TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 8, August 2018, Pages 5963–5981
http://dx.doi.org/10.1090/tran/7302
Article electronically published on November 14, 2017

# THE IRREDUCIBLE MODULES AND FUSION RULES FOR THE PARAFERMION VERTEX OPERATOR ALGEBRAS

CHUNRUI AI, CHONGYING DONG, XIANGYU JIAO, AND LI REN

Abstract. The irreducible modules for the parafermion vertex operator algebra associated to any finite dimensional Lie algebra $\mathfrak{g}$ and any positive integer $k$ are classified, the quantum dimensions are computed and the fusion rules are determined.

## 1. Introduction

This paper is a continuation of [25] and [30] on the parafermion vertex operator algebra $K(\mathfrak{g},k)$ associated to any finite dimensional simple Lie algebra $\mathfrak{g}$ and positive integer $k$. In particular, we identify the irreducible modules listed in [25], compute the quantum dimensions and determine the fusion rules for $K(\mathfrak{g},k)$. In the case $\mathfrak{g}=sl_2$, these results have been obtained previously in [3] and [30].

Closely related to the $Z$-algebras [47], [48], [49], the parafermion conformal field theory [56] and the generalized vertex operator algebras [15], the parafermion vertex operator algebra $K(\mathfrak{g},k)$ is the commutant of the Heisenberg vertex operator algebra associated to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in the affine vertex operator algebra $L_{\widehat{\mathfrak{g}}}(k,0)$. The structure of the parafermion vertex operator algebras has been studied extensively in [13], [14], [27]. The representation theory of $K(\mathfrak{g},k)$ has also been understood well due to the recent work [28], [3]-[4], and [25]. Precisely, $K(\mathfrak{g},k)$ is $C_2$-cofinite [3], [54], the irreducible modules for $K(sl_2,k)$ are classsified and rationality of $K(sl_2,k)$ is obtained in [3]-[4]. The irreducible modules and the rationality of $K(\mathfrak{g},k)$ for general $\mathfrak{g}$ are determined in [25] with the help from [54], [45], and [7].

The quantum dimensions of modules for vertex operator algebras [12] or Frobenius-Perron dimensions for fusion category [31] play essential roles in this paper. According to [36], [37], the category $\mathcal{C}_V$ of modules of a rational and $C_2$-cofinite vertex operator algebra $V$ under the tensor product defined in [35, 39–41] is a braided fusion tensor category over $\mathbb{C}$. It is essentially proved in [12] that the quantum dimension of irreducible module for $V$ is exactly the Frobenius-Perron dimension of the simple object in the category $\mathcal{C}_V$. This enables us to freely use the quantum dimensions and Frobenius-Perron dimension whenever it is convenient.

Received by the editors November 1, 2016, and, in revised form, March 30, 2017.
2010 *Mathematics Subject Classification.* Primary 17B69.
The first author was supported by National Science Foundation for Postdoctoral Science of China (No. 2017M612409).
The second author was supported by NSF grant DMS-1404741 and China NSF grant 11371261.
The third author was partially supported by China NSF grants 11401213 and Natural Science Foundation of Shanghai 13DZ2260400.
The fourth author was supported by China NSF grants 11301356 and 11671277.

The quantum dimensions of irreducible $K(\mathfrak{g},k)$-modules are computed first. It is well known that the irreducible modules for rational vertex operator algebra $L_{\widehat{\mathfrak{g}}}(k,0)$ are exactly the level $k$ integral highest weight modules $L_{\widehat{\mathfrak{g}}}(k,\Lambda)$ where $\Lambda$ is a dominant weight of $\mathfrak{g}$ such that $(\Lambda,\theta)\le k$ and $\theta$ is the maximal root of $\mathfrak{g}$ and $(\theta,\theta)=2$ [34], [46]. The set of such $\Lambda$ is denoted by $P_+^k$. Let $Q$ be the root lattice of $\mathfrak{g}$ and $Q_L$ be the sublattice of $Q$ spanned by the long roots of $\mathfrak{g}$. As we will see, the dual lattice $Q_L^\circ$ is exactly the weight lattice $P$ of $\mathfrak{g}$. The lattice vertex operator algebra $V_{\sqrt{k}Q_L}$ and $V_{\sqrt{k}Q_L}\otimes K(\mathfrak{g},k)$ are subalgebras of $L_{\widehat{\mathfrak{g}}}(k,0)$ [43], [29]. Then as $V_{\sqrt{k}Q_L}\otimes K(\mathfrak{g},k)$-module, $L_{\widehat{\mathfrak{g}}}(k,\Lambda)$ has decomposition

$$L_{\widehat{\mathfrak{g}}}(k,\Lambda)=\bigoplus_{i\in Q/kQ_L} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\Lambda+\beta_i)}\otimes M^{\Lambda,\Lambda+\beta_i},$$

where $Q=\bigcup_{i\in Q/kQ_L}(kQ_L+\beta_i)$ and $M^{\Lambda,\Lambda+\beta_i}$ is an irreducible $K(\mathfrak{g},k)$-module [25]. It turns out that the quantum dimension of each $M^{\Lambda,\Lambda+\beta_i}$ as $K(\mathfrak{g},k)$-module equals to the quantum dimension of $L_{\widehat{\mathfrak{g}}}(k,\Lambda)$ as $L_{\widehat{\mathfrak{g}}}(k,0)$-module. It follows from [9] that the quantum dimension of $M^{\Lambda,\Lambda+\beta_i}$ is equal to

$$\prod_{\alpha>0}\frac{(\Lambda+\rho,\alpha)_q}{(\rho,\alpha)_q},$$

where $\alpha>0$ means that $\alpha$ is a positive root and $n_q=\frac{q^n-q^{-n}}{q-q^{-1}}$ with $q=e^{\frac{\pi i}{k+h^\vee}}$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. This result is very useful in identifying these irreducible $K(\mathfrak{g},k)$-modules.

Let $\theta=\sum_{i=1}^l a_i\alpha_i$ where $\{\alpha_1,\cdots,\alpha_l\}$ is the set of simple roots and let $\Lambda_1,...,\Lambda_l$ be the fundamental weights of $\mathfrak{g}$. According to [51], [52], if $a_i=1$, then $L_{\widehat{\mathfrak{g}}}(k,k\Lambda_i)$ is a simple current and $L_{\widehat{\mathfrak{g}}}(k,k\Lambda_i)\boxtimes L_{\widehat{\mathfrak{g}}}(k,\Lambda)=L_{\widehat{\mathfrak{g}}}(k,\Lambda^{(i)})$ for any $\Lambda\in P_+^k$ where $\Lambda^{(i)}\in P_+^k$ is uniquely determined by $\Lambda$ and $i$. One can show that $L_{\widehat{\mathfrak{g}}}(k,\Lambda)$ and $L_{\widehat{\mathfrak{g}}}(k,\Lambda^{(i)})$ are isomorphic $K(\mathfrak{g},k)$-modules. This gives a nontrivial identification between irreducible $K(\mathfrak{g},k)$-modules $M^{\Lambda,\lambda}$ and $M^{\Lambda^{(i)},\lambda+k\Lambda_i}$ for any $\lambda\in\Lambda+Q$. As a result, the set

$$\{M^{\Lambda,\Lambda+\beta_j}|\Lambda\in P_+^k, j\in Q/kQ_L\}$$

has at most $\frac{|P_+^k||Q/kQ_L|}{|P/Q|}$ inequivalent irreducible $K(\mathfrak{g},k)$-modules. Using the relation between the global dimensions of $L_{\widehat{\mathfrak{g}}}(k,0)$ and $V_{\sqrt{k}Q_L}\otimes K(\mathfrak{g},k)$ from the category theory one can conclude that the identification given in [25] is complete and $K(\mathfrak{g},k)$ has exactly $\frac{|P_+^k||Q/kQ_L|}{|P/Q|}$ inequivalent irreducible $K(\mathfrak{g},k)$-modules.

For the determination of the fusion rules, the quantum dimensions are used again. The connection between the quantum dimensions and the fusion product is the following equality:

$$\mathrm{qdim}_V(M\boxtimes N)=\mathrm{qdim}_V M\cdot\mathrm{qdim}_V N$$

for any rational and $C_2$-cofinite vertex operator algebra $V$ and its irreducible modules $M,N$ [12]. This quantum dimension equality gives an upper bound for the fusion rules among three irreducible $V$-modules. In a normal situation, one has found some intertwining operators among irreducible modules already. Using the upper bounds from the quantum dimension equality, one can check if these intertwining operators give enough fusion rules. This is exactly how the fusion product

$$M^{\Lambda_1,\Lambda_1+\beta_i} \boxtimes M^{\Lambda_2,\Lambda_2+\beta_j} = \sum_{\Lambda_3 \in P_+^k} N_{\Lambda_1,\Lambda_2}^{\Lambda_3} M^{\Lambda_3,\Lambda_1+\Lambda_2+\beta_i+\beta_j}$$

is obtained for $\Lambda_1, \Lambda_2 \in P_+^k$ and $i, j \in Q/kQ_L$ where $N_{\Lambda_1,\Lambda_2}^{\Lambda_3}$ are the fusion rules for irreducible $L_{\widehat{\mathfrak{g}}}(k,0)$-modules:

$$L_{\widehat{\mathfrak{g}}}(k,\Lambda_1) \boxtimes L_{\widehat{\mathfrak{g}}}(k,\Lambda_2) = \sum_{\Lambda_3 \in P_+^k} N_{\Lambda_1,\Lambda_2}^{\Lambda_3} L_{\widehat{\mathfrak{g}}}(k,\Lambda_3).$$

The fusion rules in [8] and [30] are computed in the same fashion.

