

WEIGHTED FLOATING BODIES AND POLYTOPAL APPROXIMATION

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ABSTRACT. Asymptotic results for weighted floating bodies are established and used to obtain new proofs for the existence of floating areas on the sphere and in hyperbolic space and to establish the existence of floating areas in Hilbert geometries. Results on weighted best and random approximation and the new approach to floating areas are combined to derive new asymptotic approximation results on the sphere, in hyperbolic space, and in Hilbert geometries.

Let K be a convex body (that is, compact convex set) in \mathbb{R}^n . For $\delta > 0$, the floating body K_δ of K is obtained by cutting off all caps that have volume less than or equal to δ . Extending results for smooth bodies (cf. [18]), Schütt and Werner [34] showed for a general convex body K that

$$(1) \quad \lim_{\delta \rightarrow 0} (\mathbb{V}_n(K) - \mathbb{V}_n(K_\delta)) \delta^{-\frac{2}{n+1}} = \alpha_n \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} dx,$$

where α_n is an explicitly known positive constant (see Section 1.1). Here \mathbb{V}_n is n -dimensional volume, $H_{n-1}(K, x)$ is the Gauss-Kronecker curvature at $x \in \partial K$, and integration is with respect to the $(n-1)$ -dimensional Hausdorff measure. The integral on the right side is the *affine surface area* of K (cf. [20, 22] and [31, Section 10.5] for more information).

Affine surface area also determines the asymptotic behavior of random polytopes. Specifically, choose m points uniformly and independently in K and denote their convex hull by K_m . The *random polytope* K_m is easily seen to converge to K in the sense that $\mathbb{E}(\mathbb{V}_n(K) - \mathbb{V}_n(K_m)) \rightarrow 0$ as $m \rightarrow \infty$, where \mathbb{E} denotes expectation. The asymptotic behavior of K_m has been studied extensively since the 1960s, starting with the seminal results by Rényi and Sulanke [28, 29] (cf. [17, 27]). Extending results of Bárány [1], Schütt [33] was able to prove the analog to (1) for the random polytope K_m in a general convex body K ,

$$(2) \quad \lim_{m \rightarrow \infty} \mathbb{E}(\mathbb{V}_n(K) - \mathbb{V}_n(K_m)) m^{\frac{2}{n+1}} = \beta_n \mathbb{V}_n(K)^{\frac{2}{n+1}} \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} dx,$$

where β_n is an explicitly known positive constant (see Section 1.2).

The aim of the paper is to extend (1) and (2) in a simple way to convex bodies on the sphere, in hyperbolic space, and in Hilbert geometries. The approach is via

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weighted floating bodies and *weighted approximation* in Euclidean space and will also be applied to random approximation by circumscribed polytopes and to asymptotic best approximation. On the sphere, the asymptotic behavior is described by the *spherical floating area* recently introduced in [6] and in hyperbolic space by the *hyperbolic floating area* introduced in [7]. In Hilbert geometries, we obtain floating areas that depend on the choice of volume and we establish a connection to centro-affine surface area.

In the following section, a theorem for weighted floating bodies is stated, and results on weighted approximation are collected. The results on polytopal random and best approximation and floating bodies on the sphere, in hyperbolic space, and in Hilbert geometries are established in Sections 2, 3, and 4. The final section contains the proof for the theorem for weighted floating bodies.

1. WEIGHTED FLOATING BODIES AND POLYTOPAL APPROXIMATION IN \mathbb{R}^n

Let $\mathcal{K}(\mathbb{R}^n)$ denote the set of convex bodies (that is, compact convex sets) in \mathbb{R}^n with non-empty interior. For $K \in \mathcal{K}(\mathbb{R}^n)$ and $\phi, \psi : K \rightarrow (0, \infty)$ integrable, define, for $A \subset \mathbb{R}^n$ measurable, the measure Φ by $\Phi(A) = \int_A \phi$ and the measure Ψ by $\Psi(A) = \int_A \psi$. If $\int_{\mathbb{R}^n} \phi = 1$, then Φ is a probability measure, and we write \mathbb{E}_Φ for the expectation with respect to Φ .

1.1. Weighted floating bodies. For $\delta > 0$, the weighted floating body K_δ^ϕ is the intersection of all closed half-spaces whose defining hyperplanes H cut off sets of Φ -measure less than or equal to δ from K , that is,

$$(3) \quad K_\delta^\phi = \bigcap \{H^- : \Phi(K \cap H^+) \leq \delta\},$$

where H^\pm are the closed half-spaces bounded by the hyperplane H . For $\phi \equiv 1$, we obtain (convex) floating bodies, which were introduced (independently) in [3, 34] as a generalization of the classical *floating bodies* (see [31, Chapter 10.6] for more information). Weighted floating bodies were introduced in [36], and generalizations of (1) were established there.

The following result generalizes the results from volume to a general measure Ψ .

Theorem 1.1. *For $K \in \mathcal{K}(\mathbb{R}^n)$ and $\phi, \psi : K \rightarrow (0, \infty)$ continuous,*

$$(4) \quad \lim_{\delta \rightarrow 0} \frac{\Psi(K) - \Psi(K_\delta^\phi)}{\delta^{\frac{2}{n+1}}} = \alpha_n \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \psi(x) dx,$$

where

$$(5) \quad \alpha_n := \frac{1}{2} \left(\frac{n+1}{v_{n-1}} \right)^{\frac{2}{n+1}}$$

and v_{n-1} is the $(n-1)$ -dimensional volume of the $(n-1)$ -dimensional unit ball.

The proof is given in Section 5.

1.2. Random polytopes. For $K \in \mathcal{K}(\mathbb{R}^n)$, let $\phi : K \rightarrow (0, \infty)$ be a probability density and let K_m^Φ be the convex hull of m independent random points chosen according to Φ . The following generalization of (2) was established by Böröczky, Fodor, and Hug [9, Theorem 3.1].

Theorem 1.2 ([9]). *Let $K \in \mathcal{K}(\mathbb{R}^n)$ and let $\psi : K \rightarrow (0, \infty)$ be continuous. If $\phi : K \rightarrow (0, \infty)$ is a continuous probability density and the random polytope K_m^Φ is the convex hull of m independent random points chosen according to Φ , then*

$$(6) \quad \lim_{m \rightarrow \infty} \mathbb{E}_\Phi(\Psi(K) - \Psi(K_m^\Phi)) m^{\frac{2}{n+1}} = \beta_n \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \psi(x) dx,$$

where

$$(7) \quad \beta_n := \frac{(n^2 + n + 2)(n^2 + 1)}{2(n + 3) \cdot (n + 1)!} \Gamma\left(\frac{n^2 + 1}{n + 1}\right) \left(\frac{n + 1}{v_{n-1}}\right)^{\frac{2}{n+1}}.$$

Efron showed that from the expected volume of a random polytope, the expected number of vertices $f_0(K_m)$ can be easily obtained. The same argument applies here and

$$\mathbb{E}_\Phi f_0(K_m^\Phi) = m(1 - \mathbb{E}_\Phi \Phi(K_{m-1}^\Phi))$$

(cf. [17]). Böröczky, Fodor, and Hug [9, Corollary 3.2] deduced the following result.

Corollary 1.3 ([9]). *Let $K \in \mathcal{K}(\mathbb{R}^n)$. If $\phi : K \rightarrow (0, \infty)$ is a continuous probability density and the random polytope K_m^Φ is the convex hull of m independent random points chosen in K according to the probability measure Φ , then*

$$\lim_{m \rightarrow \infty} \mathbb{E}_\Phi f_0(K_m^\Phi) m^{-\frac{n-1}{n+1}} = \beta_n \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \phi(x)^{\frac{n-1}{n+1}} dx,$$

where β_n is the constant defined in (7).

1.3. Random polyhedral sets. Another model for random polytopes that was also suggested by Rényi and Sulanke [30] and that can be considered as dual to the above is the following: Given a convex body K in \mathbb{R}^n , choose m random closed half-spaces that contain K in a way that is described below and denote their intersection by K^m . The *random polyhedral set* K^m may be unbounded, and therefore one usually considers K^m intersected with a bounded neighborhood of K . The classical choice is the parallel body $K + \mathbb{B}^n$ of K , where \mathbb{B}^n is the closed Euclidean unit ball; that is, $K + \mathbb{B}^n$ is the set of all points of distance at most 1 from K .

