# ITERATED MATCHED PRODUCTS
# OF FINITE BRACES AND SIMPLICITY; NEW SOLUTIONS
# OF THE YANG-BAXTER EQUATION

D. BACHILLER, F. CEDÓ, E. JESPERS, AND J. OKNIŃSKI

Abstract. Braces were introduced by Rump as a promising tool in the study of the set-theoretic solutions of the Yang-Baxter equation. It has been recently proved that, given a left brace $B$, one can construct explicitly all the non-degenerate involutive set-theoretic solutions of the Yang-Baxter equation such that the associated permutation group is isomorphic, as a left brace, to $B$. It is hence of fundamental importance to describe all simple objects in the class of finite left braces. In this paper we focus on the matched product decompositions of an arbitrary finite left brace. This is used to construct new families of finite simple left braces.

## 1. Introduction

Braces were introduced by Rump [31] to study a class of solutions of the Yang-Baxter equation, a fundamental equation in mathematical physics that has become, since its origin in a paper of Yang [40], a key ingredient in quantum groups and Hopf algebras [27]. The primary aim of this paper is to present new general constructions of finite braces, with the main focus on constructing finite simple braces. The latter is the key step in the challenging problem of a classification of finite simple braces. Our approach is based on the notion of iterated matched product of braces, which turns out to be an indispensable tool in this context.

Recall that a solution of the quantum Yang-Baxter equation is a linear map $\mathcal{R}\colon V \otimes V \longrightarrow V \otimes V$, for a vector space $V$, such that

$$(1) \qquad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$

where $\mathcal{R}_{ij}$ denotes the map $V \otimes V \otimes V \longrightarrow V \otimes V \otimes V$ acting as $\mathcal{R}$ on the $(i,j)$ tensor factor and as the identity on the remaining factor. A central and difficult open problem is to construct new families of solutions of this equation. An equivalent problem is to find solutions of the Yang-Baxter equation

$$(2) \qquad R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

In fact, if $\tau\colon V \otimes V \longrightarrow V \otimes V$ is the linear map such that $\tau(u \otimes v) = v \otimes u$ for all $u, v \in V$, then it is easy to check that $\mathcal{R}\colon V \otimes V \longrightarrow V \otimes V$ is a solution of the quantum Yang-Baxter equation (1) if and only if $R = \tau \circ \mathcal{R}$ is a solution of the Yang-Baxter equation (2). In the context of quantum groups and Hopf algebras such solutions are often referred to as R-matrices (see for example [7, 27]). Drinfeld in [16] initiated the investigations of the set-theoretic solutions of the Yang-Baxter equation; these are the maps $r\colon X \times X \longrightarrow X \times X$ such that

$$(3) \qquad\qquad r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where $r_{ij}$ denotes the map $X \times X \times X \longrightarrow X \times X \times X$ acting as $r$ on the $(i, j)$ components and as the identity on the remaining component. Note that if $X$ is a basis of the vector space $V$, then every such solution $r$ induces a solution $R\colon V \otimes V \longrightarrow V \otimes V$ of the Yang-Baxter equation (2).

Gateva-Ivanova and Van den Bergh [21] and Etingof, Schedler, and Soloviev [17] introduced a subclass of the set-theoretic of solutions, the nondegenerate involutive solutions. Recall that a set-theoretic solution $r\colon X \times X \longrightarrow X \times X$ of the Yang-Baxter equation (3), written in the form $r(x, y) = (f_x(y), g_y(x))$ for $x, y \in X$, is involutive if $r^2 = \mathrm{id}_{X^2}$, and it is nondegenerate if $f_x$ and $g_x$ are bijective maps from $X$ to $X$, for all $x \in X$. This class of solutions has received a lot of attention in recent years; see for example [8, 9, 12–14, 17–21, 25, 26, 29, 31, 39]. Braces were introduced to study this type of solution. Recall that a left brace is a set $B$ with two operations, $+$ and $\cdot$, such that $(B, +)$ is an abelian group, $(B, \cdot)$ is a group, and

$$(4) \qquad\qquad a \cdot (b + c) + a = a \cdot b + a \cdot c,$$

for all $a, b, c \in B$. A right brace is defined similarly but replacing property (4) by $(b+c)\cdot a+a = b\cdot a+c\cdot a$. If $B$ is both a left and a right brace (for the same operations), then one says that $B$ is a two-sided brace. Rump initiated the study of this new algebraic structure, though, using another but equivalent definition [30–36]. In particular, he noted that the structure group $G(X, r)$ of a nondegenerate, involutive set-theoretic solution $(X, r)$ of the Yang-Baxter equation (solution of the YBE for short) admits a natural structure of left brace, such that its additive group is the free abelian group with basis $X$ and $xy - x = f_x(y)$ for all $x, y \in X$. The structure group $G(X, r)$ was introduced and studied in [17, 21] and it is defined as the group with the following presentation:

$$G(X, r) = \langle X \mid xy = f_x(y)g_y(x) \text{ for all } x, y \in X \rangle.$$

Another important group associated to a solution $(X, r)$ of the YBE is its permutation group $\mathcal{G}(X, r)$, which is the subgroup of the symmetric group $\mathrm{Sym}_X$ on $X$ generated by $\{f_x \mid x \in X\}$. The map $x \mapsto f_x$ extends to a group epimorphism $\phi\colon G(X, r) \longrightarrow \mathcal{G}(X, r)$ with kernel $\mathrm{Ker}(\phi) = \{a \in G(X, r) \mid ab = a + b \text{ for all } b \in G(X, r)\}$. The group $\mathcal{G}(X, r)$ inherits a natural left brace structure so that $\phi$ becomes a homomorphism of left braces.

Some important open problems have been solved in [13] using braces. Several aspects of the theory of braces and their applications in the context of the Yang-Baxter equation have been recently considered in [22, 37] and [11]. It is known that every finite left brace is isomorphic to $\mathcal{G}(X, r)$ (as left braces) for some finite solution $(X, r)$ of the YBE ([13, Theorem 2]). Thus, by [17, Theorem 2.15], the multiplicative group of every finite left brace is solvable. But not all finite solvable groups are isomorphic to the multiplicative group of a left brace [2]. In fact, there

exist finite $p$-groups that are not isomorphic to the multiplicative group of any left brace. Given a left brace $B$, in [4] a method is given to construct explicitly all the solutions $(X, r)$ of the YBE such that $\mathcal{G}(X, r) \cong B$ as left braces. Therefore, the problem of constructing all the solutions of the YBE is reduced to describing all left braces. The challenging problem of classifying all finite left braces naturally splits into two parts:

    (a) Classify the simple objects in the class of finite left braces.
    (b) Develop an appropriate theory of extensions of left braces.

Note that by [15, Corollary II.6.12], a version of the Jordan-Hölder theorem holds for finite left braces, emphasizing the importance of simple left braces.

Recall that an ideal of a left brace $B$ is a normal subgroup $I$ of its multiplicative group such that $\lambda_b(a) \in I$ for all $a \in I$ and $b \in B$, where $\lambda_b$ is the automorphism of $(B, +)$ defined by $\lambda_b(c) = bc - b$, for all $b, c \in B$. For example, the socle $\mathrm{Soc}(B) = \{b \in B \mid \lambda_b = \mathrm{id}\}$ is an ideal of $B$. A left brace $B$ is said to be simple if it is nonzero and $\{0\}$ and $B$ are the only ideals of $B$. Recall that a left brace is said to be trivial if its multiplication coincides with its addition. The socle of an arbitrary left brace $B$ is a trivial brace. It is known that every simple left brace of prime power order $p^n$ is a trivial brace of cardinality $p$ [31, Corollary on page 166]. Until recently, these were the only known examples of finite simple left braces. The first finite nontrivial simple left braces have been constructed in [3, Theorem 6.3 and Section 7], the additive groups of which are isomorphic to $\mathbb{Z}/(p_1) \times (\mathbb{Z}/(p_2))^{k(p_1-1)+1}$, where $p_1, p_2$ are primes such that $p_2 \mid p_1 - 1$ and $k$ is a positive integer. We shall give a much larger class of examples based on the construction of matched products of braces, which is introduced in [3] as a natural extension of the matched product (or bicrossed product) of groups [27]. Note that matched products of groups also appear in the context of solutions of the YBE; see for example the survey of Takeuchi [38] and [22].

Every left brace $B$ admits a left action $\lambda: (B, \cdot) \longrightarrow \mathrm{Aut}(B, +)$ defined by $\lambda(b) = \lambda_b$ for all $b \in B$ (see [13, Lemma 1]). It is called the lambda map of the left brace $B$. Recall that, given the lambda map of a left brace $B$, each of the structures $(B, \cdot)$ and $(B, +)$ determines the other one uniquely.

Lambda maps are used to define the matched products of left braces.

**Definition 1.1.** Let $G$ and $H$ be two left braces. Let $\alpha: (H, \cdot) \longrightarrow \mathrm{Aut}(G, +)$ and $\beta: (G, \cdot) \longrightarrow \mathrm{Aut}(H, +)$ be group homomorphisms. One says that $(G, H, \alpha, \beta)$ is a matched pair of left braces if the following conditions hold:

(MP1) $\lambda_a^{(1)} \circ \alpha_b = \alpha_{\beta_a(b)} \circ \lambda_{\alpha_{\beta_a(b)}^{-1}(a)}^{(1)}$,

(MP2) $\lambda_b^{(2)} \circ \beta_a = \beta_{\alpha_b(a)} \circ \lambda_{\beta_{\alpha_b(a)}^{-1}(b)}^{(2)}$,

where $\alpha(b) = \alpha_b$ and $\beta(a) = \beta_a$, for all $a \in G$ and $b \in H$, with $\lambda^{(1)}$ and $\lambda^{(2)}$ denoting the lambda maps of the left braces $G$ and $H$, respectively.

Let $(G, H, \alpha, \beta)$ be a matched pair of left braces. Then by [3, Theorem 4.2] $G \times H$ is a left brace with addition

$$(a, b) + (a', b') = (a + a', b + b')$$

and with lambda map given by

$$\lambda_{(a,b)}(a', b') = \left( \alpha_b \lambda_{\alpha_b^{-1}(a)}^{(1)}(a'),\ \beta_a \lambda_{\beta_a^{-1}(b)}^{(2)}(b') \right).$$

**Definition 1.2.** Let $(G, H, \alpha, \beta)$ be a matched pair of left braces. The left brace defined as above is called the matched product of $G$ and $H$. We simply denote it by $G \bowtie H$.

Note that if $\beta$ is trivial, then we get a semidirect product $G \rtimes H$ of left braces, considered in [34] and [13, Section 6], and then $G$ is an ideal of $G \bowtie H$; and if additionally $\alpha$ is trivial, then we get the direct product $G \times H$ of left braces.

Of course, if $(G, H, \alpha, \beta)$ is a matched pair of left braces, then so is $(H, G, \beta, \alpha)$. Furthermore, it easily is verified that the map $G \bowtie H \longrightarrow H \bowtie G$ defined by $(a, b) \mapsto (b, a)$ is an isomorphism of left braces.

Recall that a left ideal of a left brace $B$ is a subgroup $S$ of its multiplicative group such that $\lambda_b(a) \in S$ for all $a \in S$ and all $b \in B$. Note that for every $b, a \in S$ we have $b - a = \lambda_a(a^{-1}b)$. Thus, in particular, $S$ is a left subbrace of $B$. If $G \bowtie H$ is a matched product of left braces, then $G \times \{0\}$ and $\{0\} \times H$ are left ideals of $G \bowtie H$. Conversely, it is not difficult to see (use for example Lemma 2 in [13] to verify conditions (MP1) and (MP2)) that if $B$ is a left brace and $B_1, B_2$ are two left ideals of $B$ such that $(B, +)$ is the inner direct sum of $(B_1, +)$ and $(B_2, +)$, then $(B_1, B_2, \alpha, \beta)$ is a matched pair of left braces, where $\alpha_b(a) = ba - b$ and $\beta_a(b) = ab - a$, for all $a \in B_1$ and $b \in B_2$. Furthermore, the map $\eta : B_1 \bowtie B_2 \longrightarrow B$ defined by $\eta(a, b) = a + b$, for all $a \in B_1$ and $b \in B_2$, is an isomorphism of left braces.

Our main starting point is the following observation, contained implicitly in [3, Section 4].

*Remark* 1.3. Let $B$ be a finite nonzero left brace. Then there exist distinct prime numbers $p_1, \ldots, p_k$ and left braces $H_1, \ldots, H_k$, with $k \geq 1$, such that $|H_i| = p_i^{n_i}$ and $B$ is an iterated matched product $B = (\ldots (H_1 \bowtie H_2) \bowtie \ldots) \bowtie H_k$ of left braces. Moreover, each $H_i$ is a left ideal of $B$.

Essentially, the key argument used in the proof is that the Sylow subgroups of $(B, +)$ are left ideals of $B$. In Section 2 we give a proof of a more general result, Theorem 2.4.

Remark 1.3 explains why one can construct finite left braces using matched products, with braces of prime power order as the building blocks. In particular, all finite simple left braces can be constructed in this way. It is the aim of this paper to construct a large class of simple braces via this method. So we focus on part (a) of the classification problem. Some partial results on the classification of "small" left braces can be found in [1, 23, 32]. Concerning part (b), i.e., developing a theory of extensions of left braces, some preliminary results can be found in [3, 5, 6, 28]. But a general theory is missing.

In Section 2, we study iterated matched products of left braces corresponding to an inner direct sum of left ideals. First we characterize the iterated matched products of left braces that are of this form. Next, we give necessary and sufficient conditions for such a matched product to be simple provided the left ideals are simple left braces.

In Section 3, we first generalize an intriguing construction of Hegedűs [24], developed in the context of regular subgroups of the affine group, which has been recently considered also in [3] and [9]. Next, within this class, we construct iterated matched products of left braces and we give necessary and sufficient conditions on the actions corresponding to these matched products for their simplicity.

In Section 4, we construct concrete examples of simple left braces of the type described in Section 3. We also show how to construct more examples of finite simple left braces using the results of Section 2 and thus indicating that this may be a very rich area to explore. In this context, as mentioned before, it is shown in [4, Theorem 3.1] how to describe for a given left brace $B$ all solutions of the YBE with associated permutation group isomorphic to $B$ (as a left brace). So the new examples constructed in Section 4 provide new families of solutions of the YBE.

Finally, in Section 5, we state two problems that are fundamental for the program of the classification of finite (simple) left braces and thus a description of all finite solutions of the YBE. The first problem is concerned with the automorphism group of a finite left brace of prime power order. The second problem deals with simplicity of left braces of orders of the form $p^n q^m$, for two primes $p, q$. In this context, Smoktunowicz in [37] recently proved the following property.

If $|B| = q^r k$ with $q$ prime, $(q, k) = 1$, $k \neq 1$, and $B$ is a simple left brace, then there exists a prime $p$ such that $q | (p^t - 1)$ for some $1 \leq t$ and $p^t | k$.

So, if $B$ is a simple left brace of order $p^n q^m$ (with $p$ and $q$ different prime numbers and $n, m$ positive integers), then $p | (q^t - 1)$ and $q | (p^s - 1)$ for some $0 < t \leq m$ and $0 < s \leq n$. We observe that these conditions are not sufficient for simplicity of $B$.

## 2. Finite braces as iterated matched products of left ideals and simplicity

In the first part of this section, motivated by Remark 1.3, we characterize left braces that are iterated matched products of subbraces that are left ideals. In the second part, we prove a simplicity result that is later used to construct several new families of simple left braces.

