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THE DEGENERATE EISENSTEIN SERIES ATTACHED TO THE HEISENBERG PARABOLIC SUBGROUPS OF QUASI-SPLIT FORMS OF $Spin_8$

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Abstract. In [J. Inst. Math. Jussieu **14** (2015), 149–184] and [Int. Math. Res. Not. IMRN **7** (2017), 2014–2099] a family of Rankin-Selberg integrals was shown to represent the twisted standard $\mathcal{L}$-function $\mathcal{L}(s,\pi,\chi,\mathfrak{st})$ of a cuspidal representation π of the exceptional group of type G_2. These integral representations bind the analytic behavior of this $\mathcal{L}$-function with that of a family of degenerate Eisenstein series for quasi-split forms of $Spin_8$ associated to an induction from a character on the Heisenberg parabolic subgroup.

This paper is divided into two parts. In Part 1 we study the poles of these degenerate Eisenstein series in the right half-plane $\mathfrak{Re}(s)>0$. In Part 2 we use the results of Part 1 to prove the conjecture, made by J. Hundley and D. Ginzburg in [Israel J. Math. **207** (2015), 835–879], for stable poles and also to give a criterion for π to be a **CAP** representation with respect to the Borel subgroup of G_2 in terms of the analytic behavior of $\mathcal{L}(s,\pi,\chi,\mathfrak{st})$ at $s=\frac{3}{2}$.

Contents

1. Introduction

In [GS15] and [Seg17] a family of integral representations for the standard twisted $\mathcal{L}$-function $\mathcal{L}\left(s,\pi,\chi,\mathfrak{st}\right)$ of a cuspidal representation π of G_2 was considered. These

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integral representations bind the analytic properties of $\mathcal{L}(s,\pi,\chi,\mathfrak{st})$ with those of a family of degenerate Eisenstein series $\mathcal{E}_E(\chi,f_s,s,g)$ attached to a degenerate principal series induced from a character of the Heisenberg parabolic subgroup of a quasi-split form H_E of $Spin_8$. In this paper, we study the possible poles of $\mathcal{E}_E(\chi,f_s,s,g)$ and draw a few corollaries connecting the analytic properties of $\mathcal{L}(s,\pi,\chi,\mathfrak{st})$ and properties of π.

More precisely, let F be a number field, let $\mathbb{A}_F$ be its ring of adeles and let $\mathcal{P}$ denote the set of places of F. The isomorphism classes of quasi-split forms of $Spin_8$ are parametrized by étale cubic algebras over F. Such an algebra E is of one of the following types.

(1) $E=F\times F\times F$; this is called the split case.
(2) $E=F\times K$, where K is a quadratic field extension of F.
(3) E is a cubic field extension of F, either Galois or non-Galois.

If E is an étale cubic algebra over F which is not a non-Galois field extension, we call it a *Galois étale cubic algebra over* F and denote by χ_E the Hecke character of $\mathbb{A}_F^\times$ associated to E by global class field theory. In particular $\chi_{F\times F\times F}=\mathbf{1}$ and $\chi_{F\times K}=\chi_K$. Here $\mathbf{1}$ denotes the trivial character.

For an étale cubic algebra E over F there exists a quasi-split form of $Spin_8$ denoted by H_E. We fix the Heisenberg parabolic subgroup $P_E=M_E\cdot U_E$ with Levi subgroup M_E and unipotent radical U_E. The Levi subgroup M_E is given by

$$M_E\cong\left(\operatorname{Res}_{E/F}GL_2\right)^0=\left\{g\in\operatorname{Res}_{E/F}GL_2\;\middle|\;\det(g)\in\mathbb{G}_m\right\},$$

where $\det:\operatorname{Res}_{E/F}GL_2\to\operatorname{Res}_{E/F}GL_1$ is a rational map and we identify $\mathbb{G}_m=GL_1$ as a subgroup of $\operatorname{Res}_{E/F}GL_1$ in the natural way. Thus a determinant $\det_{M_E}:M_E\to\mathbb{G}_m$ of M_E is defined. For a Hecke character χ of $F^\times\backslash\mathbb{A}_F^\times$ we form the unnormalized parabolic induction

$$I_{P_E}(\chi,s)=\operatorname{Ind}_{P_E(\mathbb{A}_F)}^{H_E(\mathbb{A}_F)}\left(\chi\otimes|\cdot|^{s+\frac{5}{2}}\right)\circ\det\nolimits_{M_E}. \tag{1.1}$$

For a standard section $f_s\in I_{P_E}(\chi,s)$ we define the Eisenstein series

$$\mathcal{E}_E(\chi,f_s,s,g)=\sum_{\gamma\in P_E(F)\backslash H_E(F)}f_s(\gamma g).$$

This series converges for $\mathfrak{Re}(s)\gg0$ and admits a meromorphic continuation to the whole complex plane. We say that $\mathcal{E}_E(\chi,\cdot,s,\cdot)$ admits a pole of order n at s_0 if

$$\sup\left\{\operatorname{ord}_{s=s_0}\mathcal{E}_E(\chi,f_s,s,g)\mid f_s\in I_{P_E}(\chi,s),\ g\in H_E(\mathbb{A}_F)\right\}=n, \tag{1.2}$$

where the order $\operatorname{ord}_{s=s_0}h(s)$ of a pole of a complex function $h(s)$ at s_0 is the unique integer n such that

$$\lim_{s\to s_0}(s-s_0)^n h(s)\in\mathbb{C}^\times.$$

In Part 1 of this paper we prove the following theorem.

Theorem 4.1. *The order of the poles of $\mathcal{E}_E(\chi,\cdot,s,\cdot)$ for $\mathfrak{Re}(s)>0$ are given by the following numbers:*

	$s_0=\frac{1}{2}$	$s_0=\frac{3}{2}$		$s_0=\frac{5}{2}$
	$\chi^2=\mathbf{1}$	$\chi=\mathbf{1}$	$\chi=\chi_E,\overline{\chi_E}$	$\chi=\mathbf{1}$
$E=F\times F\times F$	1	2		1
$E=F\times K$	1	1	1	1
E Galois field extension	1	0	1	1
E non-Galois field extension	1	0		1

Namely, for a datum (E,χ,s_0,n) given in the table, $\mathcal{E}_E(\chi,\cdot,s,\cdot)$ admits a pole of order precisely n at s_0. In particular, when χ is everywhere unramified a pole of the above-mentioned order is attained by the spherical vector. For any other triple (E,χ,s_0), not appearing in the table, with $\mathfrak{Re}(s_0)\geq 0$ the degenerate Eisenstein series $\mathcal{E}_E(\chi,f_s,s,g)$ is holomorphic at s_0.

Furthermore, we find out which of the associated residual representations are square-integrable. The residual representation at $s=\frac{5}{2}$ with $\chi=\mathbf{1}$ is the trivial representation. The residual representation at $s=\frac{3}{2}$ is computed in [GGJ02] for $\chi=\chi_E$ and in [GSb] for $E=F\times K$ and $\chi=\mathbf{1}$. The residual representation at $s=\frac{1}{2}$ for various χ is computed in [Seg]. In the case where E/F is a cubic field extension, the study of the (non-degenerate) residual spectrum is carried out in [Lao].

Part 1 of this paper is dedicated to the proof of Theorem 4.1. In order to determine the orders of the poles of $\mathcal{E}_E(\chi,f_s,s,g)$ we compute its constant term along the Borel subgroup. This constant term is a sum of various intertwining operators. The poles of these intertwining operators are the possible poles of $\mathcal{E}_E(\chi,\cdot,s,\cdot)$. We proceed to check the possible cancellation of the poles of these intertwining operators. We note that while usually poles of such intertwining operators are canceled in pairs, it happens that the cancellation of poles here also happens in triples or quintuples of intertwining operators.

Part 2 of this paper is devoted to applications of Theorem 4.1 to the study of cuspidal representations of G_2. For a cuspidal representation $\pi=\bigotimes_{\nu\in\mathcal{P}}\pi_\nu$ of G_2 and a Hecke character $\chi=\bigotimes_{\nu\in\mathcal{P}}\chi_\nu$ as above, we fix a finite set of places S of F such that π_ν and χ_ν are unramified for any $\nu\notin S$. For $\nu\notin S$ we let $t_{\pi_\nu}\in G_2(\mathbb{C})$ denote the Satake parameter of π_ν. We also fix $\mathfrak{st}$ to denote the standard 7-dimensional representation of $G_2(\mathbb{C})$. We define the standard twisted partial $\mathcal{L}$-function of π to be

$$\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})=\prod_{\nu\notin S}\frac{1}{\det\left(1-\chi(\varpi_\nu)\,\mathfrak{st}(t_{\pi_\nu})\,q_\nu^{-s}\right)}, \tag{1.3}$$

where ϖ_ν is a uniformizer of F_ν and q_ν is the cardinality of the residue field of F_ν.

For an irreducible cuspidal form $\varphi\in\pi$ and a standard section $f_s\in I_{P_E}(\chi,s)$ it is proven in [Seg17] that

$$\int\limits_{G_2(F)\backslash G_2(\mathbb{A}_F)}\mathcal{E}_E(\chi,f_s,s,g)\,\varphi(g)\,dg=\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})\,d_{E,S}(s,f_S,\varphi_S), \tag{1.4}$$

where $d_{E,S}$ is a given meromorphic function. Furthermore, for any π there exists an étale cubic algebra E over F such that the integral on the left hand side of equation 1.4 is non-zero. In this case, $d_{E,S} \not\equiv 0$ and for any $s_0 \in \mathbb{C}$ one can choose f_S, φ_S such that $d_{E,S}(s, f_S, \varphi_S)$ is analytic and non-vanishing in a neighborhood of s_0. This proves the meromorphic continuation of $\mathcal{L}(s, \pi, \chi, \mathfrak{st})$, and moreover we have

$$ord_{s=s_0}\left(\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})\right) \le ord_{s=s_0}\left(\mathcal{E}_E(\chi, \cdot, s, \cdot)\right). \tag{1.5}$$

The integral in equation 1.4 can be used in order to characterize the image of functorial lifts in terms of the analytic behavior of $\mathcal{L}^S(s, \pi, \chi, \mathfrak{st})$. We apply Theorem 4.1 and equation 1.5 to classify **CAP** representations with respect to the Borel subgroup B of G_2.

We recall that a cuspidal representation π of G_2 is called **CAP** (*cuspidal associated to parabolic*) with respect to B if there exists an automorphic character τ of the torus T such that π is nearly equivalent to a subquotient of $\mathrm{Ind}_{B(\mathbb{A}_F)}^{G_2(\mathbb{A}_F)} \tau$.

We also recall that non-degenerate characters of the Heisenberg parabolic $P = M \cdot U$ of G_2 are parametrized by étale cubic algebras over F as explained in [Seg17]. Given a non-degenerate character $\Psi : U(F)\backslash U(\mathbb{A}) \to \mathbb{C}^\times$ we say that a cuspidal representation π of G_2 supports the Ψ-Fourier coefficient if

$$\exists \varphi \in \pi : \int\limits_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\,\overline{\Psi(u)}\, du \not\equiv 0.$$

We denote by $\mathcal{WF}_U(\pi)$ the set of all non-degenerate étale cubic algebras E over F such that π supports the corresponding Fourier coefficient along U. We call the set $\mathcal{WF}_U(\pi)$ *the wave front of* π *along* U; by [Gan05, Theorem 3.1] $\mathcal{WF}_U(\pi)$ is non-empty.

Given an étale cubic algebra E over F we let $S_E = \mathrm{Aut}_F(E)$ and recall the dual reductive pair

$$G_2 \times S_E \hookrightarrow H_E \rtimes S_E.$$

We denote the corresponding θ-lift from G_2 to S_E by θ_{S_E}.

For a Galois étale cubic algebra E over F let

$$n_E = \begin{cases} 2, & E = F \times F \times F, \\ 1, & \text{otherwise.} \end{cases}$$

In Section 7 we prove

Theorem 7.2. Let π be a cuspidal representation of $G(\mathbb{A})$ supporting a Fourier coefficient along U corresponding to an étale cubic algebra E over F which is not a non-Galois field extension. The following are equivalent:

(1) π is a **CAP** representation with respect to B.
(2) The partial $\mathcal{L}$-function $\mathcal{L}^S(s, \pi, \chi_E, \mathfrak{st})$ has a pole, of order n_E at $s = 2$.
(3) The θ-lift $\theta_{S_E}(\pi)$ of π to $S_E = \mathrm{Aut}_F(E)$ is non-zero. In particular π is nearly equivalent to the θ-lift $\theta_{S_E}(\mathbf{1})$, where $\mathbf{1}$ here is the automorphic trivial representation of $S_E(\mathbb{A}_F)$.

Part 1. The degenerate Eisenstein series

2. Definitions and notation

Let F be a number field and let $\mathcal{P}$ be its set of places. For any $\nu \in \mathcal{P}$ we denote by F_ν the completion of F at ν. If $\nu \nmid \infty$ we denote by $\mathcal{O}_\nu$ the ring of integers of F_ν, by ϖ_ν a uniformizer of F_ν, and by q_ν the cardinality of the residue field of F_ν. We also denote by $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of F. Also, throughout this paper we denote the trivial character of $\mathbb{A}^\times$ by $\mathbf{1}$ and the trivial character of F_ν by $\mathbf{1}_\nu$ or $\mathbf{1}$ if there is no source of confusion.

We also note that, in this paper, parabolic induction Ind_P^G for a parabolic subgroup P of a group G is unnormalized.

2.1. Quasi-split forms of $Spin_8$.

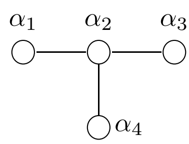
In this subsection we describe the structure of the various quasi-split forms of $Spin_8$. For more details the reader may consult the sources [Seg17, GH06, GGJ02, HMS98].

Recall from [Spr79, Section 3] the following parametrization of quasi-split forms of a split simply-connected algebraic group H defined over F:

$$\{\text{Quasi-split forms of } H \text{ over } F\} \longleftrightarrow \left\{\varphi : \mathrm{Gal}\left(\overline{F}/F\right) \to \mathrm{Aut}\left(Dyn\left(H\right)\right)\right\},$$

where $Dyn\left(H\right)$ is the Dynkin diagram of H.

The Dynkin diagram of type D_4 is given as follows:



We restrict ourselves to the case $H = Spin_8$, the split simply-connected group of type D_4. The quasi-split forms of H were described in [GH06]. Since $\mathrm{Aut}\left(Dyn\left(Spin_8\right)\right) \cong S_3$ we have

$$\left\{\begin{matrix}\text{Quasi-split forms} \\ \text{of } Spin_8 \text{ over } F\end{matrix}\right\} \longleftrightarrow \left\{\varphi : \mathrm{Gal}\left(\overline{F}/F\right) \to S_3\right\} \longleftrightarrow \left\{\begin{matrix}\text{Isomorphism classes of} \\ \text{étale cubic algebras} \\ \text{over } F\end{matrix}\right\}.$$

For any cubic algebra E let $S_E = Aut_F(E)$, which is a twisted form of S_3. An action of S_E on the algebraic group $Spin_8$ determines a simply-connected quasi-split form $H_E = Spin_8^E$ of the split group $Spin_8$ over F. As in [GGJ02, Section 3] we fix a Borel subgroup B_E containing a maximal torus T_E (both defined over F). Let Φ_{D_4} denote the set roots of $H_E \otimes \overline{F} \cong Spin_8\left(\overline{F}\right)$ with respect to $B_E\left(\overline{F}\right)$; the simple roots in Φ_{D_4} are $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Let Φ_E denote the relative root system of H_E with respect to B_E and by Δ_E the set of simple roots in Φ_E. For any γ in the relative root system of H_E we denote by F_γ the field of definition of γ. We denote the cardinality of the residue field of F_γ by q_γ. We denote by W_{H_E} the Weyl group of H_E with respect to B_E.

We now recall, from [HMS98, eq. (1.8)], that an étale cubic algebra over F is one of the following:

(1) $F \times F \times F$.
(2) $F \times K$, where K is a quadratic field extension of F.
(3) E, where E is a cubic Galois field extension of F.
(4) E, where E is a cubic non-Galois field extension of F.

We call the first three **Galois étale cubic algebras over** F. We also refer to $F\times F\times F$ as the **split cubic algebra over** F.

For a Galois étale cubic algebra E we attach an automorphic character χ_E of $F^\times\backslash\mathbb{A}_F^\times$ as follows:

(1) If $E=F\times F\times F$, then $\chi_E=\mathbf{1}$.
(2) If $E=F\times K$ when K is a field, then $\chi_E=\chi_K$, where χ_K is the quadratic automorphic character attached to K by global class field theory.
(3) If E is a field, then χ_E is the cubic automorphic character attached to E by global class field theory. Note that $\chi_E^2=\overline{\chi_E}$ satisfies the same properties, and indeed throughout this paper all statements regarding χ_E are also stated for χ_E^2.

Remark 2.1. Note that whenever E/F is a non-Galois field extension the character χ_E is not defined. Namely, if E/F is a non-Galois extension and $\chi\circ\mathrm{Nm}_{E/F}=\mathbf{1}$, then $\chi=\mathbf{1}$.

Indeed, assuming the existence of such a character, its kernel would be a subgroup of index 3 in $F^\times$ which would, due to [Mil, Theorem V.5.5, Theorem VIII.4.8], correspond to an Abelian cubic extension L of F such that $F\subset L\subset E$, which brings us to a contradiction.

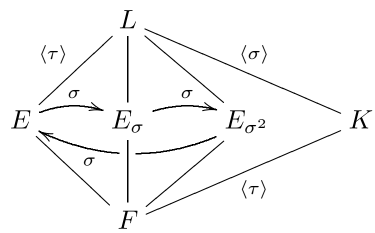
We now give a more detailed description of H_E for the different kinds of étale cubic algebras over F in terms of the action of $\mathrm{Gal}\left(\overline{F}/F\right)$ on $H_E\left(\overline{F}\right)$. This action factors through $\Gamma_E=\mathrm{Aut}_F\left(E\right)$.

(1) $\underline{E=F\times F\times F}$: In this case H_E is the split reductive simply-connected group of type D_4 over F. It corresponds to the trivial action of $\mathrm{Gal}\left(\overline{F}/F\right)$. In this case we have $\Gamma_E=\{1\}$. Also in this case
$$F_{\alpha_1}=F_{\alpha_2}=F_{\alpha_3}=F_{\alpha_4}=F.$$
(2) $\underline{E=F\times K}$: This is the case where $E=F\times K$ with K a quadratic (and hence Galois) extension of F. It is enough to define an action of $\Gamma_E=\mathrm{Gal}\left(K/F\right)=\langle\sigma\rangle$ on $Spin_8\left(K\right)$. This action is determined by
$$\begin{aligned}\sigma\left(x_{\alpha_1}\left(k\right)\right)&=x_{\alpha_1}\left(\sigma\left(k\right)\right),\\ \sigma\left(x_{\alpha_2}\left(k\right)\right)&=x_{\alpha_2}\left(\sigma\left(k\right)\right),\\ \sigma\left(x_{\alpha_3}\left(k\right)\right)&=x_{\alpha_4}\left(\sigma\left(k\right)\right),\\ \sigma\left(x_{\alpha_4}\left(k\right)\right)&=x_{\alpha_3}\left(\sigma\left(k\right)\right),\end{aligned}$$
for $k\in K$. In this case
$$F_{\alpha_1}=F_{\alpha_2}=F,\quad F_{\alpha_3}=F_{\alpha_4}=K.$$
Here, we single out the root α_1 from α_3 and α_4.
(3) $\underline{E\text{ is a cubic Galois field extension}}$: It is enough to define an action of $\Gamma_E=\mathrm{Gal}\left(E/F\right)=\left\langle\sigma\mid\sigma^3=1\right\rangle$ on $Spin_8\left(E\right)$. This action is determined by
$$\begin{aligned}\sigma\left(x_{\alpha_2}\left(k\right)\right)&=x_{\alpha_2}\left(\sigma\left(k\right)\right),\\ \sigma\left(x_{\alpha_1}\left(k\right)\right)&=x_{\alpha_3}\left(\sigma\left(k\right)\right),\\ \sigma\left(x_{\alpha_3}\left(k\right)\right)&=x_{\alpha_4}\left(\sigma\left(k\right)\right),\\ \sigma\left(x_{\alpha_4}\left(k\right)\right)&=x_{\alpha_1}\left(\sigma\left(k\right)\right),\end{aligned}$$

for $k \in E$. In this case

$$F_{\alpha_2} = F, \quad F_{\alpha_1} = F_{\alpha_3} = F_{\alpha_4} = E.$$

(4) <u>E is a cubic non-Galois field extension:</u> Here we assume that E is a cubic non-Galois extension of F. In order to define $H_E(F)$ we first consider the Galois closure L of E over F; this is a Sextic Galois extension with $\mathrm{Gal}(L/F) = \left\langle \sigma,\ \tau \;\middle|\; \sigma^3 = 1,\ \tau^2 = 1,\ (\sigma\tau)^2 = 1 \right\rangle$. Note that L is also a Galois extension of E. We consider the following tower of extensions:



where $K = L^{\langle\sigma\rangle}$ and E, $E_\sigma = L^{\langle\sigma\tau\sigma^2\rangle}$ and $E_{\sigma^2} = L^{\langle\sigma^2\tau\sigma\rangle}$ are the σ-conjugates of E in L.

The action of $\Gamma_E = \mathrm{Gal}(L/F)$ on $Spin_8(L)$ is determined by

$$\begin{aligned}
\sigma(x_{\alpha_2}(l)) &= x_{\alpha_2}(\sigma(l)), & \tau(x_{\alpha_2}(l)) &= x_{\alpha_2}(\tau(l)),\\
\sigma(x_{\alpha_1}(l)) &= x_{\alpha_3}(\sigma(l)), & \tau(x_{\alpha_1}(l)) &= x_{\alpha_3}(\tau(l)),\\
\sigma(x_{\alpha_3}(l)) &= x_{\alpha_4}(\sigma(l)), & \tau(x_{\alpha_3}(l)) &= x_{\alpha_1}(\tau(l)),\\
\sigma(x_{\alpha_4}(l)) &= x_{\alpha_1}(\sigma(l)), & \tau(x_{\alpha_4}(l)) &= x_{\alpha_4}(\tau(l)).
\end{aligned}$$

Here we singled out α_4 from α_1 and α_3, which is akin to distinguishing τ from $\sigma\tau\sigma^2$ and $\sigma^2\tau\sigma$. In this case

$$F_{\alpha_2} = F, \quad F_{\alpha_1} = E, \quad F_{\alpha_3} = E^\sigma = \sigma(E), \quad F_{\alpha_4} = E^{\sigma^2} = \sigma^2(E).$$

Let P_E be the (standard) Heisenberg parabolic subgroup of H_E with Levi decomposition $P_E = M_E \cdot U_E$ such that

$$M_E \cong \left(\mathrm{Res}_{E/F}\, GL_2\right)^0 = \left\{ g \in \mathrm{Res}_{E/F}\, GL_2 \;\middle|\; \det(g) \in \mathbb{G}_m \right\}$$

is generated by the simple roots α_1, α_3, and α_4. In particular, the determinant character $\det_{M_E}$ associated to the Levi subgroup M_E is well defined over F. Restricted to the torus, $det_{M_E}\Big|_{T_E}$ equals the highest root in Φ_{H_E}.

About the notation of roots and Weyl group elements in the quasi-split case: Fix an étale cubic algebra E over F. The absolute root system of H_E is of type D_4 with simple roots α_1, α_2, α_3, and α_4 as above. Let W_{D_4} denote the Weyl group of type D_4. We denote by $\widetilde{w_{\alpha_i}}$ the simple reflection in W_{D_4} associated to α_i. For what follows it would be more convenient to fix notation for the roots and Weyl elements in the various quasi-split groups H_E. Recall that $H_E(F) = H_E\left(\overline{F}\right)^{\Gamma_E} \subset H_E\left(\overline{F}\right)$.

