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MALLIAVIN CALCULUS FOR NON-GAUSSIAN DIFFERENTIABLE MEASURES AND SURFACE MEASURES IN HILBERT SPACES

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ABSTRACT. We construct surface measures in a Hilbert space endowed with a probability measure ν. The theory fits for invariant measures of some stochastic partial differential equations such as Burgers and reaction–diffusion equations. Other examples are weighted Gaussian measures and special product measures ν of non-Gaussian measures. In any case we prove integration by parts formulae on sublevel sets of good functions (including spheres and hyperplanes) that involve surface integrals.

CONTENTS

1. INTRODUCTION

Let X be a separable infinite dimensional Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, endowed with a nondegenerate Borel probability measure ν.

In this paper we define Sobolev spaces with respect to ν, we construct surface measures naturally associated to ν, and we describe their main properties. In particular, we aim at integration by parts formulae for Sobolev functions that involve traces of Sobolev functions on regular surfaces and at an infinite dimensional (non-Gaussian) version of the Divergence Theorem.

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The surfaces considered here are level surfaces of a Borel function g satisfying some regularity and nondegeneracy assumptions which guarantee that such level surfaces are smooth enough.

In the case of Gaussian measures this problem has been extensively studied by different approaches. We quote here [Sk74, Ug79, AiMa88, FePr92, Ma97, Bo98, Ug00, Hi10, CCKO10, AMMP10, DaLuTu14]; for an extensive bibliography see the review paper [Bo17].

The approach initiated by Airault and Malliavin in [AiMa88] for the Wiener measure in the space $X = \{f \in C([0,1];\mathbb{R}) : \ f(0) = 0\}$ is naturally extendible to many other settings. It consists of the study of the function

$$F_\varphi(r) = \int_{\{x:\, g(x)\le r\}} \varphi(x)\nu(dx), \quad r \in \mathbb{R},$$

which is well defined for every $\varphi \in L^1(X,\nu)$. If F_φ is differentiable at r, its derivative $F'_\varphi(r)$ is the candidate to be a surface integral,

$$F'_\varphi(r) = \int_X \varphi\, d\sigma^g_r. \tag{1.1}$$

It turns out that F_φ is differentiable for good enough functions φ, and the second step of the construction is to show that there exists a measure σ^g_r such that (1.1) holds. Then, one needs to show that for every $r \in \mathbb{R}$, σ^g_r is supported in $g^{-1}(r)$ for a suitable version of g, and to clarify the dependence on g. The equality (1.1) is also a useful tool to prove an infinite dimensional version of the Divergence Theorem (or of integration by parts formulae). This approach was followed, e.g., in [Bo98, DaLuTu14] for Gaussian measures in Banach spaces and in [BoMa16] for general differentiable measures. Notice that if $\varphi \equiv 1$ and $g(x)$ is the distance of x from a given ν-negligible hypersurface Σ, $F'_1(0)$ is just the Minkowski content of Σ.

A completely different approach is the one by Feyel and de La Pradelle, who constructed an infinite dimensional Hausdorff–Gauss surface measure by approximation with finite dimensional Hausdorff–Gauss surface measures [FePr92]. It uses in a very important way the structure of Gaussian measures and it seems to be hardly extendible to non-Gaussian settings, especially in the case of nonproduct measures.

A third approach comes from the general geometric measure theory and relies on the theory of the BV functions (functions with bounded variation). BV functions for Gaussian measures in Banach spaces were studied, e.g., in [Fu00, FuHi01, AMMP10]. By definition, a Borel set B has finite perimeter if its characteristic function is BV; in this case the perimeter measure is defined and its support is contained in the boundary of B. For good enough sets B, the perimeter measure coincides with the restriction to the boundary of B of the surface measure of Feyel and de La Pradelle; for a proof see [CeLu14].

In our general framework we shall follow the first approach, and we are particularly interested in the case where ν is the invariant measure of some nonlinear stochastic PDE. In the case of linear equations, ν is a Gaussian measure and we refer to our paper [DaLuTu14]. In fact, if ν is a nondegenerate Gaussian measure, our construction of surface measures coincides with the one of [DaLuTu14]; see subsection 6.1.

Let us describe our procedure. As usual, we denote by $C^1_b(X)$ the space of the bounded and continuously Fréchet differentiable functions $f : X \mapsto \mathbb{R}$ having gradient with bounded norm, by $\nabla f(x)$ the gradient of f at x, and by $\partial_z f(x) = \langle \nabla f(x), z\rangle$ the derivative of f at x along any $z \in X$.

Our starting assumption is the following.

Hypothesis 1.1. *There exists a linear bounded operator $R \in \mathcal{L}(X)$ such that $R\nabla$: $\mathrm{dom}\,(R\nabla) = C^1_b(X) \mapsto L^p(X,\nu;X)$ is closable in $L^p(X,\nu)$, for any $p \in (1,+\infty)$.*

Then we denote by $W^{1,p}(\nu)$ the domain of the closure M_p of $R\nabla$ in $L^p(X,\nu)$. $W^{1,p}(\nu)$ is a Banach space with the graph norm

$$\|f\|_{W^{1,p}(\nu)} = \left(\int_X |f(x)|^p \nu(dx)\right)^{1/p} + \left(\int_X \|M_p f(x)\|^p \nu(dx)\right)^{1/p}. \tag{1.2}$$

So, by definition an element $f \in L^p(X,\nu)$ belongs to $W^{1,p}(\nu)$ iff there exists a sequence of C^1_b functions (f_n) such that $\lim_{n\to\infty} f_n = f$ in $L^p(X,\nu)$ and the sequence $(R\nabla f_n)$ converges in $L^p(X,\nu;X)$; the limit of the latter is just $M_p f$.

Different choices of R give rise to different Sobolev spaces. For instance, if ν is the Gaussian measure $N_{0,Q}$ with mean 0 and covariance Q, Hypothesis 1.1 is satisfied by $R = Q^\alpha$, for every $\alpha \geq 0$. Taking $\alpha = 0$ and $R = I$ we obtain the Sobolev spaces studied in [DPZ02]. Taking $\alpha = 1/2$ we obtain $W^{1,p}(\nu) = \mathbb{D}^{1,p}(X,\nu)$, the usual Sobolev spaces of Malliavin calculus ([Bo98, Nu95]).

For general results ensuring that Hypothesis 1.1 holds we quote [AlRo90]. An easy sufficient condition for $R\nabla$ to be closable in $L^p(X,\nu)$ is the following one.

Hypothesis 1.2. *For any $p > 1$ and $z \in X$ there exists $C_{p,z} > 0$ such that*

$$\left|\int_X \langle R\nabla\varphi, z\rangle\, d\nu\right| \leq C_{p,z}\, \|\varphi\|_{L^p(X,\nu)}, \quad \varphi \in C^1_b(X). \tag{1.3}$$

After the canonical identifications of the dual spaces $(L^p(X,\nu))'$, $(L^p(X,\nu;X))'$ with $L^{p'}(X,\nu)$, $L^{p'}(X,\nu;X)$ respectively, with $p' = p/(p-1)$ (see, e.g., [DU77]), we denote by $M_p^* : D(M_p^*) \subset L^{p'}(X,\nu;X) \to L^{p'}(X,\nu)$ the adjoint of M_p. So, we have

$$\int_X \langle M_p\varphi, F\rangle\, d\nu = \int_X \varphi\, M_p^*(F)\, d\nu, \quad \varphi \in D(M_p),\ F \in D(M_p^*). \tag{1.4}$$

In the case that ν is the Gaussian measure $N_{0,Q}$, taking $R = Q^{1/2}$, M_2 can be seen as a Malliavin derivative and $-M_2^*$ is the Gaussian divergence or Skorohod integral. See, e.g., [Bo98, Nu95, Sa05]. In any case, the operator $-M_p^*$ plays the important role of (generalized) divergence.

Hypothesis 1.2 is equivalent to the assumption that for every $z \in X$ the constant vector field $F_z(x) := z$ belongs to $D(M_p^*)$ for every $p > 1$. Indeed, fix any $p > 1$, $F_z \in D(M_p^*)$ iff the function $W^{1,p}(\nu) \mapsto \mathbb{R}$, $\varphi \to \int_X \langle M_p\varphi, z\rangle d\nu$ has a linear continuous extension to the whole $L^p(X,\nu)$. Since $C^1_b(X)$ is dense in $W^{1,p}(\nu)$, this is equivalent to the existence of $C_{p,z}$ such that (1.3) holds, and in this case (1.4), with $F = F_z$ and $M_p^*(F_z) =: v_z$, reads as

$$\int_X \langle M_p\varphi, z\rangle\, d\nu = \int_X \varphi\, v_z\, d\nu, \quad \varphi \in D(M_p). \tag{1.5}$$

This is a natural generalization of the integration formula that holds for the Gaussian measure $N_{0,Q}$, in which case taking $R = Q^{1/2}$, (1.3) holds for every $z \in X$. Moreover v_z is an element of $L^q(X,\nu)$ for every $q \in (1,+\infty)$; it coincides with $\langle Q^{-1}z, \cdot\rangle$ if in addition $z \in Q(X)$.

We recall that a probability Borel measure μ on a Hilbert space is called Fomin differentiable along an element $y \in X$ if for every cylindrical smooth function (namely, every $\varphi : X \mapsto \mathbb{R}$ of the type $\varphi(x) = f(\langle x, x_1\rangle, \ldots, \langle x, x_n\rangle)$ with $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $f \in C_b^\infty(\mathbb{R}^n)$) there exists $\beta_y \in L^1(X, \mu)$ such that

$$\int_X \frac{\partial \varphi}{\partial y}\, d\mu = \int_X \varphi \beta_y \, d\mu.$$

Since $\langle M_p\varphi(x), z\rangle = \langle \nabla\varphi(x), R^* z\rangle$ for every smooth cylindrical φ and for every x, $z \in X$, formula (1.5) implies that ν is Fomin differentiable along any $y \in R^*(X)$. We refer to [Bo10] for a general treatment of differentiable measures.

Under Hypothesis 1.2, formula (1.1) is a useful tool to prove an integration formula,

$$\int_{\{g<r\}} \langle M_p\varphi, z\rangle \, d\nu = \int_{\{g<r\}} v_z \varphi \, d\nu + \int_{\{g=r\}} \varphi \langle \frac{M_p g}{\|M_p g\|}, z\rangle \, d\rho_r, \tag{1.6}$$

for all $z \in X$, $\rho_r = \|M_p g\| \sigma_r^g$, and for good enough φ and g. The normalized measure ρ_r is particularly meaningful, since it is independent of the choice of g within a large class of functions, being a sort of perimeter measure relevant to the set $\Omega := g^{-1}(-\infty, r)$ (see Section 5).

We already mentioned that we need some regularity/nondegeneracy conditions on g. Specifically, our assumption on g is

Hypothesis 1.3. *$g \in W^{1,p}(\nu)$ and $M_p g \, \|M_p g\|^{-2}$ belongs to the domain of the adjoint M_p^*, for every $p > 1$.*

So, regularity is meant as Sobolev regularity. The nondegeneracy condition is hidden in the condition that $M_p g \, \|M_p g\|^{-2}$ belongs to $D(M_p^*)$ for every $p > 1$. Indeed, if a vector field F belongs to $D(M_p^*)$, then $\|F\| \in L^{p'}(X, \nu)$. If g satisfies Hypothesis 1.3, taking $F = M_p g \, \|M_p g\|^{-2}$, we obtain that $1/\|M_p g\| \in L^{p'}(X, \nu)$, for every $p' > 1$. This condition is a generalization of the nondegeneracy condition of [AiMa88]. We recall that if g is smooth, its level surfaces are smooth near every point x such that $\nabla g(x) \neq 0$. Here what replaces the gradient of g is $M_p g$. $M_p g$ is allowed to vanish at some points, but not too much; otherwise $1/\|M_p g\|$ cannot belong to all $L^{p'}(X, \nu)$ spaces.

Let us describe the content of the paper.

In Section 2 we define Sobolev spaces and we prove their basic properties and their properties that are useful for the construction of surface measures.

In Section 3 we construct surface measures under Hypotheses 1.1 and 1.3.

In Section 4 we introduce and discuss the p-capacities that are used to obtain further properties of the surface measures. In particular, we show that Borel sets with null p-capacity for some $p > 1$ are negligible with respect to our surface measures.

Section 5 deals with a comparison with a geometric measure theory approach and to the proof of a variational result. Indeed, we show that for every $\varphi \in C_b^1(X)$ with nonnegative values, the integral of φ with respect to ρ_r is equal to the maximum of

$$\int_\Omega M_p^*(F\varphi)\, d\nu,$$

where $\Omega = g^{-1}(-\infty, r)$ and F runs among suitably smooth X-valued vector fields such that $\|F(x)\| = 1$ for ν-a.e $x \in X$.

Sections 6, 7, and 8 are devoted to examples. In all of them we show that Hypothesis 1.2 holds, and therefore Hypothesis 1.1 holds. Moreover, in all of them we prove that the functions $g(x) = \|x\|^2$ and $g(x) = \langle b, x\rangle$, with any $b \in X \setminus \{0\}$, satisfy Hypothesis 1.3.

In Section 6 we consider a nondegenerate centered Gaussian measure μ and a weighted Gaussian measure, $\nu(dx) = w(x)\mu(dx)$. In the case of a Gaussian measure, we make a detailed comparison with notation and results of [DaLuTu14]. In the case of a weighted Gaussian measure, under suitable conditions on the weight w and on g we show that for every $r \in (\operatorname{ess\,inf} g, \operatorname{ess\,sup} g)$, ρ_r coincides with the restriction of the weighted measure $w(x)\rho(dx)$ to the surface $g^{-1}(r)$, where ρ is the above-mentioned Gauss–Hausdorff measure of Feyel and de La Pradelle. Here we consider precise versions of w and g that are elements of Sobolev spaces without a continuous version in general. The results of Section 6 rely on [Fe16], where weighted Gaussian measures in Banach spaces are studied.

In Section 7 we introduce an infinite product of non-Gaussian measures on $\mathbb{R}$, which is one of the simplest generalizations of a Gaussian measure in a separable Hilbert space. In this toy example we have explicit formulae for all the objects involved: ν, v_z.

In Section 8 we consider the invariant measures of two particular stochastic PDEs. The first one is a reaction-diffusion equation with a polynomial nonlinearity, and the second one is a Burgers equation. In both cases a unique invariant measure ν exists, but it is not explicit in general. It is not a product measure or a Gaussian measure with known weight (except in the case of reaction-diffusion equations, for a particular value of a parameter). However, Hypothesis 1.2 is satisfied for every $p > 1$ thanks to recent results ([DaDe16, DaDe17]) that allow our machinery to work, taking as R a suitable power of the negative Dirichlet Laplacian.

The verification of Hypothesis 1.3 may be nontrivial, since ν is not explicit. (In fact, it may be nontrivial even for Gaussian measures if g is particularly nasty). It is reduced to showing that $1/\|M_p g\|$ belongs to $L^p(X, \nu)$ for every p, and this is difficult to check, except for hyperplanes, in which case $g(x) = \langle b, x\rangle$ for some $b \in X \setminus \{0\}$ and $M_p g$ is constant. We show that it holds also in the case of spherical surfaces when $g(x) = \|x\|^2$. In this case, the problem is reduced to showing that $x \mapsto \|R\nabla g(x)\|^{-1} = \|2Rx\|^{-1}$ belongs to $L^p(X, \nu)$ for every $p > 1$. To show it we need some technical tools; namely, we approximate $\|R\nabla g\|^{-1}$ by a sequence of cylindrical functions φ_n belonging to the domain of the infinitesimal generator L of the transition semigroup in $L^2(X, \nu)$. For functions $\varphi \in D(L)$ we know that $\int_X L\varphi \, d\nu = 0$, and we use this equality to estimate the L^p norm of φ_n by a constant independent of n.

Section 9 contains some comments and bibliographical remarks.

2. Notation and preliminaries, Sobolev spaces

As mentioned in the introduction, we consider a separable Hilbert space X with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, endowed with a Borel nondegenerate probability measure ν.

We recall that for Fréchet differentiable functions $\varphi : X \mapsto \mathbb{R}$ we denote by $\nabla\varphi(x)$ the gradient of φ at x and by $\partial_z\varphi(x) = \langle\nabla\varphi(x), z\rangle$ its derivative along z, for every $z \in X$.

By $C_b(X)$ (resp. $UC_b(X)$) we mean the space of all real continuous (resp. uniformly continuous) and bounded mappings $\varphi : X \to \mathbb{R}$, endowed with the sup norm $\|\cdot\|_\infty$. Moreover, $C_b^1(X)$ is the subspace of $C_b(X)$ of all continuously Fréchet differentiable functions, with bounded (resp. uniformly continuous and bounded) gradient.

For $p > 1$ we set as usual $p' = p/(p-1)$.

Throughout the paper we assume that Hypothesis 1.1 holds. The spaces $W^{1,p}(\nu)$ and the operators M_p are defined in the introduction. Here we collect some of their basic properties.

Lemma 2.1. *Let* $1 < p < \infty$.

(i) *If* $\varphi \in W^{1,p}(\nu)$, $\psi \in C_b^1(X)$, *then the product* $\varphi\psi$ *belongs to* $W^{1,p}(\nu)$ *and* $M_p(\varphi\psi) = \psi M_p\varphi + \varphi M_p\psi$. *More generally, if* $\varphi \in W^{1,p_1}(\nu)$, $\psi \in W^{1,p_2}(\nu)$, *and* $1/p_1 + 1/p_2 < 1$, *then the product* $\varphi\psi$ *belongs to* $W^{1,p}(\nu)$ *and* $M_p(\varphi\psi) = \psi M_{p_1}\varphi + \varphi M_{p_2}\psi$, *with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

(ii) *Let* $h \in C_b^1(\mathbb{R})$ *and* $\varphi \in W^{1,p}(\nu)$. *Then* $h \circ \varphi \in W^{1,p}(\nu)$ *and we have*

$$M_p(h \circ \varphi) = h'(\varphi) M_p\varphi. \tag{2.1}$$

(iii) *If* $\varphi \in W^{1,p}(\nu)$, $\varphi(x) \geq 0$ *for* ν*-a.e.* $x \in X$, *then* $x \mapsto (\varphi(x))^s \in W^{1,p/s}(\nu)$, *for every* $s \in (1, p)$, *and*

$$\|\varphi^s\|_{W^{1,p/s}(\nu)} \leq \|\varphi\|^s_{L^p(X,\nu)} + s\|M_p\varphi\|_{L^p(X,\nu;X)} \|\varphi\|^{s-1}_{L^p(X,\nu)}. \tag{2.2}$$

(iv) *For* $1 < p < \infty$, $W^{1,p}(\nu)$ *is reflexive.*

(v) *If* $p \in (1, +\infty)$ *and* $f_n \in W^{1,p}(\nu)$, $n \in \mathbb{N}$, *are such that* $f_n \to f$ *in* $L^p(X,\nu)$ *and* $M_p f_n$ *is bounded in* $L^p(X,\nu;X)$, *then* $f \in W^{1,p}(\nu)$.

(vi) $W^{1,p}(\nu) \subset W^{1,q}(\nu)$ *and* $M_p\varphi = M_q\varphi$ *for every* $\varphi \in W^{1,p}(\nu)$, *for* $1 < q < p$.

Proof. The proof of statement (i) follows by approaching $\varphi\psi$ by $\varphi_n\psi_n$, for any couple of sequences (φ_n), $(\psi_n) \subset C_b^1(X)$ that approach φ, ψ in $W^{1,p_1}(\nu)$, $W^{1,p_2}(\nu)$, respectively. Of course if $\psi \in C_b^1(X)$ we take $\psi_n = \psi$ for every n.

Concerning statement (ii) we have just to approach $h \circ \varphi$ by $h \circ \varphi_n$, for any sequence $(\varphi_n) \subset C_b^1(X)$ that approaches φ in $W^{1,p}(\nu)$.

Let us prove (iii). For every sequence $(\varphi_n) \subset C_b^1(X)$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ in $W^{1,p}(\nu)$, the sequence $\psi_n(x) := \sqrt{\varphi_n(x)^2 + 1/n}$ has a subsequence (ψ_{n_k}) such that $\psi_{n_k}(x) \to \varphi(x)$, $R\nabla\psi_{n_k}(x) \to M_p\varphi(x)$ for ν-a.e. $x \in X$, and it is easily seen that $\psi^s_{n_k} \to \varphi^s$ in $L^{p/s}(X,\nu)$ and $M_p\psi^s_{n_k} = s\psi^{s-1}_{n_k} M_p\psi_{n_k} = s(\varphi^2_{n_k} + 1/n)^{s-3/2}\varphi_{n_k} M_p\varphi_{n_k}$ converges to $s\varphi^{s-1}M_p\varphi$ in $L^{p/s}(X,\nu;X)$. Therefore, $\varphi^s \in W^{1,p/s}(\nu)$, and the Hölder inequality yields estimate (2.2).

Let us prove statement (iv). The mapping $u \mapsto Tu := (u, M_p u)$ is an isometry from $W^{1,p}(\nu)$ to the product space $E := L^p(X,\nu) \times L^p(X,\nu;X)$, which implies that the range of T is closed in E. Now, $L^p(X,\nu)$ and $L^p(X,\nu;X)$ are reflexive (for the latter statement (see, e.g., [DU77, Ch. IV])) so that E is reflexive, and $T(W^{1,p}(\nu))$ is reflexive too. Being isometric to a reflexive space, $W^{1,p}(\nu)$ is reflexive.

Statement (v) is a consequence of (iv). Since (f_n) is bounded in $W^{1,p}(\nu)$, which is reflexive, there exists a subsequence that weakly converges to an element of $W^{1,p}(\nu)$. Since $f_n \to f$ in $L^p(X,\nu)$, the weak limit is f. Therefore, $f \in W^{1,p}(\nu)$.

Statement (vi) is an immediate consequence of the definition. □

We shall use the following extension of Lemma 2.1(ii) to compositions with piecewise linear functions.

Lemma 2.2. *Let $\alpha < \beta \in \mathbb{R}$, and set*

$$h(r) = \int_{-\infty}^{r} \mathbb{1}_{[\alpha,\beta]}(s)ds = \begin{cases} 0 & \text{if } r \le \alpha, \\ r-\alpha & \text{if } \alpha \le r \le \beta, \\ \beta-\alpha & \text{if } r \ge \beta. \end{cases} \tag{2.3}$$

Then $h \circ \varphi \in W^{1,p}(\nu)$ for every $\varphi \in W^{1,p}(\nu)$, and we have

$$M_p(h\circ\varphi) = \mathbb{1}_{[\alpha,\beta]}(\varphi) M_p\varphi. \tag{2.4}$$

Proof. We approach h by a sequence of C_b^1 functions, choosing a sequence of smooth compactly supported functions $\theta_n : \mathbb{R} \mapsto \mathbb{R}$ such that $\theta_n(\xi) \to \mathbb{1}_{[\alpha,\beta]}(\xi)$ for every $\xi \in \mathbb{R}$, $0 \le \theta_n(\xi) \le 1$ for every $\xi \in \mathbb{R}$, and setting

$$h_n(r) = \int_{-\infty}^{r} \theta_n(s)ds, \quad r \in \mathbb{R}.$$

Since $h_n \in C_b^1(\mathbb{R})$, by Lemma 2.1(ii), $h_n \circ \varphi \in W^{1,p}(\nu)$ and

$$M_p(h_n \circ \varphi) = (h_n' \circ \varphi) M_p\varphi.$$

By the Dominated Convergence Theorem, $h_n \circ \varphi$ converges to $h \circ \varphi$ in $L^p(X,\nu)$. Moreover, $M_p(h_n \circ \varphi)$ converges pointwise to $\mathbb{1}_{[\alpha,\beta]}(\varphi)M_p\varphi$. Since $\|M_p(h_n\circ\varphi)(x)\| \le \|\theta_n\|_\infty \|M_p\varphi(x)\| \le \|M_p\varphi(x)\|$, still by the Dominated Convergence Theorem, $M_p(h_n \circ \varphi)$ converges to $\mathbb{1}_{[\alpha,\beta]}(\varphi)M_p\varphi$ in $L^p(X,\nu;X)$, and the statement follows. □

Corollary 2.3. *For every $\varphi \in W^{1,p}(\nu)$, the positive part φ_+ of φ, the negative part φ_- of φ, and $|\varphi|$ belong to $W^{1,p}(\nu)$, and we have*

$$M_p(\varphi_+) = \mathbb{1}_{\varphi^{-1}(0,+\infty)} M_p\varphi, \quad M_p(\varphi_-) = -\mathbb{1}_{\varphi^{-1}(-\infty,0)} M_p\varphi, \\ M_p(|\varphi|) = \text{sign}\,\varphi\, M_p\varphi. \tag{2.5}$$

Moreover, $M_p\varphi$ vanishes ν-a.e. in the level set $\varphi^{-1}(c)$, for each $c \in \mathbb{R}$.

