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A RANDOM WALK ON A NON-INTERSECTING TWO-SIDED RANDOM WALK TRACE IS SUBDIFFUSIVE IN LOW DIMENSIONS

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ABSTRACT. Let $(\overline{S}^1, \overline{S}^2)$ be the two-sided random walks in $\mathbb{Z}^d$ $(d = 2, 3)$ conditioned so that $\overline{S}^1[0,\infty) \cap \overline{S}^2[1,\infty) = \emptyset$, which was constructed by the author in 2012. We prove that the number of *global* cut times up to n grows like $n^{\frac{3}{8}}$ for $d = 2$. In particular, we show that each $\overline{S}^i$ has infinitely many global cut times with probability one. Using this property, we prove that the simple random walk on $\overline{S}^1[0,\infty) \cup \overline{S}^2[0,\infty)$ is subdiffusive for $d = 2$. We show the same result for $d = 3$.

1. Introduction and main results

1.1. **Introduction.** Let $S = (S(n))$ be the simple random walk in $\mathbb{Z}^d$ $(d = 2, 3)$ starting at the origin. Take integers $k < n$. A time k is called the cut time up to n if

$$S[0,k] \cap S[k+1,n] = \emptyset, \tag{1.1}$$

where $S[0,k] = \{S(j) : 0 \le j \le k\}$. We call $S(k)$ a cut point if k is a cut time. Let $R(n)$ be the number of cut times up to n. Lawler [13] proved that there exist $\zeta_d > 0$ and $c > 0$ such that

$$E\big(R(n)\big) \asymp n^{1-\zeta_d} \quad \text{for } d = 2, 3, \tag{1.2}$$

$$P\big(R(n) \ge c n^{1-\zeta_2}\big) \ge c \quad \text{for } d = 2, \tag{1.3}$$

$$\lim_{n\to\infty} \frac{\log R(n)}{\log n} = 1 - \zeta_3 \quad \text{with probability one for } d = 3. \tag{1.4}$$

Here $\asymp$ means "within multiplicative constants of" (see (1.10) below). We call ζ_d the intersection exponent. For the value of ζ_2, Lawler, Schramm and Werner [15] showed that

$$\zeta_2 = \frac{5}{8}, \tag{1.5}$$

by using the SLE techniques. Consequently, the expected number of cut times up to time n grows like $n^{\frac{3}{8}}$ for $d = 2$. The exact value of ζ_3 is not known. The best rigorous estimates for ζ_3 [4, 14] are

$$\frac{1}{4} < \zeta_3 < \frac{1}{2}. \tag{1.6}$$

While the understanding of the number of cut times has been advanced, to our knowledge there are no results about the geometrical structure of the path

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around the cut points. In order to investigate the structure, the following problem was considered in [17]: if we condition that $S[0,n] \cap S[n+1,2n] = \emptyset$, then what does the path look like around $S(n)$? Let S^1, S^2 be independent simple random walks starting at the origin. Then, thanks to the translation invariance and the reversibility of the simple random walk, our problem may be reduced to clarifying the structure of S^1, S^2 around the origin when we condition that $S^1[0,n] \cap S^2[1,n] = \emptyset$. To tackle this problem, the non-intersecting two-sided random walk paths were constructed for $d = 2, 3$ in [17]. Namely the following limit exists:

$$\lim_{n\to\infty} P(\cdot \mid S^1[0,\tau^1(n)] \cap S^2[1,\tau^2(n)] = \emptyset) =: P^\sharp(\cdot), \tag{1.7}$$

where $\tau^i(n) = \inf\{k \geq 0 : |S^i(k)| \geq n\}$. Let $\overline{S}^1, \overline{S}^2$ be the associated two-sided random walks whose probability law is $P^\sharp$. It was also proved in [17] that the speed of convergence in (1.7) is fast (see [17] for details). So, our problem is reduced to the following: what does $\overline{S}^1[0,\infty) \cup \overline{S}^2[0,\infty)$ look like?

In this paper, we will consider this problem, mainly for $d = 2$. Assume $d = 2$. We will study the difference between $\overline{S}^1[0,\infty) \cup \overline{S}^2[0,\infty)$ and $S^1[0,\infty) \cup S^2[0,\infty) = \mathbb{Z}^2$. Intuitively, since $\overline{S}^1$ and $\overline{S}^2$ do not intersect, one may expect that those paths are sparse. However, it will be proved (see Remark 4.5 below) that there exists $\beta < \infty$ such that for each i,

$$\# \overline{S}^i[0,\overline{\tau}^i(n)] \geq n^2(\log n)^{-\beta}, \ P^\sharp\text{-a.s.} \tag{1.8}$$

for large n, where we write

$$\overline{\tau}^i(n) = \inf\{k \geq 0 : |\overline{S}^i(k)| \geq n\}.$$

This shows that the path of $\overline{S}^i$ is not so sparse. (1.8) is due to the so-called separation lemma (see Proposition 2.1 below), which roughly says that two paths conditioned not to intersect are likely to be far apart. Once they are far apart, then each $\overline{S}^i$ forgets the conditioning and behaves like the usual simple random walk for a while. (Note that for the usual simple random walk, it is known ([6]–[8]) that $\# S^i[0,\tau^i(n)]$ is of order $n^2(\log n)^{-1}$.)

One of the most significant differences between S^i and $\overline{S}^i$ is the recurrence/transience property. Let $\mathcal{B}(m) = \{z \in \mathbb{Z}^2 : |z| < m\}$. For any $m < n$, it is clear that

$$P\Big(S^i\big[\tau^i(n),\infty\big) \cap \mathcal{B}(m) \neq \emptyset\Big) = 1.$$

On the other hand, it will be proved (see Lemma 3.8 below) that there is a constant $c < \infty$ such that

$$P^\sharp\Big(\overline{S}^i\big[\overline{\tau}^i(n),\infty\big) \cap \mathcal{B}(m) \neq \emptyset\Big) \leq c\Big(\frac{n}{m}\Big)^{-\frac{1}{2}} \tag{1.9}$$

for each $i = 1, 2$. By using this transience of $\overline{S}^i$, we will prove that each $\overline{S}^i$ has infinitely many *global* cut times with probability one (Theorem 1.1). Here a time n is called global cut time for $\overline{S}^i$ if $\overline{S}^i[0,n] \cap \overline{S}^i[n+1,\infty) = \emptyset$. Obviously, the usual simple random walk S^i has no global cut times. Moreover, we will show that the number of global cut times for $\overline{S}^i$ less than n grows like $n^{\frac{3}{8}}$ with probability one for each $i = 1, 2$ (Theorem 1.1).

To state another difference between $\overline{S}^1[0,\infty) \cup \overline{S}^2[0,\infty)$ and $\mathbb{Z}^2$, we will follow Kesten, who constructed the incipient infinite cluster (IIC) in two dimensional critical bond percolation [10] and proved that the simple random walk on IIC is subdiffusive [11]. For this purpose, we consider $\overline{\mathcal{G}} = \overline{S}^1[0,\infty) \cup \overline{S}^2[0,\infty)$ to be a random subgraph of $\mathbb{Z}^2$ with vertex set

$$V(\overline{\mathcal{G}}) = \{\overline{S}^i(n) \ : \ n \geq 0, \ i = 1,2\}$$

and edge set

$$E(\overline{\mathcal{G}}) = \{\{\overline{S}^i(n), \overline{S}^i(n+1)\} \ : \ n \geq 0, \ i = 1,2\}$$

(see Figure 1). Let $X = (X(n))$ be the simple random walk on $\overline{\mathcal{G}}$ starting at the origin. We will show that X is subdiffusive at the quenched level. More precisely, if we write

$$T(n) = \inf\{k \geq 0 \ : \ |X(k)| \geq n\},$$

then the expectation (with respect to the quenched law of X) of $T(n)$ is larger than $n^{2+\delta}$ for some $\delta > 0$, $P^\sharp$-almost surely (Theorem 1.2). From this, we see that $\overline{\mathcal{G}}$ has an anomalous structure compared to $\mathbb{Z}^2$.

We give a heuristic reason of this subdiffusivity here. Although $\overline{\mathcal{G}}$ has many vertices as in (1.8), because of global cut points, its connectivity is bad when viewed as an electrical network. Let $t < t'$ be two global cut times for $\overline{S}^i$ and assume that $X(k) \in \overline{S}^i[t,t']$ for a certain time k. Then both $\{\overline{S}^i(t), \overline{S}^i(t+1)\}$ and $\{\overline{S}^i(t'), \overline{S}^i(t'+1)\}$ play the role of bottleneck edges. In other words, X must pass through either of the two edges in order to go far away. It takes a long time for X to make it if $\overline{S}^i[t,t']$ is a big graph. So, the strategy for the proof of Theorem 1.2 is to find a long enough sequence of global cut times $t_1 < t_2 < \cdots < t_k < \overline{\tau}^i(n)$ such that each $t_{j+1} - t_j$ is large enough. We will find such a sequence for $P^\sharp$-a.s. $\overline{\mathcal{G}}$ so that X has subdiffusive behavior.

For the proof of Theorem 1.2, with the help of recent progress on the planar simple random walk [15] and the loop-erased random walk ([2], [9]), we will establish a number of estimates for global cut times and the graph distance on $\overline{\mathcal{G}}$ (Proposition 4.3, Lemma 4.6, etc.), which are of independent interest.

It is natural to investigate whether or not X has subdiffusive behavior for $d = 3$. We will show that X is also subdiffusive in this case (Theorem 1.3).

Throughout the paper, we use $c, c', c_1, \cdots$ to denote arbitrary positive constants which may change from line to line. If a constant is to depend on some other quantity, this will be made explicit. For example, if c depends on ϵ, we write c_ϵ. We write $a_n \asymp b_n$ if there exist constants c_1, c_2 such that

$$c_1 b_n \leq a_n \leq c_2 b_n. \tag{1.10}$$

To avoid complication of notation, we don't use $\lfloor r \rfloor$ (the largest integer $\leq r$), even though it is necessary to carry it.

1.2. **Framework and main results.** For $x \in \mathbb{Z}^d$ ($d = 2,3$), let

$$\mathcal{B}(x,n) = \{z \in \mathbb{Z}^d : |z - x| < n\}$$

and

$$\partial\mathcal{B}(x,n) = \{z \in \mathbb{Z}^d \backslash \mathcal{B}(x,n) : |z-y| = 1 \text{ for some } y \in \mathcal{B}(x,n)\}.$$

We write $\mathcal{B}(n) = \mathcal{B}(0,n)$ and $\partial\mathcal{B}(n) = \partial\mathcal{B}(0,n)$.

A sequence of points $\gamma = [\gamma(0), \gamma(1), \cdots, \gamma(l)] \subset \mathbb{Z}^d$ is called a path if $|\gamma(j) - \gamma(j-1)| = 1$ for each $j = 1,2,\cdots,l$. Let $\mathrm{len}\gamma = l$ be the length of the path. (For the case where part of the path is repeated, we count the overlap. For example, if

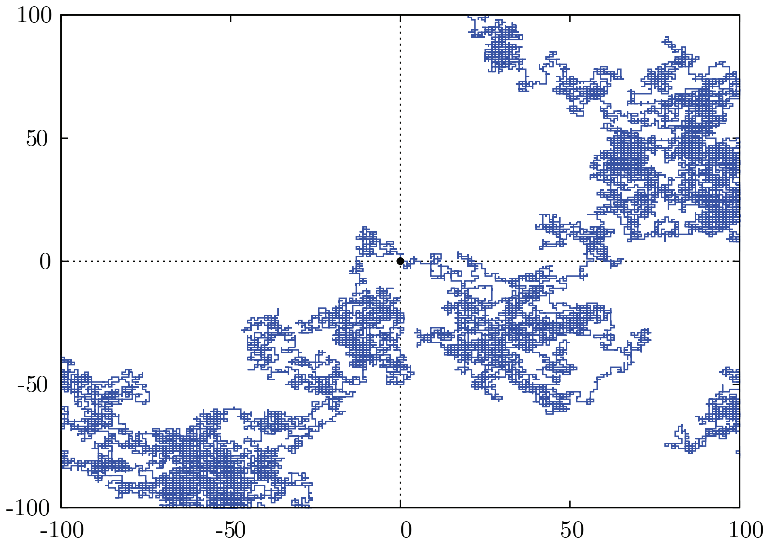

FIGURE 1. A non-intersecting two-sided random walk trace $\overline{\mathcal{G}}$ for $d = 2$.

$\gamma = [x, y, z, w, x, y, z, w]$, then $\mathrm{len}\gamma = 7$.) Let $\Lambda(n)$ be the set of paths satisfying that

$$\gamma(0) = 0, \gamma(j) \in \mathcal{B}(n) \text{ for all } j = 0, 1, \cdots, \mathrm{len}\gamma - 1,$$
$$\gamma(\mathrm{len}\gamma) \in \partial\mathcal{B}(n).$$

Let

$$\Gamma(n) = \{\overline{\gamma} = (\gamma^1, \gamma^2) \in \Lambda(n) \times \Lambda(n) : \gamma^1(i) \neq \gamma^2(j) \text{ for all } (i, j) \neq (0, 0)\} \tag{1.11}$$

and $\Gamma(\infty) = \bigcap_{n=1}^{\infty} \Gamma(n)$.

Let S^1, S^2 be the independent simple random walks in $\mathbb{Z}^d$. For any $x^1, x^2 \in \mathbb{Z}^d$, we let P^{x^1,x^2} be the probability measure associated with S^1 and S^2 with $S^1(0) = x^1$ and $S^2(0) = x^2$. If $x^1 = x^2 = 0$, we just write P instead of $P^{0,0}$. Let

$$\tau^i(n) = \inf\{k \geq 0 : S^i(k) \in \partial\mathcal{B}(n)\}$$

and

$$A_n = \big\{\big(S^1[0, \tau^1(n)], S^2[0, \tau^2(n)]\big) \in \Gamma(n)\big\}. \tag{1.12}$$

In [17], it was proved that for each $L \in \mathbb{N}$ and $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma(L)$, the limit

$$\lim_{n\to\infty} P\Big(\big(S^1[0, \tau^1(L)], S^2[0, \tau^2(L)]\big) = \overline{\gamma} \mid A_n\Big) \tag{1.13}$$

exists. If we denote the value of (1.13) by $P^\sharp(\overline{\gamma})$, then $P^\sharp$ extends uniquely to a probability measure on $\Gamma(\infty)$. We denote this probability space by $(\Omega, \mathcal{F}, P^\sharp)$.

