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RESTRICTING INVARIANTS OF UNITARY REFLECTION GROUPS

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To the memory of Robert Steinberg

Abstract. Suppose that G is a finite, unitary reflection group acting on a complex vector space V and X is the fixed point subspace of an element of G. Define N to be the setwise stabilizer of X in G, Z to be the pointwise stabilizer, and $C = N/Z$. Then restriction defines a homomorphism from the algebra of G-invariant polynomial functions on V to the algebra of C-invariant functions on X. Extending earlier work by Douglass and Röhrle for Coxeter groups, we characterize when the restriction mapping is surjective for arbitrary unitary reflection groups G in terms of the exponents of G and C and their reflection arrangements. A consequence of our main result is that the variety of G-orbits in the G-saturation of X is smooth if and only if it is normal.

1. Introduction

Let G be a finite unitary reflection group acting on a finite dimensional complex vector space V and let X be a subspace of V. Define

$$N_X = \{\, g \in G \mid g(X) = X \,\},$$

the setwise stabilizer of X in G,

$$Z_X = \{\, g \in G \mid g(x) = x \ \ \forall x \in X \,\},$$

the pointwise stabilizer of X in G, and

$$C_X = N_X/Z_X.$$

Then C_X acts faithfully on X. We frequently identify C_X with its image in $\mathrm{GL}(X)$ and consider C_X as a subgroup of $\mathrm{GL}(X)$. In this situation, restriction defines a surjective, degree preserving, algebra homomorphism $\widetilde{\rho}_X \colon \mathbb{C}[V] \to \mathbb{C}[X]$ from the algebra of polynomial functions on V to the algebra of polynomial functions on X with $\widetilde{\rho}_X(\mathbb{C}[V]^G) \subseteq \mathbb{C}[X]^{C_X}$. Let

$$\rho_X \colon \mathbb{C}[V]^G \to \mathbb{C}[X]^{C_X}$$

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be the algebra homomorphism from G-invariant polynomials on V to C_X-invariant polynomials on X obtained by restriction. Note that ρ_X is surjective if and only if every C_X-invariant polynomial function on X extends to a G-invariant polynomial function on V. The main result of this paper, Theorem 2.1, is an elementary combinatorial characterization of when the map ρ_X is surjective in the case when X is the fixed point space of an element (or equivalently a subgroup) of G. This characterization is in terms of (1) the exponents of G and the exponents of the subgroup of C_X generated by the elements that act on X as reflections and (2) the restriction of the arrangement of G to X. In addition, we show that when ρ_X is surjective, the group C_X always acts on X as a unitary reflection group. In the course of the proof of Theorem 2.1 we classify all pairs (G, X) where G is an irreducible unitary reflection group acting faithfully on V, X is the fixed point set of an element of G, and C_X acts on X as a unitary reflection group. A corollary of the main result is a characterization of the surjectivity of ρ_X in terms of the smoothness of the variety of G-orbits in the G-saturation of X in V.

Theorem 2.1 in the present paper was stated and proved for finite Coxeter groups in [6]. The arguments in this paper are almost entirely independent of that paper and give a more uniform proof of [6, Theorem 2.3].

Lehrer and Springer [10] show that if G acts on V as a reflection group and X is a subspace of V that is maximal among the eigenspaces of elements of G with a fixed eigenvalue, then the conclusions of Theorem 2.1 hold. Because we consider fixed point subspaces, that is, 1-eigenspaces, of elements of G, the only subspace covered by both our arguments and those in [10] is V itself.

The rest of this paper is organized as follows. In §2 we set notation and state the main theorems. We make some reductions and prove some preliminary results in §3; then we complete the proof of Theorem 2.1 for the infinite family $G(r, p, n)$ in §4 and for the exceptional unitary reflection groups in §5. Tables containing the results of our computations for the exceptional groups that are used in the proof of the theorem are given in an appendix.

2. Statement of the main results

Suppose V is an n-dimensional complex vector space and G is a finite subgroup of the general linear group $\mathrm{GL}(V)$. A linear transformation $r \in \mathrm{GL}(V)$ is a reflection if its fixed point subspace, $\mathrm{Fix}(r)$, is a hyperplane in V. Let G^{ref} be the subgroup of G generated by the set of reflections in G. If $G = G^{\mathrm{ref}}$, then we say that G acts on V as a unitary reflection group or simply as a reflection group. The group $\mathrm{GL}(V)$ acts on the algebra $\mathbb{C}[V]$ of polynomial functions on V with

$$(g \cdot f)(v) = f(g^{-1}v) \quad \text{for } g \in \mathrm{GL}(V),\ f \in \mathbb{C}[V], \text{ and } v \in V,$$

and one may consider the subalgebra $\mathbb{C}[V]^G$ of G-invariant polynomial functions on V. Obviously a polynomial function f is G-invariant if and only if it is constant on G-orbits in V. It follows from the well-known theorem of Chevalley-Shephard-Todd that the following statements are equivalent:

- G acts on V as a unitary reflection group.
- $\mathbb{C}[V]^G$ is a polynomial algebra.
- The orbit variety V/G is smooth.

When these conditions hold, it is known that the multiset of degrees of a set of homogeneous generators of $\mathbb{C}[V]^G$ does not depend on the chosen set of generators.

These positive integers are the degrees of G. If $d_1, \dots, d_n$ are the degrees of G, then $d_1 - 1, \dots, d_n - 1$ are the exponents of G. In general, define the exponents of a finite subgroup G of $\mathrm{GL}(V)$ to be the multiset of exponents of the unitary reflection group G^{ref} and denote this multiset by $\exp(G)$.

By a hyperplane arrangement in V we mean a finite set of hyperplanes in V. The reflections in G determine a hyperplane arrangement in V, namely,

$$\mathcal{A}(V, G) = \{ \operatorname{Fix}(r) \mid r \text{ is a reflection in } G \}.$$

The arrangement $\mathcal{A}(V, G)$ is called a reflection arrangement. Note that G^{ref} acts on V as a reflection group, $\mathcal{A}(V, G) = \mathcal{A}(V, G^{\mathrm{ref}})$ is a reflection arrangement, and $\exp(G) = \exp(G^{\mathrm{ref}})$.

For a subspace X of V there are two natural hyperplane arrangements arising from the action of G on V. These are

(1) the reflection arrangement $\mathcal{A}(X, C_X) = \mathcal{A}(X, C_X^{\mathrm{ref}})$ and
(2) the restricted arrangement $\mathcal{A}(V, G)^X$, which consists of the intersections $H \cap X$ for H in $\mathcal{A}(V, G)$ with $X \not\subseteq H$.

By definition $\mathcal{A}(X, C_X)$ is a reflection arrangement, but in general $\mathcal{A}(V, G)^X$ is not necessarily a reflection arrangement.

We can now state our main result.

Theorem 2.1. *Suppose G acts on V as a unitary reflection group and X is the space of fixed points of an element of G. Then the restriction map*

$$\rho_X \colon \mathbb{C}[V]^G \to \mathbb{C}[X]^{C_X} \ \textit{is surjective}$$

if and only if

$$\mathcal{A}(X, C_X) = \mathcal{A}(V, G)^X \quad \textit{and} \quad \exp(C_X) \subseteq \exp(G).$$

Furthermore, when these conditions hold, $C_X = C_X^{\mathrm{ref}}$ acts on X as a reflection group.

Examples show that the theorem is sharp in the sense that there are spaces of fixed points such that $\mathcal{A}(X, C_X) = \mathcal{A}(V, G)^X$ and $\exp(C_X) \not\subseteq \exp(G)$ and there are spaces of fixed points such that $\mathcal{A}(X, C_X) \neq \mathcal{A}(V, G)^X$ and $\exp(C_X) \subseteq \exp(G)$.

The map ρ_X is the comorphism of the finite morphism from the quotient variety X/C_X to the quotient V/G that maps a C_X-orbit in X to its G-orbit in V. The image of this morphism is the variety GX/G, the orbit variety of the G-saturation of X in V, and it factors as the composition

$$X/C_X \to GX/G \to V/G,$$

where the first morphism is surjective and the second is injective. The surjective morphism $X/C_X \to GX/G$ is the normalization of the affine variety GX/G. Richardson [14, 2.2.1] has shown that if G is a unitary reflection group and X is any subspace of V, then ρ_X is surjective if and only if GX/G is a normal variety or, equivalently, if and only if $X/C_X \cong GX/G$. It follows from the Chevalley-Shephard-Todd Theorem that C_X acts on X as a reflection group if and only if X/C_X is a smooth variety. Together with Theorem 2.1, this proves the following corollary.

Corollary 2.2. *Suppose G acts on V as a unitary reflection group and X is the space of fixed points of an element of G. Then the following are equivalent:*

(1) $\rho_X : \mathbb{C}[V]^G \to \mathbb{C}[X]^{C_X}$ *is surjective.*
(2) $\mathcal{A}(X, C_X) = \mathcal{A}(V, G)^X$ *and* $\exp(C_X) \subseteq \exp(G)$.
(3) $X/C_X \cong GX/G$.
(4) GX/G *is a normal variety.*
(5) GX/G *is a smooth variety.*

The lattice of a hyperplane arrangement $\mathcal{A}$, denoted by $L(\mathcal{A})$, is the set of subspaces of the form $H_1 \cap \cdots \cap H_m$, where $\{H_1, \ldots, H_m\}$ is a subset of $\mathcal{A}$. The next lemma is well-known.

Lemma 2.3. *Suppose that G acts on V as a unitary reflection group and that X is a subspace of V. Then the following statements are equivalent:*

(1) *X is in the lattice of $\mathcal{A}(V, G)$.*
(2) *X is the fixed point space of an element of G.*
(3) *X is the fixed point space of a subgroup of G.*

Proof. It is shown in [13, Theorem 6.27] that the first two statements are equivalent. The second statement implies the third because $\mathrm{Fix}(g) = \mathrm{Fix}(\langle g \rangle)$ for $g \in G$. Finally, the third statement implies the first because if H is a subgroup of G, then $\mathrm{Fix}(H) = \bigcap_{h \in H} \mathrm{Fix}(h)$, and by the second statement each subspace $\mathrm{Fix}(h)$ is in $L(\mathcal{A}(V, G))$. □

Using the lemma, in the following, we frequently refer to subspaces in the lattice of $\mathcal{A}(V, G)$ instead of fixed point subspaces of elements, or subgroups, of G. To simplify the notation, in the rest of this paper set

$$\mathcal{A}(G) = \mathcal{A}(V, G) \quad \text{and} \quad \mathcal{A}(C_X) = \mathcal{A}(X, C_X)$$

when G and V are fixed.

In §3 it is shown that under the assumptions of the theorem, if ρ_X is surjective, then $\mathcal{A}(C_X) = \mathcal{A}(G)^X$. This focuses attention on subspaces X such that $\mathcal{A}(C_X) = \mathcal{A}(G)^X$. The subspaces in the lattice of $\mathcal{A}(G)$ with the property that $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ are classified for the infinite family of irreducible unitary reflection groups in §4 and for the thirty-four exceptional unitary reflection groups in §5. One consequence of the classification is Theorem 3.5, which states that if either $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ or $\exp(C_X) \subseteq \exp(G)$, then C_X acts on X as a reflection group. A second consequence of the computations in §4 is an elementary arithmetic characterization of the subspaces $X \in L(\mathcal{A}(G))$ such that C_X acts on X as a reflection group when $G = G(r, p, n)$ is in the infinite family of reflection groups (see Corollary 4.8). For the exceptional reflection groups, the subspaces X such that C_X acts on X as a reflection group are listed in an appendix.

3. Reductions and preliminary results

For general information about hyperplane arrangements and reflection groups we refer the reader to [13].

As above, suppose V is a finite dimensional complex vector space, G is a finite subgroup of $\mathrm{GL}(V)$, and X is a subspace of V. We assume also that a positive definite, hermitian form on V is given and that G is a subgroup of the unitary group of V with respect to this form.

In this section we make several reductions; state and prove Theorem 3.2, a preliminary result that may be viewed as a strengthening of [6, Proposition 3.1]; state a second preliminary result, Theorem 3.5; and complete the proof of the forward implication in Theorem 2.1, assuming the validity of Theorem 3.5. The proof of Theorem 3.5 and the completion of the proof of Theorem 2.1 are given in §4 for the infinite family of irreducible unitary reflection groups and in §5 for the exceptional irreducible unitary reflection groups.

