta]_\alpha$, where $\eta = e + \varphi_\eta$ with $\|\varphi_\eta\|_{1,\alpha} < \delta$. Similarly, we have $\|\psi\circ\eta^{-1}\|_\infty = \|\psi\|_\infty$, and from (2.8) and $D\eta^{-1} = (D\eta)^{-1} \circ \eta^{-1}$ we get

$$\|D(\psi \circ \eta^{-1})\|_\infty \lesssim \|D\psi\|_\infty\|D\eta\|_\infty$$

and

$$[(D\eta)^{-1}]_\alpha = \big[(\det D\eta)^{-1}\mathrm{adj}(D\eta)\big]_\alpha \lesssim (1 + \|D\eta\|_\infty^2)[D\eta]_\alpha \tag{2.11}$$

which, in turn, with the help of (2.10) yields

$$[D(\psi \circ \eta^{-1})]_\alpha \le [D\psi]_\alpha\|D\eta\|_\infty^{1+\alpha} + \|D\psi\|_\infty(1 + \|D\eta\|_\infty^2)[D\eta]_\alpha. \tag{2.12}$$

From these bounds we obtain the second of the inequalities in (2.9).

Finally, observe that if $\xi \in \mathscr{U}_\delta$ with $\xi = e + \varphi_\xi$, then $\xi \circ \eta = e + \varphi_{\xi\circ\eta}$, where $\varphi_{\xi\circ\eta} = \varphi_\eta + \varphi_\xi \circ \eta$. Therefore, using (2.9) we get

$$\|\varphi_{\xi\circ\eta}\|_{1,\alpha} \lesssim \|\varphi_\eta\|_{1,\alpha} + \|\varphi_\xi\|_{1,\alpha},$$

and combining (2.9) with (2.10) and the vanishing condition (2.3) we conclude that $\varphi_{\xi\circ\eta} \in c^{1,\alpha}(\mathbb{R}^n)$. Similarly, we also have $\eta^{-1} = e + \varphi_{\eta^{-1}}$, where $\varphi_{\eta^{-1}} = -\varphi_\eta \circ \eta^{-1}$. Applying the second of the estimates in (2.9) together with (2.12) and (2.3), we find again that $\varphi_{\eta^{-1}} \in c^{1,\alpha}(\mathbb{R}^n)$. □

Lemma 2.2. *Let $0 < \alpha < 1$. Suppose that η, ξ, and ζ are in $\mathscr{U}_\delta$. Then*

$$\|\xi \circ \eta - \zeta \circ \eta\|_{1,\alpha} \lesssim C\|\varphi_\xi - \varphi_\zeta\|_{1,\alpha}, \tag{2.13}$$

and for any $\psi \in c^{2,\alpha}(\mathbb{R}^n)$ we have

$$\|\psi \circ \eta - \psi \circ \xi\|_{1,\alpha} \lesssim C\|\psi\|_{2,\alpha}\|\varphi_\eta - \varphi_\xi\|_{1,\alpha}, \tag{2.14}$$

where $C > 0$ depends only on δ and α. Furthermore, the functions $\xi, \eta \to \xi \circ \eta$ and $\eta \to \eta^{-1}$ are continuous in the Hölder norm topology.

Proof. From the estimates of Lemma 2.1 we obtain as before

$$\begin{aligned}\|\xi \circ \eta - \zeta \circ \eta\|_{1,\alpha} &\lesssim \|\varphi_\xi - \varphi_\zeta\|_\infty + \|D\eta\|_\infty \|D(\varphi_\xi - \varphi_\zeta)\|_\infty \\ &\quad + [D\eta]_\alpha \|D(\varphi_\xi - \varphi_\zeta)\|_\infty + \|D\eta\|_\infty^{1+\alpha}[D(\varphi_\xi - \varphi_\zeta)]_\alpha \\ &\lesssim \Big(1 + \|D\eta\|_\infty + \|D\eta\|_\infty^{1+\alpha} + [D\eta]_\alpha\Big)\|\varphi_\xi - \varphi_\zeta\|_{1,\alpha},\end{aligned}$$

which implies the estimate in (2.13). On the other hand, using (2.9) and the algebra property of Hölder functions we have

$$\|\psi \circ \eta - \psi \circ \xi\|_{1,\alpha} \le \int_0^1 \|D\psi\big(r\eta + (1-r)\xi\big)(\eta - \xi)\|_{1,\alpha} dr \lesssim \|D\psi\|_{1,\alpha}\|\eta - \xi\|_{1,\alpha},$$

which gives (2.14) since $\eta - \xi = \varphi_\eta - \varphi_\xi$. From (2.13) and (2.14) we conclude that composition of diffeomorphisms in $\mathscr{U}_\delta$ is continuous with respect to ξ and η.

Finally, using the second of the inequalities in (2.9) we have

$$\|\xi^{-1} - \eta^{-1}\|_{1,\alpha} \lesssim \|\xi^{-1} \circ \eta - e\|_{1,\alpha}.$$

By density, given any $\varepsilon > 0$ pick a smooth $\zeta : \mathbb{R}^n \to \mathbb{R}^n$ such that $\|\zeta - \xi^{-1}\|_{1,\alpha} < \varepsilon$ and estimate the above expression further by

$$\|\xi^{-1} \circ \eta - \zeta \circ \eta\|_{1,\alpha} + \|\zeta \circ \eta - \zeta \circ \xi\|_{1,\alpha} + \|\zeta \circ \xi - \xi^{-1} \circ \xi\|_{1,\alpha}.$$

The first and the third of these terms can be bounded using the first inequality in (2.9) by $C\varepsilon$. For the middle term we use (2.14) to bound it by $\|\zeta\|_{2,\alpha}\|\eta - \xi\|_{1,\alpha}$. □

3. A 3D shear flow in $C^{1,\alpha}$

In this section we prove Theorem 1.1 by constructing a $C^{1,\alpha}$ shear flow for which the data-to-solution map of (1.1) fails to be continuous. Shear flow solutions were introduced in [14]. They were recently used in [3] to exhibit instantaneous loss of smoothness of the Euler equations in C^α for any $0 < \alpha < 1$.

Proof of Theorem 1.1. Let $t \ge 0$ and consider

$$u(t,x) = \big(f(x_2), 0, h(x_1 - tf(x_2))\big) \quad \text{and} \quad v(t,x) = \big(g(x_2), 0, h(x_1 - tg(x_2))\big),$$

where f, g, and h are bounded real-valued functions of one variable of class $C^{1,\alpha}$ with any $0 < \alpha < 1$. It is not difficult to verify that both u and v satisfy the Euler equations with initial conditions

$$u_0(x) = \big(f(x_2), 0, h(x_1)\big) \quad \text{and} \quad v_0(x) = \big(g(x_2), 0, h(x_1)\big).$$

Given any $\varepsilon > 0$ we can arrange it so that f and g satisfy

$$\|u_0 - v_0\|_{1,\alpha} = \|f - g\|_{1,\alpha} < \varepsilon$$

and then choose h such that

$$h'(x_1) = |x_1|^\alpha \qquad \text{for all} \quad -2a \le x_1 \le 2a,$$

where $a = \max\big\{\|f\|_\infty, \|g\|_\infty\big\}$ and assume that $f(x) \ne g(x)$ on $(-a, a)$.

