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RELATIVE MORITA EQUIVALENCE OF CUNTZ–KRIEGER ALGEBRAS AND FLOW EQUIVALENCE OF TOPOLOGICAL MARKOV SHIFTS

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Abstract. We will introduce a relative version of imprimitivity bimodule and a relative version of strong Morita equivalence for pairs of C^*-algebras $(\mathcal{A}, \mathcal{D})$ such that $\mathcal{D}$ is a C^*-subalgebra of $\mathcal{A}$ satisfying certain conditions. We will then prove that two pairs $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ are relatively Morita equivalent if and only if their relative stabilizations are isomorphic. In particular, for two pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ of Cuntz–Krieger algebras with their canonical masas, they are relatively Morita equivalent if and only if their underlying two-sided topological Markov shifts $(\overline{X}_A, \bar{\sigma}_A)$ and $(\overline{X}_B, \bar{\sigma}_B)$ are flow equivalent. We also introduce a relative version of the Picard group $\mathrm{Pic}(\mathcal{A}, \mathcal{D})$ for the pair $(\mathcal{A}, \mathcal{D})$ of C^*-algebras and study them for the Cuntz–Krieger pair $(\mathcal{O}_A, \mathcal{D}_A)$.

Contents

1. Introduction

In [38] M. Rieffel introduced the notion of an imprimitivity bimodule for C^*-algebras as a Hilbert C^*-bimodule satisfying certain conditions from a viewpoint of representation theory of groups, so that he defined the notion of strong Morita equivalence in C^*-algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. An $\mathcal{A}$–$\mathcal{B}$-bimodule X

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means a Hilbert C^*-bimodule with a left $\mathcal{A}$-module structure and an $\mathcal{A}$-valued inner product ${}_{\mathcal{A}}\langle \ \mid \ \rangle$ and with a right $\mathcal{B}$-module structure and a $\mathcal{B}$-valued inner product $\langle \ \mid \ \rangle_{\mathcal{B}}$ satisfying some comparability conditions (see [34], [38], [19], [35], etc.). It is said to be full if the ideals spanned by $\{{}_{\mathcal{A}}\langle x \mid y\rangle \mid x, y \in X\}$ and $\{\langle x \mid y\rangle_{\mathcal{B}} \mid x, y \in X\}$ are dense in $\mathcal{A}$ and in $\mathcal{B}$, respectively. If a full $\mathcal{A}$–$\mathcal{B}$-bimodule X further satisfies the condition

$$ {}_{\mathcal{A}}\langle x \mid y\rangle z = x\langle y \mid z\rangle_{\mathcal{B}} \quad \text{for } x, y, z \in X, $$

it is called an $\mathcal{A}$–$\mathcal{B}$-imprimitivity bimodule. Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be strongly Morita equivalent if there exists an $\mathcal{A}$–$\mathcal{B}$-imprimitivity bimodule, which means that $\mathcal{A}$ and $\mathcal{B}$ have the same representation theory. Brown, Green, and Rieffel in [5] have shown that two σ-unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent if and only if they are stably isomorphic; that is, $\mathcal{A} \otimes \mathcal{K}$ is isomorphic to $\mathcal{B} \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on the separable infinite-dimensional Hilbert space $\ell^2(\mathbb{N})$.

In this paper we will study Morita equivalence of C^*-algebras from a viewpoint of symbolic dynamical systems. For an irreducible and non-permutation matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0, 1\}$, the two-sided topological Markov shift $(\overline{X}_A, \bar{\sigma}_A)$ is defined as a topological dynamical system on the shift space $\overline{X}_A$ consisting of two-sided sequences $(x_n)_{n\in\mathbb{Z}}$ of $x_n \in \{1, \dots, N\}$ such that $A(x_n, x_{n+1}) = 1$ for all $n \in \mathbb{Z}$ with the shift homeomorphism $\bar{\sigma}_A((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ on the compact Hausdorff space $\overline{X}_A$. J. Cuntz and W. Krieger introduced a C^*-algebra $\mathcal{O}_A$ associated to the matrix A ([14]). The C^*-algebra is called the Cuntz–Krieger algebra, which is a universal unique C^*-algebra generated by partial isometries $S_1, \dots, S_N$ subject to the relations

$$ (1.1) \qquad \sum_{j=1}^N S_j S_j^* = 1, \qquad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad i = 1, \dots, N. $$

