TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 8, August 2018, Pages 5341–5357
http://dx.doi.org/10.1090/tran/7097
Article electronically published on December 29, 2017

HYPERBOLICITY OF CYCLIC COVERS AND COMPLEMENTS

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Abstract. We prove that a cyclic cover of a smooth complex projective variety is Brody hyperbolic if its branch divisor is a generic small deformation of a large enough multiple of a Brody hyperbolic base-point-free ample divisor. We also show the hyperbolicity of complements of those branch divisors. As an application, we find new examples of Brody hyperbolic hypersurfaces in $\mathbb{P}^{n+1}$ that are cyclic covers of $\mathbb{P}^n$.

1. Introduction

A complex analytic space X is called *Brody hyperbolic* if there are no non-constant holomorphic maps from $\mathbb{C}$ to X. Lang's conjecture [Lan86] predicts that a projective variety X is Brody hyperbolic if every subvariety of X is of general type. More generally, the Green-Griffiths-Lang conjecture [GG80, Lan86] predicts that if a projective variety X is of general type, then there exists a proper Zariski closed subset $Z \subsetneq X$ such that any non-constant holomorphic map $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Z$. For hyperbolicity of hypersurfaces, Kobayashi [Kob70, Kob98] proposed the following conjecture:

Conjecture 1 (Kobayashi)**.** *For $n \geq 3$, a general hypersurface $X \subset \mathbb{P}^n$ of degree $\geq (2n-1)$ is Brody hyperbolic.*

It is easy to see that Lang's conjecture follows from the Green-Griffiths-Lang conjecture by a Noetherian induction argument. Based on results by Clemens [Cle86], Ein [Ein88, Ein91], and Xu [Xu94], Voisin [Voi96] showed that a general hypersurface X of degree $\geq (2n-1)$ in $\mathbb{P}^n$ with $n \geq 3$ satisfies that every subvariety of X is of general type. Therefore, Lang's conjecture implies Conjecture 1 by Voisin's result.

A lot of work has been done toward Conjecture 1; see [McQ99, DEG00, Rou07b, Pău08, DMR10, Siu15, Dem15, Bro16, Den16]. Examples of hyperbolic hypersurfaces are constructed in [MN96, SY97, Fuj01, SZ02, CZ03, Duv04, ZS05, CZ13, Huy15, Huy16].

In this paper, we first study the hyperbolicity of cyclic covers. For curves we know that if the branch divisor has large degree, then the cyclic cover will be Brody hyperbolic for a generic choice of the branch divisor. (See Section 3.1 for

Received by the editors May 26, 2016, and, in revised form, October 13, 2016.
2010 *Mathematics Subject Classification.* Primary 32Q45; Secondary 14J70, 14J29.
Key words and phrases. Brody hyperbolicity, cyclic covers, hypersurfaces.
The author was partially supported by NSF grant DMS-0968337.

hyperbolicity of cyclic covers of $\mathbb{P}^1$.) Our first main result is a higher dimensional generalization:

Theorem 2. *Let X be a smooth projective variety with $\dim X = n$. Let L be a globally generated ample line bundle on X. Suppose that there exists a smooth hypersurface $H \in |L|$ that is Brody hyperbolic. Let $m, d \geq 2$ be positive integers such that m is a multiple of d. For a generic small deformation S of $mH \in |L^{\otimes m}|$, let Y be the degree d cyclic cover of X branched along S. Then Y is Brody hyperbolic if $m \geq d\lceil \frac{n+2}{d-1} \rceil$.*

Here the assumption that H is Brody hyperbolic is crucial for our discussion. If $X \setminus H$ is also Brody hyperbolic, we have slightly better lower bounds on m to settle the hyperbolicity of Y:

Theorem 3. *With the notation of Theorem 2, assume in addition that $X \setminus H$ is Brody hyperbolic. Then Y is Brody hyperbolic if $m \geq d\lceil \frac{n+1}{d-1} \rceil$.*

In fact, we prove stronger results from which Theorems 2 and 3 follow (see Theorems 16 and 18).

The following theorem is an application of Theorem 2 which gives new examples of Brody hyperbolic hypersurfaces in $\mathbb{P}^{n+1}$. These hypersurfaces are cyclic covers of $\mathbb{P}^n$ via linear projections.

Theorem 4. *Suppose D is a smooth hypersurface of degree k in $\mathbb{P}^n$ that is Brody hyperbolic. For $d \geq n+3$ and a generic small deformation S of dD in $|\mathcal{O}_{\mathbb{P}^n}(dk)|$, let W be the degree dk cyclic cover of $\mathbb{P}^n$ branched along S. Then W is a Brody hyperbolic hypersurface in $\mathbb{P}^{n+1}$ of degree dk.*

In [RR13], Roulleau-Rousseau showed that a double cover of $\mathbb{P}^2$ branched along a very general curve of degree at least 10 is algebraically hyperbolic. Thus Green-Griffiths-Lang conjecture predicts that these surfaces are also Brody hyperbolic. As an application of Theorem 3, we give some evidence supporting this prediction:

Theorem 5. *Let $l \geq 3$, $k \geq 5$ be two positive integers. Let D be a smooth plane curve of degree k such that $\mathbb{P}^2 \setminus D$ is Brody hyperbolic.* (*The existence of such D was shown by Zaidenberg in* [Zau88].) *Let S be a generic small deformation of $2lD$. Then the double cover of $\mathbb{P}^2$ branched along S is Brody hyperbolic.*

Note that the minimal degree of S is 30.

For hyperbolicity of complements, the logarithmic Kobayashi conjecture and related problems have been studied in [Gre77, Zaĭ87, Zau88, SY96, SZ00, EG03, Rou07a, Rou09, IT15].

The cyclic cover being Brody hyperbolic clearly implies that the complement of the branch locus is also Brody hyperbolic. More precisely, with the notation of Theorem 2 we have that $X \setminus S$ is Brody hyperbolic if $m \geq d\lceil \frac{n+2}{d-1} \rceil$. In fact, we can still reach the same conclusion with the slightly weaker condition $m \geq n+2$, as the following theorem states.

Theorem 6. *Let X be a smooth projective variety with $\dim X = n$. Let L be a globally generated ample line bundle on X. Suppose that there exists a smooth hypersurface $H \in |L|$ that is Brody hyperbolic. Let $m \geq n+2$ be a positive integer. Then for a generic small deformation S of $mH \in |L^{\otimes m}|$, both S and $X \setminus S$ are Brody hyperbolic. Moreover, $X \setminus S$ is complete hyperbolic and hyperbolically embedded in X.*

Recall that *complete hyperbolicity* is defined in [Kob98, p. 60] and *hyperbolical embeddedness* is defined in [Kob98, p. 70].

Structure of the paper. The proofs of the theorems are mostly based on the degeneration to the normal cone (Section 2.1) and deformation type theorems of hyperbolicity (Theorems 15, 17 and 20).

In Section 2, we construct a family $\mathcal{X} \to \mathbb{A}^1$ with the general fiber X_t isomorphic to X and the special fiber X_0 being a projective cone over H. For technical reasons, we first introduce a smooth model $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ (Proposition 8). Then by taking a cyclic cover of the total space $\mathcal{X}$, we get a family of cyclic covers Y_t of X which degenerates to a cyclic cover Y_0 of X_0 (Proposition 11).

