ON SELF-AFFINE MEASURES WITH EQUAL HAUSDORFF AND LYAPUNOV DIMENSIONS

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ABSTRACT. Let μ be a self-affine measure on $\mathbb{R}^d$ associated to a self-affine IFS $\{\varphi_\lambda(x) = A_\lambda x + v_\lambda\}_{\lambda \in \Lambda}$ and a probability vector $p = (p_\lambda)_\lambda > 0$. Assume the strong separation condition holds. Let $\gamma_1 \geq \cdots \geq \gamma_d$ and D be the Lyapunov exponents and dimension corresponding to $\{A_\lambda\}_{\lambda \in \Lambda}$ and $p^{\mathbb{N}}$, and let $\mathbf{G}$ be the group generated by $\{A_\lambda\}_{\lambda \in \Lambda}$. We show that if $\gamma_{m+1} > \gamma_m = \cdots = \gamma_d$, if $\mathbf{G}$ acts irreducibly on the vector space of alternating m-forms, and if the Furstenberg measure μ_F satisfies $\dim_H \mu_F + D > (m+1)(d-m)$, then μ is exact dimensional with $\dim \mu = D$.

1. INTRODUCTION

Let $d \geq 2$ and let Λ be a finite index set. Fix a family of matrices $\{A_\lambda\}_{\lambda \in \Lambda} = \mathbf{A} \subset Gl(d, \mathbb{R})$ with $\|A_\lambda\| < 1$ for $\lambda \in \Lambda$, let $\{v_\lambda\}_{\lambda \in \Lambda} \subset \mathbb{R}^d$, and fix a probability vector $p = \{p_\lambda\}_{\lambda \in \Lambda} > 0$. Let $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ be the self-affine IFS with

$$(1.1) \qquad \varphi_\lambda(x) = A_\lambda x + v_\lambda \text{ for } \lambda \in \Lambda \text{ and } x \in \mathbb{R}^d .$$

Denote by μ the self-affine measure on $\mathbb{R}^d$ which corresponds to $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ and p, i.e., μ is the unique probability measure with

$$\mu = \sum_{\lambda \in \Lambda} p_\lambda \cdot \varphi_\lambda \mu .$$

The Lyapunov dimension D of μ (see Section 2 below) is an upper bound for the dimension of μ, but it is in general difficult to verify whether there is equality. The purpose of this paper is to present verifiable conditions under which

$$(1.2) \qquad \mu \text{ is exact dimensional with } \dim \mu = D .$$

1.1. Background for the problem.
Let us mention some notable results regarding self-affine measures and sets. From Theorem 1.9 in [JPS] it follows that D is the 'typical' value of $\dim_H \mu$, where $\dim_H$ stands for the Hausdorff dimension. More precisely, it is shown that if $\|A_\lambda\| < \frac{1}{2}$ for $\lambda \in \Lambda$ and if the translations $\{v_\lambda\}_{\lambda \in \Lambda}$ are drawn according to the Lebesgue measure, then $\dim_H \mu = \min\{D, d\}$ almost surely. The inequality $\dim_H \mu \leq D$ is always satisfied.

Analogous to this is the following classical result, due to Falconer, regarding the typical dimension of self-affine sets. Let K be the attractor of $\{\varphi_\lambda\}_{\lambda \in \Lambda}$. In [F3] it is shown that if $\|A_\lambda\| < \frac{1}{3}$ for $\lambda \in \Lambda$, then

$$\dim_H K = \min\{\dim_A K, d\} \text{ for Lebesgue almost all } \{v_\lambda\}_{\lambda \in \Lambda} .$$

Here $\dim_A K$ stands for the affinity dimension of K, which is defined in terms of the matrices in $\mathbf{A}$. This was later improved in [S] by replacing the constant $\frac{1}{3}$ by $\frac{1}{2}$. The inequality $\dim_H K \le \dim_A K$ is always true.

For fixed translations $\{v_\lambda\}_{\lambda \in \Lambda}$ the exact value of $\dim_H K$ has been found for several specific classes of self-affine sets. See the survey [F4] and the references therein. Much attention has been given to fractal carpets, where members of $\mathbf{A}$ preserve horizontal and vertical directions (see [M1] for instance).

Here we establish (1.2) in the opposite situation, in which there is no proper subspace invariant under all members of $\mathbf{A}$. This makes it possible to consider the Furstenberg measure μ_F on the Grassmannian manifold (see Section 2 below). The measure μ_F allows us to control the distribution of the orientation of cylinder sets at small scale.

For $d = 2$ this idea was already used in [FK] and [B1], in order to obtain (1.2) under assumptions different from ours. In Section 1.4 below we describe these results and compare them with the work presented here. A notable advantage in our result is that we do not require a lower bound on $\dim_H \mu$, but rather only on D which is at least as large and independent of the translations $\{v_\lambda\}_{\lambda \in \Lambda}$.

1.2. **The main result.** We shall consider only the case where the IFS $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ satisfies the strong separation condition (SSC). Denote by $\gamma_1 \ge \cdots \ge \gamma_d$ the Lyapunov exponents corresponding to the Bernoulli measure $p^{\mathbb{N}}$ and the matrices $\mathbf{A}$, and set

$$m = \max\{1 \le i \le d \,:\, \gamma_{d-i+1} = \cdots = \gamma_d\}.$$

If $m = d$ and the SSC is satisfied, then (1.2) follows directly from Theorem 2.6 in [FH]. Hence assume $m < d$. Let $\mathbf{G} \subset Gl(d, \mathbb{R})$ be the closure of the group generated by $\mathbf{A}$. We assume that $\mathbf{G}$ is m-irreducible, which means that it acts irreducibly on the vector space of alternating m-forms. A precise definition is given in Section 2. When $m = 1$ or $d-1$, and in particular when $d = 2$ or 3, this condition reduces to the absence of a proper subspace of $\mathbb{R}^d$ which is invariant under all members of $\mathbf{A}$ (see Remark 2 below).

Let $G_{d,m}$ denote the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^d$. Each $M \in Gl(d, \mathbb{R})$ defines a map from $G_{d,m}$ onto itself, which takes $W \in G_{d,m}$ to $M(W)$. From $m < d$, the irreducibility assumption, and results found in [BL2], it follows that there exists a unique probability measure μ_F on $G_{d,m}$ with

$$\mu_F = \sum_{\lambda \in \Lambda} p_\lambda \cdot A_\lambda^{-1} \mu_F,$$

and moreover that $\dim_H \mu_F > 0$ (see Proposition 3 in Section 2). The measure μ_F is called the Furstenberg measure on $G_{d,m}$ corresponding to $\mathbf{A}^{-1} := \{A_\lambda^{-1}\}_{\lambda \in \Lambda}$ and p. The following theorem is our main result.

Theorem. *Assume the following conditions:*

    (i) $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ *satisfies the SSC,*
   (ii) m *is strictly smaller than* d,
  (iii) $\mathbf{G}$ *is m-irreducible, and*
  (iv) *the measure* μ_F *satisfies*

$$\dim_H \mu_F + D > (m+1)(d-m).$$

Then (1.2) holds true, i.e., μ is exact dimensional with $\dim \mu = D$.

1.3. **Explicit examples.** The theorem just stated can be used to compute the dimension of many concrete self-affine measures. In order to do so one needs to bound $\dim_H \mu_F$ from below, which is not a trivial problem. Let us mention some results which are relevant for this task. Here we assume the elements of $\mathbf{A}$ are distinct, i.e., $A_{\lambda_1} \neq A_{\lambda_2}$ for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$. Also, we shall have no need for the matrices in $\mathbf{A}$ to be contractions. Indeed, the Furstenberg measure is unaffected if we multiply members of $\mathbf{A}$ by non-zero scalars.

In [HS] it is shown that if $\mathbf{A} \subset Gl(2, \mathbb{R})$ and p are such that elements in $\mathbf{A}$ have algebraic entries and determinant 1, $\mathbf{A}$ generates a free group, γ_1 is strictly greater than γ_2, and $\mathbf{G}$ acts irreducibly on $\mathbb{R}^2$, then

$$\dim_H \mu_F = \min\{\frac{H(p)}{-2 \cdot \gamma_1}, 1\} \, .$$

Here $H(p)$ stands for the entropy of p. For example, this can be applied when $p > 0$ and

$$(1.3) \qquad \mathbf{A} = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\} \, .$$

In Section VI.5 of [BL2] it is shown that $\dim_H \mu_F = \frac{H(p)}{-2 \cdot \gamma_1}$ whenever $|\mathbf{A}| > 1$, $p > 0$, and

$$\mathbf{A}^{-1} \subset \left\{ \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} \, : \, n \geq 1 \right\} \, .$$

For $E, L \in \mathbb{R}$ with $|E| + |L| < 2$, denote by $\mu_F^{E,L}$ the Furstenberg measure corresponding to

$$(1.4) \qquad \mathbf{A}^{-1} = \left\{ \begin{pmatrix} E - L & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} E + L & -1 \\ 1 & 0 \end{pmatrix} \right\} \text{ and } p = (\frac{1}{2}, \frac{1}{2}) \, .$$

In [B2] it is shown that there exists a constant $\delta > 0$ with

$$\lim_{L \to 0} \dim_H \mu_F^{E,L} = 1 \text{ for all } \delta < |E| < 2 - \delta \, .$$

In [B3] an example is given, for the case $d = 2$, of $\mathbf{A}$ and p for which $\gamma_1 > \gamma_2$, the action of $\mathbf{G}$ is irreducible, and μ_F is absolutely continuous with respect to the Lebesgue measure. For $d \geq 3$ an example of $\mathbf{A}$ and p with these properties is obtained in [BQ2].

1.4. **Comparison with recent work.** As mentioned above, for $d = 2$ the validity of (1.2) was established in two recent papers under conditions different from ours. From the arguments found in [FK], it follows that if the matrices in $\mathbf{A}$ have strictly positive entries, $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ satisfies the SSC, and

$$\dim_H \mu_F + \dim_H \mu > 2,$$

then (1.2) holds. This is actually done more generally, in the sense that the self-affine measure μ can be replaced by the projection of a Gibbs measure into $\mathbb{R}^2$.

Given $M \in Gl(2, \mathbb{R})$ let $\alpha_1(M) \geq \alpha_2(M) > 0$ denote the singular values of M. It is said that $\mathbf{A}$ satisfies the dominated splitting condition if there exist constants $0 < C, \delta < \infty$ with

$$\frac{\alpha_1(A_1, \cdots, A_n)}{\alpha_2(A_1, \cdots, A_n)} \geq C \cdot e^{\delta n} \text{ for all } n \geq 1 \text{ and } A_1, \ldots, A_n \in \mathbf{A} \, .$$

For example, this is satisfied when the matrices in $\mathbf{A}$ have strictly positive entries. It is shown in [B1] that if $\mathbf{A}$ satisfies dominated splitting, $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ satisfies the SSC, and

$$\dim_H \mu_F + \dim_H \mu > 2 \ \ \text{or} \ \ \dim_H \mu_F \geq \min\{1, D\},$$

then (1.2) holds.

Note that since $D \geq \dim_H \mu$, the condition $\dim_H \mu_F + D > 2$, which appears in our result when $d = 2$, is weaker than $\dim_H \mu_F + \dim_H \mu > 2$. This is important because D, as opposed to $\dim_H \mu$, is independent of the choice of translations $\{v_\lambda\}_{\lambda \in \Lambda}$. Observe also, that if the closure of the set

$$\{A_1, \cdots, A_n \ : \ n \geq 1 \text{ and } A_1, \ldots, A_n \in \mathbf{A}\}$$

contains an element $A \in Gl(2, \mathbb{R})$ for which $\frac{\alpha_1(A^n)}{\alpha_2(A^n)}$ does not increase exponentially fast as $n \to \infty$, then the results from [B1] and [FK] don't apply but our result can. This is in fact the case in examples (1.3) and (1.4) mentioned above. This is also true for the example obtained in [B3], since in that case $A^{-1} \in \mathbf{A}$ whenever $A \in \mathbf{A}$.

By using the aforementioned results about measures, results about the dimension of certain self-affine sets are obtained in [B1] and [FK]. More precisely, conditions for $\dim_H K = \dim_A K$ are given, where we recall that K is the attractor of $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ and $\dim_A K$ is the affinity dimension of K. We do not pursue this here, although it seems reasonable to believe that our work can also be applied in order to obtain this equality for new classes of self-affine sets.

Remark. In the last stages of writing up this research the author became aware of preprint [BK]. When $d = 2$ it is shown in [BK] that μ is always exact dimensional, and that $\dim \mu = D$ if the SSC holds and

$$\dim_H \mu_F > \min\{\dim \mu, \ 2 - \dim \mu\} \, .$$

As mentioned above, since $D \geq \dim_H \mu$ our result may be easier to use in some cases. For $d > 2$ results are proven in [BK] under an assumption on $\mathbf{A}$, termed totally dominated splitting, which is a multi-dimensional analogue of the dominated splitting condition previously mentioned. Hence for $d > 2$ our work applies in many situations that are untreated by [BK].

1.5. **About the proof.** We now make the dependency in the translations explicit. Given $(v_\lambda)_{\lambda \in \Lambda} = v \in \mathbb{R}^{d|\Lambda|}$ denote by $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$ the IFS satisfying (1.1), and let μ_v be the self-affine measure corresponding to $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$ and p. Let $\mathcal{V} \subset \mathbb{R}^{d|\Lambda|}$ be the set of all $v \in \mathbb{R}^{d|\Lambda|}$ for which $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$ satisfies the SSC. In the proofs found in [B1] and [FK], some $v \in \mathcal{V}$ is fixed and linear projections and sections of the measure μ_v are studied. In our proof we shall also examine linear sections of measures, but we shall consider the entire collection $\{\mu_v\}_{v \in \mathcal{V}}$ at once.