Since the stable isomorphism class of $\mathcal{O}_A$ does not have complete information about the underlying dynamical system $(\overline{X}_A, \bar{\sigma}_A)$, we need some extra structure to $\mathcal{O}_A$ to study $(\overline{X}_A, \bar{\sigma}_A)$. In this paper, we focus on the pair $(\mathcal{O}_A, \mathcal{D}_A)$ where $\mathcal{D}_A$ is the C^*-subalgebra of $\mathcal{O}_A$ generated by the projections of the form $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$, $i_1, \dots, i_n = 1, \dots, N$. We call $(\mathcal{O}_A, \mathcal{D}_A)$ the Cuntz–Krieger pair. As in [26] the isomorphism class of the pair $(\mathcal{O}_A, \mathcal{D}_A)$ is a complete invariant of the continuous orbit equivalence class of the underlying one-sided topological Markov shift (X_A, σ_A). As one of the remarkable relationships between symbolic dynamics and Cuntz–Krieger algebras, Cuntz and Krieger showed in [14] that if topological Markov shifts $(\overline{X}_A, \bar{\sigma}_A)$ and $(\overline{X}_B, \bar{\sigma}_B)$ are flow equivalent, then there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$, where $\mathcal{C}$ denotes the maximal commutative C^*-subalgebra of $\mathcal{K}$ consisting of the diagonal operators on $\ell^2(\mathbb{N})$. Recently H. Matui and the author have proved that the converse implication also holds, so that $(\overline{X}_A, \bar{\sigma}_A)$ and $(\overline{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ ([30]). We call $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ the stabilized Cuntz–Krieger pair or the relative stabilization of $(\mathcal{O}_A, \mathcal{D}_A)$, so that the isomorphism class of the relative stabilization of $(\mathcal{O}_A, \mathcal{D}_A)$ is a complete invariant for the flow equivalence class of the underlying two-sided topological Markov shift $(\overline{X}_A, \bar{\sigma}_A)$.

In this paper we will introduce a relative version of imprimitivity bimodule and a relative version of strong Morita equivalence for pairs of C^*-algebras $(\mathcal{A}, \mathcal{D})$ such that $\mathcal{D}$ is a C^*-subalgebra of $\mathcal{A}$ for which $\mathcal{D}$ has an orthogonal countable approximate unit for $\mathcal{A}$. Such a pair is said to be relative σ-unital. If $\mathcal{D}$ contains the unit of $\mathcal{A}$, the pair is relative σ-unital. Two relative σ-unital pairs $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ of C^*-algebras are said to be relatively Morita equivalent, written $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$, if there exists an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-relative imprimitivity bimodule. We will first show the following theorem for relative σ-unital pairs $(\mathcal{A}, \mathcal{D})$ of C^*-algebras.

Theorem 1.1 (Lemma 3.9, Theorem 4.7 and Theorem 5.5). *Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras. Then the following assertions are mutually equivalent:*

(1) *$(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ are relatively Morita equivalent.*
(2) *$(\mathcal{A}_1 \otimes \mathcal{K}, \mathcal{D}_1 \otimes \mathcal{C})$ and $(\mathcal{A}_2 \otimes \mathcal{K}, \mathcal{D}_2 \otimes \mathcal{C})$ are relatively Morita equivalent.*
(3) *There exists an isomorphism $\Phi : \mathcal{A}_1 \otimes \mathcal{K} \longrightarrow \mathcal{A}_2 \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_1 \otimes \mathcal{C}) = \mathcal{D}_2 \otimes \mathcal{C}$.*
(4) *$(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ are complementary relative full corners.*

We will next apply the theorem above to the Cuntz–Krieger pair $(\mathcal{O}_A, \mathcal{D}_A)$ and clarify relationships between relative Morita equivalence and flow equivalence of underlying topological dynamical systems.