- If $E = F \times F \times F$, then H_E is split over F. We denote the generators of W_{H_E} by

$$w_i = \widetilde{w_{\alpha_i}}.$$

The set of simple roots in this case is

$$\Delta_E = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$$

Furthermore, we write $[a_1, a_2, a_3, a_4]$ for

$$[a_1, a_2, a_3, a_4] = \sum_{i=1}^{4} a_i \alpha_i.$$

We note here that for $t = h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) h_{\alpha_3}(t_3) h_{\alpha_4}(t_4)$, where $t_1, t_2, t_3, t_4 \in F^\times$, it holds that

$$[a_1, a_2, a_3, a_4](t) = |t_1|_F^{2a_1 - a_2} |t_2|_F^{2a_2 - a_1 - a_3 - a_4} |t_3|_F^{2a_3 - a_2} |t_4|_F^{2a_4 - a_2}.$$

- If $E = F \times K$, then the relative root system of H_E with respect to T_E is of type B_3, and we denote the generators of W_{H_E} by

$$w_1 = \widetilde{w_{\alpha_1}},\ w_2 = \widetilde{w_{\alpha_2}},\ w_3 = \widetilde{w_{\alpha_3}} \widetilde{w_{\alpha_4}}.$$

The set of simple roots in this case is

$$\Delta_E = \{\alpha_1, \alpha_2, (\alpha_3 + \alpha_4)\}.$$

Furthermore, we write $[a_1, a_2, a_3]$ for

$$[a_1, a_2, a_3] = a_1\alpha_1 + a_2\alpha_2 + a_3(\alpha_3 + \alpha_4) = a_1\alpha_1 + a_2\alpha_2 + a_3\left(\mathrm{Nm}_{K/F} \circ \alpha_3\right).$$

We note here that for

$$t = h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) h_{\alpha_3\alpha_4^\sigma}(t_3) = h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) h_{\alpha_3}(t_3) h_{\alpha_3}(t_3^\sigma),$$

where $t_1, t_2 \in F^\times, t_3 \in K^\times$, it holds that

$$[a_1, a_2, a_3](t) = |t_1|_F^{2a_1 - a_2} |t_2|_F^{2a_2 - a_1 - 2a_3} |t_3|_K^{2a_3 - a_2}.$$

Remark 2.2. Note that given a place ν of F such that $K_\nu = F_\nu \times F_\nu$, the Weyl element $w_3 \in W_{H_E(\mathbb{A})}$ should be interpreted as $w_3 w_4 \in W_{H_E(F_\nu)}$.

- If E/F is a field extension, then the relative root system of H_E with respect to T_E is of type G_2, and we denote the generators of W_{H_E} by

$$w_1 = \widetilde{w_{\alpha_1}} \widetilde{w_{\alpha_3}} \widetilde{w_{\alpha_4}},\ w_2 = \widetilde{w_{\alpha_2}}.$$

The set of simple roots in this case is

$$\Delta_E = \{(\alpha_1 + \alpha_3 + \alpha_4), \alpha_2\}.$$

Furthermore, we write $[a_1, a_2]$ for

$$[a_1, a_2] = a_1(\alpha_1 + \alpha_3 + \alpha_4) + a_2\alpha_2 = a_1\left(\mathrm{Nm}_{E/F} \circ \alpha_1\right) + a_2\alpha_2.$$

We note here that for

$$t = h_{\alpha_1\alpha_3^\sigma\alpha_4^{\sigma^2}}(t_1) h_{\alpha_2}(t_2) = h_{\alpha_1}(t_1) h_{\alpha_2}(t_2) h_{\alpha_3}(t_1^\sigma) h_{\alpha_4}\left(t_1^{\sigma^2}\right),$$

where $t_1 \in E^\times, t_2 \in F^\times$, it holds that

$$[a_1, a_2](t) = |t_1|_E^{2a_1 - a_2} |t_2|_F^{2a_2 - 3a_1}.$$

Remark 2.3. Note that given a place ν of F such that $E_\nu = F_\nu \times F_\nu \times F_\nu$, the Weyl element $w_1 \in W_{H_E(\mathbb{A})}$ should be interpreted as $w_1 w_3 w_4 \in W_{H_E(F_\nu)}$. At a place where $E_\nu = F_\nu \times K_\nu$ (in particular E/F is a non-Galois extension), $w_1 \in W_{H_E(\mathbb{A})}$ should be interpreted as $w_1 w_3 \in W_{H_E(F_\nu)}$.

For any such quasi-split form of $Spin_8$ we denote $w_{i_1,...,i_k}$ or $w[i_1, ..., i_k]$ for $w_{\alpha_{i_1}} \cdots w_{\alpha_{i_k}}$.

2.2. **The degenerate Eisenstein series.** Fix a finite order Hecke character $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$. We consider the induced representation

$$I_{P_E}(\chi, s) = \mathrm{Ind}_{P_E(\mathbb{A})}^{H_E(\mathbb{A})} (\chi \circ \det_{M_E}) \otimes |\det_{M_E}|^{s+\frac{5}{2}}, \tag{2.1}$$

where, as mentioned above, the induction on the right hand side is unnormalized. We note that $|\det_{M_E}|^{\frac{5}{2}}$ is the normalization factor[1] of the parabolic induction, and hence the induced representation on the left hand side is normalized.

For any holomorphic section $f_s \in I_{P_E}(\chi, s)$ we define the following degenerate Eisenstein series:

$$\mathcal{E}_E(\chi, f_s, s, g) = \sum_{\gamma \in P_E(F)\backslash H_E(F)} f_s(\gamma g). \tag{2.2}$$

This series converges for $\mathfrak{Re}(s) \gg 0$ and admits a meromorphic continuation to the whole complex plane. For any $s_0 \in \mathbb{C}$ we write the Laurent expansion of $\mathcal{E}_E(\chi, f_s, s, g)$:

$$\mathcal{E}_E(\chi, f_s, s, g) = \sum_{k=-\infty}^{\infty} (s - s_0)^k [\Lambda_k(\chi, s_0) f_s](g),$$

where for each k the coefficient $\Lambda_k(\chi, s_0)$ is an intertwining map,

$$\Lambda_k(\chi, s_0) : I_{P_E}(\chi, s_0) \to \mathcal{A}(H_E) / Im(\Lambda_{k-1}(\chi, s_0)),$$

where $\mathcal{A}(H_E)$ is the space of automorphic forms of $H_E(\mathbb{A})$. In particular, if the order of $\mathcal{E}_E(\chi, \cdot, s, \cdot)$ at s_0 is n, then $\Lambda_{-n}(\chi, s_0)$ is an intertwining map from $I_{P_E}(\chi, s_0)$ to $\mathcal{A}(H_E)$.

Part 1 of this paper is devoted to the study of the analytic properties of $\mathcal{E}_E(\chi, f_s, s, g)$ in the right half-plane $\mathfrak{Re}(s) > 0$.

3. Background theory on Eisenstein series and intertwining operators

In this section we recall some general information regarding the theory of Eisenstein series. Most of the results quoted in this section can be found in [MW95].

3.1. **Intertwining operators and the constant term.** We start by noting that

$$I_{P_E}(\chi, s) \hookrightarrow I_{B_E}(\chi_s) = \mathrm{Ind}_{B_E(\mathbb{A})}^{H_E(\mathbb{A})} \delta_{B_E}^{\frac{1}{2}} \chi_s, \tag{3.1}$$

where

$$\chi_s = \delta_{B_E}^{-\frac{1}{2}} \otimes (\chi \circ \det_{M_E}) \otimes |\det_{M_E}|^{s+\frac{5}{2}}.$$

Note that, as above, the induction on the right hand side is unnormalized, while the induced representation on the left hand side is normalized.

For any $w \in W$ we define the intertwining operator

$$M(w, \chi_s) : I_{B_E}(\chi_s) \to I_{B_E}\left(w^{-1} \cdot \chi_s\right)$$

by

$$M(w, \chi_s) f_s(g) = \int_{N_E(\mathbb{A}) \cap w^{-1} N_E(\mathbb{A}) w \backslash N_E(\mathbb{A})} f_s(wng)\, dn. \tag{3.2}$$

---

[1]For the modulus character of P_E it holds that $\delta_{P_E}\Big|_{M_E} = |\det_{M_E}|^5$.

This integral converges for $\mathfrak{Re}(s) \gg 0$ and admits a meromorphic continuation to $\mathbb{C}$. When there is no source of confusion we denote $M(w, \chi_s)$ by $M(w)$ or M_w.

Remark 3.1. Note that the definition here is slightly different from the definition given in [MW95]. As a consequence, if $w = w'w''$, then a cocycle equation is satisfied:

$$M(w, \chi_s) = M\left(w'', w'^{-1} \cdot \chi_s\right) \circ M(w', \chi_s). \tag{3.3}$$

The constant term of $\mathcal{E}_E(\chi, f_s, s, g)$ along N_E is defined to be

$$\mathcal{E}_E(\chi, f_s, s, t)_{B_E} = \int\limits_{N_E(F)\backslash N_E(\mathbb{A})} \mathcal{E}_E(\chi, f_s, s, ut)\, du \quad \forall t \in T_E(\mathbb{A}). \tag{3.4}$$

By a standard computation as in [GRS97], we obtain

$$\mathcal{E}_E(\chi, f_s, s, t)_{B_E} = \sum_{w \in W(P_E, H_E)} \left(M(w, \chi_s) f_s\right)\Big|_{T_E}(t) \quad \forall t \in T_E(\mathbb{A}), \tag{3.5}$$

where $W(P_E, H_E) = \left\{w \in W_{H_E} \mid w^{-1}(\gamma) > 0\ \forall \gamma \in \Delta_E \setminus \{\alpha_2\}\right\}$ is a set of distinguished representatives for $P_E\backslash H_E/B_E \cong W_{P_E}\backslash W_{H_E}$, given by the shortest representative of each coset.

Theorem 3.2. *The degenerate Eisenstein series $\mathcal{E}_E(\chi, f_s, s, g)$ admits a pole of order n at (χ, s_0) if and only if its constant term $\mathcal{E}_E(\chi, f_s, s, g)_{B_E}$ admits a pole of order n at (χ, s_0).*

Indeed, in Section 4 we study the poles of $\mathcal{E}_E(\chi, f_s, s, g)$ via the poles of $\mathcal{E}_E(\chi, f_s, s, g)_{B_E}$, using equation 3.5.

3.2. **Rank-one intertwining operators and local factors.** In many instances, the study of Eisenstein series and intertwining operators relies on reduction to the rank-one case via the functional equation, equation 3.3. In this section, we recall some useful facts about the rank-one case and the reduction to it.

We fix a number field extension L/F and let D_L be the discriminant of $L/\mathbb{Q}$. Also let $\mathbb{A}_L$ be the ring of adeles of L, let $\mathcal{P}_L$ be the set of places of L, and for a finite place ν of L, denote by $\mathcal{O}_\nu$ the ring of integers of L_ν, by ϖ_ν a uniformizer of $\mathcal{O}_\nu$, and by q_ν the cardinality of the residue field of L_ν.

Let $\zeta_L(s)$ be the completed ζ-function of L. Following [Ike92], we define

$$\xi_L(s) = |D_L|^{\frac{s}{2}} \zeta_L(s).$$

The normalized function ξ_L then satisfies the functional equation

$$\xi_L(s) = \xi_L(1-s). \tag{3.6}$$

Let $\mathcal{B} = \mathcal{T} \cdot \mathcal{N}$ be the Borel subgroup of SL_2 with torus $\mathcal{T}$ and unipotent radical $\mathcal{N}$. Also let $w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the generator of the Weyl group of SL_2. We recall some facts about the intertwining operator M_{w_0} defined on representations of SL_2. Fix a Hecke character $\sigma = \bigotimes_{\nu \in \mathcal{P}} \sigma_\nu$ of $\mathcal{T}(\mathbb{A}_L)$; it can be considered as a representation of $\mathcal{B}(\mathbb{A}_L)$. For a section $f_s \in \operatorname{Ind}_{\mathcal{B}(\mathbb{A}_L)}^{SL_2(\mathbb{A}_L)} \sigma \delta_{\mathcal{B}}^{s+\frac{1}{2}}$ we let

$$M(w_0, \sigma, s) f_s(g) = \int\limits_{\mathcal{N}(\mathbb{A}_L) \cap w_0^{-1} \mathcal{N}(\mathbb{A}_L) w_0 \backslash \mathcal{N}(\mathbb{A}_L)} f_s(w_0 n g)\, dn = \int\limits_{\mathcal{N}(\mathbb{A}_L)} f_s(w_0 n g)\, dn. \tag{3.7}$$

This integral converges for $\mathfrak{Re}(s) \gg 0$ and admits meromorphic continuation to the whole complex plane. The intertwining operator M_{w_0} is factorizable in the sense that if $f_s = \bigotimes f_{s,\nu}$, then $M(w_0, \sigma, s) f_s = \bigotimes_{\nu \in \mathcal{P}} M(w_0, \sigma_\nu, s) f_{s,\nu}$, where for $\mathfrak{Re}(s) \gg 0$,

$$M(w_0, \sigma_\nu, s) f_{s,\nu}(g) = \int\limits_{\mathcal{N}(L_\nu)} f_{s,\nu}(w_0 n g)\, dn. \tag{3.8}$$

This integral admits a meromorphic continuation to $\mathbb{C}$. For a spherical section $f^0_{s,\nu}$ (in particular σ_ν is unramified) of $\mathrm{Ind}_{\mathcal{B}(L_\nu)}^{SL_2(L_\nu)} \sigma_\nu \delta_{\mathcal{B}}^{s+\frac{1}{2}}$ it holds that

$$M(w_0, \sigma_\nu, s) f^0_{s,\nu} = \frac{\mathcal{L}_{L_\nu}(2s, \sigma_\nu)}{\mathcal{L}_{L_\nu}(2s+1, \sigma_\nu)} f^0_{-s,\nu}, \tag{3.9}$$

where:

- For $\nu \nmid \infty$ and σ_ν unramified
$$\mathcal{L}_{L_\nu}(s, \sigma_\nu) = \frac{1}{1 - \sigma_\nu(\varpi_\nu) q_\nu^{-s}}.$$
This function is a non-vanishing meromorphic function on $\mathbb{C}$ with simple poles at $s = \frac{\log(\sigma_\nu(\varpi_\nu)) + 2\pi i n}{\log(q_\nu)}$ for all $n \in \mathbb{Z}$.
- As in [Kud03, eq. 3.16], if $\nu \nmid \infty$ and σ_ν is ramified, we define $\mathcal{L}_{L_\nu}(s, \sigma_\nu) = 1$.
- The only finite order characters σ_ν of $\mathbb{R}^\times$ are either the trivial one or the sign character. Let
$$\epsilon_\nu = \begin{cases} 0, & \sigma_\nu = \mathbf{1}, \\ 1, & \sigma_\nu = \mathrm{sgn} \end{cases}$$
and
$$\mathcal{L}_{\mathbb{R}}(s, \sigma_\nu) = \pi^{-\frac{s+\epsilon_\nu}{2}} \Gamma\left(\frac{s+\epsilon_\nu}{2}\right).$$
- The only finite order character σ_ν of $\mathbb{C}^\times$ is the trivial one. For $n \in \mathbb{Z}$ let
$$\sigma_{n,\nu}(z) = \left(\frac{z}{|z|}\right)^n.$$
Note that any continuous complex character of $\mathbb{C}^\times$ is of the form $\sigma_n(z)|z|^s$ for some $n \in \mathbb{Z}$ and $s \in \mathbb{C}$. Let
$$\mathcal{L}_{\mathbb{C}}(s, \sigma_{n,\nu}) = 2(2\pi)^{-\left(s+\frac{|n|}{2}\right)} \Gamma\left(s + \frac{|n|}{2}\right).$$

Recall that $\Gamma(z)$ is a non-vanishing meromorphic function on $\mathbb{C}$ whose only poles are simple, appearing at the points $z = -n$ for $n \geq 0$.

We fix an additive character $\psi = \bigotimes_{\nu \in \mathcal{P}} \psi_\nu : L \backslash \mathbb{A}_L \to \mathbb{C}^\times$. For simplicity, we assume that ψ_ν has conductor 0 at all finite places.We also fix a global measure $dx = \prod\limits_{\nu \in \mathcal{P}_L} dx_\nu$ such that

$$\int\limits_{\mathcal{O}_\nu} dx_\nu = 1 \quad \forall \nu \in \mathcal{P}_\infty.$$

Let $\epsilon_{L_\nu}(s,\sigma_\nu,\psi_\nu)$ be the local ϵ-factor as defined in [Kud03, Corollary 3.7]. We recall a few facts regarding ϵ_{L_ν}:

- $\epsilon_{L_\nu}(s,\sigma_\nu,\psi_\nu)$ is entire in s.
- For any finite ν such that σ_ν is unramified, it holds that $\epsilon_{L_\nu}(s,\sigma_\nu,\psi_\nu)=1$.
- For any ν it holds that $\epsilon_{L_\nu}(s,\sigma_\nu,\psi_\nu)\,\epsilon_{L_\nu}(1-s,\widetilde{\sigma_\nu},\psi_\nu)=1$.

Remark 3.3. We recall the global functional equation

$$\mathcal{L}_L(s,\sigma)=\epsilon_L(s,\sigma)\,\mathcal{L}_L(1-s,\overline{\sigma}), \tag{3.10}$$

where

$$\mathcal{L}_L(s,\sigma)=\prod_{\nu\in\mathcal{P}_L}\mathcal{L}_{L_\nu}(s,\sigma_\nu),\quad \epsilon_L(s,\sigma)=\prod_{\nu\in\mathcal{P}_L}\epsilon_{L_\nu}(s,\sigma_\nu,\psi_\nu).$$

Note that if σ is unitary, then $\sigma^{-1}=\overline{\sigma}$. Also note that fixing a finite subset $S\subset\mathcal{P}$ such that all data is unramified outside S, it holds that

$$\epsilon_L(s,\sigma)=\prod_{\nu\in S}\epsilon_{L_\nu}(s,\sigma_\nu,\psi_\nu).$$

Remark 3.4. We also recall the local functional equation [Kud03, eq. 3.26]

$$\mathcal{L}_{L_\nu}(s,\sigma_\nu)=\frac{\epsilon_{L_\nu}(s,\sigma_\nu,\psi_\nu)}{\gamma_{L_\nu}(s,\sigma_\nu,\psi_\nu)}\mathcal{L}_{L_\nu}\left(1-s,\sigma_\nu^{-1}\right), \tag{3.11}$$

where $\gamma_{L_\nu}(s,\sigma_\nu,\psi_\nu)$ is the local γ-factor as defined in [Kud03]. In particular

$$\prod_{\nu\in\mathcal{P}_L}\gamma_{L_\nu}(s,\sigma_\nu,\psi_\nu)=1.$$

Studying the analytic behavior of $M(w_0,\sigma_\nu,s)$, we have the following lemma ([Win78] for $\nu\nmid\infty$ and [Sha80] for $\nu|\infty$):

Lemma 3.5. *For any $\sigma_\nu: L_\nu^\times\to\mathbb{C}^\times$ the operator $\frac{1}{\mathcal{L}_{L_\nu}(2s,\sigma_\nu)}M(w_0,\sigma_\nu,s)$ is entire and non-vanishing.*

The *normalized intertwining operator* is defined to be

$$N(w_0,\sigma_\nu,s)=\frac{\mathcal{L}_{L_\nu}(2s+1,\sigma_\nu)}{\mathcal{L}_{L_\nu}(2s,\sigma_\nu)\,\epsilon_{L_\nu}(2s,\sigma_\nu,\psi_\nu)}M(w_0,\sigma_\nu,s).$$

It follows from equation 3.9 that

$$N(w_0,\sigma_\nu,s)\,f^0_{s,\nu}=f^0_{-s,\nu}. \tag{3.12}$$

For $\nu\nmid\infty$ it holds that (from the above, [Tad12, Section 11], and [Tad94, Section 5]):

- The operator $M(w_0,\sigma_\nu,s)$ is entire for σ_ν ramified and so is the normalized intertwining operator $N(w_0,\sigma_\nu,s)$.
- If σ_ν is unramified, then $M(w_0,\sigma_\nu,s)$ is meromorphic with a simple pole at $\frac{\log(\sigma_\nu(\varpi_\nu))+2\pi in}{\log(q_\nu)}$ for all $n\in\mathbb{Z}$. The normalized intertwining operators $N(w_0,\sigma_\nu,s)$ admit a simple pole at $s=-\frac{1}{2}+\frac{\log(\sigma_\nu(\varpi_\nu))+2\pi in}{\log(q_\nu)}$ for all $n\in\mathbb{Z}$.
- Furthermore, when $\sigma_\nu=\mathbf{1}$ then $M(w_0,\sigma_\nu,s)$ is not injective at $s=\frac{1}{2}$ and $s=-\frac{1}{2}$. The normalized intertwining operator $N(w_0,\sigma_\nu,s)$ is not injective at $s=\frac{1}{2}$, and its residue is not injective at $s=-\frac{1}{2}$.

- In particular, for $\sigma_\nu = \mathbf{1}$ we have

$$0 \longrightarrow \mathbf{1} \longrightarrow \mathrm{Ind}_{\mathcal{B}(L_\nu)}^{SL_2(L_\nu)} \mathbf{1} \overset{M_{w_0}}{\longrightarrow} St \longrightarrow 0,$$

$$0 \longrightarrow St \longrightarrow \mathrm{Ind}_{\mathcal{B}(L_\nu)}^{SL_2(L_\nu)} \delta_{\mathcal{B}}^{1} \overset{M_{w_0}}{\longrightarrow} \mathbf{1} \longrightarrow 0,$$

  where St is the Steinberg representation.
- If σ_ν is a non-trivial quadratic character, then $\mathrm{Ind}_{\mathcal{B}(L_\nu)}^{SL_2(L_\nu)} \sigma\delta_{\mathcal{B}}^{\frac{1}{2}}$ is reducible. In this case, $\mathrm{Ind}_{\mathcal{B}(L_\nu)}^{SL_2(L_\nu)} \sigma\delta_{\mathcal{B}}^{\frac{1}{2}} = \pi_\nu^{(1)} \oplus \pi_\nu^{(-1)}$ where $\pi_\nu^{(1)}$ and $\pi_\nu^{(-1)}$ are irreducible, and if σ_ν is unramified, then $\pi_\nu^{(1)}$ is also unramified. On the other hand, $\pi_\nu^{(-1)}$ is an irreducible representation unramified with respect to the compact subgroup $d \cdot SL_2(\mathcal{O}_\nu) \cdot d^{-1}$, where

$$d = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_\nu \end{pmatrix}.$$

  Furthermore, $M_\nu(w_0, \sigma_\nu, 0)$ is bijective and acts as multiplication by a scalar on $\pi_\nu^{(1)}$ and $\pi_\nu^{(-1)}$. The normalized intertwining operator $N_\nu(w_0, \sigma_\nu, s)$ acts on $\pi^{(\epsilon)}$ as ϵId.
- The only reducible principal series representations are those described in the previous bullets.

We now discuss the case $\nu|\infty$ (from the above and [Kna01, Chapters II and VII]).

If $L_\nu = \mathbb{R}$, then $\Pi_{\epsilon_\nu,s} = \mathrm{Ind}_{\mathcal{B}(L_\nu)}^{SL_2(L_\nu)} \sigma_\nu \delta_B^{s+\frac{1}{2}}$ is reducible if and only if $2s = n \in \mathbb{Z}$ and

$$\epsilon_\nu \equiv n + 1 \mod 2,$$

in which case, the decomposition series for $\Pi_{\epsilon_\nu,s}$ is as follows:

- For $s = 0$ it holds that

$$\Pi_{\epsilon_\nu,s} = \mathcal{D}_1^+ \oplus \mathcal{D}_1^-,$$

  where $\mathcal{D}_1^+$ and $\mathcal{D}_1^-$ are irreducible representations known as the holomorphic and non-holomorphic limits of discrete series (respectively).
- For $2s = n \in \mathbb{N}$ it holds that

$$\mathcal{D}_{n-1}^+ \oplus \mathcal{D}_{n-1}^- \hookrightarrow \Pi_{\epsilon_\nu,s} \twoheadrightarrow \Phi_{n-1},$$

  where Φ_{n-1} is the unique irreducible representation of $SL_2(\mathbb{R})$ of dimension $n-1$ and $\mathcal{D}_{n-1}^+$, $\mathcal{D}_{n-1}^-$ are the irreducible representations known as the holomorphic and non-holomorphic discrete series of highest weight $n-1$.
- For $-2s = n \in \mathbb{N}$ it holds that

$$\Phi_{n-1} \hookrightarrow \Pi_{\epsilon_\nu,s} \twoheadrightarrow \mathcal{D}_{n-1}^+ \oplus \mathcal{D}_{n-1}^-.$$

If $F_\nu = \mathbb{C}$, then $\Pi_{n,s} = \mathrm{Ind}_{\mathcal{B}(L_\nu)}^{SL_2(L_\nu)} \sigma_{n,\nu} |\cdot|^s$ is reducible if and only if $n = l - k$ and $4s = 2 + k + l$ or $n = k - l$ and $4s = -(2 + k + l)$ for $k, l \in \mathbb{N} \cup \{0\}$, in which case

- If $n = l - k$ and $4s = 2 + k + l$, then

$$\Phi_{k,l} \hookrightarrow \Pi_{n,s} \twoheadrightarrow \mathcal{E}_{n-1}^+ \oplus \mathcal{E}_{n-1}^-,$$

  where $\Phi_{k,l}$ is the finite-dimensional representation realized as polynomials in the complex variables $(z_1, z_2, \overline{z_1}, \overline{z_2})$, homogeneous of degree k in (z_1, z_2)

and homogeneous of degree l in $(\overline{z_1},\overline{z_2})$. $\mathcal{E}^+_{n-1}$ and $\mathcal{E}^-_{n-1}$ are analogous to $\mathcal{D}^+_{n-1}$ and $\mathcal{D}^-_{n-1}$.

The following lemma follows from the above discussion.