More precisely, it will be shown that there exists an upper semi-continuous function $F : \mathcal{V} \to [0, \infty)$, such that for every $v \in \mathcal{V}$ and $\mu_v \times \mu_F$-a.e. $(x, W) \in \mathbb{R}^d \times G_{d,m}$ the sliced measure, obtained from μ_v and supported on $x + W$, has exact dimension $F(v)$. The proof of this uses ergodic theory and results from the random matrix theory presented in [BL2]. From the result of [JPS] mentioned above, and from results found in [M2] regarding the dimension of exceptional sets of sections, it will follow that $F(v) \geq D - d + m$ for $\mathcal{L}eb$-a.e. $v \in \mathcal{V}$. The semi-continuity of F implies that this inequality holds in fact for all $v \in \mathcal{V}$. Now by fixing $v \in \mathcal{V}$ and using estimates on the dimension of exceptional sets of projections, it will follow that

$\dim_H \mu_v \geq D$. The inequality $\dim \mu_v \leq D$ in not hard to prove, and completes the proof.

1.6. **Outline of the paper.** In Section 2 we give some necessary definitions and state Theorem 4 which is our main result. Is Section 3 we carry out the proof, while relying on Proposition 6 and Lemmas 7 to 12, whose proofs are deferred to subsequent sections. In Section 4 we state and prove some required results, which follow from the theory of random matrices. In Section 5 we prove Proposition 6, which is the main ingredient in the proof of Theorem 4. In Section 6 we prove all auxiliary lemmas which were priorly used without proof.

2. STATEMENT OF THE MAIN RESULT

Fix some integer $d \geq 2$ and for $x \in \mathbb{R}^d$ denote by $|x|$ the euclidean norm of x. For a $d \times d$ matrix M (or operator on $\mathbb{R}^d$) denote by $\|M\|$ the operator norm of M with respect to the euclidean norm. Let Λ be a finite set with $|\Lambda| > 1$, and fix $\{A_\lambda\}_{\lambda \in \Lambda} \subset Gl(d, \mathbb{R})$ with $\|A_\lambda\| < 1$ for each $\lambda \in \Lambda$. Let $\mathbf{G} \subset Gl(d, \mathbb{R})$ be the closure of the group generated by $\{A_\lambda\}_{\lambda \in \Lambda}$. For $(v_\lambda)_{\lambda \in \Lambda} = v \in \mathbb{R}^{d|\Lambda|}$ let $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$ be the self-affine IFS with $\varphi_{v,\lambda}(x) = A_\lambda x + v_\lambda$ for $\lambda \in \Lambda$ and $x \in \mathbb{R}^d$. Let $K_v \subset \mathbb{R}^d$ be the attractor of $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$, i.e., K_v is the unique non-empty compact subset of $\mathbb{R}^d$ with $K_v = \bigcup_{\lambda \in \Lambda} \varphi_{v,\lambda}(K_v)$. We say that the strong separation condition (SSC) holds for $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$ if the union $\bigcup_{\lambda \in \Lambda} \varphi_{v,\lambda}(K_v)$ is disjoint, and we denote by $\mathcal{V} \subset \mathbb{R}^{d|\Lambda|}$ the set of all $v \in \mathbb{R}^{d|\Lambda|}$ for which the SSC holds. It is easy to see that $\mathcal{V}$ is an open subset of $\mathbb{R}^{d|\Lambda|}$, and we assume it to be non-empty.

Let $p = (p_\lambda)_{\lambda \in \Lambda}$ be a probability vector with $p_\lambda > 0$ for each $\lambda \in \Lambda$. Set $\Omega = \Lambda^{\mathbb{N}}$, equip Λ with the discrete topology, and equip Ω with the product topology. Let $\mathcal{F}$ be the Borel σ-algebra of Ω, and let μ be the Bernoulli measure on $(\Omega, \mathcal{F})$ which corresponds to p (i.e., $\mu = p^{\mathbb{N}}$). For each $v \in \mathbb{R}^{d|\Lambda|}$ and $\omega \in \Omega$ set

$$\pi_v(\omega) = \lim_n \varphi_{v,\omega_0} \circ \cdots \circ \varphi_{v,\omega_n}(0) .$$

Since the mappings $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$ are contractions this limit always exists and $\pi_v : \Omega \to \mathbb{R}^d$ is continuous. Note that $\pi_v \mu := \mu \circ \pi_v^{-1}$ is the unique Borel probability measure on $\mathbb{R}^d$ for which the relation $\pi_v \mu = \sum_{\lambda \in \Lambda} p_\lambda \cdot \varphi_{v,\lambda} \pi_v \mu$ is satisfied.

Given $M \in Gl(d, \mathbb{R})$ let $\alpha_1(M) \geq \cdots \geq \alpha_d(M) > 0$ be the singular values of M. Let $0 > \gamma_1 \geq \cdots \geq \gamma_d > -\infty$ be the Lyapunov exponents corresponding to μ and $\{A_\lambda\}_{\lambda \in \Lambda}$ (see chapter III.5 in [BL2]), i.e., for μ-a.e. $\omega \in \Omega$

$$(2.1) \qquad \gamma_i = \lim_n \frac{1}{n} \log \alpha_i(A_{\omega_0} \cdots A_{\omega_{n-1}}) \text{ for } 1 \leq i \leq d .$$

Denote the entropy of μ by h_μ (i.e., $h_\mu = \sum_{\lambda \in \Lambda} -p_\lambda \cdot \log p_\lambda$), set

$$(2.2) \qquad k(\mu) = \max\{0 \leq i \leq d : 0 < h_\mu + \gamma_1 + \cdots + \gamma_i\},$$

and set

$$D(\mu) = \begin{cases} k(\mu) - \frac{h_\mu + \gamma_1 + \cdots + \gamma_{k(\mu)}}{\gamma_{k(\mu)+1}} & \text{if } k(\mu) < d, \\ -d \cdot \frac{h_\mu}{\gamma_1 + \cdots + \gamma_d} & \text{if } k(\mu) = d. \end{cases}$$

The number $D(\mu)$ is called the Lyapunov dimension of μ with respect to the family $\{A_\lambda\}_{\lambda \in \Lambda}$.

Given a metric space X we denote the collection of all compactly supported Borel probability measures on X by $\mathcal{M}(X)$. For $\theta \in \mathcal{M}(X)$ we write

$$\dim_H \theta = \inf\{\dim_H E \,:\, E \subset X \text{ is a Borel set with } \theta(E) > 0\}$$

and

$$\dim_H^* \theta = \inf\{\dim_H E \,:\, E \subset X \text{ is a Borel set with } \theta(X \setminus E) = 0\},$$

where $\dim_H E$ stands for the Hausdorff dimension of the set E. For $x \in \mathbb{R}^d$ and $\epsilon > 0$ denote by $B(x, \epsilon)$ the closed ball in $\mathbb{R}^d$ with centre x and radius ϵ. Given $\theta \in \mathcal{M}(\mathbb{R}^d)$ we say that θ has exact dimension $s \geq 0$ if

$$\lim_{\epsilon \downarrow 0} \frac{\log \theta(B(x, \epsilon))}{\log \epsilon} = s \text{ for } \theta\text{-a.e. } x \in \mathbb{R}^d,$$

in which case we write $\dim \theta = s$. It is well known (see chapter 10 of [F1]) that

$$(2.3) \qquad \dim_H \theta = \operatorname{essinf}_\theta \{ \liminf_{\epsilon \downarrow 0} \frac{\log \theta(B(x, \epsilon))}{\log \epsilon} \,:\, x \in \mathbb{R}^d \}.$$

Given $1 \leq m < d$ let $G_{d,m}$ denote the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^d$. For a subspace $W \subset \mathbb{R}^d$ let $P_W : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal projection onto W. For $W, U \in G_{d,m}$ set $d_{G_{d,m}}(W, U) = \|P_W - P_U\|$; then $d_{G_{d,m}}$ is a metric on $G_{d,m}$ which we shall use. For $M \in Gl(d, \mathbb{R})$ and $W \in G_{d,m}$ set $M \cdot W = M(W) \in G_{d,m}$, which defines an action of $Gl(d, \mathbb{R})$ on $G_{d,m}$.

For $1 \leq m \leq d$ let $\mathcal{A}^m(\mathbb{R}^d)$ denote the vector space of alternating m-linear forms on $(\mathbb{R}^d)^*$. Given $x_1, \ldots, x_m \in \mathbb{R}^d$ let $x_1 \wedge \cdots \wedge x_m \in \mathcal{A}^m(\mathbb{R}^d)$ be such that

$$x_1 \wedge \cdots \wedge x_m(f_1, \ldots, f_m) = \det[\{f_i(x_j)\}_{i,j=1}^m] \text{ for } f_1, \ldots, f_m \in (\mathbb{R}^d)^*.$$

If $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^d$, then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_m} \,:\, 1 \leq i_1 < \cdots < i_m \leq d\}$$

is a basis for $\mathcal{A}^m(\mathbb{R}^d)$. For $M \in Gl(d, \mathbb{R})$ we define an automorphism $\mathcal{A}^m M$ of $\mathcal{A}^m(\mathbb{R}^d)$ by

$$\mathcal{A}^m M(x_1 \wedge \cdots \wedge x_m) = M x_1 \wedge \cdots \wedge M x_m \text{ for } x_1, \ldots, x_m \in \mathbb{R}^d.$$

Definition 1. Given $1 \leq m < d$ and $\mathbf{S} \subset Gl(d, \mathbb{R})$ we say that $\mathbf{S}$ is m-irreducible if there does not exist a proper linear subspace W of $\mathcal{A}^m(\mathbb{R}^d)$ with $\mathcal{A}^m M(W) = W$ for each $M \in \mathbf{S}$. When $m = 1$ we say that $\mathbf{S}$ is irreducible.

Remark 2. Clearly $\mathbf{S}$ is irreducible if and only if there does not exist a proper linear subspace W of $\mathbb{R}^d$ with $M(W) = W$ for each $M \in \mathbf{S}$. It is also easy to show that $\mathbf{S}$ is m-irreducible if and only if it is $d - m$-irreducible (see page 86 in [BL2]). Hence when $d = 2$ or 3 the m-irreducibility condition reduces to the absence of a proper subspace of $\mathbb{R}^d$ which is M-invariant for all $M \in \mathbf{S}$.

The following proposition follows from results found in [BL2], and shall be proven in Section 4. From now on we set

$$m = \max\{1 \leq i \leq d \,:\, \gamma_{d-i+1} = \cdots = \gamma_d\}.$$

Proposition 3. *Assume $m < d$ and that $\mathbf{G}$ is m-irreducible; then there exists a unique $\mu_F \in \mathcal{M}(G_{d,m})$ with $\mu_F = \sum_{\lambda \in \Lambda} p_\lambda \cdot A_\lambda^{-1} \mu_F$. It also holds that $\dim_H \mu_F > 0$.*

The measure μ_F is called the Furstenberg measure on $G_{d,m}$ corresponding to $\{A_\lambda^{-1}\}_{\lambda \in \Lambda}$ and p. We can now state our main result.

Theorem 4. *If $m < d$, if $\mathbf{G}$ is m-irreducible, and if*

$$\dim_H^* \mu_F + D(\mu) > (m+1)(d-m),$$

then $\pi_v \mu$ is exact dimensional with $\dim \pi_v \mu = D(\mu)$ for each $v \in \mathcal{V}$.

Remark 5. As mentioned in the introduction, if $m = d$, then it follows from Theorem 2.6 in [FH] that $\dim \pi_v \mu = D(\mu)$ for all $v \in \mathcal{V}$.

3. Proof of the main result

For the remainder of this paper we assume $m < d$, $\mathbf{G}$ is m-irreducible, and $\dim_H^* \mu_F + D(\mu) > (m+1)(d-m)$.

3.1. **Disintegration of measures.** For the proof of Theorem 4 we shall need to disintegrate the measures μ and $\{\pi_v \mu\}_{v \in \mathcal{V}}$. We now define these disintegrations and state some of their properties; for further details see chapter 3 of [FH].

Let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}^d$, let X be a metric space, let $\theta \in \mathcal{M}(X)$, let K be the support of θ, and let $f : X \to \mathbb{R}^d$ be continuous. Then there exists a family $\{\theta_x\}_{x \in X} \subset \mathcal{M}(X)$, which will be called the disintegration of θ with respect to $f^{-1}\mathcal{B}$, such that:

(a) For θ-a.e. $x \in X$ the measure θ_x is supported on $K \cap f^{-1}(f(x))$.

(b) For each $g \in L^1(\theta)$ and θ-a.e. $x \in X$ we have

$$\int g \, d\theta_x = \lim_{\epsilon \downarrow 0} \frac{1}{f\theta(B(fx, \epsilon))} \cdot \int_{f^{-1}(B(fx, \epsilon))} g \, d\theta = \frac{d(f\theta^g)}{d(f\theta)}(fx),$$

where $\theta^g(E) = \int_E g \, d\theta$ for each Borel set $E \subset X$. Here $\frac{d(f\theta^g)}{d(f\theta)}$ stands for the Radon–Nikodym derivative of $f\theta^g$ with respect to $f\theta$.