Two Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are said to be *elementary corner isomorphic* if there exists a projection $P \in \mathcal{D}_B$ and an isomorphism $\Phi : P\mathcal{O}_B P \longrightarrow \mathcal{O}_A$ such that $\Phi(\mathcal{D}_B P) = \mathcal{D}_A$. The equivalence relation in Cuntz–Krieger pairs generated by elementary corner isomorphisms is said to be *corner isomorphic*. In [9] T. M. Carlsen, E. Ruiz, and A. Sims have studied diagonal preserving stable isomorphisms of graph C^*-algebras. Among other things, they have shown that the graph C^*-algebras with their diagonals are corner isomorphic in the above sense if and only if their underlying groupoids are Kakutani equivalent in the sense of Matui [31], which is equivalent to an existence of diagonal preserving stable isomorphism of the graph C^*-algebras ([9, Corollary 4.5]). Hence the equivalence between (3) and (4) in Theorem 1.2 below follows from their result. Their technique is due to groupoid method. In this paper, we will give its proof by a functional analytic technique (Theorem 6.3). By applying Theorem 1.1 to Cuntz–Krieger pairs, we see the following result.

Theorem 1.2 (Theorem 6.4). *Let A, B be irreducible and non-permutation matrices with entries in $\{0, 1\}$. Let $(\mathcal{O}_A, \mathcal{D}_A), (\mathcal{O}_B, \mathcal{D}_B)$ be the associated Cuntz–Krieger pairs. Then the following assertions are mutually equivalent:*

(1) *$(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are relatively Morita equivalent.*
(2) *$(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C})$ are relatively Morita equivalent.*
(3) *There exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$.*
(4) *$(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are corner isomorphic.*
(5) *The two-sided topological Markov shifts $(\overline{X}_A, \bar{\sigma}_A)$ and $(\overline{X}_B, \bar{\sigma}_B)$ are flow equivalent.*

By using J. Franks' s theorem [15] (*cf.* [3], [33]), *the last assertion* (5) *is equivalent to the following* (6)*:*

(6) *The groups* $\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N$ *and* $\mathbb{Z}^M/(\mathrm{id}-B)\mathbb{Z}^M$ *are isomorphic and* $\det(\mathrm{id}-A) = \det(\mathrm{id}-B)$*, where* N *is the size of the matrix* A *and* M *is that of* B.

Hence the group $\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N$ with the value $\det(\mathrm{id}-A)$ is a complete invariant of the relative Morita equivalence class of the Cuntz–Krieger pair $(\mathcal{O}_A, \mathcal{D}_A)$. We note that related results are seen in the paper [6] by N. Brownlowe, T. M. Carlsen and M. F. Whittaker. They are studying Morita equivalence of graph algebras in terms of groupoids (cf. [17], [18], [32], [40], etc.).

In [5] Brown, Green, and Rieffel introduced the notion of the Picard group $\mathrm{Pic}(\mathcal{A})$ for a C^*-algebra $\mathcal{A}$ to study equivalence classes of imprimitivity bimodules of C^*-algebras. Natural isomorphism classes $[X]$ of imprimitivity bimodules X over $\mathcal{A}$ form a group under the relative tensor product $[X]\cdot[Y] = [X \otimes_{\mathcal{A}} Y]$. The group is called the Picard group for the C^*-algebra $\mathcal{A}$ and is denoted by $\mathrm{Pic}(\mathcal{A})$, and is considered a sort of generalization of the automorphism group $\mathrm{Aut}(\mathcal{A})$ of $\mathcal{A}$. We will introduce a relative version of the Picard group $\mathrm{Pic}(\mathcal{A}, \mathcal{D})$ as the group of $(\mathcal{A}, \mathcal{D})$–$(\mathcal{A}, \mathcal{D})$-relative imprimitivity bimodules and study their structure for the Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$. Let

$$\mathrm{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A) = \{\alpha \in \mathrm{Aut}(\mathcal{O}_A) \mid \alpha(\mathcal{D}_A) = \mathcal{D}_A, \alpha_* = \mathrm{id} \text{ on } K_0(\mathcal{O}_A)\}.$$