Let $B$ be a left brace. Suppose that there exist left ideals $B_1, \ldots, B_n$ of $B$ such that $n \geq 2$ and the additive group of $B$ is the direct sum of the additive groups of the left ideals $B_i$. Denote by $\lambda^{(i)}$ the lambda map corresponding to $B_i$, which is the restriction to $B_i$ of the lambda map $\lambda$ of $B$. For $1 \leq j < n$, denote by $\lambda^{(1,\ldots,j)}$ the lambda map corresponding to $B_1 + \cdots + B_j$, which is the restriction to $B_1 + \cdots + B_j$ of the lambda map of $B$. Since $B_1 + \cdots + B_j$ and $B_{j+1}$ are left ideals of $B$, the maps

$$\alpha^{((1,\ldots,j),j+1)} \colon (B_1 + \cdots + B_j, \cdot) \longrightarrow \operatorname{Aut}(B_{j+1}, +) \quad \text{and}$$

$$\alpha^{(j+1,(1,\ldots,j))} \colon (B_{j+1}, \cdot) \longrightarrow \operatorname{Aut}(B_1 + \cdots + B_j, +)$$

defined by $\alpha^{((1,\ldots,j),j+1)}(x) = \alpha_x^{((1,\ldots,j),j+1)}$, $\alpha^{(j+1,(1,\ldots,j))}(y) = \alpha_y^{(j+1,(1,\ldots,j))}$, $\alpha_x^{((1,\ldots,j),j+1)}(y) = xy - x$, and $\alpha_y^{(j+1,(1,\ldots,j))}(x) = yx - y$, for all $x \in B_1 + \cdots + B_j$ and $y \in B_{j+1}$, are homomorphisms of groups. Since $\lambda_a \circ \lambda_{\lambda_a^{-1}(b)} = \lambda_b \circ \lambda_{\lambda_b^{-1}(a)}$ (see [13, Lemma 2]), we have that

(i) $\lambda_y^{(j+1)} \circ \alpha_{(\alpha_y^{(j+1,(1,\ldots,j))})^{-1}(x)}^{((1,\ldots,j),j+1)} = \alpha_x^{((1,\ldots,j),j+1)} \circ \lambda_{(\alpha_x^{((1,\ldots,j),j+1)})^{-1}(y)}^{(j+1)}$ and

(ii) $\lambda_x^{(1,\ldots,j+1)} \circ \alpha_{(\alpha_x^{((1,\ldots,j),j+1)})^{-1}(y)}^{(j+1,(1,\ldots,j))} = \alpha_y^{(j+1,(1,\ldots,j))} \circ \lambda_{(\alpha_y^{(j+1,(1,\ldots,j))})^{-1}(x)}^{(1,\ldots,j+1)}$.

Therefore $(B_1 + \cdots + B_j, B_{j+1}, \alpha^{(j+1,(1,\ldots,j))}, \alpha^{((1,\ldots,j),j+1)})$ is a matched pair of left braces, and, by [3, Theorem 4.2], the map

$$\eta_{1,\ldots,j+1} \colon (B_1 + \cdots + B_j) \bowtie B_{j+1} \longrightarrow B_1 + \cdots + B_{j+1},$$

defined by $\eta_{1,\ldots,j+1}(x, y) = x + y$, is an isomorphism of left braces.

We define the maps $\beta^{(2,(1))} = \alpha^{(2,(1))}$, $\beta^{((1),2)} = \alpha^{((1),2)}$, and, for $1 < j < n$,

$$\beta^{((1,\ldots,j),j+1)} \colon B_1 \times \cdots \times B_j \longrightarrow \mathrm{Aut}(B_{j+1},+) \quad \text{and}$$

$$\beta^{(j+1,(1,\ldots,j))} \colon B_{j+1} \longrightarrow \mathrm{Aut}(B_1 \times \cdots \times B_j, +)$$

by $\beta^{((1,\ldots,j),j+1)}(a_1,\ldots,a_j) = \beta^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)}$, $\beta^{(j+1,(1,\ldots,j))}(a_{j+1}) = \beta^{(j+1,(1,\ldots,j))}_{a_{j+1}}$,

$$\beta^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)}(a_{j+1}) = \alpha^{((1,\ldots,j),j+1)}_{a_1+\cdots+a_j}(a_{j+1}), \quad \text{and}$$

$$\beta^{(j+1,(1,\ldots,j))}_{a_{j+1}}(a_1,\ldots,a_j) = \eta_j^{-1}\alpha^{(j+1,(1,\ldots,j))}_{a_{j+1}}\eta_j(a_1,\ldots,a_j),$$

where $\eta_k \colon B_1 \times \cdots \times B_k \longrightarrow B_1 + \cdots + B_k$ is defined by $\eta_k(a_1,\ldots,a_k) = a_1 + \cdots + a_k$, for $1 < k \leq n$.

**Proposition 2.1.** *With the above notation,*

$$((\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_j, B_{j+1}, \beta^{(j+1,(1,\ldots,j))}, \beta^{((1,\ldots,j),j+1)})$$

*is a matched pair of left braces, and the map* $\eta_{j+1} \colon (\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_{j+1} \longrightarrow B_1 + \cdots + B_{j+1}$ *is an isomorphism of left braces, for all* $1 \leq j < n$.

*Proof.* We will prove the result by induction on $j$. For $j = 1$, the result follows because $\beta^{(2,(1))} = \alpha^{(2,(1))}$ and $\beta^{((1),2)} = \alpha^{((1),2)}$. Suppose that $j > 1$ and the result is true for $j - 1$. By the induction hypothesis, $\eta_j$ is an isomorphism of left braces. Thus, since $\alpha^{((1,\ldots,j),j+1)}$ and $\alpha^{(j+1,(1,\ldots,j))}$ are homomorphisms of groups, we have that $\beta^{((1,\ldots,j),j+1)}$ and $\beta^{(j+1,(1,\ldots,j))}$ are homomorphisms of groups. Let $\tilde{\lambda}^{(1,\ldots,j)}$ be the lambda map of the left brace $(\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_j$ corresponding to the matched pair of left braces $((\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_{j-1}, B_j, \beta^{(j,(1,\ldots,j-1))}, \beta^{((1,\ldots,j-1),j)})$. Since $\eta_j$ is an isomorphism of left braces,

$$\tilde{\lambda}^{(1,\ldots,j)}_{(a_1,\ldots,a_j)} = \eta_j^{-1} \lambda^{(1,\ldots,j)}_{a_1+\cdots+a_j} \eta_j.$$

To show that $((\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_j, B_{j+1}, \beta^{(j+1,(1,\ldots,j))}, \beta^{((1,\ldots,j),j+1)})$ is a matched pair of left braces, we should check the following equalities:

(i) $\tilde{\lambda}^{(1,\ldots,j)}_{(a_1,\ldots,a_j)} \circ \beta^{(j+1,(1,\ldots,j))}_{(\beta^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)})^{-1}(a_{j+1})} = \beta^{(j+1,(1,\ldots,j))}_{a_{j+1}} \circ \tilde{\lambda}^{(1,\ldots,j)}_{(\beta^{(j+1,(1,\ldots,j))}_{a_{j+1}})^{-1}(a_1,\ldots,a_j)}$,

(ii) $\lambda^{(j+1)}_{a_{j+1}} \circ \beta^{((1,\ldots,j),j+1)}_{(\beta^{(j+1,(1,\ldots,j))}_{a_{j+1}})^{-1}(a_1,\ldots,a_j)} = \beta^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)} \circ \lambda^{(j+1)}_{(\beta^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)})^{-1}(a_{j+1})}$.

We first prove equality (i):

$$\tilde{\lambda}^{(1,\ldots,j)}_{(a_1,\ldots,a_j)} \circ \beta^{(j+1,(1,\ldots,j))}_{(\beta^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)})^{-1}(a_{j+1})}$$

$$= \eta_j^{-1} \circ \lambda^{(1,\ldots,j)}_{a_1+\cdots+a_j} \circ \alpha^{(j+1,(1,\ldots,j))}_{(\alpha^{((1,\ldots,j),j+1)}_{a_1+\cdots+a_j})^{-1}(a_{j+1})} \circ \eta_j$$

$$= \eta_j^{-1} \circ \alpha^{(j+1,(1,\ldots,j))}_{a_{j+1}} \circ \lambda^{(1,\ldots,j)}_{(\alpha^{(j+1,(1,\ldots,j))}_{a_{j+1}})^{-1}(a_1+\cdots+a_j)} \circ \eta_j$$

$$= \beta^{(j+1,(1,\ldots,j))}_{a_{j+1}} \circ \tilde{\lambda}^{(1,\ldots,j)}_{(\beta^{(j+1,(1,\ldots,j))}_{a_{j+1}})^{-1}(a_1,\ldots,a_j)}.$$

Now we prove equality (ii):

$$\lambda_{a_{j+1}}^{(j+1)} \circ \beta_{(\beta_{a_{j+1}}^{(j+1,(1,\dots,j))})^{-1}(a_1,\dots,a_j)}^{((1,\dots,j),j+1)}$$

$$= \lambda_{a_{j+1}}^{(j+1)} \circ \alpha_{(\alpha_{a_{j+1}}^{(j+1,(1,\dots,j))})^{-1}(a_1+\dots+a_j)}^{((1,\dots,j),j+1)}$$

$$= \alpha_{a_1+\dots+a_j}^{((1,\dots,j),j+1)} \circ \lambda_{(\alpha_{a_1+\dots+a_j}^{((1,\dots,j),j+1)})^{-1}(a_{j+1})}^{(j+1)}$$

$$= \beta_{(a_1,\dots,a_j)}^{((1,\dots,j),j+1)} \circ \lambda_{(\beta_{(a_1,\dots,a_j)}^{((1,\dots,j),j+1)})^{-1}(a_{j+1})}^{(j+1)}.$$

Hence $((\dots(B_1 \bowtie B_2) \bowtie \dots) \bowtie B_j, B_{j+1}, \beta^{(j+1,(1,\dots,j))}, \beta^{((1,\dots,j),j+1)})$ is a matched pair of left braces. Now $\eta_{j+1} = \eta_{1,\dots,j+1} \circ (\eta_j \times \mathrm{id})$. Since clearly $\eta_{j+1}$ is an isomorphism of the additive groups and $\eta_{1,\dots,j+1}$ is an isomorphism of left braces, it is enough to prove that

$$(\eta_j \times \mathrm{id})(\tilde{\lambda}_{(a_1,\dots,a_{j+1})}^{(1,\dots,j+1)}(b_1,\dots,b_{j+1})) = \lambda_{(a_1+\dots+a_j,a_{j+1})}^{((1,\dots,j),j+1)}(b_1+\dots+b_j,b_{j+1}),$$

where $\lambda^{((1,\dots,j),j+1)}$ is the lambda map of the left brace $(B_1+\dots+B_j) \bowtie B_{j+1}$. We have that

$$(\eta_j \times \mathrm{id})(\tilde{\lambda}_{(a_1,\dots,a_{j+1})}^{(1,\dots,j+1)}(b_1,\dots,b_{j+1}))$$

$$= (\eta_j \times \mathrm{id})(\beta_{a_{j+1}}^{(j+1,(1,\dots,j))}\tilde{\lambda}_{(\beta_{a_{j+1}}^{(j+1,(1,\dots,j))})^{-1}(a_1,\dots,a_j)}^{(1,\dots,j)}(b_1,\dots,b_j),$$

$$\beta_{(a_1,\dots,a_j)}^{((1,\dots,j),j+1)}\lambda_{(\beta_{(a_1,\dots,a_j)}^{((1,\dots,j),j+1)})^{-1}(a_{j+1})}^{(j+1)}(b_{j+1}))$$

$$= (\eta_j \times \mathrm{id})(\eta_j^{-1}\alpha_{a_{j+1}}^{(j+1,(1,\dots,j))}\lambda_{(\alpha_{a_{j+1}}^{(j+1,(1,\dots,j))})^{-1}(a_1+\dots+a_j)}^{(1,\dots,j)}\eta_j(b_1,\dots,b_j),$$

$$\alpha_{(a_1+\dots+a_j)}^{((1,\dots,j),j+1)}\lambda_{(\alpha_{(a_1+\dots+a_j)}^{((1,\dots,j),j+1)})^{-1}(a_{j+1})}^{(j+1)}(b_{j+1}))$$

$$= (\alpha_{a_{j+1}}^{(j+1,(1,\dots,j))}\lambda_{(\alpha_{a_{j+1}}^{(j+1,(1,\dots,j))})^{-1}(a_1+\dots+a_j)}^{(1,\dots,j)}(b_1+\dots+b_j),$$

$$\alpha_{(a_1+\dots+a_j)}^{((1,\dots,j),j+1)}\lambda_{(\alpha_{(a_1+\dots+a_j)}^{((1,\dots,j),j+1)})^{-1}(a_{j+1})}^{(j+1)}(b_{j+1}))$$

$$= \lambda_{(a_1+\dots+a_j,a_{j+1})}^{((1,\dots,j),j+1)}(b_1+\dots+b_j,b_{j+1}).$$

Therefore the result follows. □

Note that with the above notation

$$\{0\} \times \dots \times \{0\} \times B_i \times \{0\} \times \dots \times \{0\}$$

is a left ideal of $(\dots(B_1 \bowtie B_2) \bowtie \dots) \bowtie B_n$. This motivates the following definition.

**Definition 2.2.** Let $B_1,\dots,B_n$ be left braces. We say that an iterated matched product of left braces $(\dots(B_1 \bowtie B_2) \bowtie \dots) \bowtie B_n$ is an iterated matched product of left ideals if each $\{0\} \times \dots \times \{0\} \times B_i \times \{0\} \times \dots \times \{0\}$ is a left ideal of it.

Note that if $B = (\dots(B_1 \bowtie B_2) \bowtie \dots) \bowtie B_n$ is an iterated matched product of left ideals and $\sigma \in \mathrm{Sym}_n$, by Proposition 2.1 we know that $B$ is isomorphic to $(\dots(C_{\sigma(1)} \bowtie C_{\sigma(2)}) \bowtie \dots) \bowtie C_{\sigma(n)}$, where $C_i = \{0\} \times \dots \times \{0\} \times B_i \times \{0\} \times \dots \times \{0\}$ is a left ideal of $B$. Hence, in an iterated matched product $(\dots(B_1 \bowtie B_2) \bowtie \dots) \bowtie B_n$

of left ideals the order of the factors is irrelevant, and this allows us to write it simply as $B_1 \bowtie B_2 \bowtie \ldots \bowtie B_n$.

As mentioned before, if $B_1 \bowtie B_2$ is a matched product of two left braces $B_1$ and $B_2$, then it easily is verified that both factors are left ideals. However, the following example shows that a factor of an arbitrary iterated matched product is not necessarily a left ideal of the left brace. It shows, in particular, that not every iterated matched product of left braces is a matched product with the defining factors as left ideals.

**Example 2.3.** Let $A = \mathbb{Z}/(p)$, where $p$ is a prime. Then $A$ is a trivial brace. Consider the trivial brace $A \times A$ and the maps $\alpha \colon (A, +) \longrightarrow \mathrm{Aut}(A \times A, +)$, defined by $\alpha(a) = \alpha_a$ and $\alpha_a(b, c) = (b + ac, c)$ (here $ac$ is the multiplication in the field $\mathbb{Z}/(p)$). Clearly $\alpha$ is a homomorphism of groups. The semidirect product $(A \times A) \rtimes A$ with respect to $\alpha$ is a left brace with the sum defined componentwise. This is a particular case of matched product of left braces. The direct product $A \times A$ is also a particular case of matched product of left braces. But $\{0\} \times A \times \{0\}$ is not a left ideal of $(A \times A) \rtimes A$; in fact $\lambda_{(0,0,1)}(0, 1, 0) = (\alpha_1(0, 1), 0) = (1, 1, 0)$.