Let $\overline{S}^1, \overline{S}^2$ be the associated two-sided random walks whose probability law is $P^\sharp$. Define the trace of $\overline{S}^i$ to be the graph $\overline{\mathcal{G}}^i = \big(V(\overline{\mathcal{G}}^i), E(\overline{\mathcal{G}}^i)\big)$ with vertex set

$$V(\overline{\mathcal{G}}^i) = \big\{\overline{S}^i(n) \ : \ n \geq 0\big\}$$

and edge set

$$E(\overline{\mathcal{G}}^i) = \big\{\{\overline{S}^i(n), \overline{S}^i(n+1)\} \ : \ n \geq 0\big\}.$$

We denote the trace of two-sided random walks by $\overline{\mathcal{G}} = \big(V(\overline{\mathcal{G}}), E(\overline{\mathcal{G}})\big)$, i.e.,

$$V(\overline{\mathcal{G}}) = V(\overline{\mathcal{G}}^1) \cup V(\overline{\mathcal{G}}^2) \text{ and } E(\overline{\mathcal{G}}) = E(\overline{\mathcal{G}}^1) \cup E(\overline{\mathcal{G}}^2).$$

Let

$$X = \big((X(n))_{n\geq 0}, P_x^{\overline{\mathcal{G}}}, x \in V(\overline{\mathcal{G}})\big)$$

be the simple random walk on $\overline{\mathcal{G}}$. We let $E_x^{\overline{\mathcal{G}}}$ be the expectation with respect to $P_x^{\overline{\mathcal{G}}}$. Let

$$T(n) = \inf\{k \geq 0 \ : \ |X(k)| \geq n\}. \tag{1.14}$$

A time k is called a global cut time for $\overline{S}^i$ if

$$\overline{S}^i[0,k] \cap \overline{S}^i[k+1,\infty) = \emptyset. \tag{1.15}$$

We write

$$\overline{K}^i(j) = \mathbf{1}\big\{\overline{S}^i[0,j] \cap \overline{S}^i[j+1,\infty) = \emptyset\big\}$$

to denote the indicator function of the event that j is a global cut time for $\overline{S}^i$.

The following theorems are our main results in this paper.

Theorem 1.1. *Let $d = 2, 3$. It follows that for each $i = 1, 2$,*

$$\lim_{n\to\infty} \frac{\log\Big(\sum_{j=0}^{n} \overline{K}^i(j)\Big)}{\log n} = 1 - \zeta_d, \tag{1.16}$$

$P^\sharp$-almost surely. In particular, both $\overline{S}^1$ and $\overline{S}^2$ have infinitely many global cut times, $P^\sharp$-almost surely.

Theorem 1.2. *Let $d = 2$. For every $\epsilon \in (0, \frac{1}{100})$, there exists $\Omega_1 \subset \Omega$ with $P^\sharp(\Omega_1) = 1$ satisfying the following: for each $\omega \in \Omega_1$, there exists $N_1(\omega) < \infty$ such that*

$$E_0^{\overline{\mathcal{G}}(\omega)}\big(T(n)\big) \geq n^{\frac{81}{40}-\epsilon} \tag{1.17}$$

for all $n \geq N_1(\omega)$.

Let $\xi_d := 2\zeta_d$. Note that by (1.6), we see that $\frac{1}{2} < \xi_3 < 1$, so that $4 - 2\xi_3 > 2$.

Theorem 1.3. *Let $d = 3$. There exists $\rho < \infty$ such that the following holds: there exists $\Omega_2 \subset \Omega$ with $P^\sharp(\Omega_2) = 1$ satisfying the following: for each $\omega \in \Omega_2$, there exists $N_2(\omega) < \infty$ such that*

$$E_0^{\overline{\mathcal{G}}(\omega)}\big(T(n)\big) \geq n^{4-2\xi_3}(\log n)^{-\rho} \tag{1.18}$$

for all $n \geq N_2(\omega)$.

The rest of the paper is organized as follows. In Section 2, we will prove the so-called separation lemma, which plays an important role in the proof of Theorem 1.1. We will give the proof of Theorem 1.1 in Section 3, and the proofs of Theorem 1.2 and Theorem 1.3 in Section 4.

2. Separation lemma and its consequence

Throughout this section, we assume $d = 2$ or 3. Recall $\Gamma(n)$ as was defined in (1.11). For each $l < n$ and $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma(l)$, define

$$A_n(\overline{\gamma}) = \left\{ \begin{array}{l} S^1[0, \tau_n^1] \cap \gamma^2 = \emptyset, \\ S^2[0, \tau_n^2] \cap \gamma^1 = \emptyset, \\ S^1[0, \tau_n^1] \cap S^2[0, \tau_n^2] = \emptyset \end{array} \right\}. \tag{2.1}$$

Let $w^i = \gamma^i(\mathrm{len}\gamma^i)$. We assume $S^i(0) = w^i$ when we consider $A_n(\overline{\gamma})$. There are many ways to define the "separation" event; we will make one arbitrary choice. Let

$$I(r) = \{(x_1, \cdots, x_d) \in \mathbb{Z}^d : x_1 \geq r\}, \quad I'(r) = \{(x_1, \cdots, x_d) \in \mathbb{Z}^d : x_1 \leq -r\}.$$

For each $l \in \mathbb{N}$, let $\mathsf{Sep}(l)$ denote the event

$$\mathsf{Sep}(l) = \Big\{ S^1[0, \tau^1(2l)] \subset \mathcal{B}\big(\frac{3l}{2}\big) \cup I\big(\frac{4l}{3}\big) \Big\} \cap \Big\{ S^2[0, \tau^2(2l)] \subset \mathcal{B}\big(\frac{3l}{2}\big) \cup I'\big(\frac{4l}{3}\big) \Big\}. \tag{2.2}$$

A typical pair (S^1, S^2) which satisfies $A_{2l}(\overline{\gamma}) \cap \mathsf{Sep}(l)$ is pictured in Figure 2.

Proposition 2.1. *There exists $c > 0$ such that for all $l \in \mathbb{N}$ and $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma(l)$,*

$$P^{w^1, w^2}\big(\mathsf{Sep}(l) \;\big|\; A_{2l}(\overline{\gamma})\big) \geq c, \tag{2.3}$$

where $w^i = \gamma^i(\mathrm{len}\gamma^i)$.

Proof. The proof of this proposition is similar to the proof of Lemma 3.1 in [16] which is stated for the Brownian case. That lemma is slightly stronger than this proposition, but it suffices to show (2.3) for our purposes. Since we could not find the discrete version in the literature, we will give the proof for completeness.

For each $l \in \mathbb{N}$ and $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma(l)$ with $w^i = \gamma^i(\mathrm{len}\gamma^i)$, let

$$D(\overline{\gamma}) = \mathrm{dist}(w^1, \gamma^2) \wedge \mathrm{dist}(w^2, \gamma^1).$$

Notice that $D(\overline{\gamma}) \geq 1$ for every $\overline{\gamma}$. Let

$$u_n = \sum_{j=n}^{\infty} j^2 2^{-j}.$$

Take N sufficiently large so that $u_N \leq \frac{1}{4}$. For $n \geq N$, let h_n be the infimum of

$$\frac{P^{w^1, w^2}\big(\mathsf{Sep}(l) \cap A_{2l}(\overline{\gamma})\big)}{P^{w^1, w^2}\big(A_{2l}(\overline{\gamma})\big)},$$

where the infimum is over $l \geq 2^{n-1}$, $0 \leq r \leq u_n$, and all $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma((1+r)l)$ such that $\frac{D(\overline{\gamma})}{l} \geq 2^{-n}$.

We first check that in order to prove (2.3) it suffices to show that

$$\inf_{n \geq N} h_n > 0. \tag{2.4}$$

For this purpose, take an arbitrary initial configuration $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma(l)$. If $l \leq 2^N$, then it is easy to see that

$$P^{w^1, w^2}\big(\mathsf{Sep}(l) \;\big|\; A_{2l}(\overline{\gamma})\big) \geq c$$

for some $c > 0$ depending only on the dimension since N is a constant. Therefore assume $l > 2^N$. Chose a unique n such that

$$2^{-n} \leq \frac{D(\overline{\gamma})}{l} < 2^{-n+1}.$$

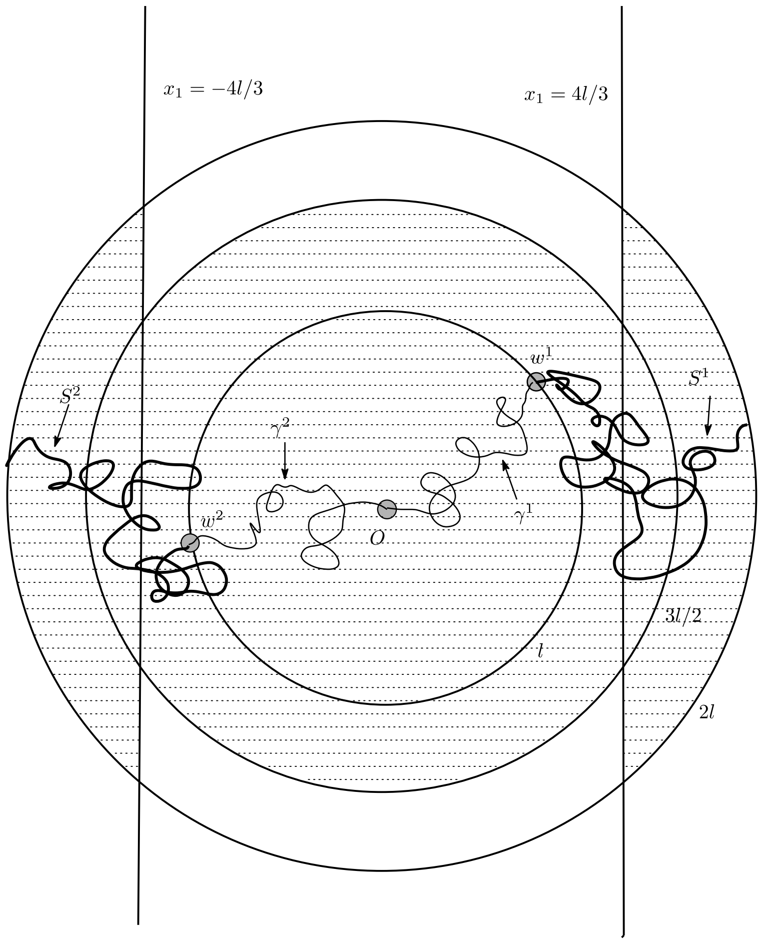

FIGURE 2. The event $A_{2l}(\overline{\gamma}) \cap \mathsf{Sep}(l)$.

If $n \leq N$, then $l > 2^N$, $\overline{\gamma} \in \Gamma(l)$ and $\frac{D(\overline{\gamma})}{l} \geq 2^{-N}$. Hence

$$P^{w^1,w^2}\big(\mathsf{Sep}(l) \bigm| A_{2l}(\overline{\gamma})\big) \geq h_N.$$

On the other hand, if $n > N$, then it follows from $D(\overline{\gamma}) \geq 1$ that

$$l > 2^{n-1}.$$

Since $\overline{\gamma} \in \Gamma(l)$ and $\frac{D(\overline{\gamma})}{l} \geq 2^{-n}$, we see that

$$P^{w^1,w^2}\big(\mathsf{Sep}(l) \bigm| A_{2l}(\overline{\gamma})\big) \geq h_n.$$

Now we return to (2.4). For this, it suffices to show that $h_n > 0$ for each $n \geq N$, and that there exists a summable sequence $\delta_n < 1$ such that

$$h_{n+1} \geq h_n(1 - \delta_n). \tag{2.5}$$

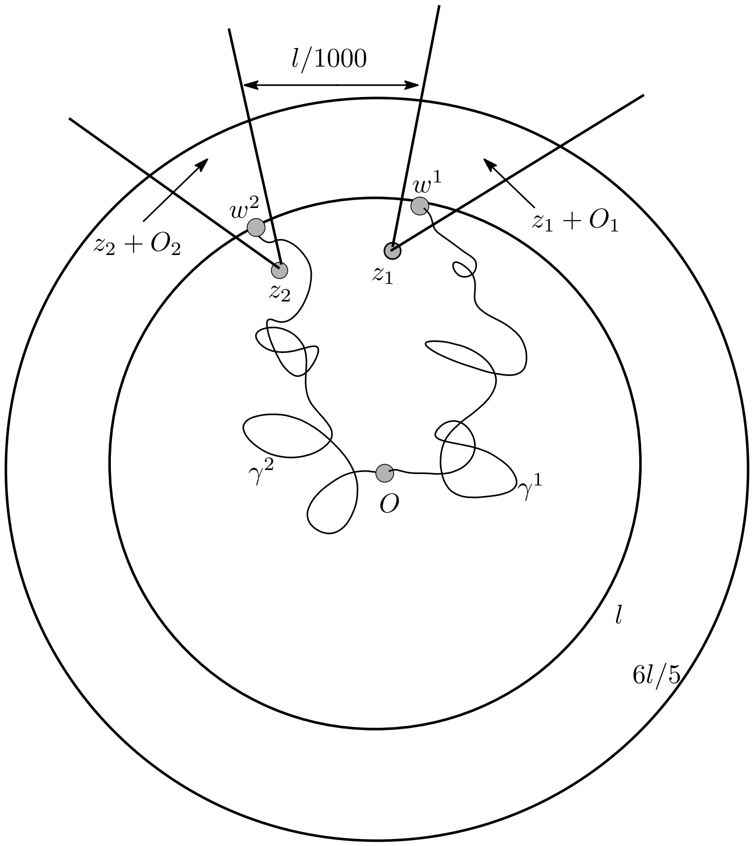

FIGURE 3. Two cones $z_j + O_j$.

Suppose U is a relatively open subset of $\{z \in \mathbb{R}^d : |z| = 1\}$. We let O denote the corresponding cone

$$O = \{rw : r > 0, \ w \in U\}. \tag{2.6}$$

Then it is easy to see that we can find infinite cones O_1, O_2 as in (2.6) and vertices $z_1, z_2 \in \mathbb{R}^d$ such that the following hold:

(a) $\dfrac{D(\overline{\gamma})}{100} \le |z_j - w^j| \le \dfrac{D(\overline{\gamma})}{20}$.

(b) $w^j \in O_j + z_j$ and $\dfrac{D(\overline{\gamma})}{100} \le \operatorname{dist}\big(w^j, \partial(z_j + O_j)\big) \le \dfrac{D(\overline{\gamma})}{20}$.

(c) $(O_j + z_j) \cap \mathcal{B}(l) \subset \mathcal{B}(w^j, \dfrac{D(\overline{\gamma})}{10})$.

(d) If $V_j = (O_j + z_j) \cap (\mathbb{Z}^2 \setminus \mathcal{B}(\dfrac{6l}{5}))$, then $\operatorname{dist}(V_1, V_2) \ge \dfrac{l}{1000}$

(see Figure 3).

