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HIGHER-DIMENSIONAL CONTACT MANIFOLDS WITH INFINITELY MANY STEIN FILLINGS

TAKAHIRO OBA

ABSTRACT. For any integer $n \geq 2$, we construct an infinite family of $(4n-1)$-dimensional contact manifolds, each of which admits infinitely many pairwise homotopy inequivalent Stein fillings.

1. INTRODUCTION

The enumeration of Stein fillings of a given contact manifold has been considered as a central problem in contact and symplectic geometry and topology. As earlier answers to this problem, there are uniqueness results for fillings of the $(2n-1)$-sphere S^{2n-1} with the standard contact structure ξ_{std}. Eliashberg, Floer, and McDuff [23] showed that any symplectically aspherical filling of (S^{2n-1}, ξ_{std}) is diffeomorphic to the disk D^{2n}. Here a symplectic manifold (W, ω) is called symplectically aspherical if $[\omega] \in H^2(W; \mathbb{R})$ vanishes on all aspherical elements in $H_2(W; \mathbb{R})$. Since a Stein domain is an exact symplectic manifold, it follows from their result that any Stein filling of (S^{2n-1}, ξ_{std}) is diffeomorphic to D^{2n}. More strongly, when $n = 2$, Eliashberg [13] (see also [9]) showed that any Stein filling of (S^3, ξ_{std}) is deformation equivalent to the disk D^4 endowed with the standard Stein structure (cf. [16, 22] and [12] for the symplectomorphism and diffeomorphism parts). Concerning other 3-dimensional contact manifolds, thanks to the seminal works of Loi and Piergallini [21] and Akbulut and Ozbagci [1], we have various answers to the above problem. They showed that a 4-dimensional Stein domain admits a Lefschetz fibration and conversely that the total space of a 4-dimensional Lefschetz fibration admits a Stein structure (cf. [27, Chapter 10.2]). This enables us to study Stein fillings of a given contact 3-manifold by Lefschetz fibrations. For example, by using Lefschetz fibrations, particularly mapping class groups of fiber surfaces, Ozbagci and Stipsicz [26] constructed an infinite family of contact 3-manifolds, each of which admits infinitely many pairwise homotopy inequivalent Stein fillings (see also [3–5, 10, 34]).

In higher dimensions, the total space of an abstract Weinstein Lefschetz fibration admits a Weinstein structure, and the contact structure on the convex boundary is supported by the open book induced by the Lefschetz fibration. Moreover, according to a result in [11] (see also [9]), this Weinstein filling can be turned into a Stein filling of the same contact manifold (see Section 2.4 for details). Hence we can construct a Stein domain via Lefschetz fibration. However, its fiber is a higher-dimensional Weinstein domain, so we have to deal with higher-dimensional

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symplectic mapping class groups. Little is known about them, and thus we cannot apply group-theoretical arguments, used in [26] for example, to them directly.

Our main result is the following.

Theorem 1.1. *For any integer $n \geq 2$, there is an infinite family $\{(M_l, \xi_l)\}_{l \in \mathbb{Z}_{>0}}$ of Stein fillable contact $(4n-1)$-manifolds such that:*

(1) *each (M_l, ξ_l) admits infinitely many pairwise homotopy inequivalent Stein fillings;*
(2) *(M_l, ξ_l) and $(M_{l'}, \xi_{l'})$ are contactomorphic if and only if $l = l'$.*

In the proof of Theorem 1.1, we use open books and Lefschetz fibrations to obtain contact manifolds and their Stein fillings. The pages and fibers of our open books and Lefschetz fibrations are symplectomorphic to the Milnor fiber V_4 of the singularity of type A_4, called the A_4-Milnor fiber, endowed with the canonical symplectic structure. There is an anti-homomorphism from the braid group B_5 to the symplectic mapping class group of V_4, which is familiar as the Birman-Hilden correspondence in dimension 2. This helps us to deal with the symplectic mapping class group combinatorially and also contributes to computation of homology groups of contact manifolds and Stein fillings, coupled with the Picard-Lefschetz formula. Thus we will provide contact manifolds and their Stein fillings via mapping class groups generalizing the arguments in low dimensions.

This article is organized as follows: Section 2 consists of five subsections, where we mainly review Lefschetz fibrations, open books, and related material. In particular, in Section 2.3 we review the A_m-Milnor fiber, examine a Lefschetz fibration on it, and present the anti-homomorphism mentioned above. Also, we exhibit an explicit formula of the homology group of a manifold endowed with a Lefschetz fibration or open book in Section 2.5. Section 3 is divided into four subsections. After reviewing the Picard-Lefschetz formula and braids given by Baykur and Van Horn-Morris in Sections 3.1 and 3.2, respectively, we prove Theorem 1.1 in Section 3.3. Finally, in Section 3.4, we conclude this article by explaining why we can obtain different Stein fillings from the surgical point of view and why we put the assumption about the dimensions of contact manifolds in the main theorem.

2. Lefschetz fibrations and open books

2.1. **Dehn twists.** Let T^*S^n be the cotangent bundle of S^n and let λ be the canonical Liouville form. Consider T^*S^n as the set

$$\{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |q| = 1, q \cdot p = 0\}.$$

In our coordinates, the Liouville form λ can be written as pdq, and the zero-section S^n, which is Lagrangian in $(T^*S^n, d\lambda)$, corresponds to the set $\{(p, q) \in T^*S^n \mid p = 0\}$. For the Hamiltonian function $\mu(p, q) = |p|$ on $T^*S^n \setminus S^n$, the Hamiltonian vector field associated to μ is

$$X_\mu := |p|^{-1} \sum_j^{n+1} p_j \frac{\partial}{\partial q_j} - |p| \sum_j^{n+1} q_j \frac{\partial}{\partial p_j}.$$

The flow of X_μ has periodic orbits. To see this, project its orbit of a given point $(p, q) \in T^*S^n \setminus S^n$ onto the base space and check that its image equals the unit-speed geodesic on S^n through q with the tangent vector $p/|p|$. Since all geodesics are 2π-periodic closed circles, the flow determines the Hamiltonian S^1-action on $T^*S^n \setminus S^n$ by

$$\sigma(e^{it})(p,q) := (\cos(t)q + |p|^{-1}\sin(t)p, -|p|\sin(t)q + \cos(t)p).$$

One can extend the time-π map $\sigma(e^{i\pi})(p,q) = (-p,-q)$ to an involution of T^*S^n. The involution restricts to the antipodal map on S^n, denoted by A. Take a function $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\psi(t)+\psi(-t) = 2\pi$ for all t and $\psi(t) = 0$ for $t \gg 0$. Then the map $\tau : T^*S^n \to T^*S^n$ defined by

$$\tau(x) := \begin{cases} \sigma(e^{i\psi(|x|)})(x) & (x \notin S^n), \\ A(x) & (x \in S^n) \end{cases}$$

is called a *modeled Dehn twist*. By definition, its support is compact. Furthermore, it is a symplectomorphism of $(T^*S^n, d\lambda)$ (cf. [29, Section 6]).

Let (W,ω) be a symplectic $2n$-manifold and let $L \subset W$ be a Lagrangian n-sphere. A *framing* of L is a diffeomorphism $v : S^n \to L$. For simplicity, we drop the framing from the notation. Two Lagrangian spheres L_k $(k = 0,1)$ with framings $v_k : S^n \to L_k$ are *isotopic* if there exists a *framed Lagrangian isotopy* $I = (i, j_0, j_1)$ between them, which consists of a smooth family of Lagrangian embeddings $i^s : S^n \to W$ $(0 \le s \le 1)$ and two isometries $j_k : S^n \to S^n$ such that $v_k \circ i^k = j_k$. Suppose that L is a framed Lagrangian sphere. By the Weinstein tubular neighborhood theorem, there is $\epsilon > 0$ and a symplectic embedding $\iota : D_\epsilon T^*S^n \to W$ such that $\iota | S^n$ equals the given framing of L, particularly $\iota(S^n) = L$. Here $D_\epsilon T^*S^n = \{(p,q) \in T^*S^n \mid |p| \le \epsilon\}$. Take a function $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\psi(t)+\psi(-t) = 2\pi$ for all t and $\psi(t) = 0$ for $t > \epsilon/2$, and let τ be the modeled Dehn twist associated to this ψ. The symplectomorphism $\tau_L : (W,\omega) \to (W,\omega)$ defined by

$$\tau_L(x) := \begin{cases} \iota \circ \tau \circ \iota^{-1} & (x \in \operatorname{Im}\iota), \\ x & (x \notin \operatorname{Im}\iota) \end{cases}$$

is called a (*generalized*) *Dehn twist* along L. The symplectic isotopy class $[\tau_L] \in \pi_0(\mathrm{Symp}(W,\omega))$ is independent of the choices of ι, ψ, where $\mathrm{Symp}(W,\omega)$ denotes the group of symplectomorphisms of (W,ω) and $\pi_0(\mathrm{Symp}(W,\omega))$ denotes the group of symplectic isotopy classes of elements in $\mathrm{Symp}(W,\omega)$. We have

$$\varphi \circ \tau_L \circ \varphi^{-1} = \tau_{\varphi(L)} \tag{1}$$

for a symplectomorphism $\varphi : (W,\omega) \to (W,\omega)$ by the definition of τ_L.