(c) For each $g \in L^1(\theta)$ the map that takes $x \in X$ to $\int g \, d\theta_x$ is $f^{-1}\mathcal{B}$ measurable and

$$\int g \, d\theta_x = E_\theta[g \mid f^{-1}\mathcal{B}](x) \text{ for } \theta\text{-a.e. } x \in X .$$

Here $E_\theta[g \mid f^{-1}\mathcal{B}]$ is the conditional expectation of g given $f^{-1}\mathcal{B}$ with respect to θ.

We shall use the following notation for the disintegrations of μ and $\{\pi_v \mu\}_{v \in \mathcal{V}}$. For a subspace $W \subset \mathbb{R}^d$ set $\mathcal{B}_W = P_{W^\perp}^{-1}(\mathcal{B})$, and for $\theta \in \mathcal{M}(\mathbb{R}^d)$ let $\{\theta_{W,x}\}_{x \in \mathbb{R}^d}$ be the disintegration of θ with respect to $\mathcal{B}_W$. Given $v \in \mathbb{R}^{d|\Lambda|}$ set $\mathcal{F}_{v,W} = \pi_v^{-1} \circ P_{W^\perp}^{-1}(\mathcal{B})$ and let $\{\mu_{v,W,\omega}\}_{\omega \in \Omega}$ be the disintegration of μ with respect to $\mathcal{F}_{v,W}$.

3.2. **Statement of auxiliary claims.** We now state some auxiliary claims which will be used in the proof of Theorem 4. The proofs are deferred to subsequent sections in order to make the argument for Theorem 4 more transparent. First we state Proposition 6 whose proof, which is given in Section 5 below, requires ergodic theory and some results from the random matrix theory presented in [BL2].

Define $F : \mathcal{V} \to [0, \infty)$ by

$$F(v) = -\frac{1}{\gamma_d} \cdot \int_{G_{d,m}} H_\mu(\mathcal{P} \mid \mathcal{F}_{v,W}) \, d\mu_F(W) \text{ for } v \in \mathcal{V},$$

where

$$\mathcal{P} = \{\{\omega \in \Omega : \omega_0 = \lambda\} \in \mathcal{F} : \lambda \in \Lambda\}$$

and $H_\mu(\mathcal{P} \mid \mathcal{F}_{v,W})$ is the conditional entropy of $\mathcal{P}$ given $\mathcal{F}_{v,W}$ with respect to μ.

Proposition 6. *For each $v \in \mathcal{V}$ and for $\mu \times \mu_F$-a.e. $(\omega, W) \in \Omega \times G_{d,m}$ the measure $\pi_v \mu_{v,W,\omega}$ is exact dimensional with $\dim(\pi_v \mu_{v,W,\omega}) = F(v)$.*

The rest of the auxiliary lemmas will be proven in Section 6.

Lemma 7. *Let* $v \in \mathbb{R}^{d|\Lambda|}$ *and* $W \in G_{d,m}$*; then* $(\pi_v \mu)_{W,\pi_v(\omega)} = \pi_v \mu_{v,W,\omega}$ *for* μ*-a.e.* $\omega \in \Omega$.

The following semi-continuity lemma makes it possible to utilize Proposition 6.

Lemma 8. *The function* F *is upper semi-continuous.*

Lemma 9. *For* $v \in \mathcal{V}$ *we have* $\pi_v \mu \perp \mathcal{L}eb_d$*, where* $\mathcal{L}eb_d$ *is the Lebesgue measure of* $\mathbb{R}^d$.

The proof of the following lemma relies on results found in [M2], which are obtained by the use of Fourier analytic techniques. This lemma makes it possible to use the assumption $\dim_H^* \mu_F + D(\mu) > (m+1)(d-m)$.

Lemma 10. *Let* $\theta \in \mathcal{M}(\mathbb{R}^d)$*, let* $1 \le l < d$ *be an integer, and set* $s = \dim_H \theta$.
  (a) If $s \le d - l$*, then for* $0 \le t \le s$

$$\dim_H \{W \in G_{d,l} \,:\, \text{essinf}_\theta \{\dim_H(\theta_{W,x}) \,:\, x \in \mathbb{R}^d\} > s - t\} \le (l-1)(d-l) + t \,.$$

  (b) If $s > d - l$*, then for* $s - l(d-l) \le t \le d - l$

$$\dim_H \{W \in G_{d,l} \,:\, \text{essinf}_\theta \{\dim_H(\theta_{W,x}) \,:\, x \in \mathbb{R}^d\} > s - t\} \le l(d-l) + t - s \,.$$

  (c) If $s > d - l$*, then*

$$\dim_H \{W \in G_{d,l} \,:\, \text{essinf}_\theta \{\dim_H(\theta_{W,x}) \,:\, x \in \mathbb{R}^d\} < s - d + l\} \le (l+1)(d-l) - s \,.$$

The proof for the following lemma is an adaptation of an argument given in the proof of part (a) of Theorem 4.3 from [JPS].

Lemma 11. *For each* $v \in \mathbb{R}^{d|\Lambda|}$ *and for* $\pi_v \mu$*-a.e.* $x \in \mathbb{R}^d$

$$\limsup_{\epsilon \downarrow 0} \frac{\log \pi_v \mu(B(x,\epsilon))}{\log \epsilon} \le D(\mu) \,.$$

Let Λ^* be the set of finite words over Λ. Given a set of transformations (or matrices) $\{f_\lambda\}_{\lambda \in \Lambda}$, that can be composed with one another, we set $f_w = f_{\lambda_1} \circ \cdots \circ f_{\lambda_k}$ for $k \ge 1$ and $\lambda_1, \cdots, \lambda_k = w \in \Lambda^*$. Given a set of real numbers $\{a_\lambda\}_{\lambda \in \Lambda}$ we set $a_w = a_{\lambda_1}, \cdots, a_{\lambda_k}$. We also set $f_\emptyset = Id$ and $a_\emptyset = 1$, where $\emptyset \in \Lambda^*$ is the empty word.

Lemma 12. *Let* $n \ge 1$*, let* $\mathbf{G}' \subset Gl(d,\mathbb{R})$ *be the closure of the group generated by* $\{A_w\}_{w \in \Lambda^n}$*, set* $p' = (p_w)_{w \in \Lambda^n}$*, set* $\mu' = (p')^{\mathbb{N}}$*, and let* $0 > \gamma_1' \ge \cdots \ge \gamma_d' > -\infty$ *be the Lyapunov exponents corresponding to* μ' *and* $\{A_w\}_{w \in \Lambda^n}$*. Then* $\mathbf{G}'$ *is* m*-irreducible,* $\gamma_i' = n \cdot \gamma_i$ *for* $1 \le i \le d$*, and* $\mu_F' = \mu_F$*, where* μ_F' *is the Furstenberg measure corresponding to* $\{A_w^{-1}\}_{w \in \Lambda^n}$ *and* p' *(see Proposition 3 above).*

3.3. **Proof of Theorem 4.** By using Proposition 6 and Lemmas 7 to 12 we shall now prove Theorem 4.

Lemma 13. *If* $\|A_\lambda\| < \frac{1}{2}$ *for each* $\lambda \in \Lambda$*, then* $D(\mu) \in (d-m, d]$ *and* $F(v) \ge D(\mu) - d + m$ *for each* $v \in \mathcal{V}$.

Proof of Lemma 13. Since $\mathcal{V}$ is non-empty (by assumption) and since it is an open subset of $\mathbb{R}^{d|\Lambda|}$, it follows that $\mathcal{L}eb_{d|\Lambda|}(\mathcal{V}) > 0$. From part (b) of Theorem 1.9 in

[JPS] it follows that if $D(\mu) > d$, then for $\mathcal{L}eb_{d|\Lambda|}$-a.e. $v \in \mathcal{V}$ we have $\pi_v\mu \ll \mathcal{L}eb_d$. This together with Lemma 9 shows that $D(\mu) \leq d$. Since

$$\dim_H^* \mu_F \leq \dim_H G_{d,m} = m(d - m)$$

and

$$\dim_H^* \mu_F + D(\mu) > m(d - m) + d - m,$$

it follows that $D(\mu) \in (d - m, d]$. From this and from part (a) of Theorem 1.9 in [JPS] we get that $\dim_H \pi_v\mu = D(\mu)$ for $\mathcal{L}eb_{d|\Lambda|}$-a.e. $v \in \mathbb{R}^{d|\Lambda|}$. Since $\mathcal{V}$ is open it follows that the set

$$\mathcal{Q} = \{v \in \mathcal{V} \ : \ \dim_H \pi_v\mu = D(\mu)\}$$

is dense in $\mathcal{V}$.

Fix $v \in \mathcal{Q}$; then from Proposition 6, from Lemma 7, and from (2.3), it follows that for μ_F-a.e. $W \in G_{d,m}$ we have for $\pi_v\mu$-a.e. $x \in \mathbb{R}^d$ that $\dim_H(\pi_v\mu)_{W,x} = F(v)$. Set

$$\mathcal{E} = \{W \in G_{d,m} \ : \ \text{essinf}_{\pi_v\mu}\{\dim_H(\pi_v\mu)_{W,x} \ : \ x \in \mathbb{R}^d\} < D(\mu) - d + m\};$$

then from $\dim_H \pi_v\mu = D(\mu) > d - m$ and from part (c) of Lemma 10 we get

$$\dim_H(\mathcal{E}) \leq (m + 1)(d - m) - D(\mu) .$$

Since $\dim_H^* \mu_F > (m + 1)(d - m) - D(\mu)$ it follows that $\mu_F(G_{d,m} \setminus \mathcal{E}) > 0$, and so there exist $W \in G_{d,m}$ and $x \in \mathbb{R}^d$ with

$$F(v) = \dim_H(\pi_v\mu)_{W,x} \geq D(\mu) - d + m .$$

Since this holds for each $v \in \mathcal{Q}$ and since $\mathcal{Q}$ is dense in $\mathcal{V}$, it follows from Lemma 8 that $F(v) \geq D(\mu) - d + m$ for each $v \in \mathcal{V}$. $\qquad \square$

Proof of Theorem 4. Let $v \in \mathcal{V}$ be given. Assume first that $\|A_\lambda\| < \frac{1}{2}$ for each $\lambda \in \Lambda$; then from Lemma 13 we get $F(v) \geq D(\mu) - d + m \in (0, m]$. From this, from Proposition 6, and from Lemma 7 it follows that

(3.1) $\qquad \dim_H(\pi_v\mu)_{W,x} \geq D(\mu) - d + m$ for $\pi_v\mu \times \mu_F$-a.e. $(x, W) .$

Set $s = \dim_H(\pi_v\mu)$. If $s < D(\mu) - d + m$, then clearly

$$\text{essinf}_{\pi_v\mu}\{\dim_H(\pi_v\mu)_{W,x} \ : \ x \in \mathbb{R}^d\} < D(\mu) - d + m$$

for each $W \in G_{d,m}$, and so we must have $s \geq D(\mu) - d + m$. Assume by contradiction that $D(\mu) - d + m \leq s < D(\mu)$, let

$$0 < \epsilon < \min \left\{ \begin{array}{c} D(\mu) - d + m, \\ D(\mu) - s, \\ \dim_H^* \mu_F + D(\mu) - (m + 1)(d - m) \end{array} \right\},$$

set

$$t = \begin{cases} \min\{2(d - m) - D(\mu) + \epsilon, s\} & \text{if } s \leq d - m, \\ d - m + s - D(\mu) + \epsilon & \text{if } s > d - m, \end{cases}$$

and set

$$\mathcal{E} = \{W \in G_{d,m} \ : \ \text{essinf}_{\pi_v\mu}\{\dim_H(\pi_v\mu)_{W,x} \ : \ x \in \mathbb{R}^d\} > s - t\} .$$

If $s \leq d - m$, then

$$D(\mu) - d + m \leq s \leq d - m,$$

so $0 \leq t \leq s$, and so from part (a) of Lemma 10

$$\dim_H(\mathcal{E}) \leq (m - 1)(d - m) + t \leq (m + 1)(d - m) - D(\mu) + \epsilon < \dim_H^* \mu_F .$$

If $s > d - m$, then

$$t - (s - m(d - m)) > d - m - D(\mu) + m(d - m) \geq m(d - m) - m \geq 0$$

and

$$d - m - t = D(\mu) - s - \epsilon > 0,$$

so $s - m(d - m) \leq t \leq d - m$, and so from part (b) of Lemma 10

$$\dim_H(\mathcal{E}) \leq m(d - m) + t - s = (m + 1)(d - m) - D(\mu) + \epsilon < \dim_H^* \mu_F .$$

In any case we have $\dim_H(\mathcal{E}) < \dim_H^* \mu_F$, so $\mu_F(G_{d,m} \setminus \mathcal{E}) > 0$, and so

$$\pi_v\mu \times \mu_F\{(x, W) \; : \; \dim(\pi_v\mu)_{W,x} \leq s - t + \frac{\epsilon}{2}\} > 0 .$$

But this gives a contradiction to (3.1) since if $s \leq d - m$, then

$$s - t + \frac{\epsilon}{2} = \max\{s - (2(d - m) - D(\mu) + \epsilon), 0\} + \frac{\epsilon}{2}$$

$$\leq \max\{D(\mu) - d + m - \epsilon, 0\} + \frac{\epsilon}{2} = D(\mu) - d + m - \frac{\epsilon}{2},$$

and if $s > d - m$, then

$$s - t + \frac{\epsilon}{2} = D(\mu) - d + m - \frac{\epsilon}{2} .$$

It follows that we must have $\dim_H(\pi_v\mu) = s \geq D(\mu)$, and so from Lemma 11 and (2.3) we obtain that $\pi_v\mu$ is exact dimensional with $\dim \pi_v\mu = D(\mu)$. This proves the theorem if $\|A_\lambda\| < \frac{1}{2}$ for each $\lambda \in \Lambda$.