Reductions. In this subsection we show that it is enough to prove Theorem 2.1 when X is chosen from a set of orbit representatives for the action of the normalizer of G in $\mathrm{GL}(V)$ on $L(\mathcal{A}(G))$ and that it is enough to prove Theorem 2.1 when G acts faithfully on V as an irreducible reflection group.

Suppose $h \in \mathrm{GL}(V)$ normalizes G. Then $hZ_Xh^{-1} = Z_{h(X)}$ and $hN_Xh^{-1} = N_{h(X)}$, so conjugation by h induces an isomorphism

$$c_h\colon C_X \xrightarrow{\cong} C_{h(X)}.$$

Also, the linear transformation h determines an algebra automorphism

$$h^{\#}\colon \mathbb{C}[V] \xrightarrow{\cong} \mathbb{C}[V] \quad \text{by} \quad h^{\#}(f) = f \circ h.$$

The proof of the next proposition is straightforward and is omitted.

Proposition 3.1. *Suppose $G \subseteq \mathrm{GL}(V)$ acts on V as a finite reflection group, $h \in \mathrm{GL}(V)$ normalizes G, and $X \in L(\mathcal{A}(G))$. Then*

1. *$c_h\colon C_X \to C_{h(X)}$ is an isomorphism that restricts to an isomorphism of reflection groups $C_X^{\mathrm{ref}} \cong C_{h(X)}^{\mathrm{ref}}$,*
2. *c_h determines bijections between $\mathcal{A}(C_X)$ and $\mathcal{A}(C_{h(X)})$ and between $L(\mathcal{A}(C_X))$ and $L(\mathcal{A}(C_{h(X)}))$,*
3. *$h|_X\colon X \to h(X)$ determines a bijection between $\mathcal{A}(G)^X$ and $\mathcal{A}(G)^{h(X)}$, and*
4. *ρ_X is surjective if and only if $\rho_{h(X)}$ is surjective.*

Consequently, the conclusions of Theorem 2.1 *hold for X if and only if they hold for $h(X)$.*

In the rest of this subsection we assume that G is a finite unitary reflection group and X is in the lattice of $\mathcal{A}(G)$.

Set $V_f = \mathrm{Fix}(G)$ and let V_r be the orthogonal complement of V_f in V. Obviously $V_f \subseteq X$. Let X_r be the orthogonal complement of V_f in X. The restriction maps $\widetilde{\rho}_X$ and ρ_X may be identified with

$$\widetilde{\rho}_{X_r} \otimes id\colon \mathbb{C}[V_r] \otimes \mathbb{C}[V_f] \to \mathbb{C}[X_r] \otimes \mathbb{C}[V_f]$$

and

$$\rho_{X_r} \otimes id\colon \mathbb{C}[V_r]^G \otimes \mathbb{C}[V_f] \to \mathbb{C}[X_r]^{C_X} \otimes \mathbb{C}[V_f],$$

respectively, where $\widetilde{\rho}_{X_r}$ and ρ_{X_r} are given by restriction. Clearly, ρ_X is surjective if and only if ρ_{X_r} is surjective.

Because V_f is contained in every reflecting hyperplane of G in V, there are canonical bijections between $\mathcal{A}(V,G)$ and $\mathcal{A}(V_r,G)$ and between $\mathcal{A}(V,G)^X$ and $\mathcal{A}(V_r,G)^{X_r}$. Let O_f denote the multiset containing 0 with multiplicity $\dim V_f$. Then

$$\exp(G,V) = \exp(G,V_r) \amalg O_f \quad \text{and} \quad \exp(C_X,X) = \exp(C_X,X_r) \amalg O_f.$$

Thus, $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and $\exp(C_X) \subseteq \exp(G)$ if and only if $\mathcal{A}(X_r, C_X) = \mathcal{A}(V_r, G)^{X_r}$ and $\exp(C_X, X_r) \subseteq \exp(G, V_r)$.

It follows from the observations in the preceding two paragraphs that it is enough to prove Theorem 2.1 when G acts faithfully on V.

Finally, suppose G acts faithfully on V as a reducible reflection group. Then there are subgroups G_1 and G_2 of G, complementary orthogonal subspaces V_1 and V_2 of V, and complementary orthogonal subspaces X_1 and X_2 of X such that (1) G_i acts faithfully on V_i as a unitary reflection group for $i = 1, 2$; (2) $G \cong G_1 \times G_2$, the action of G on V may be identified with the action of $G_1 \times G_2$ on $V_1 \oplus V_2$, and $X \cong X_1 \oplus X_2$. It is straightforward to check that Theorem 2.1 holds for V, G, and X if and only if it holds for V_i, G_i, and X_i for $i = 1, 2$. Thus, it is enough to prove Theorem 2.1 when G acts on V as an irreducible reflection group.

Recall that the irreducible unitary reflection groups are classified in [16] as one infinite, three parameter family; the groups $G(r, p, n)$; and thirty-four exceptional groups denoted by $G_4, \dots, G_{37}$.

Proof that if $\rho_{\mathbf{X}}$ is surjective, then $\mathcal{A}(\mathbf{C_X}) = \mathcal{A}(\mathbf{G})^{\mathbf{X}}$. In [6, §3] it is shown that if G and C_X act as unitary reflection groups, $\mathcal{A}(C_X) \subseteq \mathcal{A}(G)^X$, and ρ_X is surjective, then $\mathcal{A}(G)^X = \mathcal{A}(C_X)$ and $\exp(C_X) \subseteq \exp(G)$. In this subsection we prove a variant of this result that does not include the hypothesis that C_X acts on X as a reflection group or the conclusion that $\exp(C_X) \subseteq \exp(G)$.

Theorem 3.2. *Suppose G acts on V as a unitary reflection group, X is in the lattice of $\mathcal{A}(G)$, and the restriction map $\rho_X \colon \mathbb{C}[V]^G \to \mathbb{C}[X]^{C_X}$ is surjective. Then $\mathcal{A}(C_X) = \mathcal{A}(G)^X$.*

Before proving the theorem we need some preliminary results.

First, suppose Y and Z are complex, affine varieties and $\varphi \colon Y \to Z$ is a finite, surjective morphism. Let $\varphi^{\#} \colon \mathbb{C}[Z] \to \mathbb{C}[Y]$ be the comorphism. Then $\varphi^{\#}$ is injective, and we may consider $\mathbb{C}[Z]$ as a subalgebra of $\mathbb{C}[Y]$. Suppose $\mathfrak{p}$ is a prime ideal in $\mathbb{C}[Y]$ and $\mathfrak{q} = \mathfrak{p} \cap \mathbb{C}[Z]$ is the contraction of $\mathfrak{p}$ to $\mathbb{C}[Z]$. Following [2], we say that $\mathfrak{p}$ in $\mathbb{C}[Y]$ is unramified over $\mathbb{C}[Z]$ if firstly $\mathfrak{q}$ generates the maximal ideal $\mathfrak{p}\mathbb{C}[Y]_{\mathfrak{p}}$ of $\mathbb{C}[Y]_{\mathfrak{p}}$, where $\mathbb{C}[Y]_{\mathfrak{p}}$ is the localization of $\mathbb{C}[Y]$ at $\mathfrak{p}$, and secondly $\mathbb{C}[Y]_{\mathfrak{p}}/\mathfrak{p}\mathbb{C}[Y]_{\mathfrak{p}}$ is a separable field extension of $\mathbb{C}[Z]_{\mathfrak{q}}/\mathfrak{q}\mathbb{C}[Z]_{\mathfrak{q}}$. If $\mathfrak{p}$ is not unramified, then it is ramified. Let $y \in Y$ and let $\mathfrak{m}_{Y,y}$ be the maximal ideal of functions in $\mathbb{C}[Y]$ that vanish at y. The morphism φ is said to be ramified or unramified at y if $\mathfrak{m}_{Y,y}$ is ramified or unramified over $\mathbb{C}[Z]$.

Lemma 3.3. *Suppose $\varphi \colon Y \to Z$ is a finite, surjective morphism of affine varieties.*

(1) *Let $y \in Y$. Then φ is unramified at y if and only if the map on Zariski tangent spaces $d\varphi_y \colon T_yY \to T_{\varphi(y)}Z$ is injective.*
(2) *Let $\mathfrak{p}$ be a prime ideal in $\mathbb{C}[Y]$ and let $\mathcal{Z}(\mathfrak{p})$ denote the zero set of $\mathfrak{p}$ in Y. Then $\mathfrak{p}$ is ramified over $\mathbb{C}[Z]$ if and only if φ is ramified at y for every $y \in \mathcal{Z}(\mathfrak{p})$.*

Proof. It follows from [1, Proposition 3.6(i)] that φ is unramified at y if and only if the cotangent map $T^*_{\varphi(y)}Z \to T^*_yY$ is surjective. This implies the first statement.

Let $\mathfrak{H} = \mathfrak{H}_{\mathbb{C}[Y]/\mathbb{C}[Z]}$ denote the homological different defined by Auslander and Buchsbaum [2, §2]. It is shown in [2, Theorem 2.7] that a prime ideal $\mathfrak{r} \subseteq \mathbb{C}[Y]$ is ramified over $\mathbb{C}[Z]$ if and only if $\mathfrak{H} \subseteq \mathfrak{r}$. The second statement now follows from the fact that $\mathfrak{p} = \bigcap_{y \in \mathcal{Z}(\mathfrak{p})} \mathfrak{m}_{Y,y}$. ∎

Returning to the setup in the statement of the theorem, suppose that $\dim X = a$, choose $g \in G$ such that $X = \mathrm{Fix}(g)$, and choose a basis $\{b_1, \dots, b_n\}$ of V consisting of eigenvectors for g, indexed so that $\{b_1, \dots, b_a\}$ is a basis of X. Let $\{x_1, \dots, x_n\}$ denote the dual basis of V^*. Then the restrictions of $x_1, \dots, x_a$ to X form a basis of X^*.

Lemma 3.4. *Suppose $a+1 \le j \le n$. Then $\frac{\partial f}{\partial x_j}|_X = 0$ for all $f \in \mathbb{C}[V]^G$.*

Proof. Note that $X = \mathrm{Fix}(g)$ is the 1-eigenspace of g. Denef and Loeser [5] have shown that if $h \in G$, v_1 and v_2 are eigenvectors for h with eigenvalues λ_1 and λ_2 respectively, and $f \in \mathbb{C}[V]^G$ is homogeneous with degree d, then $\lambda_2 D_{v_2}(f)(v_1) = \lambda_1^{1-d} D_{v_2}(f)(v_1)$, where $D_v(f)$ denotes the directional derivative of f in the direction of v. Taking $h = g$, $v_2 = b_j$, and $v_1 = x \in X$, we have $\lambda_2 D_{b_j}(f)(x) = D_{b_j}(f)(x)$ where $\lambda_2 \neq 1$. It follows that $0 = D_{b_j}(f)(x) = \frac{\partial f}{\partial x_j}|_x$ for all $x \in X$ and all $f \in \mathbb{C}[V]^G$. ☐

Proof of Theorem 3.2. As shown in [6], it is always the case that $\mathcal{A}(C_X) \subseteq \mathcal{A}(G)^X$: Suppose K is in $\mathcal{A}(C_X)$. By assumption there is a g in N_X so that $\mathrm{Fix}(g) \cap X = K$. Then $\mathrm{Fix}(g)$ is in the lattice of $\mathcal{A}$, say $\mathrm{Fix}(g) = H_1 \cap \cdots \cap H_k$, where $H_1, \dots, H_k$ are in $\mathcal{A}$. Thus $K = H_1 \cap \cdots \cap H_k \cap X$. Since $\dim K = \dim X - 1$, it follows that $K = H_i \cap X$ for some i and so K is in $\mathcal{A}(G)^X$.