The paper is organized as follows. We review the basics on quantum dimensions from the theory of vertex operator algebras [12] and the Frobenius-Perron dimensions from the fusion tensor category [31] and discuss their properties and connections in Section 2. We recall the construction of the parafermion vertex operator algebras $K(\mathfrak{g}, k)$ and their representations [25] in Section 3. We also give an elementary result on the weight lattice $P$ and the dual lattice $Q_L^\circ$ for a simple Lie algebra $\mathfrak{g}$. In Section 4 we discuss the computation of quantum dimensions of the irreducible $K(\mathfrak{g}, k)$-modules. We finish the identification of the irreducible $K(\mathfrak{g}, k)$-modules and the determination of the fusion rules in Section 5.

## 2. The Frobenius-Perron dimensions and the quantum dimensions

In this section we review the basic properties of the Frobenius-Perron dimensions from the fusion category and the quantum dimensions from the vertex operator algebras. In the case the fusion category is the module category for a rational, $C_2$-cofinite vertex operator algebra, we discuss the connection between these dimensions.

We first collect basics of the fusion categories and the Frobenius-Perron dimensions from [5], [31], and [10].

Let $\mathcal{C}$ be a fusion category [5]. That is, $\mathcal{C}$ is a semisimple rigid monoidal category with finite dimensional spaces of morphisms, finitely many irreducible objects and an irreducible unit object $\mathbf{1}_\mathcal{C}$. We use $K(\mathcal{C})$ to denote the Grothendieck ring of $\mathcal{C}$. According to [31] we have

**Theorem 2.1.** *There exists a unique ring homomorphism* $\mathrm{FPdim} : K(\mathcal{C}) \to \mathbb{R}$ *such that* $\mathrm{FPdim}_\mathcal{C}(X) > 0$ *for all* $X \in \mathcal{C}$.

The $\mathrm{FPdim}_\mathcal{C}(X)$ is called the Frobenius-Perron dimension of $X \in \mathcal{C}$. One can also define the Frobenius-Perron dimension for the category $\mathcal{C}$ :

$$\mathrm{FPdim}\,(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \mathrm{FPdim}_\mathcal{C}(X)^2,$$

where $\mathcal{O}(\mathcal{C})$ is the equivalence classes of the simple objects in $\mathcal{C}$.

An *algebra* in a monoidal category $\mathcal{C}$ is an object $A \in \mathcal{C}$ which is an associative algebra [55]. Let $\mathcal{C}_A$ be the right $A$-module category. An algebra $A \in \mathcal{C}$ is said to be étale if it is commutative and the $\mathcal{C}_A$ is semisimple. We say that an étale algebra $A$ is connected if $\dim \mathrm{Hom}_\mathcal{C}(\mathbf{1}, A) = 1$. Using the braiding we can define two left $A$-module structures on a right $A$-module $M$ by

$$A \otimes M \stackrel{c_{A,M}}{\to} M \otimes A \to M$$

or

$$A \otimes M \overset{c_{M,A}^{-1}}{\to} M \otimes A \to M,$$

where $c_{A,M}$ is the braiding. As a result we get two $A$-bimodules $M_+$ and $M_-$. An $A$-module $M$ is called dyslectic if the identity map on $M$ gives an isomorphism of $A$-bimodules $M_+$ and $M_-$ [10]. Let $\mathcal{C}_A^0$ be the subcategory of $\mathcal{C}_A$ consisting of dyslectic modules. The following result from [31] and [10] is important in this paper.

**Theorem 2.2.** *Let $\mathcal{C}$ be a fusion category and let $A$ be a connected étale algebra in $\mathcal{C}$. Then*

$$(\mathrm{FPdim}_{\mathcal{C}} A)\mathrm{FPdim}\, \mathcal{C}_A = \mathrm{FPdim}\, \mathcal{C}$$

*and*

$$(\mathrm{FPdim}_{\mathcal{C}} A)^2 \mathrm{FPdim}\, \mathcal{C}_A^0 = \mathrm{FPdim}\, \mathcal{C}.$$

We now turn to the theory of vertex operator algebra. Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (cf. [6] and [33]). Here are some basics on vertex operator algebras.

A vertex operator algebra $V$ is of *CFT type* if $V$ is simple, $V = \bigoplus_{n\geq 0} V_n$ and $V_0 = \mathbb{C}\mathbf{1}$ [22].

A vertex operator algebra $V$ is called $C_2$-cofinite if $\dim V/C_2(V) < \infty$ where $C_2(V)$ is the subspace of $V$ spanned by $u_{-2}v$ for $u, v \in V$ [57].

A *weak* $V$-module $M = (M, Y_M)$ is a vector space equipped with a linear map

$$\begin{aligned}
&V \to (\mathrm{End}\, M)[[z^{-1}, z]] \\
&v \mapsto Y_M(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \mathrm{End}\, M) \text{ for } v \in V
\end{aligned}$$

satisfying the following conditions for $u, v \in V$, $w \in M$:

$$\begin{aligned}
&v_n w = 0 \quad \text{for} \quad n \in \mathbb{Z} \;\text{ sufficiently large;} \\
&Y_M(\mathbf{1}, z) = 1;
\end{aligned}$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2) Y_M(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2).$$

An (*ordinary*) $V$-module is a weak $V$-module $M$ which is $\mathbb{C}$-graded

$$M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$$

such that $\dim M_\lambda$ is finite and $M_{\lambda+n} = 0$ for fixed $\lambda$ and $n \in \mathbb{Z}$ small enough, where $M_\lambda$ is the eigenspace for $L(0)$ with eigenvalue $\lambda$ :

$$L(0)w = \lambda w = (\mathrm{wt}\, w)w, \quad w \in M_\lambda.$$

An *admissible* $V$-module is a weak $V$-module $M$ which carries a $\mathbb{Z}_+$-grading

$$M = \bigoplus_{n\in\mathbb{Z}_+} M(n)$$

($\mathbb{Z}_+$ is the set of all nonnegative integers) such that if $r, m \in \mathbb{Z}, n \in \mathbb{Z}_+$ and $a \in V_r$, then

$$a_m M(n) \subseteq M(r + n - m - 1).$$

Note that any ordinary module is an admissible module.

A vertex operator algebra $V$ is called *rational* if any admissible module is a direct sum of irreducible admissible modules [17]. It was proved in [18] that if $V$ is rational, then there are only finitely many inequivalent irreducible admissible modules $V = M^0, ..., M^p$ and each irreducible admissible module is an ordinary module. Each $M^i$ has weight space decomposition

$$M^i = \bigoplus_{n \geq 0} M^i_{\lambda_i + n},$$

where $\lambda_i \in \mathbb{C}$ is a complex number such that $M^i_{\lambda_i} \neq 0$ and $M^i_{\lambda_i+n}$ is the eigenspace of $L(0)$ with eigenvalue $\lambda_i + n$. The $\lambda_i$ is called the conformal weight of $M^i$. If $V$ is both rational and $C_2$-cofinite, then $\lambda_i$ and central charge $c$ are rational numbers [19].

In the rest of this paper we assume the following:

(V1) $V = \bigoplus_{n \geq 0} V_n$ is a vertex operator algebra of CFT type,
(V2) $V$ is $C_2$-cofinite and rational,
(V3) The conformal weight $\lambda_i$ is nonnegative and $\lambda_i = 0$ if and only if $i = 0$.

Let $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ be a $V$-module. Set $M' = \bigoplus_{\lambda \in \mathbb{C}} M^*_\lambda$, the restricted dual of $M$. It is proved in [32] that $M' = (M', Y')$ is naturally a $V$-module such that

$$\langle Y'(a, z)u', v \rangle = \langle u', Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v \rangle,$$

for $a \in V, u' \in M'$ and $v \in M$, and $(M')' \cong M$. Moreover, if $M$ is irreducible, so is $M'$. A $V$-module $M$ is said to be self dual if $M$ and $M'$ are isomorphic.

Recall from [32] the notion of intertwining operator and fusion rule. Let $W^i = (W^i, Y_{W^i})$ for $i = 1, 2, 3$ be $V$-modules. An intertwining operator $\mathcal{Y}(\cdot, z)$ of type $\begin{pmatrix} W^3 \\ W^1 \ W^2 \end{pmatrix}$ is a linear map

$$\mathcal{Y}(\cdot, z) : W^1 \to \mathrm{Hom}(W^2, W^3)\{z\},\ v^1 \mapsto \mathcal{Y}(v^1, z) = \sum_{n \in \mathbb{C}} v^1_n z^{-n-1}$$

such that

(i) For any $v^1 \in W^1, v^2 \in W^2$ and $\lambda \in \mathbb{C}, v^1_{n+\lambda} v^2 = 0$ for $n \in \mathbb{Z}$ sufficiently large.

(ii) For any $a \in V, v^1 \in W^1$,

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_{W^3}(a, z_1)\mathcal{Y}(v^1, z_2) - z_0^{-1}\delta(\frac{z_1 - z_2}{-z_0})\mathcal{Y}(v^1, z_2)Y_{W^2}(a, z_1)$$

$$= z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})\mathcal{Y}(Y_{W^1}(a, z_0)v^1, z_2).$$

(iii) For $v^1 \in W^1$, $\frac{d}{dz}\mathcal{Y}(v^1, z) = \mathcal{Y}(L(-1)v^1, z)$.

The intertwining operators of type $\begin{pmatrix} W^3 \\ W^1 \ W^2 \end{pmatrix}$ form a vector space denoted by

$$I_V \begin{pmatrix} W^3 \\ W^1 \ W^2 \end{pmatrix}.$$

The dimension $N_{W^1,W^2}^{W^3}$ of $I_V\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$ is called the fusion rule of type $\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix}$. It is proved in [1] that the fusion rules for three irreducible modules are finite.

The following two propositions can be found in [2].