Next, we estimate the norm of the difference of the corresponding solutions. For any $0 < t \leq 1$ we have

$$\begin{aligned}\|u(t) - v(t)\|_{1,\alpha} &= \|f - g\|_{1,\alpha} + \big\|h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot))\big\|_{1,\alpha} \\ &\geq \big\|\nabla\big(h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot))\big)\big\|_{0,\alpha} \\ &= \big\|h'(\cdot - tf(\cdot)) - h'(\cdot - tg(\cdot))\big\|_{0,\alpha}.\end{aligned}$$

It is clear that the norm on the right hand side can be bounded below by

$$\sup_{\substack{x \neq y \\ x,y \in [-a,a]^2}} \frac{\Big|\big(|x_1 - tf(x_2)|^\alpha - |x_1 - tg(x_2)|^\alpha\big) - \big(|y_1 - tf(y_2)|^\alpha - |y_1 - tg(y_2)|^\alpha\big)\Big|}{|x-y|^\alpha}.$$

Evaluating this expression at $x_2 = y_2 = c$ with $-a < c < a$ we get a further estimate from below by

$$\sup_{\substack{x_1 \neq y_1 \\ x,y \in [-a,a]^2}} \frac{|(|x_1 - tf(c)|^\alpha - |x_1 - tg(c)|^\alpha) - (|y_1 - tf(c)|^\alpha - |y_1 - tg(c)|^\alpha)|}{|x_1 - y_1|^\alpha},$$

and evaluating once again at the points $x_1 = tg(c)$ and $y_1 = tf(c)$ we obtain a final lower bound

$$\geq \frac{t^\alpha |g(c) - f(c)|^\alpha + t^\alpha |f(c) - g(c)|^\alpha}{t^\alpha |f(c) - g(c)|^\alpha} = 2.$$

Since this inequality holds for all t in an open interval, we conclude that the essential supremum of the norm $\|u(t) - v(t)\|_{1,\alpha}$ is bounded away from zero, which proves Theorem 1.1. ☐

4. Local well-posedness in $c^{1,\alpha}$

We turn to the question of well-posedness of (1.1) in the sense of Hadamard. As mentioned in the Introduction, local existence and uniqueness results in Hölder spaces are well known and our contribution here concerns only the continuity property of the solution map. To this end we will make some adjustments in the approach based on the particle-trajectory method of [30]. We first state Theorem 1.2 more precisely as follows.

Theorem 4.1. *Let $0 < \alpha < 1$. For any divergence-free vector field $u_0 \in c^{1,\alpha}(\mathbb{R}^n)$ with compactly support vorticity there exist $T > 0$ and a unique solution u of* (1.1) *such that the map $u_0 \to u$ is continuous from $c^{1,\alpha}(\mathbb{R}^n)$ to $C([0,T), c^{1,\alpha}(\mathbb{R}^n))$.*

Proof of Theorem 4.1. We will concentrate on the two-dimensional case since the arguments in the three-dimensional case are very similar (the necessary modifications will be described below). We begin by constructing the Lagrangian flow as a unique solution of an ordinary differential equation in $\mathscr{U}_\delta$.

Since in two dimensions the vorticity is conserved by the flow,[2] we can rewrite equations (2.4) in the form

$$\begin{aligned}\frac{d\eta}{dt}(t,x) &= \int_{\mathbb{R}^2} K_2\big(\eta(t,x) - \eta(t,y)\big)\omega_0(y)\,dy =: F_{u_0}(\eta_t)(x), \\ \eta(0,x) &= x,\end{aligned} \tag{4.1}$$

[2]That is, $\omega(t, \eta(t,x)) = \omega_0(x)$.

where $\omega_0 = \nabla^\perp \cdot u_0$ and K_2 is given by (2.5). In order to apply Picard's method of successive approximations it is sufficient to show that the right hand side of (4.1) is locally Lipschitz continuous in $\mathscr{U}_\delta$. The first task is thus to establish that F_{u_0} maps into $c^{1,\alpha}$.

Changing variables in the integral we have

$$F_{u_0}(\eta)(x) = (K_2 * \tilde\omega_0) \circ \eta(x), \tag{4.2}$$

where $\tilde\omega_0 = \omega_0 \circ \eta^{-1} \det D\eta^{-1}$ and $\eta \in \mathscr{U}_\delta$. Using (2.5) and (2.8) and estimating directly we obtain

$$\|F_{u_0}(\eta)\|_\infty = \|K_2 * \tilde\omega_0\|_\infty \le C\|\omega_0\|_\infty, \tag{4.3}$$

where C depends on the size of the support of ω_0. Next, differentiating F_{u_0} in (4.2) with respect to the x variable gives

$$\begin{aligned} DF_{u_0}(\eta)(x) &= D(K_2 * \tilde\omega_0) \circ \eta(x) D\eta(x) \\ &= \Big(T\tilde\omega_0 - \frac{1}{2}\tilde\omega_0 J \Big) \circ \eta(x) D\eta(x), \end{aligned} \tag{4.4}$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the standard 2×2 symplectic matrix and T is a singular integral operator of the Calderon–Zygmund type with the matrix kernel $DK_2(x) = \Omega(x)/|x|^2$, where Ω is homogeneous of degree zero

$$Tf(x) = \frac{1}{2\pi}\ p.v. \int_{\mathbb{R}^2} \frac{\Omega(x-y)}{|x-y|^2} f(y)\, dy, \qquad \Omega(x) = |x|^{-2} \begin{pmatrix} 2x_1x_2 & x_2^2 - x_1^2 \\ x_2^2 - x_1^2 & -2x_1x_2 \end{pmatrix}.$$

Standard estimates in Hölder spaces for such operators[3] give

$$\|DF_{u_0}(\eta)\|_\infty \le C\big(\|T\tilde\omega_0 \circ \eta\|_\infty + \|\tilde\omega_0 \circ \eta\|_\infty\big)\|D\eta\|_\infty \le C\big(\|\tilde\omega_0\|_\infty + [\tilde\omega_0]_\alpha\big) \tag{4.5}$$

and

$$\begin{aligned} [DF_{u_0}(\eta)]_\alpha &\le \|\big(T\tilde\omega_0 - 1/2\tilde\omega_0 J\big) \circ \eta\|_\infty [D\eta]_\alpha + \big[\big(T\tilde\omega_0 - 1/2\tilde\omega_0 J\big) \circ \eta\big]_\alpha \|D\eta\|_\infty \\ &\le C\big(\|\tilde\omega_0\|_\infty + [\tilde\omega_0]_\alpha\big)[D\varphi_\eta]_\alpha + C\|D\eta\|_\infty^{1+\alpha}\big[T\tilde\omega_0 - 1/2\tilde\omega_0 J\big]_\alpha \\ &\le C\big(\|\tilde\omega_0\|_\infty + [\tilde\omega_0]_\alpha\big)[D\varphi_\eta]_\alpha + C(1 + \|D\varphi_\eta\|_\infty)^{1+\alpha}[\tilde\omega_0]_\alpha. \end{aligned} \tag{4.6}$$

Furthermore, since from a direct computation using (2.8) we have

$$[\tilde\omega_0]_\alpha = [\omega_0 \circ \eta^{-1} \det D\eta^{-1}]_\alpha \lesssim [D\varphi_\eta]_\alpha \|\omega_0\|_\infty + [\omega_0]_\alpha, \tag{4.7}$$

combining these estimates with (4.3) and the fact that $\eta \in \mathscr{U}_\delta$ we get

$$\|F_{u_0}(\eta)\|_{1,\alpha} \lesssim C\big(\|\omega_0\|_\infty + [\omega_0]_\alpha\big). \tag{4.8}$$

To show that F_{u_0} maps $\mathscr{U}_\delta$ into $c^{1,\alpha}(\mathbb{R}^2)$ it suffices now to observe that (4.6) together with (4.7) yield

$$\lim_{h\to 0} \sup_{0<|x-y|<h} \frac{|DF_{u_0}(\eta)(x) - DF_{u_0}(\eta)(y)|}{|x-y|^\alpha} = 0, \tag{4.9}$$

since both φ_η and ω_0 are in $c^{1,\alpha}(\mathbb{R}^2)$ by assumption.

[3]See, e.g., [30, Chap. 4].