Most of the theorems are proved in Section 3. As a preparation, we study the hyperbolicity of cyclic covers of $\mathbb{P}^1$ and $\mathbb{A}^1$ in Section 3.1. We give lower bounds on the size of the reduced branch loci to get hyperbolicity of those cyclic covers (Lemmas 12 and 13). Section 3.2 is devoted to proving Theorems 2, 3 and 6. By dimension counting, we give a lower bound for the size of the reduced branch loci among all generators of X_0 (Lemma 14). When the branch locus has large degree, we prove the hyperbolicity of the cyclic cover of each generator of X_0, which gives the hyperbolicity of Y_0 since the base H is also Brody hyperbolic. Then Theorem 2 follows by applying a deformation type theorem (see Theorem 15). Theorems 3 and 6 are proved in similar ways with minor changes.

We first apply our methods to hypersurfaces in $\mathbb{P}^n$ in Section 4.1. We prove a stronger result that essentially implies Theorem 4 (see Theorem 21). We give a new proof to the main result in [Zaĭ09] by applying Mori's degeneration method (see Theorem 22). We also improve [Zai93, p. 147, Corollary of Theorem II.2] (see Theorem 23). In Section 4.2, we apply our methods to surfaces. We prove Theorem 5. We also obtain hyperbolicity of the complement of a smooth curve in some polarized K3 surface of large degree (Example 26).

Notation. Throughout this paper, we work over the complex numbers $\mathbb{C}$. We will follow the terminology of [Kob98] for various notions of hyperbolicity.

2. Construction of families

2.1. **Degeneration to the normal cone.** From now on X will be a smooth projective variety of dimension n. Let L be a globally generated ample line bundle on X. Let H be a smooth hypersurface in $|L|$.

Let $\rho : \widetilde{\mathcal{X}} \to X \times \mathbb{A}^1$ be the blowup of $X \times \mathbb{A}^1$ along $H \times \{0\}$, with exceptional divisor E. Denote the two projections from $X \times \mathbb{A}^1$ by p_1 and p_2. The line bundle $\widetilde{\mathcal{L}}$ on $\widetilde{\mathcal{X}}$ is defined by

$$\widetilde{\mathcal{L}} := (p_1 \circ \rho)^* L \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}(-E).$$

Let $\tilde{\pi} := p_2 \circ \rho$ be the composite of the projections $\widetilde{\mathcal{X}} \to X \times \mathbb{A}^1 \to \mathbb{A}^1$.

Proposition 7. *With the above notation, the line bundle $\widetilde{\mathcal{L}}$ is globally generated.*

Proof. Let $\mathcal{I}$ be the ideal sheaf of $H \times \{0\}$ in $X \times \mathbb{A}^1$. Then $\mathcal{O}_{\widetilde{\mathcal{X}}}(-E) = \rho^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{\mathcal{X}}}$. Therefore, it suffices to show that $p_1^* L \cdot \mathcal{I}$ is globally generated.

Let us choose a basis $s_1, \cdots, s_N$ of the vector space $H^0(X, L)$ with $N := \dim H^0(X, L)$. Let $s_H \in H^0(X, L)$ be a defining section of H, i.e., $H = (s_H = 0)$.

We may define sections $\sigma_0, \sigma_1, \cdots, \sigma_N \in H^0(X \times \mathbb{A}^1, p_1^* L)$ as follows:

$$\begin{aligned} \sigma_0(x,t) &= s_H(x), \\ \sigma_i(x,t) &= t s_i(x) \quad \text{for any } 1 \leq i \leq N. \end{aligned}$$

Since $s_1, \cdots, s_N$ generate L, the sections $\sigma_0, \cdots, \sigma_N$ generate the subsheaf $p_1^* L \cdot \mathcal{I}$ of $p_1^* L$. Hence we prove the proposition. □

Denote the lifting of σ_i to $\widetilde{\mathcal{X}}$ by $\tilde{\sigma}_i \in H^0(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$. The proof above implies that $\tilde{\phi} := [\tilde{\sigma}_0, \cdots, \tilde{\sigma}_N]$ defines a morphism $\tilde{\phi} : \widetilde{\mathcal{X}} \to \mathbb{P}^N$, such that $\widetilde{\mathcal{L}} \cong \tilde{\phi}^* \mathcal{O}(1)$.

Since $\widetilde{\mathcal{L}}$ is globally generated, it is also $\tilde{\pi}$-globally generated. By [Laz04, 2.1.27], we may define the ample model $\mathcal{X}$ of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ over $\mathbb{A}^1$ by

$$\mathcal{X} := \mathbf{Proj}_{\mathbb{A}^1} \bigoplus_{i \geq 0} \tilde{\pi}_* (\widetilde{\mathcal{L}}^{\otimes i}),$$

where $\psi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is an algebraic fiber space with $\mathcal{X}$ normal. Then $\widetilde{\mathcal{L}}$ descends to a globally generated ample line bundle $\mathcal{L}$ on $\mathcal{X}$, i.e., $\widetilde{\mathcal{L}} = \psi^* \mathcal{L}$. Since $\tilde{\phi}$ is induced by a base-point-free sublinear system of $|\widetilde{\mathcal{L}}|$, $\tilde{\phi}$ descends to a morphism $\phi : \mathcal{X} \to \mathbb{P}^N$, i.e., $\tilde{\phi} = \phi \circ \psi$.

Denote the fibers of $\tilde{\pi} : \widetilde{\mathcal{X}} \to \mathbb{A}^1$ and $\pi : \mathcal{X} \to \mathbb{A}^1$ by $\widetilde{X}_t$ and X_t, respectively. Then as a Cartier divisor, $\widetilde{X}_0$ can be written as

$$\widetilde{X}_0 = \hat{X}_0 + E,$$

where $\hat{X}_0$ is the birational transform of $X \times \{0\}$ under ρ.

Proposition 8. *With the above notation, we have the following properties.*

(1) *The Stein factorization of $(\tilde{\phi}, \tilde{\pi}) : \widetilde{\mathcal{X}} \to \mathbb{P}^N \times \mathbb{A}^1$ is given by the commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \xrightarrow{\ \psi\ } & \mathcal{X} \\ & \searrow^{(\tilde{\phi}, \tilde{\pi})} & \downarrow{\scriptstyle (\phi,\pi)} \\ & & \mathbb{P}^N \times \mathbb{A}^1 \end{array}$$

where $\psi_* \mathcal{O}_{\widetilde{\mathcal{X}}} = \mathcal{O}_{\mathcal{X}}$, and (ϕ, π) is finite.

(2) *The morphism ψ is birational. More precisely, ψ is an isomorphism away from $\hat{X}_0$, and it contracts $\hat{X}_0$ to a point v_0 in $\mathcal{X}$.*

(3) *Let $\rho_E : E \to H$ be the $\mathbb{P}^1$-bundle structure on E. Then $\tilde{\phi}$ sends each fiber of ρ_E isomorphically onto a line in $\mathbb{P}^N$.*

Proof.

(1) It follows from the relative version of [Laz04, 2.1.28].

(2) Since $\rho : \widetilde{\mathcal{X}} \setminus \widetilde{X}_0 \to X \times (\mathbb{A}^1 \setminus \{0\})$ is an isomorphism, the restriction $\widetilde{\mathcal{L}}|_{\widetilde{\mathcal{X}} \setminus \widetilde{X}_0}$ is $\tilde{\pi}$-ample over $\mathbb{A}^1 \setminus \{0\}$. Hence ψ is an isomorphism away from $\widetilde{X}_0$, which implies that ψ is birational.

