

## A SPECIAL CAYLEY OCTAD

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ABSTRACT. We prove that a special blowup of projective space of dimension 3 in characteristic 2 is a Mori dream space. This confirms a prediction of the Morrison–Kawamata cone conjecture.

A *Cayley octad* is a set of 8 points in  $\mathbf{P}^3$  which are the base locus of a net of quadrics. Blowing up the points of the octad gives a morphism to  $\mathbf{P}^2$  defined by the net; the fibres of this morphism are intersections of two quadrics in the net, hence curves of genus 1. The generic fibre therefore has a group structure, and the action of this group on itself extends to a birational action on the whole variety. In particular, if the generic fibre has a large group of rational points, the birational automorphism group, and hence the birational geometry, of the variety must be complicated. It is natural to ask whether the converse is true: if the generic fibre has only a small group of rational points, is the birational geometry of the variety correspondingly simple?

In this paper we study a special Cayley octad with the property that the generic fibre has finitely many rational points. In Section 1 we find that such an octad only exists in characteristic 2 and is unique up to projective transformations. Our main results then show that the simplicity of the generic fibre is indeed reflected in the simplicity of the birational geometry of our blowup. In Section 2 we show that the cones of nef and movable divisors are rational polyhedral, as predicted by the Morrison–Kawamata conjecture. Finally, in Section 3 we prove that our blowup has the “best possible” birational geometric properties: it is a Mori dream space.

### 1. NETS AND FIBRATIONS

Throughout the paper we will consider a subset  $\{p_1, \dots, p_8\} \subset \mathbf{P}^3$  of 8 distinct points which are the intersection  $Q_1 \cap Q_2 \cap Q_3$  of three quadrics. In particular, this implies that the intersections are transverse at each  $p_i$ . The *net* spanned by the  $Q_i$  is the 2-dimensional linear system  $N = \{\sum_i \lambda_i Q_i \mid \lambda_i \in k\}$ .

We start by reproducing some lemmas from Totaro’s paper [Tot]. Since the proofs are short and elementary, we include them.

**Lemma 1.1.** *No 3 of the points  $\{p_1, \dots, p_8\}$  are collinear, and no 5 are coplanar.*

*Proof.* Suppose  $L$  is a line containing 3 of the points. Then for any quadric  $Q \in N$ , we have  $|Q \cap L| \geq 3$ , so by Bézout  $L$  must be contained in  $Q$ . Since this is true for an arbitrary  $Q \in N$ , this contradicts the hypothesis that  $Q_1 \cap Q_2 \cap Q_3 = \{p_1, \dots, p_8\}$ .

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Now let  $\Pi$  be a plane containing 5 of the points. By the previous paragraph, no 3 are collinear. Then any quadric  $Q \in N$  must intersect  $\Pi$  in the unique conic through the 5 points, again contradicting the hypothesis  $Q_1 \cap Q_2 \cap Q_3 = \{p_1, \dots, p_8\}$ .  $\square$

**Lemma 1.2.** *Four of the basepoints  $\{p_1, \dots, p_8\}$  are coplanar if and only if the other four are too.*

*Proof.* Suppose that 4 of the points lie in a plane  $\Pi$ . Since no 3 of the points lie on a line, there is a pencil  $L \simeq \mathbf{P}^1$  of conics in  $\Pi$  containing these 4 points. If no quadric in  $N$  contained  $\Pi$ , restriction to  $\Pi$  would give a morphism  $N \rightarrow L$ . But any such morphism must be constant, so any quadric  $Q \cap N$  must intersect  $\Pi$  in the same conic. Again this contradicts  $Q_1 \cap Q_2 \cap Q_3 = \{p_1, \dots, p_8\}$ . So there is a quadric  $Q$  containing  $\Pi$ , which must then be of the form  $Q = \Pi \cup \Pi'$ . Since neither plane can contain 5 basepoints, each must contain exactly 4.  $\square$

Now let  $X$  be the blowup of  $\mathbf{P}^3$  at the points  $\{p_1, \dots, p_8\}$ . Then  $\text{Pic}(X) \cong \mathbf{Z}^9$  is freely generated by  $H$ , the pullback of the hyperplane class, together with the exceptional divisors  $E_i$  for  $i = 1, \dots, 8$ .

If  $Q$  is a quadric in the net  $N$ , then its proper transform on  $X$  has class  $2H - \sum_i E_i = -\frac{1}{2}K_X$ . Since the quadrics in the net  $N$  intersect transversely at each  $p_i$ , their proper transforms on  $X$  form a basepoint-free linear system. So we get a morphism  $f : X \rightarrow \mathbf{P}^2$  given by divisors in the linear system  $-\frac{1}{2}K_X$ .

**Lemma 1.3.** *The generic fibre of  $f$  is a regular curve of genus 1.*

*Proof.* The generic fibre of a surjective morphism between nonsingular varieties is always regular.

Fibres of  $f$  are proper transforms on  $X$  of intersections of two distinct quadrics in the net  $N$ . Since the  $Q_i$  intersect transversely at the points  $p_i$ , these intersections are nonsingular at each  $p_i$ . So each such intersection is isomorphic to its proper transform on  $X$ , which is the corresponding fibre of  $f$ . By adjunction, the intersection of two quadrics in  $\mathbf{P}^3$  is a curve of genus 1. So every fibre of  $f$  over a closed point of  $\mathbf{P}^2$  has genus 1, which implies the same for the generic fibre.  $\square$

If  $k$  has characteristic 0, then we can replace “regular” by smooth in the statement of the lemma. However, this is false in positive characteristic, as we will see in our main example, which lives in characteristic 2. Fibrations of this kind, with generic fibre regular but not smooth and of arithmetic genus 1, are called *quasi-elliptic*.

From now on we write  $\eta$  for the generic point of  $\mathbf{P}^2$  and  $X_\eta$  for the generic fibre of  $f$ . Restriction of divisors gives a surjection  $\text{Pic}(X) \rightarrow \text{Pic}(X_\eta)$  whose kernel is spanned by prime divisors on  $X$  which do not surject onto  $\mathbf{P}^2$ . This shows that  $\text{Pic}(X_\eta)$  is a quotient of  $\text{Pic}(X)$ ; in particular, it is a finitely generated abelian group. Denote by  $\text{Pic}^0(X_\eta)$  the kernel of the degree homomorphism  $\text{Pic}(X_\eta) \rightarrow \mathbf{Z}$ . We call it the *Mordell–Weil group* of the octad  $\{p_1, \dots, p_8\}$  or of the net  $N$  or of the map  $f$ .

**Definition 1.4.** The *Mordell–Weil rank*  $\rho$  of the octad  $\{p_1, \dots, p_8\}$  (or of the net  $N$  or of the map  $f$ ) is defined as the rank of the Mordell–Weil group  $\text{Pic}^0(X_\eta)$ . An octad (or net or map) with Mordell–Weil rank 0 is called *extremal*.

Let us explain the significance of this condition for the birational geometry of  $X$ . The group  $\text{Pic}^0(X_\eta)$  acts on  $X_\eta$  by automorphisms, and this extends to an action

on  $X$  by birational automorphisms. By a standard fact of minimal model theory [KM, Theorem 3.52], since  $K_X$  has degree 0 on all curves contracted by  $f$ , these birational automorphisms are in fact isomorphisms in codimension 1. So we get an action of  $\text{Pic}^0(X_\eta)$  on the space  $N^1(X)$  of numerical classes of divisors, preserving the cones of effective and movable divisors. If we want to find examples of blowups  $X$  whose cones of divisors are finitely generated, we therefore need the fibration  $f$  to be extremal.

To find our example of an extremal fibration, we will use the following formula.

**Theorem 1.5** (Totaro [Tot]). *The Mordell–Weil rank of the octad  $\{p_1, \dots, p_8\}$  is given by*

$$\rho = 7 - \frac{a}{2},$$

where  $a$  is the number of planes in  $\mathbf{P}^3$  containing 4 of the points  $\{p_1, \dots, p_8\}$ .

In [PS] we used this formula to deduce the following theorem.

**Theorem 1.6.** *If the characteristic of  $k$  is not 2, there are no extremal Cayley octads in  $\mathbf{P}_k^3$ . If the characteristic of  $k$  is 2, up to projective equivalence there is a unique extremal Cayley octad  $O$  in  $\mathbf{P}_k^3$ , for example given by the  $\mathbf{F}_2$ -rational points of one standard affine patch  $\mathbf{A}_k^3 \subset \mathbf{P}_k^3$ .*

The first part of the following proof was suggested by Igor Dolgachev; this simplifies the proof given in [PS]. We reproduce a proof here since it gives us detailed information about the extremal octad that we need in later sections.

*Proof.* Let  $\{p_1, \dots, p_8\}$  be an extremal Cayley octad. The basic idea is to study this octad by projecting away from one of the points. First we need some basic facts about the combinatorics of such an octad.

Since the octad is extremal, Theorem 1.5 says that there are 14 planes each containing four points, so the average number of planes through a point is  $\frac{14 \cdot 4}{8} = 7$ . On the other hand, by Lemma 1.2 for each plane  $\Pi$  containing 4 of the points including  $p_i$ , there is a plane  $\Pi'$  containing the other 4 points (and not  $p_i$ ). So  $p_i$  can lie on at most 7 of the 14 planes. Together these statements imply that each point  $p_i$  lies on exactly 7 of the 14 planes.