Lemma 3.6. *For a place $\nu\in\mathcal{P}$, it holds that*

$$N(w_0,s,\sigma_\nu)\circ N(w_0,-s,\overline{\sigma_\nu})=\mathrm{Id}\quad \forall s\in\mathbb{C}.$$

For $s_0\in\mathbb{R}$ such that $N(w_0,s,\sigma_\nu)$ admits a pole at s_0 or $-s_0$ this should be understood as

$$\lim_{s\to s_0} N(w_0,s,\sigma_\nu)\circ N(w_0,-s,\overline{\sigma_\nu})=\mathrm{Id}\,.$$

We finish the discussion of the rank-one case by recalling two results regarding the global intertwining operator; one of them is the global analog of Lemma 3.6. As $M(w_0,s,\sigma)\circ M(w_0,-s,\sigma)$ is an endomorphism of irreducible representations for all $s\in\mathbb{C}$ such that $2s\notin\mathbb{Z}$, it equals a constant.

Lemma 3.7 ([Lan76], Lemma 6.3)**.** *It holds that*

$$M(w_0,s,\sigma)\circ M(w_0,-s,\overline{\sigma})=\mathrm{Id}\quad \forall s\in\mathbb{C}.$$

For $s=\pm\frac{1}{2}$ with $\sigma=\mathbf{1}$ this should be understood as

$$\lim_{s\to\pm\frac{1}{2}} M(w_0,s,\sigma)\circ M(w_0,-s,\overline{\sigma})=\mathrm{Id}\,.$$

We would also like to recall [Ike92, Lemma 1.5]:

Lemma 3.8. *For $\sigma=\mathbf{1}$, the operator $M(w_0,s,\mathbf{1})$ is holomorphic at $s_0=0$ and is equal to the scalar multiplication by -1 at $s_0=0$.*

3.3. **Intertwining operators for induced representations of H_E.** At this point, it will be beneficial to consider a more general point of view. Let $\mathfrak{a}^*_{\mathbb{C}}=X^*(T_E)\otimes\mathbb{C}$. We equip $\mathfrak{a}^*_{\mathbb{C}}$ with the following system of coordinates:

- If $E=F\times F\times F$ we have $\mathfrak{a}^*_{\mathbb{C}}\cong\mathbb{C}^4$ and we write $\lambda=(s_1,s_2,s_3,s_4)\in\mathfrak{a}^*_{\mathbb{C}}$ for

$$\lambda\left(h_{\alpha_1}(t_1)h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)h_{\alpha_4}(t_4)\right)=|t_1|_F^{s_1}|t_2|_F^{s_2}|t_3|_F^{s_3}|t_4|_F^{s_4}\quad \forall t_1,t_2,t_3,t_4\in F^\times.$$

- If $E=F\times K$ we have $\mathfrak{a}^*_{\mathbb{C}}\cong\mathbb{C}^3$ and we write $\lambda=(s_1,s_2,s_3)\in\mathfrak{a}^*_{\mathbb{C}}$ for

$$\lambda\left(h_{\alpha_1}(t_1)h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)h_{\alpha_4}(t_3^\sigma)\right)=|t_1|_F^{s_1}|t_2|_F^{s_2}|t_3|_K^{s_3}\quad \forall t_1,t_2\in F^\times,\ \forall t_3\in K^\times.$$

- If E is a field we have $\mathfrak{a}^*_{\mathbb{C}}\cong\mathbb{C}^2$ and we write $\lambda=(s_1,s_2)\in\mathfrak{a}^*_{\mathbb{C}}$ for

$$\lambda\left(h_{\alpha_1}(t_1)h_{\alpha_2}(t_2)h_{\alpha_3}(t_1^\sigma)h_{\alpha_4}\left(t_1^{\sigma^2}\right)\right)=|t_1|_E^{s_1}|t_2|_F^{s_2}\quad \forall t_2\in F^\times,\ \forall t_1\in E^\times.$$

For any finite order character $\chi=\bigotimes_{\nu\in\mathcal{P}}\chi_\nu$ of $T_E(\mathbb{A})$ and any $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ we let

$$I_{B_E}(\chi,\lambda)=\mathrm{Ind}_{B_E(\mathbb{A})}^{H_E(\mathbb{A})}(\chi\circ\det_{M_E})\cdot(\lambda+\rho_{B_E})=\bigotimes_{\nu\in\mathcal{P}} I_{B_E}(\chi_\nu,\lambda),$$

$$I_{B_E}(\chi_\nu,\lambda)=\mathrm{Ind}_{B_E(F_\nu)}^{H_E(F_\nu)}(\chi_\nu\circ\det_{M_E})\cdot(\lambda+\rho_{B_E}),$$

where ρ_{B_E} is half the sum of positive roots in H_E with respect to B_E. We note, as above, that the induction on the right hand side is unnormalized, while the induced representation on the left hand side is normalized. This is not the most general principal series representation, but it will suffice for our needs. We note that

$$I_{P_E}(\chi,s)\hookrightarrow I_{B_E}(\chi_s)=I_{B_E}(\chi,\lambda_s),$$

where

$$\lambda_s = \begin{cases} \left(-1, s+\frac{3}{2}, -1, -1\right), & E = F\times F\times F, \\ \left(-1, s+\frac{3}{2}, -1\right), & E = F\times K, \\ \left(-1, s+\frac{3}{2}\right) & E/F \text{ is a cubic field extension.} \end{cases} \tag{3.13}$$

For $w\in W$ and a holomorphic section $f_\lambda \in I_B(\chi,\lambda)$ let

$$M(w,\chi,\lambda) f_\lambda(g) = \int\limits_{N_E(\mathbb{A})\cap w^{-1}N_E(\mathbb{A})w\backslash N_E(\mathbb{A})} f_\lambda(wng)\, dn. \tag{3.14}$$

This integral converges absolutely to an analytic function in the positive Weyl chamber

$$C^+ = \{\lambda\in\mathfrak{a}_{\mathbb{C}}^* \mid \mathfrak{Re}\langle\lambda,\alpha^\vee\rangle > 0\ \forall\alpha>0\}$$

and admits a meromorphic continuation to $\mathfrak{a}_{\mathbb{C}}^*$.

Remark 3.9. Due to the choice of representatives in $W(P_E,H_E)$, the intertwining operators $M(w,\chi_s)$ defined in equation 3.2 are generically (at points of holomorphy) restrictions of $M(w,\chi,\lambda)$ to the line λ_s as above.

Note that, by abuse of notation, for a Hecke character χ we identify χ and $\chi\circ\det_{M_E}$.

We recall that $M(w,\chi,\lambda)$ and $M(w,\chi_s)$ can be decomposed as

$$\begin{aligned} M(w,\chi,\lambda) &= \bigotimes_{\nu\in\mathcal{P}} M_\nu(w,\chi_\nu,\lambda), \\ M(w,\chi_s) &= \bigotimes_{\nu\in\mathcal{P}} M_\nu(w,\chi_{\nu,s}), \end{aligned} \tag{3.15}$$

where for any $\nu\in\mathcal{P}$, $\lambda\in C^+$, and $\mathfrak{Re}(s)\gg 0$, the local intertwining operators $M(w,\chi_\nu,\lambda)$ and $M_\nu(w,\chi_s)$ are defined via

$$\begin{aligned} M(w,\chi_\nu,\lambda) f_{\lambda,\nu}(g) &= \int\limits_{N_E(F_\nu)\cap w^{-1}N_E(F_\nu)w\backslash N_E(F_\nu)} f_{\lambda,\nu}(wng)\, dn, \\ M(w,\chi_{s,\nu}) f_{s,\nu}(g) &= \int\limits_{N_E(F_\nu)\cap w^{-1}N_E(F_\nu)w\backslash N_E(F_\nu)} f_{s,\nu}(wng)\, dn. \end{aligned} \tag{3.16}$$

These integrals converge for $\lambda\in C^+$ and $\mathfrak{Re}(s)\gg 0$ respectively and admit a meromorphic continuation to $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathbb{C}$ respectively.

We now recall the connection between the rank-one case and the intertwining operators M_{w_α}, where w_α is the simple reflection with respect to a simple root α. For any simple root α, we have an embedding $\iota_\alpha : SL_2\to H_E$, defined over F_α, so that

$$\iota_\alpha\left(\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}\right) = h_\alpha(t), \quad \iota_\alpha\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right) = x_\alpha(x), \quad \iota_\alpha\left(\begin{pmatrix} 1 & 0\\ x & 1\end{pmatrix}\right) = x_{-\alpha}(x),$$

$$\iota_\alpha\left(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\right) = w_\alpha.$$

We denote by T_α the image of h_α.

Lemma 3.10. *The following diagram is commutative:*

$$\begin{array}{ccc} I_{B_E}(\chi_\nu,\lambda) & \xrightarrow{M_\nu(w_\alpha,\chi_\nu,\lambda)} & I_{B_E}(w_\alpha\cdot\chi_\nu, w_\alpha\cdot\lambda) \\ \downarrow{\scriptstyle \iota_\alpha^*} & & \downarrow{\scriptstyle \iota_\alpha^*} \\ \mathrm{Ind}_{\mathcal{B}(F_\nu)}^{SL_2(F_\nu)}\left(\left.[\chi_\nu\otimes\lambda]\right|_{T_\alpha}\right) & \xrightarrow{M_{w_0}} & \mathrm{Ind}_{\mathcal{B}(F_\nu)}^{SL_2(F_\nu)}\left(\left.w_\alpha\cdot[\chi_\nu\otimes\lambda]\right|_{T_\alpha}\right) \end{array}$$

where the vertical maps should be understood as the pull-back map. By restriction to $I_{P_E}(\chi,s)$, this is also true for $M(w_\alpha,\chi_s)$.

Proof. We note that

$$N_E(F_\nu)\cap w_\alpha^{-1}N_E(F_\nu)w_\alpha\backslash N_E(F_\nu)=\iota_\alpha\left(\mathcal{N}(F_\nu)\cap w_0^{-1}\mathcal{N}(F_\nu)w_0\backslash\mathcal{N}(F_\nu)\right)$$

and that

$$\mathcal{N}(F_\nu)\cap w_0^{-1}\mathcal{N}(F_\nu)w_0\backslash\mathcal{N}(F_\nu)\cong\mathcal{N}(F_\nu).$$

Consequently, for $f_{s,\nu}\in I_{B_E}(\chi_{\nu,s})$ and $g\in SL_2(F_\nu)$ it holds that

$$\begin{aligned} M_{w_0}\iota_\alpha^*(f_{s,\nu})(g) &= \int\limits_{\mathcal{N}(F_\nu)} \iota_\alpha^*(f_{s,\nu})(w_0ng)\,dn \\ &= \int\limits_{\mathcal{N}(F_\nu)} (f_{s,\nu})(\iota_\alpha(w_0ng))\,dn \\ &= \int\limits_{N_E(F_\nu)\cap w_\alpha^{-1}N_E(F_\nu)w_\alpha\backslash N_E(F_\nu)} (f_{s,\nu})(w_\alpha n'\iota_\alpha(g))\,dn' \\ &= (M_\nu(w_\alpha,\chi_{s,\nu})f)(\iota_\alpha(g)) = \iota_\alpha^*(M_\nu(w_\alpha,\chi_{s,\nu})f)(g). \end{aligned}$$

□

The following is a corollary of the previous lemma, equation 3.3, and equation 3.9.

Corollary 3.11 (The Gindikin-Karpelevich formula)**.** *Let $\nu\in\mathcal{P}$ be a place such that χ_ν is unramified. Also, let $w\in W$.*

- *Let $f_\nu^0\in I_{B_E}(\chi_\nu,\lambda)$ be an unramified vector. It then holds that*

$$M_\nu(w,\chi_\nu,\lambda)f_\nu^0=\prod_{\alpha>0,\ w^{-1}\alpha<0}\frac{\mathcal{L}_{F_{\alpha,\nu}}\left(\langle\lambda,\alpha^\vee\rangle,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee\right)}{\mathcal{L}_{F_{\alpha,\nu}}\left(\langle\lambda,\alpha^\vee\rangle+1,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee\right)}f_\nu^0. \tag{3.17}$$

- *Let $f_\nu^0\in I_{B_E}(\chi_{s,\nu})$ be an unramified vector. It then holds that*

$$M_\nu(w,\chi_{s,\nu})f_\nu^0=\prod_{\alpha>0,\ w^{-1}\alpha<0}\frac{\mathcal{L}_{F_{\alpha,\nu}}\left(\chi_{s,\nu}\circ\alpha^\vee\right)}{\mathcal{L}_{F_{\alpha,\nu}}\left(q_{\alpha,\nu}^{-1}\chi_{s,\nu}\circ\alpha^\vee\right)}f_\nu^0. \tag{3.18}$$

We denote the Gindikin-Karpelevich term by

$$\begin{aligned} J_\nu(w,\chi,\lambda) &= \prod_{\alpha>0,\ w^{-1}\alpha<0}\frac{\mathcal{L}_{F_{\alpha,\nu}}\left(\langle\lambda,\alpha^\vee\rangle,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee\right)}{\mathcal{L}_{F_{\alpha,\nu}}\left(\langle\lambda,\alpha^\vee\rangle+1,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee\right)}, \\ J_\nu(w,\chi_s) &= \prod_{\alpha>0,\ w^{-1}\alpha<0}\frac{\mathcal{L}_{F_{\alpha,\nu}}\left(\chi_{s,\nu}\circ\alpha^\vee\right)}{\mathcal{L}_{F_{\alpha,\nu}}\left(q_{\alpha,\nu}^{-1}\chi_{s,\nu}\circ\alpha^\vee\right)}. \end{aligned} \tag{3.19}$$

Denote

$$J(w,\chi_s)=\prod_{\nu\in\mathcal{P}}J_\nu(w,\chi_s)\,. \tag{3.20}$$

We list the various Gindikin-Karpelevich terms and their poles in the tables in Appendix A.

The following is a corollary of Lemma 3.8 and Lemma 3.10. We note that it can also be viewed as the application of [KS88, Proposition 6.3] to a simple reflection associated to a simple root.

Corollary 3.12. *Let α be a simple root and let w_α be the associated simple reflection and fix $w\in W$. Further assume that*

$$w_\alpha^{-1}\cdot\left[w^{-1}\cdot\lambda_0\right]=w^{-1}\cdot\lambda_0$$

and

$$\left[w^{-1}\cdot(\chi\circ\det_{M_E})\right](h_\alpha(t))=1\quad\forall t\in\mathbb{A}^\times.$$

Then $M\left(w_\alpha,w^{-1}\cdot\chi,w^{-1}\cdot\lambda_0\right)$ is holomorphic at λ_0, and

$$\begin{aligned}M\left(w_\alpha,w^{-1}\cdot\chi,w^{-1}\cdot\lambda_0\right):\,&Ind_{B_E(\mathbb{A})}^{H_E(\mathbb{A})}w^{-1}\cdot(\lambda_0\otimes\chi\circ\det_{M_E})\\ &\to Ind_{B_E(\mathbb{A})}^{H_E(\mathbb{A})}w^{-1}\cdot(\lambda_0\otimes\chi\circ\det_{M_E})\end{aligned}$$

acts as a scalar multiplication by -1.

Given a standard section f_s it generates a finite-dimensional $\mathbf{K}$-representation, where $\mathbf{K}$ is a fixed maximal compact subgroup of $H_E(\mathbb{A})$ as in [MW95, Section I.1.1]. We let $\mathfrak{F}$ denote the finite set of $\mathbf{K}$-types determining the finite-dimensional subspace of $\mathrm{Ind}_{B_E(\mathbb{A})\cap\mathbf{K}}^{\mathbf{K}}\left(\chi_s\big|_{B_E(\mathbb{A})\cap\mathbf{K}}\right)$ generated by $f_s\big|_{\mathbf{K}}$. Note that both $\mathrm{Ind}_{B_E(\mathbb{A})\cap\mathbf{K}}^{\mathbf{K}}\left(\chi_s\big|_{B_E(\mathbb{A})\cap\mathbf{K}}\right)$ and $f=f_s\big|_{\mathbf{K}}$ are independent of s.

For $w\in W(P_E,H_E)$ we let $M_{\mathfrak{F}}(w,\chi_s)$ be the associated intertwining operator on $\mathrm{Ind}_{B_E(\mathbb{A})\cap\mathbf{K}}^{\mathbf{K}}\left(\chi_s\big|_{B_E(\mathbb{A})\cap\mathbf{K}}\right)$.

Lemma 3.13. *For any $w\in W(P_E,H_E)$, $b\in B_E(\mathbb{A})$, and $k\in\mathbf{K}$ it holds that*

$$M(w,\chi_s)f_s(bk)=\chi_s(b)\,M_{\mathfrak{F}}(w,\chi_s)f(k)\,.$$

Furthermore:

- *For a simple reflection w_i, the operator $M_{\mathfrak{F}}(w_i,\chi_s)$ depends only on $\chi_s\circ h_{\alpha_i}$.*
- *For two commuting simple reflections w_i and w_j the operators $M_{\mathfrak{F}}(w_i,\chi_s)$ and $M_{\mathfrak{F}}(w_i,\chi_s)$ commute.*

3.4. **Normalized intertwining operators.** It is customary to define the *normalized intertwining operator* to be

(3.21)

$$N_\nu(w,\chi_\nu,\lambda)=\frac{J_\nu(w,\chi_s)^{-1}}{\prod\limits_{\alpha>0,\ w^{-1}\alpha<0}\epsilon_{F_{\alpha,\nu}}\left(\langle\lambda,\alpha^\vee\rangle,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee,\psi_\nu\right)}M(w,\chi_\nu,\lambda)\,.$$

Lemma 3.14. *The normalized intertwining operators satisfy the local functional equation*

$$N_\nu(ww',\chi_\nu,\lambda)=N_\nu\left(w',w^{-1}\cdot\chi_\nu,w^{-1}\cdot\lambda\right)\circ N_\nu(w,\chi_\nu,\lambda)\quad\forall w,w'\in W_{H_E}.$$

For simplicity we write

$$N_\nu\left(w,\chi_s\right)=N_\nu\left(w,\chi_\nu,\lambda_s\right).$$

By Corollary 3.11 and Remark 3.3, it holds that

$$\begin{aligned} M\left(w,\chi_s\right)f_s &= \left(\bigotimes_{\nu\in S} M_\nu\left(w,\chi_s\right)f_{s,\nu}\right)\otimes\left(\bigotimes_{\nu\notin S} J_\nu\left(w,\chi_s\right)f^0_{s,\nu}\right)\\ &= J\left(w,\chi_s\right)\left(\bigotimes_{\nu\in S} J_\nu\left(w,\chi_s\right)^{-1}M_\nu\left(w,\chi_s\right)f_{s,\nu}\right)\otimes\left(\bigotimes_{\nu\notin S} f^0_{s,\nu}\right)\\ &= \left(\prod_{\alpha>0,\ w^{-1}\alpha<0}\epsilon_{F_\alpha}\left(\langle\lambda,\alpha^\vee\rangle,\chi_\nu\circ\det_{M_E}\circ\alpha^\vee\right)\right)\\ &\quad\times J\left(w,\chi_s\right)\left(\bigotimes_{\nu\in S} N_\nu\left(w,\chi_s\right)f_{s,\nu}\right)\otimes\left(\bigotimes_{\nu\notin S} f^0_{s,\nu}\right). \end{aligned} \tag{3.22}$$

Hence the analytic behavior of $M\left(w,\chi_s\right)f_s\left(g\right)$ is governed by that of $J\left(w,\chi_s\right)$ and $N_\nu\left(w,\chi_s\right)f_{s,\nu}\left(g\right)$ for $\nu\in S$. Note that according to Lemma 3.5, $N_\nu\left(w,\chi_s\right)f_{s,\nu}$ is holomorphic whenever $\mathfrak{Re}\left(\langle\chi_s,\alpha^\vee\rangle\right)>-1$ for all $\alpha>0$ such that $w\cdot\alpha<0$. In light of Tables 5, 9, and 13 and the discussion in Subsection 3.2 the following holds:

Lemma 3.15. *For any $\mathfrak{Re}\left(s_0\right)>0$ and $\nu\in\mathcal{P}$ it holds that $N\left(w,\chi_{s,\nu}\right)f_{s,\nu}$ is analytic at s_0. Moreover, there exists an $f_{s,\nu}$ such that $N\left(w,\chi_{s,\nu}\right)f_{s,\nu}$ is non-zero at s_0.*

4. Poles of the Eisenstein series

In this section we use equation 3.5 to study the poles of $\mathcal{E}_E\left(\chi,f_s,s,g\right)_{B_E}$. By Theorem 3.2, these are the poles of $\mathcal{E}_E\left(\chi,f_s,s,g\right)$. We start by considering the poles of the various intertwining operators, thus getting a bound on the order of the poles. In the following table we list the possible triples $\left(E,\chi,s_0\right)$ for which $\mathcal{E}_E\left(\chi,f_s,s,g\right)$ might admit a pole at s_0 and give bounds on the orders of the poles at these points. Here E is an étale cubic algebra over F, χ is a Hecke character of $F^\times\backslash\mathbb{A}^\times$, and $s_0\in\mathbb{C}$ with $\mathfrak{Re}\left(s_0\right)>0$. More precisely, due to Theorem 3.2 and equation 3.5, for a given étale cubic algebra E over F and a finite order automorphic character χ we have

$$\begin{aligned}\left\{\text{Poles of }\mathcal{E}_E\left(\chi,\cdot,s,\cdot\right)\right\} &= \left\{\text{Poles of }\mathcal{E}_E\left(\chi,\cdot,s,\cdot\right)_{B_E}\right\}\\ &\subseteq\left\{\text{Poles of }M\left(w,\chi_s\right)\mid w\in W\left(P_E,H_E\right)\right\}.\end{aligned}$$

We note that for $\mathfrak{Re}\left(s\right)>0$ the poles of $M\left(w,\chi_s\right)$ for various values of $w\in W\left(P_E,H_E\right)$ and χ can occur only at $s_0\in\left\{\frac{1}{2},\frac{3}{2},\frac{5}{2}\right\}$. For such triples $\left(E,\chi,s_0\right)$, the following table lists

$$\max\left\{\operatorname{ord}_{s=s_0}M\left(w,\chi_s\right)f_s\left(g\right)\mid w\in W\left(P_E,H_E\right),\ f_s\in I_{P_E}\left(\chi,s\right),\ g\in H_E\left(\mathbb{A}\right)\right\}.$$

If this has positive value, we list this value in Table 1 in the cell corresponding to $\left(E,\chi,s_0\right)$. The orders of poles of the intertwining operators are given by Tables 4, 8, and 12 in Appendix A. For a **Galois** étale cubic algebra recall the definition of χ_E from Subsection 2.1.

TABLE 1. Trivial bounds on the order of poles of $\mathcal{E}_E(\chi, f_s, s, g)$

	$s_0 = \frac{1}{2}$			$s_0 = \frac{3}{2}$		$s_0 = \frac{5}{2}$
	$\chi = \mathbf{1}$	$\chi = \chi_E, \overline{\chi_E}$	$\chi \neq \mathbf{1}, \chi_E, \overline{\chi_E}$ and $\chi^2 = \mathbf{1}$	$\chi = \mathbf{1}$	$\chi = \chi_E, \overline{\chi_E}$	$\chi = \mathbf{1}$
$E = F \times F \times F$	4		1	3		1
$E = F \times K$	3	2	1	2	1	1
E Galois field extension	2	1	1	1	1	1
E non-Galois field extension	2		1	1		1

Theorem 4.1. *The order of the poles of $\mathcal{E}_E(\chi, \cdot, s, \cdot)$ for $\mathfrak{Re}(s) > 0$ are given by the following numbers:*

TABLE 2. Orders of poles of $\mathcal{E}_E(\chi, f_s, s, g)$

	$s_0 = \frac{1}{2}$	$s_0 = \frac{3}{2}$		$s_0 = \frac{5}{2}$
	$\chi^2 = \mathbf{1}$	$\chi = \mathbf{1}$	$\chi = \chi_E, \overline{\chi_E}$	$\chi = \mathbf{1}$
$E = F \times F \times F$	1	2		1
$E = F \times K$	1	1	1	1
E Galois field extension	1	0	1	1
E non-Galois field extension	1	0		1

Namely, the Eisenstein series admits the following poles:

- *A simple pole at $s_0 = \frac{1}{2}$ for $\chi^2 = \mathbf{1}$.*
- *A simple pole at $s_0 = \frac{3}{2}$ if E/F is a non-split Galois étale cubic algebra and $\chi = \chi_E$.*
- *A double pole at $s_0 = \frac{3}{2}$ if $E = F \times F \times F$ and $\chi = \mathbf{1}$.*
- *A simple pole at $s_0 = \frac{3}{2}$ if $E = F \times K$ is non-split and $\chi = \mathbf{1}$.*
- *A simple pole at $s_0 = \frac{5}{2}$ if $\chi = \mathbf{1}$.*

For all other triples (E, χ, s_0), the series $\mathcal{E}_E(\chi, f_s, s, g)$ is holomorphic at s_0.

The orders described above can indeed be realized by sections of $I_{P_E}(\chi, s)$, and in particular, when χ is everywhere unramified a pole of the above-mentioned order is obtained for the spherical vector.