We leave it to the reader to see that such cones can be found. Moreover, it is also easy to see that there exist $c > 0$ and $\alpha < \infty$ such that

$$P^{w^1,w^2}\big(S^i[0,\tau^i(\frac{5l}{4})] \subset O_j + z_j, \text{ for } i = 1,2\big) \geq c(\frac{D(\overline{\gamma})}{l})^\alpha.$$

Let $F_l = \{S^i[0,\tau^i(\frac{5l}{4})] \subset O_j + z_j, \text{ for } i = 1,2\}$. Then it is not hard to convince oneself that

$$P^{w^1,w^2}\big(\mathsf{Sep}(l) \cap A_{2l}(\overline{\gamma}) \bigm| F_l\big) \geq c$$

for some $c > 0$. Therefore, we have

$$P^{w^1,w^2}\big(\mathsf{Sep}(l) \cap A_{2l}(\overline{\gamma})\big) \geq c(\frac{D(\overline{\gamma})}{l})^\alpha$$

and

$$h_n \geq c2^{-\alpha n}. \tag{2.7}$$

Next we will prove (2.5). Assume that $l \geq 2^n$, $0 \leq r \leq u_{n+1}$, and $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma((1+r)l)$ with $\frac{D(\overline{\gamma})}{l} \geq 2^{-n-1}$. Recall that $w^i = \gamma^i(\text{len}\gamma^i) \in \partial\mathcal{B}((1+r)l)$. We define a sequence of balls $\{\mathcal{B}^j\}_{j\geq 0}$ as follows:

$$\mathcal{B}^j = \mathcal{B}(a_j),$$

where $a_j = (1+r)l + 4j2^{-n}l$. Let

$$\begin{aligned}\rho' = \inf\Big\{j:\ &\text{dist}\Big(S^1(\tau^1(a_j)), (S^2[0,\tau^2(a_j)] \cup \gamma^2)\Big)\\ &\wedge\ \text{dist}\Big(S^2(\tau^2(a_j)), (S^1[0,\tau^1(a_j)] \cup \gamma^1)\Big) \geq 2^{-n}l\Big\}\end{aligned}$$

and $\rho = \rho' \wedge \frac{n^2}{4}$. Set

$$D_j = \text{dist}\Big(S^1(\tau^1(a_j)), (S^2[0,\tau^2(a_j)] \cup \gamma^2)\Big) \wedge \text{dist}\Big(S^2(\tau^2(a_j)), (S^1[0,\tau^1(a_j)] \cup \gamma^1)\Big).$$

It is easy to see that there is a $p > 0$ such that given $S^1[0,\tau^1(a_j)]$ and $S^2[0,\tau^2(a_j)]$, the probability that $D_{j+1} \geq 2^{-n}l$ is at least p for every j. Iterating this, we see that there exist c, δ such that

$$P^{w^1,w^2}(\rho = \frac{n^2}{4}) \leq c2^{-\delta n^2}. \tag{2.8}$$

In the event $\{\rho < \frac{n^2}{4}\} \cap A_{a_\rho}(\overline{\gamma})$, we have

$$\begin{aligned}&l > 2^{n-1},\\ &\big(S^1[0,\tau^1(a_\rho)] \cup \gamma^1, S^2[0,\tau^2(a_\rho)] \cup \gamma^2\big) \in \Gamma(a_\rho),\\ &0 \leq r + 4\rho 2^{-n} \leq u_n,\\ &D_\rho \geq 2^{-n}l.\end{aligned}$$

Using the definition of h_n, we see that

$$\begin{aligned}P^{w^1,w^2}\big(\mathsf{Sep}(l) \cap A_{2l}(\overline{\gamma})\big) &\geq P^{w^1,w^2}\big(\mathsf{Sep}(l) \cap A_{2l}(\overline{\gamma}) \cap \{\rho < \frac{n^2}{4}\}\big)\\ &\geq h_n P^{w^1,w^2}\big(A_{2l}(\overline{\gamma}) \cap \{\rho < \frac{n^2}{4}\}\big).\end{aligned}$$

However, (2.7) and (2.8) imply that

$$P^{w^1,w^2}\big(A_{2l}(\overline{\gamma}) \cap \{\rho < \frac{n^2}{4}\}\big) \geq P^{w^1,w^2}\big(A_{2l}(\overline{\gamma})\big) - c2^{-\delta n^2}$$
$$\geq P^{w^1,w^2}\big(A_{2l}(\overline{\gamma})\big)\big(1 - c2^{-\delta n^2+\alpha n}\big).$$

Therefore, (2.5) follows with $\delta_n = c2^{-\delta n^2+\alpha n}$. □

Here we establish a corollary of Proposition 2.1. Recall that $\xi_d = 2\zeta_d$.

Corollary 2.2. *There exist c_1, c_2 such that for all l, n with $2l < n$ and all $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma(l)$ with $w^i = \gamma^i(len\gamma^i) \in \partial\mathcal{B}(l)$,*

$$c_1(\frac{n}{l})^{-\xi_d} P^{w^1,w^2}(A_{2l}(\overline{\gamma})) \leq P^{w^1,w^2}(A_n(\overline{\gamma})) \leq c_2(\frac{n}{l})^{-\xi_d} P^{w^1,w^2}(A_{2l}(\overline{\gamma})). \tag{2.9}$$

Proof. The upper bound of (2.9) follows immediately from Corollary 4.6 in [13] and the strong Markov property. For the lower bound, let

$$G = \Big\{S^i[\tau^i(2l), \tau^i(n)] \cap \mathcal{B}(2l) \subset \mathcal{B}\Big(S^i(\tau^i(2l)), \frac{l}{10}\Big),\ i = 1, 2\Big\},$$
$$H = \big\{S^1[\tau^1(2l), \tau^1(n)] \cap S^2[\tau^2(2l), \tau^2(n)] = \emptyset\big\}.$$

Note that if $A_{2l}(\overline{\gamma}) \cap \mathsf{Sep}(l) \cap G \cap H$ holds, then $A_n(\overline{\gamma})$ holds. By Corollary 4.2 in [13], we see that there exists $c > 0$ such that

$$P^{w^1,w^2}\big(G \cap H \mid \mathsf{Sep}(l)\big) \geq c(\frac{n}{l})^{-\xi_d}.$$

By Proposition 2.1,

$$P^{w^1,w^2}\big(\mathsf{Sep}(l) \cap A_{2l}(\overline{\gamma})\big) \geq cP^{w^1,w^2}\big(A_{2l}(\overline{\gamma})\big).$$

Therefore,

$$P^{w^1,w^2}(A_n(\overline{\gamma})) \geq P^{w^1,w^2}\big(A_{2l}(\overline{\gamma}) \cap \mathsf{Sep}(l) \cap G \cap H\big) \geq c(\frac{n}{l})^{-\xi_d} P^{w^1,w^2}\big(A_{2l}(\overline{\gamma})\big),$$

and the proof is finished. □

3. Estimate of global cut times

In this section, we will prove Theorem 1.1. Again assume $d = 2$ or 3 throughout this section. Recall that $\overline{S}^1, \overline{S}^2$ are the associated two-sided random walks whose probability law is $P^\sharp$. Let

$$\overline{\tau}^i(n) = \inf\{k \geq 0 : |\overline{S}^i(k)| \geq n\},$$

for $i = 1, 2$. Let

$$\overline{K}^i(j, n) = \mathbf{1}\big\{\overline{S}^i[0, j] \cap \overline{S}^i[j+1, \overline{\tau}^i(n)] = \emptyset\big\} \tag{3.1}$$

and

$$\overline{K}^i(j) = \mathbf{1}\Big\{\overline{S}^i[0, j] \cap \overline{S}^i[j+1, \infty) = \emptyset\Big\}. \tag{3.2}$$

Define

$$\begin{aligned} \overline{L}^i(n) &= \Big\{\sum_{j=\overline{\tau}^i(\frac{2}{3}2^n)}^{\overline{\tau}^i(\frac{5}{6}2^n)} \overline{K}^i(j, 2^n) \geq c(2^n)^{2-\xi_d}\Big\}, \\ \overline{V}^i(n) &= \Big\{\overline{S}^i[\overline{\tau}^i(2^n), \infty) \cap \mathcal{B}(\frac{11}{12}2^n) = \emptyset\Big\}. \end{aligned} \tag{3.3}$$

Note that in the event $\overline{L}^i(n) \cap \overline{V}^i(n)$,

$$\sum_{j=\overline{\tau}^i(\frac{2}{3}2^n)}^{\overline{\tau}^i(\frac{5}{6}2^n)} \overline{K}^i(j) \geq c(2^n)^{2-\xi_d}.$$

In order to get the lower bound for Theorem 1.1, we will first show that

$$P^\sharp\big(\overline{L}^i(n) \cap \overline{V}^i(n)\big) \geq c, \tag{3.4}$$

for some $c > 0$ (Proposition 3.1). Then, by the iteration argument, we will show that there exist $c, \alpha < \infty$ such that

$$P^\sharp\Big(\bigcup_{j=n}^{n+\alpha \log n} \big(\overline{L}^i(j) \cap \overline{V}^i(j)\big)\Big) \geq 1 - cn^{-2}, \tag{3.5}$$

for each $i = 1, 2$ (Proposition 3.6). This gives the lower bound of Theorem 1.1. We then give the upper bound (which is easier) and prove the theorem in Section 3.3.

3.1. **Proof of** (3.4)**.** In this subsection, we will prove the following proposition.

Proposition 3.1. *For each $i = 1, 2$, there exists $c > 0$ such that*

$$P^\sharp\Big(\sum_{j=\overline{\tau}^i(\frac{2n}{3})}^{\overline{\tau}^i(\frac{5n}{6})} \overline{K}^i(j, n) \geq cn^{2-\xi_d},\ \overline{S}^i[\overline{\tau}^i(n), \infty) \cap \mathcal{B}(\tfrac{11n}{12}) = \emptyset\Big) \geq c. \tag{3.6}$$

In order to establish this proposition, we need several lemmas below. So we will show these lemmas first, and then Proposition 3.1.

Fix n and take $N \geq 2n$. We define five events $F_1, \cdots, F_5$ as follows. Let

$$F_1 = \Big\{\big(S^1[0, \tau^1(\frac{2n}{3})], S^2[0, \tau^2(\frac{2n}{3})]\big) \in \Gamma(\frac{2n}{3}),\ \mathsf{Sep}(\frac{n}{3})\Big\}. \tag{3.7}$$

Let

$$x_n = \begin{cases} (n, 0) & (\text{if } d = 2), \\ (n, 0, 0) & (\text{if } d = 3) \end{cases}$$

and

$$D_2 = \Big\{z \in \mathbb{R}^d :\ \operatorname{dist}\Big(z, l_{S^2(\tau^2(\frac{2n}{3})), -x_n}\Big) \leq \frac{n}{20}\Big\},$$

where $l_{S^2(\tau^2(\frac{2n}{3})), -x_n}$ denotes the line segment between $S^2(\tau^2(\frac{2n}{3}))$ and $-x_n$. Define

$$F_2 = \Big\{S^2[\tau^2(\frac{2n}{3}), \tau^2(n)] \subset D_2\Big\}. \tag{3.8}$$

Let

$$\sigma = \inf\big\{k \geq \tau^1(\frac{2n}{3}) :\ \big|S^1(k) - S^1(\tau^1(\frac{2n}{3}))\big| \geq \frac{n}{10}\big\},$$

and for each $\frac{n^2}{200} \leq j \leq \frac{n^2}{100}$, let

$$Y_j^1 = \mathbf{1}\left\{\begin{array}{c} S^1[\tau^1(\frac{2n}{3}), \tau^1(\frac{2n}{3}) + j] \cap S^1[\tau^1(\frac{2n}{3}) + j + 1, \sigma] = \emptyset, \\ S^1\big(\tau^1(\frac{2n}{3}) + j\big) \in D_3, \\ S^1[\tau^1(\frac{2n}{3}), \tau^1(\frac{2n}{3}) + j] \subset \mathcal{B}\big(S^1(\tau^1(\frac{2n}{3})), \frac{n}{15}\big), \\ S^1[\tau^1(\frac{2n}{3}) + j, \sigma] \subset D_3' \end{array}\right\}, \tag{3.9}$$

where we let

$$D_3 = \Big\{ z \in \mathbb{R}^d : \ \frac{n}{30} \le \big| z - S^1(\tau^1(\frac{2n}{3}))\big| \le \frac{n}{15}, \frac{\overrightarrow{l}_{0,S^1(\tau^1(\frac{2n}{3}))}}{|\overrightarrow{l}_{0,S^1(\tau^1(\frac{2n}{3}))}|} \cdot \frac{\overrightarrow{l}_{S^1(\tau^1(\frac{2n}{3})),z}}{|\overrightarrow{l}_{S^1(\tau^1(\frac{2n}{3})),z}|} \ge \frac{\sqrt{3}}{2} \Big\}$$

and

$$D_3' = \Big\{ z \in \mathbb{R}^d : \ \frac{n}{60} \le \big| z - S^1(\tau^1(\frac{2n}{3}))\big| \le \frac{n}{10}, \frac{\overrightarrow{l}_{0,S^1(\tau^1(\frac{2n}{3}))}}{|\overrightarrow{l}_{0,S^1(\tau^1(\frac{2n}{3}))}|} \cdot \frac{\overrightarrow{l}_{S^1(\tau^1(\frac{2n}{3})),z}}{|\overrightarrow{l}_{S^1(\tau^1(\frac{2n}{3})),z}|} \ge \frac{1}{2} \Big\}.$$

Here we write $\overrightarrow{l}_{x,y}$ to represent the vector and $(\overrightarrow{l}_{x,y}, \overrightarrow{l}_{z,w})$ to represent the inner product (see Figure 5). Define

$$F_3 = \Big\{ \sum_{j=\frac{n^2}{200}}^{\frac{n^2}{100}} Y_j^1 \ge c n^{2-\xi_d} \Big\}. \tag{3.10}$$

Let

$$D_4 = \Big\{ z \in \mathbb{R}^d : \ \mathrm{dist}\Big(z, l_{S^1(\sigma),x_n} \Big) \le \frac{n}{50} \Big\},$$

and define

$$F_4 = \Big\{ S^1[\sigma, \tau^1(n)] \subset D_4 \Big\}. \tag{3.11}$$

Finally, let

$$\begin{aligned} F_5 = & \Big\{ S^1[\tau^1(n), \tau^1(N)] \cap S^2[\tau^2(n), \tau^2(N)] = \emptyset \Big\} \\ & \cap \Big\{ S^i[\tau^i(n), \tau^i(N)] \cap \mathcal{B}(n) \subset \mathcal{B}\Big(S^i\big(\tau^i(n)\big), \frac{n}{20} \Big) \text{ for } i = 1, 2 \Big\}. \end{aligned} \tag{3.12}$$

Notice that

$$\bigcap_{i=1}^{5} F_i \subset \left\{ \begin{array}{c} \Big(S^1[0, \tau^1(N)], S^2[0, \tau^2(N)] \Big) \in \Gamma(N), \\ \displaystyle\sum_{j=\tau^1(\frac{2n}{3})}^{\tau^1(\frac{5n}{6})} \mathbf{1}\{ S^1[0,j] \cap S^1[j+1, \tau^1(N)] = \emptyset \} \ge c n^{2-\xi_d}, \\ S^1[\tau^1(n), \tau^1(N)] \cap \mathcal{B}(\frac{11n}{12}) = \emptyset \end{array} \right\} =: G_{n,N} \tag{3.13}$$

(see Figure 4). So we will give a lower bound of $P\Big(\bigcap_{i=1}^{5} F_i \Big)$ to prove Proposition 3.1.