We often will use the following useful formula, valid in any left brace $B$:

$$
\begin{aligned}
b_1 + \cdots + b_s &= (b_1 + \cdots + b_{s-1})\lambda^{-1}_{b_1 + \cdots + b_{s-1}}(b_s) \\
&= (b_1 + \cdots + b_{s-2})\lambda^{-1}_{b_1 + \cdots + b_{s-2}}(b_{s-1})\lambda^{-1}_{b_1 + \cdots + b_{s-1}}(b_s) \\
&\ \ \vdots \\
&= b_1\lambda^{-1}_{b_1}(b_2)\lambda^{-1}_{b_1+b_2}(b_3)\cdots\lambda^{-1}_{b_1+\cdots+b_{s-1}}(b_s),
\end{aligned}
$$
(5)

for any $s \geq 1$ and $b_i \in B$, $i = 1, \ldots, s$.

**Theorem 2.4.** *Let $B_1, \ldots, B_n$ be left braces with $n \geq 2$. An iterated matched product $B = (\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_n$ of left braces is an iterated matched product of left ideals if and only if there exist homomorphisms of groups*

$$
\alpha^{(j,i)} \colon (B_j, \cdot) \longrightarrow \mathrm{Aut}(B_i, +)
$$

*satisfying the following conditions:*

(IM1) $\lambda^{(i)}_a \circ \alpha^{(j,i)}_{(\alpha^{(i,j)}_a)^{-1}(b)} = \alpha^{(j,i)}_b \circ \lambda^{(i)}_{(\alpha^{(j,i)}_b)^{-1}(a)}$ *and*

(IM2) $\alpha^{(k,i)}_c \circ \alpha^{(j,i)}_{(\alpha^{(k,j)}_c)^{-1}(b)} = \alpha^{(j,i)}_b \circ \alpha^{(k,i)}_{(\alpha^{(j,k)}_b)^{-1}(c)}$,

*for all $a \in B_i$, $b \in B_j$, $c \in B_k$, $i, j, k \in \{1, 2, \ldots, n\}$, $j \neq i$, $k \neq i$, and $k \neq j$, where $\lambda^{(i)}$ is the lambda map of the left brace $B_i$ and $\alpha^{(i,j)}(a) = \alpha^{(i,j)}_a$, and furthermore $(\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_{j+1}$ is the matched product corresponding to the matched pair of left braces*

$$
((\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_j, B_{j+1}, \alpha^{(j+1,(1,\ldots,j))}, \alpha^{((1,\ldots,j),j+1)}),
$$

*where*

$$
\alpha^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)} = \alpha^{(1,j+1)}_{a_1}\alpha^{(2,j+1)}_{(\alpha^{(1,2)}_{a_1})^{-1}(a_2)}\cdots\alpha^{(j,j+1)}_{(\alpha^{((1,\ldots,j-1),j)}_{(a_1,\ldots,a_{j-1})})^{-1}(a_j)},
$$
(6)

$$
\alpha^{(j+1,(1,\ldots,j))}_{a_{j+1}}(a_1,\ldots,a_j) = (\alpha^{(j+1,1)}_{a_{j+1}}(a_1),\ldots,\alpha^{(j+1,j)}_{a_{j+1}}(a_j)).
$$
(7)

*Proof.* Denote the lambda map of $B_i$ by $\lambda^{(i)}$ and the lambda map of $(\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_{j+1}$ by $\lambda^{(1,\ldots,j+1)}$, for all $1 \leq j < n$. Suppose first that there exist homomorphisms of groups $\alpha^{(j,i)} \colon (B_j, \cdot) \longrightarrow \operatorname{Aut}(B_i, +)$ satisfying all the conditions in the statement. We shall prove that $\{0\} \times \cdots \times \{0\} \times B_k \times \{0\} \times \cdots \times \{0\}$ is a left ideal of $B$ by induction on $n$. For $n = 2$, the result follows by [3, Theorem 4.2]. Suppose that $n > 2$ and that $\{0\} \times \cdots \times \{0\} \times B_i \times \{0\} \times \cdots \times \{0\}$ is a left ideal of $(\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_{n-1}$ for all $i = 1, \ldots, n-1$. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in B_1 \times \cdots \times B_n$. We have that

$$\lambda^{(1,\ldots,n)}_{(a_1,\ldots,a_n)}(0,\ldots,0,b_n) = (0,\ldots,0, \alpha^{((1,\ldots,n-1),n)}_{(a_1,\ldots,a_{n-1})} \lambda^{(n)}_{(\alpha^{((1,\ldots,n-1),n)}_{(a_1,\ldots,a_{n-1})})^{-1}(a_n)}(b_n))$$

and for $i < n$,

$$\begin{aligned}
&\lambda^{(1,\ldots,n)}_{(a_1,\ldots,a_n)}(0,\ldots,0,b_i,0,\ldots,0) \\
&\quad = (\alpha^{(n,(1,\ldots,n-1))}_{a_n} \lambda^{(1,\ldots,n-1)}_{(\alpha^{(n,(1,\ldots,n-1))}_{a_n})^{-1}(a_1,\ldots,a_{n-1})}(0,\ldots,0,b_i,0,\ldots,0),0).
\end{aligned}$$

By the induction hypothesis, there exists $c_i \in B_i$ such that

$$\lambda^{(1,\ldots,n-1)}_{(\alpha^{(n,(1,\ldots,n-1))}_{a_n})^{-1}(a_1,\ldots,a_{n-1})}(0,\ldots,0,b_i,0,\ldots,0) = (0,\ldots,0,c_i,0,\ldots,0).$$

Hence

$$\begin{aligned}
&\lambda^{(1,\ldots,n)}_{(a_1,\ldots,a_n)}(0,\ldots,0,b_i,0,\ldots,0) \\
&\quad = (\alpha^{(n,(1,\ldots,n-1))}_{a_n}(0,\ldots,0,c_i,0,\ldots,0),0) \\
&\quad = ((0,\ldots,0,\alpha^{(n,i)}_{a_n}(c_i),0,\ldots,0),0) \quad \text{(by (7))}.
\end{aligned}$$

Thus $\{0\} \times \cdots \times \{0\} \times B_k \times \{0\} \times \cdots \times \{0\}$ is a left ideal of $B$ for every $k = 1, \ldots, n$. Therefore $B = (\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_n$ is an iterated matched product of left ideals.

Suppose now that $B = (\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_n$ is an iterated matched product of left ideals. Let $\pi_k$ be the natural projection $\pi_k \colon B_1 \times \cdots \times B_n \longrightarrow B_k$. We define

$$\alpha^{(j,i)}_{a_j}(a_i) = \pi_i \lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j,0,\ldots,0)}(0,\ldots,0,a_i,0,\ldots,0),$$

for all $a_i \in B_i$, $a_j \in B_j$, $i,j \in \{1,2,\ldots,n\}$, and $i \neq j$. Note that since $\{0\} \times \cdots \times \{0\} \times B_i \times \{0\} \times \cdots \times \{0\}$ is a left ideal of $B$, we have that

$$\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j,0,\ldots,0)}(0,\ldots,0,a_i,0,\ldots,0) = (0,\ldots,0,\alpha^{(j,i)}_{a_j}(a_i),0,\ldots,0).$$

Now it is clear that $\alpha^{(j,i)}_{a_j} \in \operatorname{Aut}(B_i, +)$ and

$$(\alpha^{(j,i)}_{a_j})^{-1}(a_i) = \pi_i(\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j,0,\ldots,0)})^{-1}(0,\ldots,0,a_i,0,\ldots,0).$$

We shall prove by induction on $n$ that

$$\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0) = (0,\ldots,0,\lambda^{(i)}_{a_i}(b_i),0,\ldots,0),$$

for all $a_i, b_i \in B_i$, and $i = 1, \ldots, n$. For $n = 2$, this follows easily by the definition of $\lambda^{(1,2)}$. Suppose that $n > 2$ and that

$$\lambda^{(1,\ldots,n-1)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0) = (0,\ldots,0,\lambda^{(i)}_{a_i}(b_i),0,\ldots,0),$$

for all $a_i, b_i \in B_i$ and $i = 1, \ldots, n-1$. Suppose that $i < n$. In this case

$$
\begin{aligned}
\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_i,0,\ldots,0)}&(0,\ldots,0,b_i,0,\ldots,0) \\
&= (\lambda^{(1,\ldots,n-1)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0),0) \\
&= ((0,\ldots,0,\lambda^{(i)}_{a_i}(b_i),0,\ldots,0),0).
\end{aligned}
$$

For $i = n$ we have

$$
\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_n)}(0,\ldots,0,b_n) = (0,\ldots,0,\lambda^{(n)}_{a_n}(b_n)).
$$

Note that

$$
\begin{aligned}
(0,\ldots,0,a_i,0,\ldots,0)&(0,\ldots,0,b_i,0,\ldots,0) \\
&= (0,\ldots,0,a_i,0,\ldots,0) + \lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0) \\
&= (0,\ldots,0,a_i + \lambda^{(i)}_{a_i}(b_i),0,\ldots,0) \\
&= (0,\ldots,0,a_i b_i,0,\ldots,0).
\end{aligned}
$$

Hence

$$
\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j b_j,0,\ldots,0)} = \lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j,0,\ldots,0)}\lambda^{(1,\ldots,n)}_{(0,\ldots,0,b_j,0,\ldots,0)}.
$$

Therefore we get that $\alpha^{(j,i)}_{a_j b_j} = \alpha^{(j,i)}_{a_j}\alpha^{(j,i)}_{b_j}$. Hence the map $\alpha^{(j,i)}\colon (B_j, \cdot) \longrightarrow \text{Aut}(B_i, +)$, defined by $\alpha^{(j,i)}(a_j) = \alpha^{(j,i)}_{a_j}$, is a homomorphism of groups. Now we shall check condition (IM1). Note that

$$
\begin{aligned}
(0,\ldots,0,&\lambda^{(i)}_{a_i}\alpha^{(j,i)}_{(\alpha^{(i,j)}_{a_i})^{-1}(a_j)}(b_i),0,\ldots,0) \\
&= \lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,\alpha^{(j,i)}_{(\alpha^{(i,j)}_{a_i})^{-1}(a_j)}(b_i),0,\ldots,0) \\
&= \lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_i,0,\ldots,0)}\lambda^{(1,\ldots,n)}_{(0,\ldots,0,(\alpha^{(i,j)}_{a_i})^{-1}(a_j),0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0) \\
&= \lambda^{(1,\ldots,n)}_{\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,(\alpha^{(i,j)}_{a_i})^{-1}(a_j),0,\ldots,0)} \\
&\quad \lambda^{(1,\ldots,n)}_{(\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,(\alpha^{(i,j)}_{a_i})^{-1}(a_j),0,\ldots,0))^{-1}(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0) \\
&= \lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j,0,\ldots,0)}\lambda^{(1,\ldots,n)}_{(\lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j,0,\ldots,0)})^{-1}(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0) \\
&= \lambda^{(1,\ldots,n)}_{(0,\ldots,0,a_j,0,\ldots,0)}\lambda^{(1,\ldots,n)}_{(0,\ldots,0,(\alpha^{(j,i)}_{a_j})^{-1}(a_i),0,\ldots,0)}(0,\ldots,0,b_i,0,\ldots,0) \\
&= (0,\ldots,0,\alpha^{(j,i)}_{a_j}\lambda^{(i)}_{(\alpha^{(j,i)}_{a_j})^{-1}(a_i)}(b_i),0,\ldots,0),
\end{aligned}
$$

where in the third equality [13, Lemma 2] is used. Hence $\lambda^{(i)}_{a_i}\alpha^{(j,i)}_{(\alpha^{(i,j)}_{a_i})^{-1}(a_j)} = \alpha^{(j,i)}_{a_j}\lambda^{(i)}_{(\alpha^{(j,i)}_{a_j})^{-1}(a_i)}$ and (IM1) is proved. Similarly one can check that condition (IM2) is satisfied. Before proving (6) we claim that

$$
\lambda^{(1,\ldots,j+1)}_{(a_1,\ldots,a_k,0,\ldots,0)}(0,\ldots,0,a_{k+1},0,\ldots,0) = (0,\ldots,0,\alpha^{((1,\ldots,k),k+1)}_{(a_1,\ldots,a_k)}(a_{k+1}),0,\ldots,0),
$$

for all $1 \le k \le j < n$. We will prove the claim by induction on $j$. For $j = 1$, we have

$$
\lambda^{(1,2)}_{(a_1,0)}(0,a_2) = (0,\alpha^{(1,2)}_{a_1}(a_2)),
$$

by the definition of $\lambda^{(1,2)}$. Suppose that $j > 1$ and that the claim is true for $j - 1$. For $k = j$ we have that

$$\lambda^{1,\ldots,j+1}_{(a_1,\ldots,a_j,0)}(0,\ldots,0,a_{j+1}) = (0,\ldots,0,\alpha^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)}(a_{j+1})),$$

by the definition of $\lambda^{(1,\ldots,j+1)}$. For $k < j$ we have

$$\lambda^{(1,\ldots,j+1)}_{(a_1,\ldots,a_k,0,\ldots,0)}(0,\ldots,0,a_{k+1},0,\ldots,0)$$
$$= (\lambda^{(1,\ldots,j)}_{(a_1,\ldots,a_k,0,\ldots,0)}(0,\ldots,0,a_{k+1},0,\ldots,0),0)$$
$$= (0,\ldots,0,\alpha^{((1,\ldots,k),k+1)}_{(a_1,\ldots,a_k)}(a_{k+1}),0,\ldots,0),$$

where the first equality is by the definition of $\lambda^{(1,\ldots,j+1)}$, and the second is by the induction hypothesis. Hence the claim follows. Now we will prove condition (6). We have

$$(0,\ldots,0,\alpha^{((1,\ldots,j),j+1)}_{(a_1,\ldots,a_j)}(a_{j+1}))$$
$$= \lambda^{(1,\ldots,j+1)}_{(a_1,\ldots,a_j,0)}(0,\ldots,0,a_{j+1})$$
$$= \lambda^{(1,\ldots,j+1)}_{(a_1,0,\ldots,0)}(\lambda^{(1,\ldots,j+1)}_{(a_1,0,\ldots,0)})^{-1}(0,a_2,0,\ldots,0)\cdots\lambda^{(1,\ldots,j+1)}_{(a_1,\ldots,a_{j-1},0,0)})^{-1}(0,\ldots,0,a_j,0)}(0,\ldots,0,a_{j+1})$$
$$= \lambda^{(1,\ldots,j+1)}_{(a_1,0,\ldots,0)}((0,\alpha^{(1,2)}_{a_1})^{-1}(a_2),0,\ldots,0)\cdots(0,\ldots,0,\alpha^{((1,\ldots,j-1),j)}_{(a_1,\ldots,a_{j-1})})^{-1}(a_j),0)}(0,\ldots,0,a_{j+1})$$
$$= \lambda^{(1,\ldots,j+1)}_{(a_1,0,\ldots,0)}\lambda^{(1,\ldots,j+1)}_{(0,(\alpha^{(1,2)}_{a_1})^{-1}(a_2),0,\ldots,0)}\cdots\lambda^{(1,\ldots,j+1)}_{(0,\ldots,0,(\alpha^{((1,\ldots,j-1),j)}_{(a_1,\ldots,a_{j-1})})^{-1}(a_j),0)}(0,\ldots,0,a_{j+1})$$
$$= (0,\ldots,0,\alpha^{(1,j+1)}_{a_1}\alpha^{(2,j+1)}_{(\alpha^{(1,2)}_{a_1})^{-1}(a_2)}\cdots\alpha^{(j,j+1)}_{(\alpha^{((1,\ldots,j-1),j)}_{(a_1,\ldots,a_{j-1})})^{-1}(a_j)}(a_{j+1})),$$

where the first equality is by the definition of $\lambda^{(1,\ldots,j+1)}$, the second follows from (5), the third follows by the claim, the fourth is because of the properties of the lambda maps, and the last follows because $\lambda^{(1,\ldots,j+1)}_{(0,\ldots,0,a_i,0,\ldots,0)}(0,\ldots,0,a_k,0,\ldots,0) = (0,\ldots,0,\alpha^{(i,k)}_{a_i}(a_k),0,\ldots,0)$ for all $i \neq k$. Therefore (6) follows. To prove (7), note that

$$(\alpha^{(j+1,(1,\ldots,j))}_{a_{j+1}}(a_1,\ldots,a_j),0)$$
$$= \lambda^{(1,\ldots,j+1)}_{(0,\ldots,0,a_{j+1})}(a_1,\ldots,a_j,0)$$
$$= \sum_{i=1}^{j}\lambda^{(1,\ldots,j+1)}_{(0,\ldots,0,a_{j+1})}(0,\ldots,0,a_i,0,\ldots,0)$$
$$= \sum_{i=1}^{j}(0,\ldots,0,\alpha^{(j+1,i)}_{a_{j+1}}(a_i),0,\ldots,0)$$
$$= (\alpha^{(j+1,1)}_{a_{j+1}}(a_1),\alpha^{(j+1,2)}_{a_{j+1}}(a_2),\ldots,\alpha^{(j+1,j)}_{a_{j+1}}(a_j),0).$$