To describe our choice of random half-spaces, we first consider the set \mathcal{H} of all closed half-spaces in \mathbb{R}^n . We parametrize closed half-spaces $H^-(u, t)$ by their normal $u \in \mathbb{S}^{n-1}$ and the distance t from the origin, i.e.,

$$H^-(u, t) := \{x \in \mathbb{R}^n : x \cdot u \leq t\}.$$

The *support function* h_K of K is defined, for $u \in \mathbb{R}^n$, by $h_K(u) = \max\{u \cdot x : x \in K\}$. For $u \in \mathbb{S}^{n-1}$, the support function measures the signed distance between the origin and a hyperplane with outer normal u that touches K , and the width of K in direction u is given by $h_K(u) + h_K(-u)$. The average width $W(K)$, also known as *mean width* of K , is

$$(8) \quad W(K) = \frac{1}{nv_n} \int_{\mathbb{S}^{n-1}} (h_K(u) + h_K(-u)) du = \frac{2}{nv_n} \int_{\mathbb{S}^{n-1}} h_K(u) du.$$

On \mathcal{H} there is a uniquely determined rigid motion invariant Borel measure μ such that

$$\mu(\{H^- \in \mathcal{H} : 0 < V_n(K \cap H^-)/V_n(K) < 1\}) = W(K).$$

For a Borel subset A of \mathcal{H} , it is defined by

$$\mu(A) = \frac{1}{nv_n} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \mathbf{1}[H^-(u, t) \in A] dt du,$$

where $\mathbf{1}[P]$ is the indicator function of the proposition P ; that is, $\mathbf{1}[P] = 1$ if P holds and $\mathbf{1}[P] = 0$ otherwise. For $K \in \mathcal{K}(\mathbb{R}^n)$, we consider the set of all half-spaces that contain K and whose boundary hyperplanes meet $K + \mathbb{B}^n$, i.e.,

$$\mathcal{H}_K = \{H^-(u, t) : u \in \mathbb{S}^{n-1}, h_K(u) \leq t \leq h_K(u) + 1\}.$$

This yields $\mu(\mathcal{H}_K) = 1$, and therefore the restriction μ_K of μ to \mathcal{H}_K is a probability measure. Write \mathbb{E}_{μ_K} for the expectation with respect to μ_K . Böröczky, Fodor, and Hug [9] obtained the following result, which can be seen as dual to Theorem 1.2 and Corollary 1.3.

Theorem 1.4 ([9]). *Let $K \in \mathcal{K}(\mathbb{R}^n)$. If the random polyhedral set K^m is the intersection of m independent random half-spaces chosen from \mathcal{H}_K according to μ_K , then*

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{E}_{\mu_K} \left(W(K^m \cap (K + \mathbb{B}^n)) - W(K) \right) m^{\frac{2}{n+1}} \\ = 2\beta_n (nv_n)^{-\frac{n-1}{n+1}} \int_{\partial K} H_{n-1}(K, x)^{\frac{n}{n+1}} dx \end{aligned}$$

and

$$\lim_{m \rightarrow \infty} \mathbb{E}_{\mu_K} f_{n-1}(K^m) m^{-\frac{n-1}{n+1}} = \beta_n (nv_n)^{-\frac{n-1}{n+1}} \int_{\partial K} H_{n-1}(K, x)^{\frac{n}{n+1}} dx,$$

where $f_{n-1}(K^m)$ is the number of facets of K^m and β_n is the constant from (7).

1.4. Weighted best approximation. Problems of asymptotic best approximation have been extensively studied since the 1940s (cf. [14]). We restrict our attention to two problems and just remark that further notions of distance and approximation by inscribed and circumscribed polytopes with a given number of faces have also been studied (cf. [14]). For $K, P \subset \mathbb{R}^n$, write $K \Delta P$ for the symmetric difference of K and P . Set

$$\text{dist}_\Psi(K, \mathcal{P}_m) = \inf \{ \Psi(K \Delta P) : P \text{ polytope with at most } m \text{ vertices} \}$$

and

$$\text{dist}_\Psi(K, \mathcal{P}_{(m)}) = \inf \{ \Psi(K \Delta P) : P \text{ polytope with at most } m \text{ facets} \}.$$

Extending results by L. Fejes Tóth [11] and Gruber [15], the following asymptotic result was established in [19] for convex bodies with positive curvature, and in [8] the curvature condition was relaxed to include all convex bodies with C^2 boundary.

Theorem 1.5 ([8, 19]). *For $K \in \mathcal{K}(\mathbb{R}^n)$ with C^2 boundary and $\psi : K \rightarrow (0, \infty)$ continuous,*

(9)

$$\lim_{m \rightarrow \infty} \text{dist}_\Psi(K, \mathcal{P}_m) m^{\frac{2}{n-1}} = \frac{1}{2} \text{ldel}_{n-1} \left(\int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \psi(x)^{\frac{n-1}{n+1}} dx \right)^{\frac{n+1}{n-1}}$$

and

$$\lim_{m \rightarrow \infty} \text{dist}_\Psi(K, \mathcal{P}_{(m)}) m^{\frac{2}{n-1}} = \frac{1}{2} \text{ldiv}_{n-1} \left(\int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \psi(x)^{\frac{n-1}{n+1}} dx \right)^{\frac{n+1}{n-1}},$$

where lrel_{n-1} and ldiv_{n-1} are positive constants.

The exact values of lrel_{n-1} and ldiv_{n-1} are only known for $n = 2$ and $n = 3$ (see [10]). Weighted best approximation was first considered by Glasauer (see [13]).

2. SPHERICAL SPACE

Let \mathbb{S}^n denote the unit sphere in \mathbb{R}^{n+1} . A set $K \subset \mathbb{S}^n$ is a proper convex body if it is closed, contained in an open hemisphere, and its positive hull $\text{pos } K = \{\lambda x : x \in K, \lambda \geq 0\}$ is a convex set in \mathbb{R}^{n+1} . Let $\mathcal{K}(\mathbb{S}^n)$ denote the set of proper convex bodies in \mathbb{S}^n with non-empty interior. A hypersphere in \mathbb{S}^n is a set $H = \{x \in \mathbb{S}^n : x \cdot e = 0\}$ with $e \in \mathbb{S}^n$, where “ \cdot ” is the inner product in \mathbb{R}^{n+1} . Let H^\pm be the closed hemispheres bounded by H . For $\delta > 0$, the spherical floating body K_δ was introduced in [6] by

$$(10) \quad K_\delta = \bigcap \{H^- : \text{vol}_n(K \cap H^+) \leq \delta\},$$

where vol_n is spherical volume, i.e., the n -dimensional Hausdorff measure on \mathbb{S}^n .

Without loss of generality, we may restrict our attention to convex bodies contained in the hemisphere $\mathbb{S}_+^n = \{x \in \mathbb{S}^n : x \cdot e_{n+1} > 0\}$, where e_{n+1} is a vector of an orthonormal basis of \mathbb{R}^{n+1} . The gnomonic (or central) projection $g : \mathbb{S}_+^n \rightarrow \mathbb{R}^n$ is defined by

$$g(x) = \frac{x}{x \cdot e_{n+1}} - e_{n+1},$$

where we identify \mathbb{R}^n with $\{x \in \mathbb{R}^{n+1} : x \cdot e_{n+1} = 0\}$ (cf. [5, Section 4]). We write $\bar{x} = g(x)$ and $\bar{K} = g(K)$. Note that $g^{-1} : \mathbb{R}^n \rightarrow \mathbb{S}^n$ maps the point \bar{x} to $(1 + \|\bar{x}\|^2)^{-1/2}(\bar{x} + e_{n+1})$ and therefore has the Jacobian $(1 + \|\bar{x}\|^2)^{-(n+1)/2}$ (cf. [6, Proposition 4.2]). Thus the pushforward of vol_n under g is the measure Ψ_n with density $\psi_n(\bar{x}) = (1 + \|\bar{x}\|^2)^{-(n+1)/2}$. For the spherical Gauss-Kronecker curvature, we have

$$H_{n-1}^{\mathbb{S}^n}(K, x) = H_{n-1}(\bar{K}, \bar{x}) \left(\frac{1 + \|\bar{x}\|^2}{1 + (\bar{x} \cdot n_{\bar{K}}(\bar{x}))^2} \right)^{\frac{n+1}{2}}$$

(cf. [6, Lemma 4.4]), where $n_{\bar{K}}(\bar{x})$ is the outer unit normal vector to \bar{K} at \bar{x} , and consequently

$$(11) \quad \int_{\partial K} H_{n-1}^{\mathbb{S}^n}(K, x)^{\frac{1}{n+1}} dx = \int_{\partial \bar{K}} H_{n-1}(\bar{K}, \bar{x})^{\frac{1}{n+1}} (1 + \|\bar{x}\|^2)^{-\frac{n-1}{2}} d\bar{x}$$

(cf. [6, p. 897]). These transformation rules allow us to translate the results from Section 1 to spherical space.