Lemma 3.2. *There exists $c > 0$ such that*

$$P\Big(F_5 \ \Big| \ \bigcap_{i=1}^{4} F_i \Big) \ge c (\frac{N}{n})^{-\xi_d}. \tag{3.14}$$

Proof. Note that in the event $F_2 \cap F_4$, we have

$$\Big| S^1\big(\tau^1(n)\big) - S^2\big(\tau^2(n)\big) \Big| \ge n.$$

Hence by Corollary 4.2 in [13], it follows that there exists $c > 0$ such that

$$P\Big(F_5 \ \Big| \ \bigcap_{i=1}^{4} F_i \Big) \ge c (\frac{N}{n})^{-\xi_d}.$$

□

It is easy to show the following lemma, so we omit the proof.

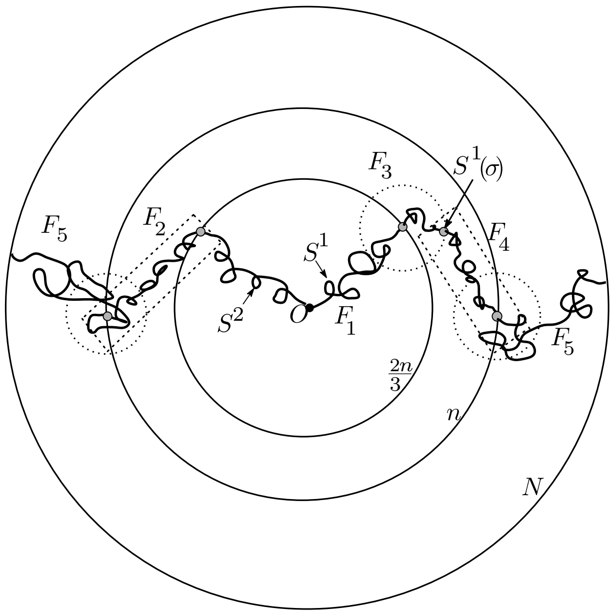

FIGURE 4. The event $\bigcap_{i=1}^{5} F_i$.

Lemma 3.3. *There exists $c > 0$ such that*

$$P(F_2) \geq c, \ P(F_4) \geq c. \tag{3.15}$$

Next we will estimate $P(F_3)$.

Lemma 3.4. *There exists $c > 0$ such that*

$$P(F_3) \geq c. \tag{3.16}$$

Proof. By applying the argument used in the proof of Corollary 4.12 in [13], we see that there exists $c > 0$ such that

$$E(Y_j^1) \geq c n^{-\xi_d}$$

for each $\frac{n^2}{200} \leq j \leq \frac{n^2}{100}$. Therefore,

$$E\Big(\sum_{j=\frac{n^2}{200}}^{\frac{n^2}{100}} Y_j^1 \Big) \geq c n^{2-\xi_d}. \tag{3.17}$$

On the other hand, it follows from Lemma 5.1 in [13] that there exists $c' < \infty$ such that

$$E\Big(\Big(\sum_{j=\frac{n^2}{200}}^{\frac{n^2}{100}} Y_j^1 \Big)^2\Big) \leq c' n^{2(2-\xi_d)}. \tag{3.18}$$

Therefore, using the second moment method, we see that

$$P\Big(\sum_{j=\frac{n^2}{200}}^{\frac{n^2}{100}} Y_j^1 \geq c n^{2-\xi_d} \Big) \geq c \tag{3.19}$$

for some $c > 0$, and the proof is finished. □

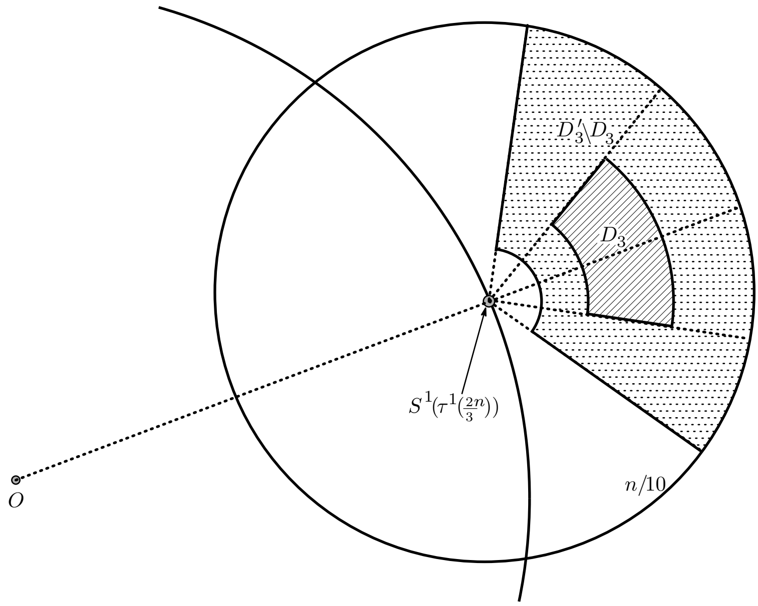

FIGURE 5. The sets D_3 and D_3'.

Finally, by using Proposition 2.1, we get the following lemma.

Lemma 3.5. *There exists* $c > 0$ *such that*

$$P(F_1) \geq cn^{-\xi_d}. \tag{3.20}$$

Proof of Proposition 3.1. Recall that the event $G_{n,N}$ was defined in (3.13). By Lemmas 3.2, 3.3, 3.4, and 3.5 and (3.13),

$$P\big(G_{n,N}\big) \geq P\Big(\bigcap_{i=1}^{5} F_i\Big) \geq cN^{-\xi_d}$$

for some $c > 0$. It follows from Theorem 1.3 in [13] that there exist c_1, c_2 such that

$$c_1 N^{-\xi_d} \leq P\Big(\big(S^1[0,\tau^1(N)], S^2[0,\tau^2(N)]\big) \in \Gamma(N)\Big) \leq c_2 N^{-\xi_d}.$$

Therefore, we have

$$P\Big(\sum_{j=\tau^1(\frac{2n}{3})}^{\tau^1(\frac{5n}{6})} \mathbf{1}\{S^1[0,j] \cap S^1[j+1,\tau^1(n)] = \emptyset\}$$
$$\geq cn^{2-\xi_d},\ S^1[\tau^1(n),\tau^1(N)] \cap \mathcal{B}(\frac{11n}{12}) = \emptyset \Big| \ A_N\Big) \geq c,$$

and letting $N \to \infty$, we get the proposition. □

3.2. **Proof of** (3.5)**.** For each $\alpha \in (0,\infty)$, let

$$\overline{\Lambda}^i(n) = \overline{\Lambda}^i(n,\alpha) = \bigcup_{n \leq j \leq n+\alpha \log n} \big(\overline{L}^i(j) \cap \overline{V}^i(j)\big).$$

In order to get the lower bound of $\sum_{j=0}^{n} \overline{K}^i(j)$ at the quenched level, we need to show the following proposition.

Proposition 3.6. *There exist $\alpha, c \in (0,\infty)$ such that*

$$P^\sharp\big(\overline{\Lambda}^i(n,\alpha)\big) \geq 1 - \frac{c}{n^2} \tag{3.21}$$

for each $i = 1, 2$.

It suffices to show (3.21) for $i = 1$. In order to prove that, we need some lemmas. We will first show them and then Proposition 3.6.

Fix n and define a random sequence $s_0, s_1, s_2, \cdots$ inductively as follows. Let $s_0 = n$. Suppose s_i has been defined. If $s_i = \infty$, then $s_{i+1} = \infty$. Suppose $s_i = s < \infty$. In the event $\big(\overline{L}^1(s)\big)^c$, we set $s_{i+1} = s + 2$. In the event $\overline{L}^1(s)$, let

$$\eta = \inf\big\{m \geq \overline{\tau}^1(2^s) : \ |\overline{S}^1(m)| \leq \frac{11}{12} 2^s\big\},$$

where $\eta = \infty$ if no such m exists. Let

$$s_{i+1} = \inf\big\{k : \ \overline{S}^1[0,\eta] \subset \mathcal{B}(2^{k-2})\big\},$$

and $s_{i+1} = \infty$ if $\eta = \infty$. Let

$$s^\star = \sup\{s_i : \ s_i < \infty\}$$

(see Figure 6 for the definition of s_i). Note that this choice of $\{s_i\}$ is the same as that used in the proof of Theorem 1.2 in [13]. It follows that the event $\{s_i = s\}$ is $\overline{S}^1[0, \overline{\tau}^1(2^{s-2})]$-measurable for each i, and $\overline{L}^1(s^\star) \cap \overline{V}^1(s^\star)$ holds. Therefore, in order to prove (3.21), it suffices to show that there exist c and α such that for all n,

$$P^\sharp(s^\star \geq n + \alpha \log n) \leq \frac{c}{n^2}. \tag{3.22}$$

Lemma 3.7. *There exists $c > 0$ such that for every i,*

$$P^\sharp\big(s_{i+1} = \infty \mid s_0, \cdots, s_i\big) \geq c. \tag{3.23}$$

Proof. It suffices to prove that there is a $c > 0$ such that for all i and $s \in [n, \infty)$,

$$P^\sharp\big(s_{i+1} = \infty \mid s_i = s\big) \geq c. \tag{3.24}$$

Since $s_i = s$ is $\overline{S}^1[0, \overline{\tau}^1(2^{s-2})]$-measurable, we have

$$P^\sharp\big(s_{i+1} = \infty, \ s_i = s\big) = \sum_{\overline{\gamma}} P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})], \overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big) = \overline{\gamma}, \ s_{i+1} = \infty\Big),$$

where the summation is over all possible $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma(2^{s-2})$ such that

$$\{\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})], \overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big) = \overline{\gamma}\} \cap \{s_i = s\}$$

is possible. Let $\Psi \subset \Gamma(2^{s-2})$ denote the set of such $\overline{\gamma}$. Fix $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Psi$. By definition of s_i,

$$\begin{aligned} &P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})], \overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big) = \overline{\gamma}, \ s_{i+1} = \infty\Big) \\ &= P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})], \overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big) = \overline{\gamma}, \ \overline{L}^1(s), \ \overline{V}^1(s)\Big). \end{aligned}$$

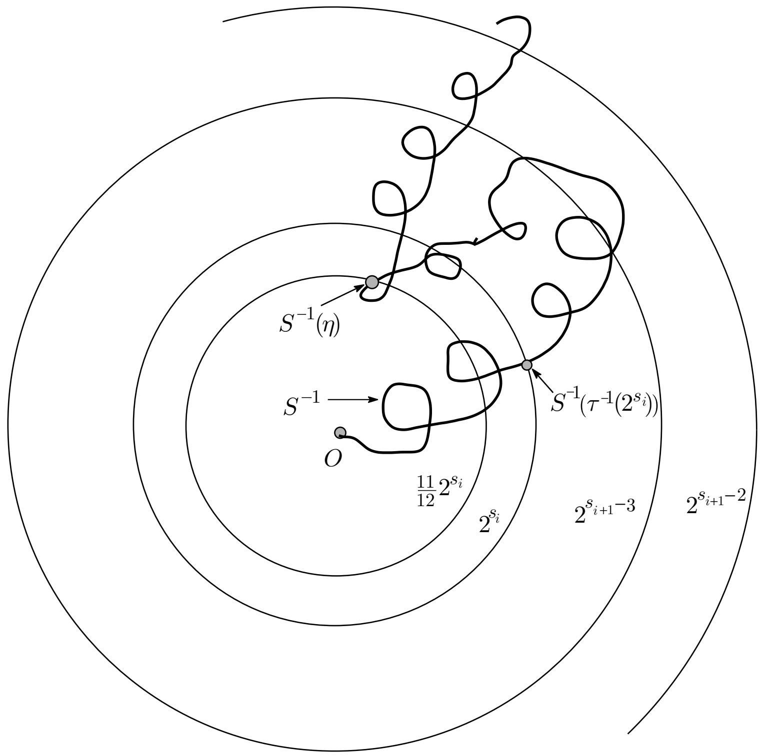

FIGURE 6. The sequence $\{s_i\}$.

Let $w^i = \gamma^i(\mathrm{len}\gamma^i)$. Applying a similar argument as in the proof of Proposition 3.1, we see that

$$\begin{aligned} &P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma},\ L^1(s),\ V^1(s),\ A_{2^N}\Big) \\ (3.25)\quad &\geq c(2^{N-s})^{-\xi_d} P^{w^1,w^2}\big(A_{2^{s-1}}(\overline{\gamma})\big) P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma}\Big), \end{aligned}$$

where $L^i(n)$ and $V^i(n)$ are the events as follows:

$$L^i(n) = Big\{\sum_{j=\tau^i(\frac{2}{3}2^n)}^{\tau^i(\frac{5}{6}2^n)} K^i(j,2^n) \geq c(2^n)^{2-\xi_d}\Big\},$$

$$V^i(n) = \Big\{S^i[\tau^i(2^n),\infty) \cap \mathcal{B}(\frac{11}{12}2^n) = \emptyset\Big\}.$$

Here we write

$$K^i(j,m) = \mathbf{1}\Big\{S^i[0,j] \cap S^j[j+1,\tau^i(m)] = \emptyset\Big\},$$

and recall that the event A_n was defined as in (1.12). By the strong Markov property,

$$P^{w^1,w^2}\big(A_{2^N}(\overline{\gamma})\big) \leq c(2^{N-s})^{-\xi_d} P^{w^1,w^2}\big(A_{2^{s-1}}(\overline{\gamma})\big).$$

Therefore, the right hand side of (3.25) is bounded below by

$$cP\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma},\ A_{2^N}\Big).$$

Letting $N \to \infty$, we see that

$$P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})],\overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big)=\overline{\gamma},\ \overline{L}^1(s),\ \overline{V}^1(s)\Big)$$
$$\geq cP^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})],\overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big)=\overline{\gamma}\Big).$$

By summing over all $\overline{\gamma}\in\Psi$, we get

$$P^\sharp(s_{i+1}=\infty, s_i=s)\geq cP^\sharp(s_i=s),$$

and the proof is finished. □

Next we will show the following lemma.