Its quotient group $\mathrm{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A)/\mathrm{Int}(\mathcal{O}_A, \mathcal{D}_A)$ by $\mathrm{Int}(\mathcal{O}_A, \mathcal{D}_A)$ is denoted by $\mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A)$. Let $\mathrm{Aut}_1(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N)$ be the subgroup of the automorphism group $\mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N)$ of the abelian group $\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N$ defined by

$$\mathrm{Aut}_1(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N) = \{\xi \in \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N) \mid \xi([1]) = [1]\},$$

where $[1] \in \mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N$ denotes the class of the vector $(1, \dots, 1)$ in $\mathbb{Z}^N$. It is well known that there exists a canonical isomorphism $\epsilon_A : K_0(\mathcal{O}_A) \longrightarrow \mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N$ such that $\epsilon([1_{\mathcal{O}_A}]) = [1]$ ([13]). We will obtain the following structure theorem for $\mathrm{Pic}(\mathcal{O}_A, \mathcal{D}_A)$.

Theorem 1.3 (Theorems 8.8 and 8.9)**.** *Let* A *be an irreducible and non-permutation matrix. Then there exist short exact sequences*

$$1 \longrightarrow \mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\bar{\Psi}} \mathrm{Pic}(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{K_*} \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N) \longrightarrow 1,$$

$$1 \longrightarrow \mathrm{Out}(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\bar{\Psi}} \mathrm{Pic}(\mathcal{O}_A, \mathcal{D}_A)$$
$$\xrightarrow{K_*} \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N)/\mathrm{Aut}_1(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N) \longrightarrow 1.$$

In Appendix A, we refer to the ordinary Picard groups $\mathrm{Pic}(\mathcal{O}_A)$ for Cuntz–Krieger algebras $\mathcal{O}_A$ and especially for the ordinary Picard groups $\mathrm{Pic}(\mathcal{O}_N)$ for Cuntz algebras $\mathcal{O}_N$ (Theorem 9.4 and Corollary 9.5).

In Appendix B, we will present concrete construction of relative imprimitivity bimodules from flow equivalent topological Markov shifts, which we regard as a functional analytic proof of (5) $\Longrightarrow$ (1) of Theorem 1.2.

2. Relative σ-unital C^*-algebras

For a C^*-algebra $\mathcal{A}$ we denote by $M(\mathcal{A})$ its multiplier C^*-algebra (cf. [41]). The locally convex topology on $M(\mathcal{A})$ generated by the seminorms $x \longrightarrow \|xa\|$, $x \longrightarrow \|ax\|$ for $a \in \mathcal{A}$ is called the strict topology. Throughout the paper, we denote by $\{e_{i,j}\}_{i,j\in\mathbb{N}}$ the matrix units on the separable infinite-dimensional Hilbert space $\ell^2(\mathbb{N})$. The C^*-algebra generated by them is denoted by $\mathcal{K}$ which is the C^*-algebra of all compact operators on $\ell^2(\mathbb{N})$. The C^*-subalgebra of $\mathcal{K}$ generated by diagonal projections $\{e_{i,i}\}_{i\in\mathbb{N}}$ is denoted by $\mathcal{C}$.

A C^*-algebra is said to be σ-unital if it has a countable approximate unit. We will first introduce the notion of a relative version of a σ-unital C^*-algebra.