Let $\hat{\rho}_0 := (p_1 \circ \rho)|_{\hat{X}_0}$ be the isomorphism from $\hat{X}_0$ to X. Recall that $\widetilde{\mathcal{L}} = (p_1 \circ \rho)^* L \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}(-E)$. Then

$$\begin{aligned}\widetilde{\mathcal{L}}|_{\hat{X}_0} &\cong \hat{\rho}_0^* L \otimes \mathcal{O}_{\hat{X}_0}(-E|_{\hat{X}_0}) \\ &\cong \hat{\rho}_0^* L \otimes \hat{\rho}_0^* \mathcal{O}_X(-H) \\ &\cong \hat{\rho}_0^*(L \otimes \mathcal{O}_X(-H)) \\ &\cong \mathcal{O}_{\hat{X}_0}.\end{aligned}$$

Hence $\widetilde{\mathcal{L}}|_{\hat{X}_0}$ is trivial, which implies that ψ contracts $\hat{X}_0$ to a point v_0 in $\mathcal{X}$.

Since $N_{H/X} \cong \mathcal{O}_X(H)|_H \cong L|_H$, we have $E \cong \mathbb{P}_H(L|_H^\vee \oplus \mathcal{O}_H)$. It is clear that $H_0 := \hat{X}_0|_E$ is the section of ρ_E corresponding to the first projection $L|_H \oplus \mathcal{O}_H \to L|_H$. Denote by H_1 the other section of ρ_E corresponding to the second projection $L|_H \oplus \mathcal{O}_H \to \mathcal{O}_H$. Then

$$\begin{aligned}\widetilde{\mathcal{L}}|_E &\cong (p_1 \circ \rho)^*(L)|_E \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}(-E)|_E \\ &\cong \rho_E^*(L|_H) \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}(\hat{X}_0)|_E \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}(\widetilde{X}_0)|_E \\ &\cong \rho_E^*(L|_H) \otimes \mathcal{O}_E(H_0) \\ &\cong \mathcal{O}_E(H_1).\end{aligned}$$

Since $L|_H$ is ample, for sufficiently large k the linear system $|\widetilde{\mathcal{L}}|_E^{\otimes k}|$ gives a birational morphism $E \to C_p(H, L|_H)$, where $C_p(H, L|_H)$ is the projective cone in the sense of [Kol13, Section 3.1]. In particular, any curve contracted by $\psi|_E$ is contained in H_0. Thus $\psi|_E$ is an isomorphism away from H_0, and we prove (2).

(3) As we have seen in the proof of (2), $\widetilde{\mathcal{L}}|_E \cong \mathcal{O}_E(H_1)$. Hence $(\tilde{\phi}^*\mathcal{O}(1) \cdot \rho_E^{-1}(x)) = (\widetilde{\mathcal{L}} \cdot \rho_E^{-1}(x)) = 1$ for any $x \in H$, and we prove (3).

□

Remark 9. By the proof of Proposition 8, the Stein factorization of $\psi|_E : E \to X_0$ is given by $E \to C_p(H, L|_H) \to X_0$. Thus $C_p(H, L|_H)$ is isomorphic to the normalization X_0^ν of X_0. In general, X_0 is not necessarily normal. According to [Kol13, 3.10], X_0 is normal if and only if $H^1(X, L^{\otimes k}) = 0$ for any $k \geq 0$.

2.2. **Constructing families of cyclic covers.** Let $m, d \geq 2$ be positive integers such that m is a multiple of d. It is clear that the linear system $\phi^*(|\mathcal{O}_{\mathbb{P}^N}(m)|)$ is base-point-free. Hence by Bertini's theorem, the following property holds for a general hypersurface $T \in |\mathcal{O}_{\mathbb{P}^N}(m)|$:

(∗) $\phi^*(T)$ is smooth, does not contain v_0, and intersects X_0 transversally.

Fix a general hypersurface $T \in |\mathcal{O}_{\mathbb{P}^N}(m)|$ satisfying property (∗). Let $\mu : \mathcal{Y} \to \mathcal{X}$ and $\tilde{\mu} : \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{X}}$ be the degree d cyclic covers of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ branched along $\phi^*(T)$ and $\tilde{\phi}^*(T)$, respectively. Let $\pi_{\widetilde{\mathcal{Y}}} := \tilde{\pi} \circ \tilde{\mu}$ and $\pi_{\mathcal{Y}} := \pi \circ \mu$ be the composition maps. Let $\psi_1 : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be the lifting of $\psi : \widetilde{\mathcal{X}} \to \mathcal{X}$. Then $\widetilde{\mathcal{Y}}$ and $\mathcal{Y}$ are proper flat families over $\mathbb{A}^1$. Denote the fibers of $\pi_{\widetilde{\mathcal{Y}}}$ and $\pi_{\mathcal{Y}}$ by $\widetilde{Y}_t$ and Y_t, respectively.

For $t \neq 0$ we notice that $(p_1 \circ \rho)|_{\widetilde{X}_t}$ maps $\widetilde{X}_t$ isomorphically onto X. Hence we may define a family of maps $f_t : X \to \mathbb{P}^N$ by $f_t := \tilde{\phi} \circ (p_1 \circ \rho)|_{\widetilde{X}_t}^{-1}$ for $t \neq 0$. In projective coordinates, we have

$$f_t(x) = [s_H(x), ts_1(x), \cdots, ts_N(x)].$$

Denote by $S_t := f_t^*(T)$ the pullback of T under f_t. Then $S_t \in |L^{\otimes m}|$ for all but finitely many t. In projective coordinates, let $F = F(z_0, \cdots, z_N)$ be a degree m homogeneous polynomial such that $T = (F = 0)$. Expanding F as a polynomial of the single variable z_0 yields

$$F(z_0, z_1, \cdots, z_N) = F_0 z_0^m + F_1 z_0^{m-1} + \cdots + F_{m-1} z_0 + F_m,$$

where F_i is a homogeneous polynomial in $z_1, \cdots, z_N$ of degree i for $0 \le i \le m$. Then S_t is the zero locus of the following section in $H^0(X, L^{\otimes m})$:

$$F(s_H, ts_1, \cdots, ts_N) = F_0 s_H^m + \sum_{i=1}^m F_i(s_1, \cdots, s_N) s_H^{m-i} t^i.$$

Since T does not contain $\phi(v_0) = [1, 0, \cdots, 0]$, we have that $F_0 \neq 0$. For simplicity we may assume that F is a monic polynomial in z_0, i.e., $F_0 = 1$. Thus for any $t \neq 0$ we have

$$S_t = \left(s_H^m + \sum_{i=1}^m F_i(s_1, \cdots, s_N) s_H^{m-i} t^i = 0 \right).$$

Definition 10. With the above notation, we say that $S \in |L^{\otimes m}|$ is a *generic small deformation* of mH if S is the zero locus of the section

$$s_H^m + \sum_{i=1}^m F_i(s_1, \cdots, s_N) s_H^{m-i} t^i$$

for generic choices of degree i polynomials F_i and for some $t \in \mathbb{A}^1 \setminus \{0\}$ with $|t| \le \epsilon$, where $\epsilon = \epsilon(\{F_i\}) \in \mathbb{R}_{>0}$ depends on the choice of $\{F_i\}$.

Notice that Definition 10 does not depend on the choice of the basis $s_1, \cdots, s_N$.

From our constructions we see that S_t is automatically a generic small deformation of mH for $|t|$ sufficiently small.

Proposition 11. *With the above notation, we have the following properties.*

(1) *The variety $\widetilde{\mathcal{Y}}$ is smooth.*
(2) *The birational morphism $\psi_1 : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ is an isomorphism away from $\tilde{\mu}^{-1}(\hat{X}_0)$, where $\tilde{\mu}^{-1}(\hat{X}_0)$ is a disjoint union of d isomorphic copies $\hat{X}_{0,1}, \cdots, \hat{X}_{0,d}$ of $\hat{X}_0$. Besides, ψ_1 contracts $\hat{X}_{0,i}$ to a point q_i in $\mathcal{Y}$, with $\mu^{-1}(v_0) = \{q_1, \cdots, q_d\}$.*
(3) *Let $\overline{Y}_0 := \tilde{\mu}^{-1}(E)$. Then $\psi_1|_{\overline{Y}_0} : \overline{Y}_0 \to Y_0$ is birational. Besides, the irreducible components of $\widetilde{Y}_0$ are $\overline{Y}_0, \hat{X}_{0,1}, \cdots, \hat{X}_{0,d}$.*
(4) *For any $t \neq 0$, the fibers $\widetilde{Y}_t$ and Y_t are both isomorphic to the degree d cyclic cover of X branched along S_t, where S_t is a generic small deformation of mH for $|t|$ sufficiently small. Besides, S_t is smooth for t sufficiently small.*

Proof.