Conversely, we need to show that $\mathcal{A}(G)^X \subseteq \mathcal{A}(C_X)$. Set

$$X_{\mathrm{ram}} = \bigcup_{K \in \mathcal{A}(G)^X} K$$

and choose a set of homogeneous generators $\{f_1 \dots, f_n\}$ of $\mathbb{C}[V]^G$. Consider the commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\pi_1} & X/C_X & \xrightarrow[\cong]{\pi_2} & GX/G & \xrightarrow[\cong]{\overline{F}} & F(GX/G) \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\
V & & \xrightarrow{\pi} & & V/G & \xrightarrow[\cong]{F} & \mathbb{C}^n
\end{array}$$

where the maps are defined as follows: π, π_1, and π_2 are the natural quotient maps; i, i_1, and i_2 are the inclusions; $F(Gv)_i = f_i(v)$ for $1 \le i \le n$; and $\overline{F}$ is obtained from F by restriction. Note that ρ_X is the comorphism of $i_1 \circ \pi_2$. Richardson [14, 2.2.1] has shown that ρ_X is surjective if and only if π_2 is an isomorphism, and so in our situation, π_2 is an isomorphism. It follows from the theorem of Chevalley-Shephard-Todd that F is an isomorphism; hence, so is $\overline{F}$.

Suppose $x \in X$. Then we have a commutative diagram of Zariski tangent spaces:

$$\begin{array}{ccccccc}
T_x X & \xrightarrow{d(\pi_1)_x} & T_{\pi_1(x)} X/C_X & \longrightarrow & T_{\pi(x)} GX/G & \longrightarrow & T_{F(\pi(x))} F(GX/G) \\
\downarrow{\scriptstyle di_x} & & & & \downarrow & & \downarrow \\
T_x V & & \xrightarrow{d\pi_x} & & T_{\pi(x)} V/G & \xrightarrow{dF_{\pi(x)}} & T_{F(\pi(x))} \mathbb{C}^n .
\end{array}$$

The matrix of the composition $dF_{\pi(x)} \circ d\pi_x$ with respect to suitable bases is given by the $n \times n$ Jacobian matrix $J(F\pi)(x)$, whose (i,j)-entry is $\frac{\partial f_i}{\partial x_j}|_x$. Steinberg [18] has shown that the nullity of $J(F\pi)(x)$ is the maximum number of linearly independent hyperplanes in $\mathcal{A}(G)$ that contain x. Thus the rank of $J(F\pi)(x)$ is at most a and is equal to a if and only if $x \notin X_{\mathrm{ram}}$. It follows from Lemma 3.4 that the last $n-a$ columns of $J(F\pi)(x)$ are zero and so the matrix of the composition $dF_{\pi(x)} \circ d\pi_x \circ di_x$ with respect to suitable bases is the $n \times a$ matrix obtained from $J(F\pi)(x)$ by deleting the last $n-a$ columns. It follows that the rank of $d(\pi_1)_x$ is at most a and equal to a if and only if $x \notin X_{\mathrm{ram}}$. Therefore, $d(\pi_1)_x$ is injective if and only if $x \notin X_{\mathrm{ram}}$ and so it follows from Lemma 3.3(1) that π_1 is ramified at x if and only if $x \in X_{\mathrm{ram}}$.

Now suppose that $K \in \mathcal{A}(G)^X$ and let $\mathcal{I}(K)$ be the ideal of K in $\mathbb{C}[X]$. Then $\mathcal{I}(K)$ is a prime ideal in $\mathbb{C}[X]$ with height one, and it follows from Lemma 3.3(2) that $\mathcal{I}(K)$ is ramified over $\mathbb{C}[X/C_X] = \mathbb{C}[X]^{C_X}$. Benson [3, §3.9] has shown that if $\mathfrak{p} \subseteq \mathbb{C}[X]$ is a height one prime ideal, then $\mathfrak{p}$ is ramified over $\mathbb{C}[X]^{C_X}$ if and only if $\mathfrak{p}$ is the ideal of a reflecting hyperplane of C_X. Thus, $K \in \mathcal{A}(C_X)$ as claimed. □

The subgroups $\mathbf{C_X^{ref}}$. The subgroups C_X^{ref} are determined in Proposition 4.7 for the groups $G(r,p,n)$ and are listed in the tables in the appendix for the exceptional irreducible unitary reflection groups, and the subspaces $X \in L(\mathcal{A}(G))$ such that $C_X^{\mathrm{ref}} = C_X$ are characterized in Corollary 4.8 for the infinite family of irreducible unitary reflection groups and are listed in the tables in the appendix for the exceptional irreducible unitary reflection groups.

Theorem 3.5. *Suppose G acts on V as a unitary reflection group and X is in the lattice of $\mathcal{A}(G)$. If either $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ or $\exp(C_X) \subseteq \exp(G)$, then $C_X^{\mathrm{ref}} = C_X$, and thus C_X acts on X as a unitary reflection group.*

Proof. As in the previous subsection it is enough to prove the theorem when G acts faithfully on V as an irreducible reflection group. If $G = G(r,p,n)$ for some r,p,n, then the result follows from Corollaries 4.9 and 4.11. If G is one of the exceptional unitary reflection groups, then the result is proved in Proposition 5.1. □

Examples show that the converse of the theorem is false; see Remark 5.2.

Proof of the forward implication in Theorem 2.1. Fix X in $L(\mathcal{A}(G))$ and suppose that $\rho_X\colon \mathbb{C}[V]^G \to \mathbb{C}[X]^{C_X}$ is surjective. We may assume that G acts faithfully on V. By Theorem 3.2, $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and so by Theorem 3.5, C_X acts on X as a reflection group. Thus $\mathbb{C}[V]^G$ and $\mathbb{C}[X]^{C_X}$ are polynomial algebras, and by [14, Lemma 4.1], if $\dim X = a$, there are homogeneous generators $\{f_1, \dots, f_n\}$ of $\mathbb{C}[V]^G$ so that $\{\rho_X(f_1), \dots, \rho_X(f_a)\}$ is a set of homogeneous generators of $\mathbb{C}[X]^{C_X}$. It follows that $\exp(C_X) \subseteq \exp(G)$. Assuming the validity of Theorem 3.5, this completes the proof that if ρ_X is surjective, then $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and $\exp(C_X) \subseteq \exp(G)$. □

4. The infinite family of irreducible unitary reflection groups

In this section we suppose that G is in the infinite family of irreducible unitary reflection groups and we classify the subspaces $X \in L(\mathcal{A}(G))$ such that $\mathcal{A}(C_X) = \mathcal{A}(G)^X$, we classify the subspaces $X \in L(\mathcal{A}(G))$ such that C_X acts on X as a reflection group, and we complete the proof of Theorem 2.1 for these groups.

For a positive integer k, let μ_k denote the group of k^{th} roots of unity in $\mathbb{C}$ and set $[k] = \{1, 2, \dots, k\}$. Throughout this section r is a positive integer, p is a positive divisor of r, and $q = r/p$.

The presentation in the next two subsections is a reformulation of the constructions and results in [13, §6.4] and [20, §3]. Proofs of the assertions are straightforward and are omitted.

The groups $\mathbf{G}(\mathbf{r},\mathbf{p},\mathbf{n})$ and the arrangements $\mathcal{A}_{\mathbf{n}}^{\mathbf{k}}$. Let $V = \mathbb{C}^n$. Using the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ we identify $\mathrm{GL}(V)$ with the matrix group $\mathrm{GL}_n(\mathbb{C})$. For $1 \le i \le n$, let $x_i\colon \mathbb{C}^n \to \mathbb{C}$ be a projection on the i^{th} coordinate. For σ in the symmetric group S_n, let p_σ be the permutation matrix defined by $p_\sigma e_i = e_{\sigma(i)}$ for $1 \le i \le n$. Let W denote the set of permutation matrices in $\mathrm{GL}_n(\mathbb{C})$ and let D denote the group of diagonal matrices in $\mathrm{GL}(V)$ with entries in μ_r. For $t \in D$ and $1 \le i \le n$, let t_i denote the (i,i) entry of t. Similarly, for $v \in V$, v_i denotes the i^{th} coordinate of v.

Define
$$\omega = e^{2\pi\sqrt{-1}/r}.$$
Then ω is a generator of μ_r and ω^p is a generator of μ_q. Note that the determinant map, $\det\colon D \to \mu_r$, is a group homomorphism. Define D_p to be the preimage of the subgroup μ_q. Then
$$D_p = \{\, t \in D \mid \det t \in \mu_q \,\}$$
and W normalizes D_p. Set
$$G(r,p,n) = WD_p.$$
Note that every element in $G(r,p,n)$ has a unique factorization of the form $p_\sigma t$ where $\sigma \in S_n$ and $t \in D_p$. As n and r are fixed, we denote $G(r,p,n)$ simply by G_p. Obviously,
$$G_r \subseteq G_p \subseteq G_1.$$

Suppose $1 \le i \ne j \le n$ and ζ is in μ_r. Define $r_{ij}(\zeta)$ to be the $n \times n$ matrix whose (k,l)-entry is
$$r_{ij}(\zeta)_{kl} = \begin{cases} 1 & k = l,\ k \ne i,\ k \ne j, \\ \zeta & (k,l) = (i,j), \\ \zeta^{-1} & (k,l) = (j,i), \\ 0 & \text{otherwise.} \end{cases}$$
It is easy to check that the characteristic polynomial of $r_{ij}(\zeta)$ is $(x-1)^{n-1}(x+1)$. It follows that $r_{ij}(\zeta)$ is a reflection with order two. Define
$$H_{ij}(\zeta) = \ker(x_i - \zeta x_j) = \{\, v \in V \mid v_i = \zeta v_j \,\}.$$
It is easy to see that $H_{ij}(\zeta)$ is the reflecting hyperplane of $r_{ij}(\zeta)$.

Suppose $1 \le i \le n$ and ζ is in μ_r with $\zeta \ne 1$. Define $r_{ii}(\zeta)$ to be the diagonal $n \times n$ matrix whose (k,k)-entry is
$$r_{ii}(\zeta)_{kk} = \begin{cases} \zeta & k = i, \\ 1 & \text{otherwise.} \end{cases}$$
Then $r_{ii}(\zeta)$ is obviously a reflection. Define
$$H_i = \ker x_i = \{\, v \in V \mid v_i = 0 \,\}.$$
It is clear that H_i is the reflecting hyperplane of $r_{ii}(\zeta)$ for all ζ in μ_r with $\zeta \ne 1$.

Obviously, $r_{ij}(\zeta)$ is in G_p for all $\zeta \in \mu_r$ and $r_{ii}(\zeta)$ is in G_p if and only if $\zeta \in \mu_q$. It can be shown that these are all the reflections in G_p. Note that G_r contains no reflections of the form $r_{ii}(\zeta)$. When $p < r$, the reflections in G_p and their reflecting hyperplanes may be arranged in the arrays

(4.1)
$$\begin{matrix} r_{11}(\zeta_1) & r_{12}(\zeta_{12}) & r_{13}(\zeta_{13}) & \dots & r_{1n}(\zeta_{1n}) \\ & r_{22}(\zeta_2) & r_{23}(\zeta_{23}) & \dots & r_{2n}(\zeta_{2n}) \\ & & r_{33}(\zeta_3) & \dots & r_{3n}(\zeta_{3n}) \\ & & & \ddots & \vdots \\ & & & & r_{nn}(\zeta_n) \end{matrix} \quad \text{and} \quad \begin{matrix} H_1 & H_{12}(\zeta_{12}) & H_{13}(\zeta_{13}) & \dots & H_{1n}(\zeta_{1n}) \\ & H_2 & H_{23}(\zeta_{23}) & \dots & H_{2n}(\zeta_{2n}) \\ & & H_3 & \dots & H_{3n}(\zeta_{3n}) \\ & & & \ddots & \vdots \\ & & & & H_n \end{matrix}$$

where $\zeta_{ij} \in \mu_r$ for $1 \leq i < j \leq n$ and $\zeta_i \in \mu_q \setminus \{1\}$ for $1 \leq i \leq n$. Similarly, when $p = r$, the reflections in G_r and their reflecting hyperplanes are

$$\{ r_{ij}(\zeta_{ij}) \mid 1 \leq i < j \leq n,\ \zeta_{ij} \in \mu_r \} \quad \text{and} \quad \{ H_{ij}(\zeta_{ij}) \mid 1 \leq i < j \leq n,\ \zeta_{ij} \in \mu_r \},$$

respectively.