Now we prove the general case. Let $n \geq 1$ be such that $\|A_w\| < \frac{1}{2}$ for each $w \in \Lambda^n$. Since the SSC holds for $\{\varphi_{v,\lambda}\}_{\lambda \in \Lambda}$ it clearly holds for $\{\varphi_{v,w}\}_{w \in \Lambda^n}$. For $\omega \in (\Lambda^n)^{\mathbb{N}}$ set $\pi'_v(\omega) = \lim_n \varphi_{v,\omega_0} \circ \cdots \circ \varphi_{v,\omega_n}(0)$, set $p' = (p_w)_{w \in \Lambda^n}$, set $\mu' = (p')^{\mathbb{N}}$, let $0 > \gamma'_1 \geq \cdots \geq \gamma'_d > -\infty$ be the Lyapunov exponents corresponding to μ' and $\{A_w\}_{w \in \Lambda^n}$, and let $\mathbf{G}' \subset Gl(d, \mathbb{R})$ be the closure of the group generated by $\{A_w\}_{w \in \Lambda^n}$. From Lemma 12 we get that $\mathbf{G}'$ is m-irreducible, $\gamma'_i = n \cdot \gamma_i$ for $1 \leq i \leq d$, and $\mu'_F = \mu_F$, where μ'_F is the Furstenberg measure corresponding to $\{A_w^{-1}\}_{w \in \Lambda^n}$ and p'. Let $h_{\mu'}$ be the entropy of μ' (i.e., $h_{\mu'} = \sum_{w \in \Lambda^n} -p_w \cdot \log p_w$), and let $D(\mu')$ be the Lyapunov dimension of μ' with respect to the family $\{A_w\}_{w \in \Lambda^n}$ (see the definition in Section 2 above). Since $h_{\mu'} = n \cdot h_\mu$ it follows from the definition of the Lyapunov dimension that $D(\mu') = D(\mu)$, hence

$$\dim_H^* \mu'_F + D(\mu') = \dim_H^* \mu_F + D(\mu) > (m + 1)(d - m) .$$

Now from the first part of the proof we get that $\pi'_v\mu'$ is exact dimensional with $\dim \pi'_v\mu' = D(\mu') = D(\mu)$. This completes the proof since $\pi_v\mu = \pi'_v\mu'$. $\qquad \square$

4. Auxiliary results from the theory of random matrices

In this section we translate results found in [BL2] to suit our needs. These results will be used in the proofs of Propositions 3 and 6.

Definition 14. Given $q \geq 2$, $1 \leq l < q$, and $\mathbf{S} \subset Gl(q, \mathbb{R})$, we say that $\mathbf{S}$ is l-strongly irreducible if there does not exist a finite family of proper linear subspaces $W_1, \ldots, W_k$ of $\mathcal{A}^l(\mathbb{R}^q)$ with

$$\mathcal{A}^l M(W_1 \cup \cdots \cup W_k) = W_1 \cup \cdots \cup W_k \text{ for each } M \in \mathbf{S} .$$

When $l = 1$ we say that $\mathbf{S}$ is strongly irreducible.

Remark 15. Given $q \geq 2$, $1 \leq l < q$, and linear subspaces $W_1, \ldots, W_k$ of $\mathcal{A}^l(\mathbb{R}^q)$, the set

$$\{M \in Gl(q, \mathbb{R}) \,:\, \mathcal{A}^l M(W_1 \cup \cdots \cup W_k) = W_1 \cup \cdots \cup W_k\}$$

is a closed subgroup of $Gl(q, \mathbb{R})$.

Definition 16. Given $q \geq 2$, $1 \leq l < q$, and $\mathbf{S} \subset Gl(q, \mathbb{R})$, we say that $\mathbf{S}$ is *l-contracting* if there exists a sequence $\{M_n\}_{n=1}^{\infty} \subset \mathbf{S}$ such that

$$\{\|\mathcal{A}^l M_n\|^{-1} \cdot \mathcal{A}^l M_n \,:\, n \geq 1\}$$

converges to a rank-one matrix. When $l = 1$ we say that $\mathbf{S}$ is *contracting*.

Throughout this section $\mathbf{T} \subset Gl(d, \mathbb{R})$ will denote the closure of the semigroup generated by $\{A_\lambda^{-1}\}_{\lambda \in \Lambda}$. Let $q \geq 1$ be the dimension of $\mathcal{A}^m(\mathbb{R}^d)$; then given $M \in Gl(d, \mathbb{R})$ we may view $\mathcal{A}^m M$ as a member of $Gl(q, \mathbb{R})$. Let $\widetilde{\mathbf{T}} \subset Gl(q, \mathbb{R})$ be the closure of the semigroup generated by $\{\mathcal{A}^m A_\lambda^{-1}\}_{\lambda \in \Lambda}$. Recall that we assume $m < d$ and $\mathbf{G}$ is m-irreducible.

Lemma 17. $\widetilde{\mathbf{T}}$ *is contracting and strongly irreducible, and* $\mathbf{T}$ *is m-contracting and m-strongly irreducible.*

Proof of Lemma 17. Since $\mathbf{G}$ is m-irreducible it follows from Remark 15 that $\{A_\lambda^{-1}\}_{\lambda \in \Lambda}$ is m-irreducible, and so $\widetilde{\mathbf{T}}$ is irreducible. Let $\infty > \gamma_1' \geq \cdots \geq \gamma_d' > 0$ be the Lyapunov exponents corresponding to μ and $\{A_\lambda^{-1}\}_{\lambda \in \Lambda}$; then $\gamma_i' = -\gamma_{d-i+1}$ for $1 \leq i \leq d$. Let $\eta_1 \geq \eta_2$ be the the two upper Lyapunov exponents corresponding to μ and $\{\mathcal{A}^m A_\lambda^{-1}\}_{\lambda \in \Lambda}$. From an argument given in the proof of Theorem IV.1.2 in [BL2] we get

$$\eta_1 = \sum_{i=1}^{m} \gamma_i' \quad \text{and} \quad \eta_2 = \sum_{i=1}^{m-1} \gamma_i' + \gamma_{m+1}',$$

hence from the definition of m

$$\eta_1 = \sum_{i=1}^{m} \gamma_i' = -\sum_{i=1}^{m} \gamma_{d-i+1} > -\sum_{i=1}^{m-1} \gamma_{d-i+1} - \gamma_{d-m} = \sum_{i=1}^{m-1} \gamma_i' + \gamma_{m+1}' = \eta_2 \,.$$

From this, from the irreducibility of $\widetilde{\mathbf{T}}$, and from Theorem III.6.1 in [BL2], we get that $\widetilde{\mathbf{T}}$ is contracting and strongly irreducible. From this and Remark 15 it follows that $\{\mathcal{A}^m A_\lambda^{-1}\}_{\lambda \in \Lambda}$ is strongly irreducible, and so $\mathbf{T}$ is m-strongly irreducible. Since $\widetilde{\mathbf{T}}$ is contracting and since $\{\mathcal{A}^m A_w^{-1} \,:\, w \in \Lambda^*\}$ is dense in $\widetilde{\mathbf{T}}$, it follows that $\{\mathcal{A}^m A_w^{-1} \,:\, w \in \Lambda^*\}$ is contracting. This shows that $\mathbf{T}$ is m-contracting. $\qquad\square$

Let $\langle \cdot, \cdot \rangle$ be the usual scalar product on $\mathbb{R}^d$. As in Section III.5 of [BL2] we define a scalar product on $\mathcal{A}^m(\mathbb{R}^d)$ by the formula

$$\langle x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_m \rangle = \det\left[\{\langle x_i, y_j \rangle\}_{i,j=1}^{m}\right].$$

Let $P(\mathcal{A}^m(\mathbb{R}^d))$ be the projective space of $\mathcal{A}^m(\mathbb{R}^d)$. Given $\bar{\xi}, \bar{\eta} \in P(\mathcal{A}^m(\mathbb{R}^d))$ set

$$d_{P(\mathcal{A}^m(\mathbb{R}^d))}(\bar{\xi}, \bar{\eta}) = \left(1 - \langle \xi, \eta \rangle^2\right)^{1/2},$$

where ξ and η are unit vectors in $\mathcal{A}^m(\mathbb{R}^d)$ with directions $\bar{\xi}$ and $\bar{\eta}$. As shown in Section III.4 of [BL2], $d_{P(\mathcal{A}^m(\mathbb{R}^d))}$ is a metric on $P(\mathcal{A}^m(\mathbb{R}^d))$.

Given independent sets $\{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\} \subset \mathbb{R}^d$, there exists a constant $a \in \mathbb{R} \setminus \{0\}$ with

$$y_1 \wedge \cdots \wedge y_m = a \cdot x_1 \wedge \cdots \wedge x_m$$

if and only if

$$\text{span}\{y_1, \ldots, y_m\} = \text{span}\{x_1, \ldots, x_m\} \,.$$

Define a map $\psi : G_{d,m} \to P(\mathcal{A}^m(\mathbb{R}^d))$ by

$$\psi(W) = \mathbb{R} \cdot x_1 \wedge \cdots \wedge x_m \ \text{ if span}\{x_1, \ldots, x_m\} = W \in G_{d,m} \,.$$

It is not hard to check that there exists a constant $C \in (1, \infty)$ with

$$(4.1) \qquad C^{-1} \cdot d_{G_{d,m}}(W, U) \le \left(d_{P(\mathcal{A}^m(\mathbb{R}^d))}(\psi(W), \psi(U)) \right)^2 \le C \cdot d_{G_{d,m}}(W, U)$$

for all $W, U \in G_{d,m}$, where $d_{G_{d,m}}$ is the metric defined above in Section 2. Hence ψ is an embedding of $G_{d,m}$ into $P(\mathcal{A}^m(\mathbb{R}^d))$. Now we can prove Proposition 3.

Proof of Proposition 3. From Lemma 17 and Theorem IV.1.2 in [BL2] it follows that there exists a unique $\theta \in \mathcal{M}(P(\mathcal{A}^m(\mathbb{R}^d)))$ with $\theta = \sum_{\lambda \in \Lambda} p_\lambda \cdot \mathcal{A}^m A_\lambda^{-1} \theta$. Since $\psi(G_{d,m})$ is compact and $\mathcal{A}^m M(\psi(G_{d,m})) = \psi(G_{d,m})$ for each $M \in Gl(d, \mathbb{R})$, it follows from Lemma I.3.5 in [BL2] that there exists $\theta' \in \mathcal{M}(\psi(G_{d,m}))$ with $\theta' = \sum_{\lambda \in \Lambda} p_\lambda \cdot \mathcal{A}^m A_\lambda^{-1} \theta'$. By the uniqueness of θ it follows that $\theta = \theta'$, and so θ is supported on $\psi(G_{d,m})$.

Set $\mu_F = \psi^{-1}\theta$; then

$$\mu_F = \psi^{-1}\theta = \sum_{\lambda \in \Lambda} p_\lambda \cdot \psi^{-1} \circ \mathcal{A}^m A_\lambda^{-1} \theta = \sum_{\lambda \in \Lambda} p_\lambda \cdot A_\lambda^{-1} \circ \psi^{-1}\theta = \sum_{\lambda \in \Lambda} p_\lambda \cdot A_\lambda^{-1} \mu_F \,.$$

Since ψ is an embedding the uniqueness of μ_F follows from the uniqueness of θ. From Corollary VI.4.2 in [BL2] and the remarks following it it follows that $\dim_H \theta > 0$. From this and from (4.1) we obtain $\dim_H \mu_F > 0$. This completes the proof of the lemma. $\square$

Given $a_1, \ldots, a_d \in \mathbb{R}$ let $diag(a_1, \ldots, a_d)$ denote the $d \times d$ matrix D with

$$D_{i,j} = \begin{cases} a_i & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases} \ \text{ for } 1 \le i, j \le d \,.$$

Given $M \in Gl(d, \mathbb{R})$ there exist orthogonal matrices $U, V \in Gl(d, \mathbb{R})$ with $M = UDV$, where $D = diag(\alpha_1(M), \ldots, \alpha_d(M))$. We call the product UDV a singular value decomposition of M. Note that $V^* e_i$ is an eigenvector of $M^* M$ with eigenvalue $\alpha_i(M)^2$ for each $1 \le i \le d$. Here $\{e_i\}_{i=1}^d$ is the standard basis of $\mathbb{R}^d$ and M^* is the transpose of M.