Proof. The proof of the first statement is just a minor modification of the proof of Lemma 2.2; it is sufficient to take $\alpha = 0$, $\beta = +\infty$, and approaching functions θ_n of $\mathbb{1}_{[0,+\infty)}$ that vanish on some left half-line. The other statements are consequences of the first one. □

We remark that taking $\alpha = 0$, $\beta = 1$, and $p = 2$ in Lemma 2.2, we obtain that for every $\varphi \in W^{1,2}(\nu)$, the function $\varphi_+ \wedge 1$ belongs to $W^{1,2}(\nu)$, and $\|\varphi_+ \wedge 1\|_{W^{1,2}(\nu)} \leq \|\varphi\|_{W^{1,2}(\nu)}$. Namely, the quadratic form

$$\mathcal{E}(\varphi, \psi) := \int_X (\varphi\psi + \langle M_2\varphi, M_2\psi\rangle) d\nu, \quad \varphi, \psi \in W^{1,2}(\nu),$$

is a Dirichlet form.

In the next lemma we exhibit a class of regular functions that belong to the Sobolev spaces.

Lemma 2.4. *Let $\varphi \in C^1(X)$ be such that $\|\nabla\varphi\|$ is bounded in $\varphi^{-1}(-r, r)$ for every $r > 0$, and*

$$\int_X (|\varphi|^p + \|R\nabla\varphi\|^p) d\nu < \infty.$$

Then $\varphi \in W^{1,p}(\nu)$ for every $p \in (1, +\infty)$, and $M_p\varphi = R\nabla\varphi$.

Proof. We approach φ by regularized truncations, introducing $\theta \in C_b^1(\mathbb{R})$ such that $\theta(\xi) = \xi$ for $|\xi| \leq 1$ and $\theta =$ constant for $\xi \geq 2$ and for $\xi \leq -2$. The functions $\varphi_n(x) := n\theta(\varphi(x)/n)$ belong to $C_b^1(X)$; they approach φ pointwise and in $L^p(X, \nu)$ by the Dominated Convergence Theorem. Moreover, $R\nabla\varphi_n(x) = \theta'(\varphi(x)/n)R\nabla\varphi(x)$, which coincides with $R\nabla\varphi(x)$ if $|\varphi(x)| \leq n$ and vanishes if $|\varphi(x)| \geq 2n$. Still by the Dominated Convergence Theorem, $R\nabla\varphi_n$ converges to $R\nabla\varphi$ in $L^p(X, \nu; X)$.

Notice that the assumption that $\|\nabla\varphi\|$ is bounded in $\varphi^{-1}(-r, r)$ for every $r > 0$ guarantees that $\|\nabla\varphi_n\|$ is bounded in X, so that $\varphi_n \in C_b^1(X)$, for every $n \in \mathbb{N}$. $\square$

Some properties of the operators M_p^* are in the next lemma.

Lemma 2.5. *Let $1 < p < \infty$.*

(i) *For any $F \in D(M_p^*)$ and any $\varphi \in C_b^1(X)$, the product φF belongs to $D(M_p^*)$ and*

$$M_p^*(\varphi F) = \varphi M_p^*(F) - \langle M_p\varphi, F\rangle. \tag{2.6}$$

More generally, for any $F \in D(M_p^*)$ and any $\varphi \in W^{1,q}(\nu)$ with $q > p$, the product φF belongs to $D(M_s^*)$ with $s = pq/(q-p)$ and (2.6) holds with s replacing p.

(ii) *For any $F \in D(M_p^*)$,*

$$\int_X M_p^* F \, d\nu = 0. \tag{2.7}$$

(iii) *$D(M_p^*) \supset D(M_q^*)$ and $M_p^* F = M_q^* F$ for every $F \in D(M_q^*)$, for $1 < q < p$.*

Proof. Let $\psi \in C_b^1(X)$. From the identity $M_p(\varphi\psi) = \varphi M_p\psi + \psi M_p\varphi$ we obtain

$$\int_X \langle M_p\psi, \varphi F\rangle d\nu = \int_X \langle M_p(\varphi\psi) - \psi M_p\varphi, F\rangle d\nu = \int_X \psi(\varphi M_p^* F - \langle M_p\varphi, F\rangle) d\nu, \tag{2.8}$$

and the first part of statement (i) follows from the definition of M_p^*. The argument is similar if $\varphi \in W^{1,q}(\nu)$; in this case $\varphi\psi \in W^{1,q}(\nu) \subset W^{1,p}(\nu)$ since $p < q$, and we have $M_p(\varphi\psi) = \varphi M_p\psi + \psi M_q\varphi$, while $M_p\psi = M_s\psi$. Formula (2.8) reads as

$$\int_X \langle M_s\psi, \varphi F\rangle d\nu = \int_X \psi \, g \, d\nu,$$

where now $g := \varphi M_p^* F - \langle M_q\varphi, F\rangle \in L^{s'}(X,\nu)$.

Since $1 \in W^{1,p}(\nu)$ and $M_p 1 = 0$, statement (ii) follows from the definition of M_p^*.

Statement (iii) is an obvious consequence of Lemma 2.1(vi). □

We state below some consequences of Hypothesis 1.2.

Lemma 2.6. *Let Hypothesis* 1.2 *hold.*

(i) *For every $z \in X$ the function v_z in formula* (1.5) *is independent of p and belongs to $L^p(X,\nu)$ for every $p \in (1,+\infty)$.*

(ii) *For every $z \in X$ and $f \in W^{1,q}(\nu)$, the vector field*

$$F(x) := f(x)z, \quad x \in X,$$

belongs to $D(M_p^*)$ for every $p > q'$, and

$$M_p^* F(x) = -\langle M_p f(x), z\rangle + v_z(x) f(x), \quad x \in X. \tag{2.9}$$

Proof. Statement (i) is an immediate consequence of Lemma 2.5(iii) and of Hypothesis 1.2. Concerning statement (ii), for every $\varphi \in C_b^1(X)$ we have

$$\begin{aligned}\int_X \langle R\nabla\varphi, F\rangle\, d\nu &= \int_X \langle f M\varphi, z\rangle\, d\nu \\ &= \int_X (\langle M_p(f\varphi) - \varphi M_p f, z\rangle)\, d\nu = \int_X (f v_z - \langle M_p f, z\rangle)\varphi\, d\nu\end{aligned} \tag{2.10}$$

by Lemma 2.1(i) and formula (1.5). Since $v_z \in L^s(X,\nu)$ for every $s \in (1,+\infty)$, the function $(f v_z - \langle M_p f, z\rangle)$ belongs to $L^s(X,\nu)$ for every $s < q$.

Approaching every $\varphi \in W^{1,p}(\nu)$ by a sequence (φ_n) of C_b^1 functions, the left-hand side of (2.10) converges to $\int_X \langle M_p\varphi, F\rangle\, d\nu$. Since $q > p'$, there exists $s \in (1,q)$ such that $s > p'$. So, also the right-hand side converges, and we get

$$\int_X \langle M_p\varphi, F\rangle\, d\nu = \int_X (f v_z - \langle M_q f, z\rangle)\varphi\, d\nu$$

for every $\varphi \in W^{1,p}(\nu)$. (2.9) follows from the definition of M_p^*. □

3. Construction of surface measures

We recall that Hypothesis 1.1 holds throughout the paper. Moreover, from now on, $g : X \mapsto \mathbb{R}$ is a Borel function that satisfies Hypothesis 1.3.

The elements of $W^{1,p}(\nu)$ are equivalence classes of functions. If g is a given function, by $g \in W^{1,p}(\nu)$ we mean as usual that g is a fixed version of an element of $W^{1,p}(\nu)$. The results of this section are independent of the particular chosen version g. Instead, in the next section the choice of the version will be important.

We recall that $M_p\varphi = M_q\varphi$ for every $\varphi \in W^{1,p}(\nu)$ (Lemma 2.1(vi)), and $M_p^* F = M_q^* F$ for every $F \in D(M_q^*)$ (Lemma 2.5(iii)) if $p > q$. To simplify notation we shall write M instead of M_p and M^* instead of M_p^* on $\bigcap_{p>1} W^{1,p}(\nu)$ and on $\bigcap_{p>1} D(M_p^*)$, respectively. Moreover we set

$$\Psi := \frac{Mg}{\|Mg\|^2}. \tag{3.1}$$

We start our analysis by introducing the function

$$F_\varphi(r) := \int_{\{g\le r\}} \varphi(x)\nu(dx), \quad r\in\mathbb{R}, \quad \varphi\in L^1(X,\nu). \tag{3.2}$$

We recall that the image measure $(\varphi\nu)\circ g^{-1}$ is defined on the Borel sets $B\subset\mathbb{R}$ by

$$(\varphi\nu)\circ g^{-1}(B) = \int_{g^{-1}(B)} \varphi(x)\nu(dx).$$

So, $F_\varphi(r) = (\varphi\nu)\circ g^{-1}((-\infty, r])$. It is easy to see that F_φ is continuously differentiable if and only if $(\varphi\nu)\circ g^{-1}$ is absolutely continuous with respect to the Lebesgue measure λ, with continuous density q_φ. In this case we have

$$F_\varphi'(r) = q_\varphi(r), \quad r\in\mathbb{R}.$$

So, our next step is to show that $(\varphi\nu)\circ g^{-1} \ll \lambda$, for all φ belonging either to $UC_b(X)$ or to $W^{1,p}(\nu)$ for some $p>1$. Also, we shall show that the density

$$\frac{d(\varphi\nu)\circ g^{-1}}{d\lambda}(r) =: q_\varphi(r) \tag{3.3}$$

is Hölder continuous if $\varphi\in W^{1,p}(\nu)$ for some $p>1$.

It will follow easily that for any $r\in\mathbb{R}$ the mapping $\varphi\mapsto F_\varphi'(r)$ is a linear positive functional on $UC_b(X)$, and by results of general measure theory it is indeed the integral of φ with respect to a Borel measure. We shall see that such a measure is concentrated on the surface $\{g=r\}$ if g is continuous and on the surface $\{g^*=r\}$ if g is not continuous, where g^* is a suitable version of g.

The next lemma is the starting point of most sublevel sets' approach to surface measures. Its proof is an abstract version of a well-known procedure; see, e.g., [Nu95, First Edition, Prop. 2.1.1].

Lemma 3.1. *Assume that Hypotheses* 1.1 *and* 1.3 *are fulfilled. Then for any* $p>1$ *and* $\varphi\in W^{1,p}(\nu)$*, the measure* $(\varphi\nu)\circ g^{-1}$ *is absolutely continuous with respect to the Lebesgue measure* λ*. Its density*

$$\frac{d[(\varphi\nu)\circ g^{-1}]}{d\lambda}(r) =: q_\varphi(r), \quad r\in\mathbb{R},$$

is given by

$$q_\varphi(r) = \int_{\{g<r\}} \left(\langle M_p\varphi, \frac{Mg}{\|Mg\|^2}\rangle - \varphi M_p^*\left(\frac{Mg}{\|Mg\|^2}\right)\right) d\nu, \tag{3.4}$$

and it is bounded and θ*-Hölder continuous in* $\mathbb{R}$ *for every* $\theta<1-1/p$*. There exists* $K_p>0$*, independent of* φ*, such that*

$$|q_\varphi(r)| \le K_p\|\varphi\|_{W^{1,p}(\nu)}, \quad r\in\mathbb{R}. \tag{3.5}$$

Proof. Fix any interval $[\alpha,\beta]\subset\mathbb{R}$ and consider the function h defined in (2.3). By Lemma 2.2, $h\circ g\in W^{1,p}(\nu)$ for every $p>1$, and

$$M(h\circ g) = \mathbb{1}_{[\alpha,\beta]}(g)Mg.$$

Therefore,

$$\mathbb{1}_{[\alpha,\beta]}\circ g = \frac{\langle M(h\circ g), Mg\rangle}{\|Mg\|^2} = \langle M(h\circ g), \Psi\rangle,$$

where Ψ is defined in (3.1) and belongs to $D(M_p^*)$ for every $p > 1$ by Hypothesis 1.3. Let $\varphi \in C_b^1(X)$. Then $\varphi\Psi \in D(M_p^*)$ for every $p > 1$. Multiplying both sides by φ and integrating yields

$$\int_X \mathbb{1}_{[\alpha,\beta]}(g(x))\varphi(x)\nu(dx) = \int_X (h \circ g)\, M_p^*(\varphi\Psi)\, d\nu.$$

On the other hand, by Lemma 2.5(i), $M_p^*(\varphi\Psi) = M^*(\Psi)\varphi - \langle M_p\varphi, \Psi\rangle$, and therefore

$$\int_X \mathbb{1}_{[\alpha,\beta]}(g(x))\varphi(x)\nu(dx) = \int_X (h \circ g)\,(M^*(\Psi)\varphi - \langle M_p\varphi, \Psi\rangle)\, d\nu. \tag{3.6}$$

Approaching any $\varphi \in W^{1,p}(\nu)$ by a sequence of C_b^1 functions, we see that formula (3.6) holds for every $\varphi \in W^{1,p}(\nu)$. The right-hand side may be rewritten as

$$\int_X \int_{\mathbb{R}} \mathbb{1}_{(-\infty,g(x)]}(r)\,\mathbb{1}_{[\alpha,\beta]}(r)dr\,(M^*(\Psi)\varphi - \langle M_p\varphi, \Psi\rangle)\, d\nu,$$

so that by the Fubini Theorem,

$$(\varphi\nu)(\alpha \le g \le \beta) = \int_\alpha^\beta dr \int_{\{g\ge r\}} (M^*(\Psi)\varphi - \langle M_p\varphi, \Psi\rangle)\, d\nu.$$

Therefore $(\varphi\nu) \circ g^{-1}$ has density q_φ given by

$$q_\varphi(r) = \int_{\{g\ge r\}} (M^*(\Psi)\varphi - \langle M_p\varphi, \Psi\rangle)\, d\nu = \int_{\{g<r\}} (\langle M_p\varphi, \Psi\rangle - M^*(\Psi)\varphi)\, d\nu,$$

where the last equality follows from Lemma 2.5(ii). Since $\Psi \in L^q(X,\nu;X)$ and $M^*\Psi \in L^q(X,\nu)$ for every $q > 1$, the function $\langle M_p\varphi, \Psi\rangle - M^*(\Psi)\varphi$ belongs to $L^s(X,\nu)$ for every $s \in [1,p)$ and there is $C_{p,s} > 0$ such that

$$\|\langle M_p\varphi, \Psi\rangle - M^*(\Psi)\varphi\|_{L^s(X,\nu)} \le C_{p,s}\|\varphi\|_{W^{1,p}(\nu)}.$$

Taking $s = 1$, estimate (3.5) is immediate.

Let us prove that q_φ is Hölder continuous. For $r_2 > r_1$ and for every $s \in (1,p)$ we have

$$\begin{aligned}|q_\varphi(r_2) - q_\varphi(r_1)| &= \left|\int_{\{r_1<g\le r_2\}} (\langle M_p\varphi, \Psi\rangle - M^*(\Psi)\varphi)d\nu\right| \\ &\le \|\langle M_p\varphi, \Psi\rangle - M^*(\Psi)\varphi\|_{L^s(X,\nu)}\left(\int_{r_1}^{r_2} q_1(r)dr\right)^{1/s'} \\ &\le C_{p,s}\|\varphi\|_{W^{1,p}(\nu)}(\|q_1\|_\infty(r_2-r_1))^{1-1/s}.\end{aligned}$$

Therefore, q_φ is Hölder continuous with any exponent less than $1 - 1/p$. □

Taking in particular $\varphi \equiv 1$, we obtain that $\nu(g^{-1}(r_0)) = \int_{r_0}^{r_0} d\nu = 0$ for every $r_0 \in \mathbb{R}$. Therefore, all the level surfaces of g are ν-negligible. In particular,

$$F_\varphi(r) = \int_{\{g\le r\}} \varphi\, d\nu = \int_{\{g<r\}} \varphi\, d\nu, \quad \varphi \in L^1(X,\nu).$$

Moreover, by Lemma 2.5, for every $\varphi \in W^{1,p}(\nu)$ the product $\varphi\, Mg/\|Mg\|^2$ belongs to $D(M_s^*)$ for every $s > p'$, and we have

$$q_\varphi(r) = -\int_{\{g<r\}} M_s^*\left(\varphi\frac{Mg}{\|Mg\|^2}\right)d\nu. \tag{3.7}$$

Let us now consider bounded and uniformly continuous functions φ. The proof of the next proposition is a modification of the proof of [DaLuTu14, Prop. 3.4], which deals with Gaussian measures. We use the following disintegration theorem, whose proof may be found, e.g., in [DaLuTu14, Theorem A1].

Theorem 3.2. *Let $\Gamma : X \to \mathbb{R}$ be a Borel function, and set $\lambda := \nu \circ \Gamma^{-1}$. Then there exists a family of Borel probability measures $\{m_s : s \in \mathbb{R}\}$ on X such that*

$$\int_X \varphi(x)\mu(dx) = \int_{\mathbb{R}} \left(\int_X \varphi(x) m_s(dx) \right) \lambda(ds), \tag{3.8}$$

for all $\varphi : X \to \mathbb{R}$ bounded and Borel measurable.

Moreover the support of m_s is contained in $\Gamma^{-1}(s)$ for λ-almost all $s \in \mathbb{R}$.

Proposition 3.3. *For any $\varphi \in UC_b(X)$, F_φ is continuously differentiable.*

Proof. Let $\varphi : X \to \mathbb{R}$ be bounded and Borel measurable, and let $r \in \mathbb{R}$. Taking $\Gamma = g$ and applying formula (3.8) to the function $\varphi \mathbb{1}_{g^{-1}(-\infty,r)}$ we obtain

$$F_\varphi(r) = \int_{-\infty}^{r} \left(\int_X \varphi(x) m_s(dx) \right) q_1(s) ds, \tag{3.9}$$

since $\nu \circ g^{-1}(ds) = q_1(s)ds$ (see Lemma 3.1).

If $\varphi \in UC_b(X)$ there exists a sequence $(\varphi_n) \subset C^1_b(X)$ convergent to φ in $C_b(X)$ (e.g., [LaLi86]). By (3.9) we get

$$\|F_{\varphi_n} - F_\varphi\|_{L^\infty(\mathbb{R})} \le \|\varphi_n - \varphi\|_{L^\infty(\mathbb{R})} \|q_1\|_{L^1(\mathbb{R})}, \quad n \in \mathbb{N}. \tag{3.10}$$

We recall that F_{φ_n} is continuously differentiable for every $n \in \mathbb{N}$ by Lemma 3.1. Still by (3.9), for every n, $m \in \mathbb{N}$ and for a.e. $r \in \mathbb{R}$ we have

$$F'_{\varphi_n}(r) - F'_{\varphi_m}(r) = q_1(r) \int_X (\varphi_n(x) - \varphi_m(x)) m_r(dx),$$

so that

$$\|F'_{\varphi_n} - F'_{\varphi_m}\|_{L^\infty(\mathbb{R})} \le \|\varphi_n - \varphi\|_{L^\infty(\mathbb{R})} \|q_1\|_{L^\infty(\mathbb{R})}, \quad n, m \in \mathbb{N}.$$

Therefore, (F'_{φ_n}) is a Cauchy sequence in $C_b(\mathbb{R})$. Recalling (3.10), the conclusion follows. □

The main result of this section is the following.

Theorem 3.4. *Let Hypotheses 1.1 and 1.3 hold. Then the function F_φ is differentiable for every $\varphi \in C_b(X)$. For every $r \in \mathbb{R}$ there exists a Borel measure σ^g_r on X such that*

$$F'_\varphi(r) = \int_X \varphi(x)\, \sigma^g_r(dx), \quad \varphi \in C_b(X). \tag{3.11}$$

In particular, for $\varphi \equiv 1$ we obtain $\sigma^g_r(X) = F'_1(r) = q_1(r)$. Therefore, σ^g_r is nontrivial iff $F'_1(r) > 0$.

Proof. Fix $r \in \mathbb{R}$ and $\varphi \in C_b(X)$. To show that F_φ is differentiable at r, we shall show that for every vanishing sequence (ε_n) of nonzero numbers the incremental ratio $(F_\varphi(r+\varepsilon_n) - F_\varphi(r))/\varepsilon_n$ converges to a real limit independent of the sequence, as $n \to \infty$.

Consider the measures m_n defined by

$$m_n = \begin{cases} \dfrac{1}{\varepsilon_n} \mathbb{1}_{g^{-1}(r,r+\varepsilon_n)}\nu & \text{if } \varepsilon_n > 0, \\ \\ -\frac{1}{\varepsilon_n} \mathbb{1}_{g^{-1}(r+\varepsilon_n,r)}\nu & \text{if } \varepsilon_n < 0. \end{cases}$$

Then (m_n) is a sequence of nonnegative finite Borel (and since X is separable, Radon) measures, and we have

$$\int_X \varphi\, dm_n = \frac{1}{\varepsilon_n}(F_\varphi(r+\varepsilon_n) - F_\varphi(r)).$$

In particular, if φ is Lipschitz continuous and bounded by Proposition 3.3 F_φ is differentiable, and therefore

$$\lim_{n\to\infty} \int_X \varphi\, dm_n = F'_\varphi(r) = q_\varphi(r). \tag{3.12}$$

So, the sequence $\int_X \varphi\, dm_n$ converges to $q_\varphi(r)$. By a corollary of the Prokhorov Theorem (e.g., [Bo07, Cor. 8.6.3]), if a sequence of nonnegative Radon measures (m_n) is such that $\int_X \varphi\, dm_n$ converges in $\mathbb{R}$ for every Lipschitz continuous and bounded φ, there exists a limiting Borel measure such that (m_n) converges weakly to it. The weak limit is independent of the chosen vanishing sequence, because for every Lipschitz continuous and bounded φ equality (3.12) holds, so that denoting by m the weak limit obtained through a sequence (ε_n) and by $\widetilde{m}$ the weak limit obtained through another sequence $(\widetilde{\varepsilon_n})$, we have $\int_X \varphi\, dm = \int_X \varphi\, d\widetilde{m}$ for every Lipschitz continuous and bounded φ, and this implies that $m = \widetilde{m}$. So, there exists a Borel measure that we denote by σ_r^g, such that for every vanishing sequence (ε_n) of nonzero numbers and for every $\varphi \in C_b(X)$ we have

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n}(F_\varphi(r+\varepsilon_n) - F_\varphi(r)) = \int_X \varphi\, d\sigma_r^g.$$

This means that for every $\varphi \in C_b(X)$ the function F_φ is differentiable at r, and (3.11) holds. □

Remark 3.5. *From the proof of Theorem* 3.4 *it follows easily that if g is continuous, then σ_r^g has support in $g^{-1}(r)$. Indeed, for every $\varepsilon > 0$ and $\varphi \in C_b(X)$ with support contained in $g^{-1}(-\infty, r-\varepsilon) \cup g^{-1}(r+\varepsilon, +\infty)$, the function F_φ is constant in $(r-\varepsilon, r+\varepsilon)$, and therefore $F'_\varphi(r) = 0$. By* (3.11), *$\int_X \varphi\, d\sigma_r^g = 0$. So, the support of σ_r^g is contained in $\bigcap_{\varepsilon>0} g^{-1}[r-\varepsilon, r+\varepsilon] = g^{-1}(r)$.*

If g is not continuous, the existence of $\varphi \in C_b(X)$ with support contained in $g^{-1}(-\infty, r-\varepsilon) \cup g^{-1}(r+\varepsilon, +\infty)$ is not guaranteed, and this argument does not work. However, the argument in Remark 3.6 of [DaLuTu14] *shows that $\sigma_r^g = q_1(r) m_r$ for a.e. $r \in \mathbb{R}$ such that $q_1(r) > 0$, where m_r are the measures used in the proof of Proposition* 3.3. *Since the support of m_r is contained in $g^{-1}(r)$ for almost all $r \in \mathbb{R}$, the support of σ_r^g is contained in $g^{-1}(r)$ for almost all $r \in \mathbb{R}$ with $q_1(r) > 0$.*

In the next section we will show that for every $r \in \mathbb{R}$ the support of σ_r^g is contained in $g^{*-1}(r)$, for a suitable version g^* of g (Proposition 4.5*).*

Theorem 3.4 asserts that σ_r^g is nontrivial iff $q_1(r) > 0$. So, it is important to know whether $q_1(r) > 0$. An obvious sufficient condition for $q_1(r) > 0$ (in view of the identity $q_1(r) = \int_{g^{-1}(r,+\infty)} M^*\Psi\, d\nu$) is that $\nu(g^{-1}(r, +\infty)) > 0$, and $M^*\Psi \geq 0$ on

$g^{-1}(r,+\infty)$, $M^*\Psi > 0$ on a subset of $g^{-1}(r,+\infty)$ with positive measure. However, this is not easy to check.