Notation 2.1. In this article, we will use the usual functional notation for the products in $\mathrm{Symp}(W,\omega)$ and $\pi_0(\mathrm{Symp}(W,\omega))$; i.e., $\varphi \circ \psi$ means that we apply ψ first and then φ.

2.2. **Exact Lefschetz fibrations.** Let $(W, d\lambda)$ denote an exact symplectic manifold, where λ is a 1-form on W such that $d\lambda$ is a symplectic form on W.

Definition 2.2. Let $(W, d\lambda)$ be an oriented compact exact symplectic manifold with corners, and let $\mathbb{D}^2$ be the closed unit disk in $\mathbb{C}$. An *exact Lefschetz fibration* is a smooth map $f : (W, d\lambda) \to \mathbb{D}^2$ such that:

(i) (Submersion) f is a submersion whose fibers are smooth manifolds with boundary, except at finitely many critical points in $\operatorname{Int} W$, and $f|_{\partial_v W} : \partial_v W \to \partial\mathbb{D}^2$, $f|_{\partial_h W} :\to \mathbb{D}^2$ are fibrations, where $\partial_v W := f^{-1}(\partial\mathbb{D}^2)$, $\partial_h W := \bigcup_{q\in\mathbb{D}^2} \partial(f^{-1}(q))$.

(ii) (Lefschetz type singularities) The critical points have distinct critical values in $\mathrm{Int}\,\mathbb{D}^2$, and around each critical point and the corresponding critical value, there exist complex coordinates $(z_1, \ldots, z_n)$, w such that f can be written as

$$w = f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2,$$

with the symplectic form $d\lambda$ identified with the standard Kähler form in these coordinates.

(iii) (Exact symplectic fibers) $(F_q := f^{-1}(q), d\lambda|_{F_q})$ is an exact symplectic submanifold in $(W, d\lambda)$ for each regular value q of f.

(iv) (Horizontality of $\partial_h W$) If $p \in \partial_h W$, then $T_p^h W \subset T_p \partial_h W$, where $T_p^h W$ is the symplectic complement of $T_p^v W := \mathrm{Ker}\, df_p$ with respect to $d\lambda$.

Remark 2.3. A *smooth Lefschetz fibration* is a smooth map $f : W \to D^2$ which satisfies the conditions (i) and (ii) of Definition 2.2 except the Kählerness condition.

Here we will briefly review some basic material about exact Lefschetz fibrations. Let $f : (W, d\lambda) \to \mathbb{D}^2$ be an exact Lefschetz fibration, and let $\mathrm{Crit}(f)$ (resp. $\mathrm{Critv}(f)$) be the set of critical points (resp. critical values) of f. For a fixed base point $q_0 \in \partial\mathbb{D}^2$, a *vanishing path* $\gamma : [0, 1] \to \mathbb{D}^2$ for $q \in \mathrm{Critv}(f)$ is an embedded path such that

$$\gamma(0) = q_0,\ \gamma(1) = q, \ \text{ and } \ \gamma^{-1}(\mathrm{Int}D^2 \setminus \mathrm{Critv}(f)) = (0, 1).$$

Since f is a symplectic fiber bundle over $\mathbb{D}^2 \setminus \mathrm{Critv}(f)$, the tangent space at any point $p \in f^{-1}(\mathbb{D}^2 \setminus \mathrm{Critv}(f))$ is equipped with the canonical splitting

$$T_p W \cong T_p^v W \oplus T_p^h W.$$

This leads to a connection of the symplectic fiber bundle. Let $h_{\gamma|_{[t_0,t_1]}}\colon F_{\gamma(t_0)} \to F_{\gamma(t_1)}$ be the parallel transport along the restriction $\gamma|_{[t_0,t_1]}$ of γ for $0 \le t_0 < t_1 < 1$. To the path γ we can associate the unique Lagrangian disk Δ_γ, called the *Lefschetz thimble*, such that $f(\Delta_\gamma) = \gamma([0, 1])$, $f(\partial\Delta_\gamma) = \gamma(0)$. The boundary $V_\gamma := \partial\Delta_\gamma$ is a Lagrangian sphere in $(F_{q_0}, d\lambda|_{F_{q_0}})$. This is called the *vanishing cycle* associated to γ. Using the metric around the critical point and the parallel transport again, we may equip the vanishing cycle with a framing, so after this we consider the vanishing cycle as a framed Lagrangian sphere. Suppose that p is the critical point of f with $f(p) = q$. Then in terms of vanishing cycles the Lefschetz thimble can be expressed as

$$\Delta_\gamma = \Big(\bigcup_{0 \le t_0 < 1} V_{\gamma|_{[t_0,1]}}\Big) \cup \{p\}.$$

Next, take a small loop around $q \in \mathrm{Critv}(f)$ and orient it counterclockwise. Connect it to the base point q_0 using the vanishing path γ and let us denote the resulting oriented loop by l. The fibration f restricts to a symplectic S^1-bundle over l. According to the *symplectic Picard-Lefschetz theory* (cf. [30, Section 1]), its monodromy is symplectically isotopic to the Dehn twist along the vanishing cycle V_γ.

Definition 2.4. Let $f : (W, d\lambda) \to \mathbb{D}^2$ be an exact Lefschetz fibration. A *matching path* for f is an embedded path $\gamma : [-1, 1] \to \mathbb{D}^2$ such that:

(i) $\gamma^{-1}(\mathrm{Critv}(f)) = \{\pm 1\}$ and $\gamma(-1) \neq \gamma(1)$;

(ii) the vanishing cycles $V_{\gamma_\pm} \subset f^{-1}(\gamma(0))$ for the vanishing paths $\gamma_\pm : [0, 1] \to \mathbb{D}^2$ given by $\gamma_\pm(t) := \gamma(\pm t)$ coincide as framed Lagrangian spheres of $f^{-1}(\gamma(0))$.

For a matching path $\gamma : [-1, 1] \to \mathbb{D}^2$ for f, let $\Delta_{\gamma_+}, \Delta_{\gamma_-}$ be the Lefschetz thimbles for γ_+, γ_-, respectively. Since the framings of the vanishing cycles $V_{\gamma_+}, V_{\gamma_-}$ are the same, $\Delta_{\gamma_+} \cup \Delta_{\gamma_-}$ is a Lagrangian sphere in $(W, d\lambda)$ (see Figure 1). This is called the *matching cycle* associated to γ. One can equip the matching cycle $\Delta_{\gamma_+} \cup \Delta_{\gamma_-}$ with a framing by combining the two framings of Δ_{γ_+} and Δ_{γ_-} (see [31, Section (16g)]).

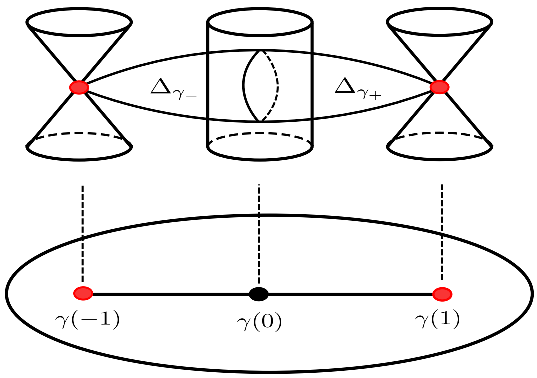

FIGURE 1. Matching cycle $\Delta_{\gamma_+} \cup \Delta_{\gamma_-}$.

2.3. **Exact symplectic Lefschetz fibrations on A_m-Milnor fibers.** We recall the definition of a Stein domain. A *strictly plurisubharmonic function* on a complex manifold is a smooth function whose complex Hessian matrix is positive definite at any point. In this article, we will deal only with the "strict case", and hence we will omit the word "strictly".

Definition 2.5. A *Stein domain* is a compact complex manifold (W, J) with boundary which admits a proper and bounded below plurisubharmonic function $\phi : W \to \mathbb{R}$ with maximal level set ∂W.

Let (W, J) be a Stein domain with a plurisubharmonic function ϕ, and let $\lambda_\phi := -d^{\mathbb{C}}\phi = -d\phi \circ J$. Since the 2-form $d\lambda_\phi = -dd^{\mathbb{C}}\phi$ is an exact symplectic form on W compatible with J, one can obtain from (W, J) with ϕ the exact symplectic manifold $(W, d\lambda_\phi)$.