Each of the 14 planes contains  $\binom{4}{2} = 6$  pairs of the points  $\{p_1, \dots, p_8\}$ . Since there are  $\binom{8}{2} = 28$  such pairs in total, the average number of planes containing a given pair of points is  $\frac{14 \cdot 6}{28} = 3$ . On the other hand given any pair of points  $\{p_i, p_j\}$ , there is at most 1 plane containing these two and any other point  $p_k$ , since two distinct planes cannot share 3 noncollinear points. So there are at most 3 planes containing  $p_i$  and  $p_j$ . Together these statements imply that there are precisely 3 planes containing each pair  $\{p_i, p_j\}$ .

Now fix one of the points, say  $p_1$ . Projecting from  $p_1$  gives a rational map  $\mathbf{P}^3 \dashrightarrow \mathbf{P}^2$ . By the previous two paragraphs, the images of the other  $p_i$  and the planes containing  $p_1$  give 7 points and 7 lines in  $\mathbf{P}^2$  with each line passing through 3 points and each point lying on 3 lines. This configuration is the Fano plane, which is realised in  $\mathbf{P}_k^2$  if and only if  $k$  has characteristic 2. This proves the first claim.

Next we turn to the uniqueness statement. First I claim that given any triple  $\{p_i, p_j, p_k\}$  of basepoints, the plane containing these points must contain a fourth basepoint  $p_l$ . This follows from the argument two paragraphs back: we know there are 3 planes containing  $\{p_i, p_j\}$  and 2 other basepoints, and no two of these planes

can share any of the other 6 basepoints. So each of the 6 points must appear on precisely one of the 3 planes.

Now take any 3 of the points, and label them  $\{p_1, p_2, p_3\}$ . The plane spanned by these contains a fourth basepoint, which we call  $p_4$ . Let  $p_5$  be any other basepoint; by Lemma 1.1, it does not lie on the plane containing  $p_1, p_2, p_3, p_4$ . Similarly, let  $p_6$  denote the fourth basepoint in the plane spanned by  $\{p_1, p_2, p_5\}$  and let  $p_7$  denote the fourth basepoint in the plane spanned by  $\{p_1, p_3, p_5\}$ .

With this labelling, the facts that no two planes can share 3 basepoints and that 4 of the basepoints are coplanar if and only if the other 4 are too now completely determine the configuration of 14 coplanar quadruples of basepoints. Writing  $i$  in place of  $p_i$ , the quadruples are:

$$\begin{aligned} &\{1, 2, 3, 4\} \quad \{1, 2, 5, 6\} \quad \{1, 2, 7, 8\} \quad \{1, 3, 5, 7\} \quad \{1, 3, 6, 8\} \quad \{1, 4, 5, 8\} \\ &\{1, 4, 6, 7\} \quad \{5, 6, 7, 8\} \quad \{3, 4, 7, 8\} \quad \{3, 4, 5, 6\} \quad \{2, 4, 6, 8\} \quad \{2, 4, 5, 7\} \\ &\{2, 3, 6, 7\} \quad \{2, 3, 5, 8\} \end{aligned}$$

So far we have determined the combinatorics of an extremal fibration; now we turn to the geometry. Denote by  $X, Y, Z, W$  the homogeneous coordinates on  $\mathbf{P}^3$ . By projective transformations we can put any 4 noncoplanar points at the 4 coordinate points of  $\mathbf{P}^3$ , so let us declare that we have

$$p_1 = [1, 0, 0, 0], \quad p_2 = [0, 1, 0, 0], \quad p_3 = [0, 0, 1, 0], \quad p_5 = [0, 0, 0, 1].$$

By changing coordinates in the planes  $\{Y = 0\}$ ,  $\{Z = 0\}$ , and  $\{W = 0\}$  we can also move the points  $p_4, p_6, p_7$  while keeping the points above fixed. In this way we can further arrange to have

$$p_4 = [1, 1, 1, 0], \quad p_6 = [1, 1, 0, 1], \quad p_7 = [1, 0, 1, 1].$$

Now let  $p_8 = [a, b, c, d]$ . Since  $p_8$  lies in the plane spanned by  $p_2, p_3$ , and  $p_5$ , we get  $a = 0$ . Now coplanarity of the points  $p_1, p_2, p_7, p_8$  means the following matrix must have determinant zero:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 1 & d \end{pmatrix}$$

This yields  $c = d$ . A similar calculation with the quadruple  $\{p_5, p_6, p_7, p_8\}$  yields  $b = c$ , so we conclude that

$$p_8 = [0, 1, 1, 1].$$

This proves uniqueness of the extremal octad  $O$  up to projective equivalence. Finally, to prove the last claim that the octad we found is projectively equivalent to the  $\mathbf{F}_2$ -points of an affine patch  $\mathbf{A}^3 \subset \mathbf{P}^3$ , observe that the hyperplane  $\{X + Y + Z + W = 0\}$  does not pass through any of the points  $p_1, \dots, p_8$ .  $\square$

*Remark 1.7.* The explicit description of our octad obtained in the proof gives us several useful pieces of information. By changing coordinates in  $\mathbf{P}^3$  we can take

our octad to the set of  $\mathbf{F}_2$ -points of the affine patch  $\{W \neq 0\}$ . In these coordinates there is a particularly convenient spanning set of quadrics for our net, namely

$$Q_1 = X(X - W), \quad Q_2 = Y(Y - W), \quad Q_3 = Z(Z - W).$$

One can easily check that every quadric in the net is singular, so this gives an example where Bertini’s theorem fails. Similarly, the intersection of any two distinct quadrics is a cuspidal curve; this justifies our earlier claim that the map  $f : X \rightarrow \mathbf{P}^2$  is a quasi-elliptic fibration.

Finally, although we will not use it again, let us mention that one can use the combinatorial information in the proof to calculate the Mordell–Weil group: we find  $\text{Pic}^0(X_\eta) \cong (\mathbf{Z}/2\mathbf{Z})^3$ .

To conclude this section, we record some consequences of Theorem 1.6 for use in later sections. The first one follows immediately from linear algebra over  $\mathbf{F}_2$ .

**Corollary 1.8.** *There is an action of the group  $\text{Aff}(3, 2)$  of affine transformations of  $\mathbf{F}_2^3$  on  $\mathbf{P}^3$  which preserves the set  $\{p_1, \dots, p_8\}$  and is transitive on points, pairs, and triples in that set. This action lifts to an action of  $\text{Aff}(3, 2)$  on  $X$ .  $\square$*

**Corollary 1.9.** *Any subset of 5 points in  $\{p_1, \dots, p_8\}$  contains a coplanar quadruple.*

*Proof.* Given 5 points, let  $\{p_i, p_j, p_k\}$  be the complementary set of 3 points. As explained in the proof of the theorem, this set is contained in a coplanar quadruple  $\{p_i, p_j, p_k, p_l\}$ . By Lemma 1.2, the complement of this quadruple is then another coplanar quadruple, contained in our original set of 5 points.  $\square$

Finally we say something about reducible fibres of  $f$ . Recall that the fibres of  $f$  are isomorphic to intersections of two quadrics in  $\mathbf{P}^3$ , so a reducible fibre must contain either a line or a conic. We will need to understand the first kind of component.

**Corollary 1.10.** *A line in  $\mathbf{P}^3$  is the proper transform of a component of a reducible fibre of  $f$  if and only if it is the line through two of the points  $\{p_1, \dots, p_8\}$ .*

*Let  $c_{ij}$  denote the proper transform on  $X$  of the line through  $p_i$  and  $p_j$ . Then  $c_{ij}$  and  $c_{kl}$  intersect on  $X$  if and only if  $\{i, j, k, l\}$  is a coplanar quadruple.*

*Proof.* For the first statement, the morphism  $f : X \rightarrow \mathbf{P}^2$  is given by the linear system of proper transforms of quadrics in the net, so the proper transform of a line  $L$  is contained in a fibre of  $f$  if and only if it is disjoint from the proper transform of a general quadric  $Q$  in the net. In  $\mathbf{P}^3$  the line and the quadric intersect transversely in two points, so blowing up makes them disjoint if and only if the two points are among the points  $\{p_1, \dots, p_8\}$ .

For the second statement, first note that if  $\{i, j, k, l\}$  is a coplanar quadruple, then the two lines corresponding to  $c_{ij}$  and  $c_{kl}$  meet in a point in the plane. Since the first line contains  $p_i$  and  $p_j$  and the second contains  $p_k$  and  $p_l$ , the intersection point is not a basepoint of the net. So after blowing up, the lines still intersect. Conversely suppose  $c_{ij}$  and  $c_{kl}$  intersect on  $X$ . Then their proper transforms on  $\mathbf{P}^3$  are a pair of intersecting lines, spanning a plane. If the lines intersect at a nonbasepoint of the net, then  $\{i, j\}$  and  $\{k, l\}$  are distinct indices, so  $\{i, j, k, l\}$  is a coplanar quadruple. If the lines intersect at a basepoint, then  $i = k$ , say. But then we have two distinct lines in  $\mathbf{P}^3$  through  $p_i$ ; after blowing up they become disjoint on  $X$ , contrary to assumption.  $\square$

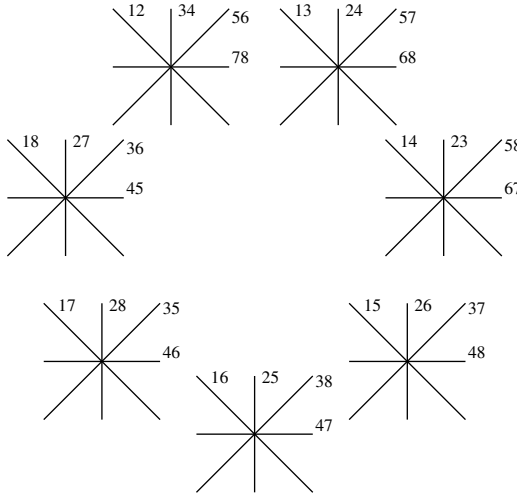


FIGURE 1. Configuration of lines in fibres. The curve  $c_{ij}$  is labelled  $ij$ .