In the Gaussian case, under reasonable assumptions on g we have $q_1(r) > 0$ if and only if $r \in (\operatorname{ess\,inf} g, \operatorname{ess\,sup} g)$ ([DaLuTu14, Lemma 3.9]). The proof is not easily extendible to our general setting, and in the next proposition we use an argument from [Nu95, Second Edition, Prop. 2.1.8]. We need a further hypothesis,

Hypothesis 3.6. *If $\varphi \in W^{1,2}(\nu)$ and $M_2\varphi = 0$, then φ is constant ν-a.e.*

For Hypothesis 3.6 to be satisfied, one needs that R be one to one. However, even in the case $R = I$, Hypothesis 3.6 is not obvious. If it holds, the Dirichlet form $\mathcal{E}(\varphi,\psi) := \int_X \langle M_2\varphi, M_2\psi\rangle\, d\nu$ is called irreducible.

Of course, a sufficient condition for Hypothesis 3.6 to be satisfied is that a Poincaré inequality holds, namely, that there exists $C > 0$ such that

$$\int_X \left(\varphi - \int_X \varphi\, d\nu\right)^2 d\nu \le C \int_X \|M_2\varphi\|^2 d\nu, \quad \varphi \in W^{1,2}(\nu). \tag{3.13}$$

We shall use the following lemma.

Lemma 3.7. *Let Hypothesis* 3.6 *hold. If B is a Borel set such that $\mathbb{1}_B \in W^{1,2}(\nu)$, then either $\nu(B) = 0$ or $\nu(B) = 1$.*

Proof. We follow [Nu95, Prop. 1.2.6]. Assume that $\mathbb{1}_B \in W^{1,2}(\nu)$ and let $\varphi \in C_c^\infty(\mathbb{R})$ be such that

$$\varphi(r) = r^2, \quad \forall\, r \in [0,1].$$

Then $\varphi \circ \mathbb{1}_B = \mathbb{1}_B$ and $\varphi' \circ \mathbb{1}_B = 2\mathbb{1}_B$, since

$$\varphi'(\mathbb{1}_B(x)) = \begin{cases} \varphi'(1) = 2 & \text{if } x \in [0,1], \\ \varphi'(0) = 0 & \text{if } x \notin [0,1]. \end{cases}$$

Now by the chain rule (Lemma 2.1(ii))

$$M_2(\mathbb{1}_B) = M_2(\varphi \circ \mathbb{1}_B) = \varphi'(\mathbb{1}_B)M_2(\mathbb{1}_B) = 2\mathbb{1}_B M_2(\mathbb{1}_B)$$

so that $M_2(\mathbb{1}_B) = 0$. By Hypothesis 3.6, $\mathbb{1}_B$ is constant a.e., and the conclusion follows. □

Proposition 3.8. *Under Hypotheses* 1.1, 1.3, *and* 3.6, *assume in addition that $M_p^*\Psi \in W^{1,p}(\nu)$ for every $p > 1$. Then for every $r \in \mathbb{R}$ we have*

$$q_1(r) > 0 \iff r \in (\operatorname{ess\,inf} g, \operatorname{ess\,sup} g).$$

Proof. The function $F_1(r) = \nu\{x : \ g(x) \le r\}$ is continuously differentiable, and it is constant in $(-\infty, \operatorname{ess\,inf} g)$ and in $(\operatorname{ess\,sup} g, +\infty)$. Therefore, for every $r \in (-\infty, \operatorname{ess\,inf} g] \cup [\operatorname{ess\,sup} g, +\infty)$ we have $F_1'(r) = q_1(r) = 0$.

To prove the converse, let us fix r_0 such that $q_1(r_0) = 0$. We shall show that the characteristic function $\mathbb{1}_{\{g>r_0\}}$ belongs to $W^{1,2}(\nu)$. We approach $\mathbb{1}_{\{g>r_0\}}$ by the functions φ_ε defined by

$$\varphi_\varepsilon(x) = \begin{cases} 0, & g(x) < r_0 - \varepsilon, \\[2ex] \dfrac{1}{2\varepsilon}(g(x) - (r_0 - \varepsilon)), & r_0 - \varepsilon \le g(x) \le r_0 + \varepsilon, \\[2ex] 1, & g(x) > r_0 + \varepsilon, \end{cases}$$

for $\varepsilon > 0$. By Lemma 2.2, $\varphi_\varepsilon \in W^{1,p}(\nu)$ for every p, and

$$M(\varphi_\varepsilon) = \frac{1}{2\varepsilon} \mathbb{1}_{\{r_0-\varepsilon \le g \le r_0+\varepsilon\}} Mg.$$

To estimate $\|M(\varphi_\varepsilon)\|_{L^2(X,\nu;X)}$, we preliminarily show that q_1' is Hölder continuous and that $q_1'(r_0) = 0$.

The Hölder continuity of q_1' follows from the regularity assumption on $M^*\Psi$. Indeed, by (3.4) we have

$$q_1(r) = -\int_{\{g<r\}} M^*\Psi \, d\nu = -F_{M^*\Psi}(r), \quad r \in \mathbb{R}.$$

By assumption, $M^*\Psi \in W^{1,p}(\nu)$ for every $p > 1$, so that by Lemma 3.1 q_1 is differentiable, and

$$q_1'(r) = -q_{M^*\Psi}(r), \quad r \in \mathbb{R}.$$

Still by Lemma 3.1, q_1' is Hölder continuous, with any exponent $\alpha \in (0,1)$.

Let us prove that $q_1'(r_0) = 0$. Since $q_1(r_0) = 0$, $\sigma^g_{r_0}(X) = 0$, and by Theorem 3.4 we get $q_\varphi(r_0) = 0$ for every $\varphi \in C_b(X)$. Approaching every $\varphi \in W^{1,p}(\nu)$ by a sequence of C^1_b functions and using estimate (3.5), we obtain $q_\varphi(r_0) = 0$ for every $\varphi \in W^{1,p}(\nu)$. In particular,

$$q_{M^*\Psi}(r_0) = -q_1'(r_0) = 0.$$

Therefore, for every $\alpha \in (0,1)$ there exists $C_\alpha > 0$ such that $|q_1(\xi)| \le C_\alpha|\xi - r_0|^{1+\alpha}$, for every $\xi \in \mathbb{R}$. It follows that there exists $K_\alpha > 0$ such that for every $\varepsilon > 0$ we have

$$\left| \int_{r_0-\varepsilon}^{r_0+\varepsilon} q_1(\xi) d\xi \right| \le K_\alpha \varepsilon^{2+\alpha}.$$

Now fix $\alpha \in (0,1)$ and take $p = 2 + 4/\alpha$, so that $(\alpha+2)(p-2)/p = 2$. By the Hölder inequality we have

$$\begin{aligned}
\int_X \|M\varphi_\varepsilon\|^2 d\nu &= \frac{1}{(2\varepsilon)^2} \int_X \|Mg\|^2 \mathbb{1}_{\{r_0-\varepsilon \le g \le r_0+\varepsilon\}} \, d\nu \\
&\le \frac{1}{(2\varepsilon)^2} \left(\int_X \|Mg\|^p d\nu \right)^{2/p} \left(\int_{r_0-\varepsilon}^{r_0+\varepsilon} q_1(\xi) d\xi \right)^{(p-2)/p} \\
&\le \frac{1}{(2\varepsilon)^2} \left(\int_X \|Mg\|^p d\nu \right)^{2/p} (K_\alpha \varepsilon^{2+\alpha})^{(p-2)/p} = C,
\end{aligned}$$

with $C := 2^{-2}(\int_X \|Mg\|^p d\nu)^{2/p} K_\alpha^{(p-2)/p}$ independent of ε.

So, $\|\varphi_\varepsilon\|_{W^{1,2}(\nu)}$ is bounded by a constant independent of ε. By Lemma 2.1(v), $\mathbb{1}_{\{g>r_0\}}$ belongs to $W^{1,2}(\nu)$. By Lemma 3.7, the measure of the set $\{x : \; g(x) > r_0\}$ is either 0 or 1, namely $r_0 \ge \operatorname{ess\,sup} g$ or $r_0 \le \operatorname{ess\,inf} g$. □

3.1. **Integration by parts formulae.** We recall that by Lemma 2.1(i), the product $\varphi\psi$ belongs to $W^{1,p}(\nu)$ provided $\varphi \in W^{1,p_1}(\nu)$, $\psi \in W^{1,p_2}(\nu)$, with $1/p_1 + 1/p_2 \le 1/p$. In this case, for all $F \in D(M_p^*)$ we may apply formula (1.4) with $\varphi\psi$ replacing φ, and we obtain

$$\int_X \langle M_{p_1}\varphi, F\rangle \, \psi \, d\nu + \int_X \langle M_{p_2}\psi, F\rangle \, \varphi \, d\nu = \int_X \varphi \, \psi \, M_p^*(F) \, d\nu. \tag{3.14}$$

The following proposition is a first basic step towards an integration by parts formula.

Proposition 3.9. *Assume that Hypotheses* 1.1 *and* 1.3 *are fulfilled. Let* $p > 1$, $F \in D(M_p^*)$, $\varphi \in W^{1,p_1}(\nu)$ *for some* $p_1 > p$, *and assume that* $\langle Mg, F\rangle\,\varphi$ *belongs to* $C_b(X)$ *or to* $W^{1,q}(\nu)$ *for some* $q > 1$. *Then*

$$\int_{\{g<r\}} \langle M_{p_1}\varphi, F\rangle\, d\nu = \int_{\{g<r\}} \varphi\, M_p^*(F)\, d\nu + q_{\langle Mg,F\rangle\varphi}(r), \quad r \in \mathbb{R}. \tag{3.15}$$

Proof. For any $\varepsilon > 0$ we set

$$\theta_\varepsilon(\xi) = \begin{cases} 1 & \text{if } \xi \le r-\varepsilon, \\ -\dfrac{1}{\varepsilon}(\xi - r) & \text{if } r-\varepsilon < \xi < r, \\ 0 & \text{if } \xi \ge r, \end{cases} \tag{3.16}$$

so that

$$\theta_\varepsilon'(\xi) = \begin{cases} 0 & \text{if } \xi < r-\varepsilon, \\ -\dfrac{1}{\varepsilon} & \text{if } r-\varepsilon < \xi < r, \\ 0 & \text{if } \xi > r. \end{cases} \tag{3.17}$$

By Lemma 2.2 (applied to $-g$), the composition $\theta_\varepsilon \circ g$ belongs to $W^{1,p_2}(\nu)$ for every $p_2 > 1$, and

$$M(\theta_\varepsilon \circ g) = (\theta_\varepsilon' \circ g) M(g).$$

Since $p_1 > p$, choosing p_2 large enough we have $1/p_1 + 1/p_2 \le 1/p$, and we may use formula (3.14) with $\psi = \theta_\varepsilon \circ g$ to obtain

$$\int_X \langle M\varphi, F\rangle\,(\theta_\varepsilon \circ g)\, d\nu = \frac{1}{\varepsilon}\int_{\{r-\varepsilon \le g \le r\}} \langle Mg, F\rangle\,\varphi\, d\nu + \int_X \varphi\,(\theta_\varepsilon \circ g)\, M_p^*(F)\, d\nu. \tag{3.18}$$

Since $\langle Mg, F\rangle\,\varphi \in C_b(X) \cup W^{1,q}(\nu)$, by Lemma 3.1 or by Theorem 3.4 we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\{r-\varepsilon \le g \le r\}} \langle Mg, F\rangle\,\varphi\, d\nu = q_{\langle Mg,F\rangle\varphi}(r).$$

On the other hand, $\theta_\varepsilon \circ g$ converges a.e. to $\mathbb{1}_{\{g \le r\}}$, and by the Dominated Convergence Theorem we have

$$\lim_{\varepsilon\to 0} \int_X \langle M\varphi, F\rangle\,(\theta_\varepsilon \circ g)\, d\nu = \int_{\{g\le r\}} \langle M\varphi, F\rangle\, d\nu,$$

$$\lim_{\varepsilon\to 0} \int_X \varphi\,(\theta_\varepsilon \circ g)\, M_p^*(F)\, d\nu = \int_{\{g\le r\}} \varphi\, M_p^*(F)\, d\nu.$$

The conclusion follows. ☐

Note that by (3.11), if $\langle Mg, F\rangle\,\varphi \in C_b(X)$, then $q_{\langle Mg,F\rangle\varphi}(r)$ is just the integral of $\langle Mg, F\rangle\varphi$ with respect to σ_r^g, and (3.15) may be rewritten as

$$\int_{\{g<r\}} \langle M\varphi, F\rangle\, d\nu = \int_{\{g<r\}} \varphi\, M_p^*(F)\, d\nu + \int_X \langle Mg, F\rangle\varphi\, d\sigma_r^g. \tag{3.19}$$

To improve formula (3.19) and extend it to a wider class of functions we have to work a bit. To this aim, in the next section we introduce the p-capacity and then we use it as a tool.

4. p-CAPACITIES

Definition 4.1. *Let Hypothesis* 1.1 *hold. For every open set $O \subset X$ and $p > 1$ we define the p-capacity of O by*

$$C_p(O) := \inf\{\|f\|_{W^{1,p}(\nu)} : \ f(x) \geq \mathbb{1}_O \ \nu - a.e., \ f \in W^{1,p}(\nu)\}.$$

If B is any Borel set, we define

$$C_p(B) := \inf\{C_p(O) : \ O \text{ is open}, \ O \supset B\}.$$

A function $f : X \mapsto \mathbb{R}$ is called C_p-quasicontinuous if for every $\varepsilon > 0$ there is an open set O such that $C_p(O) < \varepsilon$ and f is continuous in $X \setminus O$.

This is just Definition 8.13.1 of [Bo10], with the choice $\mathcal{F} = W^{1,p}(\nu)$. It follows immediately from the definition that for every Borel set A, B we have

$$C_p(A \cup B) \leq C_p(A) + C_p(B), \quad C_p(B) \geq (\nu(B))^{1/p}.$$

We recall some properties of the p-capacity taken from [Bo10, Sect. 8.13].

Proposition 4.2.

(i) *Every element $f \in W^{1,p}(\nu)$ has a C_p-quasicontinuous version f^* which satisfies*

$$C_p(\{x : \ f^*(x) > r\}) \leq \frac{1}{r}\|f\|_{W^{1,p}(\nu)}, \quad r > 0.$$

(ii) *Let (f_n)be a sequence that converges to f in $W^{1,p}(\nu)$. For every n let f_n^* be any C_p-quasicontinuous version of f_n. Then there is a subsequence $(f_{n_k}^*)$ that converges pointwise to f^*, except at most on a set with null p-capacity.*

(iii) *If O is an open set, f is C_p-quasicontinuous and $f(x) \geq 0$ for ν-a.e. $x \in O$. Then $f(x) \geq 0$ in O, except at most on a set with null p-capacity.*

We are ready to exhibit a class of sets that are negligible with respect to all the measures σ_r^g constructed in Section 3.

Proposition 4.3. *Under Hypotheses* 1.1 *and* 1.3*, let $B \subset X$ be a Borel set with $C_p(B) = 0$ for some $p > 1$. Then $\sigma_r^g(B) = 0$, for every $r \in \mathbb{R}$.*

Proof. For every $\varepsilon > 0$ let $O_\varepsilon \supset B$ be an open set such that $C_p(O_\varepsilon) < \varepsilon$. Then there exists $f_\varepsilon \in W^{1,p}(\nu)$ such that $\|f_\varepsilon\|_{W^{1,p}(\nu)} \leq \varepsilon$, $f_\varepsilon \geq 0$ ν-a.e., and $f_\varepsilon \geq 1$ ν-a.e. in O_ε. Let us fix an increasing sequence $(\theta_n) \subset C_b(X)$ that converges to $\mathbb{1}_{O_\varepsilon}$ pointwise. For instance, we can take

$$\theta_n(x) = \begin{cases} 0, & x \in X \setminus O_\varepsilon, \\ \\ n\,\text{dist}(x, X \setminus O_\varepsilon), & 0 < \text{dist}(x, X \setminus O_\varepsilon) < 1/n, \\ \\ 1, & \text{dist}(x, X \setminus O_\varepsilon) \geq 1/n. \end{cases}$$

Then, $\lim_{n\to\infty} \theta_n(x) = \mathbb{1}_{O_\varepsilon}(x)$, for every $x \in X$. Using the Dominated Convergence Theorem and then formula (3.11), we get

$$\sigma_r^g(O_\varepsilon) = \int_X \mathbb{1}_{O_\varepsilon}\, d\sigma_r^g = \lim_{n\to\infty} \int_X \theta_n\, d\sigma_r^g = \lim_{n\to\infty} q_{\theta_n}(r). \tag{4.1}$$

On the other hand, $f_\varepsilon(x) \geq \mathbb{1}_{O_\varepsilon}(x) \geq \theta_n(x)$, for ν-a.e. $x \in X$, so that the function $F_{f_\varepsilon - \theta_n}$ is increasing. In particular, $F'_{f_\varepsilon-\theta_n}(r) = q_{f_\varepsilon}(r) - q_{\theta_n}(r) \geq 0$ for every $r \in \mathbb{R}$ and $n \in \mathbb{N}$. Therefore, (4.1) yields

$$\sigma_r^g(O_\varepsilon) \leq q_{f_\varepsilon}(r), \quad r \in \mathbb{R}.$$

On the other hand, by (3.5) we have

$$|q_{f_\varepsilon}(r)| \leq K_p \|f_\varepsilon\|_{W^{1,p}(\nu)} \leq K_p \varepsilon,$$

with K_p independent of ε. Therefore, $\sigma_r^g(O_\varepsilon) \leq K_p\varepsilon$ for every $\varepsilon > 0$, which implies $\sigma_r^g(B) = 0$. □

Now we extend formula (3.11) to Sobolev functions. The procedure is similar to [CeLu14], where Gaussian measures were considered.

Theorem 4.4. *Let Hypotheses* 1.1 *and* 1.3 *hold, and let* $\varphi \in W^{1,p}(\nu)$ *for some* $p > 1$. *Fix any* $r \in \mathbb{R}$. *There exists a unique* $\psi \in L^1(X, \sigma_r^g)$ *such that every sequence of* C_b^1 *functions* (φ_n) *that converges to* φ *in* $W^{1,p}(\nu)$ *also converges in* $L^1(X, \sigma_r^g)$ *to* ψ. *Setting* $T\varphi := \psi$, *we have*

$$F'_\varphi(r) = \int_X T\varphi(x)\, \sigma_r^g(dx). \tag{4.2}$$

Moreover,

(i) (φ_n) *converges to* ψ *in* $L^q(X, \sigma_r^g)$, *and* $T \in \mathcal{L}(W^{1,p}(\nu), L^q(X, \sigma_r^g))$ *for every* $q \in [1, p)$;

(ii) *for every* p*-quasicontinuous version* φ^* *of* φ, *we have* $T\varphi(x) = \varphi^*(x)$ *for* σ_r^g*-a.e.* $x \in X$. *In particular, if* φ *is continuous, then* $T\varphi(x) = \varphi(x)$ *for* σ_r^g*-a.e.* $x \in X$;

(iii) *if* $\varphi_1 \in W^{1,p_1}(\nu)$, $\varphi_2 \in W^{1,p_2}(\nu)$, *and* $1/p_1 + 1/p_2 < 1$, *then* $T(\varphi_1\varphi_2) = T(\varphi_1)T(\varphi_2)$ *(as elements of* $L^1(X, \sigma_r^g)$*).*

Proof. By (3.11), for every $\varphi \in C_b^1(X)$ we have

$$q_{|\varphi|}(r) = \int_X |\varphi|\, \sigma_r^g(dx).$$

Take $\varphi = \varphi_n - \varphi_m$. By Corollary 2.3, $|\varphi_n - \varphi_m| \in W^{1,p}(\nu)$, and

$$\lim_{n,m\to\infty} \|\, |\varphi_n - \varphi_m|\, \|_{W^{1,p}(\nu)} = 0.$$

Using estimate (3.5) we get

$$q_{|\varphi_n-\varphi_m|}(r) = \int_X |\varphi_n - \varphi_m|\, \sigma_r^g(dx) \leq K_p \|\, |\varphi_n - \varphi_m|\, \|_{W^{1,p}(\nu)}. \tag{4.3}$$

Therefore, (φ_n) is a Cauchy sequence in $L^1(X, \sigma_r^g)$, and it converges to a limit ψ in $L^1(X, \sigma_r^g)$. The limit function ψ is apparently the same for all sequences that converge to φ in $W^{1,p}(\nu)$. Indeed, if $\varphi_n \to \varphi$, $\widetilde{\varphi}_n \to \widetilde{\varphi}$ in $W^{1,p}(\nu)$ as $n \to \infty$, the difference $\varphi_n - \widetilde{\varphi}_n$ vanishes in $L^1(X, \sigma_r^g)$ by estimate (3.5) with $\varphi_n - \varphi_m$ replaced by $\varphi_n - \widetilde{\varphi}_n$.

Still by estimate (3.5), the sequence $(q_{\varphi_n}(r))$ converges to $q_\varphi(r)$, and (4.2) follows.

To prove statement (i) we follow the above procedure, replacing $|\varphi_n - \varphi_m|$ with $|\varphi_n - \varphi_m|^q$, which belongs to $C_b^1(X)$ for $q > 1$ and vanishes in $W^{1,p/q}(\nu)$ as n,

$m \to +\infty$ by (2.2). By estimate (3.5) we have

$$q_{|\varphi_n - \varphi_m|^q}(r) \le K_{p/q} \| \, |\varphi_n - \varphi_m|^q \, \|_{W^{1,p/q}(\nu)},$$

so that (φ_n) is a Cauchy sequence in $L^q(X, \sigma^g_r)$, and its $L^1(X, \sigma^g_r)$-limit ψ belongs to $L^q(X, \sigma^g_r)$.