Hence (7) follows. This finishes the proof of the theorem. □

Using the notation of Theorem 2.4, for an iterated matched product of left ideals $B = (\ldots(B_1 \bowtie B_2) \bowtie \ldots) \bowtie B_n$, it can be checked that the $i$-th component of

$\lambda^{(1,\ldots,n)}_{(a_1,\ldots,a_n)}(b_1,\ldots,b_n)$ is of the form

$$\alpha^{(1,i)}_{a_1}\alpha^{(2,i)}_{(\alpha^{(1,i)}_{a_1})^{-1}(a_2)}\cdots\alpha^{(i-1,i)}_{(\alpha^{((1,\ldots,i-2),i-1)}_{(a_1,\ldots,a_{i-2})})^{-1}(a_{i-1})}\lambda^{(i)}_{(\alpha^{((1,\ldots,i-1),i)}_{(a_1,\ldots,a_{i-1})})^{-1}(a_i)}$$

$$(8)\qquad\qquad \cdot\alpha^{(i+1,i)}_{(\alpha^{((1,\ldots,i),i+1)}_{(a_1,\ldots,a_i)})^{-1}(a_{i+1})}\cdots\alpha^{(n,i)}_{(\alpha^{((1,\ldots,n-1),n)}_{(a_1,\ldots,a_{n-1})})^{-1}(a_n)}(b_i).$$

Note that one can interpret $B_1,\ldots,B_n$ as left ideals of $B$ such that the additive group of $B$ is the direct sum of the additive groups of the left ideals $B_i$. Then $(a_1,\ldots,a_n)$ corresponds to $a_1+\cdots+a_n$, the maps $\alpha$ correspond to some restrictions of the lambda map of $B$, and formula (8) follows from (5).

In the remainder of this section we focus on simplicity of left braces that are iterated matched products of left ideals.

We will use the following easy but useful result.

**Lemma 2.5.** *If $I$ is an ideal of a left brace $B$, then $(\lambda_b - \mathrm{id})(a) \in I$, for all $a \in B$ and $b \in I$.*

*Proof.* Let $a \in B$ and $b \in I$. Then $(\lambda_b - \mathrm{id})(a) = ba - b - a = \lambda_a(a^{-1}ba) - b \in I$, so the assertion follows. $\qquad\square$

Let $B$ be an iterated matched product of its left ideals $B_1,\ldots,B_s$ of relatively prime orders. Consider the oriented graph $\Gamma(B) = (V,E)$, defined as follows. The set of vertices $V = \{1,\ldots,s\}$ and $(i,j) \in E$ is an edge if the corresponding map $\alpha^{(i,j)} : B_i \longrightarrow \mathrm{Aut}(B_j,+)$ is nontrivial. We call $\Gamma(B)$ the graph of (nontrivial) actions of $B$.

**Theorem 2.6.** *With the above notation and assumptions, assume that every $B_i$ is a simple left brace. Then the left brace $B$ is simple if and only if $\Gamma = \Gamma(B)$ contains a full (oriented) cycle, i.e., a cycle that contains all vertices.*

*Proof.* Suppose $\Gamma$ contains a full cycle. Let $I$ be a nonzero ideal of $B$. Choose a nonzero element $b_1+\cdots+b_s \in I$, with $b_i \in B_i$. Since $B$ is a direct sum of its additive subgroups $B_i$ of relatively prime orders, it is easy to see that $0 \neq b_i \in I \cap B_i$ for some $i$. Let $j \in V$ be such that $(i,j) \in E$. Hence there exists an element $a_j \in B_j$ such that $\alpha^{(i,j)}_{b_i}(a_j) \neq a_j$. Recall that $\alpha^{(i,j)}_{a_i}(a_j) = a_i a_j - a_i = \lambda_{a_i}(a_j)$. So, by Lemma 2.5, $0 \neq \alpha^{(i,j)}_{b_i}(a_j) - a_j \in I \cap B_j$. It follows that $B_j \subseteq I$ because $B_j$ is a simple left brace. Since $\Gamma$ contains a full cycle, this easily implies that $B_k \subseteq I$ for every $k$, and consequently $I = B$.

Conversely, assume that $\Gamma$ contains no full cycle. Then there exists $i$ such that $W = \{k \in V : \text{ there exists a path in } \Gamma \text{ from } i \text{ to } k\} \neq V$. Let $I = \sum_{j \in W} B_j$. Clearly, $I$ is a left ideal of $B$. Hence $I$ is $\lambda$-invariant. We will check that $I$ is an ideal of $B$. Then the result follows. Let $b \in B$. Write $b = b_1+\cdots+b_s$, with $b_i \in B_i$. From (5) we know that $b = c_1\cdots c_s$ for some $c_i \in B_i$. Let $a \in B_j$ for $j \in W$, so that $a \in I$. Since $\lambda_c(c^{-1}ac) = \lambda_a(c) - c + a$, we get that $c^{-1}ac = \lambda_c^{-1}(\lambda_a(c) - c + a)$. If $c \in B_k$ for some $k \notin W$, then $\lambda_a(c) = c$, so that $c^{-1}ac = \lambda_c^{-1}(a) \in I$. On the other hand, if $c \in B_k$ for some $k \in W$, then $c, a \in I$ and thus also $c^{-1}ac \in I$.

We know that $a = \sum_{j \in W} a_j$, with $a_j \in B_j, j \in W$. Therefore, again by (5), we also have $a = a_{j_1}\cdots a_{j_k}$, where $j_1,\ldots,j_k \in W$. Now $c^{-1}ac = c^{-1}a_{j_1}cc^{-1}a_{j_2}c\cdots c^{-1}a_{j_k}c \in I$ for every $c \in B_i$ and any $i$. Since $b = c_1\cdots c_s$, we get that $b^{-1}ab \in I$. Hence, $I$ is an ideal of $B$ and the result follows. $\qquad\square$

Notice that the proof of the necessity in the above theorem does not require the hypothesis that $B_i$ are simple left braces. Hence, the existence of a full oriented cycle in $\Gamma(B)$ is a necessary condition for $B$ to be simple.

We shall see in Example 4.3 that the following result provides an effective way for constructing matched products of left braces.

**Proposition 2.7.** *Let $B_1, \ldots, B_s$ be left braces. Assume that $\alpha^{(i,j)} : (B_i, \cdot) \longrightarrow \mathrm{Aut}(B_j, +, \cdot)$ are group homomorphisms, for all $i, j \in \{1, \ldots, s\}$, $i \neq j$. Assume also that*

(i) $\alpha^{(i,j)}_{\alpha^{(k,i)}_{a_k}(a_i)} = \alpha^{(i,j)}_{a_i}$,

(ii) $\alpha^{(j,i)}_{a_j} \alpha^{(k,i)}_{a_k} = \alpha^{(k,i)}_{a_k} \alpha^{(j,i)}_{a_j}$,

*for all $i, j, k \in \{1, \ldots, s\}$ with $i \neq j$ and $i \neq k$, where $a_m \in B_m$ for every $m$. Then the maps $\alpha^{(i,j)}$ satisfy conditions (IM1) and (IM2), and defining $\alpha^{(j+1,(1,\ldots,j))}$ and $\alpha^{((1,\ldots,j),j+1)}$ as in (6) and (7), we get an iterated matched product of left ideals $B_1 \bowtie \ldots \bowtie B_s$.*

*Proof.* It is enough to verify conditions (IM1) and (IM2) stated in Theorem 2.4:

$$
\begin{aligned}
\lambda^{(i)}_{a_i} \alpha^{(j,i)}_{(\alpha^{(i,j)}_{a_i})^{-1}(a_j)}(b_i) &= \lambda^{(i)}_{a_i} \alpha^{(j,i)}_{a_j}(b_i) \\
&= a_i \alpha^{(j,i)}_{a_j}(b_i) - a_i \\
&= \alpha^{(j,i)}_{a_j}((\alpha^{(j,i)}_{a_j})^{-1}(a_i)b_i - (\alpha^{(j,i)}_{a_j})^{-1}(a_i)) \\
&= \alpha^{(j,i)}_{a_j} \circ \lambda^{(i)}_{(\alpha^{(j,i)}_{a_j})^{-1}(a_i)}(b_i),
\end{aligned}
$$

for $a_i, b_i \in B_i$ and $a_j \in B_j$, where $\lambda^{(i)}$ is the lambda map of $B_i$. Thus (IM1) follows. Now we verify condition (IM2):

$$
\begin{aligned}
&\alpha^{(k,i)}_{a_k} \circ \alpha^{(j,i)}_{(\alpha^{(k,j)}_{a_k})^{-1}(a_j)} \\
&= \alpha^{(k,i)}_{a_k} \circ \alpha^{(j,i)}_{a_j} = \alpha^{(j,i)}_{a_j} \circ \alpha^{(k,i)}_{a_k} = \alpha^{(j,i)}_{a_j} \circ \alpha^{(k,i)}_{(\alpha^{(j,k)}_{a_j})^{-1}(a_k)},
\end{aligned}
$$

for $a_i \in B_i$, $a_j \in B_j$, and $a_k \in B_k$. Thus the result follows. $\square$

## 3. Constructions of simple braces

In this section, we first present a family of left braces with trivial socle that generalizes the family presented by Hegedűs [24] and Catino and Rizzo in [10]. Then we use it to construct a broad family of simple left braces.

Let $p$ be a prime number, and let $r, n$ be positive integers. Assume $Q$ is a quadratic form over $(\mathbb{Z}/(p^r))^n$ (considered as a free module over the ring $\mathbb{Z}/(p^r)$) and suppose $f$ is an element in the orthogonal group of $Q$ (that is, an element $f \in \mathrm{Aut}((\mathbb{Z}/(p^r))^n)$ such that $Q(f(v)) = Q(v)$ for any $v \in (\mathbb{Z}/(p^r))^n$). Assume that $f$ has order $p^{r'}$ for some $0 \leq r' \leq r$. Consider the additive abelian group $A = (\mathbb{Z}/(p^r))^{n+1}$. The elements of $A$ will be written in the form $(\vec{x}, \mu)$, with $\vec{x} \in (\mathbb{Z}/(p^r))^n$ and $\mu \in \mathbb{Z}/(p^r)$. Consider the maps $\lambda_{(\vec{x},\mu)} : A \longrightarrow A$ defined by

(9) $$\lambda_{(\vec{x},\mu)}(\vec{y}, \mu') := (f^{q(\vec{x},\mu)}(\vec{y}), \mu' + b(\vec{x}, f^{q(\vec{x},\mu)}(\vec{y}))),$$

for $(\vec{x}, \mu), (\vec{y}, \mu') \in A$, where $q(\vec{x}, \mu) := \mu - Q(\vec{x})$, and $b$ is the bilinear form $b(\vec{x}, \vec{y}) := Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y})$ associated to $Q$. Note that $\lambda_{(\vec{x},\mu)}$ is well-defined since $q$ takes values in $\mathbb{Z}/(p^r)$ and $f$ is of order $p^{r'}$, for some $0 \leq r' \leq r$.

Recall that $Q$ is nondegenerate if and only if the matrix of $b$ in the standard basis of $(\mathbb{Z}/(p^r))^n$ is invertible.

**Theorem 3.1.** *The abelian group $A$ has a structure of a left brace with lambda map defined in (9) and with multiplication given by $a \cdot b = a + \lambda_a(b)$. Moreover, if $Q$ is nondegenerate, then the socle of this left brace is*

$$\mathrm{Soc}(A) = \{(\vec{0}, \mu) \mid \mu \in p^{r'}\mathbb{Z}/(p^r)\}.$$

*In particular, if $r' = r$, then the socle of this left brace is zero.*

*Proof.* Since $f$ is bijective, it is clear that $\lambda_{(\vec{x},\mu)}$ is bijective. By the definition of $\lambda_{(\vec{x},\mu)}$, it also is clear that it is an automorphism of the abelian group $A$.

To prove the first part of the result, by [3, Lemma 2.6], it is enough to check that $\lambda_{(\vec{x},\mu)}\lambda_{(\vec{y},\mu')} = \lambda_{(\vec{x},\mu)+\lambda_{(\vec{x},\mu)}(\vec{y},\mu')}$. On one side,

$$\lambda_{(\vec{x},\mu)}\lambda_{(\vec{y},\mu')}(\vec{z},\eta)$$
$$= \lambda_{(\vec{x},\mu)}(f^{q(\vec{y},\mu')}(\vec{z}), \eta + b(\vec{y}, f^{q(\vec{y},\mu')}(\vec{z})))$$
$$(10) \qquad = (f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z}), \eta + b(\vec{y}, f^{q(\vec{y},\mu')}(\vec{z})) + b(\vec{x}, f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z}))).$$

Note that

$$q((\vec{x},\mu) + \lambda_{(\vec{x},\mu)}(\vec{y},\mu'))$$
$$= q(\vec{x} + f^{q(\vec{x},\mu)}(\vec{y}), \mu + \mu' + b(\vec{x}, f^{q(\vec{x},\mu)}(\vec{y})))$$
$$= \mu + \mu' + b(\vec{x}, f^{q(\vec{x},\mu)}(\vec{y})) - Q(\vec{x} + f^{q(\vec{x},\mu)}(\vec{y}))$$
$$= \mu + \mu' + b(\vec{x}, f^{q(\vec{x},\mu)}(\vec{y})) - Q(\vec{x}) - Q(f^{q(\vec{x},\mu)}(\vec{y})) - b(\vec{x}, f^{q(\vec{x},\mu)}(\vec{y}))$$
$$= \mu + \mu' - Q(\vec{x}) - Q(\vec{y})$$
$$= q(\vec{x},\mu) + q(\vec{y},\mu').$$

Hence

$$(11) \qquad q((\vec{x},\mu) + \lambda_{(\vec{x},\mu)}(\vec{y},\mu')) = q(\vec{x},\mu) + q(\vec{y},\mu').$$

On the other side we have

$$\lambda_{(\vec{x},\mu)+\lambda_{(\vec{x},\mu)}(\vec{y},\mu')}(\vec{z},\eta)$$
$$= (f^{q((\vec{x},\mu)+\lambda_{(\vec{x},\mu)}(\vec{y},\mu'))}(\vec{z}), \eta + b(\vec{x} + f^{q(\vec{x},\mu)}(\vec{y}), f^{q((\vec{x},\mu)+\lambda_{(\vec{x},\mu)}(\vec{y},\mu))}(\vec{z})))$$
$$= (f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z}), \eta + b(\vec{x} + f^{q(\vec{x},\mu)}(\vec{y}), f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z}))) \quad \text{(by (11))}$$
$$= (f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z}), \eta + b(f^{q(\vec{x},\mu)}(\vec{y}), f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z})) + b(\vec{x}, f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z})))$$
$$= (f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z}), \eta + b(\vec{y}, f^{q(\vec{y},\mu')}(\vec{z})) + b(\vec{x}, f^{q(\vec{x},\mu)+q(\vec{y},\mu')}(\vec{z}))).$$

Therefore, by (10), $\lambda_{(\vec{x},\mu)}\lambda_{(\vec{y},\mu')} = \lambda_{(\vec{x},\mu)+\lambda_{(\vec{x},\mu)}(\vec{y},\mu')}$, as desired.