For later reference the configuration of the curves  $c_{ij}$  on  $X$  is shown in Figure 1.

2. NEF AND MOVABLE CONE

From now on we fix  $k = \overline{\mathbf{F}}_2$ . We write  $O$  to denote the unique extremal Cayley octad  $\{p_1, \dots, p_8\}$  in  $\mathbf{P}_k^3$  found in the previous section and  $X$  to denote the blowup of  $\mathbf{P}_k^3$  at this set of points. In this section we prove that the cones of nef and movable divisors on  $X$  are finitely generated polyhedral cones.

This result is interesting because it gives evidence for a version of the Morrison–Kawamata cone conjecture [Mo, Ka]. The most general form of this conjecture, first stated in [Tot], says that if  $Y$  is a projective variety with an effective divisor  $\Delta$  such that  $(Y, \Delta)$  is a klt pair and  $K_Y + \Delta$  is numerically trivial, then the nef and movable cones of  $X$  should have finite polyhedral fundamental domains for the actions of the groups of automorphisms and pseudo-automorphisms respectively. Let us check that this conjecture applies to our variety:

**Lemma 2.1.** *There is a  $\mathbf{Q}$ -divisor  $\Delta$  on  $X$  such that  $(X, \Delta)$  is klt and  $K_X + \Delta$  is numerically trivial. Therefore the Morrison–Kawamata conjecture applies to  $X$ .*

Since  $-K_X$  is basepoint-free, in characteristic 0 this would follow immediately from Bertini’s theorem by taking an appropriate multiple of a smooth member of  $|-mK_X|$  for any  $m > 1$ . In our example we cannot invoke Bertini, so instead we argue directly.

*Proof.* As explained in the remarks after the proof of Theorem 1.6 we can assume that  $\{p_1, \dots, p_8\}$  are the  $\mathbf{F}_2$ -points of the affine patch  $\{W \neq 0\} \subset \mathbf{P}^3$ . Let

$$Q_1 = X(X - W), \quad Q_2 = Y(Y - W), \quad Q_3 = Z(Z - W),$$

each of which is a reducible quadric in our net, and let  $D$  be the proper transform of  $Q_1 \cup Q_2 \cup Q_3$  on  $X$ . Then  $D = 6H - 3 \sum_i E_i = -\frac{3}{2}K_X$ , so  $K_X + \frac{2}{3}D$  is numerically trivial. We will show that the pair  $(X, \frac{2}{3}D)$  is klt. The singular locus of  $D$  is a union of 3 lines  $L_1 \cup L_2 \cup L_3$ , where  $L_i$  is the intersection of the two components

of  $Q_i$ . These 3 lines lie in the plane  $\{W = 0\}$ , so the divisor  $D$  has nonnormal crossing singularities precisely at the 3 intersection points  $L_i \cap L_j$ , near each of which  $D$  is a union of 4 planes. Let  $\rho : Y \rightarrow X$  be the blowup of these 3 points, with exceptional divisors denoted  $F_i$ . The proper transform  $\tilde{D}$  of  $D$  on  $Y$  is a simple normal crossing and meets the exceptional divisors transversely, so this is a log resolution of  $(X, \frac{2}{3}D)$ . We calculate

$$\begin{aligned} \rho^*(K_X + \frac{2}{3}D) &= K_Y - 2 \sum_i F_i + \frac{2}{3} \left( \tilde{D} + 4 \sum_i F_i \right) \\ &= K_Y + \frac{2}{3} \left( \tilde{D} + \sum_i E_i \right). \end{aligned}$$

The term in parentheses is then a simple normal crossing divisor and appears with all coefficients less than 1, so the pair  $(X, \frac{2}{3}D)$  is klt, as required.  $\square$

We will show that the Morrison–Kawamata conjecture is true for  $X$  for the simplest reason: the nef and movable cones are themselves finite polyhedral cones. This is interesting since most of the nontrivial examples in which the Morrison–Kawamata conjecture has been verified are over a field of characteristic 0, but the conjecture itself is equally meaningful in all characteristics. Our example therefore gives one of the few pieces of evidence for this broader version of the conjecture. (The other main example I am aware of is Totaro’s proof of the conjecture for rational elliptic surfaces [Tot, Theorem 8.2].)

We use the following notation in the rest of the paper. We denote by  $N^1(X)$  the real vector space  $\text{Pic}(X) \otimes \mathbf{R}$ , and by  $N_1(X)$  the real vector space spanned by numerical equivalence classes of curves on  $X$ . We denote by  $H$  the pullback of the hyperplane class to  $X$ , and by  $E_i$  the exceptional divisor over the point  $p_i$ . Dually, we denote by  $h$  the class in  $N_1(X)$  of the proper transform of a line in  $\mathbf{P}^3$ , and by  $e_i$  the class of a line in  $E_i$ . The intersection pairing  $N^1(X) \times N_1(X) \rightarrow \mathbf{R}$  is given by

$$\begin{aligned} H \cdot h &= 1, \\ H \cdot e_i &= E_i \cdot h = 0 \quad (i = 1, \dots, 8), \\ E_i \cdot e_j &= -\delta_{ij} \quad (i, j = 1, \dots, 8). \end{aligned}$$

Note that the action of  $\text{Aff}(3, 2)$  described in Corollary 1.8 extends to an action on  $X$  and hence on  $N^1(X)$ ; any  $\sigma \in \text{Aff}(3, 2)$  fixes the class  $H$  and sends  $E_i$  to  $E_{\sigma(i)}$ .

**2.1. The nef cone.** First we prove the statement on the nef cone of  $X$ . Recall that a divisor class  $D$  on  $X$  is *nef* if  $D \cdot C \geq 0$  for every curve  $C \subset X$ ; the *nef cone*  $\text{Nef}(X)$  is the convex cone in  $N^1(X)$  spanned by the classes of nef divisors. A class  $D$  is *semi-ample* if  $mD$  is basepoint-free for some natural number  $m$ . Any semi-ample class is evidently nef.

**Theorem 2.2.** *Let  $X$  be as above. Then  $\text{Nef}(X)$  is spanned by the semi-ample divisors*

- $H$ ;
- $H - E_i$  ( $i = 1, \dots, 8$ );
- $2H - \sum_I E_i$  where  $I \subset \{1, \dots, 8\}$  is any subset with  $|I| \geq 3$ .

*Proof.* The dual of the nef cone is the *closed cone of curves*  $\text{Curv}(X) \subset N_1(X)$ , meaning the smallest closed cone containing all classes of effective curves on  $X$ . Any cone  $C$  spanned by a set of effective curves is a subcone of  $\text{Curv}(X)$ , and so its dual cone  $C^* := \{D \in N^1(X) \mid D \cdot c \geq 0 \text{ for all } c \in C\}$  is a cone containing  $\text{Nef}(X)$ .

Now let  $C$  be the cone spanned by the classes  $e_i$  ( $i = 1 \dots, 8$ ) of lines in exceptional divisors and the classes  $c_{ij} = h - e_i - e_j$  ( $i \neq j = 1, \dots, 8$ ) of proper transforms of lines through two basepoints. These classes are all effective, so  $C$  is indeed a subcone of  $\text{Curv}(X)$ . A computer calculation shows that the dual cone  $C^*$  is indeed spanned by the divisor classes listed in the statement of the theorem. So the listed classes span a cone containing  $\text{Nef}(X)$ ; to show it equals  $\text{Nef}(X)$ , it is enough to prove that each of these classes is semi-ample.

The class  $H$  is evidently basepoint-free, hence semi-ample. The class  $H - E_i$  is represented by the proper transform of a plane in  $\mathbf{P}^3$  through  $p_i$ ; since such planes have no common tangent directions at  $p_i$ , their proper transforms have no common points in the exceptional divisor  $E_i$ , so these classes are basepoint-free also.

Now we turn to classes of the form  $D = 2H - \sum_I E_i$ . Given such a class, let  $\pi : X \rightarrow Y$  be the blow-down of the set of divisors  $\{E_i \mid i \notin I\}$ . Then  $D = \pi^*D_Y$ , where  $D_Y$  is the linear system of proper transforms on  $Y$  of quadrics through the points  $\{p_j \mid j \in I\}$ . If  $|I| \leq 6$ , the class  $D_Y$  is basepoint-free, as one can see for example by considering pairs of planes through the points of  $I$ . If  $|I| = 7$ , the base locus of  $D_Y$  is a single point  $p_i$ , so by Zariski's theorem [Laz, Remark 2.1.32]  $D_Y$  is semi-ample. Therefore in all cases,  $D = \pi^*D_Y$  is the pullback of a semi-ample class, hence is semi-ample. □

**2.2. The movable cone.** Next we prove that the movable cone of  $X$  is a rational polyhedral cone. By definition, a class  $D$  on  $X$  is *movable* if the subset  $\bigcap_{\Delta \in |D|} \text{Supp}(\Delta)$  has codimension at least 2 in  $X$ ; the *movable cone*  $\text{Mov}(X)$  is the smallest closed cone in  $N^1(X)$  containing all movable classes. For general properties of movable divisors and the movable cone, a reference is [ADHL, Section 3.3.2].