(1) Since T satisfies $(*)$, $\tilde{\phi}^*(T)$ is smooth. Hence $\widetilde{\mathcal{Y}}$ is smooth.

(2) It follows from Proposition 8, $v_0 \notin \phi^*(T)$ and its equivalent form $\tilde{\phi}^*(T) \cap \hat{X}_0 = \emptyset$.

(3) It follows from (2).

(4) From our constructions we see that $(p_1 \circ \rho)|_{\widetilde{X}_t}$ maps $(\widetilde{X}_t, \tilde{\phi}^*(T)|_{\widetilde{X}_t})$ isomorphically onto (X, S_t); hence the first statement follows. Since T satisfies $(*)$, $\phi^*(T)$ intersects X_0 transversally. In particular, $\phi^*(T)|_{X_0}$ is smooth. Hence $S_t \cong \tilde{\phi}^*(T)|_{\widetilde{X}_t} \cong \phi^*(T)|_{X_t}$ is smooth for all but finitely many t, in particular for $|t|$ sufficiently small. □

3. Proofs of the theorems

3.1. **Hyperbolicity of cyclic covers of $\mathbb{P}^1$ and $\mathbb{A}^1$.**

Firstly, let us look at the hyperbolicity of cyclic covers of $\mathbb{P}^1$.

Let $m, d \geq 2$ be positive integers such that m is a multiple of d. Denote by $f : C \to \mathbb{P}^1$ the degree d cyclic cover of $\mathbb{P}^1$ branched along an effective divisor D of degree m. We may write

$$D = a_1 p_1 + \cdots + a_l p_l,$$

where $p_1, \cdots, p_l \in \mathbb{P}^1$ are distinct closed points, and $\sum_i a_i = m$.

Let $f^\nu : C^\nu \to \mathbb{P}^1$ be the normalization of f. By the Riemann-Hurwitz formula, we have

$$\chi_{\text{top}}(C^\nu) = d \cdot \chi_{\text{top}}(\mathbb{P}^1) - \sum_{i=1}^{l} (d - \#(f^\nu)^{-1}(p_i)),$$

where $\chi_{\text{top}}(\cdot)$ is the topological Euler number. It is clear that C is locally defined by the equation $y^d = x^{a_i}$ near p_i, so $\#(f^\nu)^{-1}(p_i) = \gcd(a_i, d)$. Hence

$$\chi_{\text{top}}(C^\nu) = 2d - \sum_{i=1}^{l} (d - \gcd(a_i, d)).$$

It is easy to see that C^ν is a disjoint union of $\gcd(d, a_1, \cdots, a_l)$ isomorphic copies of an irreducible smooth projective curve. Therefore, C is Brody hyperbolic if and only if $\chi_{\text{top}}(C^\nu) < 0$.

Lemma 12. *With the above notation, assume in addition that one of the following holds:*

- *d is divisible by 2 or 3, and $l \geq \frac{m}{d} + 3$;*
- *d is relatively prime to 6, and $l \geq \frac{m}{d} + 2$.*

Then $\chi_{\text{top}}(C^\nu) < 0$; i.e., C is Brody hyperbolic.

Proof. Assume to the contrary that $\chi_{\text{top}}(C^\nu) \geq 0$. Define an index set $J := \{j \mid 1 \leq j \leq l,\ a_j \text{ is not a multiple of } d\}$. Notice that $\gcd(a_i, d) \leq d/2$ if a_i is not a multiple of d. So we have $\chi_{\text{top}}(C^\nu) < 0$ as soon as $\#J \geq 5$. Hence we only need to consider cases when $\#J \leq 4$. For simplicity, we may assume that $J = \{1, \cdots, \#J\}$.

After careful study we get Table 1, which illustrates all cases when $\chi_{\text{top}}(C^\nu) \geq 0$, i.e., when C is not Brody hyperbolic, up to permutations of $a_1, \cdots, a_l$.

Table 1. Cyclic covers of $\mathbb{P}^1$ that are not hyperbolic.

$\#J$	$\{a_1/d\}$	$\{a_2/d\}$	$\{a_3/d\}$	$\{a_4/d\}$	C_1^ν	γ	$l \leq$
0	0	0	0	0	$\mathbb{P}^1$	d	m/d
2	p/q	$(q-p)/q$	0	0	$\mathbb{P}^1$	d/q	$m/d+1$
3	1/2	1/3	1/6	0	elliptic	$d/6$	$m/d+2$
3	1/2	2/3	5/6	0	elliptic	$d/6$	$m/d+1$
3	1/2	1/4	1/4	0	elliptic	$d/4$	$m/d+2$
3	1/2	3/4	3/4	0	elliptic	$d/4$	$m/d+1$
3	1/3	1/3	1/3	0	elliptic	$d/3$	$m/d+2$
3	2/3	2/3	2/3	0	elliptic	$d/3$	$m/d+1$
4	1/2	1/2	1/2	1/2	elliptic	$d/2$	$m/d+2$

We use the following notation in Table 1. It is clear that $\#J \leq 4$ if $\chi_{\mathrm{top}}(C^\nu) \geq 0$. Let $\{x\}$ be the fractional part of a real number x. Let $p < q$ be two positive integers that are relatively prime. Denote by C_1^ν a connected component of C^ν. Let $\gamma := \gcd(d, a_1, \cdots, a_l)$ be the number of connected components of C^ν. The fact that γ is always an integer gives certain divisibility conditions on d. $\square$

Next, we will discuss the hyperbolicity of cyclic covers of $\mathbb{A}^1$.

Let us identify $\mathbb{A}^1$ with $\mathbb{P}^1 \setminus \{\infty\}$. Denote $C^0 := f^{-1}(\mathbb{A}^1)$. If D is supported away from ∞, then C^0 is the degree d cyclic cover of $\mathbb{A}^1$ branched along an effective divisor D of degree m. Since $\infty \in D$, we have $\#(f^\nu)^{-1}(\infty) = d$. Hence

$$\begin{aligned}\chi_{\mathrm{top}}((C^0)^\nu) &= \chi_{\mathrm{top}}(C^\nu) - d \\ &= d - \sum_{i=1}^{l}(d - \gcd(a_i, d)).\end{aligned}$$

It is easy to see that $(C^0)^\nu$ is a disjoint union of $\gcd(d, a_1, \cdots, a_l)$ isomorphic copies of an irreducible smooth affine curve. Therefore, C^0 is Brody hyperbolic if and only if $\chi_{\mathrm{top}}((C^0)^\nu) < 0$.

Lemma 13. *With the above notation, assume in addition that D is supported away from ∞. Moreover, assume that one of the following holds:*

- *d is even and $l \geq \frac{m}{d} + 2$;*
- *d is odd and $l \geq \frac{m}{d} + 1$.*

Then $\chi_{\mathrm{top}}((C^0)^\nu) < 0$; i.e., C^0 is Brody hyperbolic.