Let us prove (ii). By Proposition 4.2(ii), a subsequence (φ_{n_k}) converges to $\varphi^*(x)$ for every $x \in X$ except at most on a set with zero p-capacity. By Proposition 4.3, such a subsequence converges σ^g_r-a.e to φ^*. By the first part of this proposition, (φ_{n_k}) converges to $T\varphi$ in $L^1(X, \sigma^g_r)$. A further subsequence of (φ_{n_k}) converges to $T\varphi$, σ^g_r-a.e. Therefore, $T\varphi = \varphi^*$, σ^g_r-a.e.

To prove (iii) it is enough to approach φ_1 and φ_2 by sequences $(\varphi_{1,n})$, $(\varphi_{2,n})$ of C^1_b functions in $W^{1,p_1}(\nu)$, $W^{1,p_2}(\nu)$, respectively. The product $\varphi_{1,n}\varphi_{2,n}$ converges to $\varphi_1\varphi_2$ in $W^{1,s}(\nu)$, for $s = (p_1 + p_2)/p_1p_2$; therefore $T(\varphi_1\varphi_2) = \lim_{n\to\infty} \varphi_{1,n}\varphi_{2,n}$ in $L^1(X, \sigma^g_r)$. On the other hand, $T(\varphi_1) = \lim_{n\to\infty} \varphi_{1,n}$ in $L^q(X, \sigma^g_r)$ for every $q < p_1$, $T(\varphi_2) = \lim_{n\to\infty} \varphi_{1,n}$ in $L^r(X, \sigma^g_r)$ for every $r < p_2$. Choosing $q < p_1$ and $r < p_2$ such that $1/q + 1/r = 1$, we obtain $T(\varphi_1)T(\varphi_2) = \lim_{n\to\infty} \varphi_{1,n}\varphi_{2,n}$ in $L^1(X, \sigma^g_r)$, and the statement follows. □

The results that we have proved up to now are independent of the version of g that we have considered. Instead, from now on we fix a p-quasicontinuous version g^* of g, for some $p > 1$. This is because we shall consider the ν-negligible sets $(g^*)^{-1}(r)$ for $r \in \mathbb{R}$.

With the aid of Theorem 4.4 we can study the supports of the measures σ^g_r.

Proposition 4.5. *For every $r_0 \in \mathbb{R}$, the support of $\sigma^g_{r_0}$ is contained in $g^{*-1}(r_0)$.*

Proof. Fix $\varepsilon > 0$, and set $A := g^{*-1}(-\infty, r_0 - \varepsilon) \cup g^{*-1}(r_0 + \varepsilon, \infty)$. Our aim is to show that

$$\int_X \mathbb{1}_A d\sigma^g_{r_0} = 0, \tag{4.4}$$

which implies that the support of $\sigma^g_{r_0}$ is contained in $g^{*-1}([r_0 - \varepsilon, r_0 + \varepsilon])$. Since ε is arbitrary, the statement will follow.

We approach $\mathbb{1}_{(-\infty, r_0-\varepsilon)\cup(r_0+\varepsilon,+\infty)}$ by a sequence of Lipschitz functions:

$$\chi_n(\xi) = \begin{cases} 1, & \xi \le r_0 - \varepsilon - 1/n, \\ -n(\xi - (r_0 - \varepsilon)), & r_0 - \varepsilon - 1/n \le \xi \le r_0 - \varepsilon, \\ 0, & r_0 - \varepsilon \le \xi \le r_0 + \varepsilon, \\ n(\xi - (r_0 + \varepsilon)), & r_0 + \varepsilon \le \xi \le r_0 + \varepsilon + 1/n, \\ 1, & \xi \ge r_0 + \varepsilon. \end{cases}$$

We have $\lim_{n\to\infty} \chi_n(\xi) = \mathbb{1}_{(-\infty, r_0-\varepsilon)\cup(r_0+\varepsilon,+\infty)}(\xi)$, for every $\xi \in \mathbb{R}$. Consequently, $\chi_n \circ g^*$ converges pointwise, for every $x \in X$, to $\mathbb{1}_A$. Since $0 \le \chi_n \circ g^* \le 1$, by the Dominated Convergence Theorem we get

$$\int_X \mathbb{1}_A d\sigma^g_{r_0} = \lim_{n\to\infty} \int_X \chi_n \circ g^* \, d\sigma^g_{r_0}. \tag{4.5}$$

For every n, $\chi_n \circ g \in W^{1,p}(\nu)$, by Lemma 2.2. By Lemma 3.1, $q_{\chi_n \circ g}$ is continuous, so that the function $F_{\chi_n\circ g}(r) = \int_{g^{-1}(-\infty,r)} \chi_n \circ g \, d\mu$, whose derivative is $q_{\chi_n\circ g}$, is C^1. By the definition of χ_n, $F_{\chi_n\circ g}$ is constant, equal to $F_{\chi_n\circ g}(r_0 - \varepsilon)$, in the interval

$[r_0-\varepsilon, r_0+\varepsilon]$, so that the derivative $q_{\chi_n\circ g}$ vanishes in $(r_0-\varepsilon, r_0+\varepsilon)$. In particular, it vanishes at r_0.

By (4.2) we have

$$q_{\chi_n\circ g}(r_0) = \int_X T(\chi_n\circ g)\, d\sigma^g_{r_0},$$

where T is the operator defined in Theorem 4.4. Since g^* is p-quasicontinuous and χ_n is continuous, $\chi_n\circ g^*$ is p-quasicontinuous. It coincides with $\chi_n\circ g$ outside a ν-negligible set; therefore it is a p-quasicontinuous version of $\chi_n\circ g$. By Theorem 4.4, $T(\chi_n\circ g)$ coincides with $\chi_n\circ g^*$, up to $\sigma^g_{r_0}$-negligible sets. Therefore, for every $n\in\mathbb{N}$,

$$0 = q_{\chi_n\circ g}(r_0) = \int_X T(\chi_n\circ g)\, d\sigma^g_{r_0} = \int_X \chi_n\circ g^*\, d\sigma^g_{r_0},$$

and (4.4) follows from (4.5). □

Proposition 4.5 justifies the following definition.

Definition 4.6. *Let $\varphi\in W^{1,p}(\nu)$ for some $p>1$, and let $r\in\mathbb{R}$. We define the trace of φ at $g^{*-1}(r)$ as the function $T\varphi$ given by Theorem* 4.4.

Characterizing the range of the trace operator is a difficult problem that is out of reach for the moment. In the case of Gaussian measures in Banach spaces the range of the trace has been characterized only for $g\in X^*$ ([CeLu14]). Even worse, for very smooth functions in Hilbert spaces such as $g(x)=\|x\|^2$ we do not know whether the traces of elements of $W^{1,p}(\nu)$ belong to $L^p(X,\sigma^g_r)$ with the same p. See the discussion in [CeLu14].

Now we read again formula (3.15) in terms of surface integrals.

Corollary 4.7. *Assume that Hypotheses* 1.1 *and* 1.3 *are fulfilled. Let $p>1$, $F\in D(M^*_p)$, $\varphi\in W^{1,p_1}(\nu)$ for some $p_1>p$, and assume that $\langle Mg,F\rangle\,\varphi$ belongs to $C_b(X)$ or to $W^{1,q}(\nu)$ for some $q>1$. Then for every $r\in\mathbb{R}$,*

(4.6)
$$\begin{aligned}\int_{\{g<r\}}\langle M_{p_1}\varphi, F\rangle\, d\nu &= \int_{\{g<r\}}\varphi\, M^*_p(F)\, d\nu + \int_X T(\langle Mg,F\rangle\varphi)\, d\sigma^g_r\\ &= \int_{\{g<r\}}\varphi\, M^*_p(F)\, d\nu + \int_{g^{*-1}(r)} T(\langle Mg,F\rangle\varphi)\, d\sigma^g_r.\end{aligned}$$

Proof. By formula (4.2) for every $r\in\mathbb{R}$ we have

$$\int_{g^{*-1}(r)} T(\langle Mg,F\rangle\varphi)\, d\sigma^g_r = \int_X T(\langle Mg,F\rangle\varphi)\, d\sigma^g_r = q_{\langle Mg,F\rangle\varphi}(r).$$

On the other hand, Proposition 3.9 yields

$$q_{\langle Mg,F\rangle\varphi}(r) = \int_{\{g<r\}}\langle M_{p_1}\varphi, F\rangle\, d\nu - \int_{\{g<r\}}\varphi\, M^*_p(F)\, d\nu,$$

and the statement follows. □

Since the operators M_p and M^*_p play the role of the gradient and of the negative divergence, formula (4.6) is a version of the Divergence Theorem in our context.

The similarity gets better if we assume that $\|Mg\| \in W^{1,q}(\nu)$ for every $q > 1$. In this case, recalling Theorem 4.4(iii) we may rewrite (4.6) as

(4.7)
$$\begin{aligned}\int_{\{g<r\}} \langle M_{p_1}\varphi, F\rangle \, d\nu &= \int_{\{g<r\}} \varphi M_p^* F \, d\nu + \int_X T(\langle \frac{Mg}{\|Mg\|}, F\rangle \varphi) T(\|Mg\|) \, d\sigma_r^g \\ &= \int_{\{g<r\}} \varphi M_p^* F \, d\nu + \int_X T(\langle \frac{Mg}{\|Mg\|}, F\rangle \varphi) \, d\rho_r,\end{aligned}$$

where

$$\rho_r(dx) := T(\|Mg\|)(x)\sigma_r^g(dx),$$

so that $Mg/\|Mg\|$ plays the role of the exterior normal vector to the surface $g^{*-1}(r)$, and the weighted measure ρ_r plays the role of normalized surface measure. In fact ρ_r is a distinguished surface measure, and it will be discussed in the next section.

Let us consider now the case of constant vector fields F. Namely, we fix $z \in X$ and we assume that $F_z(x) \equiv z$ belongs to $D(M_p^*)$ for some $p > 1$. We recall that $F_z \in D(M_p^*)$ iff there exists $C_{p,z} > 0$ such that

$$\left| \int_X \langle R\nabla\varphi, z\rangle d\nu \right| \le C_{p,z} \|\varphi\|_{L^p(X,\nu)}, \quad \varphi \in C_b^1(X) \tag{4.8}$$

(see the Introduction). In this case, we set $v_z := M_p^*(F_z)$ and we rewrite (4.6) for every $\varphi \in W^{1,p}(\nu)$ as

$$\int_{g^{-1}(-\infty,r)} \langle M_p\varphi, z\rangle \, d\nu = \int_{g^{-1}(-\infty,r)} \varphi \, v_z \, d\nu + \int_X T(\langle Mg, z\rangle \varphi) \, d\sigma_r^g, \tag{4.9}$$

provided $\langle Mg, z\rangle\varphi$ belongs to $C_b(X)$ or to $W^{1,q}(\nu)$ for some $q > 1$.

5. Dependence on g: Comparison with the geometric measure theory approach

Even for continuous or smooth g, the measures σ_r^g constructed in the previous sections depend explicitly on the defining function g, and not only on the sets $g^{-1}(r)$ or $g^{-1}(-\infty, r)$. In particular, if we replace g by $\widetilde{g} = \theta \circ g$ with a smooth $\theta : \mathbb{R} \mapsto \mathbb{R}$ such that $\inf \theta' > 0$, it is easy to check that $\widetilde{g}$ satisfies Hypothesis 1.3, and using the definition we see that for every $r \in \mathbb{R}$, setting $\Sigma := g^{-1}(r) = \widetilde{g}^{-1}(\theta(r))$ we have

$$\int_\Sigma \varphi \, d\sigma_r^g = \theta'(r) \int_\Sigma \varphi \, d\sigma_{\theta(r)}^{\widetilde{g}}, \quad \varphi \in C_b(X).$$

So, it is desirable to modify the construction of our surface measures in order to get rid of the dependence on g and to get a surface measure with some intrinsic analytic or geometric properties.

In the case of Gaussian measures in Banach spaces, for suitably regular hypersurfaces $g^{-1}(r)$ the measure $|\nabla_H g|_H \sigma_r^g$, where H is the Cameron–Martin space, is independent of g, and it coincides with the restriction of the Hausdorff–Gauss measure of Feyel and de La Pradelle ([FePr92]) to the hypersurface and with the perimeter measure relevant to the set $\Omega = g^{-1}(-\infty, r)$ from the geometric measure theory in abstract Wiener spaces ([Fu00, FuHi01, AMMP10]). See [CeLu14].

In our setting, what plays the role of $|\nabla_H g|_H$ is $\|Mg\|$. We shall show that $\|Mg\|\sigma_r^g$ depends on g only through the set $g^{-1}(-\infty, r)$, among a class of good enough g, and it is a sort of perimeter measure.

As a first step, we notice that if $\|Mg\| \in W^{1,q}(\nu)$ for some $q > 1$, then $Mg/\|Mg\| \in D(M_s^*)$ for every $s > q/(q-1)$. This comes from Lemma 2.5, writing

$$\frac{Mg}{\|Mg\|} = \frac{Mg}{\|Mg\|^2}\|Mg\|,$$

and recalling that $Mg/\|Mg\|^2 \in D(M_p^*)$ for every $p > 1$, by Hypothesis 1.3. Lemma 2.5 also yields

$$M_s^*\Big(\varphi\frac{Mg}{\|Mg\|}\Big) = \varphi\|Mg\|M_p^*\Big(\frac{Mg}{\|Mg\|^2}\Big) - \langle M_q(\varphi\|Mg\|), \frac{Mg}{\|Mg\|^2}\rangle,$$

for every $\varphi \in C_b^1(X)$. Comparing with (3.4), we obtain

$$q_{\|Mg\|\varphi}(r) = -\int_{\{g<r\}} M_s^*\Big(\varphi\frac{Mg}{\|Mg\|}\Big)\,d\nu, \quad r \in \mathbb{R}, \tag{5.1}$$

and by Theorem 4.4,

$$\int_{g^{*-1}(r)} \varphi\, T(\|Mg\|)d\sigma_r^g = -\int_{\{g<r\}} M_s^*\Big(\varphi\frac{Mg}{\|Mg\|}\Big)\,d\nu, \quad r \in \mathbb{R}. \tag{5.2}$$

The right-hand side of (5.1) and of (5.2) is the negative integral over $g^{-1}(-\infty, r)$ of $M_s^*(\varphi F)$, where $F = Mg/\|Mg\|$ plays the role of the exterior unit normal vector to the level surfaces of g. It is indeed the exterior unit normal vector to $\partial\{x: g(x) < r\}$ if g is smooth enough and $R = I$.

To go on, it is convenient to introduce spaces of $W^{1,p}$ vector fields.

Definition 5.1. *For every $p > 1$ we denote by $W^{1,p}(X,\nu;X)$ the space of vector fields $F : X \mapsto X$ such that for a given orthonormal basis $\{e_i: i \in \mathbb{N}\}$, the functions $f_i := \langle F, e_i\rangle \in W^{1,p}(\nu)$ for every $i \in \mathbb{N}$, and $(\sum_{i=1}^\infty \|M_p f_i\|^2)^{1/2} \in L^p(X,\nu)$.*

It is easy to see that the definition does not depend on the chosen orthonormal basis. The standard proof of the following lemma is left to the reader.

Lemma 5.2.

(i) *If $F_1 \in W^{1,p_1}(X,\nu;X)$, $F_2 \in W^{1,p_2}(X,\nu;X)$, with $1/p := 1/p_1 + 1/p_2 < 1$, then $x \mapsto \langle F_1(x), F_2(x)\rangle$ belongs to $W^{1,p}(\nu)$.*

(ii) *If $F \in W^{1,p_1}(X,\nu;X)$, $\varphi \in W^{1,p_2}(\nu)$, with $1/p := 1/p_1 + 1/p_2 < 1$, then φF belongs to $W^{1,p}(X,\nu;X)$.*

(iii) *If $F \in W^{1,p}(X,\nu;X)$ for some $p > 1$, then $x \mapsto \|F(x)\|$ belongs to $W^{1,p}(\nu)$.*

The following theorem is the main result of this section.

Theorem 5.3. *Let Hypotheses 1.1 and 1.3 hold, and assume in addition that $Mg \in W^{1,q}(X,\nu;X)$ for some $q > 2$. Then for every $\varphi \in C_b^1(X)$ with nonnegative values and for any $t \in (q', q)$, $s > q'$ we have*

$$\begin{aligned}\int_X \varphi T(\|Mg\|)d\sigma_r^g \\ = \max\Big\{\int_{\{g<r\}} M_s^*(\varphi F)\,d\nu :\ F \in W^{1,t}(X,\nu;X) \cap D(M_s^*),\ \|F(x)\| \le 1 \text{ a.e.}\Big\}.\end{aligned} \tag{5.3}$$

The maximum is attained at $F = -Mg/\|Mg\|$.

Proof. By Lemma 5.2(iii), $\|Mg\| \in W^{1,q}(\nu)$ and therefore $\|Mg\|\varphi \in W^{1,q}(\nu)$. Then, by formulae (5.1) and (5.2),

$$\int_X \varphi T(\|Mg\|) d\sigma_r^g = q_{\|Mg\|\varphi}(r).$$

So, we have to show that $q_{\|Mg\|\varphi}(r)$ is equal to the right-hand side of (5.3). The proof is in two steps.

In the first step we shall prove that the vector field $F = Mg/\|Mg\|$ is one of the admissible vector fields in the right-hand side of (5.3), namely that it belongs to $D(M_s^*)$ for every $s > q' = q/(q-1)$ and to $W^{1,t}(X,\nu;X)$ for every $t \in (q', q)$.

In the second step we shall prove that for every admissible vector field F in the right-hand side of (5.3), the integral $\int_{\{g<r\}} M_s^*(\varphi F) d\nu$ is equal to $q_{\langle Mg,F\rangle\varphi}(r)$ (of course, we need to show that $\langle Mg, F\rangle\varphi$ belongs to $W^{1,p}(\nu)$ for some p). Then, using the definition of q_φ, it will be easy to see that $q_{\langle Mg,F\rangle\varphi}(r) \le q_{\|Mg\|\varphi}(r)$ if φ has nonnegative values.

In view of formula (5.1), the statement will follow.

Throughout the proof we denote by $\{e_i : i \in \mathbb{N}\}$ any orthonormal basis of X.

Step 1. F may be written as the product of $Mg/\|Mg\|^2$ which is in $D(M_p^*)$ for every p by Hypothesis 1.3 and the scalar function $\|Mg\| \in W^{1,q}(\nu)$ by Lemma 5.2(iii). Lemma 2.5(i) implies that $F \in D(M_s^*)$ for $s = pq/(q-p)$, for every $p \in (1,q)$. Letting $p \to 1$, we obtain $F \in D(M_s^*)$ for every $s > q/(q-1) = q'$.

Let us prove that $F \in W^{1,t}(X,\nu;X)$, for every $t < q$. We have $F = Mg\,\psi$, with $\psi = 1/\|Mg\|$. As easily seen approximating ψ by

$$\psi_n(x) := \left(\sum_{i=1}^n \langle Mg(x), e_i\rangle^2 + 1/n\right)^{-1/2},$$

$\psi \in W^{1,p}(\nu)$ for every $p < q$, and $M_p\psi = \|Mg\|^{-2}\sum_{k=1}^\infty M_q(\langle Mg, e_k\rangle)e_k$. Using Lemma 5.2(ii), we obtain $F \in W^{1,t}(X,\nu;X)$ if $1/t = 1/p + 1/q < 1$, so that $F \in W^{1,t}(X,\nu;X)$ for $t \in (1, q/2)$. To avoid this restriction we use the definition of the spaces $W^{1,t}(X,\nu;X)$ instead of Lemma 5.2, and we take advantage of $\|Mg\|^{-1} \in L^p(X,\nu)$ for every p which is a consequence of Hypothesis 1.3 (see the Introduction). Setting $f_i := \langle F, e_i\rangle = \langle Mg, e_i\rangle/\|Mg\|$ for $i \in \mathbb{N}$, each f_i belongs to $W^{1,p}(\nu)$ for $p \in (1,q)$ and

$$M_p f_i = \frac{M_q\langle Mg, e_i\rangle}{\|Mg\|} - \langle Mg, e_i\rangle \frac{\sum_{k=1}^\infty (M_q\langle Mg, e_k\rangle)\, e_k}{\|Mg\|^2}$$

so that

$$\|M_p f_i\| \le \frac{\|M_q\langle Mg, e_i\rangle\|}{\|Mg\|} + \frac{|\langle Mg, e_i\rangle|}{\|Mg\|^2}\left(\sum_{k=1}^\infty \|M_q\langle Mg, e_k\rangle\|^2\right)^{1/2},$$

which implies that

$$\left(\sum_{i=1}^\infty \|M_p f_i\|^2\right)^{1/2} \le 2\frac{\left(\sum_{i=1}^\infty \|M_q\langle Mg, e_i\rangle\|^2\right)^{1/2}}{\|Mg\|}.$$

Therefore, $F \in W^{1,t}(X,\nu;X)$, for every $t < q$.

Step 2. Now we show that if $F \in W^{1,t}(X,\nu;X) \cap D(M_s^*)$ for some $t > q'$, $s > q'$ is such that $\|F(x)\| \le 1$ ν-a.e., then we have

$$\int_{\{g<r\}} M_s^*(\varphi F)\, d\nu \le q_{\|Mg\|\varphi}(r). \tag{5.4}$$

To this aim, we prove that $\langle Mg, F\rangle \in W^{1,p}(\nu)$ for some $p > 1$.