**Proposition 2.3.** *Let $V$ be a vertex operator algebra and let $W^1$, $W^2$, $W^3$ be $V$-modules among which $W^1$ and $W^2$ are irreducible. Suppose that $U$ is a vertex operator subalgebra of $V$ (with the same Virasoro element) and that $N^1$ and $N^2$ are irreducible $U$-submodules of $W^1$ and $W^2$, respectively. Then the restriction map from $I_V\binom{W^3}{W^1\ W^2}$ to $I_U\binom{M^3}{N^1\ N^2}$ is injective. In particular,*

$$\dim I_V\begin{pmatrix} W^3 \\ W^1\ W^2 \end{pmatrix} \leq \dim I_U\begin{pmatrix} W^3 \\ N^1\ N^2 \end{pmatrix}.$$

Let $V^1$ and $V^2$ be vertex operator algebras, let $W^i\,(i = 1, 2, 3)$ be $V^1$-modules and let $N^i\,(i = 1, 2, 3)$ be $V^2$-modules. Then $W^i \otimes N^i\,(i = 1, 2, 3)$ are $V^1 \otimes V^2$-modules [32].

**Proposition 2.4.** *If $N_{W^1,W^2}^{W^3} < \infty$ or $N_{N^1,N^2}^{N^3} < \infty$, then*

$$N_{W^1\otimes N^1,W^2\otimes N^2}^{W^3\otimes N^3} = N_{W^1,W^2}^{W^3} N_{N^1,N^2}^{N^3}.$$

Let $W^1$ and $W^2$ be two $V$-modules. A tensor product for the ordered pair $(W^1, W^2)$ is a pair $(W, F(\cdot, z))$, which consists of a $V$-module $W$ and an intertwining operator $F(\cdot, z)$ of type $\begin{pmatrix} W \\ W^1\ W^2 \end{pmatrix}$, such that the following universal property holds: For any $V$-module $M$ and any intertwining operator $I(\cdot, z)$ of type $\begin{pmatrix} M \\ W^1\ W^2 \end{pmatrix}$, there exists a unique $V$-homomorphism $\phi$ from $W$ to $M$ such that $I(\cdot, z) = \phi \circ F(\cdot, z)$. It is clear from the definition that if a tensor product of $W^1$ and $W^2$ exsits, it is unique up to isomorphism. In this case, we denote the tensor product by $W^1 \boxtimes W^2$.

The following results are obtained in [39], [40], [41] and [35], [36], [37].

**Theorem 2.5.** *Let $V$ be a vertex operator algebra satisfying conditions (V1)-(V3).*

*1. The tensor product of any two $V$-modules $M \boxtimes N$ exists. In particular, $M^i \boxtimes M^j$ of $M^i$ and $M^j$ exists and is equal to $\sum_k N_{i,j}^k M^k$ for any $i, j \in \{0, ..., p\}$.*

*2. The $V$-module category $\mathcal{C}_V$ is a fusion category.*

The modular transformation of trace functions of irreducible modules of vertex operator algebra [57] is another important ingredient in this paper. Another vertex operator algebra structure $(V, Y[\cdot, z], \mathbf{1}, \omega - c/24)$ is defined on $V$ in [57] with grading

$$V = \bigoplus_{n\geq 0} V_{[n]}.$$

For $v \in V_{[n]}$ we write $wt[v] = n$. We denote $v_{n-1}$ by $o(v)$ for $v \in V_n$ and extend to $V$ linearly. For $i = 0, ..., p$ and $v \in V$, we set

$$Z_i(v, q) = \mathrm{tr}_{M^i}\, o(v) q^{L(0)-c/24} = \sum_{n\geq 0} (\mathrm{tr}_{M^i_{\lambda_i+n}}\, o(v)) q^{\lambda_i+n-c/24}$$

which is a formal power series in variable $q$. The constant $c$ here is the central charge of $V$. The $Z_i(\mathbf{1}, q)$ which is denoted by $\mathrm{ch}_q M^i$ sometimes is called the $q$-character of $M^i$. The $Z_i(v, q)$ converges to a holomorphic function in $0 < |q| < 1$ [57]. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid im\, \tau > 0\}$ be the upper half complex plane and $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$. Denote by $Z_i(v, \tau)$ the holomorphic function $Z_i(v, q)$ on $\mathbb{H}$.

Note that the modular group $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ in an obvious way.

**Theorem 2.6.** *Let $V$ be a vertex operator algebra satisfying (V1)-(V3).*

*(1) There is a group homomorphism $\rho : SL_2(\mathbb{Z}) \to GL_{p+1}(\mathbb{C})$ with $\rho(\gamma) = (\gamma_{ij})$ such that for any $0 \leq i \leq p$ and $v \in V_{[n]}$*

$$Z_i(v, \gamma\tau) = (c\tau + d)^n \sum_{j=0}^{p} \gamma_{ij} Z_j(v, \tau).$$

*(2) Each $Z_i(v, \gamma\tau)$ is a modular form of weight $n$ over a congruence subgroup $\Gamma(m)$ for some $m \geq 1$.*

Part (1) of the theorem was obtained in [57] and Part (2) was established in [23]. The matrices

$$S = \rho_V\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) \text{ and } T = \rho_V\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

are respectively called the *genus one* $S$ and $T$-matrices of $V$.

Finally we can define the quantum dimension. Let $V$ be as before and let $M$ be a $V$-module. Then $M = \sum_{i=0}^{p} M^i$ is a direct sum of finitely many irreducible $V$-modules. Then both $Z_V(\tau) = Z_0(\mathbf{1}, \tau)$ and $Z_M(\tau) = Z_M(1, \tau)$ exist. The quantum dimension of $M$ over $V$ is defined as

$$\mathrm{qdim}_V M = \lim_{y \to 0} \frac{Z_M(iy)}{Z_V(iy)} = \lim_{q \to 1} \frac{\mathrm{ch}_q M}{\mathrm{ch}_q V},$$

where $y$ is real and positive and $\mathrm{ch}_q M$ is the $q$-character of $M$.

The following result was given in [12].

**Theorem 2.7.** *Let $V$ be a vertex operator algebra satisfying (V1)-(V3).*

*(1) $\mathrm{qdim}_V M^i = \frac{S_{i0}}{S_{00}}$ exists and is greater than or equal to $1$ for all $i$ where $S = (S_{ij})$.*

*(2) $\mathrm{qdim}_V M^i$ is the maximal eigenvalue of the fusion matrix $N(i) = (N_{ij}^k)_{jk}$.*

*(3) $\mathrm{qdim}_V M^i \boxtimes M^j = \mathrm{qdim}_V M^i \cdot \mathrm{qdim}_V M^j$ for all $i, j$.*

*(4) $M^i$ is a simple current if and only if $\mathrm{qdim}_V M^i = 1$.*

By Theorems 2.1, 2.5 and 2.7, we see the relation between the quantum dimension and the Frobenius-Perron dimension: $\mathrm{qdim}_V M^i = \mathrm{FPdim}_{\mathcal{C}_V}(M^i)$ for all $i$. We also define the global dimension

$$\mathrm{glob}(V) = \sum_{i=0}^{p} (\mathrm{qdim}_V M^i)^2.$$

It is clear that $\mathrm{glob}(V) = \mathrm{FPdim}\,(\mathcal{C}_V)$.

An extension $U$ of $V$ is a simple vertex operator algebra containing $V$. Then $U$ is again $C_2$-cofinite [1]. Here we quote a recent result from [38].

**Theorem 2.8.** *Let $V$ be a vertex operator algebra satisfying (V1)-(V3).*

*(1) If $U$ is an extension vertex operator algebra of $V$, then $U$ induces an étale algebra $A_U$ in $\mathcal{C}_V$ such that $A_U$ is isomorphic to $U$ as a $V$-module.*

*(2) If $U$ is a $V$-module having integral conformal weight and $U$ is a commutative algebra in $\mathcal{C}_V$, then $U$ has a vertex operator algebra structure such that $U$ is an extension vertex operator algebra of $V$.*

*(3) $U$ is rational.*

**Theorem 2.9.** *Let $V$ be a vertex operator algebra satisfying (V1)-(V3) and let a simple vertex operator algebra $U$ be an extension of $V$. Then $U$ also satisfies (V1)-(V3) and*

$$\mathrm{glob}(V) = \mathrm{glob}(U)(\mathrm{qdim}_V U)^2.$$

*Proof.* The theorem is a combination of [44], Theorem 2.2, [12], and [38]. In this case, let $\mathcal{C}$ be the category of $V$-modules and $A = U$ an algebra in $\mathcal{C}$. Then $\mathcal{C}_A^0$ is the $U$-module category by [44]. □

We now consider two vertex operator algebras $V^1, V^2$ satisfying conditions (V1)-(V3). Then it is easy to see that the tensor product vertex operator algebra $V^1 \otimes V^2$ [32] also satisfies assumptions (V1)-(V3).

**Lemma 2.10.** *Let $M$ be a $V^1$-module and let $N$ be a $V^2$-module. Then*

$$\mathrm{qdim}_{V^1\otimes V^2} M \otimes N = \mathrm{qdim}_{V^1} M \cdot \mathrm{qdim}_{V^2} N,$$

$$\mathrm{glob}(V^1 \otimes V^2) = \mathrm{glob}(V^1)\mathrm{glob}(V^2).$$

*Proof.* The equality $\mathrm{qdim}_{V^1\otimes V^2} M \otimes N = \mathrm{qdim}_{V^1} M \cdot \mathrm{qdim}_{V^2} N$ follows from the fact that $Z_{M\otimes N}(\tau) = Z_M(\tau)Z_N(\tau)$ and the equality

$$\mathrm{glob}(V^1 \otimes V^2) = \mathrm{glob}(V^1)\mathrm{glob}(V^2)$$

follows from the first equality and the fact that the irreducible $V^1 \otimes V^2$-modules are exactly $M \otimes N$ where $M$ is an irreducible $V^1$-module and $N$ is an irreducible $V^2$-module [32]. □

## 3. Parafermion vertex operator algebras

In this section we recall the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ and its representations associated to any finite dimensional simple Lie algebra $\mathfrak{g}$ and positive integer $k$ from [25].