For the complex polynomial $p(z_1, \dots, z_{n+1}) = z_1^2 + \cdots + z_n^2 + z_{n+1}^{m+1}$ and a sufficiently small fixed $\epsilon > 0$, define

$$\widehat{V}_m := \{z \in \mathbb{C}^{n+1} \,|\, p(z) = \epsilon\} \text{ and } V_{m,\delta} := \widehat{V}_m \cap D^{2n+2}(\delta),$$

where $D^{2n+2}(\delta) := \{z \in \mathbb{C}^{n+1} \,|\, |z_1|^2 + \cdots + |z_{n+1}|^2 \leq \delta^2\}$ and $\delta > 1$. $V_{m,\delta}$ is the *Milnor fiber* of p, which is also called the *A_m-Milnor fiber*. Let $\phi : \widehat{V}_m \to \mathbb{R}$ be the function defined by $\phi(z) = (|z_1|^2 + \cdots + |z_{n+1}|^2)/4$. One can show that ϕ is plurisubharmonic on $V_{m,\delta}$ with $\partial V_{m,\delta} = \{\phi(z) = \delta^2/4\}$, so $V_{m,\delta}$ is a Stein domain. This implies that $(V_{m,\delta}, d\lambda_\phi|_{V_{m,\delta}})$ is an exact symplectic manifold. Here $d\lambda_\phi$ equals the restriction of the standard Kähler form $i(\Sigma_{j=1}^{n+1} dz_j \wedge d\bar{z}_j)/2$ on $\mathbb{C}^{n+1}$.

Now we construct an explicit exact Lefschetz fibration by cutting off $V_m := V_{m,2}$ in a similar way to [31, Example 15.4]. Let $k : \widehat{V}_m \to \mathbb{R}$ be the function $k(z_1, \dots, z_n, z_{n+1}) := ((|z_1|^2 + \cdots + |z_n|^2)^2 - |z_1^2 + \cdots + z_n^2|^2)/4$. For some $s > 0$, define

$$\overline{V}_{m,s} := \{z \in \widehat{V}_m \mid |z_{n+1}| \leq 1, k(z) \leq s\}.$$

One can choose a number s so that $\overline{V}_{m,s} \subset V_m$, and hence we assume this condition. We claim that the projection $f : \overline{V}_{m,s} \to \mathbb{D}^2$, $z \mapsto z_{n+1}$, provides an exact Lefschetz fibration. First, it is not difficult to see that f is a smooth Lefschetz fibration with the Kählerness condition whose critical values are the $(m+1)^{\mathrm{st}}$ roots of ϵ. Next, the restriction $\phi|_{f^{-1}(q)}$ is a plurisubharmonic function on the fiber $f^{-1}(q) = \{z_1^2 + \cdots + z_n^2 = \epsilon - q^{m+1}, k(z) \leq s\}$, and the associated exact symplectic form equals the restriction of $d\lambda_\phi$ to $f^{-1}(q)$. Thus the fiber is an exact symplectic submanifold of $(\overline{V}_{m,s}, d\lambda_\phi)$. In fact, it is symplectomorphic to the disk cotangent bundle $D_{\sqrt{s}}T^*S^{n-1}$ with the canonical symplectic form. Finally we show that $\partial_h \overline{V}_{m,s}$ satisfies condition (iv) of Definition 2.2. Observe that

$$\partial_h \overline{V}_{m,s} = \bigcup_{q \in \mathbb{D}^2} (\{z_1^2 + \cdots + z_n^2 = \epsilon - q^{m+1}, k(z) = s\}).$$

The symplectic complement $(\operatorname{Ker} df_p)^{\perp d\lambda_\phi}$ at any point $p \in \partial_h \overline{V}_{m,s}$ is spanned over $\mathbb{C}$ by

$$X := (m+1) z_{n+1}^m \sum_{j=1}^n \bar{z}_j \frac{\partial}{\partial z_j}\Big|_p - 2\Big(\sum_{j=1}^n |z_j|^2\Big) \frac{\partial}{\partial z_{n+1}}\Big|_p,$$

and $dk_p(X) = 0$, $dk_p(iX) = 0$. This implies that $(\operatorname{Ker} df_p)^{\perp d\lambda_\phi} \subset T_p \partial_h \overline{V}_{m,s}$. Thus f is an exact Lefschetz fibration.

Under the symplectic identification of $F_q := \{z_1^2 + \cdots + z_n^2 = \epsilon - q^{m+1}\}$ with T^*S^{n-1}, its zero-section corresponds to the sphere $\sqrt{\epsilon - q^{m+1}} S^{n-1} \times \{q\} \subset F_q$. Here

$$\sqrt{w} S^{n-1} = \{z \in \mathbb{C}^n \mid z = \pm\sqrt{w} x \text{ for some } x \in S^{n-1} \subset \mathbb{R}^n\} \quad \text{for } w \in \mathbb{C}.$$

We can check that this sphere is contained in the fiber $f^{-1}(q)$. Moreover, according to [18, Section 6c], the Lefschetz thimble Δ_γ for a vanishing path γ can be written explicitly as

$$\bigcup_{t \in [0,1]} (\sqrt{\epsilon - \gamma(t)^{m+1}} S^{n-1} \times \{\gamma(t)\}).$$

We see that every embedded smooth path $\gamma : [-1, 1] \to \mathbb{D}^2$ with $\gamma^{-1}(\operatorname{Critv}(f)) = \{\pm 1\}$ is a matching path for γ. Let $\delta_j : [-1, 1] \to \mathbb{D}^2$ be the path given by $\delta_j(t) := \sqrt[m+1]{\epsilon} e^{\pi i (2j + t - 1)/(m+1)}$ for $j = 1, \dots, m$ (see Figure 2). This is a matching path for f, and its matching cycle is denoted by L_j. Since L_j and L_{j+1} intersect transversely at the single critical point of f, it follows from [31, Lemma 16.13] (cf. [24, Lemma 7.1]) that two framed Lagrangian spheres $\tau_{L_{j+1}}^{-1}(L_j)$ and $\tau_{L_j}(L_{j+1})$ are framed isotopic. Hence Dehn twists along these framed Lagrangian spheres are

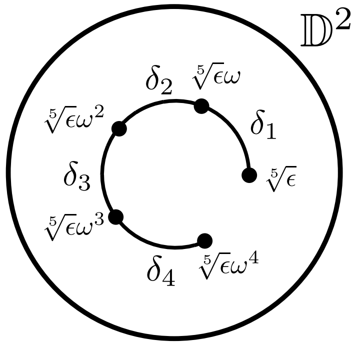

FIGURE 2. Paths $\delta_1, \delta_2, \delta_3, \delta_4$ for $m = 4$, where $\omega := e^{2\pi i/5}$.

symplectically isotopic. In particular, after embedding each $L_j \subset \overline{V}_{m,s}$ into V_m as a framed Lagrangian sphere, we have

$$[\tau_{L_j}\tau_{L_{j+1}}\tau_{L_j}] = [\tau_{L_{j+1}}\tau_{L_j}\tau_{L_{j+1}}] \in \pi_0(\mathrm{Symp}(V_m, d\lambda_\phi)).$$

This leads to the well-defined *anti*-homomorphism

$$\rho : B_{m+1} \to \pi_0(\mathrm{Symp}(V_m, d\lambda_\phi)),\ \ \rho(\sigma_j) = [\tau_{L_j}],$$

where σ_i is one of the Artin generators of B_{m+1}. Here, we use the opposite notation to the usual functional one, as mentioned in Notation 2.1, for the products in B_{m+1}, which is why ρ is not a homomorphism but an anti-homomorphism. This ρ is known as the Birman-Hilden correspondence [6] in the case $\dim V_m = 2$ and it is injective.

2.4. **Contact open books and abstract Weinstein Lefschetz fibrations.** Let $(W, d\lambda)$ be an exact symplectic manifold. A *Liouville domain* is a compact exact symplectic manifold $(W, d\lambda)$ with boundary such that the Liouville vector field X_λ defined by $\iota_{X_\lambda} d\lambda = \lambda$ is transverse to ∂W pointing outwards.

Definition 2.6. A *Weinstein domain* is a Liouville domain $(W, d\lambda)$ which admits a Morse function $\phi : W \to \mathbb{R}$ with maximal level set ∂W and whose Liouville vector field X_λ is gradient-like for ϕ.

We will rarely discuss a Liouville vector field and a Morse function associated to a Weinstein domain, so we will omit them from the notation.