The following result is a corollary to Theorem 1.1. It was first established in [6].

Theorem 2.1 ([6]). *For $K \in \mathcal{K}(\mathbb{S}^n)$,*

$$\lim_{\delta \rightarrow 0} \frac{\text{vol}_n(K) - \text{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = \alpha_n \int_{\partial K} H_{n-1}^{s_n}(K, x)^{\frac{1}{n+1}} dx,$$

where α_n is the constant from (5).

Proof. Since $g(K_\delta) = g(K)_\delta^{\psi_n}$, we have

$$\text{vol}_n(K) - \text{vol}_n(K_\delta) = \int_{g(K) \setminus g(K)_\delta^{\psi_n}} \psi_n.$$

Hence Theorem 1.1 with $\phi = \psi = \psi_n$ shows that

$$\lim_{\delta \rightarrow 0} \frac{\text{vol}_n(K) - \text{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = \alpha_n \int_{\partial \bar{K}} H_{n-1}(\bar{K}, \bar{x})^{\frac{1}{n+1}} (1 + \|\bar{x}\|^2)^{-\frac{n-1}{2}} d\bar{x}.$$

By (11), this completes the proof. □

Next, we consider random polytopes that are the spherical convex hull of points chosen uniformly according to vol_n in $K \in \mathcal{K}(\mathbb{S}^n)$. In the following, the expectation \mathbb{E}_K is with respect to the probability density $\text{vol}_n / \text{vol}_n(K)$.

Theorem 2.2. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. If K_m is the spherical convex hull of m random points chosen uniformly in K , then*

$$\lim_{m \rightarrow \infty} \mathbb{E}_K (\text{vol}_n(K) - \text{vol}_n(K_m)) m^{\frac{2}{n+1}} = \beta_n \text{vol}_n(K)^{\frac{2}{n+1}} \int_{\partial K} H_{n-1}^{s_n}(K, x)^{\frac{1}{n+1}} dx,$$

where β_n is the constant from (7).

Proof. Set $\Phi_n = \Psi_n / \Psi_n(g(K))$. Since $g(K_m) = g(K)_m^{\Phi_n}$, we have

$$\mathbb{E}_K (\text{vol}_n(K) - \text{vol}_n(K_m)) = \mathbb{E}_{\Phi_n} (\Psi_n(g(K)) - \Psi_n(g(K)_m^{\Phi_n})).$$

Thus the statement follows from Theorem 1.2 with $\psi = \psi_n$ and (11). □

Theorem 2.2 complements a recent result by Bárány, Hug, Reitzner, and Schneider [2] for random polytopes in hemispheres.

As a consequence of Corollary 1.3, we obtain the following result.

Corollary 2.3. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. If K_m is the spherical convex hull of m random points chosen uniformly in K , then*

$$\lim_{m \rightarrow \infty} \mathbb{E}_K f_0(K_m) m^{-\frac{n-1}{n+1}} = \beta_n \text{vol}_n(K)^{-\frac{n-1}{n+1}} \int_{\partial K} H_{n-1}^{s_n}(K, x)^{\frac{1}{n+1}} dx,$$

where β_n is the constant from (7).

Finally, we consider best approximation. Let

$$\text{dist}_n(K, \mathcal{P}_m^{s_n}) = \inf \{ \text{vol}_n(K \Delta P) : P \text{ spherical polytope with at most } m \text{ vertices} \}$$

and

$$\text{dist}_n(K, \mathcal{P}_{(m)}^{s_n}) = \inf \{ \text{vol}_n(K \Delta P) : P \text{ spherical polytope with at most } m \text{ facets} \}.$$

We obtain the following result.

Theorem 2.4. For $K \in \mathcal{K}(\mathbb{S}^n)$ with C^2 boundary,

$$\lim_{m \rightarrow \infty} \text{dist}_n(K, \mathcal{P}_m^{\mathbb{S}^n}) m^{\frac{2}{n-1}} = \frac{1}{2} \text{l} \text{del}_{n-1} \left(\int_{\partial K} H_{n-1}^{\mathbb{S}^n}(K, x)^{\frac{1}{n+1}} dx \right)^{\frac{n+1}{n-1}}$$

and

$$\lim_{m \rightarrow \infty} \text{dist}_n(K, \mathcal{P}_{(m)}^{\mathbb{S}^n}) m^{\frac{2}{n-1}} = \frac{1}{2} \text{l} \text{div}_{n-1} \left(\int_{\partial K} H_{n-1}^{\mathbb{S}^n}(K, x)^{\frac{1}{n+1}} dx \right)^{\frac{n+1}{n-1}},$$

where $\text{l} \text{del}_{n-1}$ and $\text{l} \text{div}_{n-1}$ are the constants from Theorem 1.5.

Proof. Note that $\text{dist}_n(K, \mathcal{P}_m^{\mathbb{S}^n}) = \text{dist}_{\Psi_n}(g(K), \mathcal{P}_m)$ and $\text{dist}_n(K, \mathcal{P}_{(m)}^{\mathbb{S}^n}) = \text{dist}_{\Psi_n}(g(K), \mathcal{P}_{(m)})$. Thus the statement follows directly from Theorem 1.5 with $\psi = \psi_n$ and (11). \square

2.1. Duality principle. Let K be a proper spherical convex body. Instead of random polytopes K_m contained in K we now consider random polytopes K^m containing K . The space of closed hemispheres \mathcal{H} of \mathbb{S}^n has a uniquely determined rotation invariant probability measure μ . For each point $x \in \mathbb{S}^n$ there is a uniquely determined hemisphere $H^-(x) = \{y \in \mathbb{S}^n : x \cdot y \leq 0\}$, and for a Borel subset A of \mathcal{H} we have

$$\mu(A) = \frac{1}{\text{vol}_n(\mathbb{S}^n)} \int_{\mathbb{S}^n} \mathbf{1}[H^-(x) \in A] dx.$$

A random polytope K^m is obtained as the intersection of m closed hemispheres chosen from $\mathcal{H}_K := \{H^- \in \mathcal{H} : K \subseteq H^-\}$ independently and according to $\mu_K := \mu/\mu(\mathcal{H}_K)$.

For $K \in \mathcal{K}(\mathbb{S}^n)$, define the *polar body* K° by

$$K^\circ = \{y \in \mathbb{S}^n : x \cdot y \leq 0 \text{ for all } x \in K\} = \bigcap_{x \in K} H^-(x)$$

(cf. [32, Section 6.5]). Since $K^{\circ\circ} = K$, a hemisphere $H^-(y)$ contains K if and only if $y \in K^\circ$. Thus we have $\mathcal{H}_K = \{H^-(y) : y \in K^\circ\}$ and $\mu(\mathcal{H}_K) = \text{vol}_n(K^\circ)$.

Let K^m be the intersection of m randomly chosen closed hemispheres in \mathcal{H}_K ; that is, there are $x_i \in K^\circ, i = 1, \dots, m$, such that $K^m = \bigcap_{i=1}^m H^-(x_i)$. We have

$$K^m = (\text{conv}\{x_1, \dots, x_m\})^\circ = (K_m^\circ)^\circ,$$

where $K_m^\circ := (K^\circ)_m$. This means that the polar of a random polytope that contains K is a polytope inside K° . In this way we can transfer results about K_m° to $(K^m)^\circ$.

Theorem 2.5. If \mathcal{F} be a non-negative measurable functional on spherical convex polyhedral sets, then

$$\mathbb{E}_{\mu_K} \mathcal{F}(K^m) = \mathbb{E}_{K^\circ} \mathcal{F}((K_m^\circ)^\circ).$$

In the Euclidean setting a similar result was obtained in [9, Proposition 5.1].