Finally, differentiating F_{u_0} in (4.1) with respect to η in the direction $w \in c^{1,\alpha}(\mathbb{R}^2)$ we obtain

$$\delta_w F_{u_0}(\eta)(x) = \frac{d}{dr} F_{u_0}(\eta + rw)(x)\Big|_{r=0} = \int_{\mathbb{R}^2} DK_2\big(\eta(x) - \eta(y)\big)(w(x) - w(y))\omega_0(y)dy, \tag{4.10}$$

which again can be bounded directly using standard Hölder estimates by

$$\|\delta_w F_{u_0}(\eta)\|_{1,\alpha} \lesssim C\big(\|\omega_0\|_\infty + [\omega_0]_\alpha\big)\|w\|_{1,\alpha}. \tag{4.11}$$

In particular, it follows that F_{u_0} has a bounded Gateaux derivative in $\mathscr{U}_\delta$ and hence is locally Lipschitz by the mean value theorem.

We now turn to the question of dependence of the solutions of (1.1) on u_0. Note that since $\omega_0 = \nabla^\perp \cdot u_0$ the initial velocity u_0 appears as a parameter on the right hand side of (4.1). Moreover, since the dependence is linear it follows that continuity (and, in fact, differentiability) of the map $u_0 \to F_{u_0}$ is an immediate consequence of the estimate in (4.8). Applying the fundamental theorem of ordinary differential equations (with parameters) for Banach spaces we find that there exist $T > 0$ and a unique Lagrangian flow $\eta \in C([0,T), \mathscr{U}_\delta)$ which depends continuously (in fact, differentiably) on u_0. Using the equations in (4.1) we find that the same is true of the time derivative $\dot\eta \in C([0,T), c^{1,\alpha}(\mathbb{R}^2))$. It follows therefore that the vector field $u = \dot\eta \circ \eta^{-1}$ belongs to $C([0,T), c^{1,\alpha}(\mathbb{R}^2)) \cap C^1([0,T), c^{\alpha}(\mathbb{R}^2))$ and a routine calculation shows that it is divergence free.

Next, suppose that u_0 and v_0 are two divergence free vector fields in $c^{1,\alpha}(\mathbb{R}^2)$ and let $\eta(t)$ and $\xi(t)$ be the corresponding Lagrangian flows solving the Cauchy problem (4.1) in $\mathscr{U}_\delta$ with initial vorticities $\nabla^\perp \cdot u_0$ and $\nabla^\perp \cdot v_0$, respectively. Given any $\varepsilon > 0$ and using the fact that smooth functions are dense in $c^{1,\alpha}$, we can choose ϕ_ε in $C^\infty([0,T) \times \mathbb{R}^2)$ such that

$$\sup_{0 \le t \le T} \|\phi_\varepsilon(t) - \dot\eta(t)\|_{1,\alpha} < \varepsilon.$$

Applying this together with (2.9) and (2.14) we estimate

$$\begin{aligned}
\|u - v\|_{1,\alpha} &= \|\dot\eta \circ \eta^{-1} - \dot\xi \circ \xi^{-1}\|_{1,\alpha} \\
&\le \|\dot\eta \circ \eta^{-1} - \phi_\varepsilon \circ \eta^{-1}\|_{1,\alpha} + \|\phi_\varepsilon \circ \eta^{-1} - \phi_\varepsilon \circ \xi^{-1}\|_{1,\alpha} \\
&\quad + \|\phi_\varepsilon \circ \xi^{-1} - \dot\xi \circ \xi^{-1}\|_{1,\alpha} \\
&\lesssim \|\dot\eta - \phi_\varepsilon\|_{1,\alpha} + \|\phi_\varepsilon\|_{2,\alpha}\|\eta^{-1} - \xi^{-1}\|_{1,\alpha} + \|\phi_\varepsilon - \dot\xi\|_{1,\alpha}.
\end{aligned}$$

The first term of the last line is clearly bounded by ε. The middle term converges to zero by Lemma 2.2 (continuity of the inversion map) and the fact that η converges to ξ in $c^{1,\alpha}$ whenever u_0 converges to v_0 since the Lagrangian flows η and ξ depend continuously on the initial velocities. To dispose of the last term we use (4.1) and the triangle inequality

$$\begin{aligned}
\|\phi_\varepsilon - \dot\xi\|_{1,\alpha} &\le \|\phi_\varepsilon - \dot\eta\|_{1,\alpha} + \|\dot\eta - \dot\xi\|_{1,\alpha} \\
&\le \varepsilon + \|F_{u_0}(\eta) - F_{v_0}(\eta)\|_{1,\alpha} + \|F_{v_0}(\eta) - F_{v_0}(\xi)\|_{1,\alpha}.
\end{aligned}$$

Applying (4.8) we obtain

$$\|F_{u_0}(\eta) - F_{v_0}(\eta)\|_{1,\alpha} = \Big\| \int_{\mathbb{R}^2} K_2\big(\eta(t,\cdot) - \eta(t,y)\big)\big(\nabla^\perp \cdot u_0(y) - \nabla^\perp \cdot v_0(y)\big)dy\Big\|_{1,\alpha}$$
$$\lesssim \|\nabla^\perp \cdot (u_0 - v_0)\|_\infty + \big[\nabla^\perp \cdot (u_0 - v_0)\big]_\alpha$$
$$\lesssim \|u_0 - v_0\|_{1,\alpha},$$

and using (4.11) we get

$$\|F_{v_0}(\eta) - F_{v_0}(\xi)\|_{1,\alpha} = \Big\| \int_0^1 \frac{d}{dr} F_{v_0}\big(r\eta + (1-r)\xi\big) dr\Big\|_{1,\alpha}$$
$$\leq \int_0^1 \|\delta_{\eta-\xi} F_{v_0}\big(r\eta + (1-r)\xi\big)\|_{1,\alpha} dr$$
$$\lesssim \|v_0\|_{1,\alpha}\|\eta - \xi\|_{1,\alpha},$$

where the latter converges to zero by continuous dependence of the flows on u_0 and v_0 as before. This completes the proof of Theorem 4.1 when $n = 2$.

In the three-dimensional case the flow equations (2.4) take only a slightly more complicated form,

$$\frac{d\eta}{dt}(t,x) = \int_{\mathbb{R}^3} K_3\big(\eta(t,x) - \eta(t,y)\big) D\eta(t,y)\omega_0(y)\, dy =: G_{u_0}(\eta_t)(x), \tag{4.12}$$
$$\eta(0,x) = x,$$

where $\omega_0 = \nabla \times u_0$, K_3 is given by (2.6), and consequently the derivative $\delta_w G_{u_0}$ in the direction $w \in c^{1,\alpha}(\mathbb{R}^3)$ has an extra term,

$$\delta_w G_{u_0}(\eta)(x) = \int_{\mathbb{R}^3} DK_3\big(\eta(x) - \eta(y)\big)(w(x) - w(y)) D\eta(y)\omega_0(y)\, dy \tag{4.13}$$
$$+ \int_{\mathbb{R}^3} K_3\big(\eta(x) - \eta(y)\big) Dw(y)\omega(y)\, dy.$$

As before, by applying standard Hölderian estimates we obtain the analogues of (4.8) and (4.11), and the proof proceeds as in the two-dimensional case. □

Remark 4.2. Theorem 4.1 remains valid if the initial vorticity has noncompact support and satisfies some suitable decay conditions at infinity. In Section 5 we will apply it under the assumption $\omega_0 \in L^1(\mathbb{R}^2)$. In this case we only have to replace the bound in (4.3) with

$$\|F_{u_0}(\eta)\|_\infty \lesssim \|\tilde\omega_0\|_\infty + \|\tilde\omega_0\|_{L^1} \tag{4.3$'$}$$

and those in (4.5), (4.6) with

$$\|DF_{u_0}(\eta)\|_\infty \lesssim \|\tilde\omega_0\|_{0,\alpha} + \|\tilde\omega_0\|_{L^1}, \tag{4.5$'$}$$
$$[DF_{u_0}(\eta)]_\alpha \lesssim \big(\|\tilde\omega_0\|_{0,\alpha} + \|\tilde\omega_0\|_{L^1}\big)[D\varphi_\eta]_\alpha + \big(1 + \|D\varphi_\eta\|_\infty\big)^{1+\alpha}[\tilde\omega_0]_\alpha \tag{4.6$'$}$$

and adjust the estimates (4.8) and (4.11) accordingly. The rest of the proof remains unchanged.