Proof. Assume to the contrary that $\chi_{\mathrm{top}}((C^0)^\nu) \geq 0$. Define an index set $J := \{j \mid 1 \leq j \leq l,\ a_j \text{ is not a multiple of } d\}$. Notice that $\gcd(a_i, d) \leq d/2$ if a_i is not a multiple of d. So we have $\chi_{\mathrm{top}}((C^0)^\nu) < 0$ as soon as $\#J \geq 3$. Hence we only need to consider cases when $\#J \leq 2$. For simplicity, we may assume that $J = \{1, \cdots, \#J\}$.

After careful study we get Table 2, which illustrates all cases when $\chi_{\mathrm{top}}((C^0)^\nu) \geq 0$, i.e., when C^0 is not Brody hyperbolic, up to permutations of $a_1, \cdots, a_l$.

TABLE 2. Cyclic covers of $\mathbb{A}^1$ that are not hyperbolic.

$\#J$	$\{a_1/d\}$	$\{a_2/d\}$	$(C^0)^\nu_1$	γ	$l \leq$
0	0	0	$\mathbb{A}^1$	d	m/d
2	1/2	1/2	$\mathbb{A}^1 \setminus \{0\}$	$d/2$	$m/d+1$

We use the following notation for Table 2. It is clear that $\#J \leq 2$ if $\chi_{\text{top}}((C^0)^\nu) \geq 0$. Denote by $(C^0)^\nu_1$ a connected component of $(C^0)^\nu$. Let $\gamma := \gcd(d, a_1, \cdots, a_l)$ be the number of connected components of $(C^0)^\nu$. The fact that γ is always an integer gives certain divisibility conditions on d. □

3.2. **Proofs.** To begin with, we will study the enumerative geometry problem of counting the intersections of a generic hypersurface with the generators of the projective cone $\tilde{\phi}(E)$ (see also [Zaĭ09, 1.3]).

A map $\alpha : \mathbb{P}^1 \to E$ is called a *ruling* if it parametrizes a fiber of the $\mathbb{P}^1$-bundle projection $\rho_E : E \to H$. We say that α *corresponds to* $x \in H$ if α parametrizes $\rho_E^{-1}(x)$.

Lemma 14. *With the above notation, the following properties hold for a general hypersurface $T \in |\mathcal{O}_{\mathbb{P}^N}(m)|$:*

(1) $\phi(v_0)$ *is not contained in* T.
(2) *For any ruling $\alpha : \mathbb{P}^1 \to E$, $(\tilde{\phi} \circ \alpha)^*(T)$ is supported at $> (m-n)$ points.*

Proof. Define $|\mathcal{O}_{\mathbb{P}^N}(m)|^\circ := \{T \in |\mathcal{O}_{\mathbb{P}^N}(m)| : \phi(v_0)$ is not contained in $T\}$. Denote by $\alpha_x : \mathbb{P}^1 \to E$ the ruling of E corresponding to $x \in H$.

We define an incidence variety Z as

$$Z := \{(T, x) \in |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ \times H : (\tilde{\phi} \circ \alpha_x)^*(T) \text{ is supported at } \leq (m-n) \text{ points}\}.$$

Denote the two projections from Z by pr_1 and pr_2. Let Z_x be the fiber of $pr_2 : Z \to H$ over x. Then

$$Z_x \cong \{T \in |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ : (\tilde{\phi} \circ \alpha_x)^*(T) \text{ is supported at } \leq (m-n) \text{ points}\}.$$

By Proposition 8(3), $(\tilde{\phi} \circ \alpha_x)$ parametrizes a line in $\mathbb{P}^N$. Therefore, the rational map

$$(\tilde{\phi} \circ \alpha_x)^* : |\mathcal{O}_{\mathbb{P}^N}(m)| \dashrightarrow |\mathcal{O}_{\mathbb{P}^1}(m)|$$

is a projection between projective spaces. In particular, $(\tilde{\phi} \circ \alpha_x)^* : |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ \to |\mathcal{O}_{\mathbb{P}^1}(m)|$ is a flat morphism for any $x \in H$.

Let $W_k := \{D \in |\mathcal{O}_{\mathbb{P}^1}(m)| : D \text{ is supported at } \leq k \text{ points}\}$. Then $\dim W_k = k$. It is clear that $Z_x = ((\tilde{\phi} \circ \alpha_x)^*)^{-1}(W_{m-n})$, so

$$\begin{aligned}\dim Z_x &= \dim((\tilde{\phi} \circ \alpha_x)^*)^{-1}(W_{m-n}) \\ &= \dim |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ - \dim |\mathcal{O}_{\mathbb{P}^1}(m)| + \dim W_{m-n} \\ &= \dim |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ - n.\end{aligned}$$

Hence $\dim Z = \dim Z_x + \dim H = \dim |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ - 1$ for a general choice of x, which implies that $\dim pr_1(Z) \leq \dim |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ - 1$. Thus the map pr_1 is not surjective, which means that a general hypersurface $T \in |\mathcal{O}_{\mathbb{P}^N}(m)|^\circ$ will satisfy property (2). The lemma then follows automatically. □

Since property $(*)$ holds for a general hypersurface $T \in |\mathcal{O}_{\mathbb{P}^N}(m)|$, Lemma 14 implies that the following property also holds for general T:

$(**)$ $\phi^*(T)$ is smooth, does not contain v_0, and intersects X_0 transversally. Besides, $(\tilde{\phi}\circ\alpha_x)^*(T)$ is supported at $\geq (m-n+1)$ points for any $x \in H$.

From now on we always fix a general hypersurface $T \in |\mathcal{O}_{\mathbb{P}^N}(m)|$. Then we may assume that T satisfies $(**)$.

The following theorem is the main tool to prove Theorem 2.

Theorem 15 ([Kob98, 3.11.1])**.** *Let $\pi : \mathcal{X} \to R$ be a proper family of connected complex analytic spaces. If there is a point $r_0 \in R$ such that the fiber X_{r_0} is Brody hyperbolic, then there exists an open neighborhood (in the Euclidean topology) $U \subset R$ of r_0 such that for each $r \in U$, the fiber X_r is Brody hyperbolic.*

We will prove the following theorem, a stronger result that implies Theorem 2.

Theorem 16. *Let X be a smooth projective variety with $\dim X = n$. Let L be a globally generated ample line bundle on X. Suppose that there exists a smooth hypersurface $H \in |L|$ that is Brody hyperbolic. Let $m, d \geq 2$ be positive integers such that m is a multiple of d. For a generic small deformation S of $mH \in |L^{\otimes m}|$, let Y be the degree d cyclic cover of X branched along S. Then Y is Brody hyperbolic if one of the following holds:*

- *d is divisible by 2 or 3, and $m \geq d\lceil \frac{n+2}{d-1} \rceil$;*
- *d is relatively prime to 6, and $m \geq d\lceil \frac{n+1}{d-1} \rceil$.*

Proof. We first show that Y_0 is Brody hyperbolic.

Proposition 11 implies that the birational morphism $\psi_1|_{\overline{Y}_0} : \overline{Y}_0 \to Y_0$ induces an isomorphism between $\overline{Y}_0 \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d})$ and $Y_0 \setminus \{q_1, \cdots, q_d\}$. Therefore, it suffices to show that $\overline{Y}_0$ is Brody hyperbolic.

Define $p_{\overline{Y}_0} : \overline{Y}_0 \to H$ to be the composition map $\overline{Y}_0 \to E \to H$. Since H is Brody hyperbolic, we only need to show that every fiber $p_{\overline{Y}_0}^{-1}(x)$ is Brody hyperbolic. It is clear that $p_{\overline{Y}_0}^{-1}(x)$ is the degree d cyclic cover of $\rho_E^{-1}(x)$ branched along $\tilde{\phi}^*(T)|_{\rho_E^{-1}(x)}$. Applying the pullback of a ruling α_x yields that $p_{\overline{Y}_0}^{-1}(x) \cong C_x$, where C_x is the degree d cyclic cover of $\mathbb{P}^1$ branched along $(\tilde{\phi}\circ\alpha_x)^*(T)$.