As an application of this theorem we consider the *spherical mean width* $U_1(K)$ of a spherical convex body K , which is defined by

$$U_1(K) = \frac{1}{2} \int_{G(n+1,n)} \chi(K \cap H) d\nu(H),$$

where χ is the Euler characteristic, $G(n + 1, n)$ is the Grassmannian of all n -dimensional linear subspaces in \mathbb{R}^{n+1} , and ν denotes the invariant probability measure on $G(n + 1, n)$. The probability that a random hypersphere hits K is equal to $2U_1(K)$. The name *spherical mean width* corresponds to the Euclidean notion of mean width $W(\bar{K})$ for $\bar{K} \in \mathcal{K}(\mathbb{R}^n)$, which can be defined as the probability of a random affine hyperplane hitting \bar{K} . Equivalently, $W(\bar{K})$ is given by (8), which, however, does not have a natural analog in the spherical setting.

Corollary 2.6. *Let $K \in \mathcal{K}(\mathbb{S}^n)$. If K^m is the intersection of m random hemispheres containing K and chosen uniformly according to μ_K , then*

$$\lim_{m \rightarrow \infty} \mathbb{E}_{\mu_K} (U_1(K^m) - U_1(K)) m^{\frac{2}{n+1}} = \frac{\beta_n}{\text{vol}_n(\mathbb{S}^n)} \text{vol}_n(K^\circ)^{\frac{2}{n+1}} \int_{\partial K} H_{n-1}^{\mathbb{S}^n}(K, x)^{\frac{n}{n+1}} dx$$

and

$$\lim_{m \rightarrow \infty} \mathbb{E}_{\mu_K} f_{n-1}(K^m) m^{-\frac{n-1}{n+1}} = \beta_n \text{vol}_n(K^\circ)^{-\frac{n-1}{n+1}} \int_{\partial K} H_{n-1}^{\mathbb{S}^n}(K, x)^{\frac{n}{n+1}} dx,$$

where β_n is the constant from (7).

Proof. By [12, equation (20)], we have

$$U_1(K) = \frac{1}{2} - \frac{\text{vol}_n(K^\circ)}{\text{vol}_n(\mathbb{S}^n)}.$$

Also, the facets of K^m correspond to the vertices of $(K^m)^\circ = K_m^\circ$. Thus $f_{n-1}(K^m) = f_0(K_m^\circ)$. Hence, by Theorem 2.5, we find that

$$\mathbb{E}_{\mu_K} (U_1(K^m) - U_1(K)) = \frac{\mathbb{E}_{K^\circ} (\text{vol}_n(K^\circ) - \text{vol}_n(K_m^\circ))}{\text{vol}_n(\mathbb{S}^n)}$$

and

$$\mathbb{E}_{\mu_K} f_{d-1}(K^m) = \mathbb{E}_{K^\circ} f_0(K_m^\circ).$$

Applying Theorem 2.2 and Corollary 2.3 on K° we obtain

$$\begin{aligned} &\lim_{m \rightarrow \infty} \mathbb{E}_{\mu_K} (U_1(K^m) - U_1(K)) m^{\frac{2}{n+1}} \\ &= \frac{\beta_n}{\text{vol}_n(\mathbb{S}^n)} \text{vol}_n(K^\circ)^{\frac{2}{n+1}} \int_{\partial K^\circ} H_{n-1}^{\mathbb{S}^n}(K^\circ, x)^{\frac{1}{n+1}} dx \end{aligned}$$

and

$$\lim_{m \rightarrow \infty} \mathbb{E}_{\mu_K} f_{n-1}(K^m) m^{-\frac{n-1}{n+1}} = \beta_n \text{vol}_n(K^\circ)^{-\frac{n-1}{n+1}} \int_{\partial K^\circ} H_{n-1}^{\mathbb{S}^n}(K^\circ, x)^{\frac{1}{n+1}} dx.$$

By [6, Theorem 7.4], we have

$$\int_{\partial K^\circ} H_{n-1}^{\mathbb{S}^n}(K^\circ, x)^{\frac{1}{n+1}} dx = \int_{\partial K} H_{n-1}^{\mathbb{S}^n}(K, x)^{\frac{n}{n+1}} dx,$$

which concludes the proof. □

3. HYPERBOLIC SPACE

Let $\mathbb{R}^{n,1}$ denote the Lorentz-Minkowski space of dimension $n + 1$, that is, \mathbb{R}^{n+1} with the indefinite inner product “ \circ ” defined by

$$x \circ x = x_1^2 + \dots + x_n^2 - x_{n+1}^2.$$

Then the hyperboloid model of hyperbolic space is given by

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} : x \circ x = -1 \text{ and } x_{n+1} > 0\}.$$

The hyperbolic distance d_H between two points $x, y \in \mathbb{H}^n$ is determined by $\cosh d_H(x, y) = -x \circ y$. A set $K \subset \mathbb{H}^n$ is a convex body if it is compact and the positive hull is a convex set in \mathbb{R}^{n+1} . Let $\mathcal{K}(\mathbb{H}^n)$ denote the set of convex bodies in \mathbb{H}^n with non-empty interior. For a hyperplane H let H^\pm be the closed half-spaces bounded by H . For $\delta > 0$, the hyperbolic floating body K_δ was introduced in [7] by

$$K_\delta = \bigcap \{H^- : \text{vol}_n(K \cap H^+) \leq \delta\},$$

where vol_n is the hyperbolic volume on \mathbb{H}^n .

We fix a Lorentz-orthonormal basis e_1, e_2, \dots, e_{n+1} in $\mathbb{R}^{n,1}$ such that e_{n+1} is in \mathbb{H}^n . The gnomonic (or central) projection $g: \mathbb{H}^n \rightarrow \mathbb{R}^n$ is defined by

$$g(x) = \frac{x}{x \circ e_{n+1}} + e_{n+1},$$

where we identify \mathbb{R}^n with $\{x \in \mathbb{R}^{n,1} : x \circ e_{n+1} = 0\}$. We write $\bar{x} = g(x)$ and $\bar{K} = g(K)$. Since

$$\|\bar{x}\|^2 = 1 - (x \circ e_{n+1})^{-2} = \tanh^2 d_H(x, e_{n+1}),$$

we have $\|\bar{x}\| \in [0, 1)$. Therefore the gnomonic projection maps \mathbb{H}^n into the open unit ball $\text{int } \mathbb{B}^n \subset \mathbb{R}^n$. Note that $g^{-1}: \text{int } \mathbb{B}^n \rightarrow \mathbb{H}^n$ maps the point \bar{x} to $(1 - \|\bar{x}\|^2)^{-1/2}(\bar{x} + e_{n+1})$. The gnomonic projection is an isometry between the hyperboloid model \mathbb{H}^n and the projective model (or Beltrami-Cayley-Klein model) $\text{int } \mathbb{B}^n$. Thus the pushforward of vol_n under g is the measure Ψ_n with density $\psi_n(\bar{x}) = (1 - \|\bar{x}\|^2)^{-(n+1)/2}$. For the hyperbolic Gauss-Kronecker curvature, we have

$$H_{n-1}^{\mathbb{H}^n}(K, x) = H_{n-1}(\bar{K}, \bar{x}) \left(\frac{1 - \|\bar{x}\|^2}{1 - (\bar{x} \cdot n_{\bar{K}}(\bar{x}))^2} \right)^{\frac{n+1}{2}}$$

(cf. [7, Corollary 3.16]), and furthermore

$$(12) \quad \int_{\partial K} H_{n-1}^{\mathbb{H}^n}(K, x)^{\frac{1}{n+1}} dx = \int_{\partial \bar{K}} H_{n-1}(\bar{K}, \bar{x})^{\frac{1}{n+1}} (1 - \|\bar{x}\|^2)^{-\frac{n-1}{2}} d\bar{x}$$

(cf. [7, (3.12)]). So again, these transformation rules allow us to translate the results from Section 1 to hyperbolic space. The proofs are identical to those in spherical space (just replace (11) by (12)) and are therefore omitted.

As a corollary to Theorem 1.1 we obtain the existence of floating area for hyperbolic space, which was originally established in [7].