To prove the second part of the statement, assume that $Q$ is nondegenerate. Let $(\vec{x},\mu)$ be an element of the socle of the left brace $A$. Then $f^{q(\vec{x},\mu)} = \mathrm{id}$ and $b(\vec{y}, f^{q(\vec{x},\mu)}(\vec{x})) = 0$ for all $\vec{y}$. Since $Q$ is nondegenerate, $\vec{x} = 0$. On the other hand, since $f^{q(\vec{x},\mu)} = \mathrm{id}$ and $f$ has order $p^{r'}$, we have $\mu = \mu - Q(\vec{x}) = q(\vec{x},\mu) \in p^{r'}\mathbb{Z}/(p^r)$. Therefore, the result follows. $\square$

**Notation.** The left brace described in Theorem 3.1 is denoted by $H(p^r, n, Q, f)$.

Let $R$ be a ring. For any matrix $A$ over $R$, we denote by $A^t$ the transpose of $A$. Sometimes we identify $R^n$ with the row matrices of length $n$ over $R$. So, for $x \in R^n$, $x^t$ is the column transpose of the row $x$.

Now we shall construct iterated matched products of left braces of the form $H(p^r, n, Q, f)$ and, as a consequence, we will give some new constructions of finite simple left braces. To do so we will make use of the existence of elements $C$ of order $p^r$ in $GL_n(\mathbb{Z}/(q^s))$ for two different primes $p$ and $q$. Note that the natural image of $C$ in $GL_n(\mathbb{Z}/(q))$ also has order $p^r$ and therefore $p^r$ has to divide $(q^n - 1) \cdots (q^n - q^{n-1})$. In particular, $p \mid q^t - 1$ for some $1 \leq t \leq n$. In light of the necessary condition for the existence of finite simple left braces mentioned in the introduction this is a natural assumption which will implicitly show up throughout the paper.

We will fix some notation. Let $s$ be an integer greater than 1 and let $p_1, p_2, \ldots, p_s$ be different prime numbers. Assume that $p_1, p_2, \ldots, p_{s-1}$ are odd. For $1 \leq i \leq s$, assume that finite left braces $H_i = H(p_i^{r_i}, n_i, Q_i, f_i)$ are constructed as in Theorem 3.1, with additive groups $(\mathbb{Z}/(p_i^{r_i}))^{n_i+1}$ ($r_i$ and $n_i$ are positive integers) and with the corresponding lambda map defined by

$$(12) \qquad \lambda_{(\vec{x}_i, \mu_i)}^{(i)}(\vec{y}_i, \mu_i') = (f_i^{q_i(\vec{x}_i, \mu_i)}(\vec{y}_i), \mu_i' + b_i(\vec{x}_i, f_i^{q_i(\vec{x}_i, \mu_i)}(\vec{y}_i))),$$

where

- $Q_i$ is a nondegenerate quadratic form over $(\mathbb{Z}/(p_i^{r_i}))^{n_i}$,
- $f_i$ is an element of order $p_i^{r_i'}$ in the orthogonal group determined by $Q_i$, for some $0 \leq r_i' \leq r_i$,
- $q_i(\vec{x}_i, \mu_i) = \mu_i - Q_i(\vec{x}_i)$ (with $\mu_i \in \mathbb{Z}/(p_i^{r_i})$),
- $b_i(\vec{x}_i, \vec{y}_i) = Q_i(\vec{x}_i + \vec{y}_i) - Q_i(\vec{x}_i) - Q_i(\vec{y}_i)$.

For $1 \leq i < s$, suppose $c_i$ is an element of order $p_{i+1}^{r_{i+1}}$ in the orthogonal group determined by $Q_i$, $c_s$ is an element of order $p_1^{r_1}$ of $Aut((\mathbb{Z}/(p_s^{r_s}))^{n_s})$, and $v_s \in (\mathbb{Z}/(p_s^{r_s}))^{n_s}$, such that

$$(13) \qquad Q_s(c_s(\vec{x})) = Q_s(\vec{x}) + v_s \vec{x}^t$$

and

$$f_i c_i = c_i f_i,$$

for $1 \leq i \leq s$. For $1 \leq i, j \leq s$, define the maps

$$\alpha^{(j,i)} : (H_j, \cdot) \longrightarrow Aut(H_i, +) : (\vec{x}_j, \mu_j) \mapsto \alpha_{(\vec{x}_j, \mu_j)}^{(j,i)},$$

with

$$\alpha_{(\vec{x}_{k+1}, \mu_{k+1})}^{(k+1,k)}(\vec{x}_k, \mu_k) = (c_k^{q_{k+1}(\vec{x}_{k+1}, \mu_{k+1})}(\vec{x}_k), \mu_k), \text{ for } 1 \leq k < s,$$

$$\alpha_{(\vec{x}_1, \mu_1)}^{(1,s)}(\vec{x}_s, \mu_s) = (c_s^{q_1(\vec{x}_1, \mu_1)}(\vec{x}_s), \mu_s + v_s((id + c_s + \cdots + c_s^{q_1(\vec{x}_1, \mu_1)-1})(\vec{x}_s))^t),$$

and $\alpha_{(\vec{x}_j, \mu_j)}^{(j,i)} = id_{H_i}$ otherwise. It is easy to check that

$$b_s(c_s(\vec{x}_s), c_s(\vec{y}_s)) = b_s(\vec{x}_s, \vec{y}_s).$$

Thus, if $p_s \neq 2$, then $v_s = 0$. Note that $\alpha_{(\vec{x}_{k+1}, \mu_{k+1})}^{(k+1,k)}$ is well-defined since $q_{k+1}$ takes values in $\mathbb{Z}/(p_{k+1}^{r_{k+1}})$ and the $c_k$ are of order $p_{k+1}^{r_{k+1}}$. Similarly $\alpha_{(\vec{x}_1, \mu_1)}^{(1,s)}$ is well-defined since $q_1$ takes values in $\mathbb{Z}/(p_1^{r_1})$ and $c_s$ is of order $p_1^{r_1}$. Note also that

$$q_i : (H_i, \cdot) \longrightarrow (\mathbb{Z}/(p_i^{r_i}), +)$$

is a homomorphism of groups, because

$$(\vec{z}, \nu) \cdot (\vec{t}, \nu') = (\vec{z}, \nu) + \lambda^{(i)}_{(\vec{z}, \nu)}(\vec{t}, \nu')$$

and by (11)

$$q_i((\vec{z}, \nu) + \lambda^{(i)}_{(\vec{z}, \nu)}(\vec{t}, \nu')) = q_i(\vec{z}, \nu) + q_i(\vec{t}, \nu').$$

Therefore, each $\alpha^{(j,i)}$ is a group homomorphism.

**Lemma 3.2.** *With the above notation, for $1 \le i, j \le s$, and $(\vec{x}_i, \mu_i) \in H_i$,*

$$q_i(\alpha^{(j,i)}_{(\vec{x}_j, \mu_j)}(\vec{x}_i, \mu_i)) = q_i(\vec{x}_i, \mu_i).$$

*Proof.* Let $(\vec{x}_k, \mu_k) \in H_k$, for $k = 1, \ldots, s$. For $i = 1, \ldots, s-1$, we have

$$
\begin{aligned}
q_i(\alpha^{(i+1,i)}_{(\vec{x}_{i+1}, \mu_{i+1})}(\vec{x}_i, \mu_i)) &= q_i(c_i^{q_{i+1}(\vec{x}_{i+1}, \mu_{i+1})}(\vec{x}_i), \mu_i) \\
&= \mu_i - Q_i(c_i^{q_{i+1}(\vec{x}_{i+1}, \mu_{i+1})}(\vec{x}_i)) \\
&= \mu_i - Q_i(\vec{x}_i) \\
&= q_i(\vec{x}_i, \mu_i).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& q_s(\alpha^{(1,s)}_{(\vec{x}_1, \mu_1)}(\vec{x}_s, \mu_s)) \\
&= q_s(c_s^{q_1(\vec{x}_1, \mu_1)}(\vec{x}_s), \mu_s + v_s((\mathrm{id} + c_s + \cdots + c_s^{q_1(\vec{x}_1, \mu_1)-1})(\vec{x}_s))^t) \\
&= \mu_s + v_s((\mathrm{id} + c_s + \cdots + c_s^{q_1(\vec{x}_1, \mu_1)-1})(\vec{x}_s))^t - Q_s(c_s^{q_1(\vec{x}_1, \mu_1)}(\vec{x}_s)) \\
&= \mu_s - Q_s(\vec{x}_s) \qquad \text{(by (13))} \\
&= q_s(\vec{x}_s, \mu_s).
\end{aligned}
$$

Therefore, the result follows. $\qquad\square$

It follows from the definitions that the map $\alpha^{(j,i)}_{(\vec{x}_j, \mu_j)}$ does not depend directly on the element $(\vec{x}_j, \mu_j)$ but just on the value $q_j(\vec{x}_j, \mu_j)$. Therefore Lemma 3.2 leads to the following consequence.

**Lemma 3.3.** *With the above notation, we have*

$$\alpha^{(j,i)}_{\alpha^{(k,j)}_{(\vec{x}_k, \mu_k)}(\vec{x}_j, \mu_j)} = \alpha^{(j,i)}_{(\vec{x}_j, \mu_j)}.$$

**Lemma 3.4.** *With the above notation, we have that $\alpha^{(j,i)}_{(\vec{x}_j, \mu_j)} \in \mathrm{Aut}(H_i, +, \cdot)$.*

*Proof.* Since $\alpha^{(j,i)}_{(\vec{x}_j, \mu_j)} \in \mathrm{Aut}(H_i, +)$, to prove the result it is enough to show that

(14) $$\alpha^{(j,i)}_{(\vec{x}_j, \mu_j)} \lambda^{(i)}_{(\vec{x}_i, \mu_i)}(\vec{y}_i, \mu_i') = \lambda^{(i)}_{\alpha^{(j,i)}_{(\vec{x}_j, \mu_j)}(\vec{x}_i, \mu_i)} \alpha^{(j,i)}_{(\vec{x}_j, \mu_j)}(\vec{y}_i, \mu_i').$$

For $1 \le i < s$ we have that

$$
\begin{aligned}
& \alpha^{(i+1,i)}_{(\vec{x}_{i+1}, \mu_{i+1})} \lambda^{(i)}_{(\vec{x}_i, \mu_i)}(\vec{y}_i, \mu_i') \\
&= \alpha^{(i+1,i)}_{(\vec{x}_{i+1}, \mu_{i+1})}(f_i^{q_i(\vec{x}_i, \mu_i)}(\vec{y}_i), \mu_i' + b_i(\vec{x}_i, f_i^{q_i(\vec{x}_i, \mu_i)}(\vec{y}_i))) \\
&= (c_i^{q_{i+1}(\vec{x}_{i+1}, \mu_{i+1})} f_i^{q_i(\vec{x}_i, \mu_i)}(\vec{y}_i), \mu_i' + b_i(\vec{x}_i, f_i^{q_i(\vec{x}_i, \mu_i)}(\vec{y}_i)))
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda^{(i)}_{\alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}(\vec{x}_i,\mu_i)} & \alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}(\vec{y}_i,\mu'_i) \\
= \; & \lambda^{(i)}_{\alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}(\vec{x}_i,\mu_i)}(c_i^{q_{i+1}(\vec{x}_{i+1},\mu_{i+1})}(\vec{y}_i),\mu'_i) \\
= \; & (f_i^{q_i(\alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}(\vec{x}_i,\mu_i))} c_i^{q_{i+1}(\vec{x}_{i+1},\mu_{i+1})}(\vec{y}_i),\mu'_i \\
& +b_i(c_i^{q_{i+1}(\vec{x}_{i+1},\mu_{i+1})}(\vec{x}_i),f_i^{q_i(\alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}(\vec{x}_i,\mu_i))} c_i^{q_{i+1}(\vec{x}_{i+1},\mu_{i+1})}(\vec{y}_i))) \\
= \; & (c_i^{q_{i+1}(\vec{x}_{i+1},\mu_{i+1})} f_i^{q_i(\vec{x}_i,\mu_i)}(\vec{y}_i),\mu'_i \\
& +b_i(\vec{x}_i,f_i^{q_i(\vec{x}_i,\mu_i)}(\vec{y}_i))) \quad \text{(by Lemma 3.2 and because } f_i c_i = c_i f_i \text{)}.
\end{aligned}
$$

Hence

$$
\alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}\lambda^{(i)}_{(\vec{x}_i,\mu_i)}(\vec{y}_i,\mu'_i) = \lambda^{(i)}_{\alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}(\vec{x}_i,\mu_i)}\alpha^{(i+1,i)}_{(\vec{x}_{i+1},\mu_{i+1})}(\vec{y}_i,\mu'_i).
$$