Furthermore, for a triple (E, χ, s_0) appearing in Table 2, the residual representation of $\mathcal{E}_E(\chi, \cdot, s, \cdot)$ is square-integrable with the exception of the following cases:

- *$E = F \times K$ where K is a field:*
  - *$s = \frac{1}{2}$ with $\chi = \mathbf{1}, \chi_K$.*
  - *$s = \frac{3}{2}$ with $\chi = \mathbf{1}$.*
- *$E = F \times F \times F$, $s = \frac{1}{2}$ with $\chi = \mathbf{1}$.*

Before proving Theorem 4.1 we wish to describe the key ideas of the proof.

In the course of the proof we use equation 3.5 to evaluate the constant term and check the cancellation of the poles of the various intertwining operators. Fix a triple (E, χ, s_0) as above such that

$$\max\{\mathrm{ord}_{s=s_0} M(w, \chi_s) f_s(g) \mid w \in W(P_E, H_E),\ f_s \in I_{P_E}(\chi, s),\ g \in H_E(\mathbb{A})\} = n > 0.$$

We denote this integer by $ord_{s=s_0} M(w, \chi_s)$. For $0 < m \leq n$ let

$$\Sigma_{(E,\chi,s_0,m)} = \{w \in W(P_E, H_E) \mid ord_{s=s_0} M(w, \chi_s) \geq m\}.$$

We say that the pole of order m cancels if

$$\lim_{s\to s_0}(s-s_0)^m \sum_{w\in W(P_E,H_E)} M(w,\chi_s)\Big|_{I_{P_E}(\chi,s)} = \lim_{s\to s_0}(s-s_0)^m \sum_{w\in \Sigma_{(E,\chi,s_0,m)}} M(w,\chi_s)\Big|_{I_{P_E}(\chi,s)} \equiv 0.$$

We describe now the reason behind the cancellation of poles.

★ **Reason for cancellation of poles**. Assume that we can decompose $\Sigma_{(E,\chi,s_0,n)}$ into a disjoint union of pairs $\{w', w'w''\}$. We then have

$$M_{w'w''} = M_{w''} \circ M_{w'}.$$

Assume that for any such pair we can further show that $M_{w''}$ is an endomorphism of the image of $M_{w'}$ acting as $-Id$; this is done using Corollary 3.12 for example. Then

$$\lim_{s\to s_0}(s-s_0)^n\left[M_{w'}+M_{w'w''}\right] \equiv 0. \tag{4.1}$$

It then follows that the pole of order n cancels.

More generally, we define an equivalence class on $\Sigma_{(E,\chi,s_0,m)}$ by

$$w \sim_{s_0} w' \iff w^{-1}\cdot\chi_{s_0} = w'^{-1}\cdot\chi_{s_0}. \tag{4.2}$$

Clearly, cancellations of poles of intertwining operators can occur only within the same equivalence class.

Assume that after decomposing $\Sigma_{(E,\chi,s_0,n)}$ into equivalence classes Σ_i and for all i,

$$\lim_{s\to s_0}(s-s_0)^n \sum_{w\in\Sigma_i} M_w \equiv 0. \tag{4.3}$$

Then the pole of order n cancels. This is done, for example, in Subsection 4.5.

After, maybe, cancellation of higher orders of a pole, we wish to determine its actual order. Namely, for $0 < m \le n$ assume that

$$\lim_{s\to s_0}(s-s_0)^{m+1} \sum_{w\in W(P_E,H_E)} M(w,\chi_s)\Big|_{I_{P_E}(\chi,s)} \equiv 0.$$

Then $\mathcal{E}_E(\chi,\cdot,s,\cdot)$ attains a pole of order m at s_0 if

$$\lim_{s\to s_0}(s-s_0)^m \sum_{w\in W(P_E,H_E)} M(w,\chi_s)\Big|_{I_{P_E}(\chi,s)} = \lim_{s\to s_0}(s-s_0)^m \sum_{w\in \Sigma_{(E,\chi,s_0,m)}} M(w,\chi_s)\Big|_{I_{P_E}(\chi,s)} \not\equiv 0.$$

In particular, for any holomorphic section $f_s \in I_{P_E}(\chi,s)$ and any $t \in T_E(\mathbb{A})$ it holds that

$$\lim_{s\to s_0}(s-s_0)^m \sum_{w\in W(P_E,H_E)} M(w,\chi_s) f_s(t) \in \mathbb{C},$$

and the limit is non-zero for some f_s and t.

We prove this using one of the following reasons:

★ **Reason 1 for non-vanishing of the leading term**. One can prove the non-vanishing of the leading term by providing a section $f_s \in I_{P_E}(\chi, s)$ such that

$$\lim_{s\to s_0} (s-s_0)^m \sum_{w\in\Sigma_{(E,\chi,s_0,m)}} M(w,\chi_s) f_s \not\equiv 0.$$

Remark 4.2. A global spherical section exists if and only if χ is everywhere unramified. In the case that χ is everywhere unramified one can check that the orders of the poles in Table 2 are realized by the global spherical section.

★ **Reason 2 for non-vanishing of the leading term**. The representation

$$\mathrm{Res}(s_0,\chi,E)_{B_E} = \mathrm{Span}_{\mathbb{C}} \left\{ \lim_{s\to s_0} (s-s_0)^m \mathcal{E}_E(\chi, f_s, s, t)_{B_E} \;\middle|\; f_s \in I_{P_E}(\chi,s) \right\}$$

decomposes into a sum of copies of one-dimensional representations of T_E:

$$\left\{ w^{-1}\cdot\chi_s \;\middle|\; w\in\Sigma_{(E,\chi,s_0,m)} \right\}.$$

The elements $M(w,\chi_s) f_s(t)$ lie in the representation $w^{-1}\cdot\chi_s$ as representations of T_E. And so, if there exists $w\in\Sigma_{(E,\chi,s_0,m)}$ such that $w \not\sim_{s_0} w'$ **for all** $w\neq w'\in\Sigma_{(E,\chi,s_0,m)}$, then the term $\lim_{s\to s_0}(s-s_0)^m M(w,\chi_s) f_s$ cannot by canceled by other terms in the sum, while it is non-zero due to equation 3.22 and Lemma 3.15.

Remark 4.3. In fact, reason 2 for non-vanishing of the leading term is contained in reason 1, but due to its usefulness it is worth noting separately.

Proof of Theorem 4.1. For any E the poles corresponding to the triple $\left(E,\chi_E,\frac{3}{2}\right)$ are treated in [GGJ02]. Also, the poles at $s=\frac{5}{2}$ and $\chi=\mathbf{1}$ arise only from the intertwining operator associated with the coset of the longest element of the Weyl group, and hence they cannot be canceled.

In what follows we treat the rest of the points in Table 1. We leave the discussion of the square integrability of the residual representations to the end of this section. In what follows, we denote by t an element of $T_E(\mathbb{A}_F)$ of the form

$$t=\begin{cases} h_{\alpha_1}(t_1)\,h_{\alpha_2}(t_2)\,h_{\alpha_3}(t_3)\,h_{\alpha_4}(t_4), & E=F\times F\times F, \quad t_1,t_2,t_3,t_4\in\mathbb{A}_F^\times,\\ h_{\alpha_1}(t_1)\,h_{\alpha_2}(t_2)\,h_{\alpha_3}(t_3)\,h_{\alpha_4}(t_3^\sigma), & E=F\times K, \quad t_1,t_2\in\mathbb{A}_F^\times,\ t_3\in\mathbb{A}_K^\times,\\ h_{\alpha_1}(t_1)\,h_{\alpha_2}(t_2)\,h_{\alpha_3}(t_1^\sigma)\,h_{\alpha_4}\left(t_1^{\sigma^2}\right), & E \text{ is a field}, \quad t_1\in\mathbb{A}_E^\times,\ t_2\in\mathbb{A}_F^\times. \end{cases}$$

4.1. E **a field,** $s=\frac{1}{2}$**,** $\chi=\mathbf{1}$**.** The intertwining operators in this case have poles at most of order 2. We show that the pole of order 2 cancels and that the pole of order 1 does not.

(1) We have

$$\Sigma_{\left(E,\mathbf{1},\frac{1}{2},2\right)} = \{w_{212}, w_{2121}\}.$$

Since $w_{212}^{-1}\cdot\chi_{\frac{1}{2}}(t) = w_{212}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \frac{1}{|t_2|_F}$ we have $w_{212}\sim_{s_0} w_{2121}$. We write $w_{2121}=w_{212}w_1$. It follows from Corollary 3.12 that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2 M(w_{2121},\chi_s) = -\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2 M(w_{212},\chi_s).$$

Following the reason for cancellation of poles stated above,

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2 \mathcal{E}_E(\chi,f_s,s,g)_{B_E} = 0 \qquad \forall f_s\in I_{P_E}(\chi,s).$$

Thus, the pole of order 2 is canceled.

(2) We have

$$\Sigma_{\left(E,\mathbf{1},\frac{1}{2},1\right)} = \left\{w_{21}, w_{212}, w_{2121}, w_{21212}\right\}.$$

We note that $w_{21}^{-1}\cdot\chi_{\frac{1}{2}}\left(t\right) = w_{21}^{-1}\cdot\chi_{\frac{1}{2}}\left(t\right) = \frac{|t_2|_F}{|t_1|_E}$. We prove that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\mathcal{E}_E\left(\chi, f_s, s, g\right)_{B_E} \not\equiv 0$$

by proving that for the global spherical section f_s^0 it holds that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left(M_{w_{21}} + M_{w_{21212}}\right) f_s^0 \neq 0,$$

thus applying both reason 1 and reason 2 for non-vanishing of the leading term. Indeed, we write

$$\xi_F\left(s\right) = \frac{\gamma_{-1}}{s-1} + \gamma_0 + \cdots, \quad \xi_E\left(s\right) = \frac{\epsilon_{-1}}{s-1} + \epsilon_0 + \cdots.$$

It holds that

$$\begin{aligned}
&\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left(M_{w_{21}} + M_{w_{21212}}\right) f_s^0 \\
&= \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right) \\
&\quad\times\left(\frac{\xi_F\left(s+\frac{3}{2}\right)\xi_E\left(s+\frac{1}{2}\right)}{\xi_F\left(s+\frac{5}{2}\right)\xi_E\left(s+\frac{3}{2}\right)} + \frac{\xi_F\left(s-\frac{3}{2}\right)\xi_F\left(s+\frac{3}{2}\right)\xi_E\left(s-\frac{1}{2}\right)\xi_F\left(2s\right)}{\xi_F\left(s-\frac{1}{2}\right)\xi_F\left(s+\frac{5}{2}\right)\xi_E\left(s+\frac{3}{2}\right)\xi_F\left(2s+1\right)}\right) \\
&= \frac{\xi_F\left(2\right)}{\xi_F\left(3\right)\xi_E\left(2\right)}\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left(\xi_E\left(s+\frac{1}{2}\right) + \frac{\xi_F\left(-1\right)}{\xi_F\left(2\right)}\frac{\xi_E\left(s-\frac{1}{2}\right)\xi_F\left(2s\right)}{\xi_F\left(s-\frac{1}{2}\right)}\right) \\
&= \frac{\xi_F\left(2\right)}{\xi_F\left(3\right)\xi_E\left(2\right)}\left(\epsilon_{-1} + \frac{\epsilon_{-1}\frac{1}{2}\gamma_{-1}}{-\gamma_{-1}}\right) = \frac{\xi_F\left(2\right)}{\xi_F\left(3\right)\xi_E\left(2\right)}\frac{3}{2}\epsilon_{-1} \neq 0.
\end{aligned}$$

Here we use the fact that $\epsilon_{-1}\neq 0$ due to the class number formula [Was97, pg. 37].

4.2. E **a field,** $s=\frac{1}{2}$**,** $\chi=\chi_E,\overline{\chi_E}$**.** Note that here we assume that E/F is a **Galois** field extension. The intertwining operators in this case have poles of order at most 1. We show that the Eisenstein series is in fact holomorphic at this point. We have

$$\Sigma_{\left(E,\chi,\frac{1}{2},1\right)} = \left\{w_{21}, w_{212}, w_{2121}, w_{21212}\right\}$$

and

$$\begin{aligned}
w_{21}^{-1}\cdot\chi_{\frac{1}{2}}\left(t\right) &= w_{21212}^{-1}\cdot\chi_{\frac{1}{2}}\left(t\right) = \overline{\chi\left(t_2\right)}\frac{|t_2|_F}{|t_1|_E}, \\
w_{212}^{-1}\cdot\chi_{\frac{1}{2}}\left(t\right) &= w_{2121}^{-1}\cdot\chi_{\frac{1}{2}}\left(t\right) = \chi\left(t_2\right)\frac{1}{|t_2|_F}.
\end{aligned}$$

We write $w_{2121} = w_{212}w_1$. It follows from Corollary 3.12 and reason 2 for cancellation of poles that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left[M\left(w_{2121},\chi_s\right) + M\left(w_{212},\chi_s\right)\right] = 0.$$

On the other hand,

$$M\left(w_{21212},\chi_s\right)=M\left(w_{212},w_{21}^{-1}\cdot\chi_s\right)\circ M\left(w_{21},\chi_s\right).$$

With notation as in Lemma 3.13 we write

$$\begin{aligned}&\left[M\left(w_{21},\chi_s\right)+M\left(w_{21212},\chi_s\right)\right]f_s\left(bg\right)\\&\quad=\left[w_{21}^{-1}\cdot\chi_s\left(b\right)+w_{21212}^{-1}\cdot\chi_s\left(b\right)M_{\mathfrak{F}}\left(w_{212},w_{21}^{-1}\right)\right]M_{\mathfrak{F}}\left(w_{21},\chi_s\right)f\left(k\right).\end{aligned}$$

Furthermore, we write

$$M_{\mathfrak{F}}\left(w_{212},w_{21}^{-1}\cdot\chi_s\right)=M_{\mathfrak{F}}\left(w_2,w_{2121}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_1,w_{212}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_2,w_{21}^{-1}\cdot\chi_s\right).$$

We note that

$$\begin{aligned}J\left(w_2,w_{21}^{-1}\cdot\chi_s\right)&=\frac{\mathcal{L}_F\left(s-\frac{3}{2},\chi\right)}{\mathcal{L}_F\left(s-\frac{1}{2},\chi\right)},\\J\left(w_1,w_{212}^{-1}\cdot\chi_s\right)&=\frac{\mathcal{L}_E\left(s-\frac{1}{2},\mathbf{1}\right)}{\mathcal{L}_E\left(s+\frac{1}{2},\mathbf{1}\right)}=\frac{\xi_E\left(s-\frac{1}{2}\right)}{\xi_E\left(s+\frac{1}{2}\right)},\\J\left(w_2,w_{2121}^{-1}\cdot\chi_s\right)&=\frac{\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(2s+1,\chi^2\right)},\end{aligned}$$

and hence $M_{\mathfrak{F}}\left(w_2,w_{21}^{-1}\cdot\chi_s\right)$, $M_{\mathfrak{F}}\left(w_1,w_{212}^{-1}\cdot\chi_s\right)$, and $M_{\mathfrak{F}}\left(w_2,w_{2121}^{-1}\cdot\chi_s\right)$ are all holomorphic at $s=\frac{1}{2}$. Also, by Corollary 3.12 and Lemma 3.7 it follows that

$$\begin{aligned}&\lim_{s\to\frac{1}{2}}M_{\mathfrak{F}}\left(w_1,w_{212}^{-1}\cdot\chi_s\right)=-\operatorname{Id},\\&\lim_{s\to\frac{1}{2}}M_{\mathfrak{F}}\left(w_2,w_{2121}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_2,w_{21}^{-1}\cdot\chi_s\right)=\operatorname{Id},\end{aligned}$$

and hence

$$\begin{aligned}&\lim_{s\to\frac{1}{2}}M_{\mathfrak{F}}\left(w_{212},w_{21}^{-1}\cdot\chi_s\right)\\&\quad=\lim_{s\to\frac{1}{2}}M_{\mathfrak{F}}\left(w_2,w_{2121}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_1,w_{212}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_2,w_{21}^{-1}\cdot\chi_s\right)=-\operatorname{Id}.\end{aligned}$$

Indeed, writing

$$\begin{aligned}M_{\mathfrak{F}}\left(w_2,w_{2121}^{-1}\cdot\chi_s\right)&=A_0+A_1\left(s-\frac{1}{2}\right)+o\left(\left(s-\frac{1}{2}\right)\right),\\M_{\mathfrak{F}}\left(w_1,w_{212}^{-1}\cdot\chi_s\right)&=B_0+B_1\left(s-\frac{1}{2}\right)+o\left(\left(s-\frac{1}{2}\right)\right),\\M_{\mathfrak{F}}\left(w_2,w_{21}^{-1}\cdot\chi_s\right)&=C_0+C_1\left(s-\frac{1}{2}\right)+o\left(\left(s-\frac{1}{2}\right)\right),\end{aligned}$$

where

$$B_0=-\operatorname{Id},\quad A_0C_0=\operatorname{Id}$$

we find that

$$\begin{aligned}&M_{\mathfrak{F}}\left(w_{212},w_{21}^{-1}\cdot\chi_s\right)\\&\quad=A_0B_0C_0+\left(s-\frac{1}{2}\right)\left[A_0B_0C_1+A_0B_1C_0+A_1B_0C_0\right]+o\left(\left(s-\frac{1}{2}\right)\right)\\&\quad=-\operatorname{Id}+o\left(1\right).\end{aligned}$$

Using the fact that $w_{21}^{-1} \cdot \chi_{\frac{1}{2}} = w_{21212}^{-1} \cdot \chi_{\frac{1}{2}}$ we apply the reason for cancellation of poles stated above and conclude that

$$\lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) \left[M\left(w_{21212}, \chi_s\right) + M\left(w_{21}, \chi_s\right)\right] = 0.$$

In conclusion,

$$\lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) \mathcal{E}_E\left(\chi, f_s, s, g\right)_{B_E} = 0,$$

and hence $\mathcal{E}_E\left(\chi, f_s, s, g\right)$ is holomorphic at $s = \frac{1}{2}$.

4.3. E **a field,** $s = \frac{1}{2}$**,** $\chi^2 = \mathbf{1}$**,** $\chi \neq \mathbf{1}$**.** The intertwining operators in this case have poles of order at most 1. We show that indeed the Eisenstein series admits a simple pole at this point. We have

$$\Sigma_{\left(E, \chi, \frac{1}{2}, 1\right)} = \left\{w_{212}, w_{2121}, w_{21212}\right\}$$

and

$$\begin{aligned}
w_{212}^{-1} \cdot \chi_{\frac{1}{2}}(t) &= \chi\left(\mathrm{Nm}_{E/F}(t_1)\right) \frac{1}{\left|t_2\right|_F}, \\
w_{2121}^{-1} \cdot \chi_{\frac{1}{2}}(t) &= \chi\left(t_2 \,\mathrm{Nm}_{E/F}(t_1)\right) \frac{1}{\left|t_2\right|_F}, \\
w_{21212}^{-1} \cdot \chi_{\frac{1}{2}}(t) &= \chi\left(t_2\right) \frac{\left|t_2\right|_F}{\left|t_1\right|_E}.
\end{aligned}$$

Following reason 2 for non-cancellation of poles, $\mathcal{E}_E\left(\chi, f_s, s, g\right)$ admits a simple pole at $s = \frac{1}{2}$.

4.4. E **a field,** $s = \frac{3}{2}$**,** $\chi = \mathbf{1}$**.** The intertwining operators in this case have poles of order at most 1. We show that the Eisenstein series is holomorphic at this point. We have

$$\Sigma_{\left(E, \mathbf{1}, \frac{3}{2}, 1\right)} = \left\{w_{2121}, w_{21212}\right\}.$$

Since $w_{2121}^{-1} \cdot \chi_{\frac{3}{2}}(t) = w_{21212}^{-1} \cdot \chi_{\frac{3}{2}}(t) = \frac{1}{\left|t_1\right|_E}$ we have $w_{2121} \sim_{s_0} w_{21212}$. We write $w_{21212} = w_{2121} w_2$. It follows from Corollary 3.12 that

$$\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right)^2 M\left(w_{21212}, \chi_s\right) = -\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right)^2 M\left(w_{2121}, \chi_s\right).$$

Following reason 1 for cancellation of poles,

$$\lim_{s \to \frac{3}{2}} \left(s - \frac{3}{2}\right)^2 \mathcal{E}_E\left(1, f_s, s, g\right)_{B_E} = 0 \qquad \forall f_s \in I_{P_E}(\chi, s).$$

Thus $\mathcal{E}_E(\mathbf{1}, f_s, s, g)$ is holomorphic at $s = \frac{3}{2}$.

4.5. $E = F \times K$**,** K **a field,** $s = \frac{1}{2}$**,** $\chi = \mathbf{1}$**.** The intertwining operators in this case have poles of order at most 3. We show that the Eisenstein series admits a simple pole at this point. We have

$$\begin{aligned}
&\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)} = \left\{w_{2132}, w_{21321}, w_{21323}, w_{213213}\right\},\\
&\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)} \setminus \Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)} = \left\{w_{213}, w_{2321}, w_{2132132}\right\},\\
&\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},1\right)} \setminus \Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)} = \left\{w_{21}, w_{23}, w_{232}\right\}
\end{aligned}$$

and also

$$\begin{aligned}
&w_{2132}^{-1}\cdot\chi_{\frac{1}{2}}(t) = w_{21321}^{-1}\cdot\chi_{\frac{1}{2}}(t) = w_{21323}^{-1}\cdot\chi_{\frac{1}{2}}(t) = w_{213213}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \frac{1}{|t_2|_F},\\
&w_{213}^{-1}\cdot\chi_{\frac{1}{2}}(t) = w_{2321}^{-1}\cdot\chi_{\frac{1}{2}}(t) = w_{2132132}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \frac{|t_2|_F}{|t_1|_F\,|t_3|_K},\\
&w_{23}^{-1}\cdot\chi_{\frac{1}{2}}(t) = w_{232}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \frac{|t_1|_F}{|t_3|_K},\\
&w_{21}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \frac{|t_3|_K}{|t_1|_F\,|t_2|_F}.
\end{aligned}$$

We note that it follows from the above that

$$\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)}/\sim_{s_0} = \left\{\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}, \Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)} \setminus \Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}\right\}.$$

After proving that the poles of orders 3 and 2 cancel, the fact that the simple pole is attained follows from reason 2 for non-vanishing of the leading term. More precisely, if f_s^0 is the global spherical vector one has

$$\lim_{s\to\frac{1}{2}} M(w_{21}, \chi_s) f_s^0 \neq 0.$$

Namely, $\mathcal{E}_{F\times K}\left(\mathbf{1}, f_s^0, s, g\right)$ admits a simple pole at $s = \frac{1}{2}$.

We now turn to proving that the poles of higher order are canceled. With notation as in Lemma 3.13 we write

$$M(w, \chi_s) f_s(bk) = \left(w^{-1}\cdot\chi_s\right)(b)\, M_{\mathfrak{F}}(w, \chi_s) f(k).$$

We start with the sum over the elements of $\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}$. Note that $\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)} = w_{2132}\cdot W_{M_E}$ and that any element in W_{M_E} is of the form $w_1^{\epsilon_1} w_3^{\epsilon_3}$, where $\epsilon_1, \epsilon_3 \in \{0,1\}$. Denote $\eta_s = w_{2132}^{-1}\cdot\chi_s$. It follows that

$$\begin{aligned}
\sum_{w\in\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}} M(w,\chi_s) f_s &= \left(\mathrm{Id} + \left(w_1^{-1}\cdot\eta_s\right) M_{\mathfrak{F}}(w_1,\eta_s)\right)\\
&\quad\times\left(\mathrm{Id} + \left(w_3^{-1}\cdot\eta_s\right) M_{\mathfrak{F}}(w_3,\eta_s)\right)\eta_s M_{\mathfrak{F}}(w_{2132},\chi_s) f.
\end{aligned}$$

It follows from Corollary 3.12 that $M_{\mathfrak{F}}\left(w_1, \eta_{\frac{1}{2}}\right) = -\mathrm{Id}$ and $M_{\mathfrak{F}}\left(w_3, \eta_{\frac{1}{2}}\right) = -\mathrm{Id}$. On the other hand, $w_1^{-1}\eta_{\frac{1}{2}} = w_3^{-1}\eta_{\frac{1}{2}} = \mathbf{1}$ and hence

$$\left(\mathrm{Id} + w_1^{-1}\cdot\eta_s M_{\mathfrak{F}}(w_1,\eta_s)\right)\left(\mathrm{Id} + w_3^{-1}\cdot\eta_s M_{\mathfrak{F}}(w_3,\eta_s)\right)$$

admits a zero of order 2. Since $M_{\mathfrak{F}}\left(w_{2132},\chi_s\right)f$ admits a pole of order 3 we conclude that

$$\sum_{w\in\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}} M\left(w,\chi_s\right)f_s$$

can admit at most a simple pole.