Definition 2.7. An *abstract contact open book* is a tuple $(\Sigma, d\lambda; \varphi)$ consisting of a Weinstein domain $(\Sigma, d\lambda)$ and a symplectomorphism φ of $(\Sigma, d\lambda)$ equal to the identity near $\partial\Sigma$.

In the above definition, $(\Sigma, d\lambda)$ is called the *page*, and φ is called the *monodromy* of the abstract contact open book $(\Sigma, d\lambda; \varphi)$.

Now we briefly explain how to obtain a contact structure adapted to a given abstract contact open book (see [14, Chapter 7.3] for more details). Let $(\Sigma, d\lambda)$ be a $2n$-dimensional Weinstein domain and let φ be a symplectomorphism of $(\Sigma, d\lambda)$ equal to the identity near $\partial\Sigma$. Giroux showed that φ is isotopic, through symplectomorphisms equal to the identity near $\partial\Sigma$, to an exact symplectomorphism φ' of $(\Sigma, d\lambda)$, i.e., a symplectomorphism such that $(\varphi')^*\lambda - \lambda$ is exact (cf. [14, Lemma 7.34]). If φ is such an exact symplectomorphism of $(\Sigma, d\lambda)$, there exists a unique smooth function $\bar{\theta} : \Sigma \to \mathbb{R}_+$, up to adding a constant, such that $\varphi^*\lambda - \lambda = d\bar{\theta}$. Note that $\bar{\theta}$ is constant near $\partial\Sigma$ because $\varphi^*\lambda$ is λ near $\partial\Sigma$. Set

$$\Sigma(\varphi) := \{(x, \theta) \in \Sigma \times \mathbb{R} \,|\, 0 \leq \theta \leq \bar{\theta}(x)\}/(x, \bar{\theta}(x)) \sim (\varphi(x), 0).$$

Although by definition it depends on the choice of $\bar{\theta}$, here we suppress $\bar{\theta}$ from the notation, and we will do the same with the following notions. The 1-form $\lambda + d\theta$ is a contact form on $\Sigma(\varphi)$. Let c be the value of $\bar{\theta}$ near $\partial\Sigma$. Define the closed $(2n+1)$-manifold

$$M(\Sigma, d\lambda; \varphi) := (\Sigma(\varphi) \sqcup \partial\Sigma \times \mathbb{D}^2)/\sim,$$

where $(x, e^{i\theta}) \in \partial(\partial\Sigma \times \mathbb{D}^2)$ is identified with $[x, c\theta/2\pi] \in \Sigma(\varphi)$. To construct a contact form on $\partial\Sigma \times \mathbb{D}^2$, let $h_1, h_2 : [0, 1] \to \mathbb{R}$ be functions, shown in Figure 3, such that

- $h_1(r) = 2$ and $h_2(r) = r^2$ near $r = 0$,
- $h_1(r) = e^{1-r}$ and $h_2(r) = 1$ for $r \in [1/2, 1]$, and
- $h_1(r)h_2'(r) - h_2(r)h_1'(r) \neq 0$ for $r \neq 0$.

Then one can define a contact form on $\partial\Sigma \times \mathbb{D}^2$ by

$$h_1(r)\lambda|_{\partial\Sigma} + h_2(r)d\theta,$$

where (r, θ) are polar coordinates on $\mathbb{D}^2$, and it extends to a contact form on $M(\Sigma, d\lambda; \varphi)$. Let us denote by $\xi_{(\Sigma, d\lambda; \varphi)}$ the corresponding contact structure. This is called *supported* by $(\Sigma, d\lambda; \varphi)$. One can prove that if two abstract contact open books have the same pages and symplectically isotopic monodromies, then the supported contact structures are isotopic.

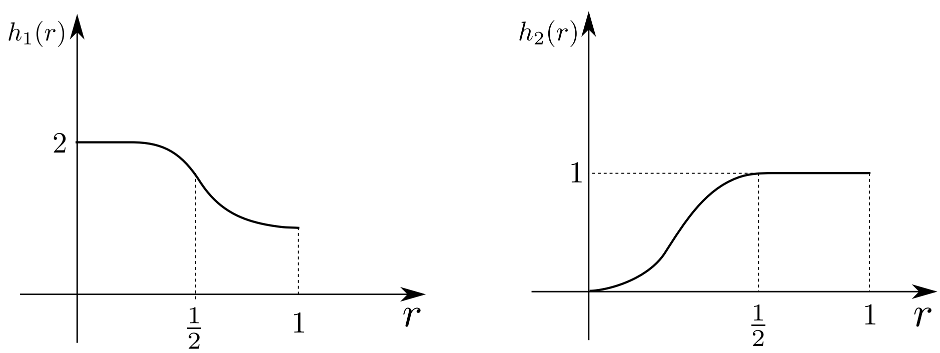

FIGURE 3. Graphs of functions h_1 and h_2.

As we associated the contact manifold to an abstract contact open book, a Weinstein domain can be obtained from the data of a symplectic manifold and an ordered collection of framed Lagrangian spheres in it.

Definition 2.8. An *abstract Weinstein Lefschetz fibration* is a tuple $(\Sigma, d\lambda; L_1, \ldots, L_m)$ consisting of a Weinstein domain $(\Sigma, d\lambda)$ and an ordered collection of framed Lagrangian spheres $L_1, \ldots, L_m$ in $(\Sigma, d\lambda)$.

In the above definition, $(\Sigma, d\lambda)$ is called the *fiber*, and L_j are called *vanishing cycles* of the abstract Weinstein Lefschetz fibration $(\Sigma, d\lambda; L_1, \ldots, L_k)$.

Given an abstract Weinstein Lefschetz fibration $(\Sigma, d\lambda; L_1, \ldots, L_m)$ of $\dim \Sigma = 2n$ ($n > 2$), we can construct a Weinstein domain. Based on [25, pp. 11-12], we briefly review the construction. Let $(\mathbb{D}^2, d\lambda_{std})$ be the standard Weinstein disk with

$$\lambda_{std} = \frac{1}{2}xdy - \frac{1}{2}ydx \quad \text{and} \quad \phi_{std}(x, y) = x^2 + y^2,$$

and let ϕ be the Morse function associated to the Weinstein domain $(\Sigma, d\lambda)$. First, take the completion $(\widehat{\Sigma}, d\widehat{\lambda})$ of $(\Sigma, d\lambda)$ with $\widehat{\phi}$ and deform $\widehat{\phi}$ into another $\tilde{\phi}$ so that $\tilde{\phi}$ is C^∞-small on Σ and $\partial\tilde{\phi}/\partial t > 0$ on $[0, \infty) \times \partial\Sigma$. Here the *completion* of a Weinstein domain $(W, d\lambda)$ with ϕ is the symplectic manifold $(\widehat{W}, d\widehat{\lambda}) := (W, d\lambda) \cup ([0, \infty) \times \partial W, d(e^t\lambda|_{\partial W}))$ with extended ϕ as $\widehat{\phi}(t, x) = e^t$ on $[0, \infty) \times \partial W$. Next, enlarge the product $(\Sigma, d\lambda) \times (\mathbb{D}^2, d\lambda_{std})$ to be $(\widehat{\Sigma} \times \mathbb{C}, d(\widehat{\lambda} + \widehat{\lambda}_{std}))$ and define a new Weinstein domain, say $(\Sigma, d\lambda) \boxtimes (\mathbb{D}^2, d\lambda_{std})$, as the sublevel set $\{\tilde{\phi} + x^2 + y^2 \leq 1\}$. We may consider the boundary of $(\Sigma, d\lambda) \boxtimes (\mathbb{D}^2, d\lambda_{std})$ as $M(\Sigma, d\lambda; id)$ and assume the contact structure on $\partial((\Sigma, d\lambda) \boxtimes (\mathbb{D}^2, d\lambda_{std}))$ to be supported by the open book $(\Sigma, d\lambda; id)$. Put each Lagrangian sphere L_j on $\Sigma \times \{e^{\frac{2\pi}{m}j}\} \subset M(\Sigma, d\lambda; id)$, where the mapping torus $\Sigma(id) \subset M(\Sigma, d\lambda; id)$ is identified with $\Sigma \times S^1$. By [33, Lemma 4.2], we may assume that L_i is a Legendrian sphere Λ_i. Attach Weinstein $(n+1)$-handles to the Weinstein domain $(\Sigma, d\lambda) \boxtimes (\mathbb{D}^2, d\lambda_{std})$ along $\Lambda_1, \ldots, \Lambda_m$ and obtain the new Weinstein domain $W(\Sigma, d\lambda; L_1, \ldots, L_m)$. We have to remark that similar discussions can be found in [8, Section 8.1] and [15, Definition 6.3]. The argument in the former is based on the Liouville setting instead of Weinstein. In the latter, Giroux and Pardon obtained a Weinstein domain corresponding to $(\Sigma, d\lambda) \boxtimes (\mathbb{D}^2, d\lambda_{std})$ in a slightly different way. To a given Weinstein domain $(\Sigma, d\lambda)$ with Morse function ϕ satisfying $\partial\Sigma = \{\phi = 0\}$, they associated the Weinstein domain $\{\phi + x^2 + y^2 \leq 0\} \subset \Sigma \times \mathbb{C}$ as the desired one. Hence, they cut the product manifold $\Sigma \times \mathbb{C}$ to get the Weinstein domain instead of enlarging $\Sigma \times \mathbb{D}$ in the above argument.