Define

$$\mathcal{A}_n^0 = \{ H_{ij}(\zeta) \mid 1 \leq i < j \leq n,\ \zeta \in \mu_r \}$$

and for $1 \leq k \leq n$ define

$$\mathcal{A}_n^k = \{ H_i \mid 1 \leq i \leq k \} \amalg \mathcal{A}_n^0.$$

It is proved in [13, §6.4] that

$$\mathcal{A}(G_p) = \mathcal{A}_n^n \quad \text{for } p < r \text{ and } \quad \mathcal{A}(G_r) = \mathcal{A}_n^0.$$

The lattice of the arrangement of $\mathbf{G_p}$ and orbit representatives. To simplify formulas later, we extend the definition of $H_{ij}(\zeta)$ to include all pairs (i, j) with $1 \leq i, j \leq n$ by defining

$$H_{ii}(\zeta) = V \quad \text{for } 1 \leq i \leq n.$$

Consider a pair $L = (L_0, \{L_1, \dots, L_a\})$ where $\{L_0, L_1, \dots, L_a\}$ is a collection of disjoint subsets of $[n]$ such that $L_i \neq \emptyset$ for $1 \leq i \leq a$ and $\bigcup_{i=0}^a L_i = [n]$. Notice that L_0 is allowed to be empty and that $\{L_1, \dots, L_a\}$ is a partition of $[n] \setminus L_0$. Define

$$X_L = \Big(\bigcap_{i \in L_0} H_i \Big) \cap \Big(\bigcap_{j=1}^a \bigcap_{k,l \in L_j} H_{k,l}(1) \Big).$$

If $v \in \mathbb{C}^n$, then v is in X_L if and only if

(1) $v_i = 0$ for all $i \in L_0$ and
(2) $v_k = v_l$ for all k, l such that $k, l \in N_j$ for some $1 \leq j \leq a$.

For $1 \leq i \leq a$, define $b_i = \sum_{k \in L_i} e_k$ and let $\mathcal{B}_L = \{ b_i \mid 1 \leq i \leq a \}$. Then $\mathcal{B}_L$ is a basis of X_L.

Suppose $t \in D$. If v is in X_L, then $(tv)_i = t_i v_i = 0$ for $i \in L_0$, and if $k, l \in L_j$ for some $j \geq 1$, then $t_k^{-1}(tv)_k = v_k = v_l = t_l^{-1}(tv)_l$. Conversely, if $v \in V$ is such that $v_i = 0$ for all $i \in L_0$ and $t_k^{-1} v_k = t_l^{-1} v_l$ for all k, l such that $k, l \in L_j$ for some $1 \leq j \leq a$, then $v \in t(X_L)$. Thus,

$$t(X_L) = \Big(\bigcap_{i \in L_0} H_i \Big) \cap \Big(\bigcap_{j=1}^a \bigcap_{k,l \in L_j} H_{k,l}(t_k/t_l) \Big).$$

Let $\mathcal{L}$ denote the set of all pairs $L = (L_0, \{L_1, \dots, L_a\})$ as above. The next theorem is a restatement of results of Orlik, Solomon, and Terao [13, §6.4].

Theorem 4.1. *The lattice of the arrangement $\mathcal{A}(G_p)$ is as follows. If $p < r$, then*
$$L(\mathcal{A}(G_p)) = L(\mathcal{A}_n^n) = \{\, t(X_L) \mid L \in \mathcal{L},\ t \in D \,\}.$$
For $p = r$,
$$L(\mathcal{A}(G_r)) = L(\mathcal{A}_n^0) = \{\, t(X_L) \mid L \in \mathcal{L},\ |L_0| \neq 1,\ t \in D \,\}.$$

The group G_p acts on $L(\mathcal{A}_n^n)$ and $L(\mathcal{A}_n^0)$ is a G_p-stable subset. In the rest of this subsection we describe a subset of $L(\mathcal{A}_n^n)$ that contains at least one representative from each G_p-orbit. The main result is Corollary 4.5. Taylor [20, Theorems 3.9 and 3.11] also describes a set of orbit representatives.

First, the subspaces $t(X_L)$ in Theorem 4.1 are not all distinct. For $L \in \mathcal{L}$ define D_L to be the setwise stabilizer of X_L in D:
$$D_L = \{\, t \in D \mid t(X_L) = X_L \,\}.$$
Obviously, $t(X_L) = X_L$ if and only if $tb_i \in X_L$ for $1 \leq i \leq a$ and so
$$D_L = \{\, t \in D \mid \forall\, 1 \leq j \leq a,\ \forall\, k, l \in L_j,\ t_k = t_l \,\}.$$
If L' is another pair, then it is easy to see that $X_L = X_{L'}$ if and only if $L = L'$ and that for $t, t' \in D$,
$$t(X_L) = t'(X_{L'}) \quad \text{if and only if} \quad L = L' \text{ and } tD_L = t'D_L.$$

Lemma 4.2. *Suppose $L = (L_0, \{L_1, \dots, L_a\})$ and $\sigma \in S_n$. Then*
$$p_\sigma(X_L) = X_{\sigma(L)},$$
where $\sigma(L) = (\sigma(L_0), \{\sigma(L_1), \dots, \sigma(L_a)\})$.

Now suppose k is an integer with $0 \leq k \leq n$ and $\lambda = (\ell_1, \dots, \ell_a)$ is a partition of k. Set $\bar{\ell}_0 = 0$ and for $i > 0$ let $\bar{\ell}_i = \ell_1 + \dots + \ell_i$ denote the i^{th} partial sum of λ. Define a pair $L_\lambda = ((L_\lambda)_0, \{(L_\lambda)_1, \dots, (L_\lambda)_a\})$ in $\mathcal{L}$ by
$$(L_\lambda)_0 = \{k+1, \dots, n\} \quad \text{and} \quad (L_\lambda)_i = \{\bar{\ell}_{i-1}+1, \bar{\ell}_{i-1}+2, \dots, \bar{\ell}_i\} \text{ for } 1 \leq i \leq a.$$
Also, define
$$\delta_\lambda = \gcd(\ell_1, \dots, \ell_a) \quad \text{and} \quad \delta_{\lambda,p} = \gcd(p, \ell_1, \dots, \ell_a)$$
and set
$$X_\lambda = X_{L_\lambda}, \quad \mathcal{B}_\lambda = \mathcal{B}_{L_\lambda}, \quad \text{and} \quad D_\lambda = D_{L_\lambda}.$$
Note that $\mathcal{B}_\lambda = \{\, b_1, \dots, b_a \,\}$ where $b_i = e_{\bar{\ell}_{i-1}+1} + \dots + e_{\bar{\ell}_i}$.

Lemma 4.3. *If $X \in L(\mathcal{A}_n^n)$, then there is an integer k with $0 \leq k \leq n$, a partition λ of k, and a permutation $\sigma \in S_n$ so that $p_\sigma(X)$ is in the D-orbit of X_λ.*

It follows from the lemma that to find representatives of the orbits of the action of G_p on $L(\mathcal{A}_n^n)$, it is enough to decompose the D-orbit of X_λ, DX_λ, into D_p-orbits when λ is a partition of k and $0 \leq k \leq n$. Now D_λ is the stabilizer of X_λ in D, and so the D_p-orbits in DX_λ are in bijection with D_p, D_λ-double cosets in D. Because D is abelian, these double cosets are simply the cosets of D_pD_λ in D.

Proposition 4.4. *Suppose $\lambda = (n_1, \dots, n_a)$ is a partition of k with $0 \leq k \leq n$.*

1. *If $k < n$, then $D_pD_\lambda = D$.*
2. *If $k = n$, then $|D : D_pD_\lambda| = \delta_{\lambda,p}$. Moreover, if for $0 \leq u < \delta_{\lambda,p}$, d_u is an element in D with $\det d_u = \omega^u$, then $\{\, d_u \mid 0 \leq u < \delta_{\lambda,p} \,\}$ is a cross-section of D_pD_λ in D.*

We now specify the elements d_u in the preceding lemma, and hence G_p-orbit representatives in $L(\mathcal{A}_n^n)$, as follows. Let t_0 be the diagonal matrix with

$$(t_0)_i = \begin{cases} \omega & i = 1, \\ 1 & i > 1. \end{cases}$$

To simplify the notation, set

$$X_{\lambda,u} = t_0^u(X_\lambda).$$

Note that $X_{\lambda,0} = X_\lambda$.

Corollary 4.5. *Suppose $X \in L(\mathcal{A}_n^n)$. Then there is an integer k with $0 \le k \le n$ and a partition λ of k such that X is in the G_1-orbit of X_λ.*

If $k < n$, then X is in the G_p-orbit of X_λ.

If $k = n$, then there is an integer u with $0 \le u < \delta_{\lambda,p}$ so that X is in the G_p-orbit of $X_{\lambda,u}$.

Note that t_0 normalizes G_p. Thus, the next corollary follows from Proposition 3.1 and Corollary 4.5.

Corollary 4.6. *Suppose $X \in L(\mathcal{A}(G_p))$ and X is in the G_p-orbit of $X_{\lambda,u}$. Then the conclusion of Theorem* 2.1 *holds for X if and only if it holds for X_λ.*

The groups $\mathbf{C_X^{ref}}$. In this subsection we determine the groups C_X^{ref} as reflection subgroups of $\mathrm{GL}(X)$, we derive enough information about the groups C_X to characterize the subspaces $X \in L(\mathcal{A}(G_p))$ such that C_X acts on X as a reflection group, and we show that if $\exp(C_X) \subseteq \exp(G_p)$, then C_X acts on X as a reflection group.

The groups N_X, Z_X, and C_X depend on the ambient group G_p. We indicate this dependence with a superscript p, so

$$N_X^p = \{\, g \in G_p \mid g(X) = X \,\}, \quad Z_X^p = \{\, g \in G_p \mid g|_X = id \,\}, \quad \text{and} \quad C_X^p = N_X^p / C_X^p.$$

Note that $N_X^p = G_p \cap N_X^1$ and $Z_X^p = G_p \cap Z_X^1$.

Suppose $0 \le k \le n$ and $\lambda = (\ell_1, \ldots, \ell_a)$ is a partition of $n - k$. Let $i_1, \ldots, i_c$ be the distinct parts of λ with $i_1 > \cdots > i_c$ and multiplicities $m_1, \ldots, m_c$, respectively, so that $\lambda = (i_1^{m_1}, \ldots, i_c^{m_c})$. Our first task is to describe the groups $N_{X_\lambda}^p$, $Z_{X_\lambda}^p$, and $C_{X_\lambda}^p$ (see (4.2), (4.3), and (4.4)). To simplify the notation, until the statement of Proposition 4.7 set $L = L_\lambda$.

Define

$$W_\lambda = \{\, p_\sigma \in W \mid \sigma(L_\lambda) = L_\lambda \,\}.$$

Then W_λ is the semidirect product of the subgroups Z_λ and C_λ, where

$$Z_\lambda = \{\, p_\sigma \in W_\lambda \mid \forall\ 0 \le i \le a,\ \sigma(L_i) = L_i \,\}$$

is a normal subgroup of W_λ and

$$C_\lambda = \{\, p_\sigma \in W_\lambda \mid \sigma(L_0) = L_0 \text{ and } \sigma|_{L_i} \text{ is increasing for } 1 \le i \le a \,\}.$$

In addition,

$$Z_\lambda \cong S_{\ell_1} \times \cdots \times S_{\ell_a} \times S_{n-k} \quad \text{and} \quad C_\lambda \cong S_{m_1} \times \cdots \times S_{m_c}.$$

Clearly $N_{X_\lambda}^1 = W_\lambda D_\lambda$ and so

$$N_{X_\lambda}^p = G_p \cap W_\lambda D_\lambda = W_\lambda (G_p \cap D_\lambda) = W_\lambda (D_p \cap D_\lambda) = C_\lambda Z_\lambda (D_p \cap D_\lambda). \tag{4.2}$$

Next consider $Z_{X_\lambda}^1$. Obviously, $g \in G_1$ is in $Z_{X_\lambda}^1$ if and only if $gb_i = b_i$ for $b_i \in \mathcal{B}_\lambda$, and by a theorem of Steinberg [19, Theorem 1.5], $Z^1_{X_\lambda}$ is generated by the reflections it contains. Referring to (4.1), it is easy to see that

(1) for $1 \le l \le n$ and $\zeta \in \mu_r \setminus \{1\}$, $r_{ll}(\zeta)$ is in $Z^1_{X_\lambda}$ if and only if $l \in L_0$,
(2) for $1 \le l < l' \le n$ and $\zeta \in \mu_r$, $r_{ll'}(\zeta)$ is in $Z^1_{X_\lambda}$ if and only if either
  (a) $l, l' \in L_0$ or
  (b) there is an $i > 0$ so that $l, l' \in L_i$ and $\zeta = 1$.