5. Proof of Theorem 1.5: Weak norm inflation in $B^2_{2,1}$

Our aim in this section is to exhibit a norm inflation type mechanism which involves a family of Besov spaces. On the one hand the result stated in Theorem 1.5 is weaker than the instantaneous blowup results obtained by Bourgain and Li. On the other hand, our method is applicable in the borderline function spaces that were left out of the analysis in [7,8]. The proof involves constructing a Lagrangian flow with a large deformation gradient and a high-frequency perturbation of the corresponding initial vorticity. It also relies on the continuity result for the solution map in the little Hölder space of Section 4.

It will be convenient to work with the vorticity equations which in two dimensions have the form

$$\begin{aligned} &\omega_t + u\cdot\nabla\omega = 0, \qquad\qquad t \geq 0,\ x\in\mathbb{R}^2,\\ &\omega(0) = \omega_0, \end{aligned} \tag{5.1}$$

where $u = \nabla^\perp\Delta^{-1}\omega$ and $\omega = \partial_1 u_2 - \partial_2 u_1$. We first proceed to choose the initial vorticity ω_0 for the Cauchy problem (5.1). Given any smooth radial bump function $0 \leq \phi \leq 1$ with support in the ball $B(0, 1/4)$ let

$$\phi_0(x_1, x_2) = \sum_{\varepsilon_1,\varepsilon_2=\pm1} \varepsilon_1\varepsilon_2\phi(x_1-\varepsilon_1, x_2-\varepsilon_2). \tag{5.2}$$

Clearly, the function ϕ_0 is odd with respect to both x_1 and x_2. Given any $M \gg 1$, $r > 0$, and $q > 0$, define

$$\omega_0(x) = \omega_0^{M,N,r,q}(x) = M^{-2}N^{-\frac{1}{q}} \sum_{0\leq k\leq N} \phi_k(x), \tag{5.3}$$

where $N = 1, 2\ldots$ and $\phi_k(x) = 2^{(-1+\frac{2}{r})k}\phi_0(2^k x)$. Note that the supports of ϕ_k are disjoint and compact with

$$\operatorname{supp}\phi_k \subset \bigcup_{\varepsilon_1,\varepsilon_2=\pm1} B\big((\varepsilon_1 2^{-k}, \varepsilon_2 2^{-k}), 2^{-(k+2)}\big). \tag{5.4}$$

Next, we have the following lemma.

Lemma 5.1. *If $1 < q < \infty$ and $2 < r < \infty$ with $q \leq r$, for any integer $N > 0$ we have*

$$\|\omega_0\|_{W^{1,r}} + \|\omega_0\|_{B^1_{r,q}} \lesssim M^{-2}. \tag{5.5}$$

Proof. We proceed to estimate the two terms on the left hand side of (5.5) separately. Observe that the supports of ϕ_k in (5.4) are disjoint, and therefore changing variables we get

$$\|\omega_0\|^r_{L^r} \simeq M^{-2r}N^{-\frac{r}{q}} \sum_{k=0}^{N} 2^{-kr} \int_{\mathbb{R}^2} |\phi_0(x)|^r dx \lesssim M^{-2r}.$$

Also, since $q \leq r$ we similarly have

$$\|\partial_l\omega_0\|^r_{L^r} \simeq M^{-2r}N^{-\frac{r}{q}} \sum_{k=0}^{N} 2^{-kr} \int_{\mathbb{R}^2} |2^k\partial_l\phi_0(x)|^r dx \lesssim M^{-2r}$$

for $l = 1, 2$, which gives the required bound for the $W^{1,r}$ term.

The estimate of the $B^1_{r,q}$ term is slightly more cumbersome. It will be convenient to work with the Fourier transform of ω_0. In this case the supports of $\hat{\phi}_k$ are no longer disjoint, nevertheless each $\hat{\phi}_k$ can be decomposed into a "bump" part and a decaying "tail" part where the bump parts have disjoint supports. From (2.1) and the calculations above we only need to estimate the homogeneous Besov norm $\|\omega_0\|_{\dot{B}^1_{r,q}}$. We have

$$\hat{\phi}_k(\xi) = 2^{(-3+\frac{2}{r})k}\hat{\phi}_0(2^{-k}\xi)$$

and, since $\hat{\phi}_0$ is a function of rapid decrease, given any $\alpha > 0$ we can find $K_1 > 1$ such that

$$|\hat{\phi}_0(\xi)| \le C|\xi|^{-\alpha} \quad \text{for} \quad |\xi| \ge K_1.$$

Let $\alpha > 3 - 2/r$. In this case we have

$$|\hat{\phi}_k(\xi)| \le 2^{(-3+\frac{2}{r})k}|\hat{\phi}_0(2^{-k}\xi)| \le 2^{(-3+\frac{2}{r}+\alpha)k}|\xi|^{-\alpha} \quad \text{for} \quad |\xi| \ge 2^k K_1.$$

Using the Hausdorff–Young inequality we get

$$\begin{aligned}
\|\omega_0\|^q_{\dot{B}^1_{r,q}} &\le \sum_\ell 2^{\ell q}\|\psi_\ell\,\hat{\omega}_0\|^q_{L^{r'}} \\
&\lesssim M^{-2q}N^{-1}\sum_\ell \Big\|\psi_\ell(\xi)\,|\xi|\sum_{k=0}^N \hat{\phi}_k(\xi)\Big\|^q_{L^{r'}_\xi} \\
&\simeq M^{-2q}N^{-1}\sum_\ell \left(\int_{\mathbb{R}^2}|\psi_\ell(\xi)|^{r'}|\xi|^{r'}\Big|\sum_{k=0}^N\hat{\phi}_k(\xi)\Big|^{r'}d\xi\right)^{q/r'},
\end{aligned}$$

where $1/r + 1/r' = 1$. Given any integers k_1, k_2, and K (with $k_2 \le K$), introduce the functions

$$\Phi^K_{k_1,k_2}(\xi) = \chi_{[2^{k_1},2^{k_1+1}]}(\xi)\,|\xi|\sum_{k=k_2}^K|\hat{\phi}_k(\xi)|.$$

A direct calculation yields

$$\begin{aligned}
\Phi^0_{j-K,-\infty}(2^{-K}\xi) &= \chi_{[2^{j-K},2^{j-K+1}]}(2^{-K}\xi)\,\big|2^{-K}\xi\big|\sum_{k=-\infty}^0\big|\hat{\phi}_k(2^{-K}\xi)\big| \\
&= 2^{-K}\chi_{[2^j,2^{j+1}]}(\xi)\,|\xi|\sum_{k=-\infty}^0 2^{(-3+\frac{2}{r})k}\big|\hat{\phi}_0(2^{-(k+K)}\xi)\big| \\
&= 2^{-K}2^{(3-\frac{2}{r})K}\chi_{[2^j,2^{j+1}]}(\xi)\,|\xi|\sum_{k=-\infty}^K 2^{(-3+\frac{2}{r})k}|\hat{\phi}_0(2^{-k}\xi)| \\
&= 2^{\frac{2K}{r'}}\Phi^K_{j,-\infty}(\xi),
\end{aligned}$$

which leads to the scaling identity

$$\Phi^K_{j,-\infty}(\xi) = 2^{-\frac{2K}{r'}}\Phi^0_{j-K,-\infty}(2^{-K}\xi)$$

for $j > K$; similarly we have

$$\Phi^\infty_{j,-\infty}(\xi) = 2^{-\frac{2j}{r'}}\Phi^\infty_{0,-\infty}(2^{-j}\xi).$$

The above will be needed below in order to control the tail parts (for both high and low frequencies).