Because $f_s c_s = c_s f_s$ and since $f_s$ is orthogonal with respect to $Q_s$, using (13) one easily verifies that $v_s f_s(\vec{y}^t) = v_s \vec{y}^t$. Hence, we also have that

$$
\begin{aligned}
\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}&\lambda^{(s)}_{(\vec{x}_s,\mu_s)}(\vec{y}_s,\mu'_s) \\
= \; & \alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s),\mu'_s + b_s(\vec{x}_s,f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s))) \\
= \; & (c_s^{q_1(\vec{x}_1,\mu_1)} f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s),\mu'_s + b_s(\vec{x}_s,f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s)) \\
& +v_s((\mathrm{id}+c_s+\cdots+c_s^{q_1(\vec{x}_1,\mu_1)-1})(f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s)))^t) \\
= \; & (c_s^{q_1(\vec{x}_1,\mu_1)} f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s),\mu'_s + b_s(\vec{x}_s,f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s)) \\
& +v_s((\mathrm{id}+c_s+\cdots+c_s^{q_1(\vec{x}_1,\mu_1)-1})(\vec{y}_s))^t) \quad \text{(by (13))}
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda^{(s)}_{\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(\vec{x}_s,\mu_s)} & \alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(\vec{y}_s,\mu'_s) \\
= \; & \lambda^{(s)}_{\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(\vec{x}_s,\mu_s)}(c_s^{q_1(\vec{x}_1,\mu_1)}(\vec{y}_s),\mu'_s \\
& +v_s((\mathrm{id}+c_s+\cdots+c_s^{q_1(\vec{x}_1,\mu_1)-1})(\vec{y}_s))^t) \\
= \; & (f_s^{q_s(\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(\vec{x}_s,\mu_s))} c_s^{q_1(\vec{x}_1,\mu_1)}(\vec{y}_s),\mu'_s \\
& +v_s((\mathrm{id}+c_s+\cdots+c_s^{q_1(\vec{x}_1,\mu_1)-1})(\vec{y}_s))^t) \\
& +b_s(c_s^{q_1(\vec{x}_1,\mu_1)}(\vec{x}_s),f_s^{q_s(\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(\vec{x}_s,\mu_s))} c_s^{q_1(\vec{x}_1,\mu_1)}(\vec{y}_s))) \\
= \; & (c_s^{q_1(\vec{x}_1,\mu_1)} f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s),\mu'_s \\
& +v_s((\mathrm{id}+c_s+\cdots+c_s^{q_1(\vec{x}_1,\mu_1)-1})(\vec{y}_s))^t) \\
& +b_s(\vec{x}_s,f_s^{q_s(\vec{x}_s,\mu_s)}(\vec{y}_s))) \quad \text{(by Lemma 3.2 and because } f_s c_s = c_s f_s \text{)}.
\end{aligned}
$$

Hence

$$
\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}\lambda^{(s)}_{(\vec{x}_s,\mu_s)}(\vec{y}_s,\mu'_s) = \lambda^{(s)}_{\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(\vec{x}_s,\mu_s)}\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)}(\vec{y}_s,\mu'_s),
$$

and the result follows. $\qquad \square$

**Lemma 3.5.** *With the above notation, we have that*

$$\alpha^{(k,i)}_{(\vec{x}_k,\mu_k)} \circ \alpha^{(j,i)}_{(\vec{x}_j,\mu_j)} = \alpha^{(j,i)}_{(\vec{x}_j,\mu_j)} \circ \alpha^{(k,i)}_{(\vec{x}_k,\mu_k)},$$

*for all $i, j, k \in \{1, 2, \ldots, s\}$, $j \neq i$, $k \neq i$, and $k \neq j$.*

*Proof.* This follows from the definition of the maps $\alpha^{(j,i)}$ (recall that $\alpha^{(j,i)}_{(\vec{x}_j,\mu_j)} = \mathrm{id}_{H_i}$ if $(j,i) \notin \{(l+1,l) \mid l = 1, \ldots, s-1\} \cup \{(s,1)\}$). □

By Lemmas 3.3, 3.4, and 3.5 the maps $\alpha^{(j,i)}$ satisfy the hypothesis of Proposition 2.7. Therefore the maps $\alpha^{(j,i)}$, with $1 \leq i, j \leq s$, define an iterated matched product $H_1 \bowtie \ldots \bowtie H_s$ of left ideals.

**Theorem 3.6.** *With the above notation, the left brace $H_1 \bowtie \ldots \bowtie H_s$ is simple if and only if $c_i - \mathrm{id}$ is an automorphism for all $1 \leq i \leq s$.*

*Proof.* Let $I$ be a nonzero ideal of $H_1 \bowtie \ldots \bowtie H_s$. Let $(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s) \in I$ be a nonzero element. Suppose that $\vec{x}_i \neq 0$ for some $i$. Since the orders of the left braces $H_i$ are pairwise coprime, we may assume that $\vec{x}_k = \vec{0}$ and $\mu_k = 0$ for each $k \neq i$ and $\vec{x}_i \neq \vec{0}$. Note that for $(\vec{y}_j, Q_j(\vec{y}_j)) \in H_j$ we have that $q_j(\vec{y}_j, Q_j(\vec{y}_j)) = 0$, for every $j$. Hence, by (12)

$$\lambda^{(i)}_{(\vec{y}_i, Q_i(\vec{y}_i))}(\vec{x}_i, \mu_i) = (\vec{x}_i, \mu_i + b_i(\vec{y}_i, \vec{x}_i)).$$

So, $(\vec{0}, 0, \ldots, \vec{0}, b_i(\vec{y}_i, \vec{x}_i), \ldots, \vec{0}, 0) \in I$ (where $b_i(\vec{y}_i, \vec{x}_i)$ is in position $2i$). Since $Q_i$ is nondegenerate and $\vec{x}_i \neq \vec{0}$, there exists $\vec{y}_i$ such that $b_i(\vec{y}_i, \vec{x}_i) \neq 0$. We may assume that $b_i(\vec{y}_i, \vec{x}_i) = p_i^{s_i}$, for some $0 \leq s_i < r_i$.

On the other hand, if $\vec{x}_k = 0$ for all $k = 1, \ldots, s$, then $\mu_i \neq 0$ for some $i$. In this case, we also get that $(\vec{0}, 0, \ldots, \vec{0}, p_i^{s_i}, \ldots, \vec{0}, 0) \in I$ (where $p_i^{s_i}$ is in position $2i$), for some $0 \leq s_i < r_i$.

Thus, without loss of generality, we may assume that $\vec{x}_l = \vec{0}$, for all $l = 1, \ldots, s$, and there exists $i$ such that $\mu_i = p_i^{s_i}$ and $\mu_k = 0$ for each $k \neq i$. By Lemma 2.5, if $i = 1$, then

$$\begin{aligned}
&(\lambda_{(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s)} - \mathrm{id})(\vec{y}_1, \mu_1', \ldots, \vec{y}_s, \mu_s') \\
&= \quad ((\lambda^{(1)}_{(\vec{x}_1, \mu_1)} - \mathrm{id})(\vec{y}_1, \mu_1'), \vec{0}, 0, \ldots, \vec{0}, 0, (\alpha^{(1,s)}_{(\vec{x}_1,\mu_1)} - \mathrm{id})(\vec{y}_s, \mu_s')) \\
&= \quad ((f_1^{\mu_1} - \mathrm{id})(\vec{y}_1), 0, \vec{0}, 0, \ldots, \vec{0}, 0, (c_s^{\mu_1} - \mathrm{id})(\vec{y}_s), \\
&\qquad v_s((\mathrm{id} + c_s + \cdots + c_s^{\mu_1 - 1})(\vec{y}_s))^t) \in I,
\end{aligned}$$

for all $(\vec{y}_1, \mu_1', \ldots, \vec{y}_s, \mu_s') \in H_1 \bowtie \ldots \bowtie H_s$. If $1 < i \leq s$, then

$$\begin{aligned}
&(\lambda_{(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s)} - \mathrm{id})(\vec{y}_1, \mu_1', \ldots, \vec{y}_s, \mu_s') \\
&= \quad (\vec{0}, 0, \ldots, (\alpha^{(i,i-1)}_{(\vec{x}_i,\mu_i)} - \mathrm{id})(\vec{y}_{i-1}, \mu_{i-1}'), (\lambda^{(i)}_{(\vec{x}_i,\mu_i)} - \mathrm{id})(\vec{y}_i, \mu_i'), \ldots, \vec{0}, 0) \\
&= \quad (\vec{0}, 0, \ldots, (c_{i-1}^{\mu_i} - \mathrm{id})(\vec{y}_{i-1}), 0, (f_i^{\mu_i} - \mathrm{id})(\vec{y}_i), 0, \ldots, \vec{0}, 0) \in I.
\end{aligned}$$

Note that if $1 < i \leq s$, then the endomorphism of $(\mathbb{Z}/(p_{i-1}))^{n_{i-1}}$ induced by $c_{i-1}^{p_i^{s_i}} - \mathrm{id}$ is nonzero and thus there exists $\vec{y}_{i-1}$ such that $(c_{i-1}^{p_i^{s_i}} - \mathrm{id})(\vec{y}_{i-1}) \notin (p_{i-1}\mathbb{Z}/(p_{i-1}^{r_{i-1}}))^{n_{i-1}}$. If $i = 1$, then the endomorphism of $(\mathbb{Z}/(p_s))^{n_s}$ induced by $c_s^{p_1^{s_1}} - \mathrm{id}$ is nonzero and thus there exists $\vec{y}_s$ such that $(c_s^{p_1^{s_1}} - \mathrm{id})(\vec{y}_s) \notin (p_s\mathbb{Z}/(p_s^{r_s}))^{n_s}$. Hence we may assume that

$$(\vec{0}, 0, \ldots, \vec{z}_k, \nu_k, \ldots, \vec{0}, 0) \in I,$$

for some $\vec{z}_k \in (\mathbb{Z}/(p_k^{r_k}))^{n_k} \setminus (p_k\mathbb{Z}/(p_k^{r_k}))^{n_k}$ and some $\nu_k \in \mathbb{Z}/(p_k^{r_k})$. As above, we get that

$$(\vec{0}, 0, \ldots, \vec{0}, b_k(\vec{y}_k, \vec{z}_k), \ldots, \vec{0}, 0) \in I$$

(where $b_k(\vec{y}_k, \vec{z}_k)$ is in position $2k$). Since $Q_k$ is nondegenerate and $\vec{z}_k \in (\mathbb{Z}/(p_k^{r_k}))^{n_k} \setminus (p_k\mathbb{Z}/(p_k^{r_k}))^{n_k}$, there exists $\vec{y}_k \in (\mathbb{Z}/(p_k^{r_k}))^{n_k} \setminus (p_k\mathbb{Z}/(p_k^{r_k}))^{n_k}$ such that $b_k(\vec{y}_k, \vec{z}_k)$ is an invertible element in $\mathbb{Z}/(p_k^{r_k})$. Hence

$$w_k = (\vec{0}, 0, \ldots, \vec{0}, 1, \ldots, \vec{0}, 0) \in I$$

(where 1 is in position $2k$). We get by the above argument that

$$w_1 = (\vec{0}, 1, \vec{0}, 0, \ldots, \vec{0}, 0), \ldots, w_s = (\vec{0}, 0, \ldots, \vec{0}, 0, \vec{0}, 1) \in I.$$

Now it is easy to see that the ideal generated by $\{w_1, \ldots, w_s\}$ is equal to

$$\{(\vec{z}_1, \nu_1, \ldots, \vec{z}_s, \nu_s) \mid \nu_i \in \mathbb{Z}/(p_i^{r_i}), \; \vec{z}_i \in V_i\},$$

where

$$V_i = \langle (f_i^{a_1} c_i^{a_2} - f_i^{a_1'} c_i^{a_2'})(\vec{y}) \mid a_k, a_k' \in \mathbb{Z}, \; \vec{y} \in (\mathbb{Z}/(p_i^{r_i}))^{n_i} \rangle_+,$$

for $1 \le i \le s$. Note that $f_i$ and $c_i$ are elements of relative prime order in the group $\mathrm{Aut}((\mathbb{Z}/(p_i^{r_i}))^{n_i})$. Hence the subgroup generated by $f_i$ and $c_i$ is $\langle f_i c_i \rangle$, for $1 \le i \le s$. Therefore

$$V_i = \mathrm{Im}(f_i c_i - \mathrm{id}),$$

for $1 \le i \le s$. Note that

$$
\begin{aligned}
(f_i c_i - \mathrm{id})^{p_i^{r_i}} &= f_i^{p_i^{r_i}} c_i^{p_i^{r_i}} - \mathrm{id} + p_i h_i \\
&= c_i^{p_i^{r_i}} - \mathrm{id} + p_i h_i \\
&= (c_i - \mathrm{id})^{p_i^{r_i}} + p_i h_i',
\end{aligned}
$$

for some $h_i, h_i' \in \mathrm{End}((\mathbb{Z}/(p_i^{r_i}))^{n_i})$. Clearly, $p_i h_i'$ is in the Jacobson radical of the ring $\mathrm{End}((\mathbb{Z}/(p_i^{r_i}))^{n_i})$. Hence $f_i c_i - \mathrm{id}$ is an automorphism if and only if $c_i - \mathrm{id}$ is an automorphism. Thus the result follows. $\square$

## 4. Realizations of constructions of simple braces

In this section concrete examples of finite simple left braces constructed as in Theorem 3.6 are given. Using Proposition 2.7 and Theorem 2.6, we then also construct more examples of simple left braces.

First, we need some computations for matrices over $\mathbb{Z}$, which we will later reduce to $\mathbb{Z}/(p^r)$. Consider the companion matrix of the polynomial $x^{n-1} + x^{n-2} + \cdots + x + 1$,

$$D = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \in GL_{n-1}(\mathbb{Z}),$$

that has multiplicative order $n$. Let $E \in M_{n-1}(\mathbb{Z})$ be the matrix given by

$$E = \frac{1}{2}(\mathrm{Id} + D^t D + (D^2)^t D^2 + \cdots + (D^{n-1})^t D^{n-1}).$$

By a straightforward computation, one can check that

$$
E = \begin{pmatrix}
n-1 & -1 & \cdots & -1 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 \\
-1 & \cdots & -1 & n-1
\end{pmatrix}.
$$

Since $D^n = \mathrm{Id}$, it follows that $D^t E D = E$. It is an easy exercise to check that $\det(E) = n^{n-2}$.

Consider the quadratic form $Q$ over $\mathbb{Z}$ defined as

$$
Q(\vec{x}) = \sum_{1 \le i < j \le n-1} x_i x_j
$$

for $\vec{x} = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{Z}^{n-1}$. One can check that

$$
(15) \qquad Q(D(\vec{x})) \;=\; Q(\vec{x}) + \binom{n-1}{2} x_{n-1}^2 - (n-1) \sum_{i=1}^{n-2} x_i x_{n-1}.
$$

Let $s$ be an integer greater than 1. Let $p_1, p_2, \ldots, p_s$ be different prime numbers and let $r_1, r_2, \ldots, r_s$ be positive integers. Assume that $p_1, \ldots, p_{s-1}$ are odd. If $p_s = 2$, then we also assume that $r_s = 1$. Consider the following matrices:

$$
D_i \equiv D \pmod{p_i^{r_i}}, \quad E_i \equiv E \pmod{p_i^{r_i}}
$$

with $D_i, E_i \in GL_{p_{i+1}^{r_{i+1}}-1}(\mathbb{Z}/(p_i^{r_i}))$, for $1 \le i < s$, and $D_s, E_s \in GL_{p_1^{r_1}-1}(\mathbb{Z}/(p_s^{r_s}))$. Recall that $\det(E_i) = n_i^{n_i-2}$, where $n_i = p_{i+1}^{r_{i+1}}$, for $1 \le i < s$, and $n_s = p_1^{r_1}$. The order of $D_j$ is $n_j$, and $D_j^t E_j D_j = E_j$. Hence, $D_j$ is an element of order $n_j$ in the orthogonal group determined by the nonsingular quadratic form corresponding to $E_j$ on the free module $(\mathbb{Z}/(p_j)^{r_j})^{n_j-1}$ if $p_j$ is odd. Moreover $D_j - \mathrm{Id}$ is invertible (because 1 is not an eigenvalue of $D_j$ modulo $(p_j)$).