Lemma 18. *For each $\omega \in \Omega$ and $n \ge 1$ set $D_{n,\omega} = diag(\alpha_1(A_{\omega|_n}), \ldots, \alpha_d(A_{\omega|_n}))$, let $U_{n,\omega} D_{n,\omega} V_{n,\omega}$ be a singular value decomposition of $A_{\omega|_n}$, and set $W_n(\omega) = \text{span}\{U_{n,\omega} e_{d-m+1}, \ldots, U_{n,\omega} e_d\}$. Then for μ-a.e. $\omega \in \Omega$ there exists $W(\omega) \in G_{d,m}$ such that $\{W_n(\omega)\}_{n=1}^\infty$ converges to $W(\omega)$ in $G_{d,m}$.*

Proof of Lemma 18. From Lemma 17 we get that $\widetilde{\mathbf{T}}$ is a contracting and strongly irreducible subset of $Gl(q, \mathbb{R})$. Hence we may apply proposition III.3.2 in [BL2]

on the i.i.d. sequence $\{\mathcal{A}^m A_{\omega_n}^{-1}\}_{n=0}^{\infty}$. For each $\omega \in \Omega$ and $n \geq 1$ set $M_{n,\omega} = A_{\omega_{n-1}}^{-1}, \cdots, A_{\omega_0}^{-1}$, set $\xi_{n,\omega} = U_{n,\omega} e_{d-m+1} \wedge \cdots \wedge U_{n,\omega} e_d$, and set

$$\widetilde{W}_n(\omega) = \{\eta \in \mathcal{A}^m(\mathbb{R}^d) \,:\, \mathcal{A}^m M_{n,\omega}^* M_{n,\omega} \eta = \alpha_1(\mathcal{A}^m M_{n,\omega}^* M_{n,\omega}) \cdot \eta\} \,.$$

From part (b) of proposition III.3.2 it follows that for μ-a.e. $\omega \in \Omega$

$$\alpha_1(\mathcal{A}^m M_{n,\omega}^* M_{n,\omega}) > \alpha_2(\mathcal{A}^m M_{n,\omega}^* M_{n,\omega})$$

for all n large enough, and so $\widetilde{W}_n(\omega)$ is 1-dimensional for all n large enough. From part (a) of proposition III.3.2 it follows that for μ-a.e. $\omega \in \Omega$ the sequence $\{\widetilde{W}_n(\omega)\}_{n=1}^{\infty}$ converges to some element in $P(\mathcal{A}^m(\mathbb{R}^d))$. For each $\omega \in \Omega$ and $n \geq 1$ we have

$$M_{n,\omega}^* M_{n,\omega} U_{n,\omega} = (A_{\omega|_n}^{-1})^* A_{\omega|_n}^{-1} U_{n,\omega}$$
$$= (V_{n,\omega}^{-1} D_{n,\omega}^{-1} U_{n,\omega}^{-1})^* (V_{n,\omega}^{-1} D_{n,\omega}^{-1} U_{n,\omega}^{-1}) U_{n,\omega} = U_{n,\omega} D_{n,\omega}^{-2},$$

and also from Lemma 5.3 in [BL2]

$$\alpha_1(\mathcal{A}^m M_{n,\omega}^* M_{n,\omega}) = \prod_{i=1}^{m} \alpha_i(M_{n,\omega}^* M_{n,\omega}) = \prod_{i=1}^{m} \alpha_i(M_{n,\omega})^2$$
$$= \prod_{i=1}^{m} \alpha_i(A_{\omega|_n}^{-1})^2 = \prod_{i=1}^{m} \alpha_{d-i+1}(A_{\omega|_n})^{-2} \,.$$

It follows that

$$\mathcal{A}^m M_{n,\omega}^* M_{n,\omega}(\xi_{n,\omega}) = U_{n,\omega} D_{n,\omega}^{-2} e_{d-m+1} \wedge \cdots \wedge U_{n,\omega} D_{n,\omega}^{-2} e_d$$
$$= \prod_{i=1}^{m} \alpha_{d-i+1}(A_{\omega|_n})^{-2} \cdot \xi_{n,\omega} = \alpha_1(\mathcal{A}^m M_{n,\omega}^* M_{n,\omega}) \cdot \xi_{n,\omega},$$

hence $\xi_{n,\omega} \in \widetilde{W}_n(\omega)$, and so for μ-a.e. $\omega \in \Omega$ we have $\mathbb{R} \cdot \xi_{n,\omega} = \widetilde{W}_n(\omega)$ for all n large enough. This shows that for μ-a.e. $\omega \in \Omega$ the sequence $\{\mathbb{R} \cdot \xi_{n,\omega}\}_{n=1}^{\infty}$ converges in $P(\mathcal{A}^m(\mathbb{R}^d))$. Now since $\{\mathbb{R} \cdot \xi_{n,\omega}\}_{n=1}^{\infty} \subset \psi(G_{d,m})$, since $\psi(G_{d,m})$ is compact, and since ψ is an embedding, it follows that

$$\{W_n(\omega)\}_{n=1}^{\infty} = \{\psi^{-1}(\mathbb{R} \cdot \xi_{n,\omega})\}_{n=1}^{\infty}$$

converges to some $W(\omega)$ in $G_{d,m}$. This completes the proof of the lemma. $\qquad\square$

Lemma 19. *Let $U \in G_{d,m}$ be given and set*

$$\mathcal{S}_U = \{W \in G_{d,m} \,:\, U^{\perp} + W \neq \mathbb{R}^d\};$$

then $\mu_F(\mathcal{S}_U) = 0$.

Proof of Lemma 19. Set $\theta = \psi \mu_F$; then $\theta \in \mathcal{M}(P(\mathcal{A}^m(\mathbb{R}^d)))$ and

$$\theta = \sum_{\lambda \in \Lambda} p_\lambda \cdot \mathcal{A}^m A_\lambda^{-1} \theta \,.$$

From the strong irreducibility of $\widetilde{\mathbf{T}}$ and from proposition III.2.3 in [BL2], it follows that

$$\theta\{\mathbb{R} \cdot z \,:\, z \in \mathcal{Q} \setminus \{0\}\} = 0$$

for every proper subspace $\mathcal{Q}$ of $\mathcal{A}^m(\mathbb{R}^d)$. Let $\{x_1, \ldots, x_{d-m}\}$ be a basis for $U^\perp$, set $\xi = x_1 \wedge \cdots \wedge x_{d-m}$, and set

$$\mathcal{Q} = \{z \in \mathcal{A}^m(\mathbb{R}^d) \, : \, \xi \wedge z = 0\};$$

then $\mathcal{Q}$ is a proper subspace of $\mathcal{A}^m(\mathbb{R}^d)$. Now since

$$\mu_F(\mathcal{S}_U)$$
$$= \mu_F\{W \in G_{d,m} : \xi \wedge w_1 \wedge \cdots \wedge w_m = 0 \text{ where } \{w_1, \ldots, w_m\} \text{ is a basis for } W\}$$
$$= \mu_F\{W \in G_{d,m} : \psi(W) = \mathbb{R} \cdot z \text{ where } z \in \mathcal{A}^m(\mathbb{R}^d) \text{ and } \xi \wedge z = 0\}$$
$$= \theta\{\mathbb{R} \cdot z : z \in \mathcal{Q} \setminus \{0\}\} = 0$$

the lemma follows. $\square$

5. Proof of Proposition 6

Fix some $v \in \mathcal{V}$ and set $\pi = \pi_v$, $K = K_v$, $\varphi_\lambda = \varphi_{v,\lambda}$ for $\lambda \in \Lambda$, and $\mathcal{F}_W = \mathcal{F}_{v,W}$ and $\{\mu_{W,\omega}\}_{\omega \in \Omega} = \{\mu_{v,W,\omega}\}_{\omega \in \Omega}$ for $W \in G_{d,m}$. For $k \geq 1$ and $\lambda_0, \cdots, \lambda_{k-1} = w \in \Lambda^k$ let

$$[w] = \{\omega \in \Omega \, : \, \omega_i = \lambda_i \text{ for } 0 \leq i < k\},$$

and let $[\emptyset] = \Omega$. Given $\omega \in \Omega$ and $k \geq 1$ set $\omega|_k = \omega_0, \cdots, \omega_{k-1} \in \Lambda^k$ and $\omega|_0 = \emptyset$.

In the proof of Proposition 6 we shall make use of the following dynamical system. Let $\sigma : \Omega \to \Omega$ be the left shift, i.e., $(\sigma\omega)_k = \omega_{k+1}$ for $\omega \in \Omega$ and $k \geq 0$. Set $X = \Omega \times G_{d,m}$, for each $(\omega, W) \in X$, set $T(\omega, W) = (\sigma(\omega), A_{\omega_0}^{-1} \cdot W)$, and set $\nu = \mu \times \mu_F$. Since μ_F is the unique member in $\mathcal{M}(G_{d,m})$ with $\mu_F = \sum_{\lambda \in \Lambda} p_\lambda \cdot A_\lambda^{-1} \mu_F$, it follows from Proposition 1.14 in [BQ1] that (X, T, ν) is measure preserving and ergodic.

Lemma 20. *Let $E \subset \Omega$ be a Borel set, let $M \in Gl(d, \mathbb{R})$, let $W \in G_{d,m}$, and set $\widetilde{B} = P_{W^\perp} \circ M(B(0,1))$; then for μ-a.e. $\omega \in \Omega$*

$$\mu_{W,\omega}(E) = \lim_{\delta \downarrow 0} \frac{\mu(\pi^{-1} \circ P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}) \cap E)}{\mu(\pi^{-1} \circ P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}))} \, .$$

Proof of Lemma 20. Let $\mu|_E$ be the restriction of μ to E, i.e., $\mu|_E(F) = \mu(F \cap E)$ for $F \in \mathcal{F}$. For $x \in W^\perp$ set $\|x\|_{\widetilde{B}} = \inf\{t > 0 \, : \, t^{-1} \cdot x \in \widetilde{B}\}$, i.e., $\|\cdot\|_{\widetilde{B}}$ is the Minkowski functional corresponding to the convex and balanced set $\widetilde{B}$. Clearly $\|\cdot\|_{\widetilde{B}}$ is a norm on $W^\perp$, and

$$\delta \cdot \widetilde{B} = \{x \in W^\perp \, : \, \|x\|_{\widetilde{B}} \leq \delta\} \text{ for } \delta > 0 \, .$$

Now from Theorem 4.2 in [BL1] and the discussion preceding it, and from property (b) in Section 3.1 above, we get that for μ-a.e. $\omega \in \Omega$

$$\lim_{\delta \downarrow 0} \frac{\mu(\pi^{-1} \circ P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}) \cap E)}{\mu(\pi^{-1} \circ P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}))}$$
$$= \lim_{\delta \downarrow 0} \frac{P_{W^\perp} \pi \mu|_E(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B})}{P_{W^\perp} \pi \mu(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B})}$$
$$= \frac{dP_{W^\perp} \pi \mu|_E}{dP_{W^\perp} \pi \mu}(P_{W^\perp} \circ \pi(\omega)) = \mu_{W,\omega}(E),$$

which proves the lemma. $\square$

Lemma 21. *For each* $W \in G_{d,m}$ *and* $k \geq 0$

$$\frac{\mu_{W,\omega}[\omega|_{k+1}]}{\mu_{W,\omega}[\omega|_k]} = \mu_{(A_{\omega|_k})^{-1} \cdot W, \sigma^k \omega}[\omega_k] \quad \text{for } \mu\text{-a.e. } \omega \in \Omega .$$

Proof of Lemma 21. For each $\lambda \in \Lambda$ and $\omega \in \Omega$ set $f_\lambda(\omega) = \lambda \cdot \omega$, i.e., $f_\lambda(\omega)$ is the concatenation of λ with ω. Let $W \in G_{d,m}$, $k \geq 0$, and $w \in \Lambda^k$ be given, and set $U = (A_w)^{-1} \cdot W$. From property (b) stated in Section 3.1 above and since $\mu(f_w(E)) = p_w \cdot \mu(E)$ for each $E \in \mathcal{F}$, it follows that for μ-a.e. $\omega \in \Omega$

$$(5.1) \quad \mu_{U,\sigma^k \omega}[\omega_k] = \lim_{\delta \downarrow 0} \frac{\mu(\pi^{-1} \circ P_{U^\perp}^{-1}(B(P_{U^\perp} \circ \pi \circ \sigma^k(\omega), \delta)) \cap [(\sigma^k \omega)|_1])}{\mu(\pi^{-1} \circ P_{U^\perp}^{-1}(B(P_{U^\perp} \circ \pi \circ \sigma^k(\omega), \delta)))}$$

$$= \lim_{\delta \downarrow 0} \frac{\mu(f_w(\pi^{-1} \circ P_{U^\perp}^{-1}(B(P_{U^\perp} \circ \pi \circ \sigma^k(\omega), \delta)) \cap [(\sigma^k \omega)|_1]))}{\mu(f_w \circ \pi^{-1} \circ P_{U^\perp}^{-1}(B(P_{U^\perp} \circ \pi \circ \sigma^k(\omega), \delta)))} .$$

Fix $\omega \in [w]$ and $\delta > 0$, and set $\widetilde{B} = P_{W^\perp} \circ A_w(B(0,1))$. Since $f_w \circ \pi^{-1}(x) = \pi^{-1} \circ \varphi_w(x)$ for $x \in K$,

$$(5.2) \quad f_w \circ \pi^{-1} \circ P_{U^\perp}^{-1}(B(P_{U^\perp} \circ \pi \circ \sigma^k(\omega), \delta))$$

$$= \pi^{-1} \circ \varphi_w(K \cap P_{U^\perp}^{-1}(B(P_{U^\perp} \circ \pi \circ \sigma^k(\omega), \delta)))$$

$$= \pi^{-1} \circ \varphi_w(K) \cap \pi^{-1} \circ \varphi_w(\pi \circ \sigma^k(\omega) + U + B(0, \delta))$$

$$= [\omega|_k] \cap \pi^{-1} \circ \varphi_w(\pi \circ \sigma^k(\omega) + U + B(0, \delta)) .$$