Let $l_x := \#\mathrm{Supp}\left((\tilde{\phi}\circ\alpha_x)^*(T)\right)$. Since T satisfies $(**)$, $l_x \geq m-n+1$ for any $x \in H$. If d is divisible by 2 or 3, then $m \geq d\lceil \frac{n+2}{d-1} \rceil \geq \frac{d}{d-1}(n+2)$. Hence $l_x \geq m-n+1 \geq \frac{m}{d}+3$. If d is relatively prime to 6, then $m \geq d\lceil \frac{n+1}{d-1} \rceil \geq \frac{d}{d-1}(n+1)$. Hence $l_x \geq m-n+1 \geq \frac{m}{d}+2$. Then Lemma 12 implies that C_x is Brody hyperbolic.

Summing up, we always have that $p_{\overline{Y}_0}^{-1}(x) \cong C_x$ is Brody hyperbolic for any $x \in H$. Therefore, Y_0 is Brody hyperbolic.

We may apply Theorem 15 to the family $\mathcal{Y} \to \mathbb{A}^1$ with $r_0 = 0$. Thus Y_t is Brody hyperbolic for $|t|$ sufficiently small. By Proposition 11, Y_t is isomorphic to the degree d cyclic cover of X branched along S_t, where S_t is a generic small deformation of mH for $|t|$ sufficiently small. The theorem then follows. □

Next we give another deformation type theorem of hyperbolicity when the special fiber has multiple irreducible components. It will be used to prove Theorem 3. Note that some cases of Theorem 17 have already been used in [ZS05, Zaĭ09].

Theorem 17. *Let $\pi : \mathcal{X} \to R$ be a proper family of connected complex analytic spaces over a non-singular complex curve R. Let $r_0 \in R$ be a point. Denote the irreducible components of the fiber X_{r_0} by $X_{r_0,1}, \cdots, X_{r_0,k}$. Suppose these data satisfy the following properties:*

(1) *$X_{r_0,i}$ is a Cartier divisor on $\mathcal{X}$ for each $1 \le i \le k$.*
(2) *For any partition of indices $I \cup J = \{1, \cdots, k\}$, $\bigcap_{i\in I} X_{r_0,i} \setminus \bigcup_{j\in J} X_{r_0,j}$ is Brody hyperbolic.*

Then there exists an open neighborhood (in the Euclidean topology) $U \subset R$ of r_0 such that for each $r \in U \setminus \{r_0\}$, the fiber X_r is Brody hyperbolic.

Proof. Assume to the contrary that there exists a sequence of points $\{r_n\}$ converging to r_0 such that X_{r_n} is not Brody hyperbolic for each n. Then there is a complex line $h_n : \mathbb{C} \to X_{r_n}$. By taking a subsequence of $\{r_n\}$ if necessary, we may assume that $\{h_n\}$ converges to a complex line $h : \mathbb{C} \to X_{r_0}$. Then by applying the generalized Hurwitz theorem [Kob98, 3.6.11] to $(\mathcal{X}, X_{r_0})$, we have that

$$h(\mathbb{C}) \subset \bigcap_{i\in I} X_{r_0,i} \setminus \bigcup_{j\in J} X_{r_0,j},$$

where $I = \{i : h(0) \in X_{r_0,i}\}$ and $J = \{j : h(0) \notin X_{r_0,j}\}$. However, $\bigcap_{i\in I} X_{r_0,i} \setminus \bigcup_{j\in J} X_{r_0,j}$ is Brody hyperbolic, and we get a contradiction! □

The following theorem is a stronger result that implies Theorem 3.

Theorem 18. *With the notation of Theorem 16, assume in addition that $X \setminus H$ is Brody hyperbolic. Then Y is Brody hyperbolic if one of the following holds:*

- *d is even and $m \ge d\lceil \frac{n+1}{d-1} \rceil$;*
- *d is odd and $m \ge d\lceil \frac{n}{d-1} \rceil$.*

Proof. We first prove that $\overline{Y}_0 \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d})$ is Brody hyperbolic.

Consider the restriction of $p_{\overline{Y}_0} : \overline{Y}_0 \to H$ on the open subset $\overline{Y}_0 \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d})$. Since H is Brody hyperbolic, we only need to show that $p_{\overline{Y}_0}^{-1}(x) \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d})$ is Brody hyperbolic for any $x \in H$.

It is clear that $\overline{Y}_0$ is the degree d cyclic cover of E branched along $\tilde{\phi}^*(T)|_E$, and $\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d} = \tilde{\mu}^{-1}(\hat{X}_0)$. Therefore, $p_{\overline{Y}_0}^{-1}(x) \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d})$ is the preimage of $\rho_E^{-1}(x) \setminus \hat{X}_0$ under the covering map $p_{\overline{Y}_0}^{-1}(x) \to \rho_E^{-1}(x)$. Applying the pullback of a ruling α_x with $\{\alpha_x(\infty)\} = \rho_E^{-1}(x) \cap \hat{X}_0$ yields that $p_{\overline{Y}_0}^{-1}(x) \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d}) \cong C_x^0$ (with the notation of Lemma 13 and proof 3.2, because $(\tilde{\phi} \circ \alpha_x)^*(T)$ is supported away from ∞).

Since T satisfies $(**)$, $l_x \ge m - n + 1$ for any $x \in H$. If d is even, then $m \ge d\lceil \frac{n+1}{d-1} \rceil \ge \frac{d}{d-1}(n+1)$. Hence $l_x \ge m - n + 1 \ge \frac{m}{d} + 2$. If d is odd, then $m \ge d\lceil \frac{n}{d-1} \rceil \ge \frac{d}{d-1} \cdot n$. Hence $l_x \ge m - n + 1 \ge \frac{m}{d} + 1$. Then Lemma 13 implies that C_x^0 is Brody hyperbolic.

Summing up, we always have that $p_{\overline{Y}_0}^{-1}(x) \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d}) \cong C_x^0$ is Brody hyperbolic for any $x \in H$. Therefore, $\overline{Y}_0 \setminus (\hat{X}_{0,1} \cup \cdots \cup \hat{X}_{0,d})$ is Brody hyperbolic.

On the other hand, $\hat{X}_{0,i} \setminus \overline{Y}_0$ is isomorphic to $\hat{X}_0 \setminus E$, which is again isomorphic to $X \setminus H$. Hence $\hat{X}_{0,i} \setminus \overline{Y}_0$ is Brody hyperbolic for any $1 \le i \le d$. We may apply Theorem 17 to the family $\widetilde{\mathcal{Y}} \to \mathbb{A}^1$ with $r_0 = 0$. Thus Y_t is Brody hyperbolic for

$|t|$ sufficiently small. By Proposition 11, Y_t is isomorphic to the degree d cyclic cover of X branched along S_t, where S_t is a generic small deformation of mH for $|t|$ sufficiently small. The theorem then follows. □

Finally, we apply our arguments to hyperbolicity of the complements. We first state a theorem which relates various notions of hyperbolicity of complements.

Theorem 19 (Green [Gre77]; Howard). *Let X be a compact complex space. Let S be an effective Cartier divisor on X. If both S and $X \setminus S$ are Brody hyperbolic, then $X \setminus S$ is complete hyperbolic and hyperbolically embedded in X.*

Next we give a deformation type theorem of hyperbolicity of complements of Cartier divisors, which will be used to prove Theorem 6.