Claim.

$$\sum_{j\geq 1}\|\Phi^0_{j,-\infty}\|^q_{L^{r'}}<\infty,\quad \sum_{j<1}\|\Phi^\infty_{j,0}\|^q_{L^{r'}}<\infty\quad \textit{and}\quad \|\Phi^\infty_{0,-\infty}\|^q_{L^{r'}}\lesssim 1.$$

Proof of Claim. We have

$$\begin{aligned}
&|\Phi^\infty_{0,-\infty}(\xi)| \\
&\quad=\chi_{[1,2]}(\xi)\,|\xi|\sum_{-\infty\leq k\leq\infty}|\hat\phi_k(\xi)| \\
&\quad\lesssim \chi_{[1,2]}(\xi)\Bigg(\sum_{-\infty\leq k\leq -\log_2 K_1}+\sum_{-\log_2 K_1\leq k\leq 1}+\sum_{1\leq k\leq\infty}\Bigg)|\hat\phi_k(\xi)| \\
&\quad\lesssim \chi_{[1,2]}(\xi)\Bigg(\sum_{-\infty\leq k\leq -\log_2 K_1}2^{(-3+\frac{2}{r}+\alpha)k}|\xi|^{-\alpha}+\text{finite sum}+\sum_{1\leq k\leq\infty}2^{(-3+\frac{2}{r})k}\Bigg) \\
&\quad\lesssim \chi_{[1,2]}(\xi)
\end{aligned}$$

so that $\|\Phi^\infty_{0,-\infty}\|^q_{L^{r'}}\lesssim 1$.

Next, since $2^kK_1<K_1$ for $k\leq 0$, we have

$$|\Phi^0_{j,-\infty}(\xi)|\lesssim \chi_{[2^j,2^{j+1}]}(\xi)\,|\xi|\sum_{k=-\infty}^{0}2^{(-3+\frac{2}{r}+\alpha)k}|\xi|^{-\alpha}\lesssim \chi_{[2^j,2^{j+1}]}(\xi)\,|\xi|^{-\alpha+1}$$

for $|\xi|>K_1$, and using this estimate we get $\sum_{j>1}\|\Phi^0_{j,-\infty}\|^q_{L^{r'}}<\infty$.

Finally, we have

$$|\Phi^\infty_{j,0}(\xi)|\lesssim \chi_{[2^j,2^{j+1}]}(\xi)\,|\xi|\sum_{k=0}^{\infty}2^{(-3+\frac{2}{r})k}\lesssim \chi_{[2^j,2^{j+1}]}(\xi)\,|\xi|,$$

and so we obtain $\sum_{j<1}\|\Phi^\infty_{j,0}\|^q_{L^{r'}}<\infty$ as before. □

We now return to the proof of Lemma 5.1. Using the fact that the supports are disjoint we have

$$\begin{aligned}
\Bigg|\sum_{k=0}^{N}|\xi|\hat\phi_k(\xi)\Bigg|^{r'}&\leq\sum_j\Bigg|\sum_{k=0}^{N}\chi_{[2^j,2^{j+1}]}(\xi)\,|\xi|\hat\phi_k(\xi)\Bigg|^{r'} \\
&\leq|\Phi^N_{1,0}(\xi)|^{r'}+|\Phi^N_{2,0}(\xi)|^{r'}+\cdots+|\Phi^N_{N,0}(\xi)|^{r'}+\sum_{j>N}|\Phi^N_{j,0}(\xi)|^{r'}+\sum_{j<1}|\Phi^N_{j,0}(\xi)|^{r'} \\
&=I_1(\xi)+I_2(\xi)+I_3(\xi),
\end{aligned}$$

and consequently (note that ψ_ℓ are also essentially disjoint)

$$\begin{aligned}
\|\omega_0\|^q_{\dot B^1_{r,q}}&\lesssim M^{-2q}N^{-1}\sum_\ell\bigg(\int_{\mathbb{R}^2}|\psi_\ell(\xi)|^{r'}I_1(\xi)d\xi\bigg)^{q/r'} \\
&\quad+M^{-2q}N^{-1}\sum_\ell\bigg(\int_{\mathbb{R}^2}|\psi_\ell(\xi)|^{r'}\big(I_2(\xi)+I_3(\xi)\big)d\xi\bigg)^{q/r'}.
\end{aligned}$$

Note that I_1 is a finite sum of bump parts while I_2 and I_3 correspond to the two decaying tails. Using the Claim and the scaling identity together with the formula for $\Phi^K_{k_1,k_2}$ we can estimate the first of the integrals on the right hand side of the expression above by

$$\begin{aligned}\sum_{\ell=1}^{N}\left(\int_{\mathbb{R}^2}|\psi_\ell(\xi)|^{r'}I_1(\xi)d\xi\right)^{q/r'} &\le \sum_{\ell=1}^{N}\left(\int|\psi_\ell(\xi)|^{r'}\Big(|\Phi^\infty_{1,-\infty}(\xi)|^{r'}+|\Phi^\infty_{2,-\infty}(\xi)|^{r'}+\cdots+|\Phi^\infty_{N,-\infty}(\xi)|^{r'}\Big)d\xi\right)^{q/r'}\\ &\lesssim \sum_{\ell=1}^{N}\left(\int|\Phi^\infty_{\ell,-\infty}(\xi)|^{r'}d\xi\right)^{q/r'}\lesssim N,\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{\ell<1,N<\ell}\left(\int_{\mathbb{R}^2}|\psi_\ell(\xi)|^{r'}\big(I_2(\xi)+I_3(\xi)\big)d\xi\right)^{q/r'} &\le \sum_{\ell<1,N<\ell}\left(\int|\psi_\ell(\xi)|^{r'}\Big(\sum_{j>N}|\Phi^N_{j,-\infty}(\xi)|^{r'}+\sum_{j<1}|\Phi^\infty_{j,0}(\xi)|^{r'}\Big)d\xi\right)^{q/r'}\\ &\lesssim \sum_{\ell<1}\left(\int|\Phi^\infty_{\ell,0}(\xi)|^{r'}d\xi\right)^{q/r'}+\sum_{N<\ell}\left(\int|\Phi^N_{\ell,-\infty}(\xi)|^{r'}d\xi\right)^{q/r'}\\ &\lesssim C,\end{aligned}$$

where $C>0$ is independent of N. Combining the above estimates we get

$$\|\omega_0\|_{\dot{B}^1_{r,q}}\lesssim M^{-2},$$

which together with the L^r bound of ω_0 gives the desired bound. ☐

In particular, since $r>2$ it follows from Lemma 5.1 that the associated velocity field $u=\nabla^\perp\Delta^{-1}\omega\in W^{2,r}$ has a C^1 smooth Lagrangian flow $\eta(t)$ obtained by solving the flow equations

$$\frac{d\eta}{dt}(t,x)=u(t,\eta(t,x)),\qquad \eta(0,x)=x. \tag{5.6}$$

Furthermore, it is not difficult to verify that $\eta(t)$ is hyperbolic with a stagnation point at the origin and preserves both x_1 and x_2 axes as well as the odd symmetries of ω_0.

Proposition 5.2. *Given $M\gg 1$ and $1<q<\infty$ we have*

$$\sup_{0\le t\le M^{-3}}\|D\eta(t)\|_\infty > M$$

for any sufficiently large integer $N>0$ in (5.3) *and any $2<r<\infty$ sufficiently close to* 2.

Proof. The proof is a repetition (with obvious adjustments) of that given in [32, Prop. 6], for the special case $q=r>2$; see also [7, Lemma 3.2] for the estimate of $R_{ii}\omega$. It will be omitted. ☐

We will also need the following simple consequence of Gronwall's inequality (see [7, Lemma 4.1] for example).