From this it is straightforward to check that if

$$D_{\ell_0} = \{\, t \in D \mid \forall\ 1 \le i \le k,\ t_i = 1 \,\},$$

then $Z^1_{X_\lambda} \cap D = D_{\ell_0}$ and

$$Z^1_{X_\lambda} = Z_\lambda D_{\ell_0} \cong S_{\ell_1} \times \cdots \times S_{\ell_a} \times G(r,1,n-k).$$

Therefore

$$Z^p_{X_\lambda} = G_p \cap Z^1_{X_\lambda} = Z_\lambda (D_p \cap D_{\ell_0}) \cong S_{\ell_1} \times \cdots \times S_{\ell_a} \times G(r,p,n-k). \tag{4.3}$$

Finally, it follows from (4.2), (4.3), and the computation of $|D_p \cap D_\lambda|$ in the proof of Proposition 4.4 that

$$|C^p_{X_\lambda}| = \begin{cases} m_1! \cdots m_c! r^a & \text{if } k < n, \\ m_1! \cdots m_c! \frac{r^a \delta_{\lambda,p}}{p} & \text{if } k = n. \end{cases} \tag{4.4}$$

Proposition 4.7. *Suppose $X \in L(\mathcal{A}(G_p))$ is in the G_p-orbit of $X_{\lambda,u}$, where $\lambda = (\ell_1, \dots, \ell_a)$ is a partition of k for some $0 \le k \le n$ and u is an integer with $0 \le u < \delta_{\lambda,p}$. Suppose $i_1, \dots, i_c$ are the distinct parts of λ with $i_1 > \cdots > i_c$ and multiplicities $m_1, \dots, m_c$, so $\lambda = (i_1^{m_1}, \dots, i_c^{m_c})$. Then as a reflection subgroup of $\mathrm{GL}(X)$,*

$$C^{\mathrm{ref}}_X \cong \begin{cases} G(r,1,m_1) \times \cdots \times G(r,1,m_c) & \text{if } k < n, \\ G(r,p_{i_1},m_1) \times \cdots \times G(r,p_{i_c},m_c) & \text{if } k = n, \end{cases}$$

where for an integer j, $p_j = \mathrm{lcm}(j,p)/j$.

Proof. By Proposition 3.1, without loss of generality we may assume that $X = X_\lambda$.

Suppose $\sigma \in S_n$ and $t \in D$. Then

$$p_\sigma t(X) = (p_\sigma t p_\sigma^{-1}) p_\sigma(X_\lambda) = (p_\sigma t p_\sigma^{-1})(X_{\sigma(L_\lambda)}).$$

It is shown in [13, Proposition 6.74] that $p_\sigma t(X_\lambda) = X_\lambda$ if and only if $\sigma(L_\lambda) = L_\lambda$ and $p_\sigma t p_\sigma^{-1} \in D_\lambda$. Now W_λ normalizes D_λ, and so

$$p_\sigma t(X) = X \quad \text{if and only if} \quad p_\sigma \in W_\lambda \text{ and } t \in D_\lambda. \tag{4.5}$$

Note that W_λ permutes the vectors in the basis $\mathcal{B}_\lambda = \{b_1, \dots, b_a\}$ of X and that each $b_i \in \mathcal{B}_\lambda$ is a common eigenvector for D_λ.

For a linear transformation $g \in \mathrm{GL}(X)$ let $[[g]]$ denote the matrix of g with respect to $\mathcal{B}_\lambda$. Then the rule $g \mapsto [[g]]$ defines a group homomorphism from N^p_X to $\mathrm{GL}_a(\mathbb{C})$ with kernel Z^p_X and hence an injection from C^p_X to $\mathrm{GL}_a(\mathbb{C})$. It follows from (4.5) that the image of this mapping is contained in the subgroup $G(r,1,a)$ of $\mathrm{GL}_a(\mathbb{C})$. In particular, every reflection in C^p_X is one of the reflections listed in (4.1).

Suppose $\zeta \in \mu_r$. If $1 \le i \ne j \le a$ and $\ell_i = \ell_j$, define $s_{ij}(\zeta) \in \mathrm{GL}(V)$ by

$$s_{ij}(\zeta) e_l = \begin{cases} \zeta e_{\bar\ell_{i-1}+\nu} & \text{if } l = \bar\ell_{j-1} + \nu,\ \nu \in [\ell_j], \\ \zeta^{-1} e_{\bar\ell_{j-1}+\nu} & \text{if } l = \bar\ell_{i-1} + \nu,\ \nu \in [\ell_i], \\ e_l & \text{otherwise}, \end{cases}$$

where, as above, $\bar{\ell}_0 = 0$ and $\bar{\ell}_i = \ell_1 + \cdots + \ell_i$ for $i > 0$. Then $s_{ij}(\zeta)$ acts on $\mathcal{B}_\lambda$ by $b_j \mapsto \zeta b_i$, $b_i \mapsto \zeta^{-1} b_j$, and $b_l \mapsto b_l$ for $l \neq i, j$, and so $s_{ij}(\zeta) \in N_X^p$ and $[[s_{ij}(\zeta)]] = r_{ij}(\zeta)$. If $1 \leq i \neq j \leq n$ and $n_i \neq n_j$, then there is no $\sigma \in S_n$ such that $p_\sigma b_i = b_j$ and no element $p_\sigma t \in N_X^p$ with $[[p_\sigma t]] = r_{ij}(\zeta)$.

Now suppose that $k < n$. For $1 \leq i \leq a$ and $\zeta \in \mu_r$ define $s_{ii}(\zeta) \in \mathrm{GL}(V)$ by

$$s_{ii}(\zeta) e_l = \begin{cases} \zeta e_l & \text{if } l \in \{\bar{\ell}_{i-1}+1, \dots, \bar{\ell}_i\}, \\ e_l & \text{if } l \in [k] \setminus \{\bar{\ell}_{i-1}+1, \dots, \bar{\ell}_i\}, \\ \zeta^{-\ell_i} e_{k+1} & \text{if } l = k+1, \\ e_l & k+1 < l \leq n. \end{cases}$$

Then $s_{ii}(\zeta)$ acts on $\mathcal{B}_\lambda$ by $b_i \mapsto \zeta b_i$ and $b_l \mapsto b_l$ for $l \neq i$, so $s_{ii}(\zeta) \in N_X^p$ and $[[s_{ii}(\zeta)]] = r_{ii}(\zeta)$ when $\zeta \neq 1$. Because $(C_X^p)^{\mathrm{ref}}$ is generated by the reflections it contains, it follows that

$$(C_X^p)^{\mathrm{ref}} \cong G(r,1,m_1) \times \cdots \times G(r,1,m_c) \quad \text{when} \quad k < n.$$

Finally, suppose $k = n$ and $1 \leq i \leq a$. If $\zeta \in \mu_r \setminus \{1\}$ and $t \in N_X^p$ is such that $[[t]] = r_{ii}(\zeta)$, then $t \in D_p$, $tb_i = \zeta b_i$, and $tb_l = b_l$ for $l \neq i$. Thus,

$$\zeta^{n_i} = \det t \in \langle \omega^{n_i} \rangle \cap \langle \omega^p \rangle = \langle \omega^{\mathrm{lcm}(n_i, p)} \rangle$$

and so $\zeta \in \langle \omega^{p_i} \rangle$. Conversely, if $\zeta \in \langle \omega^{p_i} \rangle \setminus \{1\}$ and $t \in \mathrm{GL}(V)$ is defined by

$$te_l = \begin{cases} \zeta e_l & \text{if } l = \bar{n}_{i-1} + \nu,\ \nu \in [n_i], \\ e_l & \text{if } l \in [n] \setminus \{\bar{n}_{i-1}+1, \dots, \bar{n}_i\}, \end{cases}$$

then clearly $t \in N_X^p$ and $[[t]] = r_{ii}(\zeta)$. Because $(C_X^p)^{\mathrm{ref}}$ is generated by the reflections it contains, it follows that

$$(C_X^p)^{\mathrm{ref}} \cong G(r, p_{i_1}, m_1) \times \cdots \times G(r, p_{i_c}, m_c) \quad \text{when} \quad k = n.$$

This completes the proof of the proposition. □

Krishnasamy-Taylor [12] has investigated the structure of the groups $C_{X_\lambda}^p$ and shown that when $(C_{X_\lambda}^p)^{\mathrm{ref}} = C_{X_\lambda}^p$, the elements $s_{ij}(\zeta)$ defined above generate a complement to $Z_{X_\lambda}^p$ in $N_{X_\lambda}^p$.

The next corollary follows from the preceding equation and Proposition 4.7.

Corollary 4.8. *Suppose $X \in L(\mathcal{A}(G_p))$ is in the G_p-orbit of $X_{\lambda,u}$, where λ is a partition of k for some $1 \leq k \leq n$ and u is an integer with $0 \leq u < \delta_{\lambda,p}$. Say $\lambda = (i_i^{m_1}, \dots, i_c^{m_c})$ with $i_1 > \cdots > i_c > 0$.*

(1) *If $k < n$, then $(C_X^p)^{\mathrm{ref}} = C_X^p$.*

(2) *If $k = n$, then*

$$|C_X^p : (C_X^p)^{\mathrm{ref}}| = p^{c-1} \frac{\gcd(i_1, \dots, i_c, p)}{\gcd(i_1, p) \cdots \gcd(i_c, p)}.$$

In particular,

(a) *C_X^p acts on X as a reflection group if and only if*

$$p^{c-1} \frac{\gcd(i_1, \dots, i_c, p)}{\gcd(i_1, p) \cdots \gcd(i_c, p)} = 1, \text{ and}$$

(b) *if* $\lambda = (i^m)$ *has only one distinct part, then*

$$C_X^p = (C_X^p)^{\mathrm{ref}} \cong G(r, p_i, m),$$

where $p_i = \operatorname{lcm}(i, p)/i$.

Using this corollary we can prove the following result.

Corollary 4.9. *Suppose* $X \in L(\mathcal{A}(G_p))$. *If* $\exp(C_X^p) \subseteq \exp(G_p)$, *then* $(C_X^p)^{\mathrm{ref}} = C_X^p$.

Proof. As in the proof of the proposition, we may assume that $X = X_\lambda$ where $\lambda = (i_i^{m_1}, \ldots, i_c^{m_c})$ is a partition of k for some $0 \leq k \leq n$. By Corollary 4.8, if $k < n$, then $(C_X^p)^{\mathrm{ref}} = C_X^p$. Thus we may assume that $k = n$. Then $m_1 i_1 + \cdots + m_c i_c = n$ and by Proposition 4.7, $(C_X^p)^{\mathrm{ref}} \cong G(r, p_{i_1}, m_1) \times \cdots \times G(r, p_{i_c}, m_c)$, where $p_{i_j} = p/\gcd(i_j, p)$. Using Corollary 4.8 again, it is enough to show that $c = 1$.

It is well-known that $\exp(G(r,p,n)) = \{r-1, 2r-1, \ldots, (n-1)r-1, nr/p-1\}$ (see [13, Appendix B.4]). First consider the special case when $n = p$. Then $r = qn$ and $\exp(G_n) = \{qn-1, 2qn-1, \ldots, (n-1)qn-1, qn-1\}$. Also, $m_1 q \cdot \gcd(n, i_1) - 1$ is an exponent of $(C_X^n)^{\mathrm{ref}}$ and so an exponent of G_n. But then $m_1 q \cdot \gcd(n, i_1) - 1 = sqn - 1$ for some $s \geq 1$. Thus $m_1 \cdot \gcd(n, i_1) = qn$, and so $s = 1$ and $m_1 i_1 = n$. It follows that $c = 1$ as desired.

Next, just suppose that $c > 1$ and that there exists j_1 and j_2 such that $j_1 \neq j_2$ and $m_{j_1}, m_{j_2} > 1$. Then $r-1$ occurs as an exponent of $(C_X^p)^{\mathrm{ref}}$ with multiplicity at least 2. But $r-1$ is an exponent of G_p with multiplicity greater than 1 if and only if $p = n$, and if $p = n$, then $c = 1$, a contradiction. Thus it cannot be the case that $c > 1$ and there exists j_1 and j_2 such that $j_1 \neq j_2$ and $m_{j_1}, m_{j_2} > 1$. Therefore, either $c = 1$ or $c > 1$ and there is an s such that $m_j = 1$ for $j \neq s$.