Lemma 3.8. *There exists $c<\infty$ such that for all i and k,*

$$P^\sharp\big(s_i+k\leq s_{i+1}<\infty \mid s_0,\cdots,s_i\big)\leq c2^{-\frac{k}{2}}. \tag{3.26}$$

Proof. Fix i,k and $s\in[n,\infty)$. We will prove that

$$P^\sharp\big(s_i+k\leq s_{i+1}<\infty \mid s_i=s\big)\leq c2^{-\frac{k}{2}}, \tag{3.27}$$

for some $c<\infty$. Recall that Ψ is the set of $\overline{\gamma}\in\Gamma(2^{s-2})$ such that

$$\{\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})],\overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big)=\overline{\gamma}\}\cap\{s_i=s\}$$

is possible. For $j\geq k$ and $\overline{\gamma}=(\gamma^1,\gamma^2)\in\Psi$, we will estimate

$$P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})],\overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big)=\overline{\gamma},\ s_{i+1}=s+j\Big).$$

By definition of s_i, we see that

$$P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})],\overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big)=\overline{\gamma},\ s_{i+1}=s+j\Big)$$
$$\leq P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})],\overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big)=\overline{\gamma},$$
$$\overline{S}^1[\overline{\tau}^1(2^{s+j-3}),\overline{\tau}^1(2^{s+j-2})]\cap\mathcal{B}(\frac{11}{12}2^s)\neq\emptyset\Big).$$

Hence we need to estimate

$$P\Big(\big(S^1[0,\tau^1(2^{s-2})],S^2[0,\tau^2(2^{s-2})]\big)=\overline{\gamma},$$
$$S^1[\tau^1(2^{s+j-3}),\tau^1(2^{s+j-2})]\cap\mathcal{B}(2^s)\neq\emptyset,\ A_{2^N}\Big) \tag{3.28}$$

for $N>s+j$. By the strong Markov property, the probability in (3.28) is bounded above by

$$c2^{-(N-s-j)\xi_d}P\Big(\big(S^1[0,\tau^1(2^{s-2})],S^2[0,\tau^2(2^{s-2})]\big)=\overline{\gamma},$$
$$S^1[\tau^1(2^{s+j-3}),\tau^1(2^{s+j-2})]\cap\mathcal{B}(2^s)\neq\emptyset,\ A_{2^{s+j}}\Big).$$

Let

$$\tau=\inf\big\{l\geq\tau^1(2^{s+j-3}):\ S^1(l)\in\mathcal{B}(2^s)\big\}.$$

For $d=3$, it is easy to see that

$$P(\tau<\infty)\leq c2^{-j}$$

for some $c < \infty$. Therefore,

$$P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma}, \\ S^1[\tau^1(2^{s+j-3}), \tau^1(2^{s+j-2})] \cap \mathcal{B}(2^s) \neq \emptyset,\ A_{2^{s+j}}\Big) \\ \leq c2^{-j} P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma},\ A_{2^{s+j-3}}\Big). \tag{3.29}$$

For $d = 2$, we see that

$$P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma}, \\ S^1[\tau^1(2^{s+j-3}), \tau^1(2^{s+j-2})] \cap \mathcal{B}(2^s) \neq \emptyset,\ A_{2^{s+j}}\Big) \\ \leq P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma},\ A_{2^{s+j-3}}, \\ \tau \leq \tau^1(2^{s+j-2}),\ S^1[\tau, \tau^1(2^{s+j})] \cap S^2[0, \tau^2(2^{s+j})] = \emptyset\Big). \tag{3.30}$$

By using the discrete Beurling estimate (see Theorem 2.5.2 in [12] for details), the right hand side of (3.30) is bounded above by

$$c2^{-\frac{j}{2}} P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma},\ A_{2^{s+j-3}}\Big).$$

If we write $w^i = \gamma^i(\text{len}\gamma^i)$, then

$$P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma},\ A_{2^{s+j-3}}\Big) \\ \leq c2^{-j\xi_d} P^{w^1,w^2}\big(A_{2^{s-1}}(\overline{\gamma})\big) P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma}\Big).$$

Combining these estimates, we see that (3.28) is bounded above by

$$c2^{-(N-s)\xi_d} P^{w^1,w^2}\big(A_{2^{s-1}}(\overline{\gamma})\big) P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma}\Big) 2^{-\frac{j}{2}}.$$

However, by Corollary 2.2,

$$2^{-(N-s)\xi_d} P^{w^1,w^2}\big(A_{2^{s-1}}(\overline{\gamma})\big) \leq \frac{1}{c_1} P^{w^1,w^2}\big(A_{2^N}(\overline{\gamma})\big),$$

and hence (3.28) can be bounded above by

$$P\Big(\big(S^1[0,\tau^1(2^{s-2})], S^2[0,\tau^2(2^{s-2})]\big) = \overline{\gamma},\ A_{2^N}\Big) c2^{-\frac{j}{2}}.$$

So dividing each side by $P(A_{2^N})$ first, and then by letting $N \to \infty$, we have

$$P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})], \overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big) = \overline{\gamma},\ s_{i+1} = s+j\Big) \\ \leq c2^{-\frac{j}{2}} P^\sharp\Big(\big(\overline{S}^1[0,\overline{\tau}^1(2^{s-2})], \overline{S}^2[0,\overline{\tau}^2(2^{s-2})]\big) = \overline{\gamma}\Big).$$

By summing over all $\overline{\gamma} \in \Psi$, we get

$$P^\sharp(s_i = s,\ s_{i+1} = s+j) \leq c2^{-\frac{j}{2}} P^\sharp(s_i = s).$$

Finally, by summing over all $j \geq k$, we finish the proof of the lemma. □

Proof of Proposition 3.6. As previously mentioned, in order to prove (3.21), it suffices to show that there exist c and α such that for all n,

$$P^\sharp(s^\star \geq n + \alpha \log n) \leq \frac{c}{n^2}.$$

However, once Lemma 3.7 and Lemma 3.8 have been established, then by applying the same argument used in the proof of Theorem 1.2 in [13], we conclude that (3.22) holds for some c and α. ☐

3.3. **Proof of Theorem 1.1.** In this subsection, we will prove Theorem 1.1. To establish it, we need the two lemmas below.

Lemma 3.9. *There exists* $c_1, c_2 \in (0,\infty)$ *such that*

$$P^{\sharp}\Big(\overline{\tau}^i\big(\sqrt{n}(\log n)^{-1}\big) \le n \le \overline{\tau}^i\big(\sqrt{n}(\log n)\big)\Big) \ge 1 - c_1 e^{-c_2(\log n)^2} \tag{3.31}$$

for each $i = 1, 2$.

Proof. Standard large deviation estimates give that

$$P\big(\tau^i\big(\sqrt{n}(\log n)^{-1}\big) \le n \le \tau^i\big(\sqrt{n}(\log n)\big)\Big) \ge 1 - c_1 e^{-c_2(\log n)^2}$$

for some $c_1, c_2 \in (0,\infty)$. Therefore, for each $N > n$,

$$\begin{aligned} P\Big(\tau^i\big(\sqrt{n}(\log n)^{-1}\big) > n,\ A_N\Big) &\le P\Big(\tau^i\big(\sqrt{n}(\log n)^{-1}\big) > n, \\ &\qquad S^1[\tau^1(n), \tau^1(N)] \cap S^2[\tau^2(n), \tau^2(N)] = \emptyset\Big) \\ &\le c_1 e^{-c_2(\log n)^2}\Big(\frac{N}{n}\Big)^{-\xi_d} \le c_1 e^{-\frac{c_2}{2}(\log n)^2} N^{-\xi_d}. \end{aligned}$$

Hence,

$$P\Big(\tau^i\big(\sqrt{n}(\log n)^{-1}\big) > n \mid A_N\Big) \le c e^{-c'(\log n)^2}.$$

Letting $N \to \infty$,

$$P^{\sharp}\Big(\overline{\tau}^i\big(\sqrt{n}(\log n)^{-1}\big) > n\Big) \le c e^{-c'(\log n)^2}.$$

Similarly, we see that

$$P^{\sharp}\Big(\overline{\tau}^i\big(\sqrt{n}(\log n)\big) < n\Big) \le c e^{-c'(\log n)^2},$$

and the lemma is proved. ☐

Recall that $\overline{K}^i(j)$ is the indicator function defined as in (3.2).

Lemma 3.10. *For all* $\epsilon > 0$, *there exists* $c = c_\epsilon < \infty$ *such that*

$$P^{\sharp}\Big(\sum_{j=0}^{n} \overline{K}^i(j) \ge n^{1-\zeta_d+\epsilon}\Big) \le c n^{-10}, \tag{3.32}$$

for each $i = 1, 2$.

Proof. Fix $\epsilon > 0$. Let

$$J^i(j,n) = \Big\{S^i[0,j] \cap S^i[j+1,n] = \emptyset\Big\}.$$

By Lemma 5.1 in [13] (see also Lemma 4.2 in [14]), we see that there exists a constant $c = c_\epsilon$ depending on ϵ such that

$$E\Big(\big(\sum_{j=0}^{n} J^i(j,n)\big)^{\frac{20}{\epsilon}}\Big) \le c n^{\frac{20}{\epsilon}(1-\zeta_d)}.$$

Therefore,

$$P\Big(\sum_{j=0}^{n} J^i(j,n) \geq n^{1-\zeta_d+\epsilon}\Big) \leq P\Big(\big(\sum_{j=0}^{n} J^i(j,n)\big)^{\frac{20}{\epsilon}} \geq n^{\frac{20}{\epsilon}(1-\zeta_d)+20}\Big)$$
$$\leq \frac{E\Big(\big(\sum_{j=0}^{n} J^i(j,n)\big)^{\frac{20}{\epsilon}}\Big)}{n^{\frac{20}{\epsilon}(1-\zeta_d)+20}}$$
$$\leq cn^{-20}.$$

Since $\xi_d < 2$, this implies that for each $N > n$,

$$P\Big(\sum_{j=0}^{n} J^i(j,n) \geq n^{1-\zeta_d+\epsilon} \mid A_N\Big) \leq cn^{-10}.$$

Letting $N \to \infty$,

$$P^{\sharp}\Big(\sum_{j=0}^{n} \overline{J}^i(j,n) \geq n^{1-\zeta_d+\epsilon}\Big) \leq cn^{-10},$$

where we let $\overline{J}^i(j,n)$ be the indicator function of the event

$$\Big\{\overline{S}^i[0,j] \cap \overline{S}^i[j+1,n] = \emptyset\Big\}.$$

Since $\overline{J}^i(j,n) \geq \overline{K}^i(j)$, we get the lemma. □

Proof of Theorem 1.1. For the lower bound of Theorem 1.1, we note that in the event $\overline{\Lambda}^i(n,\alpha)$, it follows that

$$\sum_{j=\overline{\tau}^i(2^{n-1})}^{\overline{\tau}^i(2^{n+\alpha \log n})} \overline{K}^i(j) \geq c(2^n)^{2-\xi_d}. \tag{3.33}$$

On the other hand, it follows from Proposition 3.6 and the Borel-Cantelli Lemma that with probability one for all n sufficiently large, $\overline{\Lambda}^i(n,\alpha)$ holds. By using Lemma 3.9, it is easy to check that if $\overline{\Lambda}^i(n,\alpha)$ holds for all sufficiently large n with probability one, then with probability one,

$$\liminf_{n\to\infty} \frac{\log\Big(\sum_{j=0}^{n} \overline{K}^i(j)\Big)}{\log n} \geq 1-\zeta_d.$$

For the upper bound, take $\epsilon > 0$. By Lemma 3.10 and the Borel-Cantelli Lemma, with probability one, for all n sufficiently large,

$$\sum_{j=0}^{n} \overline{K}^i(j) \leq n^{1-\zeta_d+\epsilon},$$

and hence

$$\limsup_{n\to\infty} \frac{\log\Big(\sum_{j=0}^{n} \overline{K}^i(j)\Big)}{\log n} \leq 1-\zeta_d+\epsilon.$$

Since ϵ is arbitrary, with probability one,

$$\limsup_{n\to\infty} \frac{\log\Big(\sum_{j=0}^{n} \overline{K}^i(j)\Big)}{\log n} \leq 1-\zeta_d$$

and

$$\lim_{n\to\infty} \frac{\log\Big(\sum_{j=0}^{n} \overline{K}^i(j)\Big)}{\log n} = 1 - \zeta_d$$

for $d = 2, 3$. □

4. Subdiffusivity

In this section, we will prove Theorem 1.2 and Theorem 1.3. We first give some notation.

For a locally finite connected graph $G = (V, E)$ with vertex set V and edge set E, let $d_G(\cdot, \cdot)$ be the shortest path graph distance on G. We define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{\substack{x,y\in V,\\ \{x,y\}\in E}} (f(x) - f(y))(g(x) - g(y)).$$

If we regard G as an electrical network with a unit resistor on each edge in E, then $\mathcal{E}(f, f)$ is the energy dissipation when the vertices of V are at a potential f. Set

$$H^2 = \{f \in \mathbb{R}^V : \mathcal{E}(f, f) < \infty\}.$$

Let A, B be disjoint subsets of V. The effective resistance between A and B is defined by

$$R_G(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{4.1}$$

Let $R_G(x, y) = R_G(\{x\}, \{y\})$.

We write $\mathcal{C}_n$ to represent the connected component of $\overline{\mathcal{G}} \cap \mathcal{B}(n)$ containing 0, and write $\mathcal{C}_n^i$ to represent the connected component of $\overline{\mathcal{G}}^i \cap \mathcal{B}(n)$ containing 0. Let $\mathcal{C}_n^c = \overline{\mathcal{G}} \backslash \mathcal{C}_n$ and $(\mathcal{C}_n^i)^c = \overline{\mathcal{G}}^i \backslash \mathcal{C}_n^i$.

Recall that $X = \big((X(n))_{n\geq 0}, P_x^{\overline{\mathcal{G}}}, x \in V(\overline{\mathcal{G}})\big)$ is the simple random walk on $\overline{\mathcal{G}}$. For a subset $A \subset V(\overline{\mathcal{G}})$, let

$$T_A = \inf\{k \geq 0 : X(k) \in A\},$$

and let $T_x = T_{\{x\}}$ for $x \in V(\overline{\mathcal{G}})$. Note that $T(n)$ in (1.14) is equal to $T_{\mathcal{C}_n^c}$. For $x, y \in \mathcal{C}_n$, we write

$$G_{\mathcal{C}_n}(x, y) = E_x^{\overline{\mathcal{G}}}\Big(\sum_{k=0}^{T_{\mathcal{C}_n^c}-1} \mathbf{1}\{X(k) = y\}\Big)$$

to denote Green's function for X in $\mathcal{C}_n$, and write $g_{\mathcal{C}_n}(x, y)$ to represent its kernel. Note that the fact

$$R_{\overline{\mathcal{G}}}(x, \mathcal{C}_n^c) = g_{\mathcal{C}_n}(x, x) \text{ for all } x \in \mathcal{C}_n$$

is well known. (For the proof of this fact, see, for example, section 3.2 in [1].)

Let $\mu_x = \#\{y \in \mathcal{C}_n : \{x,y\} \in E(\overline{\mathcal{G}})\}$ be the number of edges that contain $x \in \mathcal{C}_n$. Then,

$$\begin{aligned} E_0^{\overline{\mathcal{G}}}\big(T(n)\big) &= \sum_{x\in\mathcal{C}_n} G_{\mathcal{C}_n}(0,x) \\ &= \sum_{x\in\mathcal{C}_n} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) G_{\mathcal{C}_n}(x,x) \\ &= \sum_{x\in\mathcal{C}_n} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) g_{\mathcal{C}_n}(x,x)\mu_x \\ &\geq \sum_{x\in\mathcal{C}_n} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) R_{\overline{\mathcal{G}}}(x,\mathcal{C}_n^c), \end{aligned} \tag{4.2}$$

where we use $\mu_x \geq 1$ in the last inequality.

Since the proof of Theorem 1.3 is easy, we first give the proof.