If $p_s = 2$, then we consider the quadratic form $Q_s$ on the vector space $(\mathbb{Z}/(2))^{p_1^{r_1}-1}$ defined by $Q_s(x_1, \ldots, x_{p_1^{r_1}-1}) = \sum_{1 \le i < j \le p_1^{r_1}-1} x_i x_j$. In this case, let $v_s = (0, \ldots, 0, \binom{p_1^{r_1}-1}{2})) \in (\mathbb{Z}/(2))^{p_1^{r_1}-1}$. By (15) we have that

$$
Q_s(D_s(\vec{x})) = Q_s(\vec{x}) + \binom{p_1^{r_1}-1}{2} x_{p_1^{r_1}-1} = Q_s(\vec{x}) + v_s \vec{x}^t.
$$

Let $0 \le r_i' \le r_i$. Consider in $M_{p_i^{r_i'}(n_i-1)}(\mathbb{Z}/(p_i^{r_i}))$ the block diagonal matrices with $p_i^{r_i'}$ blocks of degree $n_i - 1$:

$$
C_i = \begin{pmatrix}
D_i & 0 & \cdots & 0 \\
0 & D_i & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & D_i
\end{pmatrix}
\quad \text{and} \quad
B_i = \begin{pmatrix}
E_i & 0 & \cdots & 0 \\
0 & E_i & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & E_i
\end{pmatrix}.
$$

Consider the following block permutation matrix:

$$F_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & J_i \\ J_i & 0 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & J_i & 0 & 0 \\ 0 & \cdots & 0 & J_i & 0 \end{pmatrix},$$

where $J_i \in M_{n_i-1}(\mathbb{Z}/(p_i^{r_i}))$ is the identity matrix. Notice that $F_i^t = F_i^{-1}$ and

$F_i^t B_i F_i$

$$= \begin{pmatrix} 0 & J_i & \cdots & 0 & 0 \\ 0 & 0 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 & J_i \\ J_i & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_i & 0 & \cdots & \cdots & 0 \\ 0 & E_i & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & E_i & 0 \\ 0 & \cdots & \cdots & 0 & E_i \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & J_i \\ J_i & 0 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & J_i & 0 & 0 \\ 0 & \cdots & 0 & J_i & 0 \end{pmatrix}$$

$$= B_i.$$

Therefore, $F_i$ is an element of order $p_i^{r_i'}$ in the orthogonal group determined by the nonsingular quadratic form corresponding to $B_i$ on the free module $(\mathbb{Z}/(p_i^{r_i}))^{p_i^{r_i'}(n_i-1)}$ if $p_i$ is odd.

For $p_s = 2$, if $r_s' = 0$, then $F_s$ is the identity matrix. If $r_s' = 1$, then we consider the quadratic form $Q_s'$ on the vector space $(\mathbb{Z}/(2))^{2(n_s-1)}$ defined by $Q_s'(x_1, \ldots, x_{2(n_s-1)}) = Q_s(x_1, \ldots, x_{n_s-1}) + Q_s(x_{n_s}, \ldots, x_{2(n_s-1)})$ and the element $v_s' = (v_s, v_s) \in (\mathbb{Z}/(2))^{2(n_s-1)}$. In this case $F_s$ is an element of order 2 in the orthogonal group determined by the nonsingular quadratic form $Q_s'$. We also have that $Q_s'(C_s(\vec{x})) = Q_s'(\vec{x}) + v_s' \vec{x}^t$.

Moreover, we have $F_i^{-1} C_i F_i = C_i$, so that $C_i F_i = F_i C_i$, and $C_i - \mathrm{Id}$ is invertible because $D_i - \mathrm{Id}$ is invertible.

By Theorem 3.6, we can construct a simple left brace with additive group $(\mathbb{Z}/(p_1^{r_1}))^{p_1^{r_1'}(p_2^{r_2}-1)+1} \times \cdots \times (\mathbb{Z}/(p_{s-1}^{r_{s-1}}))^{p_{s-1}^{r_{s-1}'}(p_s^{r_s}-1)+1} \times (\mathbb{Z}/(p_s^{r_s}))^{p_s^{r_s'}(p_1^{r_1}-1)+1}$. First take the quadratic form $Q_i$ corresponding to the matrix $B_i$ if $p_i$ is odd and $f_i$ corresponding to the matrices $F_i$. Further, take $c_i$ corresponding to the matrix $C_i$ for $1 \le i \le s$.

Note that simplicity follows from Theorem 3.6 because $c_i - \mathrm{id}$ is invertible for $i = 1, 2, \ldots, s$.

One can also construct concrete examples of simple left braces using Theorem 2.6. For this we will need the following lemma.

**Lemma 4.1.** *With the notation of Section 3, consider the left brace $H_1 \bowtie \ldots \bowtie H_s$ of Theorem 3.6. Then the map*

$$\varphi_i \colon \begin{array}{ccc} (H_1 \bowtie \ldots \bowtie H_s, \cdot) & \longrightarrow & \mathbb{Z}/(p_i^{r_i}) \\ (\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s) & \mapsto & q_i(\vec{x}_i, \mu_i) \end{array}$$

*is a homomorphism of groups.*

*Proof.* Let $(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s), (\vec{y}_1, \mu_1', \ldots, \vec{y}_s, \mu_s') \in H_1 \bowtie \ldots \bowtie H_s$. Using the formula (8) of the lambda map, we get

$$
\begin{aligned}
&\varphi_i((\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s) \cdot (\vec{y}_1, \mu_1', \ldots, \vec{y}_s, \mu_s')) \\
&= \varphi_i((\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s) + \lambda_{(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s)}(\vec{y}_1, \mu_1', \ldots, \vec{y}_s, \mu_s')) \\
&= \varphi_i((\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s) + (\lambda^{(1)}_{(\vec{x}_1, \mu_1)} \alpha^{(2,1)}_{(\vec{x}_2, \mu_2)}(\vec{y}_1, \mu_1'), \ldots, \\
&\qquad \lambda^{(s-1)}_{(\vec{x}_{s-1}, \mu_{s-1})} \alpha^{(s,s-1)}_{(\vec{x}_s, \mu_s)}(\vec{y}_{s-1}, \mu_{s-1}'), \lambda^{(s)}_{(\vec{x}_s, \mu_s)} \alpha^{(1,s)}_{(\vec{x}_1, \mu_1)}(\vec{y}_s, \mu_s'))) \\
&= \begin{cases} q_i((\vec{x}_i, \mu_i) + (\lambda^{(i)}_{(\vec{x}_i, \mu_i)} \alpha^{(i+1,i)}_{(\vec{x}_{i+1}, \mu_{i+1})}(\vec{y}_i, \mu_i'))) & \text{if } 1 \le i < s, \\ q_s((\vec{x}_s, \mu_s) + (\lambda^{(s)}_{(\vec{x}_s, \mu_s)} \alpha^{(1,s)}_{(\vec{x}_1, \mu_1)}(\vec{y}_s, \mu_s'))) & \text{if } i = s. \end{cases}
\end{aligned}
$$

Now we have that

$$
\begin{aligned}
&q_i((\vec{x}_i, \mu_i) + (\lambda^{(i)}_{(\vec{x}_i, \mu_i)} \alpha^{(i+1,i)}_{(\vec{x}_{i+1}, \mu_{i+1})}(\vec{y}_i, \mu_i'))) \\
&= q_i(\vec{x}_i, \mu_i) + q_i(\alpha^{(i+1,i)}_{(\vec{x}_{i+1}, \mu_{i+1})}(\vec{y}_i, \mu_i')) \quad \text{(by (11))} \\
&= q_i(\vec{x}_i, \mu_i) + q_i(\vec{y}_i, \mu_i') \quad \text{(by Lemma 3.2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&q_s((\vec{x}_s, \mu_s) + (\lambda^{(s)}_{(\vec{x}_s, \mu_s)} \alpha^{(1,s)}_{(\vec{x}_1, \mu_1)}(\vec{y}_s, \mu_s'))) \\
&= q_s(\vec{x}_s, \mu_s) + q_s(\alpha^{(1,s)}_{(\vec{x}_1, \mu_1)}(\vec{y}_s, \mu_s')) \quad \text{(by (11))} \\
&= q_s(\vec{x}_s, \mu_s) + q_s(\vec{y}_s, \mu_s') \quad \text{(by Lemma 3.2)}.
\end{aligned}
$$

Therefore, the result follows.                                        $\square$

**Lemma 4.2.** *With the notation of Section 3, assume that for some $i \in \{1, \ldots, s\}$ there exists a divisor $m$ of $n_i$ such that*

$$
Q_i(\vec{x}_i) = \sum_{j=1}^m Q(\vec{x}_{i,j}),
$$

$$
f_i(\vec{x}_i) = (f(\vec{x}_{i,1}), \ldots, f(\vec{x}_{i,m})), \quad \text{and} \quad c_i(\vec{x}_i) = (c(\vec{x}_{i,1}), \ldots, c(\vec{x}_{i,m})),
$$

*where $\vec{x}_i = (\vec{x}_{i,1}, \ldots, \vec{x}_{i,m})$; $\vec{x}_{i,j} \in (\mathbb{Z}/(p_i^{r_i}))^{\frac{n_i}{m}}$; $Q$ is a nonsingular quadratic form over $(\mathbb{Z}/(p_i^{r_i}))^{\frac{n_i}{m}}$; $f$ is an element in the orthogonal group determined by $Q$; if $i \ne s$, then $c$ also is an element in the orthogonal group determined by $Q$; and if $i = s$, then $c$ is an automorphism of $(\mathbb{Z}/(p_s^{r_s}))^{\frac{n_s}{m}}$, $v_s = (v, \ldots, v)$, for some $v \in (\mathbb{Z}/(p_s^{r_s}))^{\frac{n_s}{m}}$, and $Q(c(\vec{x})) = Q(\vec{x}) + v\vec{x}^t$. Let $\sigma \in \mathrm{Sym}_m$. Then the map*

$$
\psi_\sigma : H_1 \bowtie \ldots \bowtie H_s \longrightarrow H_1 \bowtie \ldots \bowtie H_s,
$$

*defined by*

$$
\begin{aligned}
&\psi_\sigma(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s) \\
&= (\vec{x}_1, \mu_1, \ldots, \vec{x}_{i-1}, \mu_{i-1}, \vec{x}_{i,\sigma(1)}, \ldots, \vec{x}_{i,\sigma(m)}, \mu_i, \vec{x}_{i+1}, \mu_{i+1}, \ldots, \vec{x}_s, \mu_s),
\end{aligned}
$$

*is an automorphism of the left brace $H_1 \bowtie \ldots \bowtie H_s$ of Theorem 3.6.*

*Proof.* Let $(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s), (\vec{y}_1, \mu'_1, \ldots, \vec{y}_s, \mu'_s) \in H_1 \bowtie \ldots \bowtie H_s$. Clearly, $\psi_\sigma$ is an automorphism of the additive group of the left brace $H_1 \bowtie \ldots \bowtie H_s$. Thus, to prove the result, it is enough to show that

$$
\psi_\sigma(\lambda_{(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s)}(\vec{y}_1, \mu'_1, \ldots, \vec{y}_s, \mu'_s))
$$

(16)
$$
= \lambda_{\psi_\sigma(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s)} \psi_\sigma(\vec{y}_1, \mu'_1, \ldots, \vec{y}_s, \mu'_s).
$$

Note that if $1 \le i < s$, then the component $i$ of $\lambda_{(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s)}(\vec{y}_1, \mu'_1, \ldots, \vec{y}_s, \mu'_s)$ is $\lambda^{(i)}_{(\vec{x}_i, \mu_i)} \alpha^{(i+1,i)}_{\vec{x}_{i+1}, \mu_{i+1}}(\vec{y}_i, \mu'_i)$.

The component $s$ of $\lambda_{(\vec{x}_1, \mu_1, \ldots, \vec{x}_s, \mu_s)}(\vec{y}_1, \mu'_1, \ldots, \vec{y}_s, \mu'_s)$ is $\lambda^{(s)}_{(\vec{x}_s, \mu_s)} \alpha^{(1,s)}_{\vec{x}_1, \mu_1}(\vec{y}_s, \mu'_s)$.

Since $q_i(\vec{x}_i, \mu_i) = \mu_i - \sum_{j=1}^m Q(\vec{x}_{i,j})$ and $b_i(\vec{x}_i, \vec{z}_i) = \sum_{j=1}^m b(\vec{x}_{i,j}, \vec{z}_{i,j})$, where $b(\vec{x}_{i,j}, \vec{z}_{i,j}) = Q(\vec{x}_{i,j} + \vec{z}_{i,j}) - Q(\vec{x}_{i,j}) - Q(\vec{z}_{i,j})$, the reader easily can check (16) using the form of $f_i$, $c_i$ (and $v_s$ in the case where $i = s$). $\qquad \square$

Because of Proposition 2.7 and Theorem 2.6, we are now in a position to construct more concrete examples of simple left braces that are iterated matched products of left ideals.

**Example 4.3.** Let $p_1, p_2, p_3, p_4$ be different prime numbers. For simplicity, assume that all are odd. We can construct as above two simple left braces $H_1 \bowtie H_2$ and $H_3 \bowtie H_4$, where

$$
H_1 = H(p_1, p_4(p_2 - 1), Q_1, \text{id}), \ H_2 = H(p_2, p_1 - 1, Q_2, \text{id}),
$$

$$
H_3 = H(p_3, p_2(p_4 - 1), Q_3, \text{id}), \ H_4 = H(p_4, p_3 - 1, Q_4, \text{id}),
$$

for some nonsingular quadratic forms $Q_j$ where

$$
Q_1(x_1, \ldots, x_{p_4(p_2-1)}) = \sum_{j=0}^{p_4-1} Q'_1(x_{1+(p_2-1)j}, \ldots, x_{p_2-1+(p_2-1)j}),
$$

$$
Q_3(y_1, \ldots, y_{p_2(p_4-1)}) = \sum_{k=0}^{p_2-1} Q'_3(y_{1+(p_4-1)k}, \ldots, y_{p_4-1+(p_4-1)k}),
$$

$Q'_1$ is a nonsingular quadratic form over $(\mathbb{Z}/(p_1))^{p_2-1}$, $Q_2$ is a nonsingular quadratic form over $(\mathbb{Z}/(p_2))^{p_1-1}$, $Q'_3$ is a nonsingular quadratic form over $(\mathbb{Z}/(p_3))^{p_4-1}$, and $Q_4$ is a nonsingular quadratic form over $(\mathbb{Z}/(p_4))^{p_3-1}$. Let $c'_1$ be an element of order $p_2$ in the orthogonal group determined by $Q'_1$, let $d_1$ be an element of order $p_1$ in the orthogonal group determined by $Q_2$, let $c'_2$ be an element of order $p_4$ in the orthogonal group determined by $Q'_3$, and let $d_2$ be an element of order $p_3$ in the orthogonal group determined by $Q_4$. Assume that the endomorphisms $c'_1 - \text{id}$, $d_1 - \text{id}$, $c'_2 - \text{id}$, and $d_2 - \text{id}$ are invertible. Note that the maps $c_1 \in \text{Aut}((\mathbb{Z}/(p_1))^{p_4(p_2-1)})$ and $c_2 \in \text{Aut}((\mathbb{Z}/(p_3))^{p_2(p_4-1)})$ defined by

$$
c_1(\vec{x}_1, \ldots, \vec{x}_{p_4}) = (c'_1(\vec{x}_1), \ldots, c'_1(\vec{x}_{p_4}))
$$

and

$$
c_2(\vec{y}_1, \ldots, \vec{y}_{p_2}) = (c'_2(\vec{y}_1), \ldots, c'_2(\vec{y}_{p_2})),
$$

for $\vec{x}_j \in (\mathbb{Z}/(p_1))^{p_2-1}$ and $\vec{y}_k \in (\mathbb{Z}/(p_3))^{p_4-1}$, are elements in the orthogonal groups determined by $Q_1$ and $Q_3$ respectively. The actions of the matched pairs