Definition 2.9. A *Stein filling* of a contact manifold (M, ξ) is a Stein domain whose boundary is contactomorphic to (M, ξ). Then ξ is called *Stein fillable*.

On the boundary $\partial((\Sigma, d\lambda) \boxtimes (\mathbb{D}^2, d\lambda_{std}))$, the above handle attachments yield Legendrian surgeries on $\Lambda_1, \ldots, \Lambda_m$, and by [33, Theorem 4.4] the resulting contact manifold is contactomorphic to

$$(M(\Sigma, d\lambda; \tau_{L_m} \circ \cdots \circ \tau_{L_1}), \xi_{(\Sigma, d\lambda; \tau_{L_m} \circ \cdots \circ \tau_{L_1})}).$$

Thanks to a result of Eliashberg [11, Theorem 1.3.2] (cf. [9, Theorem 13.5]), there is a complex structure and a plurisubharmonic function on $W(\Sigma, d\lambda; L_1, \ldots, L_m)$ such that these make it Stein, and as a symplectic manifold the resulting Stein domain is symplectomorphic to the initial Weinstein domain. From this, the contact structure on the boundary of this Stein domain is isomorphic to $\xi_{(\Sigma, d\lambda; \tau_{L_m} \circ \cdots \circ \tau_{L_1})}$, and hence the Stein domain serves as a Stein filling of the contact manifold

$$(M(\Sigma, d\lambda; \tau_{L_m} \circ \cdots \circ \tau_{L_1}), \xi_{(\Sigma, d\lambda; \tau_{L_m} \circ \cdots \circ \tau_{L_1})}).$$

2.5. **Homology groups of manifolds with Lefschetz fibrations or open books.** In this subsection, we often regard a handlebody as a CW complex and consider its homology groups.

Let $(\Sigma, d\lambda; L_1, \ldots, L_m)$ be an abstract Weinstein Lefschetz fibration, where $\dim \Sigma = 2n$, and $W(\Sigma, d\lambda; L_1, \ldots, L_m)$ is the corresponding Weinstein domain. Its homology groups are easy to read off from the collection of vanishing cycles. We omit $d\lambda$ from the collection of the notation because we focus only on the algebraic topology of the Weinstein domain. Since $(\Sigma, d\lambda)$ is Weinstein, we may take a handle decomposition of Σ without handles of index $> n$, and it yields the following handle decomposition of $\Sigma \times D^2$:

$$h^{(0)} \cup (\bigcup_j h_j^{(1)}) \cup \cdots \cup (\bigcup_j h_j^{(n)}),$$

where each $h_j^{(k)}$ is a k-handle. As mentioned before, the Weinstein domain $W(\Sigma; L_1, \ldots, L_m)$ is decomposed into $\Sigma \times D^2$ and m $(n+1)$-handles, which yields the following handle decomposition of $W(\Sigma; L_1, \ldots, L_m)$:

$$h^{(0)} \cup (\bigcup_j h_j^{(1)}) \cup \cdots \cup (\bigcup_j h_j^{(n)}) \cup (\bigcup_j h_j^{(n+1)}),$$

where each $h_j^{(n+1)}$ is the (Weinstein) $(n+1)$-handle attached along L_j. Now consider the chain complex $(C_*(W(\Sigma; L_1, \ldots, L_m)), \partial_*)$. Since $C_k(W(\Sigma; L_1, \ldots, L_m))$ is generated by the k-handles, we can easily see that $\operatorname{Ker} \partial_n$ is isomorphic to $H_n(\Sigma \times D^2; \mathbb{Z}) \cong H_n(\Sigma; \mathbb{Z})$ and write $g_1, \ldots, g_k$ for its generators. Also, $\operatorname{Im} \partial_{n+1}$ is generated by the attaching spheres L_j of the $(n+1)$-handles, which may be assumed to lie on $\Sigma \times \{pt\}$ homologically. Thus we have

$$H_n(W(\Sigma; L_1, \ldots, L_m); \mathbb{Z}) \cong \langle\, g_1, \ldots, g_k \mid [L_1], \ldots, [L_m] \,\rangle. \tag{2}$$

This is equivalent to

$$H_n(W(\Sigma; L_1, \ldots, L_m); \mathbb{Z}) \cong H_n(\Sigma; \mathbb{Z})/\langle [L_1], \ldots, [L_m] \rangle,$$

which means that the n^{th} homology group of the Lefschetz fibration is obtained as the quotient of the n^{th} homology group of the fiber Σ by the subgroup generated by the homology classes of the vanishing cycles.

Next consider a manifold endowed with an abstract contact open book and describe its homology groups in terms of the page and monodromy of the open book. For our purpose it suffices to compute the $(2n-1)^{\text{st}}$ homology group of a $(4n-1)$-manifold with an open book whose page is symplectomorphic to the Milnor fiber V_m. Hence although in general the dimension of V_m is even, after this it is assumed to be $4n-2$. Let $(V_m, d\lambda; \varphi)$ be an abstract contact open book whose page is the Milnor fiber $(V_m, d\lambda)$ of $\dim = 4n-2$ $(n \geq 2)$. As we did before, we suppress $d\lambda$ from the notation of the abstract contact open book.

To see the homology group, we examine the algebraic topology of the boundary of V_m, which is diffeomorphic to the Brieskorn $(4n-3)$-sphere

$$\Sigma(\underbrace{2, \ldots, 2}_{2n-1}, m+1) := \widehat{V}_m \cap \{|z_1|^2 + \cdots + |z_{2n}|^2 = 1\}.$$

Let $\mathbf{a} := (a_1, \ldots, a_n)$ with each $a_j \in \mathbb{Z}_{>0}$. Define the graph $G(\mathbf{a})$ for $\mathbf{a}$ whose vertices are $v_1, \ldots, v_n$ with labels $a_1, \ldots, a_n$, respectively, and whose edges lie between v_i and v_j if $i \neq j$ and $\gcd(a_i, a_j) > 1$ (e.g. Figure 4).

Proposition 2.10 (Brieskorn [7, Satz 1(ii)]). *For $n \geq 4$, the Brieskorn sphere $\Sigma(a_1, \ldots, a_n)$ is a homotopy sphere if the graph $G(\boldsymbol{a})$ associated to $\boldsymbol{a} = (a_1, \ldots, a_n)$ satisfies either of the following conditions:*

(1) *$G(\boldsymbol{a})$ has two isolated points;*
(2) *$G(\boldsymbol{a})$ has an isolated point and a connected component K consisting of an odd number of points such that if $v_i, v_j \in K$ with $i \neq j$, $\gcd(a_i, a_j) = 2$.*

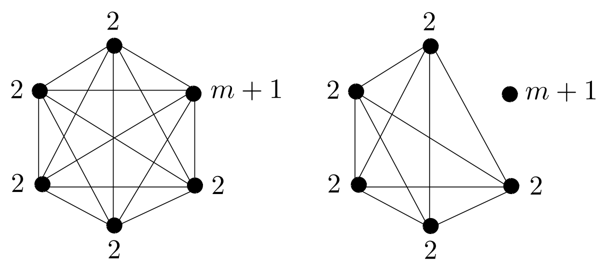

FIGURE 4. Graphs $G(\mathbf{a})$ for $\mathbf{a} = (2, 2, 2, 2, 2, m + 1)$, where m is odd (resp. even) on the left (resp. right).

Thus it follows from this proposition and Figure 4 that the Brieskorn $(4n - 3)$-sphere $\Sigma(2, \ldots, 2, m+1)$ is a homotopy sphere if m is even. Hereafter m is assumed to be even.

By definition, the manifold $M(V_m; \varphi)$ splits into the mapping torus $V_m(\varphi)$ and $\partial V_m \times \mathbb{D}^2$, and hence we first write $H_{2n+1}(V_m(\varphi); \mathbb{Z})$ and then $H_{2n+1}(M(V_m; \varphi); \mathbb{Z})$ by gluing the two parts.