Set $f_i(x) := \langle F(x), e_i\rangle$ for $i \in \mathbb{N}$, $x \in X$. Since $|\langle Mg, F\rangle| \le \|Mg\|$, $\langle Mg, F\rangle \in L^q(X,\nu)$, and the series $s_n = \sum_{i=1}^n f_i \langle Mg, e_i\rangle$ converges to $\langle Mg, F\rangle$ in $L^q(X,\nu)$. Let us prove that it converges in a Sobolev space. For every $i \in \mathbb{N}$, we have $\langle Mg, e_i\rangle \in W^{1,q}(\nu)$, $f_i \in W^{1,t}(\nu)$, and $t > q'$, so that $\langle Mg, e_i\rangle f_i \in W^{1,p}(\nu)$ with $p = qt/(q+t)$ by Lemma 2.1(i). Moreover,

$$M_p s_n = \sum_{i=1}^n f_i M_q \langle Mg, e_i\rangle + \sum_{i=1}^n \langle Mg, e_i\rangle M_t f_i,$$

so that

$$\|M_p s_n\| \le \left(\sum_{i=1}^n \|M_q\langle Mg, e_i\rangle\|^2\right)^{1/2} + \left(\sum_{i=1}^n \|M_t f_i\|^2\right)^{1/2} \|Mg\|,$$

and the series $(M_p s_n)$ converges in $L^p(X,\nu;X)$. Therefore, $\langle Mg, F\rangle\varphi \in W^{1,p}(\nu)$ for every $\varphi \in C_b^1(X)$, and Proposition 3.9 yields

$$\int_{\{g<r\}} M_s^*(\varphi F)\, d\nu = -q_{\langle Mg,F\rangle\varphi}.$$

We recall now that if $\varphi_1 \le \varphi_2$ a.e., then $q_{\varphi_1}(r) \le q_{\varphi_2}(r)$, for every r. In our case, φ has nonnegative values, so that

$$\langle Mg(x), F(x)\rangle\varphi(x) = \langle Mg(x)/\|Mg(x)\|, F(x)\rangle\varphi(x)\|Mg(x)\| \le \varphi(x)\|Mg(x)\|$$

for a.e. x, and therefore $q_{\langle Mg,F\rangle\varphi}(r) \le q_{\|Mg\|\varphi}(r)$ and (5.4) follows. □

Corollary 5.4. *Let g_1, g_2 satisfy the hypotheses of Theorem* 5.3*, and assume that for some $r \in \mathbb{R}$ we have $\{x : g_1(x) < r\} = \{x : g_2(x) < r\}$. For $i = 1, 2$, denote by $T_i \in \mathcal{L}(W^{1,p}(\nu), L^1(X, \sigma_r^{g_i}))$ the trace operator introduced in Theorem* 4.4*. Then the weighted measures $T_1(\|Mg_1\|)\, d\sigma_r^{g_1}$ and $T_2(\|Mg_2\|)\, d\sigma_r^{g_2}$ coincide.*

Proof. Set $\Omega := \{x : g_1(x) < r\} = \{x : g_2(x) < r\}$. By Theorem 4.4, for every $\varphi \in C_b^1(X)$,

$$q_{\|Mg_i\|\varphi}(r) = \int_X T_i(\varphi\|Mg_i\|)\, d\sigma_r^{g_i} = \int_X \varphi T_i(\|Mg_i\|)\, d\sigma_r^{g_i}, \quad i = 1, 2.$$

If in addition φ has nonnegative values, by Theorem 5.3 the left-hand side depends only on the set Ω. Approximating every nonnegative $\varphi \in UC_b(X)$ by a sequence of nonnegative C_b^1 functions, we obtain $\int_X \varphi T_1(\|Mg_1\|)\, d\sigma_r^{g_1} = \int_X \varphi T_2(\|Mg_2\|)\, d\sigma_r^{g_2}$; splitting every $\varphi \in UC_b(X)$ as $\varphi = \varphi^+ - \varphi^-$ we obtain $\int_X \varphi T_1(\|Mg_1\|)\, d\sigma_r^{g_1} = \int_X \varphi T_2(\|Mg_2\|)\, d\sigma_r^{g_2}$. The statement follows. □

Fix any g satisfying the assumptions of Proposition 5.3, and define

$$\rho_r(dx) := T(\|Mg\|)\sigma_r^g(dx). \tag{5.5}$$

Taking in particular $\varphi \equiv 1$, we get

$$\rho_r(g^{-1}(r)) = \sup\left\{ \int_{\{g<r\}} M_s^*(F)\, d\nu :\ F \in W^{1,t}(X) \cap D(M_s^*),\ \|F(x)\| \le 1 \text{ a.e.} \right\} < +\infty.$$

We recall that a bounded Borel set $\Omega \subset \mathbb{R}^n$ has finite perimeter if $\mathbb{1}_\Omega$ is a function with bounded variation, and in this case the perimeter measure m is defined as the total variation measure of $D\mathbb{1}_\Omega$. Equivalently, Ω has finite perimeter if and only if

$$\sup\left\{ \int_\Omega \text{div}\, F\, dx :\ F \in C_c^1(\Omega, \mathbb{R}^n),\ \|F(x)\| \le 1\, \forall x \in \Omega \right\} < +\infty,$$

and in this case for every $\varphi \in C_b^1(\mathbb{R}^n)$ with nonnegative values we have

$$\int \varphi\, dm = \sup\left\{ \int_\Omega \text{div}\, (F\varphi)\, dx :\ F \in C_c^1(\Omega, \mathbb{R}^n),\ \|F(x)\| \le 1\, \forall x \in \Omega \right\},$$

to be compared to formula (5.3). In our setting the operators $-M_s^*$ play the role of the divergence, the measure ρ_r plays the role of the perimeter measure, and $\rho_r(g^{-1}(r))$ may be called the (generalized) perimeter of the set $g^{-1}(-\infty, r)$. The vector field $Mg/\|Mg\|$ plays the role of the exterior normal vector field at $g^{-1}(r)$. It would be worth it (although it is not the aim of this paper) to develop a theory of BV functions for general differentiable measures in Hilbert or Banach spaces and to go on in the investigation of perimeter measures.

6. Gaussian and weighted Gaussian measures

6.1. **Gaussian measures: Comparison with previous results.** We refer to [Bo98] for the general theory of Gaussian measures in Banach spaces. All the results that we mention here about Sobolev spaces for general Gaussian measures are contained in Chapter 5 of [Bo98].

Let $Q \in \mathcal{L}(X)$ be a self-adjoint positive trace class operator, and let $\mu := N_{0,Q}$ be the Gaussian measure in X with mean 0 and covariance Q. Choosing $R = Q^{1/2}$, the spaces $W^{1,p}(\mu)$ used here coincide with the spaces $\mathbb{D}^{1,p}(X, \mu)$ of the standard Gaussian measure theory. To prove this fact, we recall that the Cameron–Martin space H is equal to $Q^{1/2}(X)$, with the scalar product

$$\langle h, k\rangle_H = \langle Q^{-1/2}h, Q^{-1/2}k\rangle, \quad h, k \in H. \tag{6.1}$$

For every $f \in C_b^1(X)$ and $x \in X$, the H-gradient of f at x, denoted by $\nabla_H f(x)$, is the unique $y \in H$ such that $\lim_{\|h\|_H \to 0}(f(x+h) - f(x) - \langle y, h\rangle_H)/\|h\|_H = 0$. Therefore, it is given by $\nabla_H f(x) = Q\nabla f(x)$. The space $\mathbb{D}^{1,p}(X, \mu)$ is the domain of the closure $\overline{\nabla}_H$ of $\nabla_H : D(\nabla_H) = C_b^1(X) \subset L^p(X, \mu) \mapsto L^p(X, \mu; H)$. Namely, an element $f \in L^p(X, \mu)$ belongs to $\mathbb{D}^{1,p}(X, \mu)$ if and only if there exists a sequence $(f_n) \subset C_b^1(X)$ such that $f_n \to f$ in $L^p(X, \mu)$ and $(\nabla_H f_n)$ is a Cauchy sequence in $L^p(X, \mu; H)$; in this case $\overline{\nabla}_H f = \lim_{n\to\infty} \nabla_H f_n$ in $L^p(X, \mu; H)$. Since $\|\nabla_H(f_n - f_m)\|_H = \|Q^{1/2}\nabla(f_n - f_m)\|$, $f \in \mathbb{D}^{1,p}(X, \mu)$ if and only if $f \in W^{1,p}(\mu)$, and in this case

$$M_p f = Q^{-1/2}\overline{\nabla}_H f. \tag{6.2}$$

We recall the definition of the Gaussian divergence of H-valued vector fields. For a given $\Phi \in L^1(X, \mu; H)$, a function $\beta \in L^1(X, \mu)$ is called Gaussian divergence of

Φ, and is denoted by $\mathrm{div}_\mu \Phi$, if

$$\int_X \langle \nabla_H \varphi, \Phi\rangle_H d\mu = -\int_X \varphi\, \beta\, d\mu, \quad \varphi \in C^1_b(X).$$

Recalling that $\nabla_H \varphi = Q\nabla\varphi$ for every $\varphi \in C^1_b(X)$ and using formula (6.1), this means that

$$\int_X \langle Q^{1/2}\nabla\varphi, Q^{-1/2}\Phi\rangle d\mu = -\int_X \varphi\, \beta\, d\mu, \quad \varphi \in C^1_b(X).$$

So, a vector field $\Phi \in L^{p'}(X,\mu;H)$ (namely, such that $\widetilde{\Phi} := Q^{-1/2}\Phi \in L^{p'}(X,\mu;X)$) has Gaussian divergence $\mathrm{div}_\mu\Phi \in L^{p'}(X,\mu)$ if and only if $\widetilde{\Phi}$ belongs to $D(M_p^*)$, and in this case

$$M_p^* \widetilde{\Phi} = -\mathrm{div}_\nu \Phi. \tag{6.3}$$

In the paper [DaLuTu14] surface measures have been constructed on the level surfaces of a suitable version of a Sobolev function $g \in \mathbb{D}^{1,p}(X,\mu)$ for some $p > 1$, such that the vector field $\overline{\nabla}_H g/\|\overline{\nabla}_H g\|^2_H$ belongs to $\mathbb{D}^{1,p}(X,\mu;H)$ (the version is chosen in terms of Gaussian capacities, as we did in Section 4). Since $\mathbb{D}^{1,p}(X,\mu;H)$ is contained in the domain of div_μ in $L^p(X,\mu)$ by, e.g., [Bo98, Prop. 5.8.8], the latter condition implies that $\Psi := Q^{-1/2}\overline{\nabla}_H g/\|\overline{\nabla}_H g\|^2_H = M_p g/\|M_p g\|^2$ belongs to $D(M^*_{p'})$. This is precisely the condition of Hypothesis 1.3.

The construction of the surface measures of the present paper follows the procedure of [DaLuTu14]. Therefore the surface measures of the present paper coincide with the ones of [DaLuTu14]. In particular, as proved in [DaLuTu14, Prop. 3.15], if the conditions

$$g \in \mathbb{D}^{2,p}(X,\nu), \quad \frac{1}{\|\overline{\nabla}_H g\|^2_H} \in \bigcap_{p>1} L^p(X,\nu) \tag{6.4}$$

hold, all the assumptions of [DaLuTu14] and of the present paper are satisfied, and the measures ρ_r coincide with the restrictions to $\{g^* = r\}$ of the Gauss–Hausdorff surface measure ρ of Feyel and de La Pradelle [FePr92]. This implies that the traces of Sobolev functions of Definition 4.6 coincide with the traces studied in [CeLu14]. Therefore, the "divergence theorem formula" (Remark 4.9(iii) of [CeLu14]) coincides with our formula (4.7).

Of course the proofs of [DaLuTu14] that made use of specific properties of Gaussian measures could not be adapted to the present context. In particular, we gave completely different proofs of the fact that the support of ρ_r is contained in $\{g^* = r\}$ (that worked only for continuous g in [DaLuTu14]), of the fact that ρ_r is not trivial if and only if $r \in (\operatorname{ess\,inf} g, \operatorname{ess\,sup} g)$, and of the independence of the measures ρ_r on the defining function g.

Another remark about the notation in the literature is worth noting. For every $h \in H$, $h = Q^{1/2}z$, we have

(6.5)
$$\int_X \langle Q^{1/2}\nabla\varphi(x), z\rangle \mu(dx) = \int_X \partial_h\varphi(x)\mu(dx) = \int_X \varphi(x)\hat{h}(x)\mu(dx), \quad \varphi \in C^1_b(X),$$

where $\hat{h} = R_\mu^{-1}h$, R_μ being the usual extension of Q to the closure of X^* in $L^2(X,\mu)$. In our setting the function $\hat{h}$ is called v_z (see formula (1.5)); in [DP06] it is called the "white noise function" and is denoted by W_z.

Eventually, let us consider Theorem 5.3. The space $C_b^1(X;H)$ is dense in $\mathbb{D}^{1,p}(X,\nu;H)$ for every p. A given vector field $F: X \mapsto X$ belongs to $W^{1,t}(\mu) \cap D(M_s^*)$ iff $\Phi = Q^{1/2}F$ belongs to $\mathbb{D}^{1,t}(X,\mu;H)$ and to the domain of div_μ in $L^{s'}(X,\mu)$. Since $C_b^1(X;H)$ is dense in $\mathbb{D}^{1,q}(X,\nu;H)$ for every $q>1$, it is dense in the intersection between $\mathbb{D}^{1,t}(X,\nu;H)$ and the domain of div_μ in $L^{s'}(X,\mu)$. Therefore, the maximum in the right-hand side of (5.3) is equal to

$$\sup\left\{\int_{\{g<r\}} -\mathrm{div}_\mu(\varphi\Phi)\,d\nu:\ \Phi \in C_b^1(X;H),\ \|\Phi(x)\|_H \le 1\ \forall x \in H\right\}.$$

In particular, taking $\varphi \equiv 1$ in (5.3) we get

$$\rho_r(X) = \sup\left\{\int_{\{g<r\}} \mathrm{div}_\mu\Phi\,d\nu:\ \Phi \in C_b^1(X;H),\ \|\Phi(x)\|_H \le 1\ \forall x\right\},$$

which shows that the perimeter of the set $\{g<r\}$ (according to [AMMP10]) is equal to $\rho_r(X) = \rho_r(\{g^*=r\})$. In fact under assumption (6.4) it was proved in [CeLu14] that the perimeter measure relevant to the set $\{g<r\}$ coincides with the restriction of the Gauss–Hausdorff measure ρ to $\{g^*=r\}$, and the latter coincides with our ρ_r as we already remarked.

6.2. **Weighted Gaussian measures.** We refer to paper [Fe16], where weighted Gaussian measures in Banach spaces were studied. Let $\nu(dx) = w(x)\mu(dx)$, where $\mu = N_{0,Q}$ is a centered nondegenerate Gaussian measure with covariance Q. The nonnegative weight w satisfies

$$w,\ \log w \in W^{1,s}(X,\mu)\ \ \forall s>1. \tag{6.6}$$

Of course, every C^1 weight with positive infimum and such that $w(x)$, $\|\nabla w(x)\| \le C\exp(\alpha\|x\|)$ for some C, $\alpha>0$ satisfies assumption (6.6). Examples of discontinuous weights that satisfy (6.6) are in [Fe16] (in the space $X=\ell^2$) and in [DaLu14] (in the space $X = L^2(0,1)$ with respect to the Lebesgue measure).

Since we are considering two different measures, μ and ν, it is convenient to denote by M_p^μ, M_p^ν the operators obtained by our procedure using the measures μ, ν, respectively. Instead, we consider only the covariance of μ, and we denote it by Q without superindex.

The Sobolev spaces considered in [Fe16] are the spaces $\mathbb{D}^{1,p}(X,\mu)$ that coincide with our spaces $W^{1,p}(\mu)$ with the choice $R = Q^{1/2}$. See subsection 6.1.

It is convenient to introduce an orthonormal basis of X consisting of eigenvectors of Q, $Qe_k = \mu_k e_k$ for every $k \in \mathbb{N}$. For every $z \in X$, setting $h = Q^{1/2}z$, formula (6.5) holds, and the function $\hat{h}$ is rewritten as

$$\hat{h}(x) = \sum_{k=1}^\infty \mu_k^{-1/2}\langle x, e_k\rangle\langle z, e_k\rangle, \tag{6.7}$$

the series being convergent in $L^p(X,\mu)$ for every $p>1$. Formula (6.5) is readily extended to any $\varphi \in W^{1,q}(\mu)$, with $q>1$.

Now, let us consider the weighted measure ν. For $\varphi \in C_b^1(X)$, applying (6.5) to φw which belongs to $W^{1,q}(\mu)$ for every $q>1$, we get

$$\int_X \langle Q^{1/2}\nabla\varphi(x), z\rangle\nu(dx) = \int_X \partial_h\varphi(x)\nu(dx) = \int_X \varphi(x)(\hat{h}(x) - \partial_h \log w(x))\nu(dx),$$
$$\varphi \in C_b^1(X).$$

By the Hölder inequality, $\hat{h} - \partial_h \log w \in L^q(X,\nu)$ for every $q > 1$, and applying once again the Hölder inequality we obtain that Hypothesis 1.2 is satisfied. Then, we consider the Sobolev spaces $W^{1,p}(\nu)$ defined in the Introduction, still with $R = Q^{1/2}$. They coincide with the Sobolev spaces $W^{1,p}(\nu)$ of [Fe16]. We remark that the test functions taken into consideration in [Fe16] are the smooth cylindrical functions $\mathcal{F}C_b^\infty(X)$, namely functions of the type $\varphi(x) = \theta(\langle x, v_1\rangle, \dots, \langle x, v_n\rangle)$ with $n \in \mathbb{N}$, $\theta \in C_b^\infty(\mathbb{R}^n)$, $v_1, \dots, v_k \in X$, instead of $C_b^1(X)$ as we did. However, in the basic definitions and estimates nothing changes if we replace $\mathcal{F}C_b^\infty(X)$ by $C_b^1(X)$.

The hypersurfaces considered in [Fe16] are level surfaces of functions g whose regularity and summability properties are given in terms of the Gaussian measure μ. Namely, as in [Fe01, CeLu14], $g \in \mathbb{D}^{2,p}(X,\mu)$ for every $p > 1$, and there exists $\delta > 0$ such that $1/|\overline{\nabla}_H g|_H \in L^p(g^{-1}(-\delta,\delta),\mu)$ for every $p > 1$. Here we assume for simplicity that g satisfies (6.4), so that $1/\|M^\mu g\| \in L^p(X,\mu)$ for every p, which means that $1/|\overline{\nabla}_H g|_H \in L^p(X,\mu)$ for every p. Now we prove that, under these assumptions, g satisfies Hypothesis 1.3.

Lemma 6.1. *Let g satisfy* (6.4)*. Then g satisfies Hypothesis* 1.3 *for both measures μ and ν.*

Proof. The assumption $g \in \mathbb{D}^{2,p}(X,\mu)$ is equivalent to $\overline{\nabla}_H g \in \mathbb{D}^{1,p}(X,\mu;H)$, for every $p > 1$. It follows that $\overline{\nabla}_H g/|\overline{\nabla}_H g|_H^2 \in \mathbb{D}^{1,p}(X,\mu;H)$, for every $p > 1$. Every vector field $\Phi \in \mathbb{D}^{1,p}(X,\mu;H)$ with $p > 1$ has Gaussian divergence $\mathrm{div}_\mu\Phi \in L^p(X,\mu)$, by [Bo98, Prop. 5.8.8]. By the considerations of Subsection 6.1, $\Psi = Q^{-1/2}\overline{\nabla}_H g/|\overline{\nabla}_H g|_H^2$ belongs to the domain of $M_p^{\mu *}$, for every $p > 1$. On the other hand, $Q^{-1/2}\overline{\nabla}_H g/|\overline{\nabla}_H g|_H^2 = Mg/\|Mg\|^2$. Then, g satisfies Hypothesis 1.3 for the measure μ.

Concerning the weighted measure ν, again we have to compare the divergence operator with our operators $M_p^{\nu *}$. The divergence operator is defined in [Fe16] as in the Gaussian case for vector fields $\Phi \in L^1(X,\nu;X)$. A function $\beta \in L^1(X,\nu)$ is called divergence of Φ and is denoted by $\mathrm{div}_\nu\Phi$ if

$$\int_X \langle \nabla f(x), \Phi(x)\rangle\, \nu(dx) = -\int_X f(x)\beta(x)\nu(dx), \quad f \in C_b^1(X).$$

If Φ has values in the Cameron–Martin space $Q^{1/2}(X)$, the above formula reads as

$$\int_X \langle Q^{1/2}\nabla f(x), Q^{-1/2}\Phi(x)\rangle\, \nu(dx) = -\int_X f(x)\beta(x)\nu(dx), \quad f \in C_b^1(X). \tag{6.8}$$

If $\widetilde{\Phi} := Q^{-1/2}\Phi \in L^{p'}(X,\nu;X)$ and $\beta \in L^{p'}(X,\nu)$, (6.8) means that $\widetilde{\Phi} \in D(M_p^{\nu *})$ and $M_p^{\nu *}\widetilde{\Phi} = -\beta$. Conversely, if a vector field $\widetilde{\Phi}$ belongs to $D(M_p^{\nu *})$, then $\Phi := Q^{1/2}\widetilde{\Phi}$ has divergence in the sense of [Fe16], given by $\mathrm{div}_\nu\Phi = -M_p^{\nu *}\widetilde{\Phi}$. Taking this equivalence into account, we use Proposition 5.5 of [Fe16], which states that any vector field $\Phi \in \mathbb{D}^{1,q}(X,\mu;H)$ has divergence $\mathrm{div}_\nu\Phi$ belonging to $L^r(X,\nu)$ for every $r < q$. In our case, $\Phi = \overline{\nabla}_H g/|\overline{\nabla}_H g|_H^2$ belongs to $\mathbb{D}^{1,q}(X,\mu;H)$ for every q, so that $\mathrm{div}_\nu\Phi$ belongs to $L^q(X,\nu)$ for every q. Moreover, by the Hölder inequality $\widetilde{\Phi} = Q^{-1/2}\Phi$ is in $L^{p'}(X,\nu;X)$ for every $p' > 1$. This implies that $\widetilde{\Phi}$ belongs to $D(M_p^{\nu *})$ for every p; namely, Hypothesis 1.3 holds for the measure ν. □

The weighted surface measure considered in [Fe16] is $w^*\rho$, where w^* is any C_p-quasicontinuous version of w, in the sense of the Gaussian capacity, and ρ is the

Gauss–Hausdorff measure of Feyel and de La Pradelle. Here we identify our surface measures ρ_r with $w^*\rho$ on every surface level $g^{*-1}(r)$.

Proposition 6.2. *Under the assumptions of Lemma* 6.1*, for every* $r \in \mathbb{R}$ *we have*

$$\int_X \varphi T(\|M^\nu g\|)\, d\sigma_r^g = \int_X \varphi\, d\rho_r = \int_{g^{*-1}(r)} \varphi w^* d\rho, \quad \varphi \in C_b(X). \tag{6.9}$$

Proof. Since any finite Borel measure is uniquely determined by its Fourier transform, it is sufficient to show that (6.9) holds for every $\varphi \in C_b^1(X)$. Theorem 1.3 of [Fe16] yields, for every $\Phi \in W^{1,p}(X,\nu;H)$,

$$\int_{\{g<r\}} \mathrm{div}_\nu(\varphi\Phi) d\nu = \int_{g^{*-1}(r)} \varphi \langle \mathrm{Tr}\,\Phi, \mathrm{Tr}\Big(\frac{\overline{\nabla}_H g}{|\overline{\nabla}_H g|_H}\Big)\rangle_H w^* d\rho, \tag{6.10}$$

where Tr is the trace operator considered in [Fe16]. There, traces $\mathrm{Tr}\,\varphi$ of Sobolev functions φ are defined as in the present paper, with the surface measure $w^*\rho$ replacing σ_r^g. Traces of vector fields $\Phi \in W^{1,p}(X,\nu;H)$ are defined in a natural way; namely, setting $\varphi_n(x) = \langle \Phi(x), h_n\rangle_H$, where $\{h_n : n \in \mathbb{N}\}$ is any orthonormal basis of H, then $\mathrm{Tr}\,\Phi = \sum_{n=1}^\infty \mathrm{Tr}(\varphi_n)h_n$.

Taking in particular $\Phi = \overline{\nabla}_H g/|\overline{\nabla}_H g|_H$ that belongs to $W^{1,p}(X,\nu;H)$ for every $p > 1$, we have $|\mathrm{Tr}\ \Phi|_H^2 \equiv 1$ on $g^{*-1}(r)$, and the right-hand side of (6.10) is equal to

$$\int_{g^{*-1}(r)} \varphi\, w^* d\rho.$$

Recalling that $\mathrm{div}_\nu(\varphi\Phi) = -M_p^{\nu *}(\varphi M^\nu g/\|M^\nu g\|)$, the left-hand side is equal to

$$-\int_{\{g<r\}} M_p^{\nu *}\Big(\varphi \frac{M^\nu g}{\|M^\nu g\|}\Big) d\nu,$$

which coincides with $\int_X \varphi T(\|M^\nu g\|) d\sigma_r^g$ by (5.2). ☐

Since the assumptions on g are the same as in [CeLu14, Fe16], the examples exhibited in these papers fit here. In particular, functions such as

$$g(x) = \sum_{k=1}^\infty \alpha_k \langle x - x_0, e_k\rangle^2$$

with $\alpha_k \geq 0$ for every k, not eventually vanishing, and $\sum_{k=1}^\infty \alpha_k\mu_k < \infty$ satisfy the assumptions of Lemma 6.1. Therefore, the theory may be applied to spherical surfaces and surfaces of suitable ellipsoids. The elements of the dual space $g(x) = \langle x, v\rangle$ obviously satisfy the assumptions of Lemma 6.1, so that the theory may be applied to hyperplanes. The hyperplane $\{x : \langle x, v\rangle = r\}$, with $v \in X \setminus \{0\}$, may be seen as the graph of the function $\varphi : \mathrm{span}\ \{e_k : k \neq h\} \mapsto \mathbb{R}$, $\varphi(\widetilde{x}) = (r - \sum_{k\neq h} \widetilde{x}_k v_k)/v_h$, if $v_h \neq 0$. A generalization to graphs of other functions is in [CeLu14].