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with a Cartan subalgebra $\mathfrak{h}$. We denote the corresponding root system by $\Delta$ and the root lattice by $Q$. Fix an invariant symmetric nondegenerate bilinear form $\langle , \rangle$ on $\mathfrak{g}$ such that $\langle \alpha, \alpha\rangle = 2$ if $\alpha$ is a long root, where we have identified $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\langle , \rangle$. We denote the image of $\alpha \in \mathfrak{h}^*$ in $\mathfrak{h}$ by $t_\alpha$. That is, $\alpha(h) = \langle t_\alpha, h\rangle$ for any $h \in \mathfrak{h}$. Fix simple roots $\{\alpha_1, ..., \alpha_l\}$ and let $\Delta_+$ be the set of corresponding positive roots. Denote the highest root by $\theta$.

Recall that the weight lattice $P$ of $\mathfrak{g}$ consists of $\lambda \in \mathfrak{h}^*$ such that $\frac{2\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha \in \Delta$. It is well-known that $P = \bigoplus_{i=1}^l \mathbb{Z}\Lambda_i$ where $\Lambda_i$ are the fundamental weights defined by the equation $\frac{2\langle \Lambda_i,\alpha_j\rangle}{\langle \alpha_j,\alpha_j\rangle} = \delta_{i,j}$. Let $P_+$ be the subset of $P$ consisting of the dominant weight $\Lambda \in P$ in the sense that $\frac{2\langle \Lambda,\alpha_j\rangle}{\langle \alpha_j,\alpha_j\rangle}$ is nonnegative for all $j$. For any nonnegative integer $k$ we also let $P_+^k$ be the subset of $P_+$ consisting of $\Lambda$ satisfying $\langle \Lambda, \theta\rangle \le k$.

Let $Q = \sum_{i=1}^{l} \mathbb{Z}\alpha_i$ be the root lattice and $Q_L$ be the sublattice of $Q$ spanned by the long roots. Recall that the dual lattice $Q_L^\circ$ consists $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in Q_L$.

For the purpose of identifying the irreducible $K(\mathfrak{g}, k)$-module, we need the following lemma.

**Lemma 3.1.** *For any simple Lie algebra* $\mathfrak{g}$, $Q_L^\circ = P$.

*Proof.* The result is obvious if $\mathfrak{g}$ is a Lie algebra of type $A, D, E$ as $\langle \alpha, \alpha \rangle = 2$ if $\alpha$ is a long root. We now assume that $\mathfrak{g}$ is a Lie algebra of other type. First, observe that $P \subset Q_L^\circ$. It remains to show that $Q_L^\circ \subset P$. We will do a verification case by case using the root systems given in [42].

(1) Type $B_l$. Let $\mathbb{E} = \mathbb{R}^l$ with the standard orthonormal basis $\{\epsilon_1, ..., \epsilon_l\}$. Then

$$\Delta = \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) | i \neq j\}.$$

Let $\lambda \in Q_L^\circ$. Then $\langle \lambda, (\epsilon_i \pm \epsilon_j) \rangle \in \mathbb{Z}$ for all $i \neq j$. This implies that $\langle \lambda, \epsilon_i \rangle \in \frac{1}{2}\mathbb{Z}$ and $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ if $\alpha$ is a short root. That is, $\lambda \in P$.

(2) Type $C_l$. In this case,

$$\Delta = \{\pm\sqrt{2}\epsilon_i, \pm\frac{1}{\sqrt{2}}(\epsilon_i \pm \epsilon_j) | i \neq j\}.$$

If $\lambda \in Q_L^\circ$, then $\langle \lambda, \epsilon_i \rangle \in \frac{1}{\sqrt{2}}\mathbb{Z}$ and $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ if $\alpha$ is a short root.

(3) Type $F_4$. Let $\mathbb{E} = \mathbb{R}^4$. Then

$$\Delta = \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j), \pm\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) | i \neq j\}.$$

If $\lambda \in Q_L^\circ$, then $\langle \lambda, \epsilon_i \rangle \in \frac{1}{2}\mathbb{Z}$ and $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ if $\alpha$ is a short root.

(4) Type $G_2$. Let $\mathbb{E}$ be the subspace of $\mathbb{R}^3$ orthogonal to $\epsilon_1 + \epsilon_2 + \epsilon_3$. Then

$$\Delta = \pm\frac{1}{\sqrt{3}}\{\epsilon_i - \epsilon_j, 2\epsilon_1 - \epsilon_2 - \epsilon_3, 2\epsilon_2 - \epsilon_1 - \epsilon_3, 2\epsilon_3 - \epsilon_1 - \epsilon_2 | i \neq j\}.$$

If $\lambda \in Q_L^\circ$, then

$$\langle \lambda, \frac{1}{\sqrt{3}}(2\epsilon_1 - \epsilon_2 - \epsilon_3 - 2\epsilon_2 + \epsilon_1 + \epsilon_3) \rangle \in \mathbb{Z}.$$

This gives $\langle \lambda, \epsilon_1 - \epsilon_2 \rangle \in \frac{1}{\sqrt{3}}\mathbb{Z}$ and $\frac{2\langle \lambda, \frac{1}{\sqrt{3}}(\epsilon_1 - \epsilon_2) \rangle}{\frac{1}{3}\langle \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_2 \rangle} \in \mathbb{Z}$. Similarly, one can verify that $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for any short root $\alpha$. The proof is complete. □

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Lie algebra. Fix a nonnegative integer $k$. For any $\Lambda \in P_+^k$ let $L(\Lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\Lambda$ and let $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ be the unique irreducible $\widehat{\mathfrak{g}}$-module such that $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ is generated by $L(\Lambda)$ and $\mathfrak{g} \otimes t^n L(\Lambda) = 0$ for $t > 0$ and $K$ acts as constant $k$. The following result is well known (cf. [34],[46], [57]):

**Theorem 3.2.** *The* $L_{\widehat{\mathfrak{g}}}(k, 0)$ *is a vertex operator algebra satisfying conditions (V1)-(V3). Namely,* $L_{\widehat{\mathfrak{g}}}(k, 0)$ *is a simple, rational and* $C_2$*-cofinite vertex operator algebra whose irreducible modules are* $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ *for* $\Lambda \in P_+^k$ *and the weight* $\lambda_{L_{\widehat{\mathfrak{g}}}(k,\Lambda)}$ *of* $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ *is* $\frac{\langle \Lambda + 2\rho, \Lambda \rangle}{2(k + h^\vee)}$ *where* $\rho = \sum_{i=1}^{l} \Lambda_i$ *and* $h^\vee$ *is the dual Coxeter number.*

Let $\theta = \sum_{i=1}^{l} a_i\alpha_i$. Here is a list of $a_i = 1$ using the labeling from [43]:

$$\begin{array}{rc} A_l: & a_1, ..., a_l, \\ B_l: & a_1, \\ C_l: & a_l, \\ D_l: & a_1, a_{l-1}, a_l, \\ E_6: & a_1, a_5, \\ E_7: & a_6. \end{array}$$

Denote by $I$ the set of $i$ with $a_i = 1$. It is easy to see that the cardinality of $I$ is equal to $|P/Q| - 1$ [52].

Let $M_{\widehat{\mathfrak{h}}}(k)$ be the vertex operator subalgebra of $L_{\widehat{\mathfrak{g}}}(k,0)$ generated by $h(-1)\mathbf{1}$ for $h \in \mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$, denote by $M_{\widehat{\mathfrak{h}}}(k,\lambda)$ the irreducible highest weight module for $\widehat{\mathfrak{h}}$ with a highest weight vector $e^\lambda$ such that $h(0)e^\lambda = \lambda(h)e^\lambda$ for $h \in \mathfrak{h}$. The parafermion vertex operator algebra $K(\mathfrak{g},k)$ is the commutant [34] of $M_{\widehat{\mathfrak{h}}}(k)$ in $L_{\widehat{\mathfrak{g}}}(k,0)$. We have the following decomposition:

$$L_{\widehat{\mathfrak{g}}}(k,\Lambda) = \bigoplus_{\lambda \in Q+\Lambda} M_{\widehat{\mathfrak{h}}}(k,\lambda) \otimes M^{\Lambda,\lambda}$$

as $M_{\widehat{\mathfrak{h}}}(k) \otimes K(\mathfrak{g},k)$-module. Moreover, $M^{0,0} = K(\mathfrak{g},k)$ and $M^{\Lambda,\lambda}$ is an irreducible $K(\mathfrak{g},k)$-module [25].

It is proved in [29] that the lattice vertex operator algebra $V_{\sqrt{k}Q_L}$ is a vertex operator subalgebra of $L_{\widehat{\mathfrak{g}}}(k,0)$ and the parafermion vertex operator algebra $K(\mathfrak{g},k)$ is also a commutant of $V_{\sqrt{k}Q_L}$ in $L_{\widehat{\mathfrak{g}}}(k,0)$. This gives us another decomposition

$$L_{\widehat{\mathfrak{g}}}(k,\Lambda) = \bigoplus_{i \in Q/kQ_L} V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda+\beta_i)} \otimes M^{\Lambda,\Lambda+\beta_i}$$

as modules for $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g},k)$ where $M^{\Lambda,\lambda}$ is as before and $Q = \bigcup_{i \in Q/kQ_L}(kQ_L + \beta_i)$.

Here are the main results on $K(\mathfrak{g},k)$.

**Theorem 3.3.** *Let $\mathfrak{g}$ be a simple Lie algebra and let $k$ be a positive integer.*

*(1) The $K(\mathfrak{g},k)$ is a vertex operator algebra satisfying conditions (V1)-(V3).*

*(2) For any $\Lambda \in P_+^k$, $\lambda \in \Lambda + Q$ and $\alpha \in Q_L$, $M^{\Lambda,\lambda} = M^{\Lambda,\lambda+k\alpha}$.*

*(3) For each $i \in I$, $\Lambda \in P_+^k$ there exists a unique $\Lambda^{(i)} \in P_+^k$ such that for any $\lambda \in \Lambda + Q$, $M^{\Lambda,\lambda} = M^{\Lambda^{(i)},\lambda+k\Lambda_i}$.*

*(4) Any irreducible $K(\mathfrak{g},k)$-module is isomorphic to $M^{\Lambda,\lambda}$ for some $\Lambda \in P_+^k$ and $\lambda \in \Lambda + Q$.*

The $C_2$-cofiniteness of $K(\mathfrak{g},k)$ was obtained in [3] (also see [54]) and the rest of the results in the theorem can be found in [4] and [25].