Theorem 3.1 ([7]). *For $K \in \mathcal{K}(\mathbb{H}^n)$,*

$$\lim_{\delta \rightarrow 0} \frac{\text{vol}_n(K) - \text{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = \alpha_n \int_{\partial K} H_{n-1}^{\mathbb{H}^n}(K, x)^{\frac{1}{n+1}} dx,$$

where α_n is defined in Theorem 1.1.

Next, we consider random polytopes that are the hyperbolic convex hull of points chosen uniformly according to vol_n in $K \in \mathcal{K}(\mathbb{H}^n)$. In the following, the expectation \mathbb{E}_K is with respect to the density $\text{vol}_n / \text{vol}_n(K)$.

Theorem 3.2. *Let $K \in \mathcal{K}(\mathbb{H}^n)$. If K_m is the hyperbolic convex hull of m random points chosen uniformly in K , then*

$$\lim_{m \rightarrow \infty} \mathbb{E}_K (\text{vol}_n(K) - \text{vol}_n(K_m)) m^{\frac{2}{n+1}} = \beta_n \text{vol}_n(K)^{\frac{2}{n+1}} \int_{\partial K} H_{n-1}^{\mathbb{H}^n}(K, x)^{\frac{1}{n+1}} dx,$$

where β_n is the constant from (7).

As a consequence, we obtain the following result.

Corollary 3.3. *Let $K \in \mathcal{K}(\mathbb{H}^n)$. If K_m is the hyperbolic convex hull of m random points chosen uniformly in K , then*

$$\lim_{m \rightarrow \infty} \mathbb{E}_K f_0(K_m) m^{-\frac{n-1}{n+1}} = \beta_n \text{vol}_n(K)^{-\frac{n-1}{n+1}} \int_{\partial K} H_{n-1}^{\mathbb{H}^n}(K, x)^{\frac{1}{n+1}} dx,$$

where β_n is the constant from (7).

Finally, we consider best approximation. Let

$$\text{dist}_n(K, \mathcal{P}_m^{\mathbb{H}^n}) = \inf \{ \text{vol}_n(K \triangle P) : P \text{ hyperbolic polytope with at most } m \text{ vertices} \}$$

and

$$\text{dist}_n(K, \mathcal{P}_{(m)}^{\mathbb{H}^n}) = \inf \{ \text{vol}_n(K \triangle P) : P \text{ hyperbolic polytope with at most } m \text{ facets} \}.$$

We obtain the following result.

Theorem 3.4. *For $K \in \mathcal{K}(\mathbb{S}^n)$ with C^2 boundary,*

$$\lim_{m \rightarrow \infty} \text{dist}_n(K, \mathcal{P}_m^{\mathbb{H}^n}) m^{\frac{2}{n-1}} = \frac{1}{2} \text{l}del_{n-1} \left(\int_{\partial K} H_{n-1}^{\mathbb{H}^n}(K, x)^{\frac{1}{n+1}} dx \right)^{\frac{n+1}{n-1}}$$

and

$$\lim_{m \rightarrow \infty} \text{dist}_n(K, \mathcal{P}_{(m)}^{\mathbb{H}^n}) m^{\frac{2}{n-1}} = \frac{1}{2} \text{l}div_{n-1} \left(\int_{\partial K} H_{n-1}^{\mathbb{H}^n}(K, x)^{\frac{1}{n+1}} dx \right)^{\frac{n+1}{n-1}},$$

where $\text{l}del_{n-1}$ and $\text{l}div_{n-1}$ are the constants from Theorem 1.5.

4. HILBERT GEOMETRIES

Hilbert’s Fourth Problem asks for a characterization of metric geometries whose geodesics are straight lines. Hilbert constructed a special class of examples, now called Hilbert geometries (see [26, Chapter 15] for more information). A Hilbert geometry (C, d_C) is defined on the interior of a convex body $C \in \mathcal{K}(\mathbb{R}^n)$ in the following way: For distinct points $x, y \in \text{int } C$, the line passing through x and y meets ∂C at two points p and q , say, such that one has p, x, y, q in that order on the line. Define the Hilbert distance of x and y by

$$d_C(x, y) = \frac{1}{2} \log [p, x, y, q],$$

where $[p, x, y, q]$ is the cross ratio of p, x, y, q , that is,

$$[p, x, y, q] = \frac{\|y - p\| \|x - q\|}{\|x - p\| \|y - q\|}.$$

Note that the invariance of the cross ratio by projective maps implies the projective invariance of d_C . Unbounded closed convex sets with non-empty interiors and not containing a straight line are projectively equivalent to convex bodies. Hence the definition of Hilbert geometry naturally extends to the interiors of such convex sets. If C is an ellipsoid, then the Hilbert geometry on $\text{int } C$ is isometric to hyperbolic space.

Straight lines are geodesics in a Hilbert geometry (C, d_C) , and if C is strictly convex, then the affine segment between two distinct points is the unique geodesic joining them (see e.g. [26, p. 60]). Hence, if C is strictly convex, then hyperplanes are the totally geodesic submanifolds of co-dimension 1. A convex body $K \in \mathcal{K}(\mathbb{R}^n)$ that is contained in $\text{int } C$ is therefore also a convex body of the Hilbert geometry (C, d_C) , and polytopes are an intrinsic notion of (C, d_C) . Thus we may consider polytopal approximation in a Hilbert geometry (C, d_C) for a strictly convex body C . In the following $\mathcal{K}(C)$ denotes the space of convex bodies $K \subset \text{int } C$.

The Hilbert metric d_C is induced by a weak Finsler structure in the following way: For $x \in \text{int } C$ define a (weak) Minkowski norm $\|\cdot\|_x$ by

$$\|v\|_x = \frac{1}{2} \left(\frac{1}{t^+} + \frac{1}{t^-} \right),$$

for $v \in \mathbb{R}^n$, where t^\pm is determined by $x \pm t^\pm v \in \partial C$. If we identify \mathbb{R}^n with the tangent space $T_x \mathbb{R}^n$, then $\|\cdot\|_x$ defines a Minkowski norm on $T_x \mathbb{R}^n$ for every $x \in \text{int } C$. The map $F_C : x \mapsto \|\cdot\|_x$ defines a (weak) Finsler structure on $\text{int } C$. The length of a C^1 curve $\gamma : [a, b] \rightarrow \text{int } C$ is defined by

$$\ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)} dt,$$

and the Hilbert metric between two distinct points $x, y \in \text{int } C$ is just the minimal length of a C^1 curve joining them. In particular, if C is C^2_+ , that is, the boundary of C is a C^2 manifold with positive curvature, then $(\text{int } C, F_C)$ defines a Finsler manifold in the classical sense.

The unit ball of the Minkowski norm $\|\cdot\|_x$ is $I_x^C = \{v \in \mathbb{R}^n : \|v\|_x \leq 1\}$. Recall that the polar body K^* of a convex body K is defined by $K^* = \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in K\}$, and the difference body DK is defined by $D(K) = \frac{1}{2}(K - K) = \frac{1}{2}\{x - y : x, y \in K\}$. For a fixed $x \in \text{int } C$ we find that

$$\|v\|_x = h(D(C - x)^*, v) \quad \text{and} \quad I_x^C = (D(C - x)^*)^*.$$

Hence I_x^C is the *harmonic symmetrization* of C in x (see [25]).

There are several good choices for volume vol_C in (C, d_C) which give a projective invariant notion of volume, for example, the Busemann volume or the Holmes-Thompson volume of the associated Finsler manifold. The Busemann volume is the n -dimensional Hausdorff volume of the metric space (C, d_C) . Its density function with respect to Lebesgue measure λ_n is given by $v_n/\lambda_n(I_x^C)$. The Holmes-Thompson volume has density $\lambda_n((I_x^C)^*)/v_n$. Both, the Busemann and the Holmes-Thompson volume have the property that the density σ_C is non-negative and continuous. This allows us to directly apply the results from Section 1 to Hilbert geometries with these volume densities.

First, we consider random polytopes that are the convex hull of points chosen uniformly according to vol_C in $K \in \mathcal{K}(C)$. In the following, the expectation \mathbb{E}_K is with respect to the density $\text{vol}_C / \text{vol}_C(K)$.