When formula (6.9) holds, Proposition 3.8 is not needed. Since ρ_r coincides with the restriction of $w^*\rho$ to $g^{*-1}(r)$, for ρ_r to be nontrivial it is sufficient that $w^*(r) \neq 0$ and that $\rho(g^{*-1}(r)) \neq 0$. Under the assumptions of Lemma 6.1, the latter condition holds iff $r \in (\mathrm{ess\,inf}\, g, \mathrm{ess\,sup}\, g)$ by [DaLuTu14, Lemma 3.9, Prop. 3.15].

7. A family of non-Gaussian product measures

For any $\mu > 0$, $m \geq 1$, we define the probability measure on $\mathbb{R}$,

$$\nu_{m,\mu}(d\xi) := a_m\, \mu^{-\frac{1}{2m}}\, e^{-\frac{|\xi|^{2m}}{2m\mu}}\, d\xi, \quad \xi \in \mathbb{R}, \tag{7.1}$$

where a_m is a normalization constant such that $\nu_{m,\mu}(\mathbb{R}) = 1$,

$$a_m = \frac{(2m)^{1-\frac{1}{2m}}}{2\,\Gamma(\frac{1}{2m})}.$$

For every $N > 0$ we have

$$\int_{\mathbb{R}} |\xi|^{2N} \nu_{m,\mu}(d\xi) = a_m \mu^{-\frac{1}{2m}} \int_{\mathbb{R}} |\xi|^{2N} e^{-\frac{|\xi|^{2m}}{2m\mu}}\, d\xi =: b_{m,N}\mu^{N/m}, \tag{7.2}$$

where

$$b_{m,N} = a_m \int_{\mathbb{R}} |\tau|^{N/m} e^{-\frac{|\tau|^{2m}}{2m}}\, d\tau = (2m)^{\frac{N}{m}}\, \frac{\Gamma(\frac{2N+1}{2m})}{\Gamma(\frac{1}{2m})}.$$

The measure $\nu_{m,\mu}$ has mean 0 and covariance $b_{m,1}\mu^{\frac{1}{m}}$. The following integration by parts formula holds:

$$\int_{\mathbb{R}} \varphi'(\xi)\, \nu_{m,\mu}(d\xi) = \frac{1}{\mu} \int_{\mathbb{R}} |\xi|^{2m-2}\xi\varphi(\xi)\, \nu_{m,\mu}(d\xi), \quad \varphi \in C_b^1(\mathbb{R}). \tag{7.3}$$

Next, we define a product measure on $\mathbb{R}^\infty$, the space of all sequences of real numbers endowed with the product topology, associated to the distance $d(x, y) = \sum_{n=1}^\infty 2^{-n}|x_n - y_n|(1 + |x_n - y_n|)^{-1}$. We set

$$\nu_m = \prod_{h=1}^\infty \nu_{m,\mu_h}, \tag{7.4}$$

where the sequence of positive numbers (μ_h) is chosen such as

$$\Lambda_m := \sum_{h=1}^\infty \mu_h^{\frac{1}{m}} < \infty. \tag{7.5}$$

As usual, we denote by ℓ^2 the space of all sequences (x_h) of real numbers such that $\sum_{h=1}^\infty x_h^2 < \infty$, endowed with the scalar product

$$\langle x, y\rangle = \sum_{h=1}^\infty x_h y_h, \quad x, y \in \ell^2.$$

One checks easily that ℓ^2 is a Borel set in $\mathbb{R}^\infty$ and that ν is concentrated on ℓ^2 because, in view of (7.2),

$$\int_{\mathbb{R}^\infty} |x|_{\ell_2}^2\, \nu(dx) = \sum_{h=1}^\infty \int_{\mathbb{R}} x_h^2 \nu_{m,\mu_h}(dx_h) = b_{m,1} \sum_{k=1}^\infty \mu_h^{\frac{1}{m}} < \infty.$$

So, from now on we may forget $\mathbb{R}^\infty$ and consider only ℓ^2, identifying it with X through the mapping $x \mapsto (x_h)$, where $x_h = \langle x, e_h\rangle$ and $\{e_h : h \in \mathbb{N}\}$ is any fixed orthonormal basis of X.

Remark 7.1. It is possible to show ν_m as an invariant measure of a transition semigroup P_t, $t \geq 0$, on X. Precisely, we consider the family of ordinary stochastic differential equations, indexed by $h \in \mathbb{N}$,

$$\begin{cases} dX_h = -\frac{1}{2\mu_h}|X_h|^{2m-2}X_h dt + dW_h(t), \\ \\ X_h(0) = x_h \in \mathbb{R}, \end{cases} \tag{7.6}$$

where (W_h) is a sequence of real mutually independent Brownian motions defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each equation has a unique solution $X_h(t, x_h)$. Setting

$$X(t,x) := \sum_{h=1}^{\infty} X_h(t,x_h)e_h, \quad t \geq 0,\ x \in X,$$

one can show that $X(t, x)$, $t \geq 0$, is a stochastic process in X. Defining the corresponding transition semigroup by

$$P_t\varphi(x) := \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in C_b(X),$$

it is not difficult to verify that ν_m is an invariant measure of P_t.

One can check easily that ν has mean 0 and that it possesses finite moments of any order. The covariance Q of ν is given by

$$Qe_h = b_{m,1}\,\mu_h^{\frac{1}{m}}\,e_h, \quad h \in \mathbb{N}. \tag{7.7}$$

Notice that if $m = 1$, then ν_1 is the Gaussian measure $N_{0,Q}$. In this case $Qe_h = \mu_h\, e_h$, for all $h \in \mathbb{N}$, and for all $\varphi \in C_b^1(X)$, $z \in Q^{1/2}(X)$ the classical integration formula (6.5) holds.

We are going to generalize formula (6.5) to any ν_m with $m \geq 1$.

Proposition 7.2. *Let $m \geq 1$, $\varphi \in C_b^1(X)$, $z \in X$. Then*

$$\int_X \langle Q^{\frac{1}{2}}\nabla\varphi(x), z\rangle\,\nu_m(dx) = \int_X v_z^m(x)\,\varphi(x)\,\nu_m(dx), \tag{7.8}$$

where

$$v_z^m(x) := b_{m,1}^{1/2}\sum_{h=1}^{\infty}\mu_h^{\frac{1}{2m}-1}\,|x_h|^{2m-2}x_h z_h, \tag{7.9}$$

the series being convergent in $L^p(X,\nu_m)$ for every $p \in (1,+\infty)$. Consequently, Hypothesis 1.2 *is satisfied, with $R = Q^{1/2}$ and $C_{p,z} = \|v_z^m\|_{L^{p'}(X,\nu)}$.*

Proof. As a first step, we prove that for every $\varphi \in C_b^1(X)$, $h \in \mathbb{N}$ we have

$$\int_X \frac{\partial\varphi}{\partial e_h}(x)\,\nu_m(dx) = \mu_h^{\frac{1}{2m}-1}\int_X |x_h|^{2m-2}x_h\varphi(x)\,\nu_m(dx). \tag{7.10}$$

To this aim we approach φ by a sequence of cylindrical functions, $\varphi_n(x) := \varphi(P_n x)$, where P_n is the orthogonal projection

$$P_n(x) = \sum_{k=1}^{n}\langle x, e_k\rangle e_k.$$

The sequence (φ_n) converges to φ in $W^{1,p}(\nu_m)$ for every $p \in (1,+\infty)$. Indeed, it converges in $L^p(X,\nu_m)$ by the Dominated Convergence Theorem, and moreover

$$Q^{1/2}\nabla\varphi_n(x) = Q^{1/2}P_n\nabla\varphi(P_n x), \quad n \in \mathbb{N},$$

so that

$$
\begin{aligned}
&\|Q^{1/2}\nabla\varphi_n - Q^{1/2}\nabla\varphi\|_{L^p(X,\nu_m;X)} \\
&\quad\leq \bigg(\int_X \|Q^{1/2}(P_n\nabla\varphi(P_nx) - P_n\nabla\varphi(x))\|^p\,\nu_m(dx)\bigg)^{1/p} \\
&\qquad + \bigg(\int_X \|Q^{1/2}(P_n\nabla\varphi(x) - \nabla\varphi(x))\|^p\,\nu_m(dx)\bigg)^{1/p} \\
&\quad\leq \|Q^{1/2}\|_{\mathcal{L}(X)}\bigg(\int_X \|\nabla\varphi(P_nx) - \nabla\varphi(x)\|^p\bigg)^{1/p} \\
&\qquad + \|Q^{1/2}\|_{\mathcal{L}(X)}\bigg(\int_X \|P_n\nabla\varphi(x) - \nabla\varphi(x)\|^p\,\nu_m(dx)\bigg)^{1/p},
\end{aligned}
$$

where both integrals in the right-hand side vanish as $n\to\infty$ by the Dominated Convergence Theorem.

So, it is enough to prove that (7.10) holds for cylindrical functions of the type $\varphi(x) = \widetilde{\varphi}(x_1,\dots,x_n)$ for some $\widetilde{\varphi}\in C^1_b(\mathbb{R}^n)$, $n\in\mathbb{N}$. For such functions,

$$
\int_X \frac{\partial\varphi}{\partial e_h}(x)\,\nu_m(dx) = \int_{\mathbb{R}^n}\frac{\partial\widetilde{\varphi}}{\partial\xi_h}\Pi_{k=1}^n\nu_{m,\mu_k}(d\xi),
$$

and (7.10) is an immediate consequence of (7.3).

Now let $\varphi\in C^1_b(X)$, $z\in X$. We have

$$
\begin{aligned}
\int_X \langle Q^{\frac12}\nabla\varphi(x), z\rangle\,\nu_m(dx) &= \lim_{n\to\infty}\int_X\sum_{h=1}^n b_{m,1}^{1/2}\mu_h^{1/2m}\frac{\partial\varphi}{\partial e_h}(x)z_h\,\nu_m(dx) \\
&= \lim_{n\to\infty} b_{m,1}^{1/2}\int_X\sum_{h=1}^n \mu_h^{\frac{1}{2m}-1}|x_h|^{2m-2}x_h\varphi(x)\,z_h\,\nu_m(dx).
\end{aligned}
$$

To conclude the proof it is enough to show that the series

$$
s_n(x) := \sum_{h=1}^n \mu_h^{\frac{1}{2m}-1}\,|x_h^{2m-2}|x_h\,z_h
$$

is convergent in $L^{2p}(X,\nu_m)$ for every $p\in\mathbb{N}$. Recalling that

$$
(a_1+\dots+a_n)^{2p} = \sum_{k_1,\dots,k_n\in\{0,\dots,2p\},\ \sum_{j=1}^n k_j=2p}\frac{(2p)!}{(k_1)!\cdot\ldots\cdot(k_n)!}a_1^{k_1}\cdot\ldots\cdot a_n^{k_n}
$$

for every l, $n\in\mathbb{N}$ we get

$$
(s_{l+n}(x) - s_l(x))^{2p}
$$

$$
= (2p)!\sum_{k_1,\dots,k_n\in\{0,\dots,2p\},\ \sum_{j=1}^n k_j=2p}\ \prod_{j=1}^n\frac{1}{(k_j)!}\mu_{l+j}^{(\frac{1}{2m}-1)k_j}\,|x_{l+j}|^{(2m-2)k_j}(x_{l+j}z_{l+j})^{k_j}.
$$

Integrating with respect to ν_m, the integrals of the terms with some odd k_j vanish. What remains are the integrals of the terms where all the $k_j = 2h_j$ are even, and recalling that

$$
\int_X x_{l+1}^{2(2m-1)h_1}\cdot\ldots\cdot x_{l+n}^{2(2m-1)h_n}\nu_m(dx) = \prod_{j=1}^n b_{m,(2m-1)h_j}\mu_{l+j}^{(2-1/m)h_j}
$$

we get

$$\int_X (s_{l+n}n(x) - s_l(x))^{2p} \nu_m(dx)$$

$$= \sum_{\substack{h_1,\dots,h_n \in \{0,\dots,p\}, \\ \sum_{j=1}^n h_j = p}} \frac{(2p)!}{(2h_1)! \cdot \ldots \cdot (2h_n)!} \int_X \prod_{j=1}^n \mu_{l+j}^{(\frac{1}{2m}-1)2h_j} |x_{l+j}|^{2(2m-1)h_j} z_{l+j}^{2h_j} \, d\nu_m$$

$$= \sum_{h_1,\dots,h_n \in \{0,\dots,p\}, \ \sum_{j=1}^n h_j = p} \frac{(2p)!}{(2h_1)! \cdot \ldots \cdot (2h_n)!} \prod_{j=1}^n b_{m,(2m-1)h_j} z_{l+j}^{2h_j}$$

$$\leq c_{m,p} \bigg(\sum_{j=1}^n z_{l+j}^2 \bigg)^p,$$

where $c_{m,p} = (\max\{b_{m,(2m-1)h} : \ h = 0,\dots,p\})^p$. So, (s_n) is a Cauchy series in $L^{2p}(X, \nu_m)$. □

Proposition 7.2 yields the following corollary.

Corollary 7.3. *Let $m \in \mathbb{N}$, and let* (7.5) *hold. For every $\varphi, \psi \in C_b^1(X)$, $z \in X$ we have*

$$\begin{aligned} \int_X \langle Q^{\frac12} \nabla \varphi(x), z \rangle \, \psi(x) \, \nu_m(dx) \quad &= \quad -\int_X \langle Q^{\frac12} \nabla \psi(x), z \rangle \, \varphi(x) \, \nu_m(dx) \\ &\quad + \int_X v_z^m(x) \, \varphi(x) \, \psi(x) \, \nu_m(dx). \end{aligned} \tag{7.11}$$

In particular,

$$\left| \int_X \langle Q^{\frac12} \nabla \varphi(x), z \rangle \, \nu_m(dx) \right| \leq \|\varphi\|_{L^p(X,\nu_m)} \|v_z^m\|_{L^{p'}(X,\nu_m)}.$$

Consequently, Hypothesis 1.1 is satisfied, and all the results of Section 2 hold.

According to the notation of Section 1, we denote by M_p the closure of $Q^{1/2}\nabla : C_b^1(X) \mapsto L^p(X, \nu_m; X)$ in $L^p(X, \nu_m)$ and by $W^{1,p}(\nu_m)$ the domain of M_p.

We shall show that our surface measures are well defined on hyperplanes and spherical surfaces. For simplicity, we consider only balls centered at the origin.

7.0.1. *Spherical surfaces.* Here we take $g(x) = \|x\|^2$, $x \in X$. Then g is smooth and $\{g < r\}$ is the open ball of center 0 and radius $\sqrt{r}$, for $r > 0$. In this case the vector field $Mg/\|Mg\|^2$ in Hypothesis 1.3 is given by

$$\Psi(x) = \frac{Q^{1/2}x}{2\|Q^{1/2}x\|^2}.$$

We have to prove that $\Psi \in D(M_p^*)$ for every $p > 1$. We approach it by the sequence of vector fields $S_n(x) = \sum_{h=1}^n \langle \Psi(x), e_h \rangle e_h$ that are sums of vector fields of the type considered in Lemma 2.6, with

$$f_h(x) = \langle \Psi(x), e_h \rangle = b_{m,1}^{1/2} \mu_h^{1/2m} x_k / 2\|Q^{1/2}x\|^2.$$

We use the following lemma.

Lemma 7.4.

(i) *The function* $x \mapsto \|Q^{1/2}x\|^{-1}$ *belongs to* $L^q(X,\nu_m)$ *for every* $q>1$.

(ii) *For every* $k\in\mathbb{N}$, *the function* $\varphi_k(x) := x_k/\|Q^{1/2}x\|^2$ *belongs to* $W^{1,q}(\nu_m)$ *for every* $q>1$, *and*

$$M_q\varphi_k = \sum_{h=1}^\infty b_{m,1}^{1/2}\mu_h^{1/2m}\left(\frac{\delta_{h,k}}{\|Q^{1/2}x\|^2} - \frac{b_{m,1}\mu_h^{1/m}x_hx_k}{(\|Q^{1/2}x\|^2)^2}\right)e_h. \tag{7.12}$$

Proof. The proof of statement (i) is the same as in the Gaussian case $m=1$ and it is left to the reader.

Let us prove statement (ii). We approach φ_k by the functions

$$\varphi_{k,n}(x) = \frac{x_k}{\|Q^{1/2}x\|^2+1/n},$$

which belong to $C_b^1(X)$ and which are easily seen to converge to φ_k in $L^q(X,\nu_m)$ for every $q>1$, taking (i) into account. Moreover we have

$$\langle Q^{1/2}\nabla\varphi_{k,n}(x), e_h\rangle = b_{m,1}^{1/2}\mu_h^{1/2m}\left(\frac{\delta_{h,k}}{\|Q^{1/2}x\|^2+1/n} - \frac{b_{m,1}\mu_h^{1/m}x_hx_k}{(\|Q^{1/2}x\|^2+1/n)^2}\right),\quad h\in\mathbb{N}.$$

Denoting by F the vector field in the right-hand side of (7.12) and using again (i), we see that $\lim_{n\to\infty}\|Q^{1/2}\nabla\varphi_{k,n} - F\| = 0$ in $L^q(X,\nu_m)$ for every $q>1$. Statement (ii) follows. □

Proposition 7.5. *The function* $g(x) = \|x\|^2$ *satisfies Hypothesis* 1.3, *and* $Mg \in W^{1,q}(X,\nu_m;X)$ *for every* $q>1$.

Proof. By Lemma 7.4 and Lemma 2.6, for every $k\in\mathbb{N}$ the vector field $f_k(x)e_k$ belongs to $D(M_p^*)$ for every $p>1$, and by (2.9) we have

$$M_p^*(f_ke_k) = -\frac{b_{m,1}\mu_k^{1/m}}{2}\left(\frac{1}{\|Q^{1/2}x\|^2} - 2b_{m,1}\frac{\mu_k^{1/m}x_k^2}{\|Q^{1/2}x\|^4}\right) + \frac{b_{m,1}}{2}\frac{\mu_k^{1/m-1}|x_k|^{2m}}{\|Q^{1/2}x\|^2}.$$

Therefore, the series $S_n(x) = \sum_{h=1}^n f_k(x)e_k$ converges pointwise to

$$\frac12\left(-\frac{\mathrm{Tr}\,Q}{\|Q^{1/2}x\|^2} + \frac{2\|Q^2x\|^2}{\|Q^{1/2}x\|^4}\right) + \frac{b_{m,1}}{2\|Q^{1/2}x\|^2}\sum_{k=1}^\infty \mu_k^{1/m-1}|x_k|^{2m}, \tag{7.13}$$

where the series $\sum_{k=1}^\infty \mu_k^{1/m-1}x_k^{2m}$ converges in $L^q(X,\nu_m)$ for every $q>1$, since $(\int_X |x_k|^{2mq}\nu_m(dx))^{1/q} = b_{m,mq}^{1/q}\mu_k$. By Lemma 7.4(i), $x\mapsto 1/\|Q^{1/2}x\|^2 \in L^s(X,\nu_m)$ for every $s>1$. Therefore, (S_n) converges to the right-hand side of (7.13) in $L^p(X,\nu_m)$ for every $p>1$. So, $\Psi\in D(M_p^*)$ and

$$M_p^*\Psi = \frac12\left(-\frac{\mathrm{Tr}\,Q}{\|Q^{1/2}x\|^2} + \frac{2\|Q^2x\|^2}{\|Q^{1/2}x\|^4}\right) + \frac{b_{m,1}}{2\|Q^{1/2}x\|^2}\sum_{k=1}^\infty \mu_k^{1/m-1}x_k^{2m}. \tag{7.14}$$

Hypothesis 1.3 is so fulfilled. Moreover, the vector field $Mg(x) = 2Q^{1/2}x$ belongs to $W^{1,q}(X,\nu_m;X)$ for every $q>1$, since every component $f_i(x) = 2b_{m,1}^{1/2}\mu_i^{1/2m}x_i$ is in $W^{1,p}(\nu_m)$, and $\sum_{i=1}^\infty \|M_qf_i(x)\|^2 = 4b_{m,1}^2\sum_{i=1}^\infty \mu_i^{1/m}$ is a real constant by assumption (7.5). Therefore, the assumptions of Proposition 5.3 are satisfied. □

For every $r > 0$, let σ_r^g be the measure given by Theorem 3.4. Setting

$$\rho_r(dx) := 2\|Q^{1/2}x\|\sigma_r^g(dx),$$

formula (4.7) reads as

$$\int_{B(0,r)} \langle M_p\varphi, F\rangle\, d\nu_m = \int_{B(0,r)} \varphi M_p^* F\, d\nu_m + \int_{\partial B(0,r)} T\Big(\varphi\langle F(x), \frac{Q^{1/2}x}{\|Q^{1/2}x\|}\rangle\Big)\rho_r(dx),$$

for every $F \in D(M_p^*)$, $\varphi \in W^{1,q}(\nu_m)$ with $q > p$. In particular, for a constant vector field $F(x) \equiv z$ and $\varphi \in C^1(X) \cap W^{1,q}(X, \nu_m)$ for some q we get

$$\int_{B(0,r)} \langle Q^{1/2}\nabla\varphi, z\rangle\, d\nu_m = \int_{B(0,r)} \varphi W_z^m\, d\nu_m + \int_{\partial B(0,r)} \varphi\langle z, \frac{Q^{1/2}x}{\|Q^{1/2}x\|}\rangle\, \rho_r(dx).$$

7.0.2. *Hyperplanes.* We take here $g(x) = \langle x, a\rangle$ where $a \in X \setminus \{0\}$ is fixed. Then

$$\nabla g(x) = a, \quad x \in X,$$

and the vector field $\Psi(x) = Mg(x)/\|Mg(x)\|^2$ of Hypothesis 1.3 is constant, equal to

$$\Psi(x) = \frac{Q^{1/2}a}{\|Q^{1/2}a\|^2}, \quad x \in X.$$

By Proposition 7.2, Hypothesis 1.2 is satisfied, and therefore $\Psi \in D(M_p^*)$ for every $p \in (1, +\infty)$. By (7.9) it follows that

$$M_p^*(\Psi)(x) = \frac{v_{Q^{1/2}a}(x)}{\|Q^{1/2}a\|^2} = \frac{b_{m,1}}{\|Q^{1/2}a\|^2}\sum_{h=1}^{\infty} \mu_h^{-1+1/m}|x_h|^{2m-2}x_h a_h. \tag{7.15}$$

Therefore, g satisfies Hypothesis 1.3. Since Mg is constant, it belongs to all $W^{1,q}(\nu_m)$ spaces, and also the hypotheses of Proposition 5.3 are satisfied. The normalized surface measure ρ_r on the hyperplane $\{x :\ \langle x, a\rangle = r\}$ is now

$$\rho_r(dx) = \|Q^{1/2}a\|\sigma_r^g(dx),$$

for every $r \in \mathbb{R}$, where σ_r^g is the measure given by Theorem 3.4. Formula (4.7) reads as

$$\begin{aligned}\int_{\{x\colon \langle x,a\rangle<r\}} &\langle M_p\varphi, F\rangle\, d\nu_m \\ &= \int_{\{x\colon \langle x,a\rangle<r\}} \varphi M_p^* F\, d\nu_m + \int_{\{x\colon \langle x,a\rangle=r\}} T\Big(\varphi\langle F(x), \frac{Q^{1/2}a}{\|Q^{1/2}a\|}\rangle\Big)\, \rho_r(dx),\end{aligned}$$

for every $F \in D(M_p^*)$, $\varphi \in W^{1,q}(\nu_m)$ with $q > p$. In particular, for a constant vector field $F(x) \equiv z$ and $\varphi \in C^1(X) \cap W^{1,q}(\nu_m)$ for some q we get

$$\begin{aligned}\int_{\{x\colon \langle x,a\rangle<r\}} &\langle Q^{1/2}\nabla\varphi, z\rangle\, d\nu_m \\ &= \int_{\{x\colon \langle x,a\rangle<r\}} \varphi W_z^m\, d\nu_m + \langle z, \frac{Q^{1/2}a}{\|Q^{1/2}a\|}\rangle \int_{\{x\colon \langle x,a\rangle=r\}} \varphi\rho_r(dx).\end{aligned}$$

8. Some invariant measures of SPDEs

Here we consider the invariant measures of a stochastic reaction–diffusion equation (section 8.1) and of the stochastic Burgers equation (section 8.2) in the space $X = L^2(0,1)$. We shall show that surface integrals can be defined in both cases on smooth surfaces such as spherical surfaces and hyperplanes of X.