We now consider the sum over the elements of $\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)}\setminus\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}$. We note that

$$\begin{aligned} w_{213} &= w_{2132132}w_{2132},\\ w_{2321} &= w_{2132132}w_{212}. \end{aligned}$$

Write $\eta_s = w_{2132132}^{-1}\cdot\chi_s$. We note that

$$\sum_{w\in\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)}\setminus\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}} M\left(w,\chi_s\right)f_s\left(bk\right) = \left[1+\left|\frac{t_2}{t_1}\right|_F^{2s-1}M_{\mathfrak{F}}\left(w_{212},\eta_s\right)\right]$$
$$\times\left[1+\frac{\left|t_2\right|_F^{3s-\frac{3}{2}}}{\left|t_1\right|_F^{s-\frac{1}{2}}\left|t_3\right|_K^{s-\frac{1}{2}}}M_{\mathfrak{F}}\left(w_{2132},\eta_s\right)\right]\eta_s\left(b\right)M_{\mathfrak{F}}\left(w_{2132132},\chi_s\right)f\left(k\right)-M_{\mathfrak{F}}\left(w_{213232},\chi_s\right)f_s$$

We further note that

- $w_{213232}=w_{121323}$ and hence
$$M\left(w_{213232},\chi_s\right)=M\left(w_{21323},w_1^{-1}\chi_s\right)M\left(w_1,\chi_s\right).$$
Recalling that $M\left(w_1,\chi_s\right)\big|_{I_{P_E}\left(\mathbf{1},s\right)}\equiv 0$ we conclude that $M\left(w_{213232},\chi_s\right)f_s = 0$.
- $M_{\mathfrak{F}}\left(w_{2132132},\chi_s\right)f\left(k\right)$ admits a pole of order 2 at $s=\frac{1}{2}$.
- $\lim\limits_{s\to\frac{1}{2}}\left|\frac{t_2}{t_1}\right|_F^{2s-1} = \lim\limits_{s\to\frac{1}{2}}\frac{\left|t_2\right|_F^{3s-\frac{3}{2}}}{\left|t_1\right|_F^{s-\frac{1}{2}}\left|t_3\right|_K^{s-\frac{1}{2}}}=1.$
- It follows from [Lao, Subsection 5.5.3] that the image
$$\left[\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2 M\left(w_{2132132},\chi_s\right)\right]\left(I_{P_E}\left(\mathbf{1},\frac{1}{2}\right)\right)$$
is an irreducible representation (since E_ν is not a field for all $\nu\in\mathcal{P}$) and that $M\left(w_{2132},\eta_s\right)$ acts on it as $-\mathrm{Id}$.

It follows that

$$\sum_{w\in\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)}\setminus\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}} M\left(w,\chi_s\right)f_s\left(bk\right) = \left[1+\left|\frac{t_2}{t_1}\right|_F^{2s-1}M_{\mathfrak{F}}\left(w_{212},\eta_s\right)\right]$$
$$\times\left[1+\frac{\left|t_2\right|_F^{3s-\frac{3}{2}}}{\left|t_1\right|_F^{s-\frac{1}{2}}\left|t_3\right|_K^{s-\frac{1}{2}}}M_{\mathfrak{F}}\left(w_{2132},\eta_s\right)\right]\eta_s\left(b\right)M_{\mathfrak{F}}\left(w_{2132132},\chi_s\right)f\left(k\right)$$

admits at most a simple pole at $s=\frac{1}{2}$. In conclusion, $\mathcal{E}_E\left(1,f_s^0,s,g\right)$ admits a pole of order 1 at $s=\frac{1}{2}$.

4.6. $E = F \times K$, **K a field,** $s = \frac{1}{2}$**,** $\chi = \chi_K$**.** This case is similar to Subsection 4.2. The intertwining operators in this case have poles of order at most 2. We show that the Eisenstein series admits a simple pole at this point. We have

$$\Sigma_{\left(F\times K,\chi_K,\frac{1}{2},2\right)} = \left\{w_{2321}, w_{2132}, w_{21321}, w_{21323}, w_{213213}, w_{2132132}\right\}$$

and

$$\begin{aligned}
w_{2321}^{-1}\cdot\chi_{\frac{1}{2}}(t) &= w_{2132132}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \chi_K(t_2)\frac{|t_2|_F}{|t_1|_F|t_3|_K},\\
w_{2132}^{-1}\cdot\chi_{\frac{1}{2}}(t) &= w_{21323}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \chi_K(t_1)\frac{1}{|t_2|_F},\\
w_{21321}^{-1}\cdot\chi_{\frac{1}{2}}(t) &= w_{213213}^{-1}\cdot\chi_{\frac{1}{2}}(t) = \chi_K(t_1t_2)\frac{1}{|t_2|_F}.
\end{aligned}$$

We write $w_{21323} = w_{2132}w_3$ and $w_{213213} = w_{21321}w_3$. It follows from Corollary 3.12 that

$$\begin{aligned}
&\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2\left[M\left(w_{21323},\chi_s\right)+M\left(w_{2132},\chi_s\right)\right] = 0,\\
&\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2\left[M\left(w_{213213},\chi_s\right)+M\left(w_{21321},\chi_s\right)\right] = 0.
\end{aligned}$$

On the other hand,

$$M\left(w_{2132132},\chi_s\right) = M\left(w_{232}, w_{2321}^{-1}\cdot\chi_s\right)\circ M\left(w_{2321},\chi_s\right).$$

With notation as in Lemma 3.13 we write

$$\begin{aligned}
&\left[M\left(w_{2321},\chi_s\right)+M\left(w_{2132132},\chi_s\right)\right]f_s(bg)\\
&=\left[w_{2321}^{-1}\cdot\chi_s(b)+\left(w_{2312132}^{-1}\cdot\chi_s\right)(b)\,M_{\mathfrak{F}}\left(w_{232},w_{2321}^{-1}\right)\right]M_{\mathfrak{F}}\left(w_{2321},\chi_s\right)f(k).
\end{aligned}$$

Furthermore, we write

$$\begin{aligned}
&M_{\mathfrak{F}}\left(w_{232}, w_{2321}^{-1}\cdot\chi_s\right)\\
&\quad = M_{\mathfrak{F}}\left(w_2, w_{232123}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_3, w_{23212}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_2, w_{2321}^{-1}\cdot\chi_s\right).
\end{aligned}$$

We note that

$$\begin{aligned}
J\left(w_2, w_{2321}^{-1}\cdot\chi_s\right) &= \frac{\mathcal{L}_F\left(s+\frac{1}{2},\chi_K\right)}{\mathcal{L}_F\left(s+\frac{3}{2},\chi_K\right)},\\
J\left(w_3, w_{23212}^{-1}\cdot\chi_s\right) &= \frac{\mathcal{L}_E\left(s-\frac{1}{2},\mathbf{1}\right)}{\mathcal{L}_E\left(s+\frac{1}{2},\mathbf{1}\right)} = \frac{\xi_E\left(s-\frac{1}{2}\right)}{\xi_E\left(s+\frac{1}{2}\right)},\\
J\left(w_2, w_{232123}^{-1}\cdot\chi_s\right) &= \frac{\mathcal{L}_F\left(s-\frac{1}{2},\chi_K\right)}{\mathcal{L}_F\left(s+\frac{1}{2},\chi_K\right)},
\end{aligned}$$

and hence $M_{\mathfrak{F}}\left(w_2, w_{2321}^{-1}\cdot\chi_s\right)$, $M_{\mathfrak{F}}\left(w_3, w_{23212}^{-1}\cdot\chi_s\right)$, and $M_{\mathfrak{F}}\left(w_2, w_{232123}^{-1}\cdot\chi_s\right)$ are all holomorphic at $s=0$. Also, by Corollary 3.12 and Lemma 3.7 it follows that

$$\begin{aligned}
&\lim_{s\to\frac{1}{2}} M_{\mathfrak{F}}\left(w_3, w_{23212}^{-1}\cdot\chi_s\right) = -\operatorname{Id},\\
&\lim_{s\to\frac{1}{2}} M_{\mathfrak{F}}\left(w_2, w_{232123}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_2, w_{2321}^{-1}\cdot\chi_s\right) = \operatorname{Id},
\end{aligned}$$

and hence

$$\begin{aligned}&\lim_{s\to\frac{1}{2}} M_{\mathfrak{F}}\left(w_{232}, w_{2321}^{-1}\cdot\chi_s\right)\\&\quad=\lim_{s\to\frac{1}{2}} M_{\mathfrak{F}}\left(w_2, w_{232123}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_3, w_{23212}^{-1}\cdot\chi_s\right)\circ M_{\mathfrak{F}}\left(w_2, w_{2321}^{-1}\cdot\chi_s\right)=-\operatorname{Id}.\end{aligned}$$

Using the fact that $w_{2321}^{-1}\cdot\chi_{\frac{1}{2}} = w_{2132132}^{-1}\cdot\chi_{\frac{1}{2}}$ we apply the reason for cancellation of poles stated above and conclude that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left[M\left(w_{21212},\chi_s\right)+M\left(w_{21},\chi_s\right)\right]=0.$$

In conclusion,

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^2\mathcal{E}_E\left(\chi, f_s, s, g\right)_{B_E}=0.$$

We now turn to proving that the simple pole does not cancel. It holds that

$$\begin{gathered}\Sigma_{\left(F\times K,\chi_K,\frac{1}{2},1\right)}=\left\{w_{23}, w_{232}, w_{213}\right\},\\ w_{23}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\chi_K\left(t_1t_2\right)\frac{|t_1|_F}{|t_3|_K},\quad w_{232}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\chi_K\left(t_2\right)\frac{|t_1|_F}{|t_3|_K},\\ w_{213}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\chi_K\left(t_1\right)\frac{|t_2|_F}{|t_1|_F\,|t_3|_K}.\end{gathered}$$

Reason 2 for non-vanishing of the leading term then implies that $\mathcal{E}_{F\times K}\left(\chi_K, f_s^0, s, g\right)$ admits a simple pole at $s=\frac{1}{2}$.

4.7. $E=F\times K$**,** K **a field,** $s=\frac{1}{2}$**,** $\chi^2=\mathbf{1}$**,** $\chi\neq\mathbf{1},\chi_K$**.** The intertwining operators in this case have poles at most of order 1. We show that indeed the Eisenstein series admits a simple pole at this point. We apply reason 1 for non-cancellation of poles; namely, we construct a section $f_s\in I_{P_E}(\chi,s)$ such that $\mathcal{E}(\chi, f_s, s, g)$ admits a simple pole at $s=\frac{1}{2}$. We have

$$\Sigma_{\left(E,\chi,\frac{1}{2},1\right)}=\left\{w_{2321}, w_{2132}, w_{21321}, w_{21323}, w_{213213}, w_{2132132}\right\}$$

and

$$\begin{aligned}&w_{2321}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2132132}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\chi\left(t_2\right)\frac{|t_2|_F}{|t_1|_F\,|t_3|_K},\\&w_{2132}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{21323}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\chi\left(t_1\right)\frac{1}{|t_2|_F},\\&w_{21321}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{213213}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\chi\left(t_1t_2\operatorname{Nm}_{K/F}\left(t_3\right)\right)\frac{1}{|t_2|_F}.\end{aligned}$$

We write $w_{21323}=w_{2132}w_3$. It follows from Corollary 3.12 that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left[M\left(w_{21323},\chi_s\right)+M\left(w_{2132},\chi_s\right)\right]=0.$$

Write w_{21321}, $w_{213213}=w_{21321}w_3$, and $w_{2132132}=w_{2321232}=w_{2321}w_{232}$. The sums over the two other equivalence classes:

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left[M\left(w_{232123},\chi_s\right)+M\left(w_{23212},\chi_s\right)\right],$$
$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left[M\left(w_{2321232},\chi_s\right)+M\left(w_{2321},\chi_s\right)\right],$$

do not vanish. We prove this for the first sum, and the proof for the second one is similar. We write

$$\begin{aligned}&\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)\left[M\left(w_{232123},\chi_s\right)+M\left(w_{23212},\chi_s\right)\right]\\&=\left(1+M\left(w_3,w_{21321}^{-1}\cdot\chi_{\frac{1}{2}}\right)\right)\circ\left[\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21321},\chi_s\right)\right].\end{aligned}$$

Our aim is to prove that $1+M\left(w_3,w_{21321}^{-1}\cdot\chi_{\frac{1}{2}}\right)$ does not vanish on the image of $\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21321},\chi_s\right)$, or in other words that there exists a section $f_s\in I_P\left(\chi,s\right)$ such that $\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21321},\chi_s\right)f_s\neq 0$ is not an eigenvector of eigenvalue -1 of $M\left(w_3,w_{21321}^{-1}\cdot\chi_{\frac{1}{2}}\right)$. Applying equation 3.22, we have

$$\begin{aligned}&\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21321},\chi_s\right)\\&=\left(\prod_{\alpha>0,\ w_{21321}^{-1}\alpha<0}\epsilon_{F_\alpha}\left(\left\langle\chi_{\frac{1}{2}},\alpha^\vee\right\rangle\right)\right)\cdot\left[\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)J\left(w_{21321},\chi_s\right)\right]\cdot N\left(w_{21321},\chi_s\right)\\&=C\cdot N\left(w_{21321},\chi_s\right),\end{aligned}$$

where C is a non-zero constant. Let $f_s=\bigotimes_{\nu\in\mathcal{P}}f_{s,\nu}$ be a pure tensor and let $S\subset\mathcal{P}$ be a finite set such that $f_{s,\nu}=f_{s,\nu}^0$ is for any $\nu\notin S$. It follows that

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21321},\chi_s\right)f_s=C\left(\bigotimes_{\nu\in S}N_\nu\left(w_{21321},\chi_{\frac{1}{2},\nu}\right)f_{\frac{1}{2},\nu}\right)\otimes\left(\bigotimes_{\nu\notin S}f_{\frac{1}{2},\nu}^0\right).$$

We show in Subsection 4.7.1 that for any $\nu\in\mathcal{P}$ there exists $f_{s,\nu}\in I_{P_E}\left(\chi_\nu,s\right)$ such that $N_\nu\left(w_{21321},\chi_{\frac{1}{2},\nu}\right)f_{\frac{1}{2},\nu}\neq 0$.

We consider the character $\chi\circ\mathrm{Nm}_{K/F}$ of $\mathrm{Res}_{K/F}\left(\mathbb{A}_F^\times\right)$, by the assumption $\chi\circ\mathrm{Nm}_{K/F}\neq\mathbf{1}$. By the Strong Multiplicity One Theorem, [PS79], there are infinitely many places ν such that $\chi_\nu\circ\mathrm{Nm}_{K_\nu/F_\nu}\neq\mathbf{1}_\nu$. We fix $\nu_0\nmid\infty$ such that $\chi_\nu\neq\mathbf{1}_\nu,\chi_{K_\nu}$.

In Subsection 4.7.1 we prove that, for such a place,

$$\mathrm{Ind}_{B_E(F_{\nu_0})}^{H_E(F_{\nu_0})}\left(w_{21321}^{-1}\cdot\left(\chi_{\nu_0}\circ\det{}_{M_E}\right)\otimes\lambda_{\frac{1}{2}}\right)=\Pi_1\oplus\Pi_{-1},$$

where $\Pi_\epsilon\neq 0$, for $\epsilon\in\{1,-1\}$, is the ϵ-eigenspace of $N_{\nu_0}\left(w_3,w_{21321}^{-1}\cdot\chi_{\frac{1}{2},\nu_0}\right)$. We further prove in Subsection 4.7.1 that there exist $v_{\nu_0}\in I_P\left(\chi_\nu,\frac{1}{2}\right)$ such that $N_\nu\left(w_{21321},\chi_{\frac{1}{2},\nu_0}\right)v_{\nu_0}$ is not an eigenvector of $N_{\nu_0}\left(w_3,w_{21321}^{-1}\cdot\chi_{\frac{1}{2},\nu_0}\right)$.

We may continue v_{ν_0} into a standard section f_{s,ν_0} of $I_{P_E}\left(\chi_{\nu_0},s\right)$. By choosing f_s so that $f_{s,\nu}$ is given as above, the claim follows.

Remark 4.4. If χ is unramified at all places, then the global spherical section exists and realizes the pole.

4.7.1. *Local calculations for Section* 4.7. We now prove a few results regarding the local intertwining operator which were used above.

The case of non-split $E_\nu = F_\nu \times K_\nu$ with $\chi_\nu^2 = \mathbf{1}_\nu$ and $\chi_\nu \neq 1, \chi_{K_\nu}$. Let P_3 be the parabolic subgroup of H_E whose Levi subgroup M_3 is generated by $\alpha_3 + \alpha_4$. The Levi M_3 is isomorphic to $GL_1 \times \left(\mathrm{Res}_{K/F} GL_2\right)^0$, where

$$\left(\mathrm{Res}_{K/F} GL_2\right)^0 (F_\nu) = \left\{ g \in GL_2(K) \mid \det g \in F_\nu^\times \right\}.$$

Note that $M_3 \cap B_E = GL_1 \times \mathcal{B}^0$, where $\mathcal{B}^0$ is the Borel subgroup of $\left(\mathrm{Res}_{K/F} GL_2\right)^0 (F_\nu)$. Also, associated to the root $\alpha_3 + \alpha_4$ is an embedding of $SL_2(K_\nu)$ into $\left(\mathrm{Res}_{K/F} GL_2\right)^0 (F_\nu)$ and of $\mathcal{B}(K_\nu)$ into $\mathcal{B}^0$.

Since

$$\left[w_{23212}^{-1} \cdot \chi_{\frac{1}{2}} \right] \left(h_{\alpha_3+\alpha_4}(t) \right) = \chi\left(\mathrm{Nm}_{K/F}(t_3) \right)$$

it follows that

$$\mathrm{Ind}_{\mathcal{B}}^{SL_2(F)} \left[w_{23212}^{-1} \cdot \chi_{\frac{1}{2}} \Big|_{\mathcal{B}(K)} \right] = \pi^{(1)} \oplus \pi^{(-1)}, \tag{4.4}$$

where $\pi^{(\epsilon)}$ are as in Subsection 3.2. Hence, by Lemma 3.10,

$$\mathrm{Ind}_{B_E(F)}^{H_E(F)} \left(w_{23212}^{-1} \cdot \chi_{\frac{1}{2}} \right) = \Pi_1 \oplus \Pi_{-1},$$

where Π_ϵ, for $\epsilon \in \{1, -1\}$, is the ϵ-eigenspace of $N_\nu\left(w_3, , w_{23212}^{-1} \cdot \chi_{\frac{1}{2}} \right)$.

Lemma 4.5. *If F is non-Archimedean, then there exists $v \in I_{P_E}\left(\chi_\nu, \frac{1}{2}\right)$ such that $N_\nu\left(w_{23212}, \chi_{\nu,\frac{1}{2}} \right) v \neq 0$ is not an eigenvector of $N_\nu\left(w_3, w_{23212}^{-1} \cdot \chi_{\nu,\frac{1}{2}} \right)$.*

Proof. For a parabolic subgroup $Q = L \cdot V$ of $H_E(F_\nu)$ and an admissible representation Ω of L, we denote by $\mathcal{J}_{T_E}^{L}\left(\mathrm{Ind}_Q^{H_E(F_\nu)} \Omega \right)$ the Jacquet functor of Ω with respect to $L \cap B_E$. We recall a corollary to the results of [Cas74, Section 6.3]:

Corollary 4.6. *Let $Q = L \cdot V$ be a standard parabolic subgroup of $H_E(F_\nu)$ and let Ω be an admissible representation of L. The Jacquet functor $\mathcal{J}_{T_E}^{H_E}\left(\mathrm{Ind}_Q^{H_E(F_\nu)} \Omega \right)$ of $\mathrm{Ind}_Q^{H_E(F_\nu)} \Omega$ (normalized induction) has a composition series with factors $w^{-1} \cdot \mathcal{J}_{T_E}^{L} \Omega$, where w runs over the set of representatives of minimal length of the cosets of $W_L \backslash W_{H_E(F_\nu)}$.*

In what follows, we write $w_{21321} = w_{23212}$ and decompose

$$N_\nu\left(w_{23212}, \chi_{\nu,\frac{1}{2}} \right) = N_\nu\left(w_2, w_{2321}^{-1} \cdot \chi_{\nu,\frac{1}{2}} \right) \circ N_\nu\left(w_1, w_{232}^{-1} \cdot \chi_{\nu,\frac{1}{2}} \right) \circ N_\nu\left(w_{232}, \chi_{\nu,\frac{1}{2}} \right).$$

We note that:

- $N_\nu\left(w_{232},\chi_\nu,\lambda_{\frac{1}{2}}\right)$ is an isomorphism and so is $N_\nu\left(w_2,w_{2321}^{-1}\cdot\chi_{\nu,\frac{1}{2}}\right)$.
- $N_\nu\left(w_1,w_{232}^{-1}\cdot\lambda_{\frac{1}{2}}\right):\mathrm{Ind}_{B(F_\nu)}^{H_E(F_\nu)}(\chi_\nu\circ\alpha_1)\otimes w_{232}^{-1}\cdot\lambda_{\frac{1}{2}}\to\mathrm{Ind}_{B(F)}^{H_E(F_\nu)}(\chi_\nu\circ\alpha_1)\otimes w_{2321}^{-1}\cdot\lambda_{\frac{1}{2}}$ is not an isomorphism. We now describe its image and kernel.

  Let P_1 be the standard parabolic subgroup of H_E whose Levi subgroup is $L_1=\langle T_E,x_{\alpha_1}(r),x_{-\alpha_1}(r)\mid r\in F_\nu\rangle$. We have a short exact sequence

$$\mathrm{Ind}_{P_1}^{H_E(F_\nu)}(St_{L_1}\otimes\Omega)\hookrightarrow\mathrm{Ind}_{B(F_\nu)}^{H_E(F_\nu)}\left(w_{232}^{-1}\cdot\chi_{\frac{1}{2}}\right)\twoheadrightarrow\mathrm{Ind}_{P_1}^{H_E(F_\nu)}(\Omega),$$

  where Ω is a character of L_1 such that $\mathcal{J}_{T_E}^{L_1}\Omega=(\chi_\nu\circ\alpha_1)\otimes w_{232}^{-1}\cdot\lambda_{\frac{1}{2}}$ and St_{L_1} is the Steinberg representation of L_1. We conclude that

$$\mathrm{Im}\left[N\left(w_{23212},\chi_{\frac{1}{2}}\right)\right]\cong\mathrm{Ind}_{P_1}^{H_E(F_\nu)}(\Omega).$$

- It follows from [Tad94, Section 7] that the Jacquet modules $\mathcal{J}_{T_E}^{H_E}\Pi_1$ and $\mathcal{J}_{T_E}^{H_E}\Pi_{-1}$ of Π_1 and Π_{-1} are isomorphic.

Let $\Lambda=(\chi\circ\omega_2)\otimes(-1,1,-1)$, where $(-1,1,-1)\in\mathfrak{a}_{\mathbb{C}}^*$ is given by the coordinates defined in Subsection 3.3. Applying Corollary 4.6 we calculate the multiplicity m_Λ of Λ in $\mathcal{J}_{T_E}^{H_E}\sigma$ for various representations:

- $m_\Lambda\left(\mathrm{Ind}_{B(F_\nu)}^{H_E(F_\nu)}\chi_{\nu,\frac{1}{2}}\right)=m_\Lambda\left(\mathrm{Ind}_{B(F)}^{H_E(F)}w_{2321}^{-1}\cdot\chi_{\nu,\frac{1}{2}}\right)=2.$
- $m_\Lambda\left(I_{P_E(F_\nu)}\left(\chi_\nu,\frac{1}{2}\right)\right)=2.$
- $m_\Lambda\left(\mathrm{Im}N_\nu\left(w_{23212},\chi_\nu,\lambda_{\frac{1}{2}}\right)\right)=m_\Lambda\left(\mathrm{Ind}_{P_1}^{H_E(F_\nu)}(\Omega)\right)=2.$
- $m_\Lambda(\Pi_1)=m_\Lambda(\Pi_{-1})=1.$

It follows that the image of $I_{P_E(F_\nu)}\left(\chi_\nu,\frac{1}{2}\right)$ under $N_\nu\left(w_{23212},\chi_{\nu,\frac{1}{2}}\right)$ cannot be contained in either Π_1 or Π_{-1}. $\square$

The case of $E=F\times F\times F$ with $\chi^2=1$ and $\chi\neq1$. Applying a similar argument as in the quasi-split case, one can show that

$$\mathrm{Ind}_{B_E(F_\nu)}^{H_E(F_\nu)}\left(w_{234212}^{-1}\cdot\chi_{\frac{1}{2}}\right)=\Pi_1\oplus\Pi_{-1},$$

where Π_ϵ, for $\epsilon\in\{1,-1\}$, is the ϵ-eigenspace of $N_\nu\left(w_{34},w_{234212}^{-1}\cdot\chi_{\nu,\frac{1}{2}}\right)$.

Lemma 4.7. *If F is non-Archimedean, then there exists $v\in I_{P_E}\left(\chi_\nu,\frac{1}{2}\right)$ such that $N_\nu\left(w_{234212},\chi_{\nu,\frac{1}{2}}\right)v\neq0$ is not an eigenvector of $N_\nu\left(w_{34},w_{234212}^{-1}\cdot\chi_{\nu,\frac{1}{2}}\right)$.*

Non-vanishing of $N_\nu\left(w_{2321},\chi_{\frac{1}{2},\nu}\right)f_{\frac{1}{2},\nu}\neq0$. Applying similar arguments as above, one can show that for any χ_ν the following holds.