Theorem 4.1. *Let $K \in \mathcal{K}(C)$. If K_m is the convex hull of m random points chosen uniformly in K with respect to vol_C , then*

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{E}_K(\text{vol}_C(K) - \text{vol}_C(K_m)) m^{\frac{2}{n+1}} \\ = \beta_n \text{vol}_C(K)^{\frac{2}{n+1}} \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \sigma_C(x)^{\frac{n-1}{n+1}} dx, \end{aligned}$$

where β_n is the constant from (7).

As a consequence of Corollary 1.3, we obtain the following result.

Corollary 4.2. *Let $K \in \mathcal{K}(C)$. If K_m is the convex hull of m random points chosen uniformly in K with respect to vol_C , then*

$$\lim_{m \rightarrow \infty} \mathbb{E}_K f_0(K_m) m^{-\frac{n-1}{n+1}} = \beta_n \text{vol}_C(K)^{-\frac{n-1}{n+1}} \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \sigma_C(x)^{\frac{n-1}{n+1}} dx,$$

where β_n is the constant from (7).

Next, we consider best approximation. Let

$$\text{dist}_C(K, \mathcal{P}_m^C) = \inf \{ \text{vol}_C(K \Delta P) : P \subset \text{int } C \text{ polytope with at most } m \text{ vertices} \}$$

and

$$\text{dist}_C(K, \mathcal{P}_{(m)}^C) = \inf \{ \text{vol}_C(K \Delta P) : P \subset \text{int } C \text{ polytope with at most } m \text{ facets} \}.$$

We obtain the following result.

Theorem 4.3. *For $K \in \mathcal{K}(C)$ with C^2 boundary,*

$$\lim_{m \rightarrow \infty} \text{dist}_C(K, \mathcal{P}_m^C) m^{\frac{2}{n-1}} = \frac{1}{2} \text{ldel}_{n-1} \left(\int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \sigma_C(x)^{\frac{n-1}{n+1}} dx \right)^{\frac{n+1}{n-1}}$$

and

$$\lim_{m \rightarrow \infty} \text{dist}_C(K, \mathcal{P}_{(m)}^C) m^{\frac{2}{n-1}} = \frac{1}{2} \text{ldiv}_{n-1} \left(\int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \sigma_C(x)^{\frac{n-1}{n+1}} dx \right)^{\frac{n+1}{n-1}},$$

where ldel_{n-1} and ldiv_{n-1} are the constants from Theorem 1.5.

Finally, we obtain the following result for the weighted floating body $K_\delta^{\sigma_C}$.

Theorem 4.4. *For $K \in \mathcal{K}(C)$,*

$$\lim_{\delta \rightarrow 0} \frac{\text{vol}_C(K) - \text{vol}_C(K_\delta^{\sigma_C})}{\delta^{\frac{2}{n+1}}} = \alpha_n \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \sigma_C(x)^{\frac{n-1}{n+1}} dx,$$

where α_n is defined in Theorem 1.1.

Note that the floating area

$$\Omega_C(K) = \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \sigma_C(x)^{\frac{n-1}{n+1}} dx$$

depends on the Hilbert geometry (C, d_C) and the choice of the volume density σ_C . Let $\mathcal{K}_{(0)}(\mathbb{R}^n)$ be the set of convex bodies in \mathbb{R}^n containing the origin in their interiors. For $C \in \mathcal{K}_{(0)}(\mathbb{R}^n)$ and $\lambda < 1$, the floating area $\Omega_C(\lambda C)$ is a centro-affine (or $\text{GL}(n)$) invariant by the definition of floating area and the projective invariance of the volume vol_C (however, note that $\Omega_C(\lambda C)$ is not a projective invariant). For the limiting case $\lambda \rightarrow 1$ and the Busemann floating area, we obtain the following

result. The proof is based on results by Berck, Bernig, and Vernicos [4], who studied the limiting behavior of the volume entropy of λC .

Theorem 4.5. *For $C \in \mathcal{K}_{(0)}(\mathbb{R}^n)$ with $C^{1,1}$ boundary,*

$$\Omega_n(C) = 2^{\frac{n-1}{2}} \lim_{\lambda \rightarrow 1^-} \Omega_C(\lambda C)(1 - \lambda)^{\frac{n-1}{2}},$$

where Ω_C is the Busemann floating area.

Here $\Omega_n(C)$ is the classical *centro-affine surface area* of C , which is defined as

$$\Omega_n(C) = \int_{\partial C} \frac{H_{n-1}(C, x)^{\frac{1}{2}}}{(x \cdot n_C(x))^{\frac{n-1}{2}}} dx.$$

Centro-affine surface area is an upper semicontinuous and $GL(n)$ invariant valuation on $\mathcal{K}_{(0)}^n(\mathbb{R}^n)$. Moreover, it is basically the only such functional (see [21]). For more information on centro-affine surface area, which is also called L^n -affine surface area, see [23, 24, 35].

Proof. Berck, Bernig, and Vernicos [4, Proposition 2.8] obtained that

$$(13) \quad \lim_{\lambda \rightarrow 1^-} \sigma_C(\lambda x)(1 - \lambda)^{\frac{n+1}{2}} = \frac{H_{n-1}(C, x)^{\frac{1}{2}}}{(2x \cdot n_C(x))^{\frac{n+1}{2}}},$$

for $x \in \partial C$. Using a version of Blaschke’s rolling theorem, they also showed in [4, Proposition 2.10] that

$$(14) \quad \sigma_C(\lambda x) \leq c(1 - \lambda)^{-\frac{n+1}{2}},$$

where the constant c does not depend on x and λ . Thus,

$$\begin{aligned} & \lim_{\lambda \rightarrow 1^-} \Omega_C(\lambda C)(1 - \lambda)^{\frac{n-1}{2}} \\ &= \lim_{\lambda \rightarrow 1^-} \lambda^n \frac{n-1}{n+1} \int_{\partial C} H_{n-1}(C, x)^{\frac{1}{n+1}} \left((1 - \lambda)^{\frac{n+1}{2}} \sigma_C(\lambda x) \right)^{\frac{n-1}{n+1}} dx \\ &= \int_{\partial C} H_{n-1}(C, x)^{\frac{1}{n+1}} \left(\frac{H_{n-1}(C, x)^{\frac{1}{2}}}{(2x \cdot n_C(x))^{\frac{n+1}{2}}} \right)^{\frac{n-1}{n+1}} dx \\ &= 2^{-\frac{n-1}{2}} \Omega_n(C), \end{aligned}$$

where the last inequality uses Lebesgue’s Dominated Convergence Theorem and (14). □

Theorem 4.5 holds true not only for the Busemann volume but also for other notions of volume. This follows, since according to Berck, Bernig, and Vernicos [4], equation (13) holds true for the volume densities of all volumes that satisfy the following very general assumptions:

- The volume measure vol_C is a Borel measure on $\text{int } C$ and absolutely continuous with respect to the Lebesgue measure.
- If $A \subset C \subset C'$ where $C, C' \in \mathcal{K}(\mathbb{R}^n)$, then $\text{vol}_C(A) \geq \text{vol}_{C'}(A)$.
- If C is an ellipsoid, then vol_C is the hyperbolic volume.

All volume measures that satisfy these conditions are equivalent; i.e., if σ_C and $\bar{\sigma}_C$ are the volume densities of two volume measures vol_C and $\bar{\text{vol}}_C$, then there exist positive real constants a, b such that

$$a\sigma_C(x) \leq \bar{\sigma}_C(x) \leq b\sigma_C(x);$$

see e.g. [26, p. 249]. Hence, by (14), we conclude that

$$\bar{\sigma}_C(\lambda x) \leq bc(1 - \lambda)^{-\frac{n+1}{2}}.$$

Therefore Theorem 4.5 also holds for any volume measure that satisfies these conditions and in particular for the Holmes-Thompson volume.

5. PROOF OF THEOREM 1.1

The first step of the proof is the following disintegration result, which follows easily from the area formula (see e.g. [7, Proposition 3.7] or [9, Lemma 4.2] for related results).

Lemma 5.1. *Let K, L be convex bodies such that $L \subseteq K$ and $0 \in \text{int } L$. For $x \in \partial K$,*

$$\Psi(K) - \Psi(L) = \int_{\partial K} n_K(x) \cdot (x\|x\|^{-n}) \int_{\|x_L\|}^{\|x\|} \psi(tx\|x\|^{-1})t^{n-1} dt dx,$$

where $\{x_L\} = \partial L \cap [0, x]$.