The following result will be useful later.

**Lemma 3.4.** *Fix $\Lambda \in P_+^k$ and $\lambda \in \Lambda + Q$. Let $A = \{\Lambda + \beta_j + kQ_L | j \in Q/kQ_L\}$. Then the set $\{(\Lambda, \lambda+kQ_L), (\Lambda^{(i)}, \lambda+k\Lambda_i+kQ_L) | i \in I\}$ gives exactly $|I|+1$ elements in $P_+^k \times A$.*

*Proof.* It is proved in [25] that $(\Lambda, \lambda + kQ_L)$ is different from $(\Lambda^{(i)}, \lambda + k\Lambda_i + kQ_L)$ for $i \in I$. Let $i, j \in I$ be distinct. We can assume that $\Lambda^{(i)} = \Lambda^{(j)}$. Then

$$(\Lambda^{(i)}, \lambda + k\Lambda_i + kQ_L) = (\Lambda^{(j)}, \lambda + k\Lambda_j + kQ_L)$$

if and only if $\Lambda_i - \Lambda_j \in Q_L$. If $\mathfrak{g}$ is of $A, D, E$ type, this cannot happen. For type $B_l$ and $C_l$ the cardinality of $I$ is 1. The proof is complete. $\square$

From Theorem 3.3 and Lemma 3.4 we immediately have:

**Corollary 3.5.** *Let $\mathfrak{g}$ be a simple Lie algebra and let $k$ be a positive integer. Then $K(\mathfrak{g}, k)$ has at most $\frac{|P_+^k||Q/kQ_L|}{|P/Q|}$ inequivalent irreducible modules.*

## 4. Quantum dimensions of the parafermion vertex operator algebras

In this section we compute the quantum dimensions of irreducible $K(\mathfrak{g}, k)$-modules. The ideas and methods here are different from those used in [28]. We do not need the $S$-matrix for the computation.

First we need a result on the quantum dimension in orbifold theory from [12]. Let $V$ be a simple vertex operator algebra and $G$ a finite automorphism group of $V$. Then $V^G$ is a vertex operator subalgebra and $V$ has a decomposition

$$V = \bigoplus_{\chi \in \widehat{G}} W_\chi \otimes V_\chi,$$

where $\widehat{G}$ is the set of irreducible characters of $G$ and $W_\chi$ is the simple $G$-module with the character $\chi$ and $V_\chi$ is an irreducible $V^G$-module [24], [16]. We need the following result from [12].

**Theorem 4.1.** *Let $V$ be a vertex operator algebra satisfying (V1)-(V3) and let $G$ be a finite automorphism group of $V$ such that $V$ is $g$-rational for every $g \in G$ and any irreducible $g$-twisted $V$-module $M = \bigoplus_{n \geq 0} M_{\lambda + \frac{n}{T}}$ has positive conformal weight $\lambda$ if $g \neq 1$ where $T$ is the order of $g$. Then $\operatorname{qdim}_{V^G} V_\chi = \dim W_\chi$.*

The next result tells us how the rationality of $V^G$ implies the $g$-rationality of all $g \in G$.

**Lemma 4.2.** *Let $V$ be a simple vertex operator algebra and let $G$ be a finite automorphism group of $V$ such that both $V$ and $V^G$ satisfies assumptions (V1)-(V3). Then $V$ is $g$-rational for all $g \in G$ and the conformal weight of any irreducible $g$-twisted $V$-module is positive.*

*Proof.* Since $V^G$ satisfies assumptions (V1)-(V3), the vertex operator subalgebra $V^{\langle g \rangle}$ of $V$ is $C_2$-cofinite [1] and rational [38]. Here we need some facts about associative algebra $A_{g,n}(V)$ for $g \in G$ and $0 \leq n \in \frac{1}{T}\mathbb{Z}$ from [20], [21] where $T$ is the order of $g$. Let $A_m(V) = A_{1,m}(V)$. The following are true: (1) $V$ is $g$-rational if and only if $A_{g,n}(V)$ is semisimple for all $n$, (2) There is an onto algebra homomorphism from $A_{[n]}(V^{\langle g \rangle})$ to $A_{g,n}(V)$ where $[n]$ is the largest integer less than or equal to $n$. Since $V^{\langle g \rangle}$ is rational, $A_m(V^{\langle g \rangle})$ is semisimple for all $m$. Thus, $A_{g,n}(V)$ is semisimple for all $n$ and $V$ is $g$-rational.

It remains to prove that the conformal weight $\lambda$ of any irreducible $g$-twisted $V$-module $M$ is positive if $g \neq 1$. Let $T$ be the order of $g$. Then $M$ has decomposition

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda + \frac{n}{T}}$$

such that $M_\lambda \neq 0$ where $M_{\lambda+\frac{n}{T}}$ is the eigenspace of $L(0)$ with eigenvalue $\lambda+\frac{n}{T}$ [18]. Moreover, for each $s=0,...,T-1$ the subspace $\bigoplus_{n\in\mathbb{Z}} M_{\lambda+n+\frac{s}{T}} \neq 0$. If $\lambda=0$, then there exists an irreducible $V^G$-submodule $W$ of $M$ isomorphic to $V^G$ as $V^G$ is the only irreducible $V^G$-module whose conformal weight is 0. Let $V=\bigoplus_s V^s$ where $V^s$ is the irreducible $V^G$-module. Then $M=\sum_s V^s \cdot W$ where $V^s\cdot W$ is a subspace spanned by $u_nW$ for $u\in V^s$ and $n\in\mathbb{Z}$. It is easy to see that $V^s\cdot W$ is isomorphic to $V^s$ as $V^G$-module. This implies that $M$ has only integral weights. This is a contradiction as $T\neq 1$. $\square$

Recall the irreducible $K(\mathfrak{g},k)$-module $M^{0,\beta_i}$ for $i\in Q/kQ_L$ from Section 3.

**Lemma 4.3.** *The $M^{0,\beta_i}$ is a simple current.*

*Proof.* We need to introduce a finite abelain group $G$ following [25]. Let $G$ be the dual group of the finite abelian group $Q/kQ_L$. Then $G$ is a group of automorphisms of $L_{\widehat{\mathfrak{g}}}(k,0)$ such that $g\in G$ acts as $g(\beta_i+kQ_L)$ on $V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i}\otimes M^{0,\beta_i}$. Clearly, each $\beta_i+kQ_L$ is an irreducible character of $G$. So $V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i}\otimes M^{0,\beta_i}$ in the decomposition

$$L_{\widehat{\mathfrak{g}}}(k,0)=\bigoplus_{i\in Q/kQ_L} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i}\otimes M^{0,\beta_i}$$

corresponds to the character $\beta_i+kQ_L$. In particular, $L_{\widehat{\mathfrak{g}}}(k,0)^G=V_{\sqrt{k}Q_L}\otimes K(\mathfrak{g},k)$.

It follows from Theorem 4.1 and Lemma 4.2, $\operatorname{qdim}_{V_{\sqrt{k}Q_L}\otimes K(\mathfrak{g},k)} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i}\otimes M^{0,\beta_i}=1$. We can also use the $g$-rationality of $L_{\widehat{\mathfrak{g}}}(k,0)$ from [50]. It is well known that every irreducible $V_{\sqrt{k}Q_L}$-module is a simple current [15]. Then by Lemma 2.10, $\operatorname{qdim}_{K(\mathfrak{g},k)} M^{0,\beta_i}=1$. Since $K(\mathfrak{g},k)$ satisfies conditions (V1)-(V3). It follows that $M^{0,\beta_i}$ is a simple current. $\square$

One can also obtain Lemma 4.3 by using the mirror extension [53].

The next result asserts that all the irreducible $K(\mathfrak{g},k)$-modules occurring in $L_{\widehat{\mathfrak{g}}}(k,\Lambda)$ for $\Lambda\in P_+^k$ have the same quantum dimension.

**Lemma 4.4.** *Let $\Lambda\in P_+^k$ Then $\operatorname{qdim}_{K(g,k)} M^{\Lambda,\lambda}=\operatorname{qdim}_{K(g,k)} M^{\Lambda,\Lambda}$ for all $\lambda\in\Lambda+Q$.*

*Proof.* By Theorem 3.3, every $M^{\Lambda,\lambda}$ for $\lambda\in\Lambda+Q$ is isomorphic to $M^{\Lambda,\Lambda+\beta_i}$ for some $i\in Q/kQ_L$. So it is sufficient to show that all the $M^{\Lambda,\Lambda+\beta_i}$ have the same quantum dimension.