**Theorem 2.3.** *The movable cone  $\text{Mov}(X)$  is the rational polyhedral cone consisting of all classes  $x \in N^1(X)$  satisfying the following conditions:*

$$\begin{aligned} x \cdot e_i &\geq 0 \text{ for } i = 1, \dots, 8; \\ x \cdot (h - e_i) &\geq 0 \text{ for } i = 1, \dots, 8; \\ x \cdot q_{ijkl} &\geq 0 \text{ for every coplanar quadruple } \{i, j, k, l\} \subset \{1, \dots, 8\}. \end{aligned}$$

Here for a coplanar quadruple  $I = \{i, j, k, l\}$  the notation  $q_{ijkl}$  means the class  $2h - e_i - e_j - e_k - e_l$  of the proper transform on  $X$  of a conic in  $\mathbf{P}^3$  passing through the four points.

*Proof.* First observe that if  $C$  is any class in  $N_1(X)$  such that there exist curves on  $X$  with class  $C$  whose unions fill up a subset of codimension  $\leq 1$ , then any class  $x \in N^1(X)$  which lies in the movable cone must satisfy  $x \cdot C \geq 0$ . (This is clear for movable divisors; since the condition is closed, it remains true for the whole movable cone.)



The inequalities in the theorem follow by applying this to various curve classes on  $X$ . First,  $e_i$  is the class of a line in the exceptional divisor  $E_i$ , and these lines cover  $E_i$ . Second,  $h - e_i$  is the class of the proper transform of a line through  $p_i$ , and these lines fill up  $\mathbf{P}^3$ . Finally,  $q_{ijkl}$  is the class of the proper transform of a conic passing through the four points of  $I$ , and these conics cover the plane  $\Pi_{ijkl}$  (since any 5 points in a plane are contained in a conic). So by the previous paragraph, these conditions are necessary for a class  $x$  to lie in the cone  $\text{Mov}(X)$ .

To see that the conditions are sufficient, we simply compute the extremal rays of the cone  $M \subset N^1(X)$  defined by our inequalities. This is easy to do using a computer algebra system: I used the package `VertexEnum` for *Mathematica*. If we can show that the extremal rays of  $M$  are indeed spanned by movable divisors, then the proof of the proposition is complete.

The output of the computer calculation tells us that  $M$  is spanned by the following classes:

- (1)  $H$ : 1 class.
- (2)  $H - E_i$ : 8 classes.
- (3)  $2H - E_{i_1} - \dots - E_{i_7}$ : 8 classes.
- (4)  $2H - E_1 - \dots - E_8$ : 1 class.
- (5)  $H - E_i - E_j$ :  $\binom{8}{2} = 28$  classes.
- (6)  $2H - 2E_i - E_j - E_k - E_l$  ( $i, j, k, l$  not coplanar):  $\binom{8}{3} \cdot 4 = 224$  classes.
- (7)  $2H - E_i - E_j - E_k - E_l - 2E_m$  ( $i, j, k, l$  coplanar):  $8 \cdot 7 = 56$  classes.
- (8)  $3H - 2E_i - 2E_j - 2E_k - 2E_l$  ( $i, j, k, l$  not coplanar):  $\binom{8}{4} - 14 = 56$  classes.
- (9)  $3H - 3E_i - E_j - E_k - E_l - 2E_m$  ( $i, j, k, l$  coplanar):  $\binom{8}{3} \cdot 4 = 224$  classes.
- (10)  $3H - E_i - E_j - E_k - E_l - 2E_m - 2E_n - 2E_p$  ( $i, j, k, l$  coplanar):  $8 \cdot 7 = 56$  classes.
- (11)  $3H - 3E_{i_1} - E_{i_2} - \dots - E_{i_8}$ : 8 classes.

The first 4 types of classes already appeared as extremal rays of the nef cone  $\text{Nef}(X)$ , and we saw that they are in fact semi-ample on  $X$ . In particular, they are movable.

Classes of the form  $H - E_i - E_j$  are represented by proper transforms on  $X$  of planes in  $\mathbf{P}^3$  passing through the points  $p_i$  and  $p_j$ . The base locus of the linear system of such planes is precisely the proper transform of the line through  $p_i$  and  $p_j$ , which has codimension 2 in  $X$ . So these classes are movable also.

For the remaining classes, we will show they are movable by decomposing them into effective divisors in different ways. (These decompositions will be used again later, when we verify the conditions of Hu–Keel’s theorem for  $X$ .) For simplicity of notation in each case we will fix a set of indices satisfying the stated conditions.

- (6)  $D = 2H - 2E_1 - E_2 - E_3 - E_5$ . We can decompose this as a sum of effective divisor classes in two different ways as follows:

$$\begin{aligned} D &= (H - E_1 - E_2 - E_3 - E_4) + E_4 + (H - E_1 - E_5) \\ &= (H - E_1 - E_2 - E_5 - E_6) + E_6 + (H - E_1 - E_3). \end{aligned}$$

We have seen that the base locus of the last class in both decompositions has codimension 2 in  $X$ . Each decomposition also has two fixed components, one a plane through 4 basepoints and the other an exceptional divisor. But none of these components is common to both decompositions, so the base locus has codimension at least 2. We conclude that this class is movable.

- (7)  $D = 2H - E_1 - E_2 - E_3 - E_4 - 2E_5$ . Again we decompose in two ways and observe that no prime divisor is common to both decompositions:

$$\begin{aligned} D &= (H - E_1 - E_2 - E_5 - E_6) + E_6 \\ &\quad + (H - E_3 - E_4 - E_5 - E_6) + E_6 \\ &= (H - E_1 - E_3 - E_5 - E_7) + E_7 \\ &\quad + (H - E_2 - E_4 - E_5 - E_7) + E_7. \end{aligned}$$

- (8)  $D = 3H - 2E_1 - 2E_2 - 2E_3 - 2E_5$ . Here the decomposition we need is

$$\begin{aligned} D &= (H - E_1 - E_2 - E_3 - E_4) + E_4 \\ &\quad + (H - E_2 - E_3 - E_5 - E_8) + E_8 \\ &\quad + (H - E_1 - E_5) \\ &= (H - E_1 - E_2 - E_5 - E_6) + E_6 \\ &\quad + (H - E_1 - E_3 - E_5 - E_7) + E_7 \\ &\quad + (H - E_2 - E_3). \end{aligned}$$

- (9)  $D = 3H - 3E_1 - E_2 - E_3 - E_4 - 2E_5$ . Here the decomposition we need is

$$\begin{aligned} D &= (2H - 2E_1 - E_2 - E_3 - E_5) \\ &\quad + (H - E_1 - E_4 - E_5 - E_6) + E_6 \\ &= (2H - 2E_1 - E_2 - E_4 - E_5) \\ &\quad + (H - E_1 - E_3 - E_5 - E_7) + E_7. \end{aligned}$$

In Case 7 we already showed that the first term in each decomposition is a movable class. Since the remaining terms are distinct prime divisors, this shows our class is movable.

- (10)  $D = 3H - E_1 - E_2 - E_3 - E_4 - 2E_5 - 2E_6 - 2E_7$ . Here the decomposition we need is the following:

$$\begin{aligned} D &= (2H - E_1 - \cdots - E_7) + (H - E_5 - E_6 - E_7 - E_8) + E_8 \\ &= (2H - E_1 - E_2 - E_5 - E_6 - 2E_7) + (H - E_3 - E_4 - E_5 - E_6). \end{aligned}$$

In the first decomposition, we already saw that the first term is semi-ample. In the second decomposition, we saw that the second term is movable in Case 7 above. Again, the remaining terms are distinct prime divisors.

- (11)  $D = 3H - 3E_1 - E_2 - \cdots - E_8$ . Here the decompositions we need are the following:

$$\begin{aligned} D &= (2H - 2E_1 - E_2 - E_4 - E_6 - E_8) + (H - E_1 - E_3 - E_5 - E_7) \\ &= (2H - 2E_1 - E_3 - E_4 - E_5 - E_6) + (H - E_1 - E_2 - E_7 - E_8). \end{aligned}$$

In each decomposition, the first term has already been shown to be movable on  $X$ . The remaining terms are distinct prime divisors, and this completes the proof.

□

3.  $X$  IS MORI DREAM SPACE

In this section we strengthen the results of the previous section to show that our variety  $X$  is a *Mori dream space* (see below for the definition). This class of varieties was defined by Hu and Keel; they showed [HK, Theorem 2.9] that Mori dream spaces are exactly the varieties with finitely generated Cox ring, implying that they have the best possible properties from the point of view of Mori theory. In particular, the minimal model programme terminates for any effective divisor on a Mori dream space. It is a fundamental issue in birational geometry to determine whether a given variety is a Mori dream space. By Cox [C], every toric variety is a Mori dream space; the deepest known result is the theorem of Birkar–Cascini–Hacon–McKernan [BCHM] that varieties of Fano type are Mori dream spaces.