From $\varphi_w \circ \pi = \pi \circ f_w$ and $\omega|_k = w$ we get

$$\varphi_w(\pi \circ \sigma^k(\omega) + U + B(0, \delta))$$

$$= \pi \circ f_w \circ \sigma^k(\omega) + A_w \cdot U + A_w(B(0, \delta))$$

$$= \pi(\omega) + W + \delta \cdot A_w(B(0,1))$$

$$= W + P_{W^\perp} \circ \pi(\omega) + \delta \cdot P_{W^\perp}(A_w(B(0,1)))$$

$$= P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}) .$$

From this and from (5.2) we obtain

$$f_w \circ \pi^{-1} \circ P_{U^\perp}^{-1}(B(P_{U^\perp} \circ \pi \circ \sigma^k(\omega), \delta)) = [\omega|_k] \cap \pi^{-1} \circ P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}),$$

for each $\omega \in [w]$ and $\delta > 0$. It now follows from (5.1) and Lemma 20 that for μ-a.e. $\omega \in [w]$

$$\mu_{U,\sigma^k \omega}[\omega_k] = \lim_{\delta \downarrow 0} \frac{\mu([\omega|_k] \cap \pi^{-1} \circ P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}) \cap f_w([(\sigma^k \omega)|_1]))}{\mu([\omega|_k] \cap \pi^{-1} \circ P_{W^\perp}^{-1}(P_{W^\perp} \circ \pi(\omega) + \delta \cdot \widetilde{B}))}$$

$$= \frac{\mu_{W,\omega}([\omega|_k] \cap f_w[(\sigma^k \omega)|_1])}{\mu_{W,\omega}[\omega|_k]} = \frac{\mu_{W,\omega}[\omega|_{k+1}]}{\mu_{W,\omega}[\omega|_k]} .$$

This proves the lemma since $U = (A_{\omega|_k})^{-1} \cdot W$ for $\omega \in [w]$, and since w is an arbitrary element of Λ^k. $\square$

Proof of Proposition 6. Recall that $\mathcal{P} = \{[\lambda] : \lambda \in \Lambda\}$. For $w \in \Lambda^*$ set $K_w = \varphi_w(K)$. Define $I : X \to \mathbb{R}$ by $I(\omega, W) = -\log \mu_{W,\omega}[\omega_0]$ for $(\omega, W) \in X$. It follows

from property (c) stated in Section 3.1, from the ergodic theorem, and from Lemma 21, that for ν-a.e. $(\omega, W) \in X$

$$(5.3) \quad \int H_\mu(\mathcal{P} \mid \mathcal{F}_U) \, d\mu_F(U)$$

$$= \int \int -\log E_\mu[1_{[\eta_0]} \mid \mathcal{F}_U](\eta) \, d\mu(\eta) \, d\mu_F(U)$$

$$= \int \int -\log \mu_{U,\eta}[\eta_0] \, d\mu(\eta) \, d\mu_F(U) = \int I(\eta, U) \, d\nu(\eta, U)$$

$$= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} I \circ T^k(\omega, W) = \lim_n -\frac{1}{n} \sum_{k=0}^{n-1} \log \mu_{(A_{\omega|_k})^{-1} \cdot W, \sigma^k \omega}[\omega_k]$$

$$= \lim_n -\frac{1}{n} \sum_{k=0}^{n-1} \log \frac{\mu_{W,\omega}[\omega|_{k+1}]}{\mu_{W,\omega}[\omega|_k]} = \lim_n \frac{-\log \mu_{W,\omega}[\omega|_n]}{n}$$

$$= \lim_n \frac{-\log \pi \mu_{W,\omega}(K_{\omega|_n})}{n} \, .$$

Let $0 < \epsilon < -\gamma_1$; then there exists a Borel set $\Omega_0 \in \Omega$ with $\mu(\Omega \setminus \Omega_0) = 0$, such that for $\omega \in \Omega_0$ there exists $N_\omega \geq 1$ for which

$$\alpha_i(A_{\omega|_n}) \in (e^{n(\gamma_i - \epsilon)}, e^{n(\gamma_i + \epsilon)}) \text{ for } n \geq N_\omega \text{ and } 1 \leq i \leq d \, .$$

Since $v \in \mathcal{V}$ there exists $\rho > 0$ with

$$\rho < \min\{d(\varphi_{\lambda_1}(K), \varphi_{\lambda_2}(K)) \, : \, \lambda_1, \lambda_2 \in \Lambda \text{ with } \lambda_1 \neq \lambda_2\} \, .$$

Let $\omega \in \Omega_0$, $n \geq N_\omega$, and $\lambda_0, \cdots, \lambda_{n-1} = w \in \Lambda^n \setminus \{\omega|_n\}$. Let $0 \leq k < n$ be such that $\lambda_k \neq \omega_k$ with $\lambda_j = \omega_j$ for $0 \leq j < k$. Since $\pi(\sigma^k \omega) \in K_{\omega_k}$ we have $B(\pi(\sigma^k \omega), \rho) \cap K_{\lambda_k} = \emptyset$, and so

$$\emptyset = \varphi_{\omega|_k}(B(\pi(\sigma^k \omega), \rho) \cap K_{\lambda_k}) \supset \varphi_{\omega|_k}(B(\pi(\sigma^k \omega), \rho)) \cap K_w \, .$$

Now since

$$\varphi_{\omega|_k}(B(\pi(\sigma^k \omega), \rho)) \supset B(\varphi_{\omega|_k} \circ \pi(\sigma^k \omega), \alpha_d(A_{\omega|_k}) \cdot \rho)$$

$$\supset B(\pi(\omega), \alpha_d(A_{\omega|_n}) \cdot \rho) \supset B(\pi(\omega), e^{n(\gamma_d - \epsilon)} \cdot \rho),$$

we get $B(\pi(\omega), e^{n(\gamma_d - \epsilon)} \cdot \rho) \cap K_w = \emptyset$. We have thus shown that

$$B(\pi(\omega), e^{n(\gamma_d - \epsilon)} \cdot \rho) \cap K_w = \emptyset \text{ for } \omega \in \Omega_0, \ n \geq N_\omega, \text{ and } w \in \Lambda^n \setminus \{\omega|_n\} \, .$$

It follows from this, from the fact that $\pi \mu_{W,\omega}$ is supported on K for ν-a.e. $(\omega, W) \in X$, and from (5.3), that for ν-a.e. $(\omega, W) \in X$

$$(5.4) \quad \liminf_{\delta \downarrow 0} \frac{\log(\pi \mu_{W,\omega}(B(\pi(\omega), \delta)))}{\log \delta}$$

$$= \liminf_n \frac{\log(\pi \mu_{W,\omega}(B(\pi(\omega), \rho \cdot e^{n(\gamma_d - \epsilon)}) \cap K))}{\log(\rho \cdot e^{n(\gamma_d - \epsilon)})}$$

$$\geq \lim_n \frac{\log(\pi \mu_{W,\omega}(K_{\omega|_n}))}{n \cdot (\gamma_d - \epsilon)} = \frac{\int H_\mu(\mathcal{P} \mid \mathcal{F}_U) \, d\mu_F(U)}{\epsilon - \gamma_d} \, .$$

For each $\omega \in \Omega$ and $n \geq 1$ set $D_{n,\omega} = diag(\alpha_1(A_{\omega|_n}), \ldots, \alpha_d(A_{\omega|_n}))$, let $U_{n,\omega} D_{n,\omega} V_{n,\omega}$ be a singular value decomposition of $A_{\omega|_n}$, and set

$$L_{n,\omega} = span\{U_{n,\omega} e_{d-m+1}, \ldots, U_{n,\omega} e_d\}.$$

From Lemma 18 it follows that for μ-a.e. $\omega \in \Omega$ there exists $L_\omega \in G_{d,m}$ such that $\{L_{n,\omega}\}_{n=1}^{\infty}$ converges to L_ω in $G_{d,m}$. Set

$$X_0 = \{(\omega, W) \in X \ : \ \omega \in \Omega_0, \text{ the limit } L_\omega = \lim_n L_{n,\omega} \text{ exists, and } L_\omega^\perp + W = \mathbb{R}^d\},$$

and for $U \in G_{d,m}$ set

$$\mathcal{S}_U = \{W \in G_{d,m} \ : \ U^\perp + W \neq \mathbb{R}^d\} \,.$$

From Fubini's theorem and Lemma 19 we get

$$\nu(X \setminus X_0) \leq \int_{\{L_\omega \text{ exists}\}} \mu_F(\mathcal{S}_{L_\omega}) \, d\mu(\omega) = 0 \,.$$

Let $b \in (0, \infty)$ be such that $K \subset B(0, b)$. Fix $(\omega, W) \in X_0$; then $L_\omega^\perp \cap W = \{0\}$, so $P_{L_\omega}(x) \neq 0$ for each $x \in W \setminus \{0\}$, and so

$$a_{\omega,W} := \min\{|P_{L_\omega}(x)| \ : \ x \in W \text{ and } |x| = 1\} > 0 \,.$$

Since $\{L_{n,\omega}\}_{n=1}^{\infty}$ converges to L_ω it follows that there exists $N_{\omega,W} \geq N_\omega$ with

$$\min\{|P_{L_{n,\omega}}(x)| \ : \ x \in W \text{ and } |x| = 1\} > \frac{a_{\omega,W}}{2} \text{ for every } n \geq N_{\omega,W} \,.$$

Let $n \geq N_{\omega,W}$, and set

$$R = \pi(\omega) + L_{n,\omega}^\perp + \{x \in L_{n,\omega} \ : \ |x| \leq 2b \cdot e^{n(\gamma_d + \epsilon)}\} \,.$$

For $d - m + 1 \leq i \leq d$ we have $\gamma_i = \gamma_d$, hence $\alpha_i(A_{\omega|_n}) \leq e^{n(\gamma_d + \epsilon)}$, and so

$$A_{\omega|_n}(B(0, 2b)) = U_{n,\omega} D_{n,\omega} V_{n,\omega}(B(0, 2b)) = U_{n,\omega} D_{n,\omega}(B(0, 2b))$$
$$\subset U_{n,\omega}(span\{e_1, \ldots, e_{d-m}\} + \{x \in span\{e_{d-m+1}, \ldots, e_d\} \ : \ |x| \leq 2b \cdot e^{n(\gamma_d + \epsilon)}\})$$
$$= L_{n,\omega}^\perp + \{x \in L_{n,\omega} \ : \ |x| \leq 2b \cdot e^{n(\gamma_d + \epsilon)}\} \,.$$

It follows that for $y \in K$

$$\varphi_{\omega|_n}(y) - \pi(\omega) = \varphi_{\omega|_n}(y) - \varphi_{\omega|_n} \circ \pi \circ \sigma^n(\omega)$$
$$= A_{\omega|_n}(y - \pi \circ \sigma^n(\omega)) \in A_{\omega|_n}(B(0, 2b))$$
$$\subset L_{n,\omega}^\perp + \{x \in L_{n,\omega} \ : \ |x| \leq 2b \cdot e^{n(\gamma_d + \epsilon)}\},$$

which shows that $K_{\omega|_n} \subset R$. Given $x \in W$ with $|x| > \frac{4b}{a_{\omega,W}} \cdot e^{n(\gamma_d + \epsilon)}$ we have

$$|P_{L_{n,\omega}}(x)| = |x| \cdot |P_{L_{n,\omega}}(\frac{x}{|x|})| > |x| \cdot \frac{a_{\omega,W}}{2} > 2b \cdot e^{n(\gamma_d + \epsilon)} \,.$$

It follows that $x + \pi(\omega) \notin R$, and so

$$(\pi(\omega) + W) \cap K_{\omega|_n} \subset (\pi(\omega) + W) \cap R \subset B(\pi(\omega), \frac{4b}{a_{\omega,W}} \cdot e^{n(\gamma_d + \epsilon)}) \,.$$

We have thus shown that
(5.5)
$$K_{\omega|_n} \cap (\pi(\omega) + W) \subset B(\pi(\omega), \frac{4b}{a_{\omega,W}} \cdot e^{n(\gamma_d + \epsilon)}) \text{ for every } (\omega, W) \in X_0 \text{ and } n \geq N_{\omega,W}.$$

From property (a) stated in Section 3.1 it follows that $\pi\mu_{W,\omega}$ is supported on $\pi(\omega) + W$ for ν-a.e. $(\omega, W) \in X$. From this, from (5.5), and from (5.3), we get that for ν-a.e. $(\omega, W) \in X$

$$(5.6) \quad \limsup_{\delta \downarrow 0} \frac{\log(\pi\mu_{W,\omega}(B(\pi(\omega), \delta)))}{\log \delta}$$

$$= \limsup_n \frac{\log(\pi\mu_{W,\omega}(B(\pi(\omega), \frac{4b}{a_{\omega,W}} \cdot e^{n(\gamma_d + \epsilon)})))}{\log(\frac{4b}{a_{\omega,W}} \cdot e^{n(\gamma_d + \epsilon)})}$$

$$\leq \lim_n \frac{\log(\pi\mu_{W,\omega}(K_{\omega|_n} \cap (\pi(\omega) + W)))}{n \cdot (\gamma_d + \epsilon)}$$

$$= \lim_n \frac{\log(\pi\mu_{W,\omega}(K_{\omega|_n}))}{n \cdot (\gamma_d + \epsilon)} = \frac{\int H_\mu(\mathcal{P} \mid \mathcal{F}_U) \, d\mu_F(U)}{-\gamma_d - \epsilon}.$$