Finally, just suppose that $c > 1$ and s is such that $m_j = 1$ for $j \neq s$. Reordering λ if necessary we may assume that $m_j = 1$ for $j > 1$. Suppose first that $p \nmid i_j$ for some $j > 1$. Then $r/p_{i_j} - 1 = r \cdot \gcd(p, i_j)/p - 1 < r - 1$. The only exponent of G_p that could be less than $r-1$ is $nr/p - 1$, so $nr/p - 1 = r/p_{i_j} - 1$. But then $n/p = 1/p_{i_j} = \gcd(p, i_j)/p$, and so $n = \gcd(p, i_j)$. Because $c > 1$, $i_j < n$, and so $n \neq \gcd(p, i_j)$, a contradiction. Suppose on the other hand that $p | i_j$ for all $j \geq 2$. Then $r-1$ is an exponent of $(C_X^p)^{\mathrm{ref}}$, and hence of G_p, with multiplicity $c > 1$. Again, $r-1$ is an exponent of G_p with multiplicity greater than 1 if and only if $p = n$, and if $p = n$, then $c = 1$, a contradiction. It follows that it must be the case that $c = 1$, as claimed. □

Classification of $\mathbf{X} \in \mathbf{L}(\mathcal{A}(\mathbf{G_p}))$ with $\mathcal{A}(\mathbf{C_X}) = \mathcal{A}(\mathbf{G_p})^{\mathbf{X}}$. In this subsection we classify all subspaces $X \in L(\mathcal{A}(G_p))$ with $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$ and show that if $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$, then C_X acts on X as a reflection group.

Theorem 4.10. *Suppose* $X \in L(\mathcal{A}(G_p))$ *is in the* G_p*-orbit of* $X_{\lambda,u}$, *where* λ *is a partition of* k *for some* $0 \leq k \leq n$ *and* u *is an integer with* $0 \leq u < \delta_{\lambda,p}$.

(1) *If* $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$, *then* $\lambda = (i^m)$ *has only one distinct part.*
(2) *If* $\lambda = (i^m)$, *then* $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$ *unless* $p = r$, $k = n$, $\gcd(r, i) = 1$, *and* $i > 1$.

Proof. As in the preceding subsection, we assume without loss of generality that $X = X_\lambda$ and that $\lambda = (n_1, \ldots, n_a) = (i_1^{m_1}, \ldots, i_c^{m_c})$, where $n_1 \geq \cdots \geq n_a > 0$ and $i_1 > \cdots > i_c > 0$. Recall that $\dim X = a$.

The restricted arrangements $\mathcal{A}(G_p)^X$ have been computed by Orlik, Solomon, and Terao [13, §6.4]. Let $\psi_1 = |\{ i \in [a] \mid n_i > 1 \}|$ denote the number of parts of λ greater than 1. Then

$$\mathcal{A}(G_p)^X = \begin{cases} \mathcal{A}_a^a & \text{if } p < r \text{ or } k < n, \\ \mathcal{A}_a^{\psi_1} & \text{if } p = r \text{ and } k = n. \end{cases} \tag{4.6}$$

To prove the first statement, suppose that $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$. There are two cases depending on k, n, and r.

One case is when $p < r$ or $k < n$. Then

$$\mathcal{A}(G_p)^X = \mathcal{A}_a^a.$$

By Proposition 4.7, $C_X^{\mathrm{ref}} \cong G(r, \hat{p}_1, m_1) \times \cdots \times G(r, \hat{p}_c, m_c)$, where for $1 \leq l \leq c$, $\hat{p}_l = 1$ if $k < n$ and $\hat{p}_l = \mathrm{lcm}(i_l, p)/i_l$ if $k = n$. Notice that $\mathrm{lcm}(i_l, p)/i_l < r$, because if $k = n$, then $p < r$. Thus $\hat{p}_l < r$ for $1 \leq l \leq c$ and so

$$\mathcal{A}(C_X^p) = \mathcal{A}_{m_1}^{m_1} \times \cdots \times \mathcal{A}_{m_c}^{m_c}.$$

Since $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$, it must be that $c = 1$ and $\lambda = (i^m)$, where $im = k$ and $m = \dim X = a$.

The second case is when $p = r$ and $k = n$. Then $\mathcal{A}(G_r)^X = \mathcal{A}_a^{\psi_1}$, where ψ_1 is as above. Because $\mathcal{A}(G_r)^X = \mathcal{A}(C_X^r)$ is a reflection arrangement and $\mathcal{A}_a^{\psi_1}$ is a reflection arrangement if and only if $\psi_1 = 0$ or $\psi_1 = a$, we have that either

$$\mathcal{A}(G_r)^X = \mathcal{A}_a^0 \quad \text{or} \quad \mathcal{A}(G_r)^X = \mathcal{A}_a^a.$$

If $\mathcal{A}(G_p)^X = \mathcal{A}_a^0$, then all parts of λ are equal to 1, so $\lambda = (1^n)$. If instead $\mathcal{A}(G_p)^X = \mathcal{A}_a^a$, then using Proposition 4.7 again we have $C_X^{\mathrm{ref}} \cong G(r, p_{i_1}, m_1) \times \cdots \times G(r, p_{i_c}, m_c)$, where for $1 \leq l \leq c$, $p_{i_l} = \mathrm{lcm}(i_l, r)/i_l = r/\gcd(i_l, r)$. Therefore,

$$\mathcal{A}(C_X^r) = \mathcal{A}_{m_1}^{m_1'} \times \cdots \times \mathcal{A}_{m_c}^{m_c'},$$

where $m_l' = 0$ if $\gcd(i_l, r) = 1$ and $m_l' = m_i$ if $\gcd(i_l, r) > 1$. Finally, because $\mathcal{A}_a^a = \mathcal{A}(G_r)^X = \mathcal{A}(C_X^r)$, it must be that $c = 1$ and $\lambda = (i^m)$, where now $i > 1$ and $\gcd(i, r) > 1$.

To prove the second statement, suppose that $\lambda = (i^m)$ has only one distinct part. Suppose also that $k \neq n$, or $p \neq r$, or $\gcd(p, i) \neq 1$. Then $i \neq 1$ and so by (4.6) and Proposition 4.7 $\mathcal{A}(C_X^p) = \mathcal{A}_m^m = \mathcal{A}(G_p)^X$ as claimed. Otherwise $k = n$, $p = r$, and $\gcd(p, i) = 1$. If $i = 1$, then $X = V$ and $C_X^p = G_p$, and so $\mathcal{A}(C_X^p) = \mathcal{A} = \mathcal{A}(G_p)^X$ as claimed. Finally, if $i > 1$, then by Proposition 4.7 $\mathcal{A}(C_X^p) = \mathcal{A}_m^0$ and by (4.6) $\mathcal{A}(G_p)^X = \mathcal{A}_m^m$, so $\mathcal{A}(C_X^p) \neq \mathcal{A}(G_p)^X$. □

The next corollary follows from the theorem and Corollary 4.8.

Corollary 4.11. *Suppose $X \in L(\mathcal{A}(G_p))$ and $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$. Then C_X^p acts on X as a reflection group.*

Proof of Theorem 2.1 for $\mathbf{G(r, p, n)}$. Suppose $X \in L(\mathcal{A}(G_p))$ is such that $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$ and $\exp(C_X^p) \subseteq \exp(G_p)$. We need to show that the restriction map $\rho_X \colon \mathbb{C}[V]^G \to \mathbb{C}[X]^{C_X^p}$ is surjective. By the assumption that $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$, Theorem 4.10, and Proposition 3.1, we may assume that $X = X_\lambda$ where $\lambda = (i^m)$. It then follows from Corollary 4.8 that $(C_X^p)^{\mathrm{ref}} = C_X^p$.

If $m = n$, then $i = 1$, and so $X = V$, $C_X^p = G_p$, and ρ_X is the identity map. Thus, in this case $\mathcal{A}(C_X^p) = \mathcal{A}(G_p)^X$, $\exp(C_X^p) \subseteq \exp(G_p)$, and ρ_X is surjective. In the rest of the proof we assume that $m < n$, and so X is a proper subspace of V.

Next, it follows from Proposition 4.7 that $C_X^p \cong G(r, \hat{p}, m)$, where $\hat{p} = 1$ if $im < n$ and $\hat{p} = p/\gcd(i,p)$ if $im = n$. Equivalently,

$$C_X^p \cong \begin{cases} G(r,1,m) & \text{if } im < n \text{ or if } im = n \text{ and } p|i, \\ G(r, p/\gcd(i,p), m) & \text{if } im = n \text{ and } \gcd(i,p) < p. \end{cases}$$

By [13, Table B.1] we have

- $\exp(G_p) = \{r-1, \dots, (n-1)r-1, nq-1\}$,
- $\exp(C_X^p) = \{r-1, \dots, (m-1)r-1, mr-1\}$ if $im < n$ or if $im = n$ and $p|i$, and
- $\exp(C_X^p) = \{r-1, \dots, (m-1)r-1, m\hat{q}-1\}$, where $\hat{q} = r \cdot \gcd(i,p)/p$ if $im = n$ and $\gcd(i,p) < p$.

Because $m < n$, if $im < n$ or if $im = n$ and $p|i$, then the assumption that $\exp(C_X^p) \subseteq \exp(G_p)$ is superfluous. On the other hand, if $im = n$ and $\gcd(i,p) < p$, then $m\hat{q} - 1 \in \exp(G_p)$, and so either $m\hat{q} = lr$ for some l with $m \le l \le n-1$ or $m\hat{q} = nq$. Just suppose that $m\hat{q} = lr$ for some l with $m \le l \le n-1$. Then $m \cdot \gcd(i,p)/p = l \ge m$, so $\gcd(i,p) \ge p$, so $p|i$, contradicting the assumption that $\gcd(i,p) < p$. Therefore, it must be the case that $m\hat{q} = nq$. Then $m \cdot \gcd(i,p) = n$ and $n = im$, so $\gcd(i,p) = i$. Thus, $i|p$.

Summarizing the preceding discussion, there are two cases. Either

(1) $im < n$ or $im = n$ and $p|i$, in which case $C_X^p \cong G(r,1,m)$ and $\exp(C_X^p) = \{r-1, \dots, (m-1)r-1, mr-1\}$ or
(2) $im = n$ and $i|p$, in which case $C_X^p \cong G(r, p/i, m)$ and $\exp(C_X^p) = \{r-1, \dots, (m-1)r-1, nq-1\}$.

Now consider the restriction map $\widetilde{\rho}_X : \mathbb{C}[V] \to \mathbb{C}[X]$. Recall that $\{x_1, \dots, x_n\}$ is the basis of V^* dual to the basis $\{e_1, \dots, e_n\}$ of V and let $\{y_1, \dots, y_m\}$ be the basis of X^* dual to $\mathcal{B}_\lambda$. Then $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]$, $\mathbb{C}[X] = \mathbb{C}[y_1, \dots, y_m]$,

$$\begin{aligned} \rho_X(x_1) = \cdots = \rho_X(x_i) &= y_1, \\ \rho_X(x_{i+1}) = \cdots = \rho_X(x_{2i}) &= y_2, \\ &\vdots \\ \rho_X(x_{(m-1)i+1}) = \cdots = \rho_X(x_{mi}) &= y_m, \end{aligned}$$

and

$$\rho_X(x_{mi+1}) = \cdots = \rho_X(x_n) = 0.$$

For $l \ge 1$ let

$$f_l = (x_1^r)^l + \cdots + (x_n^r)^l \quad \text{and} \quad \bar{f}_l = (y_1^r)^l + \cdots + (y_m^r)^l$$

denote the l^{th} power sums in $x_1^r, \dots, x_n^r$ and $y_1^r, \dots, y_m^r$, respectively. Then

$$\mathbb{C}[V]^{G_p} = \mathbb{C}[f_1, \dots, f_{n-1}, (x_1 \cdots x_n)^q],$$

$\widetilde{\rho}_X(f_l) = i\bar{f}_l$ for all $l \ge 1$, and

$$\widetilde{\rho}_X((x_1 \cdots x_n)^q) = \begin{cases} 0 & \text{if } im < n, \\ (y_1 \cdots y_m)^{iq} & \text{if } im = n. \end{cases}$$

If $im < n$ or if $im = n$ and $p|i$, then

$$\mathbb{C}[X]^{C_X^p} = \mathbb{C}[\bar{f}_1, \ldots, \bar{f}_m] = \mathbb{C}[\rho_X(f_1), \ldots, \rho_X(f_m)],$$

and so ρ_X is surjective in this case.