Beyond these classes of varieties not many nontrivial examples seem to be known. One family of examples is due to Hausen–Laface–Tironi–Ugaglia [HLTU] and to Laface–Tironi–Ugaglia [LTU]: these examples are extremal fibrations of blowups of del Pezzo manifolds of degree at most 4. Our example can be viewed as an extension of these results: indeed, the blowup of  $\mathbf{P}^3$  at a point is a del Pezzo manifold, and our variety  $X$  is then a blowup of this del Pezzo manifold, again with the structure of an extremal (quasi-)elliptic fibration. The interesting difference between the two cases is that our example has larger Picard number and much more complicated cones of nef and movable divisors: we saw in the previous section that the movable cone of  $X$  has 670 extremal rays, and in this section we will see that this cone decomposes into 78125 nef cones of small modifications of  $X$ !

We begin with the following definition [HK, Definition 1.10].

**Definition 3.1** (Hu–Keel). Let  $X$  be a  $\mathbf{Q}$ -factorial projective variety with  $\text{Pic}(X)$  finitely generated. We say  $X$  is a *Mori dream space* if the following conditions hold:

- (1)  $\text{Nef}(X)$  is rational polyhedral, spanned by semi-ample divisors.
- (2) There is a finite collection of small  $\mathbf{Q}$ -factorial modifications  $X \dashrightarrow X_i$  such that each  $X_i$  satisfies the previous condition and  $\text{Mov}(X)$  is the union of the nef cones  $\text{Nef}(X_i)$ .

By definition, a *small  $\mathbf{Q}$ -factorial modification* (SQM) of  $X$  is a rational map  $f : X \dashrightarrow Y$  to another  $\mathbf{Q}$ -factorial projective variety such that  $f$  is an isomorphism in codimension 1.

The main result of this paper is the following:

**Theorem 3.2.** *Let  $X$  be the blowup of  $\mathbf{P}_k^3$  at the unique extremal Cayley octad  $O$ . Then  $X$  is a Mori dream space.*

*Notation.* Let us fix the following notation for use in the proof.

- (1) For a pair of distinct indices  $i, j = 1, \dots, 8$ , we write  $c_{ij}$  to denote the proper transform on  $X$  of the line in  $\mathbf{P}^3$  through  $p_i$  and  $p_j$ .
- (2) For a subset  $\Gamma \subset \{c_{ij}\}$ , we denote by  $N(\Gamma)$  the rational polyhedral subcone of  $\text{Mov}(X)$  defined by the additional inequalities

$$\begin{aligned} x \cdot c_{ij} &\leq 0 \text{ if } c_{ij} \in \Gamma, \\ x \cdot c_{ij} &\geq 0 \text{ otherwise.} \end{aligned}$$

- (3) For a class  $D \in \text{Mov}(X)$ , we denote by  $\Gamma(D)$  the subset of  $\{c_{ij}\}$  consisting of curves such that  $D \cdot c_{ij} < 0$ . For simplicity we write  $N(D)$  instead of  $N(\Gamma(D))$ . Note that by definition we have  $D \in N(D)$ .

- (4) For a *disjoint* set of curves  $\Gamma \subset \{c_{ij}\}$ , we denote by  $X_\Gamma$  the space obtained by flopping all the curves  $c_{ij} \in \Gamma$ . (See Definition 3.4 for a precise definition of flop.) For simplicity we write  $X_D$  instead of  $X_{\Gamma(D)}$ .
- (5)  $\mathcal{E}$  denotes the set of primitive generators of the extremal rays of  $\text{Mov}(X)$  listed in Theorem 2.3.

Here is an outline of our proof. We already proved in Theorem 2.2 that condition (1) of Definition 3.1 holds for  $X$ , so we need to verify condition (2). The hyperplanes  $c_{ij}^\perp$  partition the movable cone  $\text{Mov}(X)$  into the finitely many rational polyhedral subcones  $N(\Gamma)$ , one for each subset  $\Gamma \subset \{c_{ij}\}$ . We remarked above that  $D \in N(D)$  by definition, so the cone  $\text{Mov}(X)$  is in fact covered by the smaller collection of cones of the form  $N(D)$ , for movable divisors  $D$ . We will show in Lemma 3.3 that for any movable divisor  $D$ , the set  $\Gamma(D) \subset \{c_{ij}\}$  consists of *pairwise disjoint* curves. Therefore we get an SQM  $X \dashrightarrow X_D$  obtained by flopping all the curves in  $\Gamma(D)$ , and we show in Lemma 3.6 that  $\text{Nef}(X_D)$  must be a subset of  $N(D)$ . Finally we show in Theorem 3.9 that each extremal ray of  $N(D)$  is semi-ample on  $X_D$ . This shows that  $N(D) = \text{Nef}(X_D)$ , and at the same time that condition (2) in Definition 3.1 is satisfied.

To put this strategy into practice we need some reductions. There are  $5^7 = 78125$  choices for  $\Gamma$ , and each of the corresponding cones  $N(\Gamma)$  may have many extremal rays. The action of the group  $\text{Aff}(3, 2)$  of order 1344 simplifies things somewhat, but there is more we can do. We will show that if a divisor spans an extremal ray of several cones  $N(\Gamma)$ , then we need only check semi-ampleness of one of the corresponding models  $X_\Gamma$ . So if we can find a convenient subset  $\mathcal{E}$  of SQMs whose nef cones include all the extremal rays of all the other cones, then it is enough to check semi-ampleness on this smaller set.

Figure 2 illustrates the idea. Here the hexagon represents a slice of the movable cone. The small cones are the nef cones  $N(D)$ , and the shaded cones are the nef cones of SQMs in the collection  $\mathcal{E}$ . The key point is that every extremal ray of every cone  $N(D)$  is an extremal ray of at least one shaded cone.

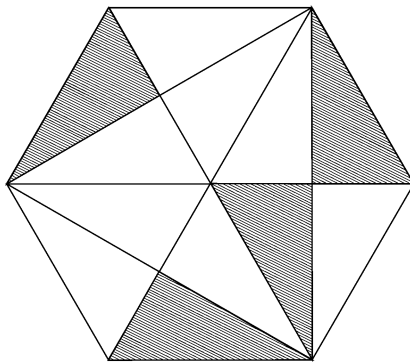


FIGURE 2. Nef cones inside  $\text{Mov}(X)$ .

We now begin the proof described above.

**Lemma 3.3.** *For a class  $D \in \text{Mov}(X)$ , any two curves in  $\Gamma(D)$  are disjoint.*

*Proof.* Suppose that  $c_{ij}$  and  $c_{kl}$  are intersecting curves on  $X$ . Then their proper transforms on  $X$  are two lines whose union contains the 4 points  $\{p_i, p_j, p_k, p_l\}$  and lies in a plane. Then  $c_{ij} + c_{kl} = 2h - e_i - e_j - e_k - e_l = q_{ijkl}$  is the class of the proper transform of a conic in the plane passing through the 4 points. By Theorem 2.3 we must then have  $D \cdot (c_{ij} + c_{kl}) \geq 0$ , so the two classes cannot both belong to  $\Gamma(D)$ .  $\square$

Figure 1 shows the configuration of the 28 curves  $c_{ij}$  on  $X$ . Any choice of a subset  $\Gamma$  containing at most 1 curve from each of the 7 sets of 4 intersecting curves gives us a nonempty subcone  $N(\Gamma) \subset \text{Mov}(X)$ , so as mentioned above there are  $5^7$  cones in our decomposition.

**Definition 3.4.** Let  $Y$  be a smooth threefold and let  $C \subset Y$  be a smooth rational curve with normal bundle  $N_{C/Y} \cong O(-1) \oplus O(-1)$ . Blowing up along  $C$  and contracting the exceptional divisor  $E \cong \mathbf{P}^1 \times \mathbf{P}^1$  in the other direction yield a smooth algebraic space  $Y'$  with a rational map  $\varphi : Y \dashrightarrow Y'$  which is an isomorphism outside  $C$ . We call  $Y'$  (or  $\varphi$ ) the *flop* of  $Y$  along  $C$ . The curve  $C$  is called the *centre* of the flop, and the curve  $C' \subset Y'$  along which  $\varphi^{-1}$  is not defined is called the *cocentre* of the flop.

Existence of flops in this setting follows from Artin’s contractibility criterion [Art, Corollary 6.11]. Note that if  $C$  and  $C'$  are disjoint smooth rational curves on  $Y$ , both with normal bundle  $O(-1) \oplus O(-1)$ , then we can flop them one after another, and the resulting space is independent of the order. The same applies to any finite set of disjoint smooth rational curves each with normal bundle  $O(-1) \oplus O(-1)$ . In particular, the notation  $X_\Gamma$  defined in the list above makes sense whenever  $\Gamma$  is a set of disjoint curves. Note also that in our example, each of the curves  $c_{ij} \subset X$  has the correct normal bundle  $O(-1) \oplus O(-1)$  to be flopped, since a line in  $\mathbf{P}^3$  has normal bundle  $O(1) \oplus O(1)$  and blowing up a point twists the normal bundle by  $O(-1)$ .

**Lemma 3.5.** *Let  $D$  be a movable divisor on  $X$ . Then  $X_D$  as defined above is a projective variety.*

*Proof.* Note that  $X_D$  is well-defined since by Lemma 3.3 any two curves in  $\Gamma(D)$  are disjoint.