Lemma 5.3. *If u and $\tilde{u}$ are smooth divergence free vector fields on $\mathbb{R}^2$ and $\eta(t)$ and $\tilde{\eta}(t)$ are the corresponding solutions of* (5.6), *then*

$$\sup_{0\le t\le 1}\|\eta(t)-\tilde{\eta}(t)\|_{C^1}\le C\sup_{0\le t\le 1}\|u(t)-\tilde{u}(t)\|_{C^1},$$

where $C>0$ depends only on the L^∞ norm of u and $\tilde{u}$ and its derivatives. □

Proof of Theorem 1.5. Let $M_j \nearrow \infty$. Choose any $N \gg 1$ and any sequences $r_j \searrow 2$ and $q_j \searrow 1$ such that the estimate of Proposition 5.2 holds for the flow $\eta_j(t)$ of $u_j = \nabla^\perp\Delta^{-1}\omega_j$, where ω_j solves the vorticity equation (5.1) with initial condition $\omega_{0,j} = \omega_0^{M_j,N,r_j,q_j}$ given by (5.3). For each $j \ge 1$ we will introduce a high-frequency perturbation of $\omega_{0,j}^n$ such that for any sufficiently large n we have

$$\|\omega_{0,j}^n\|_{B^1_{r_j,q_j}} \lesssim 1 \quad\text{and}\quad \|\omega_j^n(t^*)\|_{B^1_{r_j,q_j}} \gtrsim M_j^{1/3} \quad \text{for some } 0<t^*\le M_j^{-3}. \tag{5.7}$$

To this end observe that we may assume

$$\|\omega_j(t)\|_{B^1_{r_j,q_j}} \le M_j^{1/3} \qquad \text{for all } \ 0\le t\le M_j^{-3} \tag{5.8}$$

or else there is nothing to prove. Using Proposition 5.2 we can pick $0\le t^*\le M_j^{-3}$ and a point $x^*=(x_1^*,x_2^*)$ for which the absolute value of one of the entries in $D\eta_j(t_0,x^*)$ is at least as large as M_j and by continuity (since $r_j>2$) deduce that in a sufficiently small δ-neighbourhood of x^* we have

$$\left|\frac{\partial \eta_j^2}{\partial x_2}(t_0,x)\right| \ge M_j \qquad \text{for all} \quad |x-x^*|<\delta. \tag{5.9}$$

To construct a sequence of perturbations of $\omega_{0,j}$ in $B^1_{r_j,q_j}$ pick a smooth function $\hat{\chi}\in C_c^\infty(\mathbb{R}^2)$ with support in the unit ball such that $0\le\hat{\chi}\le 1$ and $\int_{\mathbb{R}^2}\hat{\chi}(\xi)\,d\xi=1$ and set

$$\hat{\rho}(\xi)=\hat{\chi}(\xi-\xi_0)+\hat{\chi}(\xi+\xi_0), \qquad \text{where } \ \xi\in\mathbb{R}^2 \ \text{ and } \ \xi_0=(2,0). \tag{5.10}$$

Observe that sup $\hat{\rho}\subset B(-\xi_0,1)\cup B(\xi_0,1)$ and

$$\rho(0)=\int_{\mathbb{R}^2}\hat{\rho}(\xi)\,d\xi = 2. \tag{5.11}$$

For any $k\in\mathbb{Z}_+$ and $\lambda>0$ define

$$\beta_j^{k,\lambda}(x)=\frac{\lambda^{-1+\frac{2}{r_j}}}{\sqrt{k}}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_1\varepsilon_2\rho(\lambda(x-x_\epsilon^*))\sin kx_1, \tag{5.12}$$

where $x_\epsilon^*=(\varepsilon_1x_1^*,\varepsilon_2x_2^*)$.

Lemma 5.4. *Let $1<q_j<2<r_j<\infty$, $2\le p\le\infty$, and $\sigma>0$. For any sufficiently large $k\in\mathbb{Z}^+$ and $\lambda>0$ we have*

1. $\|\beta_j^{k,\lambda}\|_{W^{1,r_j}} \lesssim \|\beta_j^{k,\lambda}\|_{B^1_{r_j,q_j}} \lesssim k^{\frac12}\lambda^{-1}$,
2. $\|\Delta^{\frac{1+\sigma}{2}}\partial_l\Delta^{-1}\beta_j^{k,\lambda}\|_{L^p} \lesssim k^{-\frac12}\lambda^{-1+\frac{2}{r_j}-\frac{2}{p}}(\lambda^\sigma+k^\sigma)$,
3. $\|\partial_l\Delta^{-1}\beta_j^{k,\lambda}\|_{L^p} \lesssim k^{-\frac12}\lambda^{-2+\frac{2}{r_j}-\frac{2}{p}}$,

where $l=1,2$.

Proof of Lemma 5.4. The proof of the first inequality is similar to the second inequality which in turn is similar to that of Lemma 5.1. To prove the second and third estimates it will be convenient to use the Fourier transform

$$\hat{\beta}_j^{k,\lambda}(\xi) = \frac{1}{2i} k^{-\frac{1}{2}} \lambda^{-3+\frac{2}{r_j}} \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \sum_{m=1}^{2} (-1)^{j+1} \varepsilon_1 \varepsilon_2 \hat{\rho}\big(\lambda^{-1}\xi_m^k\big) e^{-2\pi i \langle x_\varepsilon^*, \xi_m^k\rangle}, \tag{5.13}$$

where $\xi_m^k = \big(\xi_1 + \frac{(-1)^m}{2\pi} k, \xi_2\big)$. Applying the Hausdorff–Young inequality we obtain

$$\big\|\Delta^{\frac{1+\sigma}{2}} \partial_l \Delta^{-1} \beta_j^{k,\lambda}\big\|_{L^p} \lesssim k^{-\frac{1}{2}} \lambda^{-1+\frac{2}{r_j}} \sum_{m=1}^{2} \left(\int_{\mathbb{R}^2} \lambda^{-2p'} |\xi|^{\sigma p'} \big|\hat{\rho}(\lambda^{-1}\xi_m^k)\big|^{p'} d\xi \right)^{1/p'},$$

where $1/p + 1/p' = 1$. Changing the variables we further estimate by

$$\begin{aligned}
&\lesssim k^{-\frac{1}{2}} \lambda^{-1+\frac{2}{r_j} - 2(1-\frac{1}{p'})} \sum_{m=1}^{2} \left(\int_{\mathbb{R}^2} \left(\left(\xi_1 - \frac{(-1)^m}{2\pi} k\right)^2 + \xi_2^2 \right)^{\frac{\sigma p'}{2}} \big|\hat{\rho}(\lambda^{-1}\xi)\big|^{p'} \frac{d\xi}{\lambda^2} \right)^{1/p'} \\
&\lesssim k^{-\frac{1}{2}} \lambda^{-1+\frac{2}{r_j} - \frac{2}{p}} \sum_{m=1}^{2} \left(\int_{\mathbb{R}^2} \left(\left(\lambda\xi_1 - \frac{(-1)^m}{2\pi} k\right)^2 + (\lambda\xi_2)^2 \right)^{\frac{\sigma p'}{2}} \big|\hat{\rho}(\xi)\big|^{p'} d\xi \right)^{1/p'} \\
&\lesssim k^{-\frac{1}{2}} \lambda^{-1+\frac{2}{r_j} - \frac{2}{p}} \big(\lambda^\sigma + k^\sigma\big).
\end{aligned}$$

Similarly, we also obtain

$$\big\|\partial_l \Delta^{-1} \beta_j^{k,\lambda}\big\|_{L^p} \lesssim k^{-\frac{1}{2}} \lambda^{-2+2(\frac{1}{r_j} - \frac{1}{p})}$$

for any sufficiently large k and λ. □

Next, set $\beta_j^n = \beta_j^{k,\lambda}$, where $k = \lambda^2$, $\lambda = 3n$, and $n \gg 1$. Using (5.9) and (5.11) we now have the following.