Recall the decompositions

$$L_{\widehat{\mathfrak{g}}}(k,\Lambda)=\bigoplus_{i\in Q/kQ_L} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\Lambda+\beta_i)}\otimes M^{\Lambda,\Lambda+\beta_i}$$

and

$$L_{\widehat{\mathfrak{g}}}(k,0)=\bigoplus_{i\in Q/kQ_L} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i}\otimes M^{0,\beta_i}.$$

Since $L_{\widehat{\mathfrak{g}}}(k,\Lambda)$ is an irreducible $L_{\widehat{\mathfrak{g}}}(k,0)$-module, we see that

$$L_{\widehat{\mathfrak{g}}}(k,0)\cdot V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\Lambda+\beta_i)}\otimes M^{\Lambda,\Lambda+\beta_i}=L_{\widehat{\mathfrak{g}}}(k,\Lambda)$$

for any $i \in Q/kQ_L$. Here we use $X \cdot W$ to denote the subspace spanned by $u_n W$ for $u \in X$ and $n \in \mathbb{Z}$ where $X$ is a subspace of a vertex operator algebra and $W$ is a subset of a $V$-module. It follows that

$$\bigoplus_{j \in Q/kQ_L} V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}\beta_j} \otimes M^{0,\beta_j} \cdot V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda+\beta_i)} \otimes M^{\Lambda,\Lambda+\beta_i} = L_{\widehat{\mathfrak{g}}}(k,\Lambda).$$

This implies that

$$\begin{aligned} V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}\beta_j} \otimes M^{0,\beta_j} \cdot V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda+\beta_i)} \otimes M^{\Lambda,\Lambda+\beta_i} \\ = V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda+\beta_i+\beta_j)} \otimes M^{\Lambda,\Lambda+\beta_i+\beta_j} \end{aligned}$$

for all $j$. By Lemma 4.3, $V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}\beta_j} \otimes M^{0,\beta_j}$ is a simple current. It follows from Lemma 2.10 that

$$\begin{aligned} \operatorname{qdim}_{K(g,k)} M^{\Lambda,\Lambda+\beta_i+\beta_j} &= \operatorname{qdim}_{K(g,k)} M^{0,\beta_j} \cdot \operatorname{qdim}_{K(g,k)} M^{\Lambda,\Lambda+\beta_i} \\ &= \operatorname{qdim}_{K(g,k)} M^{\Lambda,\Lambda+\beta_i}. \end{aligned}$$

The proof is complete. □

We now can give an explicit expression for the quantum dimension of any irreducible $K(\mathfrak{g},k)$-module $M^{\Lambda,\lambda}$. Recall from [9] that the quantum dimension $\operatorname{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda) = \prod_{\alpha>0} \frac{(\Lambda+\rho,\alpha)_q}{(\rho,\alpha)_q}$ (see Introduction).

**Theorem 4.5.** *For any $\Lambda \in P_+^k$ and $\lambda \in \Lambda + Q$,*

$$\operatorname{qdim}_{K(g,k)} M^{\Lambda,\lambda} = \operatorname{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda).$$

*Proof.* The proof is a straightforward computation by noting that the irreducible modules of $V_{\sqrt{k}Q_L}$ are simple currents:

$$\begin{aligned} &\operatorname{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda) = \lim_{q\to 1} \frac{\operatorname{ch}_q L_{\widehat{\mathfrak{g}}}(k,\Lambda)}{\operatorname{ch}_q L_{\widehat{\mathfrak{g}}}(k,0)} \\ &= \lim_{q\to 1} \frac{\sum_{i\in Q/kQ_L} \operatorname{ch}_q V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\Lambda+\beta_i)} \cdot \operatorname{ch}_q M^{\Lambda,\Lambda+\beta_i}}{\sum_{i\in Q/kQ_L} \operatorname{ch}_q V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i} \cdot \operatorname{ch}_q M^{0,\beta_i}} \\ &= \lim_{q\to 1} \frac{\big(\sum_{i\in Q/kQ_L} \operatorname{ch}_q V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\Lambda+\beta_i)} \cdot \operatorname{ch}_q M^{\Lambda,\Lambda+\beta_i}\big)/\big(\operatorname{ch}_q V_{\sqrt{k}Q_L} \cdot \operatorname{ch}_q K(\mathfrak{g},k)\big)}{\big(\sum_{i\in Q/kQ_L} \operatorname{ch}_q V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i} \cdot \operatorname{ch}_q M^{0,\beta_i}\big)/\big(\operatorname{ch}_q V_{\sqrt{k}Q_L} \cdot \operatorname{ch}_q K(\mathfrak{g},k)\big)} \\ &= \frac{\sum_{i\in Q/kQ_L} \lim_{q\to 1} \frac{\operatorname{ch}_q V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\Lambda+\beta_i)}}{\operatorname{ch}_q V_{\sqrt{k}Q_L}} \lim_{q\to 1} \frac{\operatorname{ch}_q M^{\Lambda,\Lambda+\beta_i}}{\operatorname{ch}_q K(\mathfrak{g},k)}}{\sum_{i\in Q/kQ_L} \lim_{q\to 1} \frac{\operatorname{ch}_q V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i}}{\operatorname{ch}_q V_{\sqrt{k}Q_L}} \lim_{q\to 1} \frac{\operatorname{ch}_q M^{0,\beta_i}}{\operatorname{ch}_q K(\mathfrak{g},k)}} \\ &= \frac{\sum_{i\in Q/kQ_L} \lim_{q\to 1} \frac{\operatorname{ch}_q M^{\Lambda,\Lambda+\beta_i}}{\operatorname{ch}_q K(\mathfrak{g},k)}}{|Q/kQ_L|} \\ &= \frac{|Q/kQ_L| \cdot \operatorname{qdim}_{K(\mathfrak{g},k)} M^{\Lambda,\Lambda+\beta_i}}{|Q/kQ_L|} \\ &= \operatorname{qdim}_{K(\mathfrak{g},k)} M^{\Lambda,\Lambda+\beta_i} \end{aligned}$$

for any $i \in Q/kQ_L$. □

We remark that one can also use the $S$-matrix given in [43] to compute the quantum dimension of $M^{\Lambda,\lambda}$ from the definition. But it will be very complicated as we do not have a complete classification of the irreducible $K(\mathfrak{g},k)$-modules at this point.

For an arbitrary simple vertex operator algebra $V$ and a finite automorphism group $G$ such that $V^G$ is regular, the quantum dimensions of irreducible $V^G$-modules are determined in terms of the quantum dimensions of $g$ twisted modules for $g \in G$ recently in [26]. Theorem 4.5 also follows from these results easily.

## 5. Classification of the irreducible modules and the fusion rules

In this section, we classify the irreducible $K(\mathfrak{g},k)$-modules and determine the fusion rules.

By Theorem 3.3, there are at most $\frac{|P_+^k||Q/kQ_L|}{|P/Q|}$ inequivalent irreducible $K(\mathfrak{g},k)$-modules. We prove that there are exactly $\frac{|P_+^k||Q/kQ_L|}{|P/Q|}$ inequivalent irreducible $K(\mathfrak{g},k)$-modules. Namely, the identification given in [25] is complete.

**Theorem 5.1.** *Let $\mathfrak{g}$ be a simple Lie algebra and $k$ a positive integer. Then there are exactly $\frac{|P_+^k||Q/kQ_L|}{|P/Q|}$ inequivalent irreducible $K(\mathfrak{g},k)$-modules.*

*Proof.* Recall that $L_{\widehat{\mathfrak{g}}}(k,0) = \oplus_{i\in Q/kQ_L} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i} \otimes M^{0,\beta_i}$. By Theorem 2.9 and Lemmas 2.10 and 4.3 we have

$$\begin{aligned}
&\mathrm{glob}(V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g},k)) \\
&= \mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0)) \cdot \big(\mathrm{qdim}_{V_{\sqrt{k}Q_L}\otimes K(\mathfrak{g},k)} \sum_{i\in Q/kQ_L} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i} \otimes M^{0,\beta_i}\big)^2 \\
&= \mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0)) \big(\sum_{i\in Q/kQ_L} \mathrm{qdim}_{V_{\sqrt{k}Q_L}} V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\beta_i} \cdot \mathrm{qdim}_{K(\mathfrak{g},k)} M^{0,\beta_i}\big)^2 \\
&= \mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0))|Q/kQ_L|^2.
\end{aligned}$$

From Lemma 2.10 or Lemma 4.7 of [26] we have

$$\mathrm{glob}(V_{\sqrt{k}Q_L})\mathrm{glob}(K(\mathfrak{g},k)) = \mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0))|Q/kQ_L|^2.$$

We need to determine the global dimension of $V_{\sqrt{k}Q_L}$ first. Note that $V_{\sqrt{k}Q_L}$ has exactly $|(\sqrt{k}Q_L)^\circ/Q_L|$ inequivalent irreducible modules [11] and each irreducible is a simple current. It is evident that $(\sqrt{k}Q_L)^\circ = \frac{1}{\sqrt{k}}Q_L^\circ$. Then

$$\mathrm{glob}(V_{\sqrt{k}Q_L}) = \sum_{i\in \frac{1}{\sqrt{k}}Q_L^\circ/\sqrt{k}Q_L} (\mathrm{qdim}_{V_{\sqrt{k}Q_L}} V_{\sqrt{k}Q_L+\lambda_i})^2 = |Q_L^\circ/kQ_L| = |P/kQ_L|,$$

where we have used Lemma 3.1 in the last equality.

Also recall that the irreducible $L_{\widehat{\mathfrak{g}}}(k,0)$-modules are $\{L_{\widehat{\mathfrak{g}}}(k,\Lambda) \mid \Lambda \in P_+^k\}$. Using Theorem 4.5 gives

$$\begin{aligned}
\mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0)) \cdot |Q/kQ_L| &= \sum_{\Lambda \in P_+^k} |Q/kQ_L| \cdot (\mathrm{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda))^2 \\
&= \sum_{\Lambda \in P_+^k} |Q/kQ_L| \cdot (\mathrm{qdim}_{K(\mathfrak{g},k)} M^{\Lambda,\lambda})^2 \\
&= \sum_{\Lambda \in P_+^k} \sum_{i \in Q/kQ_L} (\mathrm{qdim}_{K(\mathfrak{g},k)} M^{\Lambda,\Lambda+\beta_i})^2,
\end{aligned}$$

where $\lambda$ is any fixed element in $\Lambda + Q$. So we get

$$\begin{aligned}
\mathrm{glob}(K(\mathfrak{g},k)) &= \frac{\mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0))|Q/kQ_L|^2}{\mathrm{glob}(V_{\sqrt{k}Q_L})} \\
&= \frac{\mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0))|Q/kQ_L|^2}{|P/Q||Q/kQ_L|} \\
&= \frac{\mathrm{glob}(L_{\widehat{\mathfrak{g}}}(k,0))|Q/kQ_L|}{|P/Q|} \\
&= \frac{\sum_{\Lambda \in P_+^k} \sum_{i \in Q/kQ_L} (\mathrm{qdim}_{K(\mathfrak{g},k)} M^{\Lambda,\Lambda+\beta_i})^2}{|P/Q|}.
\end{aligned}$$

It follows from Theorem 3.3, Lemma 3.4 that the identification in Theorem 3.3 is complete and $K(\mathfrak{g},k)$ has exactly $\frac{|P_+^k||Q/kQ_L|}{|P/Q|}$ inequivalent irreducible $K(\mathfrak{g},k)$-modules. □

Finally we determine the fusion rules among the irreducible modules for $K(\mathfrak{g},k)$. Let

$$L_{\widehat{\mathfrak{g}}}(k,\Lambda^1) \boxtimes L_{\widehat{\mathfrak{g}}}(k,\Lambda^2) = \sum_{\Lambda^3 \in P_+^k} N_{\Lambda^1,\Lambda^2}^{\Lambda^3} L_{\widehat{\mathfrak{g}}}(k,\Lambda_3),$$

where $\Lambda^1, \Lambda^2 \in P_+^k$ and $N_{\Lambda^1,\Lambda^2}^{\Lambda^3}$ are the fusion rules for the irreducible $L_{\widehat{\mathfrak{g}}}(k,0)$-modules.