First assume that no two curves in the set  $\Gamma = \Gamma(D)$  have disjoint index sets. Suppose that  $\Gamma_1$  is some proper subset of  $\Gamma$  such that  $X_{\Gamma_1}$  is projective. Choose a curve  $c_{kl} \in \Gamma \setminus \Gamma_1$ . Then  $H - E_k - E_l$  is an effective class on  $X$  whose base locus is  $c_{kl}$  and is represented by a divisor disjoint from all the curves  $c_{ij} \in \Gamma_1$  (by the condition on index sets). Therefore the proper transform of  $H - E_k - E_l$  on  $X_{\Gamma_1}$  has degree  $-1$  on  $c_{kl}$  and nonnegative degree on every other curve. Let  $A$  be any ample divisor  $A$  on  $X_{\Gamma_1}$  and let  $a = A \cdot c_{kl}$ . Note that since  $A$  is ample,  $a$  is a strictly positive integer. Then the divisor  $D_{kl} := (H - E_k - E_l) + \frac{1}{a}A$  is nef and big and has degree zero precisely on  $c_{kl}$ . By Keel’s Basepoint-Freeness Theorem [Ke, Theorem 0.5], the divisor  $D_{kl}$  is then semi-ample.

Putting  $\Gamma_2 = \Gamma_1 \cup \{c_{kl}\}$ , the flop of the curve  $c_{kl}$  can be seen as a commutative diagram

$$\begin{array}{ccc}
 X_{\Gamma_1} & \overset{\phi}{\dashrightarrow} & X_{\Gamma_2} \\
 \searrow f_{kl} & & \swarrow f_{kl}^+ \\
 & Y_{\Gamma_1} &
 \end{array}$$

in which  $f_{kl}$  and  $f_{kl}^+$  denote the contraction of  $c_{kl}$  and  $c'_{kl}$  (the cocentre of the flop), respectively. Since  $f_{kl}$  is defined by the semi-ample line bundle  $D_{kl}$ , we get that  $Y_{\Gamma_1}$  is projective. Moreover since the proper transform  $(H - E_k - E_l)'$  is  $(f_{kl}^+)$ -ample, we get that  $f_{kl}^+$  is a projective morphism, and so  $X_{\Gamma_2}$  is also projective. Since  $X$  is projective, induction on the cardinality of  $\Gamma$  then tells us that  $X_\Gamma$  is projective, as required.

Now we prove the general case, in which  $\Gamma$  may contain curves with disjoint index sets. We can assume first without loss of generality that  $D \cdot c_{ij} \neq 0$  for any  $i$  and  $j$ . This is valid because if  $D$  is any movable divisor, then for any ample divisor  $A$  and any sufficiently small  $\epsilon > 0$ , the divisor  $D' = D + \epsilon A$  is movable, it has the property assumed, and  $\Gamma(D') = \Gamma(D)$ .

So suppose now that  $D$  satisfies the assumption above, and  $\Gamma$  contains 2 curves  $c_{ij}$  with disjoint index sets, say, without loss of generality,  $c_{12}$  and  $c_{35}$ . (Recall that a movable divisor  $D$  cannot have  $\{c_{ij}, c_{km}\} \subset \Gamma(D)$  for  $\{i, j, k, m\}$  coplanar.) I claim that there is an SQM  $X \dashrightarrow X_1$  to a projective variety  $X_1$  (which is in fact isomorphic to  $X$ ) such that  $|\Gamma_1| < |\Gamma|$ , where  $\Gamma_1$  denotes the set of those  $c_{ij}$  whose proper transform on  $X_1$  has negative intersection with the proper transform of  $D$ . Given this, we are done by induction: repeatedly applying these SQMs, eventually we must reach a variety  $X_k$  on which either  $|\Gamma_k| = 0$ , in other words  $X_k = X_\Gamma$ , or no two curves in  $\Gamma_k$  have disjoint index sets, in which case we can flop them all to obtain  $X_\Gamma$ , by the previous argument. (This explains the need to perturb  $D$ : without this, it could be that  $X_k$  differs from  $X_\Gamma$  by flopping some  $D$ -trivial curves.)

To prove the claim, let  $X \dashrightarrow X_1 \cong X$  be the standard cubic transformation based at  $\{p_1, p_2, p_3, p_5\}$ ; this is the same thing as the flop of all the curves  $\{c_{ij} : i, j \in \{1, 2, 3, 5\}\}$ . By assumption  $D \cdot c_{12} < 0$  and  $D \cdot c_{35} < 0$ ; putting these together, we get

$$0 > D \cdot (c_{12} + c_{35}) = D \cdot (c_{13} + c_{25}) = D \cdot (c_{15} + c_{23}).$$

So at least one curve from each pair  $\{c_{13}, c_{25}\}$  and  $\{c_{15}, c_{23}\}$  is in  $\Gamma$ , so we have  $|\Gamma \cap \{c_{ij} : i, j \in \{1, 2, 3, 5\}\}| \geq 4$ . Flopping the 6 curves  $c_{ij}$ , we get  $|\Gamma_1 \cap \{c_{ij} : i, j \in \{1, 2, 3, 5\}\}| \leq 2$ . Since flopping these curves does not change the intersection number of  $D$  with any other curve  $c_{ij} \in \Gamma$ , the claim is proved.  $\square$

**Lemma 3.6.** *For any  $D \in \text{Mov}(X)$  we have  $\text{Nef}(X_D) \subset N(D)$ .*

*Proof.* The nef cone of any SQM of  $X$  is automatically a subcone of  $\text{Mov}(X)$ . We must show that any nef divisor on  $X_D$  satisfies the other defining inequalities of  $N(D)$ . The SQM  $X \dashrightarrow X_D$  is the flop of all the curves  $c_{ij}$  in  $\Gamma(D)$ , so all of the

following are classes of curves on  $X_D$ :

$$\begin{aligned} -c_{ij} & \text{ for } c_{ij} \in \Gamma(D), \\ c_{ij} & \text{ for } c_{ij} \notin \Gamma(D). \end{aligned}$$

Any nef divisor must have nonnegative intersection with all these classes, and the resulting inequalities are exactly the inequalities we used to define  $N(D)$  inside  $\text{Mov}(X)$ .  $\square$

**Theorem 3.7.** *For any  $D \in \mathcal{E}$ , the cone  $N(D)$  is spanned by semi-ample classes on  $X_D$ . In particular  $\text{Nef}(X_D) = N(D)$ .*

*Proof.* The proof is a long check similar in spirit to the proof of Theorem 2.3. Given  $D \in \mathcal{E}$  we find the set of curves  $\Gamma(D)$ , and this gives the inequalities defining  $N(D)$  inside  $\text{Mov}(X)$ . Again we use computer algebra to compute the extremal rays of  $N(D)$ , and we check by hand that each of them is semi-ample of  $X_D$ . The details are straightforward but simple, so we relegate them to Appendix A.  $\square$

**Lemma 3.8.** *If  $\varphi : Y \dashrightarrow Z$  is the flop of a curve  $C$ , and  $\Delta$  is a semi-ample class on  $Y$  such that  $\Delta \cdot C = 0$ , then the proper transform of  $\Delta$  is semi-ample on  $Z$  too.*

*Proof.* Replacing  $\Delta$  with a positive multiple if necessary, we can assume it is basepoint-free. The condition  $\Delta \cdot C = 0$  implies that  $\Delta$  has a representative which is disjoint from  $C$ . Taking proper transform gives a representative which is disjoint from the cocentre  $C'$ . Finally, since  $\varphi$  is an isomorphism on the complement of  $C$ , the proper transform of  $\Delta$  has no basepoints in  $Z \setminus C'$  either.  $\square$

**Theorem 3.9.** *For any  $D \in \text{Mov}(X)$ , the cone  $N(D)$  is spanned by semi-ample divisor classes on  $X_D$ . In particular  $\text{Nef}(X_D) = N(D)$ .*

*Proof.* Let  $\Delta$  be a divisor spanning an extremal ray of  $N(D)$ . Assume first that  $\Delta$  also spans an extremal ray of a cone  $N(D')$  for some  $D' \in \mathcal{E}$ . Then  $X_D$  is obtained from  $X_{D'}$  by flopping all the curves  $c_{ij}$  in  $\Gamma(D) \setminus \Gamma(D')$ . Since by assumption  $\Delta$  belongs to the intersection  $N(D) \cap N(D')$ , we must have  $\Delta \cdot c_{ij} = 0$  for all  $c_{ij} \in \Gamma(D) \setminus \Gamma(D')$ . By Theorem 3.7 the class  $\Delta$  is semi-ample on  $X_{D'}$ , so applying Lemma 3.8 repeatedly we get that  $\Delta$  is semi-ample on  $X_D$ , as required.

So it remains to prove that each extremal ray of each cone  $N(D)$  is also an extremal ray of a cone  $N(D')$  for some  $D' \in \mathcal{E}$ . In principle this is simple. Each cone  $N(D)$  is cut out inside  $\text{Mov}(X)$  by hyperplanes dual to curves of the form  $c_{ij}$ . So we have a set of 58 curve classes

$$e_i, \quad h - e_i, \quad c_{ij} \ (i \neq j), \quad 2h - e_i - e_j - e_k - e_l \quad (\{i, j, k, l\} \text{ coplanar})$$

and we are looking for subsets  $\{\gamma_1, \dots, \gamma_8\}$  of 8 of these classes such that the linear map

$$\begin{aligned} N^1(X) & \longrightarrow \mathbf{R}^8 \\ x & \mapsto \begin{pmatrix} \gamma_1 \cdot x \\ \vdots \\ \gamma_8 \cdot x \end{pmatrix} \end{aligned}$$

has rank 8 and kernel intersecting  $\text{Mov}(X)$  nontrivially. This is now a finite check which can in principle be carried out by computer.