The next step is to give upper and lower bounds of the weighted floating body K_δ^ϕ by a reparametrized Euclidean floating body. To be more precise, we find $\delta_1 = \delta_1(\delta)$ and $\delta_2 = \delta_2(\delta)$ such that $0 < \delta_1 \leq \delta_2$ and $K_{\delta_2} \subseteq K_\delta^\phi \subseteq K_{\delta_1}$. Before we go into the details of this proof, we need to fix a few notions.

For $v \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$, define, as before, the closed half-spaces $H^-(v, t) := \{y \in \mathbb{R}^n : y \cdot v \leq t\}$ and $H^+(v, t) := H^-(-v, -t)$. The weighted floating body K_δ^ϕ can be expressed as

$$(15) \quad K_\delta^\phi = \bigcap \left\{ H^-(v, t_\delta(v)) : v \in \mathbb{S}^{n-1} \right\},$$

where $t_\delta(v) = t(K, \phi, \delta, v)$ is determined implicitly by

$$(16) \quad \delta = \Phi \left(K \cap H^+(v, t_\delta(v)) \right) = \int_{t_\delta(v)}^{h_K(v)} \int_{K \cap H(v, s)} \phi(x) d\lambda_{H(v, s)}(x) ds.$$

Here $\lambda_{H(v, s)}$ is the Lebesgue measure in the affine hyperplane $H(v, s) = \{y \in \mathbb{R}^n : y \cdot v = s\}$. Note that there exists $\delta_0 > 0$ such that $t_\delta(v)$ is uniquely determined for all $(\delta, v) \in (0, \delta_0) \times \mathbb{S}^{n-1}$ and in this case the function $(\delta, v) \mapsto t_\delta(v)$ is continuous and $\lim_{\delta \rightarrow 0^+} t_\delta(v) = h_K(v)$.

Lemma 5.2. *Let $K \in \mathcal{K}(\mathbb{R}^n)$ and let $\varepsilon \in (0, \min_{\partial K} \phi)$. For*

$$(17) \quad \alpha := \min_{\partial K} \phi - \varepsilon, \quad \beta := \max_{\partial K} \phi + \varepsilon,$$

there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that for all $\delta \in (0, \delta_0)$, we have

$$K_{\delta/\alpha} \subseteq K_\delta^\phi \subseteq K_{\delta/\beta}.$$

Proof. Note that by our assumptions ϕ is continuous and positive on ∂K and therefore $\min_{\partial K} \phi > 0$. First we show that there is $\delta_1 = \delta_1(\varepsilon) > 0$ such that for all $\delta \in (0, \delta_1)$ and $v \in \mathbb{S}^{n-1}$ we have

$$(18) \quad K \cap H^+(v, t_\delta(v)) \subseteq \{x \in K : \phi(x) \leq \beta\}.$$

Assume the opposite. Then for all $\delta > 0$ there exists $v(\delta) \in \mathbb{S}^{n-1}$ and $y(\delta) \in K$ such that $\phi(y(\delta)) \geq \beta$ and $y(\delta) \cdot v(\delta) \geq t_\delta(v(\delta))$. By compactness there are converging subsequences with limits $v_0 \in \mathbb{S}^{n-1}$ and $y_0 \in \partial K$ such that $\phi(y_0) \geq \beta$ and $y_0 \cdot v_0 \geq t_0(v_0) = h_K(v_0)$. Thus $y_0 \in \partial K$ and therefore $\phi(y_0) \leq \max_{\partial K} \phi < \beta \leq \phi(y_0)$, a contradiction.

By (18), we have that

$$\delta = \Phi\left(K \cap H^+(v, t_\delta(v))\right) \leq \beta \lambda_n\left(K \cap H^+(v, t_\delta(v))\right),$$

which yields $t(K, 1, \delta/\beta, v) \geq t(K, \phi, \delta, v)$. Thus, by (15) and (16), $K_\delta^\phi \subseteq K_{\delta/\beta}$.

Conversely, there is $\delta_2 = \delta_2(\varepsilon) > 0$ such that for all $\delta \in (0, \delta_2)$ and $v \in \mathbb{S}^{n-1}$ we have

$$K \cap H^+(v, t(K, \phi, \delta, v)) \subseteq \{x \in K : \phi(x) \geq \alpha\}.$$

Similarly to the above we first have

$$\delta = \Phi\left(K \cap H^+(v, t_\delta(v))\right) \geq \alpha \lambda\left(K \cap H^+(v, t_\delta(v))\right),$$

and therefore $K_{\delta/\alpha} \subseteq K_\delta^\phi$. Setting $\delta_0 = \min\{\delta_1, \delta_2\}$ concludes the proof. □

For two distinct points $x, y \in \mathbb{R}^n$ the affine segment joining x and y is denoted by $[x, y]$. The previous lemma immediately implies the following result.

Corollary 5.3. *Let $K \in \mathcal{K}(\mathbb{R}^n)$, let α, β be as in (17), and let $z \in \text{int } K$. For $x \in \partial K$ we set*

$$\{x_{\delta/\alpha}\} = \partial K_{\delta/\alpha} \cap [x, z], \quad \{x_{\delta/\beta}\} = \partial K_{\delta/\beta} \cap [x, z], \quad \{x_\delta^\phi\} = \partial K_\delta^\phi \cap [x, z].$$

Then for $\delta > 0$ sufficiently small, we have

$$\|x_{\delta/\alpha} - z\| \leq \|x_\delta^\phi - z\| \leq \|x_{\delta/\beta} - z\|.$$

To complete the proof, we proceed as follows: The left-hand side of (4) can be written as an integral over ∂K by Lemma 5.1. Theorem 1.1 follows by applying Lebesgue’s Dominated Convergence Theorem and calculating the pointwise limit of the integrand. To do so, we need to bound the integrand from above by an integrable function.

We denote by $r_K : \partial K \rightarrow [0, +\infty)$ the maximal radius of a Euclidean ball that contains $x \in \partial K$ and is contained in K . It was proven in [34] that for $\alpha > -1$ we have

$$\int_{\partial K} r_K(x)^\alpha dx < +\infty.$$

Hence r_K is an integrable function and it was already used as the upper bound of the integrand for the Euclidean floating body. The following upper bound for the weighted floating body follows by the Euclidean results obtained in [34].

Lemma 5.4. *Let $K \in \mathcal{K}(\mathbb{R}^n)$ with $0 \in \text{int } K$. There exists $C = C(K) > 0$ such that for $\delta > 0$ sufficiently small*

$$\frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^\phi\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt \leq C \left(\max_K \psi \right) r_K(x)^{-\frac{n-1}{n+1}},$$

for almost all $x \in \partial K$.

Proof. Since $0 \in \text{int } K$, by Corollary 5.3 we have $\|x_\delta^\phi\| \geq \|x_{\delta/\alpha}\|$. We conclude that

$$\frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^\phi\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt \leq \left(\max_K \psi \right) \frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_{\delta/\alpha}\|}^{\|x\|} t^{n-1} dt.$$

Furthermore,

$$\frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_{\delta/\alpha}\|}^{\|x\|} t^{n-1} dt \leq \frac{x \cdot n_K(x)}{\|x\|} \frac{\|x - x_{\delta/\alpha}\|}{\delta^{2/(n+1)}} \leq C r_K(x)^{-\frac{n-1}{n+1}},$$

where the last inequality is the Euclidean result established in [34, Lemma 6]. \square

To calculate the pointwise limit of the integrand, we also use the Euclidean result to obtain the result for the weighted floating body. We recall some notions for boundary points of a convex body (see, for example, [31, Sections 2.2 and 2.5]).