Definition 2.1. A pair $(\mathcal{A}, \mathcal{D})$ of C^*-algebras $\mathcal{A}, \mathcal{D}$ is called *relative σ-unital* if it satisfies the following conditions:

(1) $\mathcal{D}$ is a C^*-subalgebra of $\mathcal{A}$.
(2) $\mathcal{D}$ contains a countable approximate unit for $\mathcal{A}$.
(3) There exists a sequence $a_n \in \mathcal{A}, n = 1, 2, \ldots,$ such that
  (a) $a_n^* d a_n$, $a_n d a_n^* \in \mathcal{D}$ for all $d \in \mathcal{D}$ and $n = 1, 2, \ldots.$
  (b) $\sum_{n=1}^{\infty} a_n^* a_n = 1$ in the strict topology of $M(\mathcal{A})$.
  (c) $a_n d a_m^* = 0$ for all $d \in \mathcal{D}$ and $n, m \in \mathbb{N}$ with $n \neq m$.

We call the sequence $\{a_n\}_{n\in\mathbb{N}}$ satisfying the three conditions (a), (b), and (c) a *relative approximate unit* for the pair $(\mathcal{A}, \mathcal{D})$.

Remark 2.2. By the above condition (2), we know that $M(\mathcal{D})$ is a C^*-subalgebra of $M(\mathcal{A})$ in natural way (cf. [41, p. 46, 2G]).

Lemma 2.3. *Assume that $(\mathcal{A}, \mathcal{D})$ is a relative σ-unital pair of C^*-algebras. Let $\{a_n\}_{n\in\mathbb{N}}$ be a relative approximate unit for $(\mathcal{A}, \mathcal{D})$. Then we have the following.*

(i) *$a_n^* a_n$, $a_n a_n^* \in \mathcal{D}$ for all $n = 1, 2, \ldots.$*
(ii) *$b_n = \sum_{k=1}^{n} a_k^* a_k$ belongs to $\mathcal{D}$ and the sequence $\{b_n\}_{n\in\mathbb{N}}$ is a countable approximate unit for $\mathcal{A}$.*

Proof. (i) Take and fix $k \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} a_n^* a_n = 1$ in $M(\mathcal{A})$, we have $0 \leq a_k^* a_k \leq 1$ so that $\|a_k\| \leq 1$. As $\mathcal{D}$ has an approximate unit for $\mathcal{A}$, for any $\epsilon > 0$, there exists $d \in \mathcal{D}$ such that $\|a_k - d a_k\| < \epsilon$, so that $\|a_k^* a_k - a_k^* d a_k\| < \epsilon$. The condition $a_k^* d a_k \in \mathcal{D}$ ensures that $a_k^* a_k$ belongs to $\mathcal{D}$. Similarly, we know that $a_k a_k^*$ belongs to $\mathcal{D}$.

(ii) Since $b_n = \sum_{k=1}^{n} a_k^* a_k$ converges to 1 in the strict topology of $M(\mathcal{A})$, $\{b_n\}_{n\in\mathbb{N}}$ is an approximate unit for $\mathcal{A}$. ☐

Lemma 2.4. *Let $\mathcal{D}$ be a C^*-subalgebra of $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{D})$ is relative σ-unital if and only if there exists a sequence $d_n \in \mathcal{D}, n = 1, 2, \ldots$ such that the following holds.*

(a) $d_n \geq 0$, $n = 1, 2, \ldots.$
(b) $\sum_{n=1}^{\infty} d_n = 1$ *in the strict topology of* $M(\mathcal{A})$.
(c) $d_n d d_m = 0$ *for all* $d \in \mathcal{D}$ *and* $n, m \in \mathbb{N}$ *with* $n \neq m$.

Proof. Suppose that $(\mathcal{A}, \mathcal{D})$ is relative σ-unital. Take a relative approximate unit $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$. Put $d_n = a_n^* a_n$. By the preceding lemma, d_n belongs to $\mathcal{D}$ and satisfies the desired properties. Conversely, suppose that there exists a sequence d_n in $\mathcal{D}$ satisfying the above three conditions. Put $a_n = \sqrt{d_n}$, which becomes a relative approximate unit for $(\mathcal{A}, \mathcal{D})$. ☐

We call the sequence $\{d_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ satisfying the conditions (a), (b), and (c) in Lemma 2.4 an *orthogonal approximate unit for* $(\mathcal{A},\mathcal{D})$.