Such equations look like

$$\begin{cases} dX(t) = [AX(t) + f(X(t))]dt + (-A)^{-\gamma/2}dW(t), \\ \\ X(0) = x, \end{cases} \tag{8.1}$$

with $\gamma \in [0,1)$. In both cases, A is the realization of the second order derivative in $X = L^2(0,1)$ with Dirichlet boundary conditions

$$D(A) = H^2(0,1) \cap H^1_0(0,1), \quad Ax(\xi) = x''(\xi).$$

W is an X-valued cylindrical Wiener process, and f is a suitable function: either it is the composition with a polynomial, $f(x)(\xi) = \sum_{k=0}^d a_k(x(\xi))^k$, or $f(x)(\xi) = x(\xi)x'(\xi)$ for $x \in H^1(0,1)$, $\xi \in (0,1)$.

We consider the complete orthonormal system in X given by

$$\{e_h(\xi) := \sqrt{2}\sin(h\pi\xi), \quad h \in \mathbb{N}\},$$

consisting of eigenfunctions of A, since

$$Ae_h = -h^2\pi^2 e_h =: -\alpha_h e_h, \quad h \in \mathbb{N}.$$

We recall that $D((-A)^\beta) = H^{2\beta}(0,1) \cap H^1_0(0,1)$ for all $\beta \in (1/2, 1]$.

As in the previous section we set

$$x_h := \langle x, e_h \rangle, \quad x \in X, \ h \in \mathbb{N},$$

and for every $n \in \mathbb{N}$ we denote by P_n the orthogonal projection on the subspace generated by $e_1, \ldots, e_n$, namely

$$P_n x := \sum_{h=1}^n x_h e_h. \tag{8.2}$$

Moreover, we consider the space $\mathcal{E}_A(X)$, consisting of the linear span of real and imaginary parts of the functions $x \mapsto e^{i\langle x,y\rangle}$ with $y \in D(A)$.

The following approximation lemma will be used in both examples.

Lemma 8.1. *Let $h \in \mathbb{N} \cup \{0\}$. For every $\varphi \in C^h_b(\mathbb{R}^n)$ there exists a sequence of trigonometric polynomials φ_k (namely, functions in the linear span of real and imaginary parts of the functions $x \mapsto \exp(i\langle x, a\rangle_{\mathbb{R}^n})$, with $a \in \mathbb{R}^n$) such that for every multi-index α with $0 \le |\alpha| \le h$ we have*

(i) $\lim_{k\to\infty} D^\alpha\varphi_k(x) = D^\alpha\varphi(x)$, *for every $x \in \mathbb{R}^n$,*
(ii) $\|D^\alpha\varphi_k\|_\infty \le C\|D^\alpha\varphi\|_\infty$,

where the constant C depends only on h and n.

Proof. The result is classical for functions that are periodic in each variable. Indeed, if φ is 1-periodic in all the variables we can take the convolutions with the Fejer kernels,

$$\varphi_N(x) = \int_{[-1/2,1/2]^n} K_N(y)\varphi(x-y)dy, \quad N \in \mathbb{N},$$

with

$$K_N(y) = \prod_{j=1}^{n} \frac{1}{N+1}\left(\frac{\sin \pi(N+1)y_j}{\sin \pi y_j}\right)^2, \quad N \in \mathbb{N}.$$

Then, $\|K_N\|_{L^1([-1/2,1/2]^n)} = 1$ for every N, and $D^\alpha K_N * \varphi = K_N * D^\alpha \varphi$ converges uniformly to $D^\alpha \varphi$, for $|\alpha| \le h$. In this case, the constant C is 1. See, e.g., [DS58, Exercise 73] or [So84] for detailed proofs.

If φ is T-periodic in all variables, the convolutions over $[-T/2, T/2]^n$ with the rescaled Fejer kernels $K_{N,T}(y) := K_N(y/T)/T^n$ make the same job. The constant C is still 1.

If φ is not periodic, there exists a sequence (φ_k) of smooth functions with compact support that satisfy (i) and (ii). In its turn, the restriction of each φ_k to its support may be approximated by a sequence of trigonometric polynomials (P^k_h), considering any extension of $\varphi_{k|supp\,\varphi_k}$ which is periodic in each variable and using the first part of the proof. The diagonal sequence (P^k_k) is the sequence that we are looking for. □

8.1. **Reaction-Diffusion equations.** Here we consider problem (8.1) where $f(x)$ is the composition of a decreasing polynomial of odd degree d greater than 1 with x,

$$f(x)(\xi) = \sum_{k=1}^{d} a_k (x(\xi))^k, \quad x \in X,\ \xi \in (0,1).$$

It is well known that for every $x \in X$ equation (8.1) has a unique generalized solution and that the associated transition semigroup $T(t)$ defined by

$$(T(t)\varphi)(x) := \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in C_b(X),\ t \ge 0,$$

possesses a unique invariant measure ν_R; see, e.g., [DP04, Ch. 4]. So, $T(t)$ may be extended to a contraction semigroup $T_p(t)$ to all spaces $L^p(X, \nu_R)$, $p \in [1, +\infty)$.

For $\gamma = 0$ the measure ν_R is an explicit weighted Gaussian measure,

$$\nu_R(dx) = \frac{1}{Z} e^{2U(x)} N_{0,Q}(dx),$$

where $N_{0,Q}$ is the Gaussian measure with mean 0 and covariance $Q = -A^{-1}/2$, the function U is defined by

$$U(x) = \begin{cases} -\displaystyle\int_0^1 f(x)d\xi, & x \in L^d(0,1), \\ -\infty, & x \notin L^d(0,1), \end{cases}$$

and $Z = \int_X e^{2U} dN_{0,Q}$. See [DaLu14, Sect. 5]. Since U, $e^{2U} \in W^{1,p}(X, N_{0,Q})$ for every $p > 1$ by [DaLu14, Sect. 5], ν_R is one of the measures considered in Section 6.

For $\gamma > 0$, ν_R is not explicit.

The following result is proved in [DaDe17, Thm. 1.2] for $\delta < 1 - \gamma$, and in [DaDe17, Thm. 10] for $\delta = 1 - \gamma$.

Theorem 8.2. *Let $\delta \in (0, 1-\gamma]$, $p \in (1,\infty)$. Then there exists $C_p > 0$ such that for all $\varphi \in C_b^1(X)$ we have*

$$\left| \int_X \langle \nabla\varphi(x), h\rangle\, \nu_R(dx)\right| \leq C_p \|\varphi\|_{L^p(X,\nu_R)}\, \|h\|_{H^{1+\delta+\gamma}(0,1)}, \qquad h \in H^{1+\delta+\gamma}(0,1) \cap H_0^1(0,1). \tag{8.3}$$

Setting $h = (-A)^{-(1+\delta+\gamma)/2}k$ with $k \in X$, formula (8.3) may be rewritten as

$$\left| \int_X \langle (-A)^{-(1+\delta+\gamma)/2}\nabla\varphi(x), k\rangle\, \nu_R(dx)\right| \leq C_p \|\varphi\|_{L^p(X,\nu_R)}\, \|k\|, \quad k \in X.$$

Therefore, fixing any $\beta \in ((1+\gamma)/2, 1]$, Hypothesis 1.2 is fulfilled with $R = (-A)^{-\beta}$. With this choice of R, Hypothesis 1.1 too is fulfilled, and we can consider the operators M_p and their adjoint operators M_p^* described in Sections 1, 2 for $p \in (1, +\infty)$. We do not know whether Hypothesis 3.6 holds.

To define surface measures on the level sets of a function $g : X \mapsto \mathbb{R}$, we need that g satisfies Hypothesis 1.3. If $g : X \mapsto \mathbb{R}$ is a twice Fréchet differentiable function, the vector field Ψ in formula (3.1) is given by

$$\Psi(x) = \frac{(-A)^{-\beta}\nabla g(x)}{\|(-A)^{-\beta}\nabla g(x)\|^2} = \frac{1}{\|(-A)^{-\beta}\nabla g(x)\|^2} \sum_{h=1}^{\infty} \alpha_h^{-\beta}\partial_{e_h} g(x) e_h, \quad x \in X. \tag{8.4}$$

We present below two examples of smooth functions g that satisfy Hypothesis 1.3, namely such that $g \in W^{1,p}(\nu_R)$ and $\Psi \in D(M_p^*)$ for every $p > 1$.

8.1.1. *Spherical surfaces.* Let $g(x) := \|x\|^2$. Theorem 4.20 of [DP04] and the Hölder inequality yield $g \in L^d(X, \nu_R)$, where d is the degree of f. The arguments of [DP04] can be easily carried on to improve this result.

Lemma 8.3.

(i) $\nu_R(L^q(0,1)) = 1$ *for every* $q \geq 2$;
(ii) $x \mapsto \|x\|^2 \in L^p(X, \nu_R)$ *for every* $p > 1$.

Proof. We follow the proof of Theorem 4.20 of [DP04], replacing $2d$ by $2m$ with $m \in \mathbb{N}$, and obtaining

$$\int_X \|x\|^{2m}_{L^{2m}(0,1)} \nu_R(dx) < \infty, \quad m \in \mathbb{N}. \tag{8.5}$$

Therefore, the function $x \mapsto \|x\|_{L^{2m}(0,1)}$ has finite values ν_R-a.e., namely $\nu_R(L^{2m}(0,1)) = 1$ for every $m \in \mathbb{N}$, which is statement (i). By the Hölder inequality, $\|x\|_X \leq \|x\|_{L^{2m}(0,1)}$ for every $x \in L^{2m}(0,1)$, and statement (ii) follows. □

Lemma 8.3 yields that $g \in L^p(X, \nu)$ for every $p > 1$.

As we mentioned in the Introduction, the verification of Hypothesis 1.3 will be reduced to checking that $\|Mg(\cdot)\|^{-1}$ belongs to $L^p(X, \nu)$ for every $p > 1$. In this case, $\|Mg(x)\|^{-1} = (2\|(-A)^{-\beta}x\|)^{-1}$, and the p-summability of this function is not obvious.

To begin with, we prove that suitable smooth cylindrical functions belong to the domain of the infinitesimal generator L of $T_2(t)$. This will be used to get estimates through the equality $\int_X L\varphi\, d\nu_R = 0$, which holds for every $\varphi \in D(L)$.

Lemma 8.4. *For every $n \in \mathbb{N}$ and $\theta \in C_b^2(\mathbb{R}^n)$ the function $\varphi(x) := \theta(\langle x, e_1\rangle, \dots, \langle x, e_n\rangle)$ belongs to the domain of the infinitesimal generator L of $T_2(t)$, and*

$$
\begin{aligned}
L\varphi(x) &= \frac{1}{2}\,\mathrm{Tr}\,[(-A)^{-\gamma}D^2\varphi] + \langle x, A\nabla\varphi(x)\rangle + \langle f(x), \nabla\varphi(x)\rangle \\
&= \frac{1}{2}\sum_{h=1}^{n}\alpha_h^{-\gamma}\frac{\partial^2\theta}{\partial\xi_h^2}(x_1,\dots,x_n) - \sum_{h=1}^{n}\alpha_h x_h\frac{\partial\theta}{\partial\xi_h}(x_1,\dots,x_n) \\
&\quad + \sum_{h=1}^{n}\langle f(x), e_h\rangle\frac{\partial\theta}{\partial\xi_h}(x_1,\dots,x_n).
\end{aligned}
\tag{8.6}
$$

Proof. By [DP04, Thm. 4.23], L is the closure of the operator $L_0 : \mathcal{E}_A(X) \mapsto L^2(X,\nu_R)$ defined by $L_0\psi(x) = \frac{1}{2}\,\mathrm{Tr}\,[(-A)^{-\gamma}D^2\psi] + \langle x, A\nabla\psi(x)\rangle + \langle f(x), \nabla\psi(x)\rangle$ for $\psi \in \mathcal{E}_A(X)$. To prove that $\varphi \in D(L)$ it is sufficient to approach φ by a sequence (ψ_k) of elements of $\mathcal{E}_A(X)$ in $L^2(X,\nu_R)$, such that the sequence $L_0\psi_k$ converges in $L^2(X,\nu_R)$.

By Lemma 8.1 there exists a sequence of trigonometric polynomials (θ_k) such that θ_k and its first and second order derivatives converge pointwise to θ and to its first and second order derivatives, respectively, and moreover $\|\theta_k\|_{C_b^2(\mathbb{R}^n)} \leq C$ independent of k. We set

$$
\psi_k(x) = \theta_k(x_1,\dots,x_n), \quad k \in \mathbb{N},\ x \in X. \tag{8.7}
$$

Then $\psi_k \in \mathcal{E}_A(X)$ for every $k \in \mathbb{N}$, and it is not hard to see that the sequence $(L_0\psi_k)$ converges to the function in the right-hand side of (8.6) in $L^2(X,\nu_R)$. □

Proposition 8.5. *If $\gamma \leq 1/2$, $x \mapsto \|(-A)^{-\beta}x\|^{-1} \in L^p(X,\nu_R)$ for every $p > 1$.*

Proof. Recalling that the sequence (α_n) is increasing, for every $n \in \mathbb{N}$ we estimate

$$
\frac{1}{\|(-A)^{-\beta}x\|^2} \leq \frac{1}{\|(-A)^{-\beta}P_n x\|^2} \leq \frac{\alpha_n^{2\beta}}{\|P_n x\|^2},
$$

where P_n is the projection on span $e_1,\dots,e_n$ defined in (8.2). So, it is enough to show that for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that

$$
x \mapsto \frac{1}{\|P_n x\|^2} \in L^{k+1}(X,\nu_R). \tag{8.8}
$$

We shall show that (8.8) holds for large enough n. To this aim we approach $1/\|P_n x\|^2$ by the smooth functions

$$
\varphi_\varepsilon(x) := \frac{1}{(\varepsilon + \|P_n x\|^2)^k}, \quad x \in X,
$$

that belong to the domain of the infinitesimal generator L of the transition semigroup by Lemma 8.4. For every h, h_1, $h_2 \in X$ we have

$$
\langle \nabla\varphi_\varepsilon(x), h\rangle = -\frac{2k\langle P_n x, P_n h\rangle}{(\varepsilon + \|P_n x\|^2)^{k+1}}
$$

and

$$
D^2\varphi_\varepsilon(x)(h_1,h_2) = -2k\frac{\langle P_n h_1, P_n h_2\rangle}{(\varepsilon + \|P_n x\|^2)^{k+1}} + 4k(k+1)\frac{\langle P_n x, P_n h_1\rangle\,\langle P_n x, P_n h_2\rangle}{(\varepsilon + \|P_n x\|^2)^{k+2}}.
$$

Therefore,

$$\frac{1}{2}\,\mathrm{Tr}\,[(-A)^{-\gamma}D^2\varphi_\varepsilon(x)] = -\frac{k\sum_{j=1}^n \alpha_j^{-\gamma}}{(\varepsilon+\|P_nx\|^2)^{k+1}} + 2k(k+1)\frac{\|(-A)^{-\gamma/2}P_nx\|^2}{(\varepsilon+\|P_nx\|^2)^{k+2}}.$$

So, (8.6) yields

$$\begin{aligned} L\varphi_\varepsilon(x) \quad &= \quad -\frac{k\sum_{j=1}^n \alpha_j^{-\gamma}}{(\varepsilon+\|P_nx\|^2)^{k+1}} + 2k(k+1)\frac{\|(-A)^{-\gamma/2}P_nx\|^2}{(\varepsilon+\|P_nx\|^2)^{k+2}} \\ &\quad -\frac{2k\langle AP_nx, x\rangle}{(\varepsilon+\|P_nx\|^2)^{k+1}} - \frac{2k\langle P_nx, f(x)\rangle}{(\varepsilon+\|P_nx\|^2)^{k+1}}. \end{aligned} \tag{8.9}$$

Since ν_R is invariant we have

$$\int_X L\varphi_\varepsilon(x)\,\nu_R(dx) = 0,$$

and therefore

$$\begin{aligned} &k\sum_{j=1}^n \alpha_j^{-\gamma}\int_H \frac{1}{(\varepsilon+\|P_nx\|^2)^{k+1}}\,\nu_R(dx) \\ &= 2k\int_H \frac{\|(-A)^{1/2}P_nx\|^2}{(\varepsilon+\|P_nx\|^2)^{k+1}}\,\nu_R(dx) \\ &\quad -2k\int_H \frac{\langle P_nx, f(x)\rangle}{(\varepsilon+\|P_nx\|^2)^{k+1}}\,\nu_R(dx) \\ &\quad +2k(k+1)\int_H \frac{\|(-A)^{-\gamma/2}P_nx\|^2}{(\varepsilon+\|P_nx\|^2)^{k+2}}\,\nu_R(dx) \\ &=: I_1+I_2+I_3. \end{aligned} \tag{8.10}$$

Let us estimate I_1. Since (α_n) is an increasing sequence,

$$\|(-A)^{1/2}P_nx\|^2 \le \alpha_n\|P_nx\|^2 \le \alpha_n(\varepsilon+\|P_nx\|^2),$$

and using the Hölder and Young inequalities we obtain that for any $\delta>0$ there is $C_1(\delta,k,n)$ such that
(8.11)

$$\begin{aligned} |I_1 &\le 2k\alpha_n\int_X \frac{1}{(\varepsilon+\|P_nx\|^2)^k}\,\nu_R(dx) \le 2k\alpha_n\left(\int_X \frac{1}{(\varepsilon+\|P_nx\|^2)^{k+1}}\,\nu_R(dx)\right)^{\frac{k}{k+1}} \\ &\le C_1(\delta,k,n)+\delta\int_X \frac{1}{(\varepsilon+\|P_nx\|^2)^{k+1}}\,\nu_R(dx). \end{aligned}$$

Let us estimate I_2. Since

$$|\langle P_nx, f(x)\rangle| \le \|P_nf(x)\|\,\|P_nx\| \le \|f(x)\|\,(\varepsilon+\|P_nx\|^2)^{1/2},$$

arguing as before and taking (8.5) into account, we see that for any $\delta > 0$ there is $C_2(\delta, k)$ such that

(8.12)
$$\begin{aligned} |I_2| &\leq 2k \int_X \frac{\|P_n f(x)\|}{(\varepsilon + \|P_n x\|^2)^{k+1/2}}\, \nu_R(dx) \\ &\leq 2k \left(\int_X \|f(x)\|^{2k+2}\, \nu_R(dx) \right)^{\frac{1}{2k+2}} \left(\int_X \frac{1}{(\varepsilon + \|P_n x\|^2)^{k+1}}\, \nu_R(dx) \right)^{\frac{2k+1}{2k+2}} \\ &\leq C_2(\delta, k) + \delta \int_X \frac{1}{(\varepsilon + \|P_n x\|^2)^{k+1}}\, \nu_R(dx). \end{aligned}$$

To estimate I_3 we recall once again that (α_n) is an increasing sequence, so that $\|(-A)^{-\gamma/2} P_n x\|^2 \leq \alpha_1^{-\gamma} \|P_n x\|^2$. Then

$$|I_3| \leq \frac{2k(k+1)}{\alpha_1^\gamma} \int_X \frac{1}{(\varepsilon + \|Px\|^2)^{k+1}}\, \nu_R(dx). \tag{8.13}$$

Estimates (8.11)–(8.13) yield

(8.14)
$$\begin{aligned} k \sum_{j=1}^n \alpha_j^{-\gamma} \int_X \frac{\nu_R(dx)}{(\varepsilon + \|P_n x\|^2)^{k+1}} &\leq C_1(\delta, k, n) + C_2(\delta, k) \\ &\quad + \left(\frac{2k(k+1)}{\alpha_1^\gamma} + 2\delta \right) \int_X \frac{\nu_R(dx)}{(\varepsilon + \|P_n x\|^2)^{k+1}}. \end{aligned}$$

Since $\gamma \leq 1/2$, the series $s_n = \sum_{j=1}^n \alpha_j^{-\gamma}$ is divergent (recall that $\alpha_j = \pi^2 j^2$). Now we choose n and δ such that

$$k \sum_{j=1}^n \alpha_j^{-\gamma} > \frac{2k(k+1)}{\alpha_1^\gamma} + 2\delta$$

and we conclude that there exists $M > 0$, independent of ε, such that

$$\int_X \frac{1}{(\varepsilon + \|P_n x\|^2)^{k+1}}\, \nu_R(dx) \leq M. \tag{8.15}$$

Letting $\varepsilon \to 0$ concludes the proof. □

With the aid of Lemma 8.3 and Proposition 8.5 we prove the main result of this section.

Proposition 8.6. *If $0 \leq \gamma \leq 1/2$, the function $g(x) = \|x\|^2$ satisfies Hypothesis* 1.3.

Proof. g is smooth and it belongs to $L^p(X, \nu_R)$ for every $p > 1$ by Lemma 8.3(ii). Moreover, $(-A)^{-\beta} \nabla g(x) = 2(-A)^{-\beta} x$ for every $x \in X$, and since $\|(-A)^{-\beta} x\| \leq \pi^{-2\beta} \|x\|$, still by Lemma 8.3(ii) $x \mapsto \|(-A)^{-\beta} \nabla g(x)\| \in L^p(X, \nu_R)$ for every $p > 1$. By Lemma 2.4, $g \in W^{1,p}(\nu_R)$ for every $p > 1$.

It remains to prove that the vector field Ψ in formula (3.1) belongs to $D(M_p^*)$ for every $p > 1$. It is given by (see (8.4))

$$\Psi(x) = \frac{(-A)^{-\beta} x}{2\|(-A)^{-\beta} x\|^2} = \lim_{n \to \infty} \Psi_n(x), \tag{8.16}$$

where

$$\Psi_n(x) = \sum_{h=1}^n \frac{\alpha_h^{-\beta} x_h}{2\|(-A)^{-\beta}x\|^2} e_h =: \sum_{h=1}^n \psi_h(x) e_h.$$

Approaching every ψ_h by the C_b^1 functions $\psi_{h,\varepsilon}(x) := \alpha_h^{-\beta} x_h / 2(\|(-A)^{-\beta}x\|^2 + \varepsilon)$ and using Proposition 8.5, one sees easily that ψ_h belongs to $W^{1,p}(\nu_R)$ for every $p > 1$, and

$$\langle M\psi_h(x), e_h\rangle = \alpha_h^{-2\beta}/2\|(-A)^{-\beta}x\|^2 - \alpha_h^{-4\beta} x_h^2/\|(-A)^{-\beta}x\|^2.$$

By Lemma 2.6, Ψ_n belongs to $D(M_p^*)$ for every $p > 1$, and by (2.9) we get

$$M_p^*\Psi_n(x) = -\sum_{h=1}^n \frac{\alpha_h^{-2\beta}}{2\|(-A)^{-\beta}x\|^2} + \sum_{h=1}^n \frac{\alpha_h^{-4\beta} x_h^2}{2\|(-A)^{-\beta}x\|^4} + \sum_{h=1}^n \frac{\alpha_h^{-\beta} x_h v_{e_h}(x)}{2\|(-A)^{-\beta}x\|^2}. \tag{8.17}$$

Recalling that the series $\sum_{h=1}^n \alpha_h^{-2\beta}$ converges, that $\|v_{e_h}\|_{L^{p'}(X,\nu_R)}$ is bounded by a constant independent of h, and using Lemma 8.3 and Proposition 8.5, we easily deduce that $(M_p^*\Psi_n)$ converges in $L^{p'}(X,\nu_R)$, for every $p > 1$. Therefore, $\Psi \in D(M_p^*)$ for every $p > 1$, and Hypothesis 1.3 is satisfied. □

8.1.2. *Hyperplanes.* Let $g(x) = \langle x, b\rangle$, where $b \in X \setminus \{0\}$. g is smooth, it has constant gradient, and $Mg(x) = (-A)^{-\beta} b$ (constant). Therefore, g belongs to all spaces $W^{1,p}(\nu_R)$, for $p > 1$, by Lemmas 8.3 and 2.4. The vector field $\Psi = Mg/\|Mg\|^2$ is also constant and it is given by

$$\Psi(x) = \frac{(-A)^{-\beta} b}{\|(-A)^{-\beta} b\|^2}, \quad x \in X.$$

Since Hypothesis 1.2 is satisfied, Ψ belongs to $D(M_p^*)$ for every $p > 1$, and we have

$$M_p^*\Psi = \frac{v_{(-A)^{-\beta} b}}{\|(-A)^{-\beta} b\|^2}.$$

Therefore, g satisfies Hypothesis 1.3.