Now since $\epsilon > 0$ can be chosen arbitrarily small the proposition follows from (5.4) and (5.6). $\qquad\square$

6. Proofs of auxiliary lemmas

Proof of Lemma 7. Given a continuous $g : \mathbb{R}^d \to \mathbb{R}$ with compact support it holds for μ-a.e. ω that

$$\int g \, d(\pi_v \mu)_{W, \pi_v(\omega)} = \lim_{\delta \downarrow 0} \frac{1}{P_{W^\perp} \pi_v \mu(B(P_{W^\perp} \pi_v(\omega), \delta))} \cdot \int_{P_{W^\perp}^{-1}(B(P_{W^\perp} \pi_v(\omega), \delta))} g \, d\pi_v \mu$$

$$= \lim_{\delta \downarrow 0} \frac{1}{P_{W^\perp} \pi_v \mu(B(P_{W^\perp} \pi_v(\omega), \delta))} \cdot \int_{\pi_v^{-1} \circ P_{W^\perp}^{-1}(B(P_{W^\perp} \pi_v(\omega), \delta))} g \circ \pi_v \, d\mu$$

$$= \int g \circ \pi_v \, d\mu_{v, W, \omega} = \int g \, d\pi_v \mu_{v, W, \omega},$$

which proves the lemma. $\qquad\square$

Proof of Lemma 8. Fix $W \in G_{d,m}$ and $v_0 \in \mathcal{V}$, and for each $v \in \mathcal{V}$ set $F_W(v) = H_\mu(\mathcal{P} \mid \mathcal{F}_{v,W})$; then it suffices to show that $F_W : \mathcal{V} \to \mathbb{R}$ is upper semi-continuous at v_0. Let $\{u_1, \ldots, u_{d-m}\}$ be an orthonormal basis for $W^\perp$, and for $1 \leq i \leq d-m$ set $U_i = span\{u_i\}$ and

$$\mathcal{Q}_i = \{t \in \mathbb{R} \, : \, P_{U_i} \pi_{v_0} \mu\{t \cdot u_i\} = 0\}.$$

Clearly $\mathbb{R} \setminus \mathcal{Q}_i$ is at most countable. For each $1 \leq i \leq d-m$ and $n \geq 1$ let $\{a_{n,k}^i\}_{k=-\infty}^\infty = \mathcal{J}_n^i \subset \mathcal{Q}_i$ be such that $2^{-n-1} \leq a_{n,k+1}^i - a_{n,k}^i \leq 2^{-n}$ for $k \in \mathbb{Z}$, and such that $\mathcal{J}_n^i \subset \mathcal{J}_{n+1}^i$. For $n \geq 1$ and $(k_1, \ldots, k_{d-m}) = \bar{k} \in \mathbb{Z}^{d-m}$ set

$$S_{n,\bar{k}} = P_{W^\perp}^{-1} \{\sum_{i=1}^{d-m} t^i \cdot u_i : (t^1, \ldots, t^{d-m}) \in [a_{n,k_1}^1, a_{n,k_1+1}^1) \times \cdots \times [a_{n,k_{d-m}}^{d-m}, a_{n,k_{d-m}+1}^{d-m})\}.$$

For $n \geq 1$ and $v \in \mathcal{V}$ let $\mathcal{G}_{v,n}$ be the σ-algebra on Ω generated by

$$\{\pi_v^{-1}(S_{n,\bar{k}}) \, : \, \bar{k} \in \mathbb{Z}^{d-m}\},$$

and set $F_{W,n}(v) = H_\mu(\mathcal{P} \mid \mathcal{G}_{v,n})$. For $v \in \mathcal{V}$ we have $\mathcal{G}_{v,1} \subset \mathcal{G}_{v,2} \subset \ldots$ and $\mathcal{F}_{v,W} = \bigvee_{n=1}^\infty \mathcal{G}_{v,n}$, hence from Theorem 6 on page 38 of [P] we get that $F_{W,1} \geq F_{W,2} \geq \ldots$ and $F_W = \lim_n F_{W,n}$. It follows that it is enough to prove that $F_{W,n} : \mathcal{V} \to \mathbb{R}$ is continuous at v_0 for $n \geq 1$. Let $n \geq 1$, $(k_1, \ldots, k_{d-m}) = \bar{k} \in \mathbb{Z}^{d-m}$ and $\lambda \in \Lambda$ be

given, and for $v \in \mathcal{V}$ set $f(v) = \mu([\lambda] \cap \pi_v^{-1}(S_{n,\bar{k}}))$. From the way $F_{W,n}$ is defined it follows that it suffices to show that f is continuous at v_0. From $a_{n,k_i}^i, a_{n,k_i+1}^i \in \mathcal{Q}_i$ for each $1 \le i \le d-m$ it follows that $\mu(\pi_{v_0}^{-1}(\partial S_{n,\bar{k}})) = 0$, and for $\omega \in \Omega \backslash \pi_{v_0}^{-1}(\partial S_{n,\bar{k}})$ we have

$$\lim_{v \to v_0} 1_{[\lambda] \cap \pi_v^{-1}(S_{n,\bar{k}})}(\omega) = 1_{[\lambda] \cap \pi_{v_0}^{-1}(S_{n,\bar{k}})}(\omega),$$

hence from the dominated convergence theorem $\lim_{v \to v_0} f(v) = f(v_0)$. This completes the proof of the lemma. □

Proof of Lemma 9. Since $\pi_v\mu$ is supported on K_v it suffices to show that $\mathcal{L}eb_d(K_v) = 0$. Let $\rho > 0$ be such that

$$\rho < \frac{1}{2} \cdot \min\{d(\varphi_{v,\lambda_1}(K_v), \varphi_{v,\lambda_2}(K_v)) \ : \ \lambda_1, \lambda_2 \in \Lambda \text{ with } \lambda_1 \neq \lambda_2\}$$

and set $U = \{x \in \mathbb{R}^d : d(x, K_v) < \rho\}$; then $\varphi_{v,\lambda_1}(U) \subset U$ and $\varphi_{v,\lambda_1}(U) \cap \varphi_{v,\lambda_2}(U) = \emptyset$ for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$. Also it is easy to see that the set $U \setminus \bigcup_{\lambda \in \Lambda} \varphi_{v,\lambda}(U)$ has a non-empty interior, hence

$$\mathcal{L}eb_d(U) > \mathcal{L}eb_d(\bigcup_{\lambda \in \Lambda} \varphi_{v,\lambda}(U)) = \sum_{\lambda \in \Lambda} \mathcal{L}eb_d(\varphi_{v,\lambda}(U)) = \mathcal{L}eb_d(U) \cdot \sum_{\lambda \in \Lambda} |\det(A_\lambda)|,$$

and so $\sum_{\lambda \in \Lambda} |\det(A_\lambda)| < 1$. In addition, for each $n \ge 1$ we have

$$\mathcal{L}eb_d(K_v) \le \mathcal{L}eb_d(\bigcup_{w \in \Lambda^n} \varphi_{v,w}(U)) = \sum_{w \in \Lambda^n} \mathcal{L}eb_d(\varphi_{v,w}(U))$$

$$= \mathcal{L}eb_d(U) \cdot \sum_{w \in \Lambda^n} |\det(A_w)| = \mathcal{L}eb_d(U) \cdot \sum_{\lambda_1, \dots, \lambda_n \in \Lambda} \prod_{i=1}^{n} |\det(A_{\lambda_i})|$$

$$= \mathcal{L}eb_d(U) \cdot (\sum_{\lambda \in \Lambda} |\det(A_\lambda)|)^n,$$

which shows that $\mathcal{L}eb_d(K_v) = 0$. □

For the proof of Lemma 10 we shall first need the following lemma regarding the dimension of exceptional sets of projections. Given $\theta \in \mathcal{M}(\mathbb{R}^d)$ and $t > 0$ let $I_t(\theta)$ be the t-energy of θ (see Section 2.5 of [M2]), and let $\dim_S \theta$ be the Sobolev dimension of θ (see Section 5.2 of [M2]). Given a Borel set $E \subset \mathbb{R}^d$ we denote the restriction of θ to E by $\theta|_E$.

Lemma 22. *Let* $\theta \in \mathcal{M}(\mathbb{R}^d)$ *and* $1 \le l < d$ *be given and set* $s = \dim_H \theta$; *then:*
    (a) If $s \le l$, *then for* $0 < t \le s$

$$\dim_H\{W \in G_{d,l} \ : \ \dim_H(P_W\theta) < t\} \le l(d - l - 1) + t.$$

    (b) If $s > l$, *then for* $s - l(d - l) \le t \le l$

$$\dim_H\{W \in G_{d,l} \ : \ \dim_H(P_W\theta) < t\} \le l(d - l) + t - s.$$

    (c) If $s > l$, *then*

$$\dim_H(G_{d,l} \setminus \{W \in G_{d,l} \ : \ P_W\theta \ll \mathcal{H}^l\}) \le l(d - l + 1) - s,$$

where $\mathcal{H}^l$ *is the l-dimensional Hausdorff measure.*

Proof of Lemma 22, *part (a).* Let $0 < t_0 < t_1 < t$, and for each $n \geq 1$ set

$$E_n = \{x \in \mathbb{R}^d \ : \ \theta(B(x,\delta)) \leq n \cdot \delta^{t_1} \text{ for each } \delta > 0\} \,.$$

From $\dim_H \theta > t_1$ and (2.3) we get $\theta(\mathbb{R}^d \setminus \bigcup_n E_n) = 0$. From an argument as the one given on page 19 of [M2] it follows that $I_{t_0}(\theta|_{E_n}) < \infty$ for each $n \geq 1$. From this, from Theorem 5.10 in [M2], and since $\dim_S \xi \leq \dim_H \xi$ for each $\xi \in \mathcal{M}(\mathbb{R}^d)$ with $\dim_H \xi < d$, we get

$$\dim_H \{W \in G_{d,l} \ : \ \dim_H(P_W \theta) < t_0\}$$
$$= \sup_{n \geq 1} \dim_H \{W \in G_{d,l} \ : \ \dim_H(P_W(\theta|_{E_n})) < t_0\} \leq l(d - l - 1) + t \,.$$

As this holds for every $0 < t_0 < t$ we obtain (a).   □

Proof of part (b). Let $l < t_0 < t_1 < s$, and for each $n \geq 1$ let E_n be as in the proof of (**a**). Since $I_{t_0}(\theta|_{E_n}) < \infty$ for each $n \geq 1$, it follows from Theorem 5.10 in [M2] that

$$\dim_H \{W \in G_{d,l} \ : \ \dim_H(P_W \theta) < t\}$$
$$= \sup_{n \geq 1} \dim_H \{W \in G_{d,l} \ : \ \dim_H(P_W(\theta|_{E_n})) < t\} \leq l(d - l) + t - t_0 \,.$$

Now by letting t_0 tend to s we obtain (b).   □

Proof of part (c). Let $l < t_2 < t_0 < t_1 < s$, and for each $n \geq 1$ let E_n be as in the proof of (**a**). Since $I_{t_0}(\theta|_{E_n}) < \infty$ for each $n \geq 1$, it follows from Theorems 5.4.b and 5.10 in [M2] that

$$\dim_H(G_{d,l} \setminus \{W \in G_{d,l} \ : \ P_W \theta \ll \mathcal{H}^l\})$$
$$= \sup_{n \geq 1} \dim_H(G_{d,l} \setminus \{W \in G_{d,l} \ : \ P_W(\theta|_{E_n}) \ll \mathcal{H}^l\})$$
$$\leq \sup_{n \geq 1} \dim_H \{W \in G_{d,l} \ : \ \dim_S(P_W(\theta|_{E_n})) < t_2\} \leq l(d - l) + t_2 - t_0 \,.$$

Now by letting t_2 tend to l and t_0 tend to s we obtain (c).   □

For the proof of Lemma 10 we shall also need the following proposition, which follows directly from Theorem 5.8 in [F2]. The proof is actually given in [F2] for the case $d = 2$, but extends to higher dimensions without difficulty.

Proposition 23. *Let* $1 \leq l < d$, $E \subset \mathbb{R}^d$, $W \in G_{d,l}$, $\emptyset \neq A \subset W^\perp$, *and* $t > 0$ *be given. If* $\dim_H(E \cap (x + W)) \geq t$ *for each* $x \in A$, *then* $\dim_H E \geq t + \dim_H A$.