Before computing the homology, we claim that V_m has a handle decomposition with one 0-handle and m $(2n - 1)$-handles. In particular, such a handle decomposition can be arranged so that the cores of these $(2n - 1)$-handles generate $H_{2n-1}(V_m; \mathbb{Z})$, and moreover each of them is represented by the matching cycle L_i in Section 2.3. To see this, consider the handle decomposition of V_m associated to the Lefschetz fibration $f : \overline{V}_{m,s} \to \mathbb{D}^2$. Note that after rounding off the corners, $\overline{V}_{m,s}$ is diffeomorphic to V_m. Fix the segments from the origin to the $(m+1)^{\mathrm{st}}$ roots of ϵ as vanishing paths for $\mathrm{Critv}(f)$. Since the regular fiber of f is diffeomorphic to the disk cotangent bundle DT^*S^{2n-2} of some radius, its handle decomposition consists of one 0-handle and one $(2n - 2)$-handle. Combined with the fact that V_m splits into $DT^*S^{2n-2} \times D^2$ and $m + 1$ $(2n - 1)$-handles, this induces the handle decomposition of V_m,

$$h^{(0)} \cup (h^{(2n-2)}) \cup (\bigcup_{j=0}^{m} h_j^{(2n-1)}),$$

where the j^{th} $(2n - 1)$-handle $h_j^{(2n-1)}$ corresponds to the critical value $\sqrt[m+1]{\epsilon} e^{2\pi i j/(m+1)}$ of f. Slide $h_j^{(2n-1)}$ over $h_{j-1}^{(2n-1)}$ in descending order for $j = 1, \ldots, m$ and write $\tilde{h}_j^{(2n-1)}$ for the resulting $(2n - 1)$-handle. The attaching sphere of $h_0^{(2n-1)}$ is the vanishing cycle with respect to the fixed vanishing path for $\sqrt[m+1]{\epsilon} \in \mathrm{Critv}(f)$, which is the image of the zero-section S_0^{2n-2} of DT^*S^{2n-2}. Here this disk cotangent bundle is considered as a fiber of the trivial fibration

$$DT^*S^{2n-2} \times \partial D^2 \to \partial D^2.$$

Let $B \subset DT^*S^{2n-2}$ be the cocore of the $(2n-2)$-handle of the handle decomposition of DT^*S^{2n-2} we took before. Obviously, B is a fiber of the disk bundle DT^*S^{2n-2}. Hence, the belt sphere of the $(2n-2)$-handle $h^{(2n-2)}$ is

$$\partial(B \times D^2) = B \times \partial D^2 \cup \partial B \times D^2 \subset \partial(DT^*S^{2n-2} \times D^2),$$

which intersects the attaching sphere S_0^{2n-1} of $h_0^{(2n-1)}$ transversely in a single point. Thus, we can cancel the pair of $h_0^{(2n-1)}$ and $h^{(2n-2)}$. Finally, we obtain the new handle decomposition of V_m,

$$h^{(0)} \cup \tilde{h}_1^{(2n-1)} \cup \cdots \cup \tilde{h}_m^{(2n-1)},$$

which is desired. This implies that the $\tilde{h}_j^{(2n-1)}$ generate $H_{2n-1}(V_m;\mathbb{Z})$, and they are represented by L_j because the cores of $h_j^{(2n-1)}$ are the Lefschetz thimbles for the fixed vanishing paths and the previous handle slides make the core of $\tilde{h}_j^{(2n-1)}$ the sum of two Lefschetz thimbles homologically.

For the mapping torus $V_m(\varphi)$, the following long exact sequence holds (see [17, Example 2.48]):

$$\begin{aligned}\cdots \to H_{2n-1}(V_m;\mathbb{Z}) &\xrightarrow{\varphi_*-id_*} H_{2n-1}(V_m;\mathbb{Z}) \xrightarrow{i_*} H_{2n-1}(V_m(\varphi);\mathbb{Z})\\ &\to H_{2n-2}(V_m;\mathbb{Z}) \to \cdots,\end{aligned}$$

where i is the inclusion map $V_m \hookrightarrow V_m \times \{0\} \subset V_m(\varphi)$ and φ_*, id_*, i_* are automorphisms of homology groups induced from φ, id, i, respectively. V_m admits the above handle decomposition without $(2n-2)$-handles, and hence $H_{2n-2}(V_m;\mathbb{Z}) = 0$. Therefore, by the above exact sequence we have

$$H_{2n-1}(V_m(\varphi);\mathbb{Z}) \cong \langle\, [L_1], \ldots, [L_m] \mid \varphi_*([L_1]) - ([L_1]), \ldots, \varphi_*([L_m]) - [L_m] \,\rangle.$$

To get a description of $H_{2n-1}(M(V_m;\varphi);\mathbb{Z})$, by checking the following Mayer-Vietoris long exact sequence:

$$\begin{aligned}\cdots \to H_{2n-1}(\partial V_m \times \partial\mathbb{D}^2;\mathbb{Z}) &\to H_{2n-1}(V_m(\varphi);\mathbb{Z}) \oplus H_{2n-1}(\partial V_m \times \mathbb{D}^2;\mathbb{Z})\\ &\to H_{2n-1}(M(V_m;\varphi);\mathbb{Z}) \to H_{2n-2}(\partial V_m \times \partial\mathbb{D}^2;\mathbb{Z}) \to \cdots,\end{aligned}$$

we conclude that $H_{2n-1}(M(V_m;\varphi);\mathbb{Z})$ is isomorphic to $H_{2n-1}(V_m(\varphi);\mathbb{Z})$ because ∂V_m is a homotopy $(4n-2)$-sphere $(n \geq 2)$. Thus,

(3)
$$H_{2n-1}(M(V_m;\varphi);\mathbb{Z}) \cong \langle\, [L_1], \ldots, [L_m] \mid \varphi_*([L_1]) - ([L_1]), \ldots, \varphi_*([L_m]) - [L_m] \,\rangle.$$

3. Construction

3.1. **Picard-Lefschetz formula.** The *Picard-Lefschetz formula* helps us compute homology groups (2) and (3). This formula was initially proven to study an action of the monodromy around a Lefschetz type singularity of a holomorphic function on the homology group of its regular fiber. Given an exact symplectic manifold $(W, d\lambda)$ and a framed Lagrangian sphere $L \subset (W, d\lambda)$, we obtain an exact Lefschetz fibration whose fiber is symplectomorphic to $(W, d\lambda)$ and vanishing cycle is L (see [31, Lemma 16.8]). Thus we can state the Picard-Lefschetz formula apart from holomorphic maps.

Theorem 3.1 (Picard-Lefschetz formula [20, 28] (cf. [19, (6.3.3)]))**.** *Let L be a framed Lagrangian n-sphere in a compact exact symplectic $2n$-manifold $(W, d\lambda)$*

with boundary. Then we have for the induced automomorphism $(\tau_L)_*: H_j(W;\mathbb{Z}) \to H_j(W;\mathbb{Z})$,

$$(\tau_L)_*(c) = \begin{cases} c + (-1)^{\frac{(n+1)(n+2)}{2}} \langle c, [L]\rangle [L] & (c \in H_n(W;\mathbb{Z})), \\ c & (c \in H_j(W;\mathbb{Z}), j \neq n). \end{cases}$$

Here, $\langle\ ,\ \rangle : H_n(W;\mathbb{Z}) \times H_n(W;\mathbb{Z}) \to \mathbb{Z}$ *denotes the intersection product.*

Although we need to fix an orientation of L temporarily to determine the homology class, the above formula still holds even if we change this orientation.

In Theorem 1.1, we will deal only with the case $\dim W = 4n-2 = 2(2n-1)$. For a Lagrangian $(2n-1)$-sphere $L \subset W$, $\chi(L) = 0$ and $\langle [L], [L]\rangle = 0$. Hence we have

$$(\tau_L)_*^m(c) = c + m(-1)^{\frac{2n(2n+1)}{2}} \langle c, [L]\rangle [L]$$

for any $c \in H_{2n-1}(W;\mathbb{Z})$ and $m \in \mathbb{Z}$.