4.1. **Proof of Theorem 1.3.** For each $\alpha > 0$, let

$$G' = \Big\{ \sum_{j=0}^{\overline{\tau}^i(n)} \overline{K}^i(j) \geq n^{2-\xi_3}(\log n)^{-\alpha}, \sum_{j=\overline{\tau}^i(n(\log n))}^{\overline{\tau}^i(n(\log n)^\alpha)} \overline{K}^i(j) \geq n^{2-\xi_3}, \text{ for } i=1,2\Big\}.$$

Then, by Proposition 3.6, it follows that there exist $c, \alpha < \infty$ such that

$$P^\sharp\big(G'\big) \geq 1 - c(\log n)^{-2}.$$

Let $b_n' = n(\log n)^\alpha$. By (4.2),

$$E_0^{\overline{\mathcal{G}}}\Big(T\big(b_n'\big)\Big) \geq \sum_{x\in\mathcal{C}_{b_n'}} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_{b_n'}^c}\big) R_{\overline{\mathcal{G}}}(x,\mathcal{C}_{b_n'}^c). \tag{4.3}$$

Note that

$$P_0^{\overline{\mathcal{G}}}\big(T_{\mathcal{C}_{b_n'}^c} = T_{(\mathcal{C}_{b_n'}^1)^c}\big) \vee P_0^{\overline{\mathcal{G}}}\big(T_{\mathcal{C}_{b_n'}^c} = T_{(\mathcal{C}_{b_n'}^2)^c}\big) \geq \frac{1}{2}.$$

By the symmetry between $\overline{S}^1$ and $\overline{S}^2$, we may assume that

$$P_0^{\overline{\mathcal{G}}}\Big(T_{\mathcal{C}_{b_n'}^c} = T_{(\mathcal{C}_{b_n'}^1)^c}\Big) \geq \frac{1}{2}. \tag{4.4}$$

Let

$$C_n^1 = \{\overline{S}^1(t) : 0 \leq t \leq \overline{\tau}^1(n), \ \overline{K}^1(t) = 1\}.$$

Then by (4.4),

$$P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_{b_n'}^c}\big) \geq \frac{1}{2}, \tag{4.5}$$

for all $x \in C_n^1$. By using the parallel law for electrical resistance, it follows that for $x \in C_n^1$,

$$R_{\overline{\mathcal{G}}}\Big(x,\mathcal{C}_{b_n'}^c\Big) = \frac{R_{\overline{\mathcal{G}}}\Big(x,(\mathcal{C}_{b_n'}^1)^c\Big) R_{\overline{\mathcal{G}}}\Big(x,(\mathcal{C}_{b_n'}^2)^c\Big)}{R_{\overline{\mathcal{G}}}\Big(x,(\mathcal{C}_{b_n'}^1)^c\Big) + R_{\overline{\mathcal{G}}}\Big(x,(\mathcal{C}_{b_n'}^2)^c\Big)}. \tag{4.6}$$

In the event G', we see that

$$R_{\overline{\mathcal{G}}}\Big(x,(\mathcal{C}_{b_n'}^i)^c\Big) \geq \sum_{j=\overline{\tau}^i(n(\log n))}^{\overline{\tau}^i(n(\log n)^\alpha)} \overline{K}^i(j) \geq n^{2-\xi_3}$$

for $x \in C_n^1$. Hence by (4.6),

$$R_{\overline{\mathcal{G}}}(x, \mathcal{C}_{b'_n}^c) \geq \frac{1}{2} n^{2-\xi_3} \tag{4.7}$$

for $x \in C_n^1$ in the event G'. Therefore,

$$\begin{aligned} E_0^{\overline{\mathcal{G}}}\Big(T(b'_n)\Big) &\geq \sum_{x \in \mathcal{C}_{b'_n}} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_{b'_n}^c}\big) R_{\overline{\mathcal{G}}}(x, \mathcal{C}_{b'_n}^c) \\ &\geq \sum_{x \in C_n^1} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_{b'_n}^c}\big) R_{\overline{\mathcal{G}}}(x, \mathcal{C}_{b'_n}^c) \\ &\geq \frac{1}{4} n^{2-\xi_3} \# C_n^1 \\ &\geq \frac{1}{4} n^{4-2\xi_3} (\log n)^{-\alpha}, \end{aligned}$$

in the event G'. By a simple reparameterisation, we conclude that

$$P^{\sharp}\Big(E_0^{\overline{\mathcal{G}}}\big(T(n)\big) \geq n^{4-2\xi_3} (\log n)^{-4\alpha}\Big) \geq 1 - c(\log n)^{-2}.$$

So, using the Borel-Cantelli Lemma, it follows that, with probability one for all k sufficiently large, the following holds:

$$E_0^{\overline{\mathcal{G}}}\big(T(2^k)\big) \geq (2^k)^{4-2\xi_3} (\log(2^k))^{-4\alpha}.$$

Take n sufficiently large and let k be such that $2^k \leq n < 2^{k+1}$. Then

$$E_0^{\overline{\mathcal{G}}}\big(T(n)\big) \geq E_0^{\overline{\mathcal{G}}}\big(T(2^k)\big) \geq (2^k)^{4-2\xi_3} (\log(2^k))^{-4\alpha} \geq c n^{4-2\xi_3} (\log n)^{-4\alpha},$$

for some $c > 0$, and the proof of Theorem 1.3 is finished. $\square$

4.2. **Loop-erased random walk.** From now on, we assume $d = 2$. Since $4 - 2\xi_2 = \frac{3}{2} < 2$, the proof of Theorem 1.3 in the previous subsection does not give subdiffusivity for $d = 2$. We first give the idea of the proof of Theorem 1.2 here. Recall that

$$E_0^{\overline{\mathcal{G}}}\big(T(n)\big) \geq \sum_{x \in \mathcal{C}_n} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) R_{\overline{\mathcal{G}}}(x, \mathcal{C}_n^c). \tag{4.8}$$

In order to prove the theorem, we will find a long enough sequence $x_j \in \mathcal{C}_n$ such that both $P_0^{\overline{\mathcal{G}}}\big(T_{x_j} < T_{\mathcal{C}_n^c}\big)$ and $R_{\overline{\mathcal{G}}}(x_j, \mathcal{C}_n^c)$ are large. Fix $\epsilon \in (0, \frac{1}{100})$. Assume that

$$P_0^{\overline{\mathcal{G}}}\Big(T_{\mathcal{C}_n^c} = T_{(\mathcal{C}_n^1)^c}\Big) \geq \frac{1}{2}.$$

Then for each global cut time t for $\overline{S}^1$ with $t < \overline{\tau}^1(\frac{n}{2})$,

$$P_0^{\overline{\mathcal{G}}}\big(T_{\overline{S}^1(t)} < T_{\mathcal{C}_n^c}\big) \geq \frac{1}{2},$$

and by a similar argument as in the proof of (4.7), we see that $R_{\overline{\mathcal{G}}}(\overline{S}^1(t), \mathcal{C}_n^c)$ can be bounded below by $n^{\frac{3}{4}} (\log n)^{-\alpha}$. The strategy for the proof of Theorem 1.2 is to find a long enough sequence of global cut times $t^1 < t^2 < \cdots < t^l < \overline{\tau}^1(\frac{n}{2})$ for $\overline{S}^1$ such that each $t^{j+1} - t^j$ is large. We will show that there are $d_n := n^{\frac{3}{10}}$ global cut times $t^1 < t^2 < \cdots < t^{d_n} < \overline{\tau}^1(\frac{n}{2})$ such that $t^{j+1} - t^j > n^{\frac{6}{5}}$ for each j (see (4.33) for details). By the theory of electrical networks, it follows that if $d_{\overline{\mathcal{G}}}\big(x, \overline{S}^1(t^j)\big)$ is small, then both

$$\big|P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) - P_0^{\overline{\mathcal{G}}}\big(T_{\overline{S}^1(t^j)} < T_{\mathcal{C}_n^c}\big)\big| \text{ and } \big|R_{\overline{\mathcal{G}}}(x, \mathcal{C}_n^c) - R_{\overline{\mathcal{G}}}(\overline{S}^1(t^j), \mathcal{C}_n^c)\big|$$

are also small. Indeed, we will show that if

$$x \in \Big\{y \in \mathcal{C}_n : \ d_{\overline{\mathcal{G}}}\big(y, \overline{S}^1(t^j)\big) \le n^{\frac{3}{4}} (\log n)^{-2\alpha}\Big\} =: V^j,$$

then

$$P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) \ge \frac{1}{4} \text{ and } R_{\overline{\mathcal{G}}}(x, \mathcal{C}_n^c) \ge n^{\frac{3}{4}} (\log n)^{-\alpha}. \tag{4.9}$$

(See (4.38) and (4.40).) Because each $t^{j+1} - t^j$ $(> n^{\frac{6}{5}})$ is large, it can be shown that V^j is disjoint. For the cardinality of V^j, we will show that

$$\# V^j \ge n^{\frac{39}{40} - \epsilon}. \tag{4.10}$$

(See Lemma 4.4 below.) Combining (4.9) and (4.10) with (4.8), we have

$$\begin{aligned} E_0^{\overline{\mathcal{G}}}\big(T(n)\big) &\ge \sum_{x \in \mathcal{C}_n} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) R_{\overline{\mathcal{G}}}(x, \mathcal{C}_n^c) \\ &\ge \sum_{x \in \bigcup_{j=1}^{d_n} V^j} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_n^c}\big) R_{\overline{\mathcal{G}}}(x, \mathcal{C}_n^c) \\ &\ge \frac{1}{4} n^{\frac{3}{4}} (\log n)^{-\alpha} \sum_{j=1}^{d_n} \# V^j \\ &\ge c n^{\frac{3}{4}} (\log n)^{-\alpha} n^{\frac{39}{40} - \epsilon} d_n \\ &= c n^{\frac{81}{40} - \epsilon} (\log n)^{-\alpha}, \end{aligned}$$

and we obtain Theorem 1.2 (see Figure 7).

To prove (4.10), we need to estimate the graph distance on $\overline{\mathcal{G}}$, which we will do in this subsection. The proof of Theorem 1.2 will be given in Section 4.3.

Now we will establish estimates of the graph distance on $\overline{\mathcal{G}}$ by using a loop-erased random walk (LERW). For this purpose, we begin with the introduction of the definition of LERW.

For a path $\lambda = [\lambda(0), \cdots, \lambda(m)]$ of length m in $\mathbb{Z}^2$, assign a self-avoiding walk path $L(\lambda)$ in the following way. Let

$$\sigma_0 = \sup\{j : \lambda(j) = \lambda(0)\},$$

and for $i > 0$,

$$\sigma_i = \sup\{j : \lambda(j) = \lambda(\sigma_{i-1} + 1)\}.$$

Let

$$l = \inf\{i : \sigma_i = m\}.$$

Now define

$$\widehat{\lambda}(i) = \lambda(\sigma_i)$$

and

$$L(\lambda) = [\widehat{\lambda}(0), \widehat{\lambda}(1), \cdots, \widehat{\lambda}(l)].$$

This self-avoiding path clearly satisfies $(L(\lambda))(0) = \lambda(0)$ and $(L(\lambda))(l) = \lambda(m)$. Let $S = (S(n))_{n \ge 0}$ be the simple random walk in $\mathbb{Z}^2$ started at 0, and let

$$\tau(n) = \inf\{k \ge 0 : \ |S(k)| \ge n\}.$$

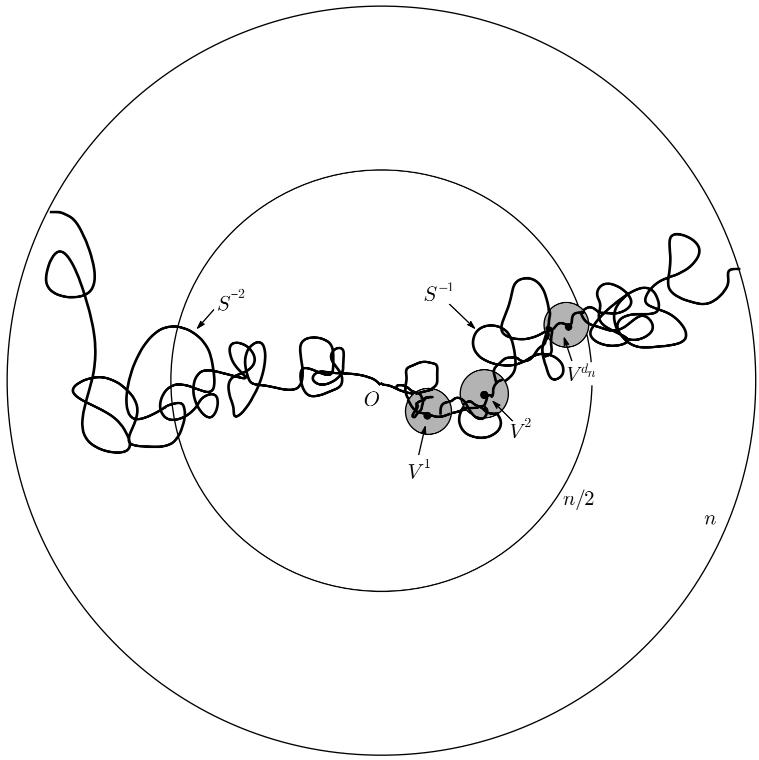

FIGURE 7. Illustration of the idea of the proof of Theorem 1.2.

We denote the length of $L\big(S[0,\tau(n)]\big)$ by M_n. Then the following two propositions have been proved.

Proposition 4.1 ([9, Theorem 3])**.** *It follows that*

$$\lim_{n\to\infty}\frac{\log E(M_n)}{\log n}=\frac{5}{4}. \tag{4.11}$$

The quantity 5/4 is called the *growth exponent* for the planar loop-erased random walk. For tail bounds on M_n, the following holds.

Proposition 4.2 ([2, Theorem 1.1])**.** *There exists $c>0$ such that for all $t\geq 0$,*

$$P\big(M_n > tE(M_n)\big)\leq 2e^{-ct}. \tag{4.12}$$

Fix $\epsilon\in(0,\frac{1}{100})$. By Proposition 4.1, we see that for all n sufficiently large,

$$E(M_n)\leq n^{\frac{5}{4}+\epsilon}. \tag{4.13}$$

From now on, assume n is large so that (4.13) holds.

For $k<l$, let $\mathcal{G}(k,l)=\big(V(\mathcal{G}(k,l)),E(\mathcal{G}(k,l))\big)$ be the graph with

$$V(\mathcal{G}(k,l))=\big\{S(j):\ k\leq j\leq l\big\},\ E(\mathcal{G}(k,l))=\big\{\{S(j),S(j+1)\}:\ k\leq j<l\big\}.$$

Let

$$N_n=\#\Big\{S(k):\ n\leq k\leq n(\log n)^8,\ d_{\mathcal{G}(0,k)}(0,S(k))\leq n^{\frac{5}{8}+\epsilon}\Big\}.$$

The key result in this subsection is the following proposition.