8.2. **Burgers equation.** We are concerned with the stochastic differential equation (8.1) with $\gamma = 0$ and

$$f(x) = 2xx', \quad x \in H_0^1(0,1),$$

where the prime denotes the weak derivative. It is well known that for every $x \in X$, equation (8.1) has a unique mild solution and that the associated transition semigroup $P(t)$, defined on $C_b(X)$ by

$$P(t)\varphi(x) := \mathbb{E}[\varphi(X(t,x))], \quad t \geq 0,\ x \in X,$$

possesses a unique invariant measure ν_B; see, e.g., [DPZ96, Thm. 14.4.4]. So, $P(t)$ may be extended to a strongly continuous semigroup $P_p(t)$ in $L^p(X,\nu_B)$, for every $p \geq 1$.

A result analogous to Theorem 8.2 was proved in [DaDe16, Thm. 2].

Theorem 8.7. *For any $p > 1$, $\delta > 0$, there exists $C > 0$ such that for all $\varphi \in C_b^1(X)$ and all $h \in H^{1+\delta}(0,1) \cap H_0^1(0,1)$, we have*

$$\left| \int_X \langle D\varphi(x), h\rangle\, \nu_B(dx) \right| \leq C\|\varphi\|_{L^p(X,\nu_B)}\, \|h\|_{H^{1+\delta}(0,1)}. \tag{8.18}$$

As in Section 8.1, it follows that Hypotheses 1.1 and 1.2 are fulfilled with $R = A^{-\beta}$ for all $\beta \in (1/2, 1)$. Also in this case, we do not know whether Hypothesis 1.3 holds. And also in this case we are going to show that our theory fits to spherical surfaces and to hyperplanes. The proofs are similar to the proofs in Section 8.1 and we only sketch them.

Let $g(x) := \|x\|^2$. It was proved in [DaDe07, Prop. 2.3] that

$$\int_X \|x\|^k_{L^q(0,1)}\, \nu_B(dx) < +\infty, \quad k \in \mathbb{N},\ q \geq 2. \tag{8.19}$$

It follows that $\nu_B(L^q(0,1)) = 1$ for every $q \geq 2$ and that $g \in L^p(X, \nu_B)$ for every $p > 1$. To prove that g satisfies Hypothesis 1.3, we argue as in Proposition 8.6. First, we remark that $g \in W^{1,p}(\nu_B)$ for every $p > 1$ by (8.19) and Lemma 2.4. Second, the vector field $\Psi = Mg/\|Mg\|^2$ is still given by formula (8.16). Proving that it belongs to $D(M_p^*)$ for every $p > 1$ amounts to showing that $x \mapsto \|(-A)^{-\beta}x\|^{-2}$ belongs to $L^p(X, \nu_B)$ for every $p > 1$. This can be proved as in the case of reaction-diffusion equations, with the aid of the following lemma.

Lemma 8.8. *For every $n \in \mathbb{N}$ and $\theta \in C_b^2(\mathbb{R}^n)$ the function $\varphi(x) := \theta(x_1, \ldots, x_n)$ belongs to the domain of the infinitesimal generator N of $P_2(t)$, and*

$$\begin{aligned} N\varphi(x) &= \tfrac{1}{2}\,\mathrm{Tr}\,[D^2\varphi] + \langle x, A\nabla\varphi(x)\rangle + \langle x^2, (\nabla\varphi(x))'\rangle \\ &= \frac{1}{2}\sum_{h=1}^n \frac{\partial^2\theta}{\partial\xi_h^2}(x_1, \ldots, x_n) - \sum_{h=1}^n \alpha_h x_h \frac{\partial\theta}{\partial\xi_h}(x_1, \ldots, x_n) \\ &\quad - \sum_{h=1}^n \frac{\partial\theta}{\partial\xi_h}(x_1, \ldots, x_n)\langle x^2, e_h'\rangle. \end{aligned} \tag{8.20}$$

Proof. By [DaDe07, §4.1], N is the closure of the operator $N_0 : \mathcal{E}_A(X) \mapsto L^2(X, \nu_R)$ defined by $N_0\psi(x) = \frac{1}{2}\,\mathrm{Tr}\,[D^2\psi] + \langle x, A\nabla\psi(x)\rangle - \langle x^2, (\nabla\psi(x))'\rangle$ for $\psi \in \mathcal{E}_A(X)$. In fact, $N_0\psi(x)$ is formally defined by

$$N_0\psi(x) = \frac{1}{2}\,\mathrm{Tr}\,[D^2\psi] + \langle x, A\nabla\psi(x)\rangle + \langle 2xx', \nabla\psi(x)\rangle,$$

which is meaningful for $x \in H^1(0,1)$. However, we do not know whether $\nu_B(H^1(0,1)) = 1$ so that the scalar product $\langle 2xx', \nabla\psi(x)\rangle$ has to be rewritten in the more convenient way $\langle x^2, (\nabla\psi(x))'\rangle$, obtained just integrating by parts.

As in Lemma 8.4, we approach φ by a sequence (ψ_k) of elements of $\mathcal{E}_A(X)$ in $L^2(X, \nu_B)$, such that the sequence $L_0\psi_k$ converges in $L^2(X, \nu_B)$. (ψ_k) is the sequence defined in (8.7), and it converges to ψ in $L^2(X, \nu_B)$ by the Dominated Convergence Theorem. Moreover,

$$\begin{aligned} N_0\psi_k(x) = \frac{1}{2}\sum_{h=1}^n \frac{\partial^2\theta_k}{\partial\xi_h^2}(x_1, \ldots, x_n) - \sum_{h=1}^n \alpha_h x_h \frac{\partial\theta_k}{\partial\xi_h}(x_1, \ldots, x_n) \\ - \sum_{h=1}^n \frac{\partial\theta_k}{\partial\xi_h}(x_1, \ldots, x_n)\langle x^2, e_h'\rangle, \end{aligned}$$

which converges pointwise to the function in the right-hand side of (8.20). Moreover, $|N_0\psi_k(x)| \leq C\|\theta\|_{C_b^2(\mathbb{R}^n)}(1 + \|x\| + \|x\|^2)$, which is in $L^2(X, \nu_B)$ by (8.19), and again by the Dominated Convergence Theorem the sequence $(N_0\psi_k)$ converges to the function in the right-hand side of (8.20) in $L^2(X,\nu_B)$. ☐

Proposition 8.9.

$$x \mapsto \frac{1}{\|(-A)^{-\beta}x\|^2} \in L^{k+1}(X,\nu_B), \quad \forall\, k \in \mathbb{N}.$$

Proof. We follow the proof of Proposition 8.5. For every $n \in \mathbb{N}$ we estimate

$$\frac{1}{\|(-A)^{-\beta}x\|^2} \le \frac{1}{\|(-A)^{-\beta}P_nx\|^2} \le \frac{\alpha_n^\beta}{\|P_nx\|^2}.$$

Then it is enough to show that for each $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that

$$\frac{1}{\|P_nx\|^2} \in L^{k+1}(X,\nu_B), \tag{8.21}$$

and to this aim we approach $1/\|P_nx\|^{2(k+1)}$ by the functions

$$\varphi_\varepsilon(x) = \frac{1}{(\varepsilon + \|P_nx\|^2)^{k+1}},$$

which belong to $D(N)$ by Lemma 8.8. Formula (8.20) (recall that now $\gamma = 0$) yields

$$\begin{aligned} N\varphi_\varepsilon(x) \;&=\; -\frac{kn}{(\varepsilon + \|P_nx\|^2)^{k+1}} + 2k(k+1)\frac{\|P_nx\|^2}{(\varepsilon + \|P_nx\|^2)^{k+2}} \\ &\quad -\frac{2k\langle AP_nx, x\rangle}{(\varepsilon + \|Px\|^2)^{k+1}} + \frac{2k\langle (P_nx)', x^2\rangle}{(\varepsilon + \|Px\|^2)^{k+1}}. \end{aligned} \tag{8.22}$$

Since

$$\int_X N\varphi_\varepsilon(x)\,\nu_B(dx) = 0$$

by the invariance of ν_B, we find that

$$\begin{aligned} kn\int_X \frac{1}{(\varepsilon + \|P_nx\|^2)^{k+1}}\,\nu_B(dx) &= 2k\int_X \frac{\|(-A)^{1/2}P_nx\|^2}{(\varepsilon + \|P_nx\|^2)^{k+1}}\,\nu_B(dx) \\ +2k\int_X \frac{\langle (P_nx)', x^2\rangle}{(\varepsilon + \|P_nx\|^2)^{k+1}}\,\nu_B(dx) &+ 2k(k+1)\int_X \frac{\|Px\|^2}{(\varepsilon + \|Px\|^2)^{k+2}}\,\nu_B(dx) \\ =: I_1 + I_2 + I_3. \end{aligned} \tag{8.23}$$

Estimates of I_1 and I_3 are identical to the corresponding ones in the proof of Proposition 8.5 with $\gamma = 0$; to estimate I_2 we need different arguments. We have

$$\langle x^2, (P_nx)'\rangle = \int_0^1 (x(\xi))^2 \sum_{h=1}^n \langle x, e_h\rangle e_h'(\xi)\,d\xi$$

so that

$$\begin{aligned} |\langle x^2, (P_nx)'\rangle| \;&\le\; \left(\int_0^1 x^4 d\xi\right)^{1/2}\left(\int_0^1 \left(\sum_{h=1}^n \langle x, e_h\rangle e_h'(\xi)\right)^2 d\xi\right)^{1/2} \\ &\le\; \|x\|^2_{L^4(0,1)} C_n \|P_nx\| \\ &\le\; \|x\|^2_{L^4(0,1)} C_n (\varepsilon + \|P_nx\|^2)^{1/2}, \end{aligned}$$

and therefore

(8.24)

$$
\begin{aligned}
|I_2| &\leq 2kC_n \int_X \frac{\|x\|^2_{L^4(0,1)}}{(\varepsilon+\|Px\|^2)^{k+1/2}}\,\nu_B(dx)\\
&\leq 2kC_n \left(\int_X \|x\|^{2k+2}_{L^4(0,1)}\,\nu_B(dx)\right)^{\frac{1}{2k+2}} \left(\int_X \frac{1}{(\varepsilon+\|Px\|^2)^{k+1}}\,\nu_B(dx)\right)^{\frac{2k+1}{2k+2}}.
\end{aligned}
$$

Since $\int_X \|x\|^{2k+2}_{L^4(0,1)}\,\nu_B(dx) < \infty$ by (8.19), there exists a constant $C(k,n) > 0$ such that

(8.25)

$$
\begin{aligned}
|I_2| &\leq C(k,n)\left(\int_X \frac{1}{(\varepsilon+\|P_n x\|^2)^{k+1}}\,\nu_B(dx)\right)^{\frac{2k+1}{2k+2}}\\
&\leq C(k,n,\delta) + \delta \int_X \frac{1}{(\varepsilon+\|P_n x\|^2)^{k+1}}\,\nu_B(dx),
\end{aligned}
$$

for any $\delta > 0$ and a suitable $C(k,n,\delta) > 0$, by Young's inequality. The conclusion follows now as in the proof of Proposition 8.5 . □

The procedure of Subsection 8.1.2 works as well in this case, without any modification. Therefore, for every $b \in X \setminus \{0\}$ the function $g(x) := \langle x, b\rangle$ satisfies Hypothesis 1.3.

9. Final remarks and bibliographical notes

9.1. **Sobolev spaces.** The theory of Sobolev spaces for differentiable measures is well developed only in the Gaussian case. See [Bo98] for Gaussian measures in general locally convex spaces, [DPZ02] for Gaussian measures in Hilbert spaces. Basic results for general differentiable measures are in [Bo10, Ch. 2].

We did not consider the space $W^{1,1}(\nu)$, which is a very special case (even for Gaussian measures) and would deserve a specific treatment. Together with $W^{1,1}(\nu)$, spaces of BV functions are still to be thoroughly investigated. Some initial results are in [RoZhZh15]. The case of weighted Gaussian measures in Hilbert spaces was considered in [AmDaGoPa12].

Sobolev spaces of functions defined in (smooth) domains rather than in the whole X are even more puzzling. Even in the case of Gaussian measures the theory is far from being complete. A major difficulty comes from the lack of a bounded extension operator from $W^{1,p}(\Omega,\nu)$ to $W^{1,p}(\nu)$; see [BoPiSh14] for a counterexample. If X is a separable infinite dimensional Hilbert space and ν is a nondegenerate Gaussian measure in X, the existence of a bounded extension operator from $W^{1,2}(B(0,1),\nu)$ to $W^{1,2}(\nu)$ is still an open question.

9.2. **Surface measures.** For a detailed account of the existing literature on surface measures in infinite dimension, we refer to the survey paper [Bo17].

Hypothesis 1.3 on the defining function g is our main assumption. It could be replaced by $Mg/\|Mg\|^2 \in D(M^*_{\overline{p}})$ for *some* $\overline{p}$, but this would lead to restrictions on the validity of several results. For instance, in Lemma 3.1 and in all its consequences we should take $\varphi \in L^p(X,\nu)$ only with $p \geq \overline{p}'$.

Checking Hypothesis 1.3 in specific examples is reduced to some regularity/summability assumptions on g, plus summability of $\|Mg\|^{-p}$ for every p. While the regularity and summability properties of Mg can be considered standard conditions and can be checked in standard ways, to prove that $\|Mg\|^{-p}$ belongs to $L^1(X,\nu)$ is much more difficult. To overcome this difficulty, we could replace the function F_φ used throughout the paper by

$$\widetilde{F}_\varphi(r) = \int_{\{g<r\}} \varphi(x)\|Mg(x)\|^2\nu(dx), \quad r\in\mathbb{R},$$

and replace Hypothesis 1.3 by $Mg \in D(M_p^*)$ for every $p>1$, as suggested in [Bo17]. Then, the procedure of Lemma 3.1 yields that the measure $(\varphi\|Mg\|\nu)\circ g^{-1}$ is absolutely continuous with respect to the Lebesgue measure, with density

$$\widetilde{q}_\varphi(r) = \int_{\{g<r\}} (\langle M_p\varphi, Mg\rangle - \varphi M^*(Mg))d\nu, \quad r\in\mathbb{R},$$

and the procedure of Theorem 3.4 gives a Borel measure $\widetilde{\sigma}^g_r$ such that $\widetilde{F}'_\varphi(r) = \int_X \varphi(x)\,\widetilde{\sigma}^g_r(dx)$, for every $\varphi\in C_b(X)$. However, as for the measures σ^g_r, these measures depend explicitly on g and have no intrinsic geometric or analytic meaning. The geometrically meaningful measure is what we called ρ_r (see Section 5), and to obtain it the assumption $\|Mg\|^{-p}\in L^1(X,\nu)$ for some p seems to be unavoidable.

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References

[AiMa88] H. Airault and P. Malliavin, *Intégration géométrique sur l'espace de Wiener* (French, with English summary), Bull. Sci. Math. (2) **112** (1988), no. 1, 3–52. MR942797

[AlRo90] S. Albeverio and M. Röckner, *Classical Dirichlet forms on topological vector spaces—closability and a Cameron-Martin formula*, J. Funct. Anal. **88** (1990), no. 2, 395–436, DOI 10.1016/0022-1236(90)90113-Y. MR1038449

[AMMP10] L. Ambrosio, M. Miranda Jr., S. Maniglia, and D. Pallara, *BV functions in abstract Wiener spaces*, J. Funct. Anal. **258** (2010), no. 3, 785–813, DOI 10.1016/j.jfa.2009.09.008. MR2558177

[AmDaGoPa12] L. Ambrosio, G. Da Prato, B. Goldys, and D. Pallara, *Bounded variation with respect to a log-concave measure*, Comm. Partial Differential Equations **37** (2012), no. 12, 2272–2290, DOI 10.1080/03605302.2012.690014. MR3005544

[Bo98] V. I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR1642391

[Bo07] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007. MR2267655

[Bo10] V. I. Bogachev, *Differentiable measures and the Malliavin calculus*, Mathematical Surveys and Monographs, vol. 164, American Mathematical Society, Providence, RI, 2010. MR2663405

[BoPiSh14] V. I. Bogachev, A. Yu. Pilipenko, and A. V. Shaposhnikov, *Sobolev functions on infinite-dimensional domains*, J. Math. Anal. Appl. **419** (2014), no. 2, 1023–1044, DOI 10.1016/j.jmaa.2014.05.020. MR3225419

[Bo17] V. I. Bogachev, *Surface measures in infinite-dimensional spaces*, Measure theory in non-smooth spaces, Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, 2017, pp. 52–97. MR3701736

[BoMa16] V. I. Bogachev and I. I. Malofeev, *Surface measures generated by differentiable measures*, Potential Anal. **44** (2016), no. 4, 767–792, DOI 10.1007/s11118-015-9530-1. MR3490549

[CeLu14] P. Celada and A. Lunardi, *Traces of Sobolev functions on regular surfaces in infinite dimensions*, J. Funct. Anal. **266** (2014), no. 4, 1948–1987, DOI 10.1016/j.jfa.2013.11.013. MR3150149

[Ce01] S. Cerrai, *Second order PDE's in finite and infinite dimension,* A probabilistic approach, Lecture Notes in Mathematics, vol. 1762, Springer-Verlag, Berlin, 2001. MR1840644

[CCKO10] S. Chaari, F. Cipriano, H.-H. Kuo, and H. Ouerdiane, *Surface measures on the dual space of the Schwartz space*, Commun. Stoch. Anal. **4** (2010), no. 3, 467–480. MR2677202

[DP04] G. Da Prato, *Kolmogorov equations for stochastic PDEs*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2004. MR2111320

[DP06] G. Da Prato, *An introduction to infinite-dimensional analysis*, revised and extended from the 2001 original by Da Prato, Universitext, Springer-Verlag, Berlin, 2006. Revised and extended from the 2001 original by Da Prato. MR2244975

[DaDe07] G. Da Prato and A. Debussche, *m-dissipativity of Kolmogorov operators corresponding to Burgers equations with space-time white noise*, Potential Anal. **26** (2007), no. 1, 31–55, DOI 10.1007/s11118-006-9021-5. MR2276524

[DaDe15] G. Da Prato, A. Debussche, Existence of the Fomin derivative of the invariant measure of a stochastic reaction–diffusion equation, preprint arXiv:1502.07490. Proceedings of the conference *Mathematical Analysis of Viscous Incompressible Fluids*, November 17–19, 2014. Research Institute for Mathematical Science, Kyoto University, 121–134, 2015.

[DaDe16] G. Da Prato and A. Debussche, *Estimate for P_tD for the stochastic Burgers equation* (English, with English and French summaries), Ann. Inst. Henri Poincaré Probab. Stat. **52** (2016), no. 3, 1248–1258, DOI 10.1214/15-AIHP685. MR3531708

[DaDe17] G. Da Prato and A. Debussche, *An integral inequality for the invariant measure of a stochastic reaction-diffusion equation*, J. Evol. Equ. **17** (2017), no. 1, 197–214, DOI 10.1007/s00028-016-0349-z. MR3630320

[DaLu14] G. Da Prato and A. Lunardi, *Sobolev regularity for a class of second order elliptic PDE's in infinite dimension*, Ann. Probab. **42** (2014), no. 5, 2113–2160, DOI 10.1214/14-AOP936. MR3262499

[DaLuTu14] G. Da Prato, A. Lunardi, and L. Tubaro, *Surface measures in infinite dimension*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **25** (2014), no. 3, 309–330, DOI 10.4171/RLM/681. MR3256212

[DPZ96] G. Da Prato and J. Zabczyk, *Ergodicity for infinite-dimensional systems*, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996. MR1417491

[DPZ02] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002. MR1985790

[DS58] N. Dunford and J. T. Schwartz, *Linear operators. Part I*, General theory, with the assistance of William G. Bade and Robert G. Bartle, reprint of the 1958 original. Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988. MR1009162

[DU77] J. Diestel and J. J. Uhl Jr., *Vector measures,* with a foreword by B. J. Pettis; Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977. MR0453964

[Fe16] S. Ferrari, Sobolev spaces with respect to weighted Gaussian measures in infinite dimension, preprint, 2016, arXiv 1510.08283.

[Fe01] D. Feyel, *Hausdorff-Gauss measures*, Stochastic analysis and related topics, VII (Kusadasi, 1998), Progr. Probab., vol. 48, Birkhäuser Boston, Boston, MA, 2001, pp. 59–76. MR1915449

[FePr92] D. Feyel and A. de La Pradelle, *Hausdorff measures on the Wiener space*, Potential Anal. **1** (1992), no. 2, 177–189, DOI 10.1007/BF01789239. MR1245885

[Fu00] M. Fukushima, *BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space*, J. Funct. Anal. **174** (2000), no. 1, 227–249, DOI 10.1006/jfan.2000.3576. MR1761369

[FuHi01] M. Fukushima and M. Hino, *On the space of BV functions and a related stochastic calculus in infinite dimensions*, J. Funct. Anal. **183** (2001), no. 1, 245–268, DOI 10.1006/jfan.2000.3738. MR1837539

[Hi10] M. Hino, *Sets of finite perimeter and the Hausdorff-Gauss measure on the Wiener space*, J. Funct. Anal. **258** (2010), no. 5, 1656–1681, DOI 10.1016/j.jfa.2009.06.033. MR2566314

[Kh12] R. Khasminskii, *Stochastic stability of differential equations,* with contributions by G. N. Milstein and M. B. Nevelson, 2nd ed., Stochastic Modelling and Applied Probability, vol. 66, Springer, Heidelberg, 2012. MR2894052

[LaLi86] J.-M. Lasry and P.-L. Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math. **55** (1986), no. 3, 257–266, DOI 10.1007/BF02765025. MR876394

[Ma97] P. Malliavin, *Stochastic analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 313, Springer-Verlag, Berlin, 1997. MR1450093

[Nu95] D. Nualart, *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer-Verlag, New York, 1995. MR1344217

[RoZhZh15] M. Röckner, R. Zhu, and X. Zhu, *BV functions in a Gelfand triple for differentiable measure and its applications*, Forum Math. **27** (2015), no. 3, 1657–1687, DOI 10.1515/forum-2012-0137. MR3341491

[Sa05] M. Sanz-Solé, *Malliavin calculus,* with applications to stochastic partial differential equations, Fundamental Sciences, EPFL Press, Lausanne; distributed by CRC Press, Boca Raton, FL, 2005. MR2167213

[Sk74] A. V. Skorohod, *Integration in Hilbert space*, translated from the Russian by Kenneth Wickwire, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 79, Springer-Verlag, New York-Heidelberg, 1974. MR0466482

[So84] P. Soardi, Serie di Fourier in più variabili (Italian), Quaderni dell'Unione Matematica Italiana **26**, Pitagora Editrice, 1984.

[Ug79] A. V. Uglanov, *Surface integrals in a Banach space* (Russian), Mat. Sb. (N.S.) **110(152)** (1979), no. 2, 189–217, 319. MR552111

[Ug00] A. V. Uglanov, *Integration on infinite-dimensional surfaces and its applications*, Mathematics and its Applications, vol. 496, Kluwer Academic Publishers, Dordrecht, 2000. MR1742722

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