If $im = n$ and $i|p$, then

$$\begin{aligned}\mathbb{C}[X]^{C_X^p} &= \mathbb{C}[\bar{f}_1, \ldots, \bar{f}_{m-1}, (y_1 \cdots y_m)^{\hat{q}}] \\ &= \mathbb{C}[\bar{f}_1, \ldots, \bar{f}_{m-1}, (y_1 \cdots y_m)^{iq}] \\ &= \mathbb{C}[\rho_X(f_1), \ldots, \rho_X(f_{m-1}), \rho_X((x_1 \cdots x_n)^q)],\end{aligned}$$

and so ρ_X is surjective in this case as well. This completes the proof of Theorem 2.1 when $G = G(r, p, n)$. □

5. The exceptional irreducible unitary reflection groups

In this section we suppose that G is one of the thirty-four exceptional unitary reflection groups labeled $G_4, \ldots, G_{37}$, we classify the subspaces $X \in L(\mathcal{A}(G))$ such that $\mathcal{A}(C_X) = \mathcal{A}(G)^X$, we classify the subspaces $X \in L(\mathcal{A}(G))$ such that C_X acts on X as a reflection group, and we complete the proof of Theorem 2.1 for these groups.

Classification of X with $\mathcal{A}(\mathbf{C_X}) = \mathcal{A}(\mathbf{G})^{\mathbf{X}}$, $\mathbf{C_X} = \mathbf{C_X^{ref}}$, or $\mathbf{exp}(\mathbf{C_X}) \subseteq \mathbf{exp}(\mathbf{G})$. First, as in §3, it is enough to consider one representative from each orbit of G on $L(\mathcal{A}(G))$. For the exceptional unitary reflection groups, orbit representatives for the action of G on $L(\mathcal{A}(G))$ are given in [13, Appendix C]. We use the labeling in those tables. Specifically, it is shown in [13, Lemma 6.88] that for $X, Y \in L(\mathcal{A}(G))$, X and Y are in the same G-orbit if and only if Z_X and Z_Y are conjugate and that in most cases the reflection type of Z_X uniquely determines X. Thus, the reflection type of the subgroups Z_X is used to index the orbit containing X. When there are two orbits whose pointwise stabilizers have the same reflection type, we label the orbits as in [13, Appendix C] with $'$ and $''$. For example, in the group G_{27} there are two orbits whose pointwise stabilizer has reflection type A_2. These are denoted by A_2' and A_2''.

Next, if $X \in L(\mathcal{A}(G))$, then whether or not $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and whether or not $C_X = C_X^{\mathrm{ref}}$ can be determined from knowledge of the reflection type of C_X^{ref} as we now describe.

It was shown in the proof of Theorem 3.2 that $\mathcal{A}(C_X) \subseteq \mathcal{A}(G)^X$, so equality holds if and only if $|\mathcal{A}(C_X)| = |\mathcal{A}(G)^X|$. Because $\mathcal{A}(C_X)$ is a reflection arrangement, by [13, Corollary 6.63] $|\mathcal{A}(C_X)|$ is the sum of the coexponents of C_X^{ref}. The coexponents of the irreducible unitary reflection groups are given in [13, Appendix B.4]. By [13, §6.4] and [8], $\mathcal{A}(G)^X$ is a free arrangement, and so by [13, Theorem 4.23] $|\mathcal{A}(G)^X|$ is the sum of the exponents of $\mathcal{A}(G)^X$. The exponents of $\mathcal{A}(G)^X$ are given in the last sections of the tables in [13, Appendix C]. Thus, whether or not $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ can be determined once the reflection type of C_X^{ref} is known.

By definition $C_X^{\mathrm{ref}} \subseteq C_X$, so equality holds if and only if $|C_X^{\mathrm{ref}}| = |C_X|$. Because C_X^{ref} is a reflection group, $|C_X^{\mathrm{ref}}|$ is the product of the degrees of C_X^{ref}. The degrees of C_X^{ref} are obtained from the exponents by adding 1, and the exponents of the irreducible unitary reflection groups are given in [13, Appendix B.4]. For $|C_X|$, as described in [13, §6.4], in the tables in [13, Appendix C], the size of the orbit $G \cdot X$ is the entry in the column indexed by the reflection type of Z_X in the first row. The size of C_X can then be computed using the orbit-stabilizer formula: $|G \cdot X| = |G|/|N_X| = |G|/(|C_X||Z_X|)$. Thus, whether or not $C_X = C_X^{\mathrm{ref}}$ can be determined once the reflection type of C_X^{ref} is known.

For example, in the group G_{34}, if the reflection type of Z_X is A_1A_2, then C_X^{ref} is isomorphic as a reflection group to the product $C_3 \times G(3,1,2)$, where C_3 is a cyclic group of order three acting on a one-dimensional vector space. Using the tables in [13] we see that the multiset of coexponents of $C_3 \times G(3,1,2)$ is $\{1,1,4\}$ and the multiset of exponents of $\mathcal{A}(G)^X$ is $\{1,13,16\}$, so $\mathcal{A}(C_X) \neq \mathcal{A}(G)^X$. Next, the multiset of exponents of $C_3 \times G(3,1,2)$ is $\{2,2,5\}$, so $|C_X^{\mathrm{ref}}| = 3 \cdot 3 \cdot 6 = 54$. On the other hand, the size of the orbit indexed by A_1A_2 is 30240, $\exp(G) = \{5,11,17,23,29,41\}$, and $\exp(Z_X) = \{1,1,2\}$, therefore, we have $|C_X| = (6 \cdot 12 \cdot 18 \cdot 24 \cdot 30 \cdot 42)/(30240 \cdot 12) = 108$ and $C_X^{\mathrm{ref}} \neq C_X$. Note also that $\exp(C_X) \not\subseteq \exp(G)$.

It remains to compute the reflection type of C_X^{ref} for each orbit. There are three cases when the reflection type of C_X^{ref} is easily determined. First, if $X = \{0\}$, then $C_X = C_X^{\mathrm{ref}}$ is the trivial group. Second, if $\dim X = 1$, then since C_X is a finite subgroup of the multiplicative group of non-zero complex numbers, it is a cyclic group, and so clearly $C_X = C_X^{\mathrm{ref}}$. Third, if $X = V$, then $C_X = C_X^{\mathrm{ref}} = G$.

The reflection types of the groups C_X^{ref} for the exceptional Coxeter groups were computed by Howlett [9].

The reflection types of the groups C_X^{ref} for the exceptional, non-Coxeter, unitary reflection groups when $1 < \dim X < \dim V$ were determined using GAP4 (release 4.12) with the aid of the `SmallGroups` library as follows. First, a representative X for each orbit was chosen, and then the elements N_X were computed as matrices of linear transformations of X. This information yielded generators of C_X^{ref}. The reflection type of C_X^{ref} was then determined using the `SmallGroups` library in GAP4, except for the case of the orbit of type A_1 in the lattice of G_{34}, where the type of C_X^{ref} was easy to determine as it is an irreducible unitary reflection group of rank 5.

The reflection types of the subgroups C_X^{ref} for all pairs (G,X), where G is an exceptional unitary reflection group with rank three or more and X is not equal to $\{0\}$ or V, are given in Tables 1–7 in the Appendix. We have included Howlett's results for exceptional Coxeter groups, because the tables in [9] contain several omissions. The columns labeled by $\mathcal{A}$ in the tables indicate whether or not $\mathcal{A}(C_X) = \mathcal{A}(G)^X$; the columns labeled by C_X indicate whether or not $C_X = C_X^{\mathrm{ref}}$, and the columns labeled by exp indicate whether or not $\exp(C_X) \subseteq \exp(G)$. For example, for the group G_{34} and the orbit with pointwise stabilizer of type B_3, $\mathcal{A}(C_X) \neq \mathcal{A}(G)^X$, $C_X^{\mathrm{ref}} \neq C_X$, and $\exp(C_X) \not\subseteq \exp(G)$.

The following proposition may be deduced from the tables.

Proposition 5.1. *Suppose G is an irreducible, exceptional unitary reflection group and $X \in L(\mathcal{A}(G))$. If $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ or if $\exp(C_X) \subseteq \exp(G)$, then $C_X^{\mathrm{ref}} = C_X$.*

Remark 5.2. It is possible to have $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and $\exp(C_X) \subseteq \exp(G)$, for example the orbit indexed by $G(3,3,3)$ in G_{34}. It is also possible to have $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and $\exp(C_X) \not\subseteq \exp(G)$, for example the orbit indexed by D_4 in E_6. Similarly, it is possible to have $\mathcal{A}(C_X) \neq \mathcal{A}(G)^X$, $C_X^{\mathrm{ref}} = C_X$, and $\exp(C_X) \subseteq \exp(G)$, for example the orbit indexed by $G(3,3,4)$ in G_{34}, and it is also possible to have $\mathcal{A}(C_X) \neq \mathcal{A}(G)^X$, $C_X^{\mathrm{ref}} = C_X$, and $\exp(C_X) \not\subseteq \exp(G)$, for example the orbit indexed by A_1^3 in G_{34}.

Proof of Theorem 2.1 for the exceptional unitary reflection groups. Suppose G is an exceptional unitary reflection group and $X \in L(\mathcal{A}(G))$ is such that $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and $\exp(C_X) \subseteq \exp(G)$. It remains to show that the restriction map $\rho_X : \mathbb{C}[V]^G \to \mathbb{C}[X]^{C_X^p}$ is surjective.

First, if G is a Coxeter group, $\dim X = 1$, and both conditions hold, then C_X is of type A_1 and acts as -1 on X. The canonical bilinear form defined on V in [4, Ch. VI, §1.1] gives a non-zero, homogeneous polynomial of degree two in $\mathbb{C}[V]^G$, say f_2. It is straightforward to check that $\rho_X(f_2) \neq 0$ (see [6, §4]), and it follows that ρ_X is surjective in these cases.

Finally, suppose that both conditions hold and that either G is not a Coxeter group or G is a Coxeter group and $1 < \dim X < \dim V$. All pairs (G, X) such that $\mathcal{A}(C_X) = \mathcal{A}(G)^X$ and $\exp(C_X) \subseteq \exp(G)$ are highlighted in Tables 1–7. The fact that ρ_X is surjective was checked directly in every case by implementing the following argument using a pre-packaged version of GAP3 (release 4.4) [15], with the packages CHEVIE (version 4) [7], for functionality on unitary reflection groups, and VKCURVE (version 1.2), for functionality on multivariate polynomials, provided by J. Michel [11].

(1) Choose a basis $\{x_1, \dots x_n\}$ of V^* such that the restrictions of $x_1, \dots, x_a$ to X are a basis of X^* and the restrictions of $x_{a+1}, \dots, x_n$ are a basis of $(X^\perp)^*$ (recall that V is endowed with a G-invariant hermitian form). Then the restriction mapping $\mathbb{C}[V] \to \mathbb{C}[X]$ is given by evaluating x_j at zero for $a + 1 \leq j \leq n$.

(2) Let $\{f_1, \dots, f_n\}$ be the basic invariants of G obtained from the CHEVIE package. If the rank of G is two, restrict f_1 and f_2 to X to obtain invariants $\rho_X(f_1)$ and $\rho_X(f_2)$ of C_X. If the rank of G is greater than 2, then in all cases the multisets of exponents of C_X and G are actually sets. Choose the numbering so that $\deg f_1, \dots, \deg f_a$ are the degrees of C_X. Restrict f_1, $\dots$, f_a to X to obtain invariants $\rho_X(f_1)$, $\rho_X(f_2)$, $\dots$, $\rho_X(f_a)$ of C_X.

(3) If the rank of G is greater than 2, compute the determinant of the Jacobian of $\rho_X(f_1)$, $\rho_X(f_2)$, $\dots$, $\rho_X(f_a)$ (with respect to the basis $x_1, \dots, x_a$ of X^*).

If the rank of G is 2, then it turns out that at least one of $\rho_X(f_1)$ or $\rho_X(f_2)$ is non-zero, so ρ_X is surjective in these cases. If the rank of G is greater than 2, then the Jacobian determinant is non-zero in all cases, and so it follows from [17, Proposition 2.3] that $\mathbb{C}[X]^{C_X} = \mathbb{C}[\rho_X(f_1), \rho_X(f_2), \dots, \rho_X(f_a)]$. Therefore, ρ_X is surjective in these cases as well. This completes the proof of Theorem 2.1. □

Appendix

Results of the computations for exceptional groups with rank greater than 2 used in the proof of Theorem 2.1 are presented in the tables below.