In practice checking all  $\binom{58}{8}$  possible subsets is not computationally feasible, so some reductions are necessary. We use the following approach: for a vector  $x \in N^1(X)$ , we can use the natural basis given by  $H$  and the  $E_i$  to write it in the form

$$x = aH - \sum_i b_i E_i$$

for some real numbers  $a, b_1, \dots, b_8$ . (We include the negative sign above for convenience so that all these numbers will be nonnegative for the classes we are interested in.) In this notation, the conditions  $x \cdot \gamma_m = 0$  cutting out our extremal rays then take the form:

$$\begin{aligned} b_i &= 0 \text{ for curve classes } e_i, \\ b_i &= a \text{ for curve classes } h - e_i, \\ b_i + b_j &= a \text{ for curve classes } c_{ij}, \\ b_i + b_j + b_k + b_l &= 2a \text{ for curve classes } 2h - e_i - e_j - e_k - e_l. \end{aligned}$$

We simplify the problem by fixing the number of  $b_i$  which equal zero and checking each case in turn.

Let us give details in the case where 3 of the coefficients  $b_i$  are zero and the others are nonzero. So  $x = aH - b_1 E_1 - \dots - b_5 E_5$ . We have exactly 3 equations  $b_i = 0$  ( $i = 6, 7, 8$ ), so we need 5 additional equations from the other 55 classes. We know the equations  $b_i = 0$  ( $i = 1, 2, 3, 4, 5$ ) and  $b_i = a$  ( $i = 6, 7, 8$ ) do not hold, so we can discard those 8 classes from the list. Next, each equation  $b_i + b_j = a$  for  $i = 6, 7, 8$  becomes identical to the equation  $b_j = a$ , so we can discard another  $7+6+5 = 18$  classes. Finally, since  $x \in \text{Mov}(X)$  we know that  $b_1 + b_2 + b_3 + b_4 \leq 2a$ , so since these coefficients are nonzero we know that  $b_i + b_j < 2a$  for any  $\{i, j\} \subset \{1, 2, 3, 4\}$ . Therefore if  $\{i, j, k, l\}$  is a coplanar quadruple not containing 5, we have  $b_i + b_j + b_k + b_l < 2a$  since two summands must come from  $\{1, 2, 3, 4\}$  and the other two from  $\{6, 7, 8\}$ . This allows us to eliminate another 7 classes. So finally we are looking for a subset of 5 equations from a list of  $55 - 8 - 18 - 7 = 22$ , and this is tractable by computer. The output tells us that the only classes satisfying 5 of these 22 equations and lying in the movable cone are (up to the action of  $\text{Aff}(3, 2)$ ) the following:

$$\begin{aligned} 2h - e_1 - e_2 - e_3 - e_4 - e_5, \quad & 2h - e_1 - e_2 - e_3 - e_4 - 2e_5, \\ 3h - e_1 - e_2 - e_3 - 3e_4 - 2e_5, \quad & 3h - e_1 - e_2 - 2e_3 - 2e_4 - 2e_5, \\ 4h - e_1 - 2e_2 - 2e_3 - 3e_4 - 3e_5, \quad & 5h - e_1 - 3e_2 - 3e_3 - 3e_4 - 4e_5. \end{aligned}$$

These all appear on our list in the proof of Theorem 3.7, as required. □

Finally we can deduce Theorem 3.2:

**Corollary 3.10.** *The variety  $X$  is a Mori dream spaces.*

*Proof.* We proved in the previous section that  $\text{Nef}(X)$  is rational polyhedral and spanned by semi-ample classes. Theorem 3.9 shows that  $\text{Mov}(X)$  is covered by the finite collection of nef cones  $\{\text{Nef}(X_D) \mid D \in \text{Mov}(X)\}$ , and each cone is spanned by semi-ample classes. Therefore both conditions of Definition 3.1 are satisfied. □

*Remark 3.11.* In Section 1 we showed that there is a unique extremal net of quadrics with distinct basepoints, but it is also interesting to consider nets of quadrics with



infinitely near basepoints. In [PS] we classified all extremal nets, including those with infinitely near basepoints (subject to some restrictions on characteristic of the base field), and found 11 other examples. It seems very likely that the techniques of the present paper can be applied to verify that these varieties are also Mori dream spaces.

APPENDIX A. PROOF OF THEOREM 3.7

In this appendix we give details of the proofs of Theorem 3.7.

We need to calculate the cones  $N(D)$  for  $D \in \mathcal{E}$ , a generator of an extremal ray of  $\text{Mov}(X)$ . Recall that  $N(D)$  is defined by the inequalities defining  $\text{Mov}(X)$  in Theorem 2.3 together with the extra conditions

$$\begin{aligned} x \cdot c_{ij} &\leq 0 \text{ if } c_{ij} \in \Gamma(D), \\ x \cdot c_{ij} &\geq 0 \text{ if } c_{ij} \notin \Gamma(D). \end{aligned}$$

We calculate the extremal rays of each of the resulting cones for each  $D \in \mathcal{E}$  listed in the proof of Theorem 2.3. The results are shown in the table below. For brevity, we make a number of reductions:

- We only list one representative of each  $\text{Aff}(3, 2)$ -orbit in  $\mathcal{E}$  in the left-hand column and one representative of each  $\text{Aff}(3, 2)$ -orbit in the set of extremal rays of  $N(D)$  in the right-hand column.
- In the left-hand column we omit elements of  $\mathcal{E}$  which lie in the nef cone of  $X$ , since the corresponding SQM  $X_D$  is  $X$  itself, and we proved that all the relevant divisors are semi-ample in Theorem 2.2.
- As in the proof of Theorem 3.9 if a class  $\Delta$  belongs to two cones  $N(D) \cap N(D')$  it suffices to prove semi-ampleness on one of the two SQMs. So for each  $D \in \mathcal{E}$  we will only list extremal rays of  $N(D)$  which did not yet appear on our list. This explains the heading of the rightmost column.

We explain why each class in the right-hand column is semi-ample on the relevant SQM  $X_D$ .

- (1)  $D = H - E_1 - E_2$ : here the only class to consider is  $D$  itself.  $D$  is represented on  $X$  by the proper transform of any plane through  $p_1$  and  $p_2$ . The base locus on  $X$  is therefore  $c_{12}$  itself. Since these planes have no common normal directions along  $c_{12}$ , after flopping to get  $X_D$  the class becomes basepoint-free.
- (2)  $D = 2H - 2E_1 - E_2 - E_3 - E_5$ : again we only need to consider  $D$  itself. As in the proof of Theorem 2.3 we write

$$\begin{aligned} 2H - 2E_1 - E_2 - E_3 - E_5 &= (H - E_1 - E_2 - E_3 - E_4) + E_4 + (H - E_1 - E_5) \\ &= (H - E_1 - E_2 - E_5 - E_6) + E_6 + (H - E_1 - E_3), \end{aligned}$$

and intersecting these representatives shows that the base locus of  $D$  is exactly the curves in  $\Gamma(D)$ . When we flop, the proper transform of  $(H - E_1 - E_2 - E_3 - E_4)$  and  $(H - E_1 - E_2 - E_5 - E_6)$  are both divisors intersecting the cocentre  $c'_{12}$  transversely in a single point, and these points are different since the planes have different normal directions along  $c_{12}$ . Moreover the proper transforms of  $(H - E_1 - E_5)$  and  $(H - E_1 - E_3)$  are disjoint from  $c'_{12}$ . So after flopping  $c_{12}$  there are no basepoints in the cocentre  $c'_{12}$ . Similarly, flopping  $c_{13}$  and  $c_{15}$  will remove all the other basepoints, so  $D$  is basepoint-free on  $X_D$ .

| $D$                                               | $\Gamma(D)$                                       | New rays of $N(D)$                                                                                                                                                                                                                                                                              |
|---------------------------------------------------|---------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $H - E_1 - E_2$                                   | $c_{12}$                                          | $H - E_1 - E_2$                                                                                                                                                                                                                                                                                 |
| $2H - 2E_1 - E_2 - E_3 - E_5$                     | $\{c_{12}, c_{13}, c_{15}\}$                      | $2H - 2E_1 - E_2 - E_3 - E_5$                                                                                                                                                                                                                                                                   |
| $2H - E_1 - E_2 - E_3 - E_4 - 2E_5$               | $\{c_{i5} \mid i = 1, 2, 3, 4\}$                  | $2H - E_1 - E_2 - E_3 - E_4 - 2E_5$                                                                                                                                                                                                                                                             |
| $3H - 2E_1 - 2E_2 - 2E_3 - 2E_5$                  | $\{c_{ij} \mid \{i, j\} \subset \{1, 2, 3, 5\}\}$ | $3H - 2E_1 - 2E_2 - 2E_3 - E_5$<br>$3H - 2E_1 - 2E_2 - 2E_3 - E_5 - E_6$<br>$3H - 2E_1 - 2E_2 - 2E_3 - E_5 - E_6 - E_7$<br>$3H - 2E_1 - 2E_2 - 2E_3 - 2E_5$<br>$4H - 3E_1 - 2E_2 - 2E_3 - E_4 - 3E_5$<br>$4H - 3E_1 - 3E_2 - 2E_3 - 2E_5 - E_7 - E_8$<br>$5H - 3E_1 - 3E_2 - 3E_3 - E_4 - 4E_5$ |
| $3H - 3E_1 - E_2 - E_3 - E_4 - 2E_5$              | $\{c_{1j} \mid j = 2, 3, 4, 5\}$                  | $3H - 3E_1 - E_2 - E_3 - E_4 - 2E_5$                                                                                                                                                                                                                                                            |
| $3H - 3E_1 - E_2 - \dots - E_8$                   | $\{c_{1j} \mid j = 2, \dots, 8\}$                 | $3H - 3E_1 - 2E_2 - E_3 - E_5 - E_7$<br>$3H - 3E_1 - E_2 - \dots - E_8$                                                                                                                                                                                                                         |
| $3H - E_1 - E_2 - E_3 - E_4 - 2E_5 - 2E_6 - 2E_7$ | $\{c_{56}, c_{57}, c_{67}\}$                      | $3H - E_1 - 2E_5 - 2E_6 - 2E_7$<br>$3H - E_1 - E_2 - 2E_5 - 2E_6 - 2E_7$<br>$3H - E_1 - E_2 - E_3 - 2E_5 - 2E_6 - 2E_7$<br>$3H - E_1 - E_2 - E_3 - E_4 - 2E_5 - 2E_6 - 2E_7$                                                                                                                    |