Lemma 5.5. *Let M_j, N, r_j, q_j, n, and t^* be as above. Then:*

1. $\|\partial_2 \beta_j^n \partial_1 \eta_j^2(t^*)\|_{L^{r_j}} \lesssim Cn^{-1}$,
2. $\|\partial_1 \beta_j^n \partial_2 \eta_j^2(t^*)\|_{L^{r_j}} \gtrsim M_j\big(1 + \mathcal{O}(n^{-\frac{1}{2}})\big) - Cn^{-1}$,

where C depends on $\|\hat{\rho}\|_{L^{r_j'}}$ and $\sup_{0\le t\le 1} \|u_j(t)\|_{C^1}$ and $1/r_j' + 1/r_j = 1$.

Proof. The proof is analogous to that in [32, Lem. 11]. □

For each $j \geq 1$ define a perturbation sequence of initial vorticities

$$\omega_{0,j}^n(x) = \omega_{0,j}(x) + \beta_j^n(x), \qquad n \gg 1.$$

By Lemma 5.1 and Lemma 5.4 (part 1) it is in $B^1_{r_j,q_j}$, which shows the first of the inequalities in (5.7). Let $\omega_j^n \in C([0,1], B^1_{r_j,q_j}(\mathbb{R}^2))$ be the solution of the vorticity equations with initial data $\omega_{0,j}^n$. Recall that $r_j > 2$ and $q_j > 1$ are already fixed. Given any $p \geq 2$ pick $0 < \sigma < 1 + 1/p - 1/r_j$ in Lemma 5.4 (parts 2 and 3) so that $\|\nabla^\perp \Delta^{-1}(\omega_{0,j}^n - \omega_{0,j})\|_{W^{1+\sigma,p}} \to 0$ as $n \to \infty$.[4] Therefore, using continuity of the

[4]More precisely, observe that the power of n (recall $k = \lambda^2 \simeq n$) on the right hand side of the inequality in part 2 of Lemma 5.4 is

$$2(-1 + 1/r_j - 1/p + \sigma) < 0$$

solution map in the little Hölder spaces of Theorem 1.2 we find[5]

$$\sup_{0\le t\le 1}\|\nabla^\perp\Delta^{-1}(\omega_j^n(t)-\omega_j(t))\|_{C^1}\lesssim \sup_{0\le t\le 1}\|\nabla^\perp\Delta^{-1}(\omega_j^n(t)-\omega_j(t))\|_{1,\alpha}\longrightarrow 0 \tag{5.14}$$

as $n\to\infty$, and from Lemma 5.3 we get

$$\theta_n=\sup_{0\le t\le 1}\|\eta_j^n(t)-\eta_j(t)\|_{C^1}\longrightarrow 0\quad \text{as } n\to\infty, \tag{5.15}$$

where $\eta_j^n(t)$ is the flow of the velocity field $\nabla^\perp\Delta^{-1}\omega_j^n$.

Remark 5.6. By following a well-known argument of Kato and Ponce [26] we can show the incompressible Euler equations are locally well-posed in the sense of Hadamard in $W^{s,p}(\mathbb{R}^2)$ with $p>2$ and $s>2+2/p$ or $p=2$ and $s>2$. For details, see the Appendix. We can also apply continuity of the solution map in $H^{1+\sigma}(\mathbb{R}^2)$ to (5.14) directly. However, continuity in $H^{1+\sigma}$ is currently known to be valid only in two dimensions due to appearance of the vortex-stretching term in the 3D case. It therefore seems that so far the little Hölder spaces are the most suitable function space to study well-posedness in the sense of Hadamard.

Using (5.15), conservation of vorticity and the fact that the flows are volume-preserving we have

$$\begin{aligned}\|\omega_j^n(t^*)\|_{B^1_{r_j,q_j}} &\gtrsim \|\nabla\omega_{0,j}^n\cdot\nabla^\perp\eta_j^{n,2}(t^*)\|_{L^{r_j}}\gtrsim \|\nabla\omega_{0,j}^n\cdot\nabla^\perp\eta_j^{2}(t^*)\|_{L^{r_j}}-\theta_n\|\nabla\omega_{0,j}^n\|_{L^{r_j}}\\ &\gtrsim \|\nabla\beta_j^n\cdot\nabla^\perp\eta_j^2(t^*)\|_{L^{r_j}}-\|\nabla\omega_{0,j}\cdot\nabla^\perp\eta_j^2(t^*)\|_{L^{r_j}}-\theta_n\|\nabla\omega_{0,j}^n\|_{L^{r_j}}\end{aligned} \tag{5.16}$$

for any $j\ge 1$. Finally, observe that by (5.8) and the embedding $\dot B^1_{r_j,q_j}\subset\dot B^1_{r_j,2}\subset\dot W^{1,r_j}$, we have

$$\|\nabla\omega_{0,j}\cdot\nabla^\perp\eta_j^2(t^*)\|_{L^{r_j}}\lesssim\|\omega_j(t^*)\|_{B^1_{r_j,q_j}}\lesssim M_j^{1/3},$$

and by Lemma 5.5 for any sufficiently large $n\gg 1$ we also have

$$\|\nabla\beta_j^n\cdot\nabla^\perp\eta_j^2(t^*)\|_{L^{r_j}}\gtrsim\|\partial_1\beta_j^n\partial_2\eta_j^2(t^*)\|_{L^{r_j}}-\|\partial_2\beta_j^n\partial_1\eta_j^2(t^*)\|_{L^{r_j}}\gtrsim M_j.$$

This establishes the second of the inequalities in (5.7). The desired sequence of velocities $\tilde u_j$ can now be obtained by selecting for each $j\ge 1$ a suitably large integer n_j and setting $\tilde u_j=\nabla^\perp\Delta^{-1}\omega_j^{n_j}$. The proof of Theorem 1.5 is completed. ☐

6. Appendix: Continuity of the solution map in $W^{s,p}(\mathbb{R}^2)$

In this section we mention the continuity of the solution map more precisely. Continuous dependence of the solution map of the Euler equations with initial data in the Sobolev space $W^{s,p}$ for $p\ge 2$ and $s>2/p+2$ is of course well known (cf., e.g., Ebin and Marsden [15], Kato and Lai [25], and Kato and Ponce [26]).

Theorem 6.1. *The incompressible Euler equations* (1.1) *are locally well-posed in the sense of Hadamard in*

(i) *the Sobolev space* $W^{s,p}(\mathbb{R}^2)$ *with* $p\ge 2$ *and* $s>2/p+2$,
(ii) *the Sobolev space* $H^s(\mathbb{R}^2)$ *with* $s>2$.

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so that the L^p-norm there goes to zero with $n\to\infty$. Furthermore, if we choose p to satisfy $p>2/\sigma$ (which is possible whenever $p>\frac{r_j}{r_j-1}$), then we have the embeddings $W^{1+\sigma,p}(\mathbb{R}^2)\subset C^{1,\sigma-\frac{2}{p}}(\mathbb{R}^2)\subset c^{1,\alpha}(\mathbb{R}^2)$ for any $0<\alpha<\sigma-2/p$.

[5]Note that by construction $\beta_j^n\in\mathscr{S}(\mathbb{R}^2)$ has noncompact support; cf. Remark 4.2.

Note that the proof of (ii) is similar to that of (i). The key is to just use the new commutator estimate in [19]. By using the new commutator estimate, continuity of the solution map (in the 2D case) is restored in $p = 2$ and $s > 2$. This is related to Parseval's identity.

In what follows we shall sketch the proof of continuity of the solution map of (1.1) in $W^{s,p}$ for $s > 2+2/p$ and $p > 2$. The problem turns out to be rather subtle. As Kato and Lai point out in [25] the first such result for the Euler equations in the Sobolev H^s setting was proved in [15] for bounded domains and integer values $s > 1 + n/2$. The general case of $W^{s,p}$ was settled in [26] for unbounded domains and fractional $s > 2 + n/p$. Alternative proofs were also developed in [22], [25] or [4].

6.1. **Proof of Theorem 6.1 (i).** The argument follows closely that given by Kato and Ponce in [26, Sect. 2 and 3]. Only a minor adjustment is needed to one of the lemmas in their paper (see Lemma 6.4 below) which we restate here with a proof.