- Orbits in $\mathcal{A}(G)$ are labeled by the reflection type of the pointwise stabilizer of a subspace in the orbit. The order of the rows is the same as in [13, Appendix C].
- C_k denotes a cyclic group of order k, and we have written $G_{r,p,n}$ instead of $G(r, p, n)$ to save (a little) space.
- In the columns labeled $\mathcal{A}$, Y indicates that $\mathcal{A}(C_X) = \mathcal{A}(G)^X$.
- In the columns labeled C_X, Y indicates that $C_X^{\mathrm{ref}} = C_X$.
- In the columns labeled exp, Y indicates that $\exp(C_X) \subseteq \exp(G)$.

- If $X \in L(\mathcal{A})$ is such that $\mathcal{A}^X = \mathcal{A}(C_X^{\mathrm{ref}})$ and $\exp(C_X) \subseteq \exp(G)$, then the row indexed by the reflection type of Z_X is highlighted. As explained in §5, the map ρ_X was checked to be surjective in all these cases.

TABLE 1. Exceptional groups of rank 3

G	Z_X	C_X^{ref}	$\mathcal{A}$	C_X	exp
G_{23}	A_0	H_3	Y	Y	Y
(H_3)	A_1	A_1^2	N	Y	N
	A_1^2	A_1	Y	Y	Y
	A_2	A_1	Y	Y	Y
	$I_2(5)$	A_1	Y	Y	Y
	H_3	A_0	Y	Y	Y
G_{24}	A_0	G_{24}	Y	Y	Y
	A_1	B_2	N	Y	N
	A_2	A_1	Y	Y	N
	B_2	A_1	Y	Y	N
	G_{24}	A_0	Y	Y	Y
G_{25}	A_0	G_{25}	Y	Y	Y
	C_3	$G_{3,1,2}$	Y	Y	N
	C_3^2	C_6	Y	Y	Y
	G_4	C_3	Y	Y	N
	G_{25}	A_0	Y	Y	Y
G_{26}	A_0	G_{26}	Y	Y	Y
	A_1	G_5	Y	Y	Y
	C_3	$G_{6,2,2}$	Y	Y	N
	A_1C_3	C_6	Y	Y	Y
	G_4	C_6	Y	Y	Y
	$G_{3,1,2}$	C_6	Y	Y	Y
	G_{26}	A_0	Y	Y	Y
G_{27}	A_0	G_{27}	Y	Y	Y
	A_1	B_2	N	N	N
	A_2'	C_6	Y	Y	Y
	A_2''	C_6	Y	Y	Y
	B_2	C_6	Y	Y	Y
	$I_2(5)$	C_6	Y	Y	Y
	G_{27}	A_0	Y	Y	Y

TABLE 2. Exceptional groups of rank 4

G	Z_X	C_X^{ref}	$\mathcal{A}$	C_X	exp
G_{28}	A_0	F_4	Y	Y	Y
(F_4)	A_1	B_3	N	Y	N
	$\tilde{A}_1$	B_3	N	Y	N
	$A_1\tilde{A}_1$	A_1^2	N	Y	N
	A_2	G_2	Y	Y	Y
	$\tilde{A}_2$	G_2	Y	Y	Y
	B_2	B_2	Y	Y	N
	C_3[a]	A_1	Y	Y	Y
	B_3	A_1	Y	Y	Y
	$A_1\tilde{A}_2$	A_1	Y	Y	Y
	$\tilde{A}_1A_2$	A_1	Y	Y	Y
	F_4	A_0	Y	Y	Y
G_{29}	A_0	G_{29}	Y	Y	Y
	A_1	B_3	N	N	N
	A_1^2	$G_{4,2,2}$	N	Y	N
	A_2	A_1^2	N	N	N
	B_2	$G_{4,1,2}$	Y	Y	Y
	A_1A_2	C_4	Y	Y	Y
	A_3'	C_4	Y	Y	Y
	A_3''	C_4	Y	Y	Y
	B_3	C_4	Y	Y	Y
	$G_{4,4,3}$	C_4	Y	Y	Y
	G_{29}	A_0	Y	Y	Y
G_{30}	A_0	H_4	Y	Y	Y
(H_4)	A_1	G_{23}	N	Y	N
	A_1^2	B_2	N	Y	N
	A_2	G_2	N	Y	N
	$I_2(5)$	C_{20}	N	Y	N
	A_1A_2	A_1	Y	Y	Y
	$A_1I_2(5)$	A_1	Y	Y	Y
	A_3	A_1	Y	Y	Y
	H_3	A_1	Y	Y	Y
	H_4	A_0	Y	Y	Y
G_{31}	A_0	G_{31}	Y	Y	Y
	A_1	$G_{4,1,3}$	N	Y	N
	A_1^2	$G_{4,1,2}$	N	Y	N
	A_2	G_2	N	N	N
	$G_{4,2,2}$	G_8	Y	Y	Y
	A_1A_2	C_4	Y	Y	N
	A_3	C_4	Y	Y	N
	$G_{4,2,3}$	C_4	Y	Y	N
	G_{31}	A_0	Y	Y	Y
G_{32}	A_0	G_{32}	Y	Y	Y
	C_3	G_{26}	Y	Y	N
	G_4	G_5	Y	Y	N
	C_3^2	$G_{6,1,2}$	Y	Y	N
	C_3G_4	C_6	Y	Y	N
	G_{25}	C_6	Y	Y	N
	G_{32}	A_0	Y	Y	Y

[a] A Coxeter group of type C_3, not a cyclic group.

TABLE 3. The exceptional group G_{33}

Z_X	C_X^{ref}	$\mathcal{A}$	C_X	exp
A_0	G_{33}	Y	Y	Y
A_1	D_4	N	N	N
A_1^2	B_3	N	Y	N
A_2	$G_{3,1,2}$	N	N	N
A_1^3	$G_{6,3,2}$	Y	Y	Y
A_1A_2	C_3	N	N	N
A_3	A_1^2	N	Y	N
$G_{3,3,3}$	G_4	Y	Y	Y
A_1A_3	A_1	Y	Y	N
A_4	A_1	Y	Y	N
D_4	C_6	Y	Y	Y
$G_{3,3,4}$	A_1	Y	Y	N
G_{33}	A_0	Y	Y	Y

TABLE 4. The exceptional group G_{34}

Z_X	C_X^{ref}	$\mathcal{A}$	C_X	exp
A_0	G_{34}	Y	Y	Y
A_1	G_{33}	N	N	N
A_1^2	F_4	N	N	N
A_2	$G_{3,1,4}$	N	N	N
A_1^3	$G_{6,3,3}$	N	Y	N
A_1A_2	$C_3G_{3,1,2}$	N	N	N
A_3	B_3	N	N	N
$G_{3,3,3}$	G_{26}	Y	Y	Y
$A_1^2A_2$	$G_{6,2,2}$	N	Y	N
A_2^2	$G_{6,2,2}$	N	Y	N
A_1A_3	A_1^2	N	N	N
A_4	A_1^2	N	N	N
D_4	$G_{6,1,2}$	Y	Y	Y
$A_1G_{3,3,3}$	G_5	Y	Y	Y
$G_{3,3,4}$	$G_{6,2,2}$	N	Y	Y
A_2A_3	C_6	Y	Y	Y
A_1A_4	C_6	Y	Y	Y
A_5'	C_6	Y	Y	Y
A_5''	C_6	Y	Y	Y
D_5	C_6	Y	Y	Y
$A_1G_{3,3,4}$	C_6	Y	Y	Y
$G_{3,3,5}$	C_6	Y	Y	Y
G_{33}	C_6	Y	Y	Y
G_{34}	A_0	Y	Y	Y

TABLE 5. The exceptional group G_{35} (E_6)

Z_X	C_X^{ref}	$\mathcal{A}$	C_X	exp
A_0	E_6	Y	Y	Y
A_1	A_5	N	Y	N
A_1^2	B_3	N	Y	N
A_2	A_2^2	N	N	N
A_1^3	A_1A_2	N	Y	N
A_1A_2	A_2	N	Y	N
A_3	B_2	N	Y	N
$A_1^2A_2$	A_1	N	Y	Y
A_2^2	G_2	Y	Y	Y
A_1A_3	A_1	N	Y	Y
A_4	A_1	N	Y	Y
D_4	A_2	Y	Y	N
$A_1A_2^2$	A_1	Y	Y	Y
A_1A_4	A_0	N	Y	N
A_5	A_1	Y	Y	Y
D_5	A_0	N	Y	N
E_6	A_0	Y	Y	Y

TABLE 6. The exceptional group G_{36} (E_7)

Z_X	C_X^{ref}	$\mathcal{A}$	C_X	exp
A_0	E_7	Y	Y	Y
A_1	D_6	N	Y	N
A_1^2	A_1B_4	N	Y	N
A_2	A_5	N	N	N
$(A_1^3)'$	F_4	Y	Y	Y
$(A_1^3)''$	A_1B_3	N	Y	N
A_1A_2	A_3	N	Y	N
A_3	A_1B_3	N	Y	N
A_1^4	B_3	N	Y	N
$A_1^2A_2$	A_1^3	N	Y	N
A_2^2	A_1G_2	N	Y	N
$(A_1A_3)'$	B_3	N	Y	N
$(A_1A_3)''$	A_1^3	N	Y	N
A_4	A_2	N	N	N
D_4	B_3	Y	Y	N
$A_1^3A_2$	G_2	Y	Y	Y
$A_1A_2^2$	A_1^2	N	Y	N
$A_1^2A_3$	A_1^2	N	Y	N
A_2A_3	A_1^2	N	Y	N
A_1A_4	A_0	N	N	N
A_5'	G_2	Y	Y	Y
A_5''	A_1^2	N	Y	N
A_1D_4	B_2	Y	Y	N
D_5	A_1^2	N	Y	N
$A_1A_2A_3$	A_1	Y	Y	Y
A_2A_4	A_1	Y	Y	Y
A_1A_5	A_1	Y	Y	Y
A_6	A_1	Y	Y	Y
A_1D_5	A_1	Y	Y	Y
D_6	A_1	Y	Y	Y
E_6	A_1	Y	Y	Y
E_7	A_0	Y	Y	Y

TABLE 7. The exceptional group G_{37} (E_8)

Z_X	C_X^{ref}	$\mathcal{A}$	C_X	exp
A_0	E_8	Y	Y	Y
A_1	E_7	N	Y	N
A_1^2	B_6	N	Y	N
A_2	E_6	N	N	N
A_1^3	A_1F_4	N	Y	N
A_1A_2	A_5	N	N	N
A_3	B_5	N	N	N
A_1^4	B_4	N	Y	N
$A_1^2A_2$	A_1B_3	N	Y	N
A_2^2	G_2^2	N	N	N
A_1A_3	A_1B_3	N	Y	N
A_4	A_4	N	N	N
D_4	F_4	Y	Y	N
$A_1^3A_2$	A_1G_2	N	Y	N
$A_1A_2^2$	A_1G_2	N	Y	N
$A_1^2A_3$	A_1B_2	N	Y	N
A_2A_3	A_1B_2	N	Y	N
A_1A_4	A_2	N	N	N
A_5	A_1G_2	N	Y	N
A_1D_4	B_3	N	Y	N
D_5	B_3	N	Y	N
$A_1^2A_2^2$	B_2	N	Y	N
$A_1A_2A_3$	A_1^2	N	Y	N
$A_1^2A_4$	A_1^2	N	Y	N
A_3^2	B_2	N	Y	N
A_2A_4	A_1^2	Y	Y	N
A_1A_5	A_1^2	N	Y	N
A_6	A_1^2	N	Y	N
A_2D_4	G_2	Y	Y	N
A_1D_5	A_1^2	N	Y	N
D_6	B_2	N	Y	N
E_6	G_2	Y	Y	N
$A_1A_2A_4$	A_1	Y	Y	Y
A_3A_4	A_1	Y	Y	Y
A_1A_6	A_1	Y	Y	Y
A_7	A_1	Y	Y	Y
A_2D_5	A_1	Y	Y	Y
D_7	A_1	Y	Y	Y
A_1E_6	A_1	Y	Y	Y
E_7	A_1	Y	Y	Y
E_8	A_0	Y	Y	Y

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