Example 2.5. 1. If a C^*-subalgebra $\mathcal{D}$ of a unital C^*-algebra $\mathcal{A}$ contains the unit 1 of $\mathcal{A}$, the pair $(\mathcal{A},\mathcal{D})$ is relative σ-unital by putting $d_1 = 1$ and $d_n = 0$ for $n = 2, 3, \dots$.

2. Let $\mathcal{A} = \mathcal{K}$ and $\mathcal{D} = \mathcal{C}$. Then the pair $(\mathcal{A},\mathcal{D})$ is relative σ-unital by putting $d_n = e_{n,n}, n \in \mathbb{N}$, where $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ is the matrix units of $\mathcal{K}$.

More generally we know the following proposition.

Proposition 2.6. *If* $(\mathcal{A},\mathcal{D})$ *is relative* σ*-unital, so is* $(\mathcal{A}\otimes\mathcal{K},\mathcal{D}\otimes\mathcal{C})$.

Proof. Take an orthogonal approximate unit $\{d_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ for the pair $(\mathcal{A},\mathcal{D})$. Put $d_{(n,m)} = d_n \otimes e_{m,m}$ for $n, m = 1, 2, \dots$. It is straightforward to see that the sequence $d_{(n,m)}, n, m = 1, 2, \dots$ becomes an orthogonal approximate unit for the pair $(\mathcal{A}\otimes\mathcal{K},\mathcal{D}\otimes\mathcal{C})$. □

We call the pair $(\mathcal{A}\otimes\mathcal{K},\mathcal{D}\otimes\mathcal{C})$ *the relative stabilization* for $(\mathcal{A},\mathcal{D})$.

Corollary 2.7. *If a* C^**-subalgebra* $\mathcal{D}$ *of* $\mathcal{A}$ *contains the unit of* $\mathcal{A}$*, both the pairs* $(\mathcal{A},\mathcal{D})$ *and* $(\mathcal{A}\otimes\mathcal{K},\mathcal{D}\otimes\mathcal{C})$ *are relative* σ*-unital.*

We remark that these kinds of pairs $(\mathcal{A},\mathcal{D})$ in this section are seen in many different contexts as in [23], [36], [37], etc.

3. Relative imprimitivity bimodules and relative Morita equivalence

In this section we first recall the definition of Rieffel's imprimitivity bimodule over C^*-algebras ([38]). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras. A left Hilbert C^*-module X over $\mathcal{A}_1$ is a $\mathbb{C}$-vector space with a left $\mathcal{A}_1$-module structure and an $\mathcal{A}_1$-valued inner product ${}_{\mathcal{A}_1}\langle\ \mid\ \rangle$ satisfying the following conditions [19, Definition 1.1](cf. [38], etc.).

(1) ${}_{\mathcal{A}_1}\langle\ \mid\ \rangle$ is left linear and right conjugate linear.
(2) ${}_{\mathcal{A}_1}\langle ax \mid y\rangle = a\,{}_{\mathcal{A}_1}\langle x \mid y\rangle$ and ${}_{\mathcal{A}_1}\langle x \mid ay\rangle = {}_{\mathcal{A}_1}\langle x \mid y\rangle a^*$ for all $x, y \in X$ and $a \in \mathcal{A}_1$.
(3) ${}_{\mathcal{A}_1}\langle x \mid x\rangle \ge 0$ for all $x \in X$, and ${}_{\mathcal{A}_1}\langle x \mid x\rangle = 0$ if and only if $x = 0$.
(4) ${}_{\mathcal{A}_1}\langle x \mid y\rangle = {}_{\mathcal{A}_1}\langle y \mid x\rangle^*$ for all $x, y \in X$.
(5) X is complete with respect to the norm $\|x\| = \|{}_{\mathcal{A}_1}\langle x \mid x\rangle\|^{\frac{1}{2}}$.