Theorem 20. *Let $\pi : \mathcal{X} \to R$ be a proper family of connected complex analytic spaces. Let $\mathcal{S}$ be an effective Cartier divisor on $\mathcal{X}$. Assume that there is a point $r_0 \in R$ satisfying the following properties:*

(1) *Both S_{r_0} and $X_{r_0} \setminus S_{r_0}$ are Brody hyperbolic.*
(2) *$\mathcal{S}$ does not contain any irreducible component of X_{r_0}.*

Then there exists an open neighborhood (in the Euclidean topology) $U \subset R$ of r_0 such that for each $r \in U$, both S_r and $X_r \setminus S_r$ are Brody hyperbolic.

Proof. Let $U_r := X_r \setminus S_r$. Since S_{r_0} is Brody hyperbolic, Theorem 15 implies that S_r is Brody hyperbolic for r sufficiently close to r_0. Therefore, it suffices to show that U_r is Brody hyperbolic for r in a small neighborhood of r_0.

Assume to the contrary that there exists a sequence of points $\{r_n\}$ converging to r_0 such that U_{r_n} is not Brody hyperbolic for each n. Hence U_{r_n} is not hyperbolically embedded in X_{r_n}. Then there is a limit complex line $h_n : \mathbb{C} \to X_{r_n}$ coming from U_n. By taking a subsequence of $\{r_n\}$ if necessary, we may assume that $\{h_n\}$ converges to a complex line $h : \mathbb{C} \to X_{r_0}$. Then by applying the generalized Hurwitz theorem [Kob98, 3.6.11] to $(\mathcal{X}, \mathcal{S})$, $h(\mathbb{C})$ is contained in either S_{r_0} or U_{r_0}. However, both S_{r_0} and U_{r_0} are Brody hyperbolic, and we get a contradiction! □

Proof of Theorem 6. We first prove that $X_0 \setminus \phi^*(T)$ is Brody hyperbolic.

It is clear that the birational morphism $\psi|_E : E \to X_0$ induces an isomorphism between $E \setminus \hat{X}_0$ and $X_0 \setminus \{v_0\}$. Therefore, it suffices to show that $E \setminus \tilde{\phi}^*(T)$ is Brody hyperbolic.

Consider the restriction of $\rho_E : E \to H$ on the open subset $E \setminus \tilde{\phi}^*(T)$. Since H is Brody hyperbolic, we only need to show that $\rho_E^{-1}(x) \setminus \tilde{\phi}^*(T)$ is Brody hyperbolic. Applying the pullback of a ruling α_x yields that $\rho_E^{-1}(x) \setminus \tilde{\phi}^*(T) \cong \mathbb{P}^1 \setminus (\tilde{\phi} \circ \alpha_x)^*(T)$.

Since T satisfies $(**)$, $l_x \geq m - n + 1$ for any $x \in H$. Then the assumption $m \geq n + 2$ implies that $l_x \geq 3$. Hence $\mathbb{P}^1 \setminus (\tilde{\phi} \circ \alpha_x)^*(T)$ is Brody hyperbolic, which means that $\rho_E^{-1}(x) \setminus \tilde{\phi}^*(T)$ is also Brody hyperbolic. Consequently, $X_0 \setminus \phi^*(T)$ is Brody hyperbolic.

On the other hand, since $\tilde{\phi}^*(T)$ is disjoint from $\hat{X}_0$, no fiber of ρ_E is contained in $\tilde{\phi}^*(T)|_E$. Hence the restriction of ρ_E on $\tilde{\phi}^*(T)|_E$ is a finite morphism onto H. Then H being Brody hyperbolic implies that $\tilde{\phi}^*(T)|_E$ is Brody hyperbolic. Recall that $\phi^*(T)|_{X_0} \cong \tilde{\phi}^*(T)|_E$, so $\phi^*(T)|_{X_0}$ is also Brody hyperbolic.

So far we have shown that both $\phi^*(T)|_{X_0}$ and $X_0 \setminus \phi^*(T)$ are Brody hyperbolic. Applying Theorem 20 to the family $\mathcal{X} \to \mathbb{A}^1$ with $\mathcal{S} = \phi^*(T)$ and $r_0 = 0$ yields that both $\phi^*(T)|_{X_t}$ and $X_t \setminus \phi^*(T)$ are Brody hyperbolic for $|t|$ sufficiently small. Since

$(X_t, \phi^*(T)|_{X_t})$ is isomorphic to (X, S_t), both S_t and $X \setminus S_t$ are Brody hyperbolic for $|t|$ sufficiently small. By Proposition 11, S_t is a generic small deformation of mH for $|t|$ sufficiently small. The first statement of the theorem then follows. The last statement follows directly from Theorem 19. ☐

4. Applications and examples

4.1. Hypersurfaces in $\mathbb{P}^n$.

Let us introduce some notation to describe the moduli spaces of hypersurfaces in $\mathbb{P}^n$ with various hyperbolic conditions.

- Let $\mathbb{P}_{n,\delta}$ be the projective space of dimension $\binom{n+\delta}{n} - 1$ whose points parametrize hypersurfaces of degree δ in $\mathbb{P}^n$.
- Let $H_{n,\delta} \subset \mathbb{P}_{n,\delta}$ be the subset corresponding to Brody hyperbolic hypersurfaces.
- Denote by $HE_{n,\delta}$ the subset of $\mathbb{P}_{n,\delta}$ consisting of the hypersurfaces of degree δ in $\mathbb{P}^n$ with hyperbolically embedded complements.

The following theorem (which essentially implies Theorem 4) produces Brody hyperbolic hypersurfaces that are cyclic covers of $\mathbb{P}^n$.

Theorem 21. *Let k, δ be positive integers such that δ is a multiple of k. Suppose one of the following conditions holds.*

1. *$H_{n,k}$ is non-empty and $\delta \geq (n+3)k$.*
2. *$HE_{n,k} \cap H_{n,k}$ is non-empty and $\delta \geq (n+2)k$.*

Then there exists a Brody hyperbolic smooth hypersurface W of degree δ in $\mathbb{P}^{n+1}$, such that W is a cyclic cover of $\mathbb{P}^n$ under some linear projection.

Proof.

(1) Choose a smooth hypersurface $D \in H_{n,k}$. Let $d := \delta/k$. For any $d \geq n+3$, applying Theorem 2 to $(X, L, H, d, m) := (\mathbb{P}^n, \mathcal{O}(k), D, d, d)$ yields that there exists a degree d cyclic cover Y of $\mathbb{P}^n$ branched along a smooth hypersurface S of degree δ such that Y is Brody hyperbolic. Let W be the degree δ cyclic cover of $\mathbb{P}^n$ branched along S. Then $W \to Y$ is a finite surjective morphism. Thus Y being Brody hyperbolic implies that W is also Brody hyperbolic.

(2) Choose a smooth hypersurface $D \in HE_{n,k} \cap H_{n,k}$. Let $d := \delta/k$. For any $d \geq n+2$, apply Theorem 3 to $(X, L, H, d, m) := (\mathbb{P}^n, \mathcal{O}(k), D, d, d)$. The rest of the proof is the same as (1). ☐

Next, we give a new proof to [Zaĭ09, 1.1] using Mori's degeneration method.