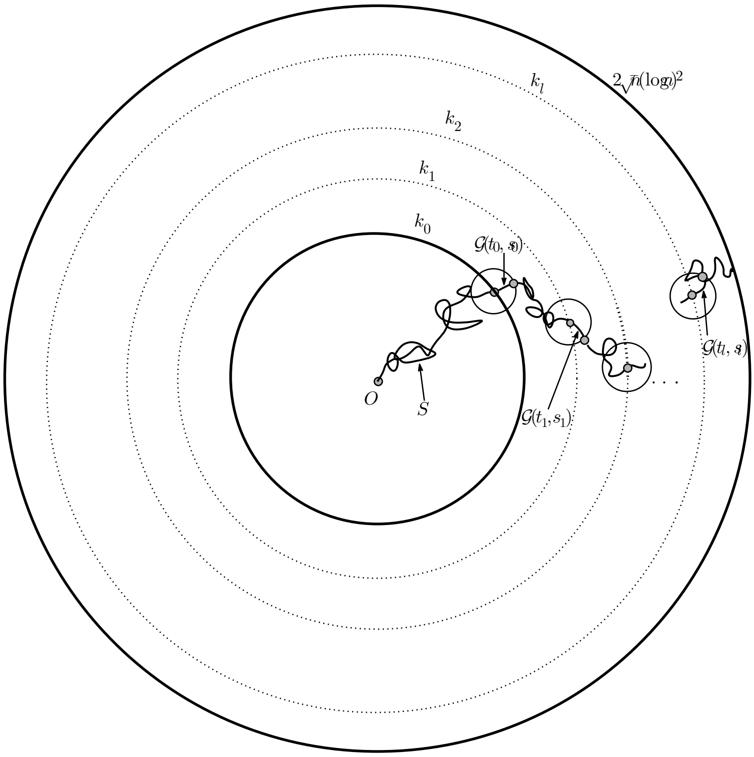

FIGURE 8. $\mathcal{G}(t_j, s_j)$ are disjoint.

Proposition 4.3. *There exists $c > 0$ such that*

$$P\Big(N_n \geq n^{\frac{13}{16}} (\log n)^{-7}\Big) \geq 1 - e^{-c(\log n)^2}. \tag{4.14}$$

Proof. By a standard large deviation estimate, we see that

$$P\Big(n < \tau\big(\sqrt{n}(\log n)^2\big) < \tau\big(3\sqrt{n}(\log n)^2\big) < n(\log n)^8\Big) \geq 1 - e^{-c(\log n)^2},$$

for some $c > 0$. For $j = 0, 1, \cdots$, define

$$k_j = \sqrt{n}(\log n)^2 + jn^{\frac{5}{16}},$$

and let

$$l = \sup\big\{j : \ k_j < 2\sqrt{n}(\log n)^2\big\}.$$

Note that

$$l \asymp \frac{\sqrt{n}(\log n)^2}{n^{\frac{5}{16}}} = n^{\frac{3}{16}} (\log n)^2.$$

For $j = 0, 1, \cdots, l$, we write

$$t_j = \tau(k_j).$$

Let

$$s_j = \inf\Big\{k \geq t_j : \ \big|S(k) - S(t_j)\big| \geq \frac{1}{3} n^{\frac{5}{16}}\Big\}$$

for $j = 0, 1, \cdots, l$. Notice that $s_{j-1} < t_j < s_j$ for $j = 1, 2, \cdots, l$ and $\mathcal{G}(t_j, s_j)$ are disjoint. Moreover, $\big\{\mathcal{G}(t_j, s_j) - S(t_j)\big\}$ is i.i.d. (see Figure 8).

Let

$$N_{j,n} = \#\Big\{S(k):\ t_j \le k \le s_j,\ d_{\mathcal{G}(t_j,k)}\big(S(t_j), S(k)\big) \le n^{\frac{5}{8}}\Big\}.$$

We will show that

$$N_{j,n} \ge n^{\frac{5}{8}}(\log n)^{-6},$$

with positive probability for each j. It follows from a standard large deviation estimate that

$$P\Big(n^{\frac{5}{8}}(\log n)^{-4} \le s_j - t_j \le n^{\frac{5}{8}}(\log n)^{4}\Big) \ge 1 - e^{-c(\log n)^2},$$

for some $c > 0$. So assume $s_j - t_j \ge n^{\frac{5}{8}}(\log n)^{-4}$; then

$$\#V\Big(\mathcal{G}\big(t_j, t_j + n^{\frac{5}{8}}(\log n)^{-4}\big)\Big) \le N_{j,n}.$$

However, by the translation invariance of the simple random walk, we see that

$$\begin{aligned} &P\Big(\#V\Big(\mathcal{G}\big(t_j, t_j + n^{\frac{5}{8}}(\log n)^{-4}\big)\Big) \ge n^{\frac{5}{8}}(\log n)^{-6}\Big) \\ &= P\Big(\#V\Big(\mathcal{G}\big(0, n^{\frac{5}{8}}(\log n)^{-4}\big)\Big) \ge n^{\frac{5}{8}}(\log n)^{-6}\Big). \end{aligned} \tag{4.15}$$

For moment estimates of $\#V\Big(\mathcal{G}\big(0, n^{\frac{5}{8}}(\log n)^{-4}\big)\Big)$, the following are known:

$$E\Big(\#V\big(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4})\big)\Big) \asymp n^{\frac{5}{8}}(\log n)^{-5}, \tag{4.16}$$

$$\mathrm{Var}\Big(\#V\big(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4})\big)\Big) \asymp n^{\frac{5}{4}}(\log n)^{-12}. \tag{4.17}$$

(4.16) is from Lemma 2.6 in [7] with the estimates (2.2) and (2.3) in [6]. (4.17) is from Theorem 4.2 in [8]. Therefore,

$$\begin{aligned} &P\Big(\#V\big(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4})\big) < n^{\frac{5}{8}}(\log n)^{-6}\Big) \\ &\le P\Big(\#V\big(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4})\big) \le E\big(\#V(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4}))\big)c(\log n)^{-1}\Big) \\ &\le P\Big(\Big|\#V\big(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4})\big) - E\big(\#V(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4}))\big)\Big| \\ &\qquad \ge \frac{1}{2}E\big(\#V(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4}))\big)\Big) \\ &\le \frac{4\mathrm{Var}\Big(\#V\big(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4})\big)\Big)}{E\Big(\#V\big(\mathcal{G}(0, n^{\frac{5}{8}}(\log n)^{-4})\big)\Big)^2} \\ &\le c(\log n)^{-2} \end{aligned}$$

for some $c < \infty$. Hence if we write

$$I_{j,n} = \mathbf{1}\{N_{j,n} \ge n^{\frac{5}{8}}(\log n)^{-6}\},$$

then

$$\begin{aligned} P(I_{j,n} = 1) \ge P\Big(s_j - t_j \ge n^{\frac{5}{8}}(\log n)^{-4},\ \#V\Big(\mathcal{G}\big(t_j, t_j + n^{\frac{5}{8}}(\log n)^{-4}\big)\Big) \\ \ge n^{\frac{5}{8}}(\log n)^{-6}\Big) \ge \frac{1}{2}. \end{aligned}$$

Since $I_{j,n}$ are i.i.d., we see that

$$P\Big(\sum_{j=0}^{(\log n)^2} I_{j,n}=0\Big)\le e^{-\delta(\log n)^2}$$

for some $\delta>0$. Therefore,

$$P\Big(\sum_{j=0}^{l} I_{j,n}\le l(\log n)^{-2}\Big)\le le^{-\delta(\log n)^2}\le e^{-\frac{\delta}{2}(\log n)^2} \tag{4.18}$$

for large n.

Next we will estimate $d_{\mathcal{G}(0,S(t_j))}\big(0,S(t_j)\big)$ for each j by using (4.11) and (4.12). Recall that

$$\sqrt{n}(\log n)^2\le k_j\le 2\sqrt{n}(\log n)^2$$

for each $j=0,1,\cdots,l$. Hence by (4.13),

$$E(M_{k_j})\le n^{\frac{5}{8}+\frac{2\epsilon}{3}}$$

for all $j=0,1,\cdots,l$ and n sufficiently large. Therefore, by (4.12),

$$\begin{aligned}P\Big(M_{k_j}\ge\frac{1}{2}n^{\frac{5}{8}+\epsilon}\text{ for some } j=0,1,\cdots,l\Big)&\le\sum_{j=0}^{l}P\Big(M_{k_j}\ge\tfrac{1}{2}n^{\frac{5}{8}+\epsilon}\Big)\\&\le\sum_{j=0}^{l}P\Big(M_{k_j}\ge\tfrac{1}{2}n^{\frac{\epsilon}{3}}E\big(M_{k_j}\big)\Big)\\&\le 2le^{-c'n^{\frac{\epsilon}{3}}}\le e^{-cn^{\frac{\epsilon}{3}}}\end{aligned}$$

for some c,c'. Since $L\big(S[0,\tau(k_j)]\big)(0)=0$ and the end point of $L\big(S[0,\tau(k_j)]\big)$ is $S(\tau(k_j))$, we see that $d_{\mathcal{G}(0,S(t_j))}\big(0,S(t_j)\big)\le M_{k_j}$. Hence,

$$P\Big(d_{\mathcal{G}(0,S(t_j))}\big(0,S(t_j)\big)\ge\frac{1}{2}n^{\frac{5}{8}+\epsilon}\text{ for some } j=0,1,\cdots,l\Big)\le e^{-cn^{\frac{\epsilon}{3}}}. \tag{4.19}$$

Combining (4.19) with (4.18), we see that

$$\begin{aligned}P\Big(d_{\mathcal{G}(0,S(t_j))}\big(0,S(t_j)\big)\le\frac{1}{2}n^{\frac{5}{8}+\epsilon}\text{ for all } j=0,\cdots,l,\text{ and }\sum_{j=0}^{l}I_{j,n}&\\ \ge l(\log n)^{-2}\Big)&\ge 1-e^{-c(\log n)^2}\end{aligned}$$

for some $c>0$. So assume

$$\begin{aligned}&n<\tau\big(\sqrt{n}(\log n)^2\big)<\tau\big(3\sqrt{n}(\log n)^2\big)<n(\log n)^8,\\&d_{\mathcal{G}(0,S(t_j))}\big(0,S(t_j)\big)\le\frac{1}{2}n^{\frac{5}{8}+\epsilon}\text{ for all } j=0,\cdots,l,\\&\sum_{j=0}^{l}I_{j,n}\ge l(\log n)^{-2}.\end{aligned}$$

Then

$$N_n\ge\sum_{j=0}^{l}N_{j,n}\ge\sum_{j=0}^{l}I_{j,n}n^{\frac{5}{8}}(\log n)^{-6}\ge l(\log n)^{-2}n^{\frac{5}{8}}(\log n)^{-6}\asymp n^{\frac{13}{16}}(\log n)^{-6},$$

and the proposition is proved. □

Let $\overline{\mathcal{G}}^i(k,l) = \big(V(\overline{\mathcal{G}}^i(k,l)), E(\overline{\mathcal{G}}^i(k,l))\big)$ be the graph with

$$V(\overline{\mathcal{G}}^i(k,l)) = \{\overline{S}^i(j): \ k \le j \le l\}, \ E(\overline{\mathcal{G}}^i(k,l)) = \{\{\overline{S}^i(j), \overline{S}^i(j+1)\}: \ k \le j < l\}$$

for $k < l$ and $i = 1,2$. Define

$$\overline{N}^i_n = \#\Big\{\overline{S}^i(k): \ n \le k \le n(\log n)^8, \ d_{\overline{\mathcal{G}}^i(0,k)}(0, \overline{S}^i(k)) \le n^{\frac{5}{8}+\epsilon}\Big\}.$$

By using a similar argument as in the proof of Lemma 3.9, the following lemma is an easy consequence of Proposition 4.3. So we omit the proof.

Lemma 4.4. *There exists $c > 0$ such that*

$$P^\sharp\Big(\overline{N}^i_n \ge n^{\frac{13}{16}}(\log n)^{-7} \text{ for } i = 1,2\Big) \ge 1 - e^{-c(\log n)^2}. \tag{4.20}$$

Remark 4.5. By using a similar idea as in the proof of Proposition 4.3, one can show that

$$P\Big(\#S^i[0, \tau^i(n)] \le n^2(\log n)^{-10}\Big) \le e^{-\delta(\log n)^2},$$

for some $\delta > 0$. Therefore, we see that

$$P^\sharp\Big(\#\overline{S}^i[0, \overline{\tau}^i(n)] \ge n^2(\log n)^{-10}\Big) \ge 1 - e^{-\frac{\delta}{2}(\log n)^2}.$$

Using the Borel-Cantelli Lemma, it follows that with probability one for all n sufficiently large,

$$\#\overline{S}^i[0, \overline{\tau}^i(n)] \ge n^2(\log n)^{-10}. \tag{4.21}$$

4.3. **Proof of Theorem 1.2.** In this subsection, we will give the proof of Theorem 1.2. For this purpose, we first define several events as follows. Let $d = 2$. Fix $\epsilon \in (0, \frac{1}{100})$. Let

$$G^{1,i} = \Big\{ \sum_{j=0}^{n^2} \overline{K}^i(j) \ge n^{\frac{3}{4}}(\log n)^{-\alpha}, \ 2n^2 < \overline{\tau}^i(n(\log n)^2), \qquad \sum_{j=\overline{\tau}^i(n(\log n)^2)}^{\overline{\tau}^i(n(\log n)^{2+\alpha})} \overline{K}^i(j) \ge n^{\frac{3}{4}}(\log n)\Big\}, \tag{4.22}$$

and $G^1 = G^{1,1} \cap G^{1,2}$. By Proposition 3.6, there exist $\alpha, c < \infty$ such that

$$P^\sharp(G^1) \ge 1 - c(\log n)^{-2}. \tag{4.23}$$

So fix such an α.