3.2. **Baykur-Van Horn-Morris' 4-braids.** A *quasipositive factorization* of a braid $\beta \in B_m$ is an ordered tuple $(\beta_1, \dots, \beta_k)$ such that $\beta = \beta_1 \cdots \beta_k$ and each β_j is conjugate to one of the Artin generators of B_m. Two quasipositive factorizations are *equivalent* if they are related by a finite sequence of Hurwitz moves, their inverses, and global conjugations:

$$(\beta_1, \dots, \beta_{j-1}, \beta_j, \beta_{j+1}, \beta_{j+2}, \dots, \beta_k) \sim (\beta_1, \dots, \beta_{j-1}, \beta_j\beta_{j+1}\beta_j^{-1}, \beta_j, \beta_{j+2}, \dots, \beta_k),$$

$$(\beta_1, \dots, \beta_k) \sim (\gamma^{-1}\beta_1\gamma, \dots, \gamma^{-1}\beta_k\gamma) \quad \text{for any } \gamma \in B_m.$$

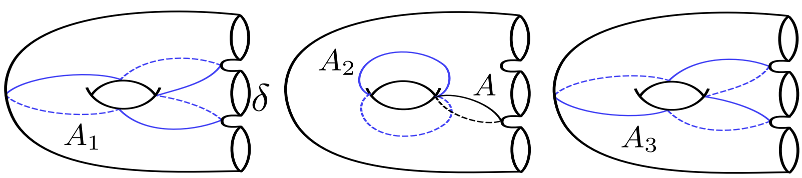

FIGURE 5

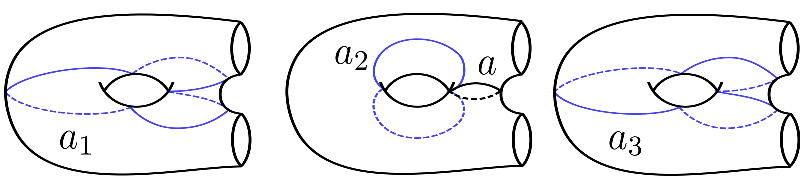

FIGURE 6

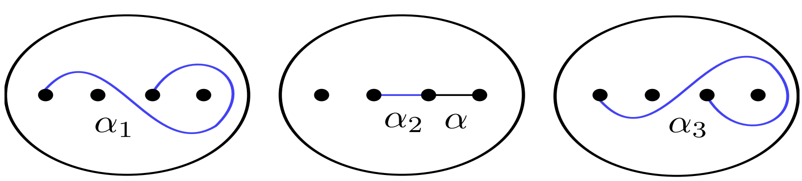

FIGURE 7

Baykur and Van Horn-Morris [5] recently constructed infinitely many 4-braids, each of which admits infinitely many inequivalent quasipositive factorizations. Their construction is based on the open book adapted to the standard contact 3-torus constructed by Van Horn-Morris [32] whose page is diffeomorphic to $\Sigma_{1,3}$ and whose monodromy is $\tau_{A_3} \circ \tau_{A_2} \circ \tau_{A_1}$. Here $\Sigma_{1,3}$ is an oriented compact surface of genus 1 with 3 boundary components, and A_1, A_2, A_3 are simple closed curves shown on $\Sigma_{1,3}$ depicted in Figure 5. Let A be a simple closed curve on $\Sigma_{1,3}$ as shown in Figure 5. Capping off the boundary δ of $\Sigma_{1,3}$, we obtain curves a_1, a_2, a_3, a depicted in Figure 6. Since these curves are preserved under the hyperelliptic involution of the capped-off surface, we obtain from a_1, a_2, a_3, a the arcs $\alpha_1, \alpha_2, \alpha_3, \alpha$ on the hyperelliptic quotient, that is, the disk (see Figure 7). We often identify the braid group B_m on m strands with the mapping class group of an m marked disk. Write β_i and β for the braids corresponding to the half twists along α_i and α, respectively. Define the braid $\beta_{k,l}$ by

$$\beta_{k,l} := (\beta^{-k}\beta_1\beta^k)(\beta^{-k}\beta_2\beta^k)(\beta^{-k}\beta_3\beta^k){\beta_2}^l \in B_4.$$

Yasui [34] pointed out that $[\tau_A]$ belongs to the centralizer of $[\tau_{A_3}] \circ [\tau_{A_2}] \circ [\tau_{A_1}]$. It follows from this result that $[\tau_a]$ belongs to the centralizer of $[\tau_{a_3}] \circ [\tau_{a_2}] \circ [\tau_{a_1}]$. Moreover, $[\tau_a]$ (resp. $[\tau_{a_j}]$) is the image of the braids β (resp. β_j) under the anti-homomorphism ρ between the braid group and the 2-dimensional mapping class group defined in Section 2.3. Hence, we have

$$\rho(\beta\beta_1\beta_2\beta_3) = [\tau_{a_3}] \circ [\tau_{a_2}] \circ [\tau_{a_1}] \circ [\tau_a] = [\tau_a] \circ [\tau_{a_3}] \circ [\tau_{a_2}] \circ [\tau_{a_1}] = \rho(\beta_1\beta_2\beta_3\beta),$$

which proves, coupled with the injectivity of ρ, that β belongs to the centralizer of $\beta_1\beta_2\beta_3$. Therefore,

$$\beta_{k,l} = \beta_{k',l} \quad \text{and} \quad \rho(\beta_{k,l}) = \rho(\beta_{k',l})$$

for any integers k, k'. For the next subsection, we describe $\beta_1, \beta_2, \beta_3, \beta$ in the Artin generators $\sigma_1, \sigma_2, \sigma_3$ of B_4:

$$\beta_1 = \sigma_3^{-2}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3^2, \quad \beta_2 = \sigma_2, \quad \beta_3 = \sigma_3^2\sigma_1\sigma_2\sigma_1^{-1}\sigma_3^{-2}, \quad \beta = \sigma_3.$$

3.3. **Proof of Theorem 1.1.** First recall that as defined in Section 2.3, L_1, L_2, L_3, L_4 are framed Lagrangian spheres realized as matching cycles of the exact Lefschetz fibration f on the cut-off Milnor fiber $\overline{V}_{4,s}$ of $\dim \overline{V}_{4,s} = 4n-2$ ($n \geq 2$). The image of the Artin generator σ_j under the anti-homomorphism $\rho : B_5 \to \pi_0(\mathrm{Symp}(V_4, d\lambda_\phi))$ is the symplectic isotopy class of the Dehn twist along L_j, that is, $\rho(\sigma_j) = [\tau_{L_j}]$. Define the Lagrangian spheres $B_{1,k}, B_{2,k}, B_{3,k}$ in $(V_4, d\lambda_\phi)$ by

$$B_{1,k} := \tau_{L_3}^{k+2} \circ \tau_{L_1}(L_2), \quad B_{2,k} := \tau_{L_3}^{k}(L_2), \quad B_{3,k} := \tau_{L_3}^{k-2} \circ \tau_{L_1}^{-1}(L_2).$$

Note that $\rho(\beta^{-k}\beta_j\beta^k) = [\tau_{B_{j,k}}]$ for $j = 1, 2, 3$. We set

$$\varphi_{k,l} := \tau_{L_4} \circ \tau_{L_2}^l \circ \tau_{B_{3,k}} \circ \tau_{B_{2,k}} \circ \tau_{B_{1,k}},$$

and we have

$$\begin{aligned}
\rho(\beta_{k,l}\sigma_4) &= \rho(\sigma_4) \circ \rho(\beta_2^l) \circ \rho(\beta^{-k}\beta_3\beta^k) \circ \rho(\beta^{-k}\beta_2\beta^k) \circ \rho(\beta^{-k}\beta_1\beta^k) \\
&= [\tau_{L_4}] \circ [\tau_{L_2}]^l \circ [\tau_{B_{3,k}}] \circ [\tau_{B_{2,k}}] \circ [\tau_{B_{1,k}}] \\
&= [\varphi_{k,l}].
\end{aligned}$$

Here $\beta_{k,l}$ is thought of as the element in B_5 through the canonical inclusion $B_4 \hookrightarrow B_5$. Consider the abstract Weinstein Lefschetz fibration

$$(V_4, d\lambda_\phi; B_{1,k}, B_{2,k}, B_{3,k}, \underbrace{L_2, \dots, L_2}_{l}, L_4)$$

for any integers $k \geq 0$ and $l > 0$, and let $W_{k,l}$ denote the Weinstein domain associated to this abstract Weinstein Lefschetz fibration. As mentioned in Section 2.4, $W_{k,l}$ may be assumed to be a Stein domain. Let $\xi_{k,l}$ be the contact structure supported by the abstract contact open book $(V_4, d\lambda_\phi; \varphi_{k,l})$. Note that the contact structure on $\partial W_{k,l}$ induced from the Stein structure on $W_{k,l}$ is isomorphic to $\xi_{k,l}$ on $M(V_4, d\lambda_\phi; \varphi_{k,l})$. Since $\beta_{k,l} = \beta_{k',l}$ and $[\varphi_{k,l}] = \rho(\beta_{k,l}\sigma_4) = \rho(\beta_{k',l}\sigma_4) = [\varphi_{k',l}]$, two abstract contact open books $(V_4, d\lambda_\phi; \varphi_{k,l})$ and $(V_4, d\lambda_\phi; \varphi_{k',l})$ support isotopic contact structures, and hence the contactomorphism class of $\xi_{k,l}$ is independent of k. Therefore, we may write (M_l, ξ_l) for $(\partial W_{k,l}, \xi_{k,l})$ and regard the Stein domain $W_{k,l}$ as a Stein filling of (M_l, ξ_l).