Let $D^s = (-\Delta)^{s/2}$ denote the fractional Laplacian as before. Recall the following classical commutator estimate.

Lemma 6.2. *If $s > 0$ and $1 < p < \infty$, then*

$$\|D^s(fg) - fD^s g\|_p \lesssim \|\nabla f\|_\infty \|D^{s-1}g\|_p + \|D^s f\|_p \|g\|_\infty$$

for any f and $g \in \mathcal{S}(\mathbb{R}^2)$.

Proof. See [27, Lem. X1]. □

Remark 6.3 ([19]). If $s > 1$, then

$$\|D^s((f\cdot\nabla)g) - (f\cdot\nabla)D^s g\|_2 \lesssim \|f\|_{H^{s+1}}\|g\|_{H^s}. \tag{6.1}$$

This is an improvement of the above lemma ($p = 2$). In their proof they essentially used Parseval's identity, so we cannot directly generalize it to the $p > 2$ case.

We have the following[6]

Lemma 6.4. *Assume $1 < p < \infty$ and $s_* > 1+2/p$. Let $a \in C([0,T), W^{s_*p}(\mathbb{R}^2))$ be a divergence free vector field on $\mathbb{R}^2$. If $y_0 \in W^{s_*,p}(\mathbb{R}^2)$, then there exists a unique solution of the Cauchy problem*

$$\begin{cases} \partial_t y + a\cdot\nabla y = 0, & x \in \mathbb{R}^2, \\ y(0,x) = y_0(x) \end{cases} \tag{6.2}$$

such that

$$\|y(t)\|_{W^{s_*,p}} \lesssim \|y_0\|_{W^{s_*,p}} \exp\left(C\int_0^t \|a(\tau)\|_{W^{s_*,p}}d\tau\right)$$

for any $0 \le t < T$.

Proof of Lemma 6.4. Since $s_* > 1 + 2/p$ by Sobolev lemma there exists a smooth flow $\xi(t)$ of volume-preserving diffeomorphisms of class C^1 with $y(t,\xi(t,x)) = y_0(x)$ so that taking L^p norms and changing variables we have

$$\|y(t)\|_{L^p} = \|y_0\|_{L^p}. \tag{6.3}$$

Applying D^{s_*} to both sides of the transport equation (6.2) we find

$$\partial_t(D^{s_*}y) + a\cdot\nabla D^{s_*}y = -D^{s_*}(a\cdot\nabla y) + a\cdot\nabla D^{s_*}y.$$

[6]Compare [26, Lem. 1.1].

Evaluating this equation along $\xi(t)$, integrating it with respect to t, and taking L^p norms as before, we obtain

(6.4)

$$\begin{aligned}\|D^{s_*}y\|_{L^p} &\le \|D^{s_*}y_0\|_{L^p} + \int_0^t \left\|D^{s_*}(a\cdot\nabla y) - a\cdot\nabla D^{s_*}y\right\|_{L^p} d\tau \\ &\lesssim \|D^{s_*}y_0\|_{L^p} + \int_0^t \Big(\|Da\|_\infty \|D^{s_*-1}\nabla y\|_{L^p} + \|D^{s_*}a\|_{L^p}\|\nabla y\|_\infty\Big) d\tau \\ &\lesssim \|D^{s_*}y_0\|_{L^p} + \int_0^t \|a\|_{W^{s_*,p}}\|y\|_{W^{s_*,p}} d\tau,\end{aligned}$$

where in the second and third line we used the Kato–Ponce estimate of Lemma 6.2 and the Sobolev embedding theorem, respectively. Combining the estimates in (6.3) and (6.4) with Gronwall's inequality we obtain the required estimate. □

Remark 6.5. For $s_* > 1 + 2/p$ and $p > 2$, with the external force case

$$\begin{cases}\partial_t y + a\cdot\nabla y = f(t), & x\in\mathbb{R}^2,\\ y(0,x) = y_0(x),\end{cases} \tag{6.5}$$

we have

$$\|y(t)\|_p \lesssim \|y_0\|_p + \int_0^t \|f(\tau)\|_p d\tau$$

and

$$\|D^{s_*}y(t)\|_p \lesssim \|D^{s_*}y_0\|_p + \int_0^t \left(\|D^{s_*}a(\tau)\|_p\|D^{s_*}y(\tau)\|_p + \|D^{s_*}f(\tau)\|_p\right) d\tau$$

for any $0 \le t < T$. For $p = 2$ and $s_* > 1$, we can get a better estimate by using Remark 6.3. We have

$$\|y(t)\|_{H^{s_*}} \lesssim \|y_0\|_{H^{s_*}} + \int_0^t \left(\|a(\tau)\|_{H^{s_*}}\|y(\tau)\|_{H^{s_*}} + \|f(\tau)\|_{H^{s_*}}\right) d\tau.$$

The rest part of the proof of Theorem 6.1 is essentially the same as Section 2 and Section 3 in [26]. The point is to estimate $\omega^j(t) - \omega^{j'}(t)$ in $W^{s-1,p}$ using a sequence of initial data $\{u_0^j\}_{j=1}^\infty$ ($u_0^j := \varphi_j * u_0$, $\varphi_j(x) = 2^{2j}\varphi(2^j x)$, $\varphi \in \mathcal{S}$) converging to u_0 in $W^{s,p}$ (the corresponding initial vorticity is $\omega_0^j := \operatorname{rot} u_0^j$, and its solution is $\omega^j(t) := \operatorname{rot} u^j(t)$). In this case we need to set $s_* = s-1$ and need separability of the function spaces. We apply Remark 6.5 with $s_* = s - 1 > 2/p + 1$ to

$$\partial_t\left(\omega^j - \omega^{j'}\right) = (u^j\cdot\nabla)(\omega^j - \omega^{j'}) + ((u^j - u^{j'})\cdot\nabla)\omega^{j'},$$

and then we have

$$\begin{aligned}\|\omega^j(t) - \omega^{j'}(t)\|_{s_*,p} \le\ & \|\omega_0^j - \omega_0^{j'}\|_{s_*,p} \\ & + \int_0^t \Big(\|u^j(\tau)\|_{s_*,p}\|\omega^j(\tau) - \omega^{j'}(\tau)\|_{s_*,p} \\ & + \|u^j(\tau) - u^{j'}(\tau)\|_{s_*,p}\|\omega^{j'}(\tau)\|_{s_*,p} \\ & + \|u^j(\tau) - u^{j'}(\tau)\|_{s_*-1,p}\|\omega^{j'}(\tau)\|_{s_*+1,p}\Big) d\tau.\end{aligned}$$

On the other hand, again we apply Remark 6.5 to the usual Euler equation with $p = D^{-2}\text{div}\,(u \cdot \nabla)u$ and

$$\|D^{-2+s_*}\nabla\text{div}\,(u \cdot \nabla)v\|_p \leq \|\nabla u\|_\infty \|D^{s_*}v\|_p + \|D^{s_*}u\|_p \|\nabla v\|_\infty,$$

and then we have

$$\|u^j(t) - u^{j'}(t)\|_{s_*-1,p} \leq \|u_0^j - u_0^{j'}\|_{s_*-1,p} \quad \text{with} \quad \sup_{j\geq 1, t\in[0,T]} \|u_0^j\|_{s_*+1,p} \leq C$$

for $t \in [0,T]$. By the density of u_0^j in $W^{s_*+1,p}$, we see

$$\|u_0^j - u_0^{j'}\|_{s_*-1,p}\|\omega^{j'}\|_{s_*+1,p} \to 0 \quad \text{as} \quad j, j' \to \infty.$$

Combining the above calculations, we can show the continuity of the solution map in $p > 2$, $2/p + 2 < s < 2$. The case $p = 2$, $s > 2$ is parallel, so we omit it (just replace the commutator estimate to (6.3)).

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