(3)  $2H - E_1 - E_2 - E_3 - E_4 - 2E_5$ : here we decompose  $D$  as

$$\begin{aligned}
 2H - E_1 - E_2 - E_3 - E_4 - 2E_5 &= (H - E_1 - E_2 - E_5 - E_6) + E_6 \\
 &\quad + (H - E_3 - E_4 - E_5 - E_6) + E_6 \\
 &= (H - E_1 - E_3 - E_5 - E_7) + E_7 \\
 &\quad + (H - E_2 - E_4 - E_5 - E_7) + E_7,
 \end{aligned}$$

and intersecting these shows that the base locus of  $D$  is exactly the curves in  $\Gamma(D)$ . Just as in the previous case, we see that flopping  $c_{i5}$  removes all basepoints on that curve, proving that  $D$  becomes basepoint-free on  $X_D$ .

(4)  $D = 3H - 2E_1 - 2E_2 - 2E_3 - 2E_5$ : here there are several extremal rays  $\Delta$  of  $N(D)$  to deal with. For brevity, from now on we will just write the necessary decompositions of  $\Delta$  and omit the details of checking that  $\Delta$  is basepoint-free on  $X_D$ , since in each case the argument is very similar to the preceding ones.

(a)  $\Delta = 3H - 2E_1 - 2E_2 - 2E_3 - E_5$ :

$$\begin{aligned}
 &= (H - E_1 - E_2 - E_5 - E_6) + E_6 + (H - E_1 - E_3) + (H - E_2 - E_3) \\
 &= (H - E_2 - E_3 - E_5 - E_8) + E_8 + (H - E_1 - E_2) + (H - E_1 - E_3)
 \end{aligned}$$

(b)  $\Delta = 3H - 2E_1 - 2E_2 - 2E_3 - E_5 - E_6$ :

$$\begin{aligned}
 &= (H - E_1 - E_3 - E_6 - E_8) + (H - E_2 - E_3 - E_5 - E_8) \\
 &\quad + 2E_8 + (H - E_1 - E_2) \\
 &= (H - E_2 - E_3 - E_6 - E_7) + (H - E_2 - E_3 - E_5 - E_7) \\
 &\quad + 2E_7 + (H - E_1 - E_2)
 \end{aligned}$$

$$\begin{aligned}
\text{(c) } \Delta &= 3H - 2E_1 - 2E_2 - 2E_3 - 2E_5: \\
&= (H - E_1 - E_2 - E_3 - E_4) + (H - E_2 - E_3 - E_5 - E_8) \\
&\quad + E_4 + E_8 + (H - E_1 - E_5) \\
&= (H - E_1 - E_2 - E_5 - E_6) + (H - E_1 - E_3 - E_5 - E_7) \\
&\quad + E_6 + E_7 + (H - E_2 - E_3)
\end{aligned}$$

$$\begin{aligned}
\text{(d) } \Delta &= 4H - 3E_1 - 2E_2 - 2E_3 - E_4 - 3E_5: \\
&= (2H - E_1 - E_2 - E_3 - E_4 - E_5) + (H - E_1 - E_2 - E_5) \\
&\quad + (H - E_1 - E_3 - E_5) \\
&= (H - E_1 - E_2 - E_3) + (H - E_1 - E_4 - E_5) + (H - E_2 - E_3 - E_5) \\
&\quad + (H - E_1 - E_5) \\
&= (H - E_1 - E_2 - E_5) + (H - E_1 - E_4 - E_5) + (H - E_1 - E_3 - E_5) \\
&\quad + (H - E_2 - E_3)
\end{aligned}$$

$$\begin{aligned}
\text{(e) } \Delta &= 4H - 3E_1 - 2E_2 - 2E_3 - 2E_5 - E_7 - E_8: \\
&= (2H - E_1 - E_2 - E_3 - E_5 - E_7 - E_8) + (H - E_1 - E_2 - E_3) \\
&\quad + (H - E_1 - E_2 - E_5) \\
&= (H - E_1 - E_2 - E_7) + (H - E_1 - E_3 - E_5) \\
&\quad + (H - E_2 - E_3 - E_5) + (H - E_1 - E_2 - E_8)
\end{aligned}$$

$$\begin{aligned}
\text{(f) } \Delta &= 5H - 3E_1 - 3E_2 - 3E_3 - E_4 - 4E_5: \\
&= 2(2H - E_1 - E_2 - E_3 - 2E_5) + (H - E_1 - E_2 - E_3 - E_4) \\
&= (H - E_1 - E_2 - E_3) + (H - E_1 - E_2 - E_5) + (H - E_1 - E_3 - E_5) \\
&\quad + (H - E_2 - E_3 - E_5) + (H - E_4 - E_5) \\
&= (H - E_1 - E_2) + (H - E_1 - E_2 - E_5) + (H - E_1 - E_3 - E_5) \\
&\quad + (H - E_2 - E_3 - E_5) + (H - E_3 - E_4 - E_5)
\end{aligned}$$

$$\begin{aligned}
\text{(5) } D &= 3H - 3E_1 - E_2 - E_3 - E_4 - 2E_5: \\
&= (2H - 2E_1 - E_2 - E_3 - E_5) + (H - E_1 - E_4 - E_5) \\
&= (2H - 2E_1 - E_2 - E_4 - E_5) + (H - E_1 - E_3 - E_5)
\end{aligned}$$

$$\begin{aligned}
\text{(6) } D &= 3H - 3E_1 - E_2 - \cdots - E_8: \\
\text{(a) } \Delta &= 3H - 3E_1 - 2E_2 - E_3 - E_5 - E_7: \\
&= 2(H - E_1 - E_2) + (H - E_1 - E_3 - E_5 - E_7) \\
&= (H - E_1 - E_2 - E_3) + (H - E_1 - E_2 - E_5) + (H - E_1 - E_7)
\end{aligned}$$

$$\begin{aligned}
\text{(b) } \Delta &= 3H - 3E_1 - E_2 - \cdots - E_8: \\
&= (2H - 2E_1 - E_2 - E_4 - E_6 - E_8) + (H - E_1 - E_3 - E_5 - E_7) \\
&= (2H - 2E_1 - E_3 - E_4 - E_5 - E_6) + (H - E_1 - E_2 - E_7 - E_8) \\
&= (2H - 2E_1 - E_3 - E_4 - E_7 - E_8) + (H - E_1 - E_2 - E_5 - E_6)
\end{aligned}$$

$$\text{(7) } D = 3H - E_1 - E_2 - E_3 - E_4 - 2E_5 - 2E_6 - 2E_7:$$

$$\begin{aligned}
\text{(a) } \Delta &= 3H - E_1 - 2E_5 - 2E_6 - 2E_7: \\
&= 2(H - E_5 - E_6 - E_7) + (H - E_1) \\
&= (H - E_5 - E_7 - E_1) + (H - E_5 - E_6) + (H - E_6 - E_7) \\
&= (H - E_5 - E_6 - E_1) + (H - E_5 - E_7) + (H - E_6 - E_7) \\
\text{(b) } \Delta &= 3H - E_1 - E_2 - 2E_5 - 2E_6 - 2E_7: \\
&= 2(H - E_5 - E_6 - E_7) + (H - E_1 - E_2) \\
&= (H - E_5 - E_6 - E_1) + (H - E_5 - E_7 - E_2) + (H - E_6 - E_7) \\
&= (H - E_5 - E_6 - E_2) + (H - E_5 - E_7 - E_1) + (H - E_6 - E_7) \\
\text{(c) } \Delta &= 3H - E_1 - E_2 - E_3 - 2E_5 - 2E_6 - 2E_7: \\
&= 2(H - E_5 - E_6 - E_7) + (H - E_1 - E_2 - E_3) \\
&= (H - E_5 - E_6 - E_1) + (H - E_5 - E_7 - E_2) + (H - E_6 - E_7 - E_3) \\
&= (H - E_5 - E_6 - E_3) + (H - E_5 - E_7 - E_1) + (H - E_6 - E_7 - E_2) \\
\text{(d) } \Delta &= 3H - E_1 - E_2 - E_3 - E_4 - 2E_5 - 2E_6 - 2E_7: \\
&= (2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7) + (H - E_5 - E_6 - E_7) \\
&= (2H - E_1 - E_2 - E_5 - E_6 - 2E_7) + (H - E_3 - E_4 - E_5 - E_6)
\end{aligned}$$

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