Lemma 4.8. *For any $\nu\in\mathcal{P}$ there exists $v\in I_{P_E(F_\nu)}(\chi_\nu,s)$ such that*

$$N_\nu\left(w_{2321},\chi_{\frac{1}{2},\nu}\right)f_{\nu\frac{1}{2}}\neq0.$$

4.8. $E = F \times K$**,** K **a field,** $s = \frac{3}{2}$**,** $\chi = \mathbf{1}$**.** The intertwining operators in this case have poles at most of order 2. We show that the Eisenstein series admits a simple pole at this point. We have

$$\Sigma_{\left(F\times K,\chi_K,\frac{1}{2},2\right)} = \{w_{213213}, w_{2132132}\}$$

and

$$w_{213213}^{-1} \cdot \chi_{\frac{1}{2}}(t) = w_{2132132}^{-1} \cdot \chi_{\frac{1}{2}}(t) = \frac{1}{|t_1|_F\,|t_3|_K}.$$

We write $w_{2132132} = w_{213213}w_2$. It follows from Corollary 3.12 that

$$\lim_{s\to\frac{3}{2}} \left(s-\frac{3}{2}\right)^2 \left[M\left(w_{2132132},\chi_s\right) + M\left(w_{213213},\chi_s\right)\right] = 0,$$

and hence $\lim\limits_{s\to\frac{3}{2}} \left(s-\frac{3}{2}\right)^2 \mathcal{E}_E\left(\mathbf{1}, f_s, s, g\right)_{B_E} = 0$.

We now turn to proving that the simple pole does not cancel. It holds that

$$\begin{aligned}
&\Sigma_{\left(F\times K,\chi_K,\frac{1}{2},1\right)} = \{w_{232}, w_{2321}, w_{21321}, w_{21323}\},\\
&w_{232}^{-1} \cdot \chi_{\frac{1}{2}}(t) = \frac{|t_1|_F^3}{|t_2|_F\,|t_3|_K},\\
&w_{2321}^{-1} \cdot \chi_{\frac{1}{2}}(t) = \frac{|t_2|_F^2}{|t_1|_F^3\,|t_3|_K},\\
&w_{21321}^{-1} \cdot \chi_{\frac{1}{2}}(t) = \frac{|t_3|_K}{|t_1t_2^2|_F},\\
&w_{21323}^{-1} \cdot \chi_{\frac{1}{2}}(t) = \frac{|t_1|_F}{|t_2|_F\,|t_3|_K}.
\end{aligned}$$

Reason 2 for non-vanishing of the leading term then implies that $\mathcal{E}_{F\times K}\left(\chi_K, f_s^0, s, g\right)$ admits a simple pole at $s = \frac{1}{2}$.

4.9. $E = F \times F \times F$**,** $s = \frac{1}{2}$**,** $\chi = \mathbf{1}$**.** This case is similar to Subsection 4.5. The intertwining operators in this case have poles at most of order 4. We show that the Eisenstein series admits a simple pole at this point. We have

$$\begin{aligned}
&\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)}\\
&\qquad = \{w_{21342}, w_{213421}, w_{213423}, w_{213424}, w_{2134213}, w_{2134214}, w_{2134234}, w_{21342134}\},\\
&\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},3\right)} \setminus \Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)} = \{w_{2134}, w_{21324}, w_{21423}, w_{23421}, w_{213421342}\},\\
&\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},2\right)} \setminus \Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},3\right)} = \{w_{213}, w_{214}, w_{234}, w_{2132}, w_{2142}, w_{2342}\},\\
&\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},1\right)} \setminus \Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},2\right)} = \{w_{21}, w_{23}, w_{24}\}
\end{aligned}$$

and also

$$w_{21342}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{213421}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{213423}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{213424}^{-1}\cdot\chi_{\frac{1}{2}}(t)$$
$$=w_{2134213}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2134214}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2134234}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{21342134}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{1}{t_2}\right|,$$
$$w_{2134}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{23421}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{21423}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{21324}^{-1}\cdot\chi_{\frac{1}{2}}(t)$$
$$=w_{213421342}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{t_2}{t_1t_3t_4}\right|,$$
$$w_{21}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{t_3t_4}{t_1t_2}\right|,\quad w_{23}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{t_1t_4}{t_2t_3}\right|,\quad w_{24}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{t_1t_3}{t_2t_4}\right|,$$
$$w_{213}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2132}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{t_4}{t_1t_3}\right|,$$
$$w_{214}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2142}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{t_3}{t_1t_4}\right|,$$
$$w_{234}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2342}^{-1}\cdot\chi_{\frac{1}{2}}(t)=\left|\frac{t_1}{t_3t_4}\right|.$$

We note that it follows from the above that

$$\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},2\right)}/\sim_{s_0}=\Big\{\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)},\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},3\right)}\setminus\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)},$$
$$\{w_{213},w_{2132}\},\{w_{214},w_{2142}\},\{w_{234},w_{2342}\}\Big\}.$$

After proving that the poles of order 4, 3, and 2 cancel, the fact that the simple pole is attained follows from reason 2 for non-vanishing of the leading term. More precisely, if f_s^0 is the global spherical vector one has

$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21},\chi_s\right)f_s^0,\ \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{23},\chi_s\right)f_s^0,$$
$$\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{24},\chi_s\right)f_s^0\neq 0.$$

Namely, $\mathcal{E}_{F\times F\times F}\left(1,f_s^0,s,g\right)$ admits a simple pole at $s=\frac{1}{2}$.

We now turn to proving that the poles of higher order are canceled. With notation as in Lemma 3.13 we write

$$M\left(w,\chi_s\right)f_s\left(bk\right)=\left(w^{-1}\cdot\chi_s\right)(b)\,M_{\mathfrak{F}}\left(w,\chi_s\right)f\left(k\right).$$

We start with the sum over the elements of $\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},4\right)}$. Note that $\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}$ $=w_{21342}\cdot W_{M_E}$ and that any element in W_{M_E} is of the form $w_1^{\epsilon_1}w_3^{\epsilon_3}w_4^{\epsilon_4}$, where $\epsilon_1,\epsilon_3,\epsilon_4\in\{0,1\}$.

Denote $\eta_s=w_{21342}^{-1}\cdot\chi_s$. It follows that

$$\sum_{w\in\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)}}M\left(w,\chi_s\right)f_s=\left(\mathrm{Id}+w_1^{-1}\cdot\eta_sM_{\mathfrak{F}}\left(w_1,\eta_s\right)\right)$$
$$\times\left(\mathrm{Id}+w_3^{-1}\cdot\eta_sM_{\mathfrak{F}}\left(w_3,\eta_s\right)\right)\left(\mathrm{Id}+w_4^{-1}\cdot\eta_sM_{\mathfrak{F}}\left(w_4,\eta_s\right)\right)\eta_sM_{\mathfrak{F}}\left(w_{21342},\chi_s\right)f.$$

It follows from Corollary 3.12 that $M_{\mathfrak{F}}\left(w_1,\eta_{\frac{1}{2}}\right)=-\mathrm{Id}$, $M_{\mathfrak{F}}\left(w_3,\eta_{\frac{1}{2}}\right)=-\mathrm{Id}$, and $M_{\mathfrak{F}}\left(w_4,\eta_{\frac{1}{2}}\right)=-\mathrm{Id}$. On the other hand, $w_1^{-1}\eta_{\frac{1}{2}}=w_3^{-1}\eta_{\frac{1}{2}}=w_4^{-1}\eta_{\frac{1}{2}}=\mathbf{1}$, and hence

$$\left(\mathrm{Id}+w_1^{-1}\cdot\eta_s M_{\mathfrak{F}}\left(w_1,\eta_s\right)\right)\left(\mathrm{Id}+w_3^{-1}\cdot\eta_s M_{\mathfrak{F}}\left(w_3,\eta_s\right)\right)\left(\mathrm{Id}+w_4^{-1}\cdot\eta_s M_{\mathfrak{F}}\left(w_4,\eta_s\right)\right)$$

admits a zero of order 3. Since $M_{\mathfrak{F}}\left(w_{21342},\chi_s\right)f$ admits a pole of order 4 we conclude that

$$\sum_{w\in\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)}} M\left(w,\chi_s\right)f_s$$

can admit at most a simple pole (in fact, one can show that it admits exactly a simple pole).

We now consider the sum over the elements of $\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}\setminus\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},4\right)}$. We note that

$$\begin{aligned}
w_{2134} &= w_{213421342}w_{21342},\\
w_{23421} &= w_{213421342}w_{2342},\\
w_{21423} &= w_{213421342}w_{2142},\\
w_{21324} &= w_{213421342}w_{2132}.
\end{aligned}$$

Write $\eta_s = w_{213421342}^{-1}\cdot\chi_s$. We note that

$$\begin{aligned}
&\sum_{w\in\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},3\right)}\setminus\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)}} M\left(w,\chi_s\right)f_s\left(bk\right)\\
&=\left(\prod_{i=1}^{3}\left[\mathrm{Id}+\left|\frac{t_2}{t_i}\right|_F^{2s-1}M_{\mathfrak{F}}\left(w_{2i2},\eta_s\right)\right]\right)\eta_s\left(b\right)M_{\mathfrak{F}}\left(w_{21342},\chi_s\right)f\left(k\right)\\
&-\left[M\left(w_{21342342},\chi_s\right)+M\left(w_{21342142},\chi_s\right)+M\left(w_{21342132},\chi_s\right)\right]f_s.
\end{aligned}$$

We further note that

- $w_{21342342}=w_{12134234}$, $w_{21342142}=w_{32134214}$, and $w_{21342132}=w_{42134213}$. Hence

$$\begin{aligned}
M\left(w_{21342342},\chi_s\right) &= M\left(w_{2134234},w_1^{-1}\chi_s\right)M\left(w_1,\chi_s\right),\\
M\left(w_{21342142},\chi_s\right) &= M\left(w_{2134214},w_3^{-1}\chi_s\right)M\left(w_3,\chi_s\right),\\
M\left(w_{21342132},\chi_s\right) &= M\left(w_{2134213},w_4^{-1}\chi_s\right)M\left(w_4,\chi_s\right).
\end{aligned}$$

  Recalling that

$$M\left(w_1,\chi_s\right)\big|_{I_{P_E}(\mathbf{1},s)}\equiv M\left(w_3,\chi_s\right)\big|_{I_{P_E}(\mathbf{1},s)}\equiv M\left(w_4,\chi_s\right)\big|_{I_{P_E}(\mathbf{1},s)}\equiv 0$$

  we conclude that

$$M\left(w_{21342342},\chi_s\right)f_s=M\left(w_{21342142},\chi_s\right)f_s=M\left(w_{21342132},\chi_s\right)f_s=0.$$

- $M_{\mathfrak{F}}\left(w_{213213},\chi_s\right)f\left(k\right)$ admits a pole of order 3 at $s=\frac{1}{2}$.
- $\lim\limits_{s\to\frac{1}{2}}\left|\frac{t_2}{t_i}\right|_F^{2s-1}=1$ for $i=1,3,4$.
- It follows from [Lao, Subsection 5.5.3] that the image

$$\left[\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)^3M\left(w_{213421342},\chi_s\right)\right]\left(I_{P_E}\left(\mathbf{1},\frac{1}{2}\right)\right)$$

  is an irreducible representation (since E_ν is not a field for all $\nu\in\mathcal{P}$) and that $M\left(w_{21342},\eta_s\right)$ acts on it as $-\mathrm{Id}$.

By an argument similar to the one used there one could prove that $M(w_{212},\eta_s)$, $M(w_{232},\eta_s)$, and $M(w_{242},\eta_s)$ will all act as $-\mathrm{Id}$ on this irreducible representation.

Alternatively, one could argue as follows: $M(w_{212},\eta_s)$, $M(w_{232},\eta_s)$, and $M(w_{242},\eta_s)$ are endomorphisms of an irreducible representation and hence act as multiplication by a scalar. By triality, they all act by the same scalar z. Since $M(w_{212},\eta_s)\,M(w_{232},\eta_s)\,M(w_{242},\eta_s) = M(w_{21342},\eta_s)$ it follows that $z^3=-1$. On the other hand, $M(w_{212},\eta_s)\,M(w_{212},\eta_s)=\mathrm{Id}$ so that $z^2=1$. It follows that $z=-1$. It follows that the product

$$\prod_{i=1}^{3}\left[\mathrm{Id}+\left|\frac{t_2}{t_i}\right|_F^{2s-1} M_{\mathfrak{F}}(w_{2i2},\eta_s)\right]$$

admits a zero of order 3.

It follows that

$$\begin{aligned}&\sum_{w\in\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},3\right)}\setminus\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},4\right)}} M(w,\chi_s)\,f_s(bk)\\ &=\left(\prod_{i=1}^{3}\left[\mathrm{Id}+\left|\frac{t_2}{t_i}\right|_F^{2s-1} M_{\mathfrak{F}}(w_{2i2},\eta_s)\right]\right)\eta_s(b)\,M_{\mathfrak{F}}(w_{21342},\chi_s)\,f(k)\end{aligned}$$

admits at most a simple pole at $s=\frac{1}{2}$.

Finally, we consider the sum over the elements of $\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},2\right)}\setminus\Sigma_{\left(F\times K,\mathbf{1},\frac{1}{2},3\right)}$. We note that

$$\begin{aligned}w_{2132}&=w_{213}w_2,\\ w_{2142}&=w_{214}w_2,\\ w_{2342}&=w_{234}w_2.\end{aligned}$$

We then note that $M\left(w_2,w_{213}^{-1}\cdot\chi_s\right)$, $M\left(w_2,w_{214}^{-1}\cdot\chi_s\right)$, and $M\left(w_2,w_{234}^{-1}\cdot\chi_s\right)$ satisfy the conditions of Corollary 3.12, and hence we have a cancellation of the pole of order two of the following sums:

$$\begin{aligned}M(w_{213},\chi_s)+M(w_{2132},\chi_s)&\equiv 0,\\ M(w_{214},\chi_s)+M(w_{2142},\chi_s)&\equiv 0,\\ M(w_{234},\chi_s)+M(w_{2342},\chi_s)&\equiv 0.\end{aligned}$$

It follows that the pole of order two of

$$\sum_{w\in\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},2\right)}\setminus\Sigma_{\left(F\times F\times F,\mathbf{1},\frac{1}{2},3\right)}} M(w,\chi_s)$$

is canceled.

In conclusion, $\mathcal{E}_E\left(1,f_s^0,s,g\right)$ admits a pole of order 1 at $s=\frac{1}{2}$.

4.10. **$E=F\times F\times F$, $s=\frac{1}{2}$, $\chi^2=\mathbf{1}$, $\chi\neq\mathbf{1}$.** The intertwining operators in this case have poles of order at most 1. We show that indeed the Eisenstein series admits a simple pole at this point. Here we apply reason 2 for non-cancellation of poles. We have

$$\Sigma_{\left(F\times F\times F,\chi,\frac{1}{2},1\right)}=\{w_{21324},w_{21423},w_{23421},w_{21342},w_{213421},w_{213423},w_{213424},\\ w_{2134213},w_{2134214},w_{2134234},w_{21342134},w_{213421342}\}$$

and

$$\begin{aligned} w_{21324}^{-1}\cdot\chi_{\frac{1}{2}}(t)&=w_{21423}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{23421}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{213421342}^{-1}\cdot\chi_{\frac{1}{2}}(t)\\ &=\chi(t_2)\left|\frac{t_2}{t_1t_3t_4}\right|,\\ w_{21342}^{-1}\cdot\chi_{\frac{1}{2}}(t)&=w_{2134213}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2134214}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{2134234}^{-1}\cdot\chi_{\frac{1}{2}}(t)\\ &=\chi(t_1t_3t_4)\left|\frac{1}{t_2}\right|,\\ w_{213421}^{-1}\cdot\chi_{\frac{1}{2}}(t)&=w_{213423}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{213424}^{-1}\cdot\chi_{\frac{1}{2}}(t)=w_{21342134}^{-1}\cdot\chi_{\frac{1}{2}}(t)\\ &=\chi(t_1t_2t_3t_4)\left|\frac{1}{t_2}\right|. \end{aligned}$$

We note that

$$\begin{aligned} &w_{213421342}=w_{21324}w_{2132}=w_{21423}w_{2142}=w_{23421}w_{2342},\\ &w_{21342134}=w_{213424}w_{13}=w_{213423}w_{14}=w_{213421}w_{34},\\ &w_{2134213}=w_{21342}w_{13},\\ &w_{2134214}=w_{21342}w_{14},\\ &w_{2134234}=w_{21342}w_{34}. \end{aligned}$$

According to Corollary 3.12, we conclude that

$$\begin{aligned} \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21342},\chi_s\right)&=\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{2134213},\chi_s\right)\\ &=\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{2134214},\chi_s\right)=\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{2134234},\chi_s\right),\\ \lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{213421},\chi_s\right)&=\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{213423},\chi_s\right)\\ &=\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{213424},\chi_s\right)=\lim_{s\to\frac{1}{2}}\left(s-\frac{1}{2}\right)M\left(w_{21342134},\chi_s\right). \end{aligned}$$

We do not treat the rest of the terms, as they have different exponents; in particular, the pole of order 1 does not cancel.

4.11. **Square integrability of the residual representations.** We now determine, for a point where $\mathcal{E}_E(\chi,f_s,s,g)$ admits a pole, whether the residual representation is square-integrable or not. Before doing so, we recall the following criterion from [Lan76, pg. 104].

For $w\in W(P_E,H_E)$ the element $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)\in\mathfrak{a}_{\mathbb{R}}^*$ is known as the *exponent of* $I_{P_E}(\chi,s)$ *corresponding to* w.

Assume $\mathcal{E}_E(\chi,f_s,s,g)$ admits a pole of order n at s_0. We recall the equivalence relation defined in equation 4.2 and define the quotient set

$$\Sigma_{s_0}=\Sigma_{(E,\chi,s_0,n)}/\sim_{s_0}.$$

Note that the exponent is well defined for equivalence classes, namely $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right) = \mathfrak{Re}\left(w'^{-1}\cdot\chi_s\right)$ when $w \sim_{s_0} w'$. And we consider the elements contributing to the residual representation at s_0, namely

$$\Sigma_{s_0}^0 = \left\{\Omega\in\Sigma_{s_0} \;\middle|\; \lim_{s\to s_0}\left(s-s_0\right)^n \sum_{w\in\Omega} M\left(w,\chi_s\right)\not\equiv 0\right\}.$$

Lemma 4.9 (Langlands' criterion for square integrability)**.** *Assume $\mathcal{E}_E\left(\chi,f_s,s,g\right)$ admits a pole of order n at s_0. The residual representation $\mathrm{Res}_{s=s_0}\,\mathcal{E}_E\left(\chi,f_s,s,g\right)$ is a square-integrable representation if and only if $\mathfrak{Re}\left(\Omega^{-1}\cdot\chi_s\right)<0$ for all $\Omega\in\Sigma_{s_0}^0$.*

Corollary 4.10. *The residual representation of $\mathcal{E}_E\left(\chi,\cdot,s,\cdot\right)$ is square-integrable with the exception of the following cases:*

- *$E=F\times K$ where K is a field:*
  - *$s=\frac{1}{2}$ with $\chi=\mathbf{1},\chi_K$.*
  - *$s=\frac{3}{2}$ with $\chi=\mathbf{1}$.*
- *$E=F\times F\times F$, $s=\frac{1}{2}$ with $\chi=\mathbf{1}$.*

This follows from the proof of Theorem 4.1, from Langlands' criterion for square integrability, and from the information in Tables 6, 10, and 14.

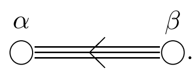
□

Part 2. **Applications**

5. The twisted standard $\mathcal{L}$-function of a cuspidal representation of G_2

In this section we recall the main result of [Seg17].

5.1. **The group G_2.** Let G be the simple, split group of type G_2 defined over F. In particular, G is adjoint and simply-connected. Let B be a Borel subgroup of G and let T be a maximal torus in B. Let α and β be the short and long simple roots of G with respect to (B,T). The Dynkin diagram of G is



We have a short exact sequence

$$1\to H_E^{ad}\to \mathrm{Aut}\left(H_E\right)\to S_E\to 1.$$

Forming the semidirect product $H_E\rtimes S_E$ it holds that $G\cong \mathrm{Cent}_{H_E\rtimes S_E}\left(S_E\right)$. This gives a natural embedding

$$G\hookrightarrow H_E.$$

Moreover, (G,S_E) forms a dual reductive pair in $H_E\rtimes S_E$. Under this embedding, it holds that B can be chosen so that $B=G\cap B_E$. The set of positive roots of G is

$$\Phi^+=\left\{\alpha,\beta,\alpha+\beta,2\alpha+\beta,3\alpha+\beta,3\alpha+2\beta\right\}.$$

For any root γ we fix a one-parameter subgroup $x_\gamma:\mathbb{G}_a\to G$. Also, let $h_\gamma:\mathbb{G}_m\to T$ be the coroot subgroup such that for any root ϵ,

$$\epsilon\left(h_\gamma\left(t\right)\right)=t^{\left\langle\epsilon,\gamma^\vee\right\rangle}.$$

The group G contains a Heisenberg maximal parabolic subgroup $P = M \cdot U$. The Levi subgroup M is isomorphic to GL_2 and contains the root subgroups attached to α and $-\alpha$, while U is a 5-dimensional Heisenberg group. It holds that $P = G \cap P_E$.

Finally, we let $\mathfrak{st} : G \hookrightarrow GL_7$ be the standard 7-dimensional embedding.

5.2. **The twisted standard $\mathcal{L}$-function and an integral representation.** The dual Langlands group ${}^L G$ of G is isomorphic to $G_2(\mathbb{C})$.

Let $\pi = \bigotimes_{\nu\in\mathcal{P}} \pi_\nu$ be an irreducible cuspidal representation of $G(\mathbb{A})$ and let $\chi = \bigotimes_{\nu\in\mathcal{P}} \chi_\nu : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a Hecke character, both unramified outside a finite subset $S \subset \mathcal{P}$. For $\nu \notin S$ we denote its Satake parameter by t_{π_ν}. We let

$$\mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) = \prod_{\nu\notin S} \frac{1}{\det\left(I - \mathfrak{st}(t_{\pi_\nu})\,\chi(\varpi_\nu)\,q_\nu^{-s}\right)}.$$

This product converges for $\mathfrak{Re}(s) \gg 0$ to an analytic function.

For factorizable data $\varphi = \bigotimes_{\nu\in\mathcal{P}} \varphi_\nu \in \pi$ and $f_s = \bigotimes_{\nu\in\mathcal{P}} f_\nu \in I_{P_E}(\chi,s)$ we consider the following integral:

$$\mathcal{Z}_E(\chi,s,\varphi,f) = \int\limits_{G(F)\backslash G(\mathbb{A})} \varphi(g)\,\mathcal{E}_E^*(\chi,s,f,g)\,dg. \tag{5.1}$$

It holds that

Theorem 5.1 ([Seg17])**.** *Given a finite subset $S \subset \mathcal{P}$ such that for any $\nu \notin S$ all data is unramified and $\nu \nmid 2,3,\infty$, then*

$$\mathcal{Z}_E(\chi,s,\varphi,f) = \mathcal{L}^S\left(s+\frac{1}{2},\pi,\chi,\mathfrak{st}\right) d_S(\chi,s,\Psi_E,\varphi_S,f_S). \tag{5.2}$$

Moreover, for any s_0 there exist vectors φ_S, f_S such that $d_S(\chi,s,\Psi_E,\varphi_S,f_S)$ is analytic in a neighborhood of s_0 and $d_S(\chi,s_0,\Psi_E,\varphi_S,f_S) \neq 0$.

In particular, the family of twisted partial $\mathcal{L}$-functions $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ admits a meromorphic continuation to the whole complex plane.

Remark 5.2. The reason to make the assumption that $\nu \nmid 2,3$ is that some of the structure constants of G_2 are divisible by 2 and 3 and the local unramified calculation of [Seg17] assumed that all structure constants are invertible in $\mathcal{O}_\nu$.

For our applications, we need only the following corollary.

Corollary 5.3. *$\mathcal{L}^S\left(s+\frac{1}{2},\pi,\chi,\mathfrak{st}\right)$ is a meromorphic function on $\mathbb{C}$ and for any $s_0 \in \mathbb{C}$ it holds that*

$$ord_{s=s_0}\left(\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})\right) \leq ord_{s=s_0}\left(\mathcal{E}_E(\chi,f_s,s,g)\right). \tag{5.3}$$

Remark 5.4. We note that the residual representation of $\mathcal{E}_E(\mathbf{1},f_s,s,g)$ at $s=\frac{5}{2}$ is the trivial representation. It follows that

$$\lim_{s\to 3}(s-3)\,\mathcal{Z}_E(\mathbf{1},s,\varphi,f) = \int\limits_{G(F)\backslash G(\mathbb{A})} \varphi(g)\,dg = 0,$$

where the integral vanish due to the cuspidality of φ. It follows that $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ is holomorphic at $s=3$ for any π and χ. In particular, for $\mathfrak{Re}(s) > 0$, poles of $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ can occur only at $s=1$ and $s=2$.