Let

$$a_n = n^{\frac{6}{5}-\epsilon},$$

and

$$k_j = ja_n$$

for $j = 0, 1, \cdots$. We write $I^j = [k_{j-1}, k_j]$. Let

$$m = \inf\{j: \ k_j > n^2\}.$$

Then,

$$m \asymp \frac{n^2}{a_n} = n^{\frac{4}{5}+\epsilon}.$$

Define the event

$$G^{2,i} = \Big\{ \sum_{t=k_{j-1}}^{k_j} \overline{K}^i(t) \le n^{\frac{9}{20}}, \ \text{for all } j = 1, 2, \cdots, m\Big\}, \tag{4.24}$$

and let $G^2 = G^{2,1} \cap G^{2,2}$. For $t \in I^j$, let

$$Z^i_j(t) = \mathbf{1}\Big\{ S^i[k_{j-1}, t] \cap S^i[t+1, k_j] = \emptyset \Big\}$$

and

$$\overline{Z}^i_j(t) = \mathbf{1}\Big\{ \overline{S}^i[k_{j-1}, t] \cap \overline{S}^i[t+1, k_j] = \emptyset \Big\}.$$

By Lemma 3.10, it follows that there exists a constant $c = c_\epsilon$ depending on ϵ such that

$$\begin{aligned} P\Big(\sum_{t=k_{j-1}}^{k_j} Z^i_j(t) \geq n^{\frac{9}{20}} \text{ for some } j = 1, 2, \cdots, m \Big) &\leq \sum_{j=1}^{m} P\Big(\sum_{t=k_{j-1}}^{k_j} Z^i_j(t) \geq n^{\frac{9}{20}} \Big) \\ &= mP\Big(\sum_{t=0}^{k_1} Z^i_1(t) \geq n^{\frac{9}{20}} \Big) \\ &\leq cmn^{-10} \leq cn^{-9}. \end{aligned}$$

Therefore, by using a similar argument as in the proof of Lemma 3.9, we see that

$$P^\sharp\Big(\sum_{t=k_{j-1}}^{k_j} \overline{Z}^i_j(t) \geq n^{\frac{9}{20}} \text{ for some } j = 1, 2, \cdots, m \Big) \leq cn^{-6} \tag{4.25}$$

for some $c = c_\epsilon < \infty$. Since $\sum_{t=k_{j-1}}^{k_j} \overline{K}^i(t) \leq \sum_{t=k_{j-1}}^{k_j} \overline{Z}^i_j(t)$, it follows that

$$P^\sharp\big((G^{2,i})^c \big) \leq cn^{-6}$$

and

$$P^\sharp(G^2) \geq 1 - cn^{-6}. \tag{4.26}$$

For each $j = 1, 2, \cdots, m$, define

$$\overline{N}^i_{j,n} = \# \overline{V}^i_{j,n},$$

where

$$\overline{V}^i_{j,n} = \Big\{ \overline{S}^i(t) : \ k_j \leq t \leq k_j + a_n (\log n)^8, \ d_{\overline{\mathcal{G}}^i(k_{j-1}, t)}(\overline{S}^i(k_{j-1}), \overline{S}^i(t)) \leq n^{\frac{3}{4}} \Big\}. \tag{4.27}$$

Let

$$G^{3,i} = \Big\{ \overline{N}^i_{j,n} \geq n^{\frac{39}{40} - \epsilon} (\log n)^{-7} \text{ for all } j = 1, 2, \cdots, m \Big\}, \tag{4.28}$$

and $G^3 = G^{3,1} \cap G^{3,2}$. Although $P^\sharp$ is not translation invariant, by using similar arguments as in the proof of (4.25), it follows from Lemma 4.4 that

$$P^\sharp(G^2) \geq 1 - e^{-c(\log n)^2} \tag{4.29}$$

for some $c > 0$. Hence we get the following lemma.

Lemma 4.6. *For every $\epsilon \in (0, \frac{1}{100})$, there exists a constant $c = c_\epsilon < \infty$ such that*

$$P^\sharp\big(G^1 \cap G^2 \cap G^3 \big) \geq 1 - c(\log n)^{-2}. \tag{4.30}$$

Now we will prove Theorem 1.2.

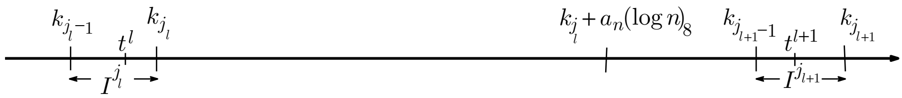

FIGURE 9. Good intervals.

Proof of Theorem 1.2. We will give a lower bound of $E_0^{\overline{\mathcal{G}}}\Big(T\big(b_n\big)\Big)$ in the event $G^1 \cap G^2 \cap G^3$, where $b_n = n(\log n)^{2+\alpha}$. So assume $G^1 \cap G^2 \cap G^3$ holds. By (4.2),

$$E_0^{\overline{\mathcal{G}}}\Big(T\big(b_n\big)\Big) \geq \sum_{x \in \mathcal{C}_{b_n}} P_0^{\overline{\mathcal{G}}}\big(T_x < T_{\mathcal{C}_{b_n}^c}\big) R_{\overline{\mathcal{G}}}(x, \mathcal{C}_{b_n}^c). \tag{4.31}$$

By the symmetry between $\overline{S}^1$ and $\overline{S}^2$, we may assume that

$$P_0^{\overline{\mathcal{G}}}\Big(T_{\mathcal{C}_{b_n}^c} = T_{(\mathcal{C}_{b_n}^1)^c}\Big) \geq \frac{1}{2}; \tag{4.32}$$

the converse can be proved similarly. In this case, the probability that X passes through all global cut points in $\mathcal{C}_{b_n}^1$ before it goes outside in $\mathcal{C}_{b_n}$ is at least $\frac{1}{2}$.

Recall that we write

$$a_n = n^{\frac{6}{5}-\epsilon},\ k_j = ja_n,\ I^j = [k_{j-1}, k_j],$$

and $m = \inf\{j:\ k_j > n^2\}$. We will say that I^j is *good* if there is at least one global cut time for $\overline{S}^1$ in I^j, i.e.,

$$\sum_{t=k_{j-1}}^{k_j} \overline{K}^1(t) \geq 1.$$

In the event $G^1 \cap G^2$, we have

$$\#\Big\{1 \leq j \leq m:\ I^j \text{ is good }\Big\} \geq \frac{n^{\frac{3}{4}}(\log n)^{-\alpha}}{n^{\frac{9}{20}}} = n^{\frac{3}{10}}(\log n)^{-\alpha}. \tag{4.33}$$

Therefore, there are $d_n := n^{\frac{3}{10}}(\log n)^{-\alpha-9}$ indexes $j_1 < j_2 < \cdots < j_{d_n}$ such that I^{j_l} is good and

$$k_{j_{l+1}} - k_{j_l} = (j_{l+1} - j_l)a_n \geq a_n(\log n)^9$$

for each l. So we write t^l to represent a global cut time for $\overline{S}^1$ in the (good) interval I^{j_l}.

Recall that $\overline{V}_{j,n}^1$ is defined as in (4.27). If $\overline{S}^1(t) \in \overline{V}_{j_l,n}^1$, then $k_{j_l} \leq t \leq k_{j_l} + a_n(\log n)^8$ and

$$d_{\overline{\mathcal{G}}^1(k_{j_l-1},t)}\big(\overline{S}^1(k_{j_l-1}), \overline{S}^1(t)\big) \leq n^{\frac{3}{4}}.$$

However, since $t^l \in I^{j_l} = [k_{j_l-1}, k_{j_l}]$ is a global cut time for $\overline{S}^1$, we see that

$$\begin{aligned} d_{\overline{\mathcal{G}}^1(k_{j_l-1},t)}\big(\overline{S}^1(k_{j_l-1}), \overline{S}^1(t)\big) = {} & d_{\overline{\mathcal{G}}^1(k_{j_l-1},t^l)}\big(\overline{S}^1(k_{j_l-1}), \overline{S}^1(t^l)\big) \\ & + d_{\overline{\mathcal{G}}^1(t^l,t)}\big(\overline{S}^1(t^l), \overline{S}^1(t)\big). \end{aligned}$$

Hence,

$$d_{\overline{\mathcal{G}}^1(t^l,t)}\big(\overline{S}^1(t^l), \overline{S}^1(t)\big) \leq n^{\frac{3}{4}} \tag{4.34}$$

for all $\overline{S}^1(t) \in \overline{V}_{j_l,n}^1$ and $l = 1, 2, \cdots, d_n$.

By using the parallel law for electrical resistance,

$$R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t^l), \mathcal{C}_{b_n}^c\Big) = \frac{R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t^l), (\mathcal{C}_{b_n}^1)^c\Big) R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t^l), (\mathcal{C}_{b_n}^2)^c\Big)}{R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t^l), (\mathcal{C}_{b_n}^1)^c\Big) + R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t^l), (\mathcal{C}_{b_n}^2)^c\Big)}. \tag{4.35}$$

In the event G^1, we have

$$R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t^l), (\mathcal{C}_{b_n}^i)^c\Big) \geq \sum_{j=\overline{\tau}^i(n(\log n)^2)}^{\overline{\tau}^i(n(\log n)^{2+\alpha})} \overline{K}^i(j) \geq n^{\frac{3}{4}}(\log n). \tag{4.36}$$

Therefore, the right hand side of (4.35) is bounded below by a constant times $n^{\frac{3}{4}}(\log n)$. By (4.34),

$$R_{\overline{\mathcal{G}}}\big(\overline{S}^1(t^l), \overline{S}^1(t)\big) \leq d_{\overline{\mathcal{G}}^1(t^l,t)}\big(\overline{S}^1(t^l), \overline{S}^1(t)\big) \leq n^{\frac{3}{4}} \tag{4.37}$$

for all $\overline{S}^1(t) \in \overline{V}^1_{j_l,n}$. Hence,

$$R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t), \mathcal{C}_{b_n}^c\Big) \geq R_{\overline{\mathcal{G}}}\Big(\overline{S}^1(t^l), \mathcal{C}_{b_n}^c\Big) - R_{\overline{\mathcal{G}}}\big(\overline{S}^1(t^l), \overline{S}^1(t)\big) \geq cn^{\frac{3}{4}}(\log n) \tag{4.38}$$

for all $\overline{S}^1(t) \in \overline{V}^1_{j_l,n}$ and $l = 1, 2, \cdots, d_n$.

To give a lower bound on the right hand side of (4.31), we will estimate

$$P_0^{\overline{\mathcal{G}}}\Big(T_{\overline{S}^1(t)} < T_{\mathcal{C}_{b_n}^c}\Big)$$

for $\overline{S}^1(t) \in \overline{V}^1_{j_l,n}$. Assume that $T_{\mathcal{C}_{b_n}^c} = T_{(\mathcal{C}_{b_n}^1)^c}$; then

$$T_{\overline{S}^1(t^l)} < T_{\mathcal{C}_{b_n}^c}.$$

Therefore, by (4.32),

$$P_0^{\overline{\mathcal{G}}}\Big(T_{\overline{S}^1(t^l)} < T_{\mathcal{C}_{b_n}^c}\Big) \geq \frac{1}{2} \tag{4.39}$$

for all $l = 1, 2, \cdots, d_n$. For $\overline{S}^1(t) \in \overline{V}^1_{j_l,n}$, we have

$$\begin{aligned} P_0^{\overline{\mathcal{G}}}\Big(T_{\overline{S}^1(t)} < T_{\mathcal{C}_{b_n}^c}\Big) &= P_0^{\overline{\mathcal{G}}}\Big(T_{\overline{S}^1(t^l)} < T_{\mathcal{C}_{b_n}^c}\Big) P_{\overline{S}^1(t^l)}^{\overline{\mathcal{G}}}\Big(T_{\overline{S}^1(t)} < T_{\mathcal{C}_{b_n}^c}\Big) \\ &\geq \frac{1}{2} P_{\overline{S}^1(t^l)}^{\overline{\mathcal{G}}}\Big(T_{\overline{S}^1(t)} < T_{\mathcal{C}_{b_n}^c}\Big). \end{aligned}$$

However, it follows from (4) in [3] that

$$P_{\overline{S}^1(t^l)}^{\overline{\mathcal{G}}}\Big(T_{\mathcal{C}_{b_n}^c} < T_{\overline{S}^1(t)}\Big) \leq \frac{R_{\overline{\mathcal{G}}}\big(\overline{S}^1(t^l), \overline{S}^1(t)\big)}{R_{\overline{\mathcal{G}}}\big(\overline{S}^1(t^l), \mathcal{C}_{b_n}^c\big)}.$$

Therefore, by (4.36) and (4.37),

$$P_0^{\overline{\mathcal{G}}}\Big(T_{\overline{S}^1(t)} < T_{\mathcal{C}_{b_n}^c}\Big) \geq \frac{1}{2}(1 - (\log n)^{-1}) \geq \frac{1}{4}. \tag{4.40}$$

So combining (4.38) and (4.40) with (4.31), we have

$$E_0^{\overline{\mathcal{G}}}\big(T(b_n)\big) \geq cn^{\frac{3}{4}}(\log n)\#\Big(\bigcup_{l=1}^{d_n} \overline{V}^1_{j_l,n}\Big). \tag{4.41}$$

Recall that

$$k_{j_l-1} \leq t^l \leq k_{j_l} < k_{j_l} + a_n(\log n)^8 < k_{j_l} + a_n(\log n)^9 \leq k_{j_{l+1}-1} \leq t^{l+1} \leq k_{j_{l+1}}$$

and $\overline{V}^1_{j_l,n} \subset \overline{\mathcal{G}}^1\big(k_{j_l}, k_{j_l} + a_n(\log n)^8\big)$ (see Figure 9). Since t^l is a global cut time, we see that $\overline{V}^1_{j_l,n}$ are disjoint. Hence, by (4.28), the right hand side of (4.41) is bounded below by

$$cn^{\frac{3}{4}}(\log n)n^{\frac{39}{40}-\epsilon}(\log n)^{-7}d_n = cn^{\frac{3}{4}+\frac{39}{40}-\epsilon+\frac{3}{10}}(\log n)^{-\alpha-15} = cn^{\frac{81}{40}-\epsilon}(\log n)^{-\alpha-15},$$

in the event $G^1 \cap G^2 \cap G^3$. Therefore, by Lemma 4.6 and a simple reparameterisation, we conclude that

$$P^\sharp\Big(E_0^{\overline{\mathcal{G}}}\big(T(n)\big) \geq n^{\frac{81}{40}-\epsilon}(\log n)^{-4\alpha-21}\Big) \geq 1 - c(\log n)^{-2}.$$

So, using the Borel-Cantelli Lemma, it follows that, with probability one for all k sufficiently large, the following holds:

$$E_0^{\overline{\mathcal{G}}}\big(T(2^k)\big) \geq (2^k)^{\frac{81}{40}-\epsilon}(\log(2^k))^{-4\alpha-21}.$$

Take n sufficiently large and let k be such that $2^k \leq n < 2^{k+1}$. Then

$$E_0^{\overline{\mathcal{G}}}\big(T(n)\big) \geq E_0^{\overline{\mathcal{G}}}\big(T(2^k)\big) \geq (2^k)^{\frac{81}{40}-\epsilon}(\log(2^k))^{-4\alpha-21} \geq cn^{\frac{81}{40}-\epsilon}(\log n)^{-4\alpha-21}$$

for some $c > 0$, and the proof of Theorem 1.2 is finished. □

Remark 4.7. It is desirable to show that X is subdiffusive with respect to the graph distance $d_{\overline{\mathcal{G}}}$. For the usual two dimensional simple random walk S, it was conjectured in [5] that

$$E\Big(d_{\mathcal{G}(0,n)}\big(0, S(n)\big)\Big) \approx E\Big(|S(n)|\Big) \asymp \sqrt{n}, \tag{4.42}$$

where $\approx$ denotes that the logarithms of the two sides are asymptotic as $n \to \infty$. From this, we expect that the difference between $d_{\overline{\mathcal{G}}}$ and the Euclidean distance is negligible for $d = 2$; more precisely, we conjecture that

$$d_{\overline{\mathcal{G}}}(0, x) \approx |x| \text{ as } |x| \to \infty, \tag{4.43}$$

with probability one. Combining (4.43) with Theorem 1.2, we expect that X is subdiffusive with respect to the graph distance. The question of whether or not (4.42) holds remains open. It is a challenging problem to prove/disprove (4.42) and (4.43).

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