**Theorem 5.2.** *Let $\Lambda^1, \Lambda^2 \in P_+^k$ and $i, j \in Q/kQ_L$. Then*

$$M^{\Lambda^1,\Lambda^1+\beta_i} \boxtimes M^{\Lambda^2,\Lambda^2+\beta_j} = \sum_{\Lambda^3 \in P_+^k} N_{\Lambda^1,\Lambda^2}^{\Lambda^3} M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}.$$

*Moreover, $M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}$ with $N_{\Lambda^1,\Lambda^2}^{\Lambda^3} \neq 0$ are inequivalent $K(\mathfrak{g},k)$-modules.*

*Proof.* We first prove that $M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}$ with $N_{\Lambda^1,\Lambda^2}^{\Lambda^3} \neq 0$ are inequivalent $K(\mathfrak{g},k)$-modules for $\Lambda^3 \in P_+^k$. Note from Theorem 3.3, Lemma 3.4 and Theorem 5.1 that

$$M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j} = M^{\bar{\Lambda},\Lambda^1+\Lambda^2+\beta_i+\beta_j}$$

for some $\Lambda^3, \bar{\Lambda} \in P_+^k$ if and only if $\bar{\Lambda} = (\Lambda^3)^{(s)}$ for some $s$ with $a_s = 1$ and

$$\Lambda^1 + \Lambda^2 + \beta_i + \beta_j + k\Lambda_s - (\Lambda^1 + \Lambda^2 + \beta_i + \beta_j) \in kQ_L.$$

That is, $\Lambda_s \in Q_L$. But this is impossible [25].

For any $\Lambda^3 \in P_+^k$, let $N_{M^{\Lambda^1,\Lambda^1+\beta_i}, M^{\Lambda^2,\Lambda^2+\beta_j}}^{M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}}$ be the fusion rules determined by the irreducible $K(\mathfrak{g},k)$-modules $M^{\Lambda^1,\Lambda^1+\beta_i}$, $M^{\Lambda^2,\Lambda^2+\beta_j}$ and $M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}$. We claim that

$$N_{M^{\Lambda^1,\Lambda^1+\beta_i}, M^{\Lambda^2,\Lambda^2+\beta_j}}^{M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}} \geq N_{\Lambda^1,\Lambda^2}^{\Lambda^3}.$$

For short we set $\lambda = \Lambda^1 + \beta_i$ and $\mu = \Lambda^2 + \beta_j$. By Proposition 2.3 we know that

$$\begin{aligned} N_{\Lambda^1,\Lambda^2}^{\Lambda^3} &\leq N_{V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\lambda}\otimes M^{\Lambda^1,\lambda}, V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\mu}\otimes M^{\Lambda^2,\mu}}^{L_{\widehat{\mathfrak{g}}}(k,\Lambda^3)} \\ &= N_{V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\lambda}\otimes M^{\Lambda^1,\lambda}, V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\mu}\otimes M^{\Lambda^2,\mu}}^{V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\lambda+\mu)}\otimes M^{\Lambda^3,\lambda+\mu}}. \end{aligned}$$

Using Proposition 2.4 and the identity

$$N_{V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\lambda}, V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}\mu}}^{V_{\sqrt{k}Q_L+\frac{1}{\sqrt{k}}(\lambda+\mu)}} = 1$$

from [15] proves the claim.

From the discussion above, we see that $\sum_{\Lambda^3\in P_+^k} N_{\Lambda^1,\Lambda^2}^{\Lambda^3} M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}$ is a $K(\mathfrak{g},k)$-submodule of $M^{\Lambda^1,\Lambda^1+\beta_i} \boxtimes M^{\Lambda^2,\Lambda^2+\beta_j}$. On the other hand, by Theorem 4.5 we have

$$\begin{aligned} &\operatorname{qdim}_{K(g,k)} M^{\Lambda^1,\Lambda^1+\beta_i} \boxtimes M^{\Lambda^2,\Lambda^2+\beta_j} \\ &= \operatorname{qdim}_{K(g,k)} M^{\Lambda^1,\Lambda^1+\beta_i} \operatorname{qdim}_{K(g,k)} M^{\Lambda^2,\Lambda^2+\beta_j} \\ &= \operatorname{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda^1) \operatorname{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda^2) \\ &= \operatorname{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda^1) \boxtimes L_{\widehat{\mathfrak{g}}}(k,\Lambda^2) \\ &= \sum_{\Lambda^3\in P_+^k} N_{\Lambda^1,\Lambda^2}^{\Lambda^3} \operatorname{qdim}_{L_{\widehat{\mathfrak{g}}}(k,0)} L_{\widehat{\mathfrak{g}}}(k,\Lambda^3) \\ &= \sum_{\Lambda^3\in P_+^k} N_{\Lambda^1,\Lambda^2}^{\Lambda^3} \operatorname{qdim}_{K(\mathfrak{g},k)} M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}. \end{aligned}$$

So the quantum dimension of the submodule $\sum_{\Lambda^3\in P_+^k} N_{\Lambda^1,\Lambda^2}^{\Lambda^3} M^{\Lambda^3,\Lambda^1+\Lambda^2+\beta_i+\beta_j}$ of $M^{\Lambda^1,\Lambda^1+\beta_i} \boxtimes M^{\Lambda^2,\Lambda^2+\beta_j}$ equals the quantum dimension of $M^{\Lambda^1,\Lambda^1+\beta_i} \boxtimes M^{\Lambda^2,\Lambda^2+\beta_j}$. The theorem follows immediately. □

## References

[1] Toshiyuki Abe, Geoffrey Buhl, and Chongying Dong, *Rationality, regularity, and $C_2$-cofiniteness*, Trans. Amer. Math. Soc. **356** (2004), no. 8, 3391–3402, DOI 10.1090/S0002-9947-03-03413-5. MR2052955

[2] Toshiyuki Abe, Chongying Dong, and Haisheng Li, *Fusion rules for the vertex operator algebra $M(1)$ and $V_L^+$*, Comm. Math. Phys. **253** (2005), no. 1, 171–219, DOI 10.1007/s00220-004-1132-5. MR2105641

[3] Tomoyuki Arakawa, Ching Hung Lam, and Hiromichi Yamada, *Zhu's algebra, $C_2$-algebra and $C_2$-cofiniteness of parafermion vertex operator algebras*, Adv. Math. **264** (2014), 261–295, DOI 10.1016/j.aim.2014.07.021. MR3250285

[4] T. Arakawa, C. H. Lam, and H. Yamada, *Parafermion vertex operator algebras and W-algebras*, arXiv:1701.06229.

[5] Bojko Bakalov and Alexander Kirillov Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, vol. 21, American Mathematical Society, Providence, RI, 2001. MR1797619

[6] Richard E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Nat. Acad. Sci. U.S.A. **83** (1986), no. 10, 3068–3071, DOI 10.1073/pnas.83.10.3068. MR843307

[7] S. Carnahan and M. Miyamoto, *Regularity of fixed-point vertex operator subalgebras*, arXiv:1603.05645.

[8] Hsian-Yang Chen and Ching Hung Lam, *Quantum dimensions and fusion rules of the VOA $V^{\tau}_{L_{\mathcal{C}\times\mathcal{D}}}$*, J. Algebra **459** (2016), 309–349, DOI 10.1016/j.jalgebra.2016.03.038. MR3503976

[9] R. Coquereaux, *Global dimensions for Lie groups at level k and their conformally exceptional quantum subgroups*, Rev. Un. Mat. Argentina **51** (2010), no. 2, 17–42. MR2840162

[10] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik, *The Witt group of non-degenerate braided fusion categories*, J. Reine Angew. Math. **677** (2013), 135–177. MR3039775

[11] Chongying Dong, *Vertex algebras associated with even lattices*, J. Algebra **161** (1993), no. 1, 245–265, DOI 10.1006/jabr.1993.1217. MR1245855

[12] Chongying Dong, Xiangyu Jiao, and Feng Xu, *Quantum dimensions and quantum Galois theory*, Trans. Amer. Math. Soc. **365** (2013), no. 12, 6441–6469, DOI 10.1090/S0002-9947-2013-05863-1. MR3105758

[13] Chongying Dong, Ching Hung Lam, and Hiromichi Yamada, *$W$-algebras related to parafermion algebras*, J. Algebra **322** (2009), no. 7, 2366–2403, DOI 10.1016/j.jalgebra.2009.03.034. MR2553685

[14] Chongying Dong, Ching Hung Lam, Qing Wang, and Hiromichi Yamada, *The structure of parafermion vertex operator algebras*, J. Algebra **323** (2010), no. 2, 371–381, DOI 10.1016/j.jalgebra.2009.08.003. MR2564844