A boundary point x of K is called *regular* if there is a unique outer unit normal $n_K(x)$ to K at x . Almost all boundary points are regular. Recall that for a convex body K the boundary ∂K is C^2 almost everywhere in the following sense: If x is a regular boundary point, there is $\varepsilon > 0$ and an open neighborhood U of x such that $U \cap \partial K$ can be described as

$$U \cap \partial K = \{x + v - f(v) n_K(x) : v \in n_K(x)^\perp \cap \varepsilon \mathbb{B}^n\},$$

where $f: n_K(x)^\perp \cap \varepsilon \mathbb{B}^n \rightarrow \mathbb{R}$ is a convex function which satisfies $f \geq 0$, $f(0) = 0$, and $n_K(x)^\perp = \{y \in \mathbb{R}^n : y \cdot n_K(x) = 0\}$. A regular boundary point $x \in \partial K$ is *normal* (or second order differentiable) if f is twice differentiable at 0 in the following sense: f is differentiable at 0 and there exists a symmetric linear map $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for $v, w \in n_K(x)^\perp$,

$$f(w) = f(v) + \nabla f(v) \cdot (w - v) + \frac{1}{2} A(w - v) \cdot (w - v) + o(\|w - v\|^2),$$

as $\|w - v\| \rightarrow 0$. Note that almost all boundary points are normal (see [31, Theorem 2.5.5]) and the (generalized) Gauss-Kronecker curvature $H_{n-1}(K, x) = \det(A)$ exists for normal boundary points.

Lemma 5.5. *Let $K \in \mathcal{K}(\mathbb{R}^n)$. If $x \in \partial K$ is a normal boundary point, then*

$$\lim_{\delta \rightarrow 0^+} \frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^\phi\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt = \alpha_n H_{n-1}(K, x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \psi(x).$$

In the proof of Lemma 5.5 we will use the following two results.

Lemma 5.6 ([7, Lemma 2.9]). *Let $K \in \mathcal{K}(\mathbb{R}^n)$ with $0 \in \text{int } K$ and $\varepsilon > 0$. If $x \in \partial K$ is a normal boundary point such that $H_{n-1}(K, x) > 0$, then there is $\delta_0 = \delta_0(\varepsilon)$ such that for all $\delta \in (0, \delta_0)$,*

$$[x, 0] \cap L_\delta^\phi = [x, 0] \cap K_\delta^\phi,$$

where $L = K \cap (x + \varepsilon \mathbb{B}^n)$. In particular, if we set $\{x_\delta^{\phi,K}\} = \partial K_\delta^\phi \cap [x, 0]$ and $\{x_\delta^{\phi,L}\} = \partial L_\delta^\phi \cap [x, 0]$, then $x_\delta^{\phi,K} = x_\delta^{\phi,L}$.

Lemma 5.7 ([34]). *Let $K \in \mathcal{K}(\mathbb{R}^n)$. If $x \in \partial K$ is a normal boundary point, then*

$$\lim_{\delta \rightarrow 0^+} \frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta\|}^{\|x\|} t^{n-1} dt = \alpha_n H_{n-1}(K, x)^{\frac{1}{n+1}}.$$

Proof of Lemma 5.5. Since x is normal, $H_{n-1}(K, x)$ exists. First, if $H_{n-1}(K, x) = 0$, then

$$\frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^\phi\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt \leq (\max_K \psi) \frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_{\delta/\alpha}\|}^{\|x\|} t^{n-1} dt.$$

By Lemma 5.7, we conclude that

$$\begin{aligned} & \limsup_{\delta \rightarrow 0^+} \frac{x \cdot n_K(x)}{\delta^{\frac{2}{n+1}} \|x\|^n} \int_{\|x_\delta^\phi\|}^{\|x\|} t^{n-1} \psi\left(\frac{tx}{\|x\|}\right) dt \\ & \leq \frac{\max_K \psi}{\alpha^{\frac{2}{n+1}}} \limsup_{\delta \rightarrow 0^+} \frac{x \cdot n_K(x)}{(\delta/\alpha)^{\frac{2}{n+1}} \|x\|^n} \int_{\|x_{\delta/\alpha}\|}^{\|x\|} t^{n-1} dt = 0. \end{aligned}$$

Now assume $H_{n-1}(K, x) > 0$ and let $\varepsilon > 0$ be arbitrary. Set $L = K \cap (x + \varepsilon \mathbb{B}^n)$. Then $H_{n-1}(L, x) = H_{n-1}(K, x)$ and $n_K(x) = n_L(x)$. Furthermore, for δ small enough, we have

$$\frac{x \cdot n_K(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^{\phi,K}\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt = \frac{x \cdot n_L(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^{\phi,L}\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt.$$

Let $z \in (\text{int } L) \cap [0, x]$. We apply Corollary 5.3 on L with ε and obtain, for $y \in \partial L$,

$$\|y_{\delta/\beta} - z\| \geq \|y_\delta^{\phi,L} - z\| \geq \|y_{\delta/\alpha} - z\|,$$

where $\beta = \max_{\partial L} \phi + \varepsilon$ and $\alpha = \min_{\partial L} \phi - \varepsilon$. Since $\|x\| = \|z\| + \|x - z\|$, this yields $\|x_{\delta/\beta}\| \geq \|x_\delta^{\phi,L}\| \geq \|x_{\delta/\alpha}\|$. We conclude that

$$\begin{aligned} & \frac{x \cdot n_L(x)}{\delta^{\frac{2}{n+1}} \|x\|^n} \int_{\|x_\delta^{\phi,L}\|}^{\|x\|} t^{n-1} \psi\left(\frac{tx}{\|x\|}\right) dt \\ & \leq \frac{x \cdot n_L(x)}{\delta^{\frac{2}{n+1}} \|x\|^n} \left(\max_{t \in [\|x_{\delta/\alpha}^L\|, \|x\|]} \psi\left(\frac{tx}{\|x\|}\right) \right) \int_{\|x_{\delta/\alpha}^L\|}^{\|x\|} t^{n-1} dt, \end{aligned}$$

and therefore

$$\limsup_{\delta \rightarrow 0^+} \frac{x \cdot n_L(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^{\phi,L}\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt \leq \alpha_n H_{n-1}(L, x)^{\frac{1}{n+1}} \frac{\psi(x)}{\alpha^{2/(n+1)}}.$$

Conversely, we have

$$\begin{aligned} & \frac{x \cdot n_L(x)}{\delta^{\frac{2}{n+1}} \|x\|^n} \int_{\|x_\delta^{\phi,L}\|}^{\|x\|} t^{n-1} \psi\left(\frac{tx}{\|x\|}\right) dt \\ & \geq \frac{x \cdot n_L(x)}{\delta^{\frac{2}{n+1}} \|x\|^n} \left(\min_{t \in [\|x_{\delta/\beta}^L\|, \|x\|]} \psi\left(\frac{tx}{\|x\|}\right) \right) \int_{\|x_{\delta/\beta}^L\|}^{\|x\|} t^{n-1} dt, \end{aligned}$$

and hence

$$\liminf_{\delta \rightarrow 0^+} \frac{x \cdot n_L(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^{\phi, L}\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt \geq \alpha_n H_{n-1}(L, x)^{\frac{1}{n+1}} \frac{\psi(x)}{\beta^{2/(n+1)}}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small and $\beta, \alpha \rightarrow \phi(x)$ for $\varepsilon \rightarrow 0$, we conclude that

$$\lim_{\delta \rightarrow 0^+} \frac{x \cdot n_L(x)}{\delta^{2/(n+1)} \|x\|^n} \int_{\|x_\delta^{\phi, L}\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt = \alpha_n H_{n-1}(L, x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \psi(x).$$

This finishes the proof, as, for $\delta > 0$ sufficiently small, we have $n_L(x) = n_K(x)$, $H_{n-1}(L, x) = H_{n-1}(K, x)$, and $x_\delta^{\phi, L} = x_\delta^{\phi, K}$. \square

The proof of (4) is now straightforward. By Lemma 5.1 we have

$$\frac{\Psi(K) - \Psi(K_\delta^\phi)}{\delta^{\frac{2}{n+1}}} = \int_{\partial K} \frac{x \cdot n_K(x)}{\delta^{(n+1)/2} \|x\|^n} \int_{\|x_\delta\|}^{\|x\|} t^{n-1} \psi(tx/\|x\|) dt dx.$$

By Corollary 5.3, there is $\delta_0 > 0$ such that the integrand is bounded by an integrable function for all $\delta < \delta_0$. By Lebesgue's Dominated Convergence Theorem and Lemma 5.5, we conclude that

$$\lim_{\delta \rightarrow 0^+} \frac{\Psi(K) - \Psi(K_\delta^\phi)}{\delta^{\frac{2}{n+1}}} = \alpha_n \int_{\partial K} H_{n-1}(K, x)^{\frac{1}{n+1}} \phi(x)^{-\frac{2}{n+1}} \psi(x) dx.$$

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