Theorem 22 (Zaidenberg [Zaĭ09])**.** *Let $X = (F(z_0, \cdots, z_n) = 0)$ be a Brody hyperbolic hypersurface of degree k in $\mathbb{P}^n$ $(n \geq 2)$. We may realize $\mathbb{P}^n$ as the hyperplane $(z_{n+1} = 0)$ in $\mathbb{P}^{n+1}$. Denote by $C(X) := (F(z_0, \cdots, z_n) = 0) \subset \mathbb{P}^{n+1}$ the projective cone over X. Let $dC(X) := (F^d = 0) \subset \mathbb{P}^{n+1}$ be the d-th thickening of $C(X)$ where $d \geq 2$ is a positive integer. Then a generic small deformation of $dC(X)$ (in the sense of* [Zaĭ09]*) is Brody hyperbolic. In particular, $H_{n,k} \neq \emptyset$ implies that $H_{n+1,dk} \neq \emptyset$ for $d \geq 2$.*

Proof. Firstly, let us recall Mori's degeneration method from [Mor75].

Let $\mathbb{P}(1^{n+2}, k)$ be a weighted projective space of dimension $(n+2)$ with coordinates $z_0, \cdots, z_{n+1}, w$. Let G be a general homogeneous polynomial of degree dk in

$z_0, \cdots, z_{n+1}$. Consider the family of complete intersections

$$Y_t := \big(tw - F(z_0, \cdots, z_n) = w^d - G(z_0, \cdots, z_{n+1}) = 0\big) \subset \mathbb{P}(1^{n+2}, k).$$

For $t \neq 0$ we can eliminate w to obtain a degree dk smooth hypersurface

$$Y_t \cong \big(F^d(z_0, \cdots, z_n) = t^d G(z_0, \cdots, z_{n+1})\big) \subset \mathbb{P}^{n+1}.$$

For $t = 0$ we see that $\mathcal{O}_{Y_0}(1)$ is not very ample but realizes Y_0 as a degree d cyclic cover

$$h : Y_0 \to C(X) = (F(z_0, \cdots, z_n) = 0) \subset \mathbb{P}^{n+1}$$

of $C(X)$ branched along $(F = G = 0)$.

Next, we will show that Y_0 is Brody hyperbolic. Let us fix a general homogeneous polynomial G from now on. By Lemma 14, a general hypersurface $T := (G = 0)$ satisfies that T does not contain the vertex $[0, \cdots, 0, 1]$ of $C(X)$ and that

$$\#(T \cap \ell) \geq dk - n$$

for any generator ℓ of $C(X)$. Applying Lemma 12 to $(d, m, l) := (d, dk, \#(T \cap \ell))$ yields that $h^{-1}(\ell)$ is Brody hyperbolic if $\#(T \cap \ell) \geq k + 3$. Since X is a Brody hyperbolic hypersurface of degree d in $\mathbb{P}^n$ with $n \geq 2$, it is clear that $k \geq n + 3$. Hence

$$\#(T \cap \ell) \geq dk - n \geq k + 3.$$

Thus $h^{-1}(\ell)$ is Brody hyperbolic, which together with X being Brody hyperbolic implies that Y_0 is Brody hyperbolic.

Finally, Theorem 15 implies that Y_t is Brody hyperbolic for $|t|$ sufficiently small. Hence we prove the theorem. □

The following theorem is an improvement of [Zai93, p. 147, Corollary of Theorem II.2], where they assumed $\delta \geq (2n + 1)k$ (without assuming δ being a multiple of k).

Theorem 23. *If $H_{n,k}$ is non-empty, then $HE_{n,\delta} \cap H_{n,\delta}$ is a non-empty open subset of $\mathbb{P}_{n,\delta}$ (in the Euclidean topology) for any $\delta \geq (n + 2)k$ with δ being a multiple of k.*

Proof. Let $m = \delta/k \geq n + 2$. Applying Theorem 6 to $X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(k)$, $H \in H_{n,k}$ yields that both S and $\mathbb{P}^n \setminus S$ are Brody hyperbolic for a generic small deformation S of $mH \in |\mathcal{O}_{\mathbb{P}^n}(\delta)|$. Theorem 19 implies that $S \in HE_{n,\delta} \cap H_{n,\delta}$ for any generic small deformation S of mH; hence $HE_{n,\delta} \cap H_{n,\delta}$ is non-empty. The openness of $HE_{n,\delta} \cap H_{n,\delta}$ follows from Theorems 19 and 20. □

4.2. **Surfaces.**

The following theorem provides new examples of hyperbolic surfaces in $\mathbb{P}^3$ of minimal degree 15.

Theorem 24. *Let $\delta = d \cdot k$ be the product of two positive integers $d \geq 3$, $k \geq 5$. Then there exists a smooth Brody hyperbolic surface X_δ of degree δ in $\mathbb{P}^3$ that is a cyclic cover of $\mathbb{P}^2$ under some linear projection.*

Proof. According to [Zau88], for any $k \geq 5$ there exists a smooth curve D in $\mathbb{P}^2$ of degree k with $\mathbb{P}^2 \setminus D$ being Brody hyperbolic. Then the proof is along the same lines as for Theorem 21(2), except that we apply Theorem 18 instead of Theorem 3 when $d = 3$. □

Next, we prove Theorem 5.

Theorem 25 (=Theorem 5). *Let $l \geq 3$, $k \geq 5$ be two positive integers. Let D be a smooth plane curve of degree k such that $\mathbb{P}^2 \setminus D$ is Brody hyperbolic.* (*The existence of such D was shown by Zaidenberg in* [Zau88].) *Let S be a generic small deformation of $2lD$. Then the double cover of $\mathbb{P}^2$ branched along S is Brody hyperbolic.*

Proof. Apply Theorem 3 to $(X, L, H, d, m) := (\mathbb{P}^2, \mathcal{O}(k), D, 2, 2l)$. □

Example 26. Let (X_0, L_0) be a primitively polarized K3 surface of degree $2l$ for $l \in \mathbb{Z}_{>0}$. For any $m \geq 4$, denote $M_0 := L_0^m$. Pick a general member $H \in |L_0|$; then H is smooth and $g(H) \geq 2$. Let S_0 be a generic small deformation of mH that is smooth. Then Theorem 6 implies that S_0 and $X_0 \setminus S_0$ are both Brody hyperbolic.

There exists a deformation $(\mathcal{X}, \mathcal{M})$ of (X_0, M_0) over Δ such that (X_t, M_t) is a primitively polarized K3 surface of degree $2lm^2$ for $t \in \Delta \setminus \{0\}$. It is clear that $h^0(X_t, M_t)$ does not depend on the choice of t in Δ. Hence Grauert's theorem implies that $\pi_*\mathcal{M}$ is a locally free sheaf on Δ, where $\pi : \mathcal{X} \to \Delta$ is the projection map. In other words, $\{H^0(X_t, M_t)\}_{t\in\Delta}$ forms a holomorphic vector bundle over Δ. Now we may deform S_0 to a family of divisors $S_t \in |M_t|$ for $|t|$ sufficiently small. By choosing a generic deformation, we may assume that S_t is smooth for $|t|$ sufficiently small. Hence Theorem 20 implies that both S_t and $X_t \setminus S_t$ are Brody hyperbolic for $|t|$ sufficiently small. Moreover, $X_t \setminus S_t$ is completely hyperbolic and hyperbolically embedded in X_t by Theorem 19.

As a consequence, for any $l \geq 1$ and $m \geq 4$ there exists a primitively polarized K3 surface (X, M) of degree $2lm^2$ and a smooth curve $S \in |M|$, such that $X \setminus S$ is completely hyperbolic and hyperbolically embedded in X. Notice that the minimal degree of (X, M) is 32.

Acknowledgments

I would like to thank my advisor, János Kollár, for his constant support, encouragement, and many inspiring conversations. I wish to thank Chi Li, Charles Stibitz, Amos Turchet, and Anibal Velozo for many helpful discussions, and Gang Tian, Xiaowei Wang, and Chenyang Xu for their interest and encouragement. I also wish to thank Mikhail Zaidenberg for his helpful comments through emails.

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