6. A conjecture of Ginzburg and Hundley

In [GH15], D. Ginzburg and J. Hundley have constructed a doubling integral representing $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$. We recall the construction.

We first recall the commuting pair $G_2\times G_2 \subseteq E_8$. Given a cuspidal representation π of G_2, $\varphi\in\pi$, and $\widetilde{\varphi}\in\widetilde{\pi}$ we consider the integral

$$\int\limits_{G_2\times G_2(F)\backslash G_2\times G_2(\mathbb{A})} \varphi(g_1)\,\widetilde{\varphi}(g)\,\mathcal{E}_{E_8}^{\Psi_1}((g_1,g_2),f_{s,\chi})\,d(g_1,g_2), \tag{6.1}$$

where $\mathcal{E}_{E_8}^{\Psi_1}$ is a certain Fourier coefficient of a degenerate Eisenstein series for E_8 associated with the maximal parabolic subgroup whose Levi factor is of type A_7. In [GH15], Ginzburg and Hundley have shown that the integral in equation 6.1 represents $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$.

Considering the normalizing factor of this integral they conjectured the following:

Conjecture A. *The twisted partial standard $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ can have at most a double pole.*

We prove the following variant of the conjecture.

Theorem 6.1. *The **stable** poles of the twisted partial standard $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ can be of order at most 2. Namely, the order of a pole of $\mathcal{L}^S(s,\pi,\chi,\mathfrak{st})$ at $\mathfrak{Re}(s)>0$, for a large enough finite subset $S\subset\mathcal{P}$, is at most 2.*

Proof. This follows immediately from Corollary 5.3 and Table 2. ∎

7. CAP representations with respect to the Borel subgroup

We recall the definition of a **CAP** representation.

Definition 7.1. Let $Q=L\cdot V\subset G$ be a parabolic subgroup, let σ be a cuspidal unitary representation of the Levi part L, and let χ be a character of L. A cuspidal representation π of $G(\mathbb{A})$ is called **CAP** (cuspidal attached to parabolic) with respect to Q, σ, and χ if π is nearly equivalent to a subquotient of $\mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})}\sigma\otimes\chi$.

CAP representations for G_2 were constructed in [GGJ02] for the Borel subgroup, in [RS89] for the Heisenberg parabolic subgroup P, and in [GG09] for the non-Heisenberg maximal parabolic subgroup. Using Corollary 5.3 and Table 2 we prove that [GGJ02] exhausts the list of **CAP** representations with respect to the Borel subgroup.

For a Galois étale cubic algebra E over F let

$$n_E=\begin{cases}2, & E=F\times F\times F,\\ 1, & \text{otherwise.}\end{cases}$$

Theorem 7.2. *Let π be a cuspidal representation of $G(\mathbb{A})$ supporting a Fourier coefficient along U corresponding to an étale cubic extension E of F which is not a non-Galois field extension. The following are equivalent:*

1. *π is a **CAP** representation with respect to B.*
2. *The partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\chi_E,\mathfrak{st})$ has a pole of order n_E at $s=2$.*
3. *$\Theta_{S_E}(\pi)\neq 0$. In particular π is nearly equivalent to $\Theta_{S_E}(\mathbf{1})$, where $\mathbf{1}$ here is the automorphic trivial representation of $S_E(\mathbb{A})$.*

In particular, for π that satisfies these conditions we have $\mathcal{WF}_U(\pi)=\{E\}$.

Proof. The fact that (3) implies (1) and (2) was proven in [GGJ02]. The fact that (2) implies (3) is proven in [Seg17]. It is left to prove that (1) implies (2).

Let π be a **CAP** representation with respect to B that supports the Fourier coefficient corresponding to an étale cubic algebra E over F. We will prove that (2) holds by proving that π is nearly equivalent to $\Theta_{H_E}\left(\mathbf{1}_{S_E}\right)$ where $\mathbf{1}_{S_E}$ is the trivial representation of $S_E\left(\mathbb{A}\right)$.

Remark 7.3. Note that all irreducible automorphic representations of $S_E\left(\mathbb{A}\right)$ are nearly equivalent to $\mathbf{1}_{S_E}$.

By the assumption, there exists an automorphic character μ such that π is nearly equivalent to a subquotient of $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\mu$, where the induction here is unitary. Let

$$\mu\left(h_{2\alpha+\beta}\left(a\right)h_{3\alpha+2\beta}\left(b\right)\right)=\mu_1\left(a\right)\mu_2\left(b\right). \tag{7.1}$$

We denote $\mu_i\left(x\right)=\eta_i\left(x\right)\left|x\right|^{z_i}$, where η_i are unitary characters and $z_i\in\mathbb{R}$. By choosing a Weyl chamber we may assume that

$$0\leq z_2\leq z_1\leq 2z_2. \tag{7.2}$$

According to [GGJ02] we need to show that:

- If $E=F\times F\times F$, then $\mu_1\left(t\right)=\mu_2\left(t\right)=\left|t\right|$ for any $t\in\mathbb{A}^\times$.
- If $E=F\times K$, then $\mu_1\left(t\right)=\left|t\right|$ and $\mu_2\left(t\right)=\chi_K\left(t\right)\left|t\right|$ for any $t\in\mathbb{A}^\times$, or vice versa.
- If E/F is a cubic Galois extension, then $\mu_1\left(t\right)=\mu_2\left(t\right)=\chi_E\left(t\right)\left|t\right|$ for any $t\in\mathbb{A}^\times$.

It holds that

$$\begin{aligned}\mathcal{L}^S\left(s,\pi,\chi,\mathfrak{st}\right)=&\,\mathcal{L}_F^S\left(\mu_1\chi,s\right)\mathcal{L}_F^S\left(\mu_1^{-1}\chi,s\right)\mathcal{L}_F^S\left(\mu_2\chi,s\right)\mathcal{L}_F^S\left(\mu_2^{-1}\chi,s\right)\mathcal{L}_F^S\left(\frac{\mu_1}{\mu_2}\chi,s\right)\\&\times\mathcal{L}_F^S\left(\frac{\mu_2}{\mu_1}\chi,s\right)\mathcal{L}_F^S\left(\chi,s\right).\end{aligned}$$

For $\chi\left(t\right)=\mu_1\left(t\right)\left|t\right|^{-1}$, $\mathcal{L}^S\left(s,\pi,\mu_1\left|\cdot\right|^{-1},\mathfrak{st}\right)$ admits a pole at $s=2$, and hence $\mathcal{E}_E\left(\mu_1\left|\cdot\right|^{-1},f_s,s,g\right)$ admits a pole at $s=\frac{3}{2}$. Similarly, $\mathcal{E}_E\left(\mu_2\left|\cdot\right|^{-1},f_s,s,g\right)$ also admits a pole at $s=\frac{3}{2}$.

We continue by considering different kinds of E.

- <u>$E=F\times F\times F$:</u> Since $\mathcal{E}_E\left(\mu_1\left|\cdot\right|^{-1},f_s,s,g\right)$ and $\mathcal{E}_E\left(\mu_2\left|\cdot\right|^{-1},f_s,s,g\right)$ admit a pole at $s=\frac{3}{2}$, it holds that

  $$\left(z_1,\eta_1\right),\left(z_2,\eta_2\right)\in\left\{\left(0,\eta\right)\mid\eta^2\equiv\mathbf{1}\right\}\cup\left\{\left(1,\mathbf{1}\right)\right\}.$$

  We assume that $z_1=0$ and hence also that $z_2=0$. In this case η_1 and η_2 are quadratic characters. If $\eta_1=\mathbf{1}$, then

  $$\mathcal{L}^S\left(s,\pi,\chi,\mathfrak{st}\right)=\mathcal{L}_F^S\left(\chi,s\right)^3\mathcal{L}_F^S\left(\mu_2\chi,s\right)^4.$$

  If $\eta_2=\mathbf{1}$, then $\mathcal{L}^S\left(s,\pi,\mathbf{1},\mathfrak{st}\right)$ admits a pole of order 7 at $s=1$, while $\mathcal{E}_E\left(\mathbf{1},f_s,s,g\right)$ admits a pole of order at most 1 at $s=\frac{1}{2}$, which brings us to a contradiction.

  Assume that $\eta_2\neq\mathbf{1}$. Then $\mathcal{L}^S\left(s,\pi,\eta_2,\mathfrak{st}\right)$ admits a pole of order 4 at $s=1$, while $\mathcal{E}_E\left(\eta_2,f_s,s,g\right)$ admits a pole of order at most 1 at $s=\frac{1}{2}$, which again brings us to a contradiction.

We now assume that $\eta_1, \eta_2 \not\equiv \mathbf{1}$ are quadratic characters. In this case

$$\mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) = \mathcal{L}_F^S(\eta_1\chi,s)^2\,\mathcal{L}_F^S(\eta_2\chi,s)^2\,\mathcal{L}_F^S(\eta_1\eta_2\chi,s)^2\,\mathcal{L}_F^S(\chi,s).$$

$\mathcal{L}^S(s,\pi,\eta_1,\mathfrak{st})$ admits a pole of order at least 2 at $s=1$, while $\mathcal{E}_E(\eta_1,f_s,s,g)$ admits a pole of order at most 1, which again brings us to a contradiction.

In conclusion, $z_1 = 1$ and hence also $z_2 \geq \frac{1}{2}$. In particular, $z_2 = 1$. We conclude that $\eta_1 \equiv \eta_2 \equiv \mathbf{1}$, which proves the assertion.

- <u>$E = F \times K$, where K/F is a quadratic extension:</u> Since $\mathcal{E}_E\left(\mu_1\left|\cdot\right|^{-1}, f_s, s, g\right)$ and $\mathcal{E}_E\left(\mu_2\left|\cdot\right|^{-1}, f_s, s, g\right)$ admit a pole at $s=2$, it holds that
$$(z_1,\eta_1),(z_2,\eta_2) \in \{(0,\mathbf{1}),(0,\chi_K),(1,\mathbf{1}),(1,\chi_K)\}.$$
The proof that $z_1, z_2 \neq 0$ is similar to the split case. It then holds that $z_1 = z_2 = 1$. We need to prove that $\eta_1 \equiv \eta_2 \equiv \mathbf{1}$ or $\eta_1 \equiv \eta_2 \equiv \chi_K$ cannot happen.

  Assume that $\eta_1 \equiv \eta_2 \equiv \mathbf{1}$. In this case
$$\mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) = \mathcal{L}_F^S(\chi,s)^3\,\mathcal{L}_F^S(\chi,s-1)^2\,\mathcal{L}_F^S(\chi,s+1)^2.$$
$\mathcal{L}^S(s,\pi,\mathbf{1},\mathfrak{st})$ would have a pole of order at least 3 at $s=1$, while $\mathcal{E}_E(\mathbf{1},f_s,s,g)$ admits a pole of order at most 1 at $s=\frac{1}{2}$, which brings us to a contradiction.

  Assume that $\eta_1 \equiv \eta_2 \equiv \chi_K$. In this case
$$\mathcal{L}^S(s,\pi,\chi,\mathfrak{st}) = \mathcal{L}_F^S(\chi,s)^3\,\mathcal{L}_F^S(\chi_K\chi,s-1)^2\,\mathcal{L}_F^S(\chi_K\chi,s+1)^2.$$
$\mathcal{L}^S(s,\pi,\mathbf{1},\mathfrak{st})$ would have a pole of order at least 3 at $s=1$, while $\mathcal{E}_E(\mathbf{1},f_s,s,g)$ admits a pole of order at most 1 at $s=\frac{1}{2}$, which brings us to a contradiction.

  In conclusion, $\mu_1 = \left|\cdot\right|$ and $\mu_2 = \left|\cdot\right|\chi_K$, or vice versa, which proves the assertion.
- <u>E/F is a cubic Galois extension:</u> Since
$$\mathcal{E}_E\left(\mu_1\left|\cdot\right|^{-1}, f_s, s, g\right) \text{ and } \mathcal{E}_E\left(\mu_2\left|\cdot\right|^{-1}, f_s, s, g\right)$$
admit a pole at $s=2$, it holds that
$$(z_1,\eta_1),(z_2,\eta_2) \in \left\{(0,\eta)\;\middle|\;\eta^2 \equiv \mathbf{1}\right\} \cup \{(1,\chi_E)\} \cup \{(1,\overline{\chi_E})\}.$$
The proof that $(z_1,\eta_1),(z_2,\eta_2) \neq (0,\eta)$ for η a quadratic character is similar to the split case. Hence, $\mu_1 \equiv \mu_2 \equiv \chi_E\left|\cdot\right|$ or $\mu_1 \equiv \mu_2 \equiv \overline{\chi_E}\left|\cdot\right|$, which proves the assertion.

$\square$

Appendix A. Tables of intertwining operators

In this section we list useful tables containing information about the local intertwining operators, poles of global Gindikin-Karpelevich factors, and the exponents of $w^{-1}\cdot\chi_s(t)$ in the various cases.

A.1. **Cubic extension case.** Assume E is a cubic field extension of F. In this case we denote

$$t=h_{\alpha_1\alpha_3^\sigma\alpha_4^{\sigma^2}}(t_1)\,h_{\alpha_2}(t_2)=h_{\alpha_1}(t_1)\,h_{\alpha_2}(t_2)\,h_{\alpha_3}(t_1^\sigma)\,h_{\alpha_4}\left(t_1^{\sigma^2}\right),$$

where $t_1\in E^\times, t_2\in F^\times$.

In Table 3 we list $w^{-1}\cdot\chi_s(t)$ for the various $w\in W(P_E,H_E)$.

Table 3. $w^{-1}\cdot\chi_s$ for $w\in W(P_E,H_E)$, E is a field

$w\in W(P_E,H_E)$	$w^{-1}\cdot\chi_s(t)$
$w[]$	$\chi(t_2)\,\frac{\lvert t_2\rvert_F^{s+\frac{3}{2}}}{\lvert t_1\rvert_E}$
$w[2]$	$\chi\left(\frac{\mathrm{Nm}_{E/F}(t_1)}{t_2}\right)\frac{\lvert t_1\rvert_E^{s+\frac{1}{2}}}{\lvert t_2\rvert_F^{s+\frac{3}{2}}}$
$w[2,1]$	$\chi\left(\frac{t_2^2}{\mathrm{Nm}_{E/F}(t_1)}\right)\frac{\lvert t_2\rvert_F^{2s}}{\lvert t_1\rvert_E^{s+\frac{1}{2}}}$
$w[2,1,2]$	$\chi\left(\frac{\mathrm{Nm}_{E/F}(t_1)}{t_2^2}\right)\frac{\lvert t_1\rvert_E^{s-\frac{1}{2}}}{\lvert t_2\rvert_F^{2s}}$
$w[2,1,2,1]$	$\chi\left(\frac{t_2}{\mathrm{Nm}_{E/F}(t_1)}\right)\frac{\lvert t_2\rvert_F^{s-\frac{3}{2}}}{\lvert t_1\rvert_E^{s-\frac{1}{2}}}$
$w[2,1,2,1,2]$	$\chi\left(\frac{1}{t_2}\right)\frac{1}{\lvert t_1\rvert_E\lvert t_2\rvert_F^{s-\frac{3}{2}}}$

In Table 4 we list the Gindikin-Karpelevich factor $J(w,\chi,\lambda)$ and the poles of the global Gindikin-Karpelevich factor $J(w,\chi_s)$ for $\mathfrak{Re}(s) > 0$.

TABLE 4. Poles of $J(w,\chi_s)$ for $w \in W(P_E,H_E)$, E is a field

		$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
$w \in W(P_E,H_E)$	$J(w,\chi_s)$	$\mathbf{1}$	χ_E	$\chi^2=\mathbf{1}$	$\mathbf{1}$	χ_E	$\mathbf{1}$
$w[]$	1	0	0	0	0	0	0
$w[2]$	$\frac{\mathcal{L}_F(s+\frac{3}{2},\chi)}{\mathcal{L}_F(s+\frac{5}{2},\chi)}$	0	0	0	0	0	0
$w[2,1]$	$\frac{\mathcal{L}_F(s+\frac{3}{2},\chi)\,\mathcal{L}_E(s+\frac{1}{2},\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_E(s+\frac{3}{2},\chi\circ\mathrm{Nm})}$	1	1	0	0	0	0
$w[2,1,2]$	$\frac{\mathcal{L}_F(s+\frac{3}{2},\chi)\,\mathcal{L}_E(s+\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_E(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	2	1	1	0	0	0
$w[2,1,2,1]$	$\frac{\mathcal{L}_F(s+\frac{3}{2},\chi)\,\mathcal{L}_E(s-\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_E(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	2	1	1	1	1	0
$w[2,1,2,1,2]$	$\frac{\mathcal{L}_F(s-\frac{3}{2},\chi)\,\mathcal{L}_F(s+\frac{3}{2},\chi)\,\mathcal{L}_E(s-\frac{1}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s,\chi^2)}{\mathcal{L}_F(s-\frac{1}{2},\chi)\,\mathcal{L}_F(s+\frac{5}{2},\chi)\,\mathcal{L}_E(s+\frac{3}{2},\chi\circ\mathrm{Nm})\,\mathcal{L}_F(2s+1,\chi^2)}$	1	1	1	1	1	1

In Table 5 we list the Gindikin-Karpelevich factor $J(w,\chi,\lambda)$. Here $\lambda(t) = |t_1|_E^{s_1}\,|t_2|_F^{s_2}$.

TABLE 5. $J(w,\chi,\lambda)$ for $w \in W(P_E,H_E)$, E is a field

$w \in W(P_E,H_E)$	$J(w,\chi,\lambda)$
$w[]$	1
$w[2]$	$\frac{\mathcal{L}_F(s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)}$
$w[2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})}$
$w[2,1,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2,\chi^2)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2+1,\chi^2)}$
$w[2,1,2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2,\chi^2)\,\mathcal{L}_E(2s_1+s_2,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2+1,\chi^2)\,\mathcal{L}_E(2s_1+s_2+1,\chi\circ\mathrm{Nm})}$
$w[2,1,2,1,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_E(s_1+s_2,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2,\chi^2)\,\mathcal{L}_E(2s_1+s_2,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_E(s_1+s_2+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+2s_2+1,\chi^2)\,\mathcal{L}_E(2s_1+s_2+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(3s_1+s_2+1,\chi)}$

In Table 6 we list the exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for all $w\in W\left(P_E,H_E\right)$ at the points $s=\frac{1}{2}$ and $\frac{3}{2}$. For the exponents, we use the notation introduced on page 5989. We further note by ✓ which $w\in W\left(P_E,H_E\right)$ satisfy the condition $w\in\Sigma^0_{s_0}$ in Lemma 4.9 for the various relevant cases. In fact, we note all elements of $\Sigma_{(E,\chi,s_0,1)}\supset\Sigma^0_{s_0}$ as we do not have full information on cancellations.

TABLE 6. The exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for $w\in W\left(P_E,H_E\right)$, E is a field

	$s=\frac{1}{2}$			$s=\frac{3}{2}$	
$w\in W\left(P_E,H_E\right)$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{1}{2}}\right)$	$\mathbf{1}$	$\chi^2=\mathbf{1}$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{3}{2}}\right)$	χ_E
$w[]$	$[0,1]$			$[1,3]$	
$w[2]$	$-[0,1]$			$[1,0]$	
$w[2,1]$	$-[1,1]$	✓		$-[1,0]$	
$w[2,1,2]$	$-[1,2]$	✓	✓	$-[1,3]$	
$w[2,1,2,1]$	$-[1,2]$	✓	✓	$-[2,3]$	✓
$w[2,1,2,1,2]$	$-[1,1]$	✓	✓	$-[2,3]$	✓

A.2. **Quadratic extension case.** Assume $E=F\times K$, where K is a field. For this case we denote

$$t=h_{\alpha_1}\left(t_1\right)h_{\alpha_2}\left(t_2\right)h_{\alpha_3\alpha_4^\sigma}\left(t_3\right)=h_{\alpha_1}\left(t_1\right)h_{\alpha_2}\left(t_2\right)h_{\alpha_3}\left(t_3\right)h_{\alpha_3}\left(t_3^\sigma\right),$$

where $t_1,t_2\in F^\times, t_3\in K^\times$.

In Table 7 we list $w^{-1}\cdot\chi_s\left(t\right)$ for the various $w\in W\left(P_E,H_E\right)$.

TABLE 7. $w^{-1}\cdot\chi_s$ for $w\in W(P_E,H_E)$. $E=F\times K$

$w\in W(P_E,H_E)$	$w^{-1}\cdot\chi_s(t)$
$w[]$	$\chi(t_2)\frac{\lvert t_2\rvert_F^{s+\frac{3}{2}}}{\lvert t_1\rvert_F\lvert t_3\rvert_K}$
$w[2]$	$\chi\left(\frac{t_1\,\mathrm{Nm}_{K/F}(t_3)}{t_2}\right)\frac{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_3\rvert_K^{s+\frac{1}{2}}}{\lvert t_2\rvert_F^{s+\frac{3}{2}}}$
$w[2,1]$	$\chi\left(\frac{\mathrm{Nm}_{K/F}(t_3)}{t_1}\right)\frac{\lvert t_3\rvert_K^{s+\frac{1}{2}}}{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_2\rvert_F}$
$w[2,3]$	$\chi\left(\frac{t_1t_2}{\mathrm{Nm}_{K/F}(t_3)}\right)\frac{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_2\rvert_F^{s-\frac{1}{2}}}{\lvert t_3\rvert_K^{s+\frac{1}{2}}}$
$w[2,1,3]$	$\chi\left(\frac{t_2^2}{t_1\,\mathrm{Nm}_{K/F}(t_3)}\right)\frac{\lvert t_2\rvert_F^{2s}}{\lvert t_1\rvert_F^{s+\frac{1}{2}}\lvert t_3\rvert_K^{s+\frac{1}{2}}}$
$w[2,3,2]$	$\chi\left(\frac{t_1^2}{t_2}\right)\frac{\lvert t_1\rvert_F^{2s}}{\lvert t_2\rvert_F^{s-\frac{1}{2}}\lvert t_3\rvert_K}$
$w[2,1,3,2]$	$\chi\left(\frac{t_1\,\mathrm{Nm}_{K/F}(t_3)}{t_2^2}\right)\frac{\lvert t_1\rvert_F^{s-\frac{1}{2}}\lvert t_3\rvert_K^{s-\frac{1}{2}}}{\lvert t_2\rvert_F^{2s}}$
$w[2,3,2,1]$	$\chi\left(\frac{t_2}{t_1^2}\right)\frac{\lvert t_2\rvert_F^{s+\frac{1}{2}}}{\lvert t_1\rvert_F^{2s}\lvert t_3\rvert_K}$
$w[2,1,3,2,1]$	$\chi\left(\frac{\mathrm{Nm}_{K/F}(t_3)}{t_1t_2}\right)\frac{\lvert t_3\rvert_K^{s-\frac{1}{2}}}{\lvert t_1\rvert_F^{s-\frac{1}{2}}\lvert t_2\rvert_F^{s+\frac{1}{2}}}$
$w[2,1,3,2,3]$	$\chi\left(\frac{t_1}{\mathrm{Nm}_{K/F}(t_3)}\right)\frac{\lvert t_1\rvert_F^{s-\frac{1}{2}}}{\lvert t_2\rvert_F\lvert t_3\rvert_K^{s-\frac{1}{2}}}$
$w[2,1,3,2,1,3]$	$\chi\left(\frac{t_2}{t_1\,\mathrm{Nm}_{K/F}(t_3)}\right)\frac{\lvert t_2\rvert_F^{s-\frac{3}{2}}}{\lvert t_1\rvert_F^{s-\frac{1}{2}}\lvert t_3\rvert_K^{s-\frac{1}{2}}}$
$w[2,1,3,2,1,3,2]$	$\chi\left(\frac{1}{t_2}\right)\frac{1}{\lvert t_1\rvert_F\lvert t_2\rvert_F^{s-\frac{3}{2}}\lvert t_3\rvert_K}$

In Table 8 we list the Gindikin-Karpelevich factor $J\left(w,\chi_s\right)$ and the poles of the global Gindikin-Karpelevich factor $J\left(w,\chi_s\right)$ for $\mathfrak{Re}\left(s\right)>0$.