$(H_1, H_2, \alpha^{(2,1)}, \alpha^{(1,2)})$ and $(H_3, H_4, \alpha^{(4,3)}, \alpha^{(3,4)})$ are defined by $\alpha^{(j,i)}(\vec{z}, \mu') = \alpha^{(j,i)}_{(\vec{z},\mu')}$ and

$$
\begin{aligned}
\alpha^{(2,1)}_{(\vec{z},\mu')}(\vec{x}, \mu) &= (c_1^{q_2(\vec{z},\mu')}(\vec{x}), \mu), \\
\alpha^{(1,2)}_{(\vec{x},\mu)}(\vec{z}, \mu') &= (d_1^{q_1(\vec{x},\mu)}(\vec{z}), \mu'), \\
\alpha^{(4,3)}_{(\vec{u},\nu')}(\vec{y}, \nu) &= (c_2^{q_4(\vec{u},\nu')}(\vec{y}), \nu), \\
\alpha^{(3,4)}_{(\vec{y},\nu)}(\vec{u}, \nu') &= (d_2^{q_3(\vec{y},\nu)}(\vec{u}), \nu').
\end{aligned}
$$

Let $\sigma_1$ and $\sigma_2$ be the cyclic permutations $\sigma_1 = (1, 2, \ldots, p_2)$ and $\sigma_2 = (1, 2, \ldots, p_4)$. By Lemma 4.2, the maps $\psi_{\sigma_1} \colon H_3 \bowtie H_4 \longrightarrow H_3 \bowtie H_4$, defined by

$$
\psi_{\sigma_1}(\vec{y}_1, \ldots, \vec{y}_{p_2}, \nu, \vec{u}, \nu') = (\vec{y}_{\sigma_1(1)}, \ldots, \vec{y}_{\sigma_1(p_2)}, \nu, \vec{u}, \nu'),
$$

and $\psi_{\sigma_2} \colon H_1 \bowtie H_2 \longrightarrow H_1 \bowtie H_2$, defined by

$$
\psi_{\sigma_2}(\vec{x}_1, \ldots, \vec{x}_{p_4}, \mu, \vec{z}, \mu') = (\vec{x}_{\sigma_2(1)}, \ldots, \vec{x}_{\sigma_2(p_4)}, \mu, \vec{z}, \mu'),
$$

are automorphisms of left braces. We define $\alpha \colon (H_3 \bowtie H_4, \cdot) \longrightarrow \mathrm{Aut}(H_1 \bowtie H_2, +, \cdot)$ and $\beta \colon (H_1 \bowtie H_2, \cdot) \longrightarrow \mathrm{Aut}(H_3 \bowtie H_4, +, \cdot)$ by

$$
\alpha(\vec{y}_1, \nu, \vec{u}, \nu') = \psi_{\sigma_2}^{q_4(\vec{u},\nu')} \quad \text{and} \quad \beta(\vec{x}_1, \mu, \vec{z}, \mu') = \psi_{\sigma_1}^{q_2(\vec{z},\mu')}.
$$

By Lemma 4.1, $\alpha$ and $\beta$ are homomorphisms of groups. Denote $\alpha(\vec{y}_1, \nu, \vec{u}, \nu')$ by $\alpha_{(\vec{y}_1,\nu,\vec{u},\nu')}$ and denote $\beta(\vec{x}_1, \mu, \vec{z}, \mu')$ by $\beta_{(\vec{x}_1,\mu,\vec{z},\mu')}$. Note that

$$
\alpha_{\beta_{(\vec{x},\mu,\vec{z},\mu')}(\vec{y},\nu,\vec{u},\nu')} = \alpha_{\psi_{\sigma_1}^{q_2(\vec{z},\mu')}(\vec{y},\nu,\vec{u},\nu')} = \psi_{\sigma_2}^{q_4(\vec{u},\nu')} = \alpha_{(\vec{y},\nu,\vec{u},\nu')}
$$

and

$$
\beta_{\alpha_{(\vec{y},\nu,\vec{u},\nu')}(\vec{x},\mu,\vec{z},\mu')} = \beta_{\psi_{\sigma_2}^{q_4(\vec{u},\nu')}(\vec{x},\mu,\vec{z},\mu')} = \psi_{\sigma_1}^{q_2(\vec{z},\mu')} = \beta_{(\vec{x},\mu,\vec{z},\mu')}.
$$

Hence, by Proposition 2.7, $(H_1 \bowtie H_2, H_3 \bowtie H_4, \alpha, \beta)$ is a matched pair of left braces, and the matched product $(H_1 \bowtie H_2) \bowtie (H_3 \bowtie H_4)$ is a simple left brace.

*Remark* 4.4. Clearly Example 4.3 can be generalized to matched products of arbitrary two simple braces of coprime orders constructed as in Theorem 3.6; namely, by replacing $H_1 \bowtie H_2$ and $H_3 \bowtie H_4$ by any braces as in Theorem 3.6. Even more, we can construct simple braces as iterated matched product of left ideals $B_1 \bowtie \ldots \bowtie B_n$, where every $B_i$ is a simple left brace as in Theorem 3.6.

## 5. COMMENTS AND QUESTIONS

In view of Remark 1.3 and the comment following Theorem 2.6, the following seems to be a crucial step in the general program of describing all finite simple left braces.

**Problem 5.1.** Describe the structure of all left braces of order $p^n$ for a prime $p$ and describe the group $\mathrm{Aut}(B, +, \cdot)$ of automorphisms for all such left braces.

Given two distinct primes $p$ and $q$, in the previous sections we constructed many simple left braces of order $p^\alpha q^\beta$ with some natural restrictions on the positive integers $\alpha$ and $\beta$. As mentioned in the introduction, these natural restrictions come from a recent result of Smoktunowicz that yields a necessary condition for a left brace of order $p^\alpha q^\beta$ to be simple. Namely, $q \mid (p^i - 1)$ and $p \mid (q^j - 1)$, for some $1 \le i \le \alpha$ and $1 \le j \le \beta$. Hence, the following seems to be an interesting problem.

**Problem 5.2.** Determine for which prime numbers $p$, $q$ and positive integers $\alpha, \beta$, there exists a simple left brace of cardinality $p^\alpha q^\beta$.

An easy observation shows that not all such orders can occur.

*Remark* 5.3. Let $G$ be a group of order $p^n q$, where $n$ is the multiplicative order of $p$ in $(\mathbb{Z}/(q))^\star$ and $p$ and $q$ are distinct prime numbers. Because of the Sylow theorems, it is easy to see that either a Sylow $p$-subgroup or a Sylow $q$-subgroup of $G$ is a normal subgroup. By Proposition 6.1 in [3], every normal Sylow subgroup of the multiplicative group of a left brace $B$ is an ideal of $B$. Therefore, $G$ does not admit a structure of a simple left brace. For instance, any group of order $2^3 \cdot 7$ has a normal Sylow 2-subgroup or a normal Sylow 7-subgroup, so there is no simple left brace of order $2^3 \cdot 7$.

## References

[1] David Bachiller, *Classification of braces of order $p^3$*, J. Pure Appl. Algebra **219** (2015), no. 8, 3568–3603, DOI 10.1016/j.jpaa.2014.12.013. MR3320237

[2] David Bachiller, *Counterexample to a conjecture about braces*, J. Algebra **453** (2016), 160–176, DOI 10.1016/j.jalgebra.2016.01.011. MR3465351

[3] D. Bachiller, Extensions, matched products and simple braces, J. Pure Appl. Algebra (2017).

[4] David Bachiller, Ferran Cedó, and Eric Jespers, *Solutions of the Yang-Baxter equation associated with a left brace*, J. Algebra **463** (2016), 80–102, DOI 10.1016/j.jalgebra.2016.05.024. MR3527540

[5] N. Ben David, *On groups of central type and involutive Yang-Baxter groups: a cohomological approach*, Ph.D. thesis, The Technion-Israel Institute of Technology, Haifa, 2012.

[6] Nir Ben David and Yuval Ginosar, *On groups of I-type and involutive Yang-Baxter groups*, J. Algebra **458** (2016), 197–206, DOI 10.1016/j.jalgebra.2016.03.025. MR3500774

[7] Ken A. Brown and Ken R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2002. MR1898492

[8] Francesco Catino, Ilaria Colazzo, and Paola Stefanelli, *On regular subgroups of the affine group*, Bull. Aust. Math. Soc. **91** (2015), no. 1, 76–85, DOI 10.1017/S000497271400077X. MR3294261

[9] Francesco Catino, Ilaria Colazzo, and Paola Stefanelli, *Regular subgroups of the affine group and asymmetric product of radical braces*, J. Algebra **455** (2016), 164–182, DOI 10.1016/j.jalgebra.2016.01.038. MR3478858

[10] Francesco Catino and Roberto Rizzo, *Regular subgroups of the affine group and radical circle algebras*, Bull. Aust. Math. Soc. **79** (2009), no. 1, 103–107, DOI 10.1017/S0004972708001068. MR2486886

[11] Ferran Cedó, T. Gateva-Ivanova, and A. Smoktunowicz, *On the Yang-Baxter equation and left nilpotent left braces*, J. Pure Appl. Algebra **221** (2017), no. 4, 751–756, DOI 10.1016/j.jpaa.2016.07.014. MR3574204

[12] Ferran Cedó, Eric Jespers, and Jan Okniński, *Retractability of set theoretic solutions of the Yang-Baxter equation*, Adv. Math. **224** (2010), no. 6, 2472–2484, DOI 10.1016/j.aim.2010.02.001. MR2652212

[13] Ferran Cedó, Eric Jespers, and Jan Okniński, *Braces and the Yang-Baxter equation*, Comm. Math. Phys. **327** (2014), no. 1, 101–116, DOI 10.1007/s00220-014-1935-y. MR3177933

[14] Ferran Cedó, Eric Jespers, and Ángel del Río, *Involutive Yang-Baxter groups*, Trans. Amer. Math. Soc. **362** (2010), no. 5, 2541–2558, DOI 10.1090/S0002-9947-09-04927-7. MR2584610

[15] P. M. Cohn, *Universal algebra*, 2nd ed., Mathematics and its Applications, vol. 6, D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1981. MR620952

[16] V. G. Drinfel'd, *On some unsolved problems in quantum group theory*, Quantum groups (Leningrad, 1990), Lecture Notes in Math., vol. 1510, Springer, Berlin, 1992, pp. 1–8, DOI 10.1007/BFb0101175. MR1183474

[17] Pavel Etingof, Travis Schedler, and Alexandre Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. **100** (1999), no. 2, 169–209, DOI 10.1215/S0012-7094-99-10007-X. MR1722951

[18] Tatiana Gateva-Ivanova, *A combinatorial approach to the set-theoretic solutions of the Yang-Baxter equation*, J. Math. Phys. **45** (2004), no. 10, 3828–3858, DOI 10.1063/1.1788848. MR2095675

[19] Tatiana Gateva-Ivanova and Peter Cameron, *Multipermutation solutions of the Yang-Baxter equation*, Comm. Math. Phys. **309** (2012), no. 3, 583–621, DOI 10.1007/s00220-011-1394-7. MR2885602

[20] Tatiana Gateva-Ivanova and Shahn Majid, *Matched pairs approach to set theoretic solutions of the Yang-Baxter equation*, J. Algebra **319** (2008), no. 4, 1462–1529, DOI 10.1016/j.jalgebra.2007.10.035. MR2383056

[21] Tatiana Gateva-Ivanova and Michel Van den Bergh, *Semigroups of I-type*, J. Algebra **206** (1998), no. 1, 97–112, DOI 10.1006/jabr.1997.7399. MR1637256

[22] T. Gateva-Ivanova, Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups, preprint arXiv: 1507.02602v2 [math.QA].

[23] L. Guarnieri and L. Vendramin, *Skew braces and the Yang–Baxter equation*, Math. Comp. **86** (2017), no. 307, 2519–2534, DOI 10.1090/mcom/3161. MR3647970

[24] Pál Hegedűs, *Regular subgroups of the affine group*, J. Algebra **225** (2000), no. 2, 740–742, DOI 10.1006/jabr.1999.8176. MR1741559

[25] Eric Jespers and Jan Okniński, *Monoids and groups of I-type*, Algebr. Represent. Theory **8** (2005), no. 5, 709–729, DOI 10.1007/s10468-005-0342-7. MR2189580

[26] Eric Jespers and Jan Okniński, *Noetherian semigroup algebras*, Algebra and Applications, vol. 7, Springer, Dordrecht, 2007. MR2301033

[27] Christian Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995. MR1321145

[28] Victoria Lebed and Leandro Vendramin, *Cohomology and extensions of braces*, Pacific J. Math. **284** (2016), no. 1, 191–212, DOI 10.2140/pjm.2016.284.191. MR3530867

[29] Wolfgang Rump, *A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation*, Adv. Math. **193** (2005), no. 1, 40–55, DOI 10.1016/j.aim.2004.03.019. MR2132760

[30] Wolfgang Rump, *Modules over braces*, Algebra Discrete Math. **2** (2006), 127–137. MR2320986

[31] Wolfgang Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra **307** (2007), no. 1, 153–170, DOI 10.1016/j.jalgebra.2006.03.040. MR2278047

[32] Wolfgang Rump, *Classification of cyclic braces*, J. Pure Appl. Algebra **209** (2007), no. 3, 671–685, DOI 10.1016/j.jpaa.2006.07.001. MR2298848

[33] Wolfgang Rump, *Addendum to "Generalized radical rings, unknotted biquandles, and quantum groups" (Colloq. Math. 109 (2007), 85–100) [MR2308827]*, Colloq. Math. **117** (2009), no. 2, 295–298. MR2550134

[34] Wolfgang Rump, *Semidirect products in algebraic logic and solutions of the quantum Yang-Baxter equation*, J. Algebra Appl. **7** (2008), no. 4, 471–490, DOI 10.1142/S0219498808002904. MR2442072

[35] Wolfgang Rump, *Addendum to "Generalized radical rings, unknotted biquandles, and quantum groups" (Colloq. Math. 109 (2007), 85–100) [MR2308827]*, Colloq. Math. **117** (2009), no. 2, 295–298. MR2550134

[36] Wolfgang Rump, *The brace of a classical group*, Note Mat. **34** (2014), no. 1, 115–144. MR3291816

[37] A. Smoktunowicz, A note on set-theoretic solutions of the Yang-Baxter equation, J. Algebra (2016), http://dx.doi.org/10.1016/j.jalgebra.2016.04.015

[38] Mitsuhiro Takeuchi, *Survey on matched pairs of groups—an elementary approach to the ESS-LYZ theory*, Noncommutative geometry and quantum groups (Warsaw, 2001), Banach Center Publ., vol. 61, Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 305–331, DOI 10.4064/bc61-0-19. MR2024436

[39] L. Vendramin, *Extensions of set-theoretic solutions of the Yang-Baxter equation and a conjecture of Gateva-Ivanova*, J. Pure Appl. Algebra **220** (2016), no. 5, 2064–2076, DOI 10.1016/j.jpaa.2015.10.018. MR3437282

[40] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312–1315, DOI 10.1103/PhysRevLett.19.1312. MR0261870

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bella-terra (Barcelona), Spain
  *Email address*: `dbachiller@mat.uab.cat`

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bella-terra (Barcelona), Spain
  *Email address*: `cedo@mat.uab.cat`

Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium
  *Email address*: `eric.jespers@vub.be`

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland
  *Email address*: `okninski@mimuw.edu.pl`