Next, we show that the contact manifold (M_l, ξ_l) admits infinitely many pairwise homotopy inequivalent Stein fillings. To see this, we prove that for $k, k' \geq 0$, $W_{k,l}$ and $W_{k',l}$ are homotopy equivalent if and only if $k = k'$ by computing the $(2n-1)^{\mathrm{st}}$ homology group of $W_{k,l}$. As we saw in Section 2.5, $[L_j]$ generate $H_{2n-1}(V_4; \mathbb{Z})$, and they also serve as generators of $H_{2n-1}(W_{k,l}; \mathbb{Z})$, where we choose the orientations of L_j so that for $i \leq j$,

$$\langle [L_i], [L_j] \rangle = \begin{cases} 1 & (j = i+1), \\ 0 & (\text{otherwise}). \end{cases}$$

According to the Picard-Lefschetz formula combined with the definitions of $B_{j,k}$, we have

$$\begin{aligned}
[B_{1,k}] &= (\tau_{L_3}^{k+2} \circ \tau_{L_1})_*([L_2]) = -(-1)^{\epsilon(n)}[L_1] + [L_2] + (k+2)(-1)^{\epsilon(n)}[L_3], \\
[B_{2,k}] &= (\tau_{L_3}^{k})_*([L_2]) = [L_2] + k(-1)^{\epsilon(n)}[L_3], \\
[B_{3,k}] &= (\tau_{L_3}^{k-2} \circ \tau_{L_1}^{-1})_*([L_2]) = (-1)^{\epsilon(n)}[L_1] + [L_2] + (k-2)(-1)^{\epsilon(n)}[L_3],
\end{aligned}$$

where $\epsilon(n) := 2n(2n+1)/2$ is the exponent appearing in the Picard-Lefschetz formula. From the equation (2),

$$\begin{aligned}
H_{2n-1}(W_{k,l}; \mathbb{Z}) &\cong \langle\ [L_1], [L_2], [L_3], [L_4] \mid [B_{1,k}], [B_{2,k}], [B_{3,k}], [L_2], \dots, [L_2], [L_4]\ \rangle \\
&\cong \langle\ [L_3] \mid k[L_3]\ \rangle \\
&\cong \begin{cases} \mathbb{Z} & (k = 0), \\ \mathbb{Z}_k & (k > 0). \end{cases}
\end{aligned}$$

The homology group depends on k, and $W_{k,l}$ and $W_{k',l}$ are mutually homotopy inequivalent if $k \neq k'$. Thus we obtain the conclusion.

Finally, we see that the infinite family $\{M_l\}_{l \in \mathbb{Z}_{>0}}$ contains infinitely many pairwise homotopy inequivalent $(4n-1)$-manifolds. Here M_l may be assumed to be equipped with the abstract contact open book $(V_4, d\lambda_\phi; \varphi_{0,l})$. Since the page V_4 of the open book is a homotopy sphere, we can apply the equation (3) to M_l. By the

Picard-Lefschetz formula again, we have

$$\begin{aligned}(\varphi_{0,l})_*([L_1]) &= [L_1] + (-1)^{\epsilon(n)} l[L_2],\\ (\varphi_{0,l})_*([L_2]) &= 3(-1)^{\epsilon(n)}[L_1] + (9l+1)[L_2] - 6(-1)^{\epsilon(n)}[L_3] - 6[L_4],\\ (\varphi_{0,l})_*([L_3]) &= -l(-1)^{\epsilon(n)}[L_2] + [L_3] + (-1)^{\epsilon(n)}[L_4],\\ (\varphi_{0,l})_*([L_4]) &= -2(-1)^{\epsilon(n)}[L_1] - 6l[L_2] + 4(-1)^{\epsilon(n)}[L_3] + 5[L_4].\end{aligned}$$

Hence

$$\begin{aligned}H_{2n-1}(M_l;\mathbb{Z}) &\cong \langle\ [L_1],[L_2],[L_3],[L_4] \mid (\varphi_{0,l})_*([L_j]) - [L_j]\ (j=1,\dots,4)\rangle\\ &\cong \langle\ [L_2],[L_3]\ | l[L_2]=0\ \rangle\\ &\cong \mathbb{Z}\oplus\mathbb{Z}_l.\end{aligned}$$

Therefore, if $l \neq l'$, M_l and $M_{l'}$ are mutually homotopy inequivalent. In particular, two contact manifolds (M_l, ξ_l) and $(M_{l'}, \xi_{l'})$ are mutually non-contactomorphic, which finishes the proof of Theorem 1.1.

3.4. **Remarks on Theorem 1.1.** To obtain distinct Stein fillings, we use inequivalent quasipositive braid factorizations constructed by Baykur and Van Horn-Morris. They took advantage of the element τ_a in the centralizer of $\tau_{a_3}\circ\tau_{a_2}\circ\tau_{a_1}$ and then conjugated the corresponding part of the factorization $\tau_{a_2}^l\circ\tau_{a_3}\circ\tau_{a_2}\circ\tau_{a_1}$ by τ_a. This twisting operation for the given factorization is called a *partial twist*, studied in [2], which corresponds to a Luttinger surgery along a Lagrangian torus in the total space of the Lefschetz fibration corresponding to the initial factorization. The curves a_1, a_2, a_3, a are symmetric with respect to the hyperelliptic involution of the surface (see Figure 6), and hence this procedure can descend to the braid group B_4. Moreover the anti-homomorphism ρ makes a partial twist valid for the symplectic mapping class group of the Milnor fiber. Similarly to the 4-dimensional case, in our case, we see that the parallel transport corresponding to $\tau_{B_{3,0}}\circ\tau_{B_{2,0}}\circ\tau_{B_{1,0}} \in \mathrm{Symp}(V_4,\lambda_\phi)$ preserves the Lagrangian sphere L_3, and this provides a Lagrangian $S^1\times S^{2n-1}$ in $W_{0,l}$. Thus our Stein filling $W_{k,l}$ is obtained from a surgery on $W_{0,l}$ along this Lagrangian $S^1\times S^{2n-1}$.

In our theorem, we assume that the dimensions of the contact manifolds are $4n-1$. The proof of this theorem is given by the algebraic argument, especially the Picard-Lefschetz formula. Let $(W, d\lambda)$ be an exact symplectic $4n$-manifold, and let L be a framed Lagrangian $2n$-sphere in $(W, d\lambda)$. Then the self-intersection number of L is non-zero, and according to the Picard-Lefschetz formula, $(\tau_L)_*^2$ acts trivially on $H_{2n}(W;\mathbb{Z})$. Hence, we have at most two distinct subgroups of $H_{2n}(V_4;\mathbb{Z})$ generated by the vanishing cycles. This is why we put the assumption of the dimensions.

However, we can construct the corresponding infinite families of contact $(4n+1)$-manifolds and their Stein fillings as we did. Let L_1, L_2, L_3, L_4 be framed Lagrangian spheres obtained as matching cycles of the exact Lefschetz fibration $f : \overline{V}_{4,s}\to\mathbb{D}^2$ of $\dim\overline{V}_{4,s} = 4n$ $(n\geq 1)$. Define the Lagrangian spheres $B_{1,k}, B_{2,k}, B_{3,k}$ in $(V_4, d\lambda_\phi)$ and the symplectomorphism $\varphi_{k,l}\in\mathrm{Symp}(V_4, d\lambda_\phi)$ as in the proof of Theorem 1.1. Let (M_l,ξ_l) be the $(4n+1)$-dimensional contact manifold obtained from the abstract contact open book $(V_4, d\lambda_\phi;\varphi_{0,l})$, and let $W_{k,l}$ be its Stein filling obtained from the abstract Weinstein Lefschetz fibration $(V_4, d\lambda_\phi; B_{1,k}, B_{2,k}, B_{3,k}, \underbrace{L_2,\dots,L_2}_{l}, L_4)$.

We would like to conclude this article with the following question:

Question 3.2. Let (M_l, ξ_l) be the $(4n+1)$-dimensional contact manifold and let $W_{k,l}$ be its Stein filling defined above. Then, does the family $\{W_{k,l}\}_{k\in\mathbb{Z}}$ of Stein fillings contain infinitely many Stein fillings up to symplectic deformation equivalent? Also, does the family $\{(M_l, \xi_l)\}_{l\in\mathbb{Z}}$ of contact manifolds contain infinitely many contact manifolds up to contactomorphism?

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Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguroku, Tokyo 152-8551, Japan

Email address: oba.t.ac@m.titech.ac.jp

Email address: takahiroohba@gmail.com