Proof of Lemma 10, *part (a).* Assume by contradiction that the claim is false for some $0 < t \leq s$; then

(6.1) $\dim_H \{W \in G_{d,l} : \text{essinf}_\theta \{\dim_H(\theta_{W,x}) : x \in \mathbb{R}^d\} > s - t\} > (l-1)(d-l) + t.$

Since the map that sends $W \in G_{d,l}$ to $W^\perp \in G_{d,d-l}$ is an isometry with respect to the metric on the Grassmannian defined in Section 2, we get from part (a) of Lemma 22 that

$$\dim_H \{W \in G_{d,l} \ : \ \dim_H(P_{W^\perp} \theta) < t\}$$
$$= \dim_H \{W \in G_{d,d-l} \ : \ \dim_H(P_W \theta) < t\} \leq (l-1)(d-l) + t \,.$$

From this and (6.1) it follows that there exists $0 < \epsilon < t$ and $W \in G_{d,l}$ such that $\dim_H(P_{W^\perp}\theta) \geq t$ and

$$\operatorname{essinf}_\theta\{\dim_H(\theta_{W,x}) \, : \, x \in \mathbb{R}^d\} > s - t + \epsilon \, .$$

Let $E \subset \mathbb{R}^d$ be a Borel set with $\theta(E) > 0$, for $x \in W^\perp$ set $E_x = E \cap (x + W)$, and set

$$A = \{x \in W^\perp \, : \, \theta_{W,x}(E_x) > 0 \text{ and } \dim_H(\theta_{W,x}) \geq s - t + \epsilon\} \, .$$

From properties stated in Section 3.1 it follows that $P_{W^\perp}\theta(A) > 0$, hence

$$\dim_H A \geq \dim_H(P_{W^\perp}\theta) \geq t \, .$$

For $x \in A$ we have

$$\dim_H E_x \geq \dim_H(\theta_{W,x}) \geq s - t + \epsilon,$$

and so from Proposition 23 we obtain $\dim_H E \geq s + \epsilon$. As this holds for every Borel set $E \subset \mathbb{R}^d$ with $\theta(E) > 0$, it follows that $s = \dim_H \theta \geq s + \epsilon$. This is clearly a contradiction, and so we obtain part (a) of the lemma. The proof of part (b) is the same, except we need to use part (b) of Lemma 22 instead of part (a). $\qquad\square$

Proof of part (c). Set

$$S = \{W \in G_{d,l} \, : \, P_{W^\perp}\theta \ll \mathcal{H}^{d-l}\};$$

then from part (c) of Lemma 22 we get

$$(6.2) \qquad \dim_H(G_{d,l} \setminus S) \leq (d-l)(l+1) - s \, .$$

Let $d - l < t_0 < t_1 < s$ and for $n \geq 1$ set

$$E_n = \{x \in \mathbb{R}^d \, : \, \theta(B(x,\delta)) \leq n \cdot \delta^{t_1} \text{ for each } \delta > 0\};$$

then as in the proof of part (a) of Lemma 22 we have $\theta(\mathbb{R}^d \setminus \bigcup_n E_n) = 0$ and $I_{t_0}(\theta|_{E_n}) < \infty$ for each $n \geq 1$. Since for each $W \in G_{d,l}$ we have $\theta_{W,x}(\mathbb{R}^d \setminus \bigcup_n E_n) = 0$ for θ-a.e. $x \in \mathbb{R}^d$, it follows that

$$(6.3) \quad \dim_H\{W \in S \, : \, \operatorname{essinf}_\theta\{\dim_H(\theta_{W,x}) \, : \, x \in \mathbb{R}^d\} < t_0 - d + l\}$$
$$= \sup_{n \geq 1} \dim_H\{W \in S \, : \, \operatorname{essinf}_\theta\{\dim_H(\theta_{W,x}|_{E_n}) \, : \, x \in \mathbb{R}^d\} < t_0 - d + l\} \, .$$

As described in Section 2 of [JM], given $W \in G_{d,l}$ and a Radon measure ξ on $\mathbb{R}^d$ with compact support, there exist Radon measures $\{\xi^{W,x}\}_{x \in W^\perp}$ on $\mathbb{R}^d$ such that for $\mathcal{H}^{d-l}$-a.e. $x \in W^\perp$

$$\int g \, d\xi^{W,x} = \lim_{\delta \downarrow 0} \frac{1}{(2\delta)^{d-l}} \cdot \int_{P_{W^\perp}^{-1}(B(x,\delta))} g \, d\xi \quad \text{for } g \in C(\mathbb{R}^d) \, .$$

For $x \in \mathbb{R}^d$ we set $\xi^{W,x} := \xi^{W,P_{W^\perp}x}$.

Fix some $n \geq 1$ with $\theta(E_n) > 0$, and let $W \in S$. From property (b) in Section 3.1 above and from Theorem 2.12 in [M3], it follows that for θ-a.e. $x \in \mathbb{R}^d$ we have for each $g \in C(\mathbb{R}^d)$

$$\int g \, d\theta^{W,x} = \lim_{\delta \downarrow 0} \frac{P_{W^\perp}\theta(B(P_{W^\perp}x,\delta))}{(2\delta)^{d-l}} \cdot \frac{\int_{P_{W^\perp}^{-1}(B(P_{W^\perp}x,\delta))} g \, d\theta}{P_{W^\perp}\theta(B(P_{W^\perp}x,\delta))}$$
$$= \frac{dP_{W^\perp}\theta}{d\mathcal{H}^{d-l}}(P_{W^\perp}x) \cdot \int g \, d\theta_{W,x},$$

which shows that

$$\theta^{W,x} = \frac{dP_{W^\perp}\theta}{d\mathcal{H}^{d-l}}(P_{W^\perp}x) \cdot \theta_{W,x} \,.$$

From this, from $0 < \frac{dP_{W^\perp}\theta}{d\mathcal{H}^{d-l}}(P_{W^\perp}x) < \infty$ for θ-a.e. $x \in \mathbb{R}^d$, and from Lemma 3.2 in [JM], we get that for θ-a.e. $x \in \mathbb{R}^d$

$$\dim_H(\theta_{W,x}|_{E_n}) = \dim_H(\theta^{W,x}|_{E_n}) = \dim_H((\theta|_{E_n})^{W,x}) \,.$$

Now from Lemma 2.22 in [JM], from $I_{t_0}(\theta|_{E_n}) < \infty$, and from Theorem 6.5 in [M2], we obtain

$$\dim_H\{W \in S \,:\, \operatorname{essinf}_\theta\{\dim_H(\theta_{W,x}|_{E_n}) \,:\, x \in \mathbb{R}^d\} < t_0 - d + l\}$$
$$= \dim_H\{W \in S \,:\, \operatorname{essinf}_\theta\{\dim_H((\theta|_{E_n})^{W,x}) \,:\, x \in \mathbb{R}^d\} < t_0 - d + l\}$$
$$\leq \dim_H\{W \in S \,:\, \int_{W^\perp} I_{t_0-d+l}((\theta|_{E_n})^{W,x}) \, d\mathcal{H}^{d-l}(x) = \infty\} \leq (d-l)(l+1) - t_0 \,.$$

This together with (6.2) and (6.3) proves part (c) of the lemma, since we can let t_0 tend to s. $\qquad\square$

Proof of Lemma 11. Fix $v \in \mathbb{R}^{d|\Lambda|}$ and set $\pi = \pi_v$, $K = K_v$, and $\varphi_\lambda = \varphi_{v,\lambda}$ for $\lambda \in \Lambda$. Let $k := k(\mu) \geq 0$ be as defined in (2.2). If $D(\mu) \geq d$, then there is nothing to prove (see Proposition 10.3 in [F1]), hence we can assume $D(\mu) < d$, and so $k < d$. For $1 \leq i \leq k$ and $w \in \Lambda^*$ set $d_{i,w} = \left\lceil \frac{\alpha_i(A_w)}{\alpha_{k+1}(A_w)} \right\rceil$, and set

$$d_w = \begin{cases} \prod_{i=1}^k d_{i,w} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

There exists a constant $a > 0$ such that for each $w \in \Lambda^*$ there exists a rectangle $R_w \subset \mathbb{R}^d$ with $\varphi_w(K) \subset R_w$ and with side lengths $s_1, \ldots, s_d > 0$, where

$$s_i = \begin{cases} a \cdot \alpha_{k+1}(A_w) \cdot d_{i,w} & \text{if } 1 \leq i \leq k, \\ a \cdot \alpha_{k+1}(A_w) & \text{if } k+1 \leq i \leq d. \end{cases}$$

For $w \in \Lambda^*$ let $\mathcal{R}_w = \{R_{w,1}, \ldots, R_{w,d_w}\}$ be a partition of R_w into disjoint squares of side length $a \cdot \alpha_{k+1}(A_w)$. For $\omega \in \Omega$ and $n \geq 1$ let $R_{\omega,n}$ be the unique member of $\mathcal{R}_{\omega|_n}$ which contains $\pi(\omega)$. For each $n \geq 1$ set

$$E_n = \{\omega \in \Omega \,:\, \pi\mu(R_{\omega,n}) \leq \frac{\mu[\omega|_n]}{d_{\omega|_n} \cdot n^2}\};$$

then

$$\mu(E_n) \leq \sum_{w \in \Lambda^n} \sum_{j=1}^{d_w} \pi\mu(R_{w,j}) \cdot 1_{\{\pi\mu(R_{w,j}) \leq \frac{\mu[w]}{d_w \cdot n^2}\}} \leq \frac{1}{n^2},$$

and so $\sum_{n=1}^\infty \mu(E_n) < \infty$. From this and the Borel-Cantelli Lemma it follows that

$$(6.4) \qquad \mu\{\omega \,:\, \#\{n \geq 1 \,:\, \omega \in E_n\} = \infty\} = 0 \,.$$

There exists a constant $a' > a$ such that

$$R_{\omega,n} \subset B(\pi(\omega), a' \cdot \alpha_{k+1}(A_{\omega|_n})) \text{ for } \omega \in \Omega \text{ and } n \geq 1,$$

hence for $\omega \in \Omega$

$$\limsup_{\delta \downarrow 0} \frac{\log \pi\mu(B(\pi(\omega), \delta))}{\log \delta}$$

$$= \limsup_{n \to \infty} \frac{\log \pi\mu(B(\pi(\omega), a' \cdot \alpha_{k+1}(A_{\omega|_n})))}{\log(a' \cdot \alpha_{k+1}(A_{\omega|_n}))}$$

$$\leq \limsup_{n \to \infty} \frac{\log \pi\mu(R_{\omega,n})}{\log(\alpha_{k+1}(A_{\omega|_n}))} \, .$$

Now from (6.4) it follows that for μ-a.e. $\omega \in \Omega$

$$\limsup_{\delta \downarrow 0} \frac{\log \pi\mu(B(\pi(\omega), \delta))}{\log \delta} \leq \limsup_{n \to \infty} \frac{\log(\frac{\mu[\omega|_n]}{d_{\omega|_n} \cdot n^2})}{\log(\alpha_{k+1}(A_{\omega|_n}))}$$

$$= \limsup_{n \to \infty} \frac{\log \mu[\omega|_n] - \sum_{i=1}^{k} \log \frac{\alpha_i(A_{\omega|_n})}{\alpha_{k+1}(A_{\omega|_n})}}{\log(\alpha_{k+1}(A_{\omega|_n}))} \, .$$

This together with (2.1) and the Shannon-McMillan-Breiman theorem gives

$$\limsup_{\delta \downarrow 0} \frac{\log \pi\mu(B(\pi(\omega), \delta))}{\log \delta} \leq k - \frac{h_\mu + \gamma_1 + \cdots + \gamma_k}{\gamma_{k+1}} = D(\mu)$$

for μ-a.e. $\omega \in \Omega$, which proves the lemma. $\qquad\square$

Proof of Lemma 12. Assume by contradiction that $\mathbf{G}'$ is not m-irreducible; then there exists a proper linear subspace W of $\mathcal{A}^m(\mathbb{R}^d)$ such that $\mathcal{A}^m M(W) = W$ for all $M \in \mathbf{G}'$. Let $W_1, \ldots, W_k$ be an enumeration of the set

$$\{\mathcal{A}^m A_w(W) \, : \, w = \lambda_1, \cdots, \lambda_l \text{ for some } 0 \leq l < n \text{ and } \lambda_1, \ldots, \lambda_l \in \Lambda\}$$

and define

$$\mathbf{H} = \{M \in Gl(d, \mathbb{R}) \, : \, \forall \, 1 \leq i \leq k \quad \exists \, 1 \leq j \leq k \ \text{ with } \mathcal{A}^m M(W_i) = W_j\};$$

then $\mathbf{H}$ is a closed subgroup of $Gl(d, \mathbb{R})$. Let $\mathbf{T}$ denote the closure of the semigroup generated by $\{A_\lambda^{-1}\}_{\lambda \in \Lambda}$. Since $\mathcal{A}^m M(W) = W$ for each $M \in \mathbf{G}'$ it follows that $\mathbf{H}$ contains the semigroup generated by $\{A_\lambda\}_{\lambda \in \Lambda}$, and so $\mathbf{T} \subset \mathbf{H}$. This implies that $\mathbf{T}$ is not m-strongly irreducible which contradicts Lemma 17, and so it must hold that $\mathbf{G}'$ is m-irreducible.

From Proposition III.5.6 in [BL2] it follows that for each $1 \leq i \leq d$

$$\gamma_i' = \lim_N \frac{1}{N} \int_{(\Lambda^n)^\mathbb{N}} \log \alpha_i(A_{\omega|_N}) \, d\mu'(\omega)$$

$$= \lim_N \frac{1}{N} \int_{\Lambda^\mathbb{N}} \log \alpha_i(A_{\omega|_{n \cdot N}}) \, d\mu(\omega) = n \cdot \gamma_i,$$

hence

$$\max\{1 \leq i \leq d \, : \, \gamma_{d-i+1}' = \cdots = \gamma_d'\} = m < d \, .$$

From this, from the m-irreducibility of $\mathbf{G}'$, and from Proposition 3, it follows that there exists a unique $\mu_F' \in \mathcal{M}(G_{d,m})$ with $\mu_F' = \sum_{w \in \Lambda^n} p_w \cdot A_w^{-1} \mu_F'$. Clearly we also have $\mu_F = \sum_{w \in \Lambda^n} p_w \cdot A_w^{-1} \mu_F$, hence $\mu_F' = \mu_F$. $\qquad\square$

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