TABLE 8. Poles of $J\left(w,\chi_s\right)$ for $w\in W\left(P_E,H_E\right)$. $E=F\times K$

		$s=\frac{1}{2}$			$s=\frac{3}{2}$		$s=\frac{5}{2}$
$w\in W\left(P_E,H_E\right)$	$J\left(w,\chi_s\right)$	$\mathbf{1}$	χ_K	$\chi^2=\mathbf{1}$	$\mathbf{1}$	χ_K	$\mathbf{1}$
$w\left[\right]$	1	0	0	0	0	0	0
$w\left[2\right]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)}$	0	0	0	0	0	0
$w\left[2,1\right]$	$\frac{\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)}$	1	0	0	0	0	0
$w[2,3]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)}$	1	1	0	0	0	0
$w[2,1,3]$	$\frac{\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)}$	2	1	0	0	0	0
$w\left[2,3,2\right]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(s-\frac{1}{2},\chi\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)}$	1	1	0	1	0	0
$w\left[2,1,3,2\right]$	$\frac{\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	3	2	1	0	0	0
$w[2,3,2,1]$	$\frac{\mathcal{L}_F\left(s+\frac{3}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(s-\frac{1}{2},\chi\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	2	2	1	1	0	0
$w\left[2,1,3,2,1\right]$	$\frac{\mathcal{L}_F\left(s-\frac{1}{2},\chi\right)\mathcal{L}_K\left(s+\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	3	2	1	1	0	0
$w[2,1,3,2,3]$	$\frac{\mathcal{L}_F\left(s+\frac{1}{2},\chi\right)\mathcal{L}_K\left(s-\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	3	2	1	1	1	0
$w[2,1,3,2,1,3]$	$\frac{\mathcal{L}_F\left(s-\frac{1}{2},\chi\right)\mathcal{L}_K\left(s-\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	3	2	1	2	1	0
$w\left[2,1,3,2,1,3,2\right]$	$\frac{\mathcal{L}_F\left(s-\frac{3}{2},\chi\right)\mathcal{L}_K\left(s-\frac{1}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s,\chi^2\right)}{\mathcal{L}_F\left(s+\frac{5}{2},\chi\right)\mathcal{L}_K\left(s+\frac{3}{2},\chi\circ\mathrm{Nm}\right)\mathcal{L}_F\left(2s+1,\chi^2\right)}$	2	2	1	2	1	1

In Table 9 we list the Gindikin-Karpelevich factor $J(w,\chi,\lambda)$. Here $\lambda(t) = |t_1|_F^{s_1} |t_2|_F^{s_2} |t_3|_K^{s_3}$.

TABLE 9. $J(w,\chi,\lambda)$ for $w \in W(P_E, H_E)$, $E = F \times K$

$w \in W(P_E,H_E)$	$J(w,\chi,\lambda)$
$w[]$	1
$w[2]$	$\frac{\mathcal{L}_F(s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)}$
$w[2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_F(s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_F(s_1+s_2+1,\chi)}$
$w[2,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})}$
$w[2,1,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)}$
$w[2,3,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3+1,\chi)}$
$w[2,1,3,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)}$
$w[2,3,2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_2+2s_3+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)}$
$w[2,1,3,2,1]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)\,\mathcal{L}_F(s_2+2s_3,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)\,\mathcal{L}_F(s_2+2s_3,\chi)}$
$w[2,1,3,2,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)} \times \frac{\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)\,\mathcal{L}_K(s_1+s_2+s_3,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)\,\mathcal{L}_K(s_1+s_2+s_3+1,\chi\circ\mathrm{Nm})}$
$w[2,1,3,2,1,3]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)} \times \frac{\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3,\chi\circ\mathrm{Nm})}{\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)\,\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3+1,\chi\circ\mathrm{Nm})}$
$w[2,1,3,2,1,3,2]$	$\frac{\mathcal{L}_F(s_2,\chi)\,\mathcal{L}_K(s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2,\chi)\,\mathcal{L}_F\left(s_1+2s_2+2s_3,\chi^2\right)}{\mathcal{L}_F(s_2+1,\chi)\,\mathcal{L}_K(s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+1,\chi)\,\mathcal{L}_F(s_1+2s_2+2s_3+1,\chi^2)} \times \frac{\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+2s_3,\chi)}{\mathcal{L}_F(s_2+2s_3,\chi)\,\mathcal{L}_K(s_1+s_2+s_3+1,\chi\circ\mathrm{Nm})\,\mathcal{L}_F(s_1+s_2+2s_3,\chi)}$

In Table 10 we list the exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for all $w \in W(P_E,H_E)$. For the exponents, we use the notation introduced on page 5989. We further note by ✓ which $w \in W(P_E,H_E)$ satisfy the condition $w \in \Sigma_{s_0}^0$ in Lemma 4.9 for the various relevant cases. Furthermore, we note by ✓⃝ elements satisfying $w \in \Sigma_{s_0}^0$, while $\mathfrak{Re}\left(\Omega^{-1}\cdot\chi_s\right) < 0$ is not satisfied. When these exponents appear in the residue it will not be square-integrable. In fact, we note all elements of $\Sigma_{(E,\chi,s_0,1)} \supset \Sigma^0_{s_0}$ as we do not have full information on cancellations.

TABLE 10. The exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for $w \in W\left(P_E, H_E\right)$, $E = F \times K$

	$s=\frac{1}{2}$				$s=\frac{3}{2}$	
$w \in W\left(P_E, H_E\right)$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{1}{2}}\right)$	$\mathbf{1}$	χ_K	$\chi^2=\mathbf{1}$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{3}{2}}\right)$	$\mathbf{1}$
$w[]$	$[0,1,0]$				$[1,3,1]$	
$w[2]$	$-[0,1,0]$				$[1,0,1]$	
$w[2,1]$	$-[1,1,0]$	✓			$[-1,0,1]$	
$w[2,3]$	$-[0,1,1]$	✓	✓		$[1,0,-1]$	
$w[2,1,3]$	$-[1,1,1]$	✓	✓		$-[1,0,1]$	
$w[2,3,2]$	$-[0,1,1]$	✓	✓		$[1,-1,-1]$	✓
$w[2,1,3,2]$	$-[1,2,1]$	✓	✓	✓	$-[1,3,1]$	
$w[2,3,2,1]$	$-[1,1,1]$	✓	✓	✓	$-[2,1,1]$	✓
$w[2,1,3,2,1]$	$-[1,2,1]$	✓	✓	✓	$-[2,3,1]$	✓
$w[2,1,3,2,3]$	$-[1,2,1]$	✓	✓	✓	$-[1,3,2]$	✓
$w[2,1,3,2,1,3]$	$-[1,2,1]$	✓	✓	✓	$-[2,3,2]$	✓
$w[2,1,3,2,1,3,2]$	$-[1,1,1]$	✓	✓	✓	$-[2,3,2]$	✓

A.3. **Split case.** Assume $E = F \times F \times F$. For this case, we denote

$$t = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)\, h_{\alpha_3}(t_3)\, h_{\alpha_4}(t_4),$$

where $t_1, t_2, t_3, t_4 \in F^\times$.

In Table 11 we list $w^{-1}\cdot\chi_s(t)$ for the various $w \in W\left(P_E, H_E\right)$ and also the resulting characters $w^{-1}\cdot\chi_s$.

TABLE 11. $w^{-1} \cdot \chi_s$ for $w \in W(P_E, H_E)$, $E = F \times F \times F$

$w \in W(P_E, H_E)$	$w^{-1} \cdot \chi_s(t)$
$w[]$	$\chi(t_2) \frac{\lvert t_2\rvert^{s+\frac{3}{2}}}{\lvert t_1 t_3 t_4\rvert}$
$w[2]$	$\chi\left(\frac{t_1 t_3 t_4}{t_2}\right) \frac{\lvert t_1 t_3 t_4\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{3}{2}}}$
$w[2,1]$	$\chi\left(\frac{t_3 t_4}{t_1}\right) \frac{\lvert t_3 t_4\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert \lvert t_1\rvert^{s+\frac{1}{2}}}$
$w[2,3]$	$\chi\left(\frac{t_1 t_4}{t_3}\right) \frac{\lvert t_1 t_4\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert \lvert t_3\rvert^{s+\frac{1}{2}}}$
$w[2,4]$	$\chi\left(\frac{t_1 t_3}{t_4}\right) \frac{\lvert t_1 t_3\rvert^{s+\frac{1}{2}}}{\lvert t_2\rvert \lvert t_4\rvert^{s+\frac{1}{2}}}$
$w[2,1,3]$	$\chi\left(\frac{t_2 t_4}{t_1 t_3}\right) \frac{\lvert t_2\rvert^{s-\frac{1}{2}} \lvert t_4\rvert^{s+\frac{1}{2}}}{\lvert t_1 t_3\rvert^{s+\frac{1}{2}}}$
$w[2,1,4]$	$\chi\left(\frac{t_2 t_3}{t_1 t_4}\right) \frac{\lvert t_2\rvert^{s-\frac{1}{2}} \lvert t_3\rvert^{s+\frac{1}{2}}}{\lvert t_1 t_4\rvert^{s+\frac{1}{2}}}$
$w[2,3,4]$	$\chi\left(\frac{t_2 t_1}{t_3 t_4}\right) \frac{\lvert t_2\rvert^{s-\frac{1}{2}} \lvert t_1\rvert^{s+\frac{1}{2}}}{\lvert t_3 t_4\rvert^{s+\frac{1}{2}}}$
$w[2,1,3,2]$	$\chi\left(\frac{t_4^2}{t_2}\right) \frac{\lvert t_4\rvert^{2s}}{\lvert t_1 t_3\rvert \lvert t_2\rvert^{s-\frac{1}{2}}}$
$w[2,1,4,2]$	$\chi\left(\frac{t_3^2}{t_2}\right) \frac{\lvert t_3\rvert^{2s}}{\lvert t_1 t_4\rvert \lvert t_2\rvert^{s-\frac{1}{2}}}$
$w[2,3,4,2]$	$\chi\left(\frac{t_1^2}{t_2}\right) \frac{\lvert t_1\rvert^{2s}}{\lvert t_3 t_4\rvert \lvert t_2\rvert^{s-\frac{1}{2}}}$
$w[2,1,3,4]$	$\chi\left(\frac{t_2^2}{t_1 t_3 t_4}\right) \frac{\lvert t_2\rvert^{2s}}{\lvert t_1 t_3 t_4\rvert^{s+\frac{1}{2}}}$
$w[2,3,4,2,1]$	$\chi\left(\frac{t_2}{t_1^2}\right) \frac{\lvert t_2\rvert^{s+\frac{1}{2}}}{\lvert t_3 t_4\rvert \lvert t_1\rvert^{2s}}$
$w[2,1,4,2,3]$	$\chi\left(\frac{t_2}{t_3^2}\right) \frac{\lvert t_2\rvert^{s+\frac{1}{2}}}{\lvert t_1 t_4\rvert \lvert t_3\rvert^{2s}}$
$w[2,1,3,2,4]$	$\chi\left(\frac{t_2}{t_4^2}\right) \frac{\lvert t_2\rvert^{s+\frac{1}{2}}}{\lvert t_1 t_3\rvert \lvert t_4\rvert^{2s}}$
$w[2,1,3,4,2]$	$\chi\left(\frac{t_1 t_3 t_4}{t_2^2}\right) \frac{\lvert t_1 t_3 t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{2s}}$
$w[2,1,3,4,2,1]$	$\chi\left(\frac{t_3 t_4}{t_1 t_2}\right) \frac{\lvert t_3 t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{1}{2}} \lvert t_1\rvert^{s-\frac{1}{2}}}$

TABLE 11. $w^{-1} \cdot \chi_s$ for $w \in W(P_E, H_E)$, $E = F \times F \times F$

$w \in W(P_E, H_E)$	$w^{-1} \cdot \chi_s(t)$
$w[2,1,3,4,2,3]$	$\chi\left(\frac{t_1t_4}{t_2t_3}\right)\frac{\lvert t_1t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{1}{2}}\lvert t_3\rvert^{s-\frac{1}{2}}}$
$w[2,1,3,4,2,4]$	$\chi\left(\frac{t_1t_3}{t_2t_4}\right)\frac{\lvert t_1t_3\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert^{s+\frac{1}{2}}\lvert t_4\rvert^{s-\frac{1}{2}}}$
$w[2,1,3,4,2,1,3]$	$\chi\left(\frac{t_4}{t_1t_3}\right)\frac{\lvert t_4\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert\lvert t_1t_3\rvert^{s-\frac{1}{2}}}$
$w[2,1,3,4,2,1,4]$	$\chi\left(\frac{t_3}{t_1t_4}\right)\frac{\lvert t_3\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert\lvert t_1t_4\rvert^{s-\frac{1}{2}}}$
$w[2,1,3,4,2,3,4]$	$\chi\left(\frac{t_1}{t_3t_4}\right)\frac{\lvert t_1\rvert^{s-\frac{1}{2}}}{\lvert t_2\rvert\lvert t_3t_4\rvert^{s-\frac{1}{2}}}$
$w[2,1,3,4,2,1,3,4]$	$\chi\left(\frac{t_2}{t_1t_3t_4}\right)\frac{\lvert t_2\rvert^{s-\frac{3}{2}}}{\lvert t_1t_3t_4\rvert^{s-\frac{1}{2}}}$
$w[2,1,3,4,2,1,3,4,2]$	$\chi\left(\frac{1}{t_2}\right)\frac{1}{\lvert t_2\rvert^{s-\frac{3}{2}}\lvert t_1t_3t_4\rvert}$

In Table 12 we list the Gindikin-Karpelevich factor $J(w,\chi_s)$ and the poles of the global Gindikin-Karpelevich factor $J(w,\chi_s)$ for $\mathfrak{Re}(s) > 0$.

TABLE 12. Poles of $J(w,\chi_s)$ for $w \in W(P_E, H_E)$, $E = F \times F \times F$

		$s=\frac{1}{2}$		$s=\frac{3}{2}$	$s=\frac{5}{2}$
$w \in W(P_E, H_E)$	$J(w,\chi_s)$	$\chi = \mathbf{1}$	$\chi^2 = \mathbf{1}$	$\chi = \mathbf{1}$	$\mathbf{1}$
$w[]$	1	0	0	0	0
$w[2]$	$\frac{\mathcal{L}(s+\frac{3}{2},\chi)}{\mathcal{L}(s+\frac{5}{2},\chi)}$	0	0	0	0
$w[2,1]$ $w[2,3]$ $w[2,4]$	$\frac{\mathcal{L}(s+\frac{1}{2},\chi)}{\mathcal{L}(s+\frac{5}{2},\chi)}$	1	0	0	0
$w[2,1,3]$ $w[2,1,4]$ $w[2,3,4]$	$\frac{\mathcal{L}(s+\frac{1}{2},\chi)^2}{\mathcal{L}(s+\frac{3}{2},\chi)\,\mathcal{L}(s+\frac{5}{2},\chi)}$	2	0	0	0
$w[2,1,3,2]$ $w[2,1,4,2]$ $w[2,3,4,2]$	$\frac{\mathcal{L}(s+\frac{1}{2},\chi)\,\mathcal{L}(s-\frac{1}{2},\chi)}{\mathcal{L}(s+\frac{5}{2},\chi)\,\mathcal{L}(s+\frac{3}{2},\chi)}$	2	0	1	0
$w[2,1,3,4]$	$\frac{\mathcal{L}(s+\frac{1}{2},\chi)^3}{\mathcal{L}(s+\frac{3}{2},\chi)^2\,\mathcal{L}(s+\frac{5}{2},\chi)}$	3	0	0	0
$w[2,3,4,2,1]$ $w[2,1,4,2,3]$ $w[2,1,3,2,4]$	$\frac{\mathcal{L}(s-\frac{1}{2},\chi)\,\mathcal{L}(s+\frac{1}{2},\chi)\,\mathcal{L}(2s,\chi^2)}{\mathcal{L}(s+\frac{3}{2},\chi)\,\mathcal{L}(s+\frac{5}{2},\chi)\,\mathcal{L}(2s+1,\chi^2)}$	3	1	1	0
$w[2,1,3,4,2]$	$\frac{\mathcal{L}(s+\frac{1}{2},\chi)^3\,\mathcal{L}(2s,\chi^2)}{\mathcal{L}(s+\frac{5}{2},\chi)\,\mathcal{L}(s+\frac{3}{2},\chi)^2\,\mathcal{L}(2s+1,\chi^2)}$	4	1	0	0
$w[2,1,3,4,2,1]$ $w[2,1,3,4,2,3]$ $w[2,1,3,4,2,4]$	$\frac{\mathcal{L}(s-\frac{1}{2},\chi)\,\mathcal{L}(s+\frac{1}{2},\chi)^2\,\mathcal{L}(2s,\chi^2)}{\mathcal{L}(s+\frac{3}{2},\chi)^2\,\mathcal{L}(s+\frac{5}{2},\chi)\,\mathcal{L}(2s+1,\chi^2)}$	4	1	1	0

TABLE 12. Poles of $J(w,\chi_s)$ for $w \in W(P_E, H_E)$, $E = F \times F \times F$

		$s=\frac{1}{2}$		$s=\frac{3}{2}$	$s=\frac{5}{2}$
$w \in W(P_E,H_E)$	$J(w,\chi_s)$	$\chi = \mathbf{1}$	$\chi^2 = \mathbf{1}$	$\chi = \mathbf{1}$	$\mathbf{1}$
$w[2,1,3,4,2,1,3]$ $w[2,1,3,4,2,1,4]$ $w[2,1,3,4,2,3,4]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)^2 \mathcal{L}\left(s+\frac{1}{2},\chi\right) \mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)^2 \mathcal{L}\left(s+\frac{5}{2},\chi\right) \mathcal{L}(2s+1,\chi^2)}$	4	1	2	0
$w[2,1,3,4,2,1,3,4]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)^3 \mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{3}{2},\chi\right)^2 \mathcal{L}\left(s+\frac{5}{2},\chi\right) \mathcal{L}(2s+1,\chi^2)}$	4	1	3	0
$w[2,1,3,4,2,1,3,4,2]$	$\frac{\mathcal{L}\left(s-\frac{1}{2},\chi\right)^2 \mathcal{L}\left(s-\frac{3}{2},\chi\right) \mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{5}{2},\chi\right) \mathcal{L}\left(s+\frac{3}{2},\chi\right)^2 \mathcal{L}(2s+1,\chi^2)}$	3	1	3	1

In Table 13 we list the Gindikin-Karpelevich factor $J(w,\chi,\lambda)$. Here $\lambda(t) = |t_1|_F^{s_1} |t_2|_F^{s_2} |t_3|_F^{s_3} |t_4|_F^{s_4}$.

TABLE 13. $J(w,\chi,\lambda)$ for $w \in W(P_E, H_E)$,$E = F \times F \times F$

$w \in W(P_E,H_E)$	$J(w,\chi,\lambda)$
$w[]$	1
$w[2]$	$\frac{\mathcal{L}(s_2,\chi)}{\mathcal{L}(s_2+1,\chi)}$
$w[2,1]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)}$
$w[2,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)}$
$w[2,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$w[2,1,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)}$
$w[2,1,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$w[2,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$w[2,1,3,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_1+s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)}$
$w[2,1,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$

TABLE 13. $J(w,\chi,\lambda)$ for $w \in W(P_E, H_E)$,$E = F\times F\times F$

$w \in W(P_E,H_E)$	$J(w,\chi,\lambda)$
$w[2,3,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)}$
$w[2,1,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)}$
$w[2,3,4,2,1]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$w[2,1,4,2,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$w[2,1,3,2,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$w[2,1,3,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)}$
$w[2,1,3,4,2,1]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\,\mathcal{L}(s_2+s_3+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)}$
$w[2,1,3,4,2,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$w[2,1,3,4,2,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\,\mathcal{L}(s_1+s_2+s_3,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)}$
$w[2,1,3,4,2,1,3]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)} \times \frac{\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\,\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$w[2,1,3,4,2,1,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)} \times \frac{\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\,\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)}{\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)}$
$w[2,1,3,4,2,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)} \times \frac{\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)\,\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$
$w[2,1,3,4,2,1,3,4]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}\left(s_1+2s_2+s_3+s_4,\chi^2\right)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)} \times \frac{\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)}{\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)}$

TABLE 13. $J(w,\chi,\lambda)$ for $w \in W(P_E,H_E)$,$E = F \times F \times F$

$w \in W(P_E,H_E)$	$J(w,\chi,\lambda)$
$w[2,1,3,4,2,1,3,4,2]$	$\frac{\mathcal{L}(s_2,\chi)\,\mathcal{L}(s_1+s_2,\chi)\,\mathcal{L}(s_2+s_3,\chi)\,\mathcal{L}(s_2+s_4,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4,\chi^2)}{\mathcal{L}(s_2+1,\chi)\,\mathcal{L}(s_1+s_2+1,\chi)\,\mathcal{L}(s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_4+1,\chi)\,\mathcal{L}(s_1+2s_2+s_3+s_4+1,\chi^2)} \times \frac{\mathcal{L}(s_1+s_2+s_3,\chi)\,\mathcal{L}(s_2+s_3+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_4,\chi)\,\mathcal{L}(s_1+s_2+s_3+s_4,\chi)}{\mathcal{L}(s_1+s_2+s_3+1,\chi)\,\mathcal{L}(s_2+s_3+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_4+1,\chi)\,\mathcal{L}(s_1+s_2+s_3+s_4+1,\chi)}$

In Table 14 we list the exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for all $w \in W(P_E,H_E)$. For the exponents, we use the notation introduced on page 5989. We further note by ✓ which $w \in W(P_E,H_E)$ satisfy the condition $w \in \Sigma^0_{s_0}$ in Lemma 4.9 for the various relevant cases. Furthermore, we note by ✓⃝ elements satisfying $w \in \Sigma^0_{s_0}$ while $\mathfrak{Re}\left(\Omega^{-1}\cdot\chi_s\right) < 0$ is not satisfied. When these exponents appear in the residue it will not be square-integrable. In fact, we note all elements of $\Sigma_{(E,\chi,s_0,1)} \supset \Sigma^0_{s_0}$, as we do not have full information on cancellations.

TABLE 14. The exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for $w \in W(P_E,H_E)$, $E = F \times F \times F$

	$s=\frac{1}{2}$			$s=\frac{3}{2}$
$w \in W(P_E,H_E)$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{1}{2}}\right)$	$\mathbf{1}$	$\chi^2=\mathbf{1}$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{3}{2}}\right)$
$w[]$	$[0,1,0,0]$			$[1,3,1,1]$
$w[2]$	$-[0,1,0,0]$			$[1,0,1,1]$
$w[2,1]$	$-[1,1,0,0]$	✓⃝		$[-1,0,1,1]$
$w[2,3]$	$-[0,1,1,0]$	✓⃝		$[1,0,-1,1]$
$w[2,4]$	$-[0,1,0,1]$	✓⃝		$[1,0,1,-1]$
$w[2,1,3]$	$-[1,1,1,0]$	✓⃝		$[-1,0,-1,1]$
$w[2,1,4]$	$-[1,1,0,1]$	✓⃝		$[-1,0,1,-1]$
$w[2,3,4]$	$-[0,1,1,1]$	✓⃝		$[1,0,-1,-1]$
$w[2,1,3,2]$	$-[1,1,1,0]$	✓⃝		$[-1,-1,-1,1]$
$w[2,1,4,2]$	$-[1,1,0,1]$	✓⃝		$[-1,-1,1,-1]$

TABLE 14. The exponents $\mathfrak{Re}\left(w^{-1}\cdot\chi_s\right)$ for $w \in W\left(P_E, H_E\right)$, $E = F\times F\times F$

	$s=\frac{1}{2}$			$s=\frac{3}{2}$
$w \in W\left(P_E, H_E\right)$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{1}{2}}\right)$	$\mathbf{1}$	$\chi^2=\mathbf{1}$	$\mathfrak{Re}\left(w^{-1}\cdot\chi_{\frac{3}{2}}\right)$
$w[2,3,4,2]$	$-[0,1,1,1]$	✓		$[1,-1,-1,-1]$
$w[2,1,3,4]$	$-[1,1,1,1]$	✓		$[-1,0,-1,-1]$
$w[2,3,4,2,1]$	$-[1,1,1,1]$	✓	✓	$[-2,-1,-1,-1]$
$w[2,1,4,2,3]$	$-[1,1,1,1]$	✓	✓	$[-1,-1,-2,-1]$
$w[2,1,3,2,4]$	$-[1,1,1,1]$	✓	✓	$[-1,-1,-1,-2]$
$w[2,1,3,4,2]$	$-[1,2,1,1]$	✓	✓	$[-1,-3,-1,-1]$
$w[2,1,3,4,2,1]$	$-[1,2,1,1]$	✓	✓	$[-2,-3,-1,-1]$
$w[2,1,3,4,2,3]$	$-[1,2,1,1]$	✓	✓	$[-1,-3,-2,-1]$
$w[2,1,3,4,2,4]$	$-[1,2,1,1]$	✓	✓	$[-1,-3,-1,-2]$
$w[2,1,3,4,2,1,3]$	$-[1,2,1,1]$	✓	✓	$[-2,-3,-2,-1]$
$w[2,1,3,4,2,1,4]$	$-[1,2,1,1]$	✓	✓	$[-2,-3,-1,-2]$
$w[2,1,3,4,2,3,4]$	$-[1,2,1,1]$	✓	✓	$[-1,-3,-2,-2]$
$w[2,1,3,4,2,1,3,4]$	$-[1,2,1,1]$	✓	✓	$[-2,-3,-2,-2]$
$w[2,1,3,4,2,1,3,4,2]$	$-[1,1,1,1]$	✓	✓	$[-2,-3,-2,-2]$

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