[15] Chongying Dong and James Lepowsky, *Generalized vertex algebras and relative vertex operators*, Progress in Mathematics, vol. 112, Birkhäuser Boston, Inc., Boston, MA, 1993. MR1233387

[16] Chongying Dong, Haisheng Li, and Geoffrey Mason, *Compact automorphism groups of vertex operator algebras*, Internat. Math. Res. Notices **18** (1996), 913–921, DOI 10.1155/S1073792896000566. MR1420556

[17] Chongying Dong, Haisheng Li, and Geoffrey Mason, *Regularity of rational vertex operator algebras*, Adv. Math. **132** (1997), no. 1, 148–166, DOI 10.1006/aima.1997.1681. MR1488241

[18] Chongying Dong, Haisheng Li, and Geoffrey Mason, *Twisted representations of vertex operator algebras*, Math. Ann. **310** (1998), no. 3, 571–600, DOI 10.1007/s002080050161. MR1615132

[19] Chongying Dong, Haisheng Li, and Geoffrey Mason, *Modular-invariance of trace functions in orbifold theory and generalized Moonshine*, Comm. Math. Phys. **214** (2000), no. 1, 1–56, DOI 10.1007/s002200000242. MR1794264

[20] Chongying Dong, Haisheng Li, and Geoffrey Mason, *Vertex operator algebras and associative algebras*, J. Algebra **206** (1998), no. 1, 67–96, DOI 10.1006/jabr.1998.7425. MR1637252

[21] Chongying Dong, Haisheng Li, and Geoffrey Mason, *Twisted representations of vertex operator algebras and associative algebras*, Internat. Math. Res. Notices **8** (1998), 389–397, DOI 10.1155/S1073792898000269. MR1628239

[22] C. Dong, H. Li, G. Mason, and P. S. Montague, *The radical of a vertex operator algebra*, The Monster and Lie algebras (Columbus, OH, 1996), Ohio State Univ. Math. Res. Inst. Publ., vol. 7, de Gruyter, Berlin, 1998, pp. 17–25. MR1650625

[23] Chongying Dong, Xingjun Lin, and Siu-Hung Ng, *Congruence property in conformal field theory*, Algebra Number Theory **9** (2015), no. 9, 2121–2166, DOI 10.2140/ant.2015.9.2121. MR3435813

[24] Chongying Dong and Geoffrey Mason, *On quantum Galois theory*, Duke Math. J. **86** (1997), no. 2, 305–321, DOI 10.1215/S0012-7094-97-08609-9. MR1430435

[25] Chongying Dong and Li Ren, *Representations of the parafermion vertex operator algebras*, Adv. Math. **315** (2017), 88–101, DOI 10.1016/j.aim.2017.05.016. MR3667581

[26] Chongying Dong, Li Ren, and Feng Xu, *On orbifold theory*, Adv. Math. **321** (2017), 1–30. MR3715704

[27] Chongying Dong and Qing Wang, *The structure of parafermion vertex operator algebras: general case*, Comm. Math. Phys. **299** (2010), no. 3, 783–792, DOI 10.1007/s00220-010-1114-8. MR2718932

[28] Chongying Dong and Qing Wang, *On $C_2$-cofiniteness of parafermion vertex operator algebras*, J. Algebra **328** (2011), 420–431, DOI 10.1016/j.jalgebra.2010.10.015. MR2745574

[29] Chongying Dong and Qing Wang, *Parafermion vertex operator algebras*, Front. Math. China **6** (2011), no. 4, 567–579, DOI 10.1007/s11464-011-0138-5. MR2819965

[30] Chongying Dong and Qing Wang, *Quantum dimensions and fusion rules for parafermion vertex operator algebras*, Proc. Amer. Math. Soc. **144** (2016), no. 4, 1483–1492, DOI 10.1090/proc/12838. MR3451226

[31] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik, *On fusion categories*, Ann. of Math. (2) **162** (2005), no. 2, 581–642, DOI 10.4007/annals.2005.162.581. MR2183279

[32] Igor B. Frenkel, Yi-Zhi Huang, and James Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. **104** (1993), no. 494, viii+64, DOI 10.1090/memo/0494. MR1142494

[33] Igor Frenkel, James Lepowsky, and Arne Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988. MR996026

[34] Igor B. Frenkel and Yongchang Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), no. 1, 123–168, DOI 10.1215/S0012-7094-92-06604-X. MR1159433

[35] Yi-Zhi Huang, *A theory of tensor products for module categories for a vertex operator algebra. IV*, J. Pure Appl. Algebra **100** (1995), no. 1-3, 173–216, DOI 10.1016/0022-4049(95)00050-7. MR1344849

[36] Yi-Zhi Huang, *Vertex operator algebras and the Verlinde conjecture*, Commun. Contemp. Math. **10** (2008), no. 1, 103–154, DOI 10.1142/S0219199708002727. MR2387861

[37] Yi-Zhi Huang, *Rigidity and modularity of vertex tensor categories*, Commun. Contemp. Math. **10** (2008), no. suppl. 1, 871–911, DOI 10.1142/S0219199708003083. MR2468370

[38] Yi-Zhi Huang, Alexander Kirillov Jr., and James Lepowsky, *Braided tensor categories and extensions of vertex operator algebras*, Comm. Math. Phys. **337** (2015), no. 3, 1143–1159, DOI 10.1007/s00220-015-2292-1. MR3339173

[39] Y.-Z. Huang and J. Lepowsky, *A theory of tensor products for module categories for a vertex operator algebra. I, II*, Selecta Math. (N.S.) **1** (1995), no. 4, 699–756, 757–786, DOI 10.1007/BF01587908. MR1383584

[40] Y.-Z. Huang and J. Lepowsky, *A theory of tensor products for module categories for a vertex operator algebra. I, II*, Selecta Math. (N.S.) **1** (1995), no. 4, 699–756, 757–786, DOI 10.1007/BF01587908. MR1383584

[41] Yi-Zhi Huang and James Lepowsky, *A theory of tensor products for module categories for a vertex operator algebra. III*, J. Pure Appl. Algebra **100** (1995), no. 1-3, 141–171, DOI 10.1016/0022-4049(95)00049-3. MR1344848

[42] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978. Second printing, revised. MR499562

[43] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219

[44] Alexander Kirillov Jr. and Viktor Ostrik, *On a $q$-analogue of the McKay correspondence and the ADE classification of $\mathfrak{sl}_2$ conformal field theories*, Adv. Math. **171** (2002), no. 2, 183–227, DOI 10.1006/aima.2002.2072. MR1936496

[45] Matthew Krauel and Masahiko Miyamoto, *A modular invariance property of multivariable trace functions for regular vertex operator algebras*, J. Algebra **444** (2015), 124–142, DOI 10.1016/j.jalgebra.2015.07.013. MR3406171

[46] James Lepowsky and Haisheng Li, *Introduction to vertex operator algebras and their representations*, Progress in Mathematics, vol. 227, Birkhäuser Boston, Inc., Boston, MA, 2004. MR2023933

[47] James Lepowsky and Mirko Primc, *Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$*, Contemporary Mathematics, vol. 46, American Mathematical Society, Providence, RI, 1985. MR814303

[48] James Lepowsky and Robert Lee Wilson, *A new family of algebras underlying the Rogers-Ramanujan identities and generalizations*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), no. 12, 7254–7258. MR638674

[49] James Lepowsky and Robert Lee Wilson, *The structure of standard modules. I. Universal algebras and the Rogers-Ramanujan identities*, Invent. Math. **77** (1984), no. 2, 199–290, DOI 10.1007/BF01388447. MR752821

[50] Hai-Sheng Li, *Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules*, Moonshine, the Monster, and related topics (South Hadley, MA, 1994), Contemp. Math., vol. 193, Amer. Math. Soc., Providence, RI, 1996, pp. 203–236, DOI 10.1090/conm/193/02373. MR1372724

[51] Haisheng Li, *The physics superselection principle in vertex operator algebra theory*, J. Algebra **196** (1997), no. 2, 436–457, DOI 10.1006/jabr.1997.7126. MR1475118

[52] Haisheng Li, *Certain extensions of vertex operator algebras of affine type*, Comm. Math. Phys. **217** (2001), no. 3, 653–696, DOI 10.1007/s002200100386. MR1822111

[53] Xingjun Lin, *Mirror extensions of rational vertex operator algebras*, Trans. Amer. Math. Soc. **369** (2017), no. 6, 3821–3840, DOI 10.1090/tran/6749. MR3624394

[54] Masahiko Miyamoto, *$C_2$-cofiniteness of cyclic-orbifold models*, Comm. Math. Phys. **335** (2015), no. 3, 1279–1286, DOI 10.1007/s00220-014-2252-1. MR3320313

[55] Victor Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups **8** (2003), no. 2, 177–206, DOI 10.1007/s00031-003-0515-6. MR1976459

[56] A. B. Zamolodchikov and V. A. Fateev, *Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in $Z_N$-symmetric statistical systems* (Russian), Zh. Èksper. Teoret. Fiz. **89** (1985), no. 2, 380–399; English transl., Soviet Phys. JETP **62** (1985), no. 2, 215–225 (1986). MR830910

[57] Yongchang Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996), no. 1, 237–302, DOI 10.1090/S0894-0347-96-00182-8. MR1317233

School of Mathematics, Sichuan University, Chengdu 610064, People's Republic of China–and–School of Mathematics and Statistics, Zhengzhou University, Henan 450001, People's Republic of China
*Email address*: aichunrui0908@126.com

Department of Mathematics, University of California, Santa Cruz, California 95064
*Email address*: dong@ucsc.edu

Department of Mathematics, East China Normal University, Shanghai 200241, People's Republic of China
*Email address*: xyjiao@math.ecnu.edu.cn

School of Mathematics, Sichuan University, Chengdu 610064, People's Republic of China
*Email address*: renl@scu.edu.cn