If the closed linear span of $\{{}_{\mathcal{A}_1}\langle x \mid y\rangle \mid x, y \in X\}$ is equal to $\mathcal{A}_1$, X is said to be left full. Similarly a right Hilbert C^*-module X over $\mathcal{A}_2$ is defined as a $\mathbb{C}$-vector space with a right $\mathcal{A}_2$-module structure and an $\mathcal{A}_2$-valued inner product $\langle\ \mid\ \rangle_{\mathcal{A}_2}$ satisfying the following conditions [19, Definition 1.2].

(1) $\langle\ \mid\ \rangle_{\mathcal{A}_2}$ is left conjugate and right linear.
(2) $\langle x \mid yb\rangle_{\mathcal{A}_2} = \langle x \mid y\rangle_{\mathcal{A}_2} b$ and $\langle xb \mid y\rangle_{\mathcal{A}_2} = b^*\langle x \mid y\rangle_{\mathcal{A}_2}$ for all $x, y \in X$ and $b \in \mathcal{A}_2$.
(3) $\langle x \mid x\rangle_{\mathcal{A}_2} \ge 0$ for all $x \in X_{\mathcal{A}_2}$, and $\langle x \mid x\rangle_{\mathcal{A}_2} = 0$ if and only if $x = 0$.
(4) $\langle x \mid y\rangle_{\mathcal{A}_2} = \langle y \mid x\rangle^*_{\mathcal{A}_2}$ for all $x, y \in X$.
(5) X is complete with respect to the norm $\|x\| = \|\langle x \mid x\rangle_{\mathcal{A}_2}\|^{\frac{1}{2}}$.

The right fullness for X is similarly defined to the left fullness. Throughout the paper, an $\mathcal{A}_1$–$\mathcal{A}_2$-Hilbert C^*-bimodule means a left Hilbert C^*-module over $\mathcal{A}_1$ and also a right Hilbert C^*-module over $\mathcal{A}_2$ in the above sense ([38], [19], [35], etc.). In

[38, Definition 6.10], M. Rieffel has defined the notion of an $\mathcal{A}_1$–$\mathcal{A}_2$-imprimitivity bimodule in the following way. An $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule X is said to be an $\mathcal{A}_1$–$\mathcal{A}_2$-imprimitivity bimodule if the three conditions below hold.

(1) X is a full left Hilbert $\mathcal{A}_1$-module with $\mathcal{A}_1$-valued left inner product ${}_{\mathcal{A}_1}\langle \ \mid \ \rangle$, and a full right Hilbert $\mathcal{A}_2$-module with $\mathcal{A}_2$-valued right inner product $\langle \ \mid \ \rangle_{\mathcal{A}_2}$.
(2) $\langle a \cdot x \mid y\rangle_{\mathcal{A}_2} = \langle x \mid a^* \cdot y\rangle_{\mathcal{A}_2}$ and ${}_{\mathcal{A}_1}\langle x \cdot b \mid y\rangle = {}_{\mathcal{A}_1}\langle x \mid y \cdot b^*\rangle$ for all $x, y \in X$ and $a \in \mathcal{A}_1$, $b \in \mathcal{A}_2$.
(3) ${}_{\mathcal{A}_1}\langle x \mid y\rangle \cdot z = x \cdot \langle y \mid z\rangle_{\mathcal{A}_2}$ for all $x, y, z \in X$.

We note that the above condition (2) implies

$$\|{}_{\mathcal{A}_1}\langle x \mid x\rangle\| = \|\langle x \mid x\rangle_{\mathcal{A}_2}\|, \qquad x \in X, \tag{3.1}$$

so that the two norms on X induced by the left-hand side and the right-hand side of (3.1) coincide (cf. [19, Corollary 1.19], [35, Proposition 3.11]).

We will introduce a relative version of the above imprimitivity bimodule. Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras.

Definition 3.1. Let X be an $\mathcal{A}_1$–$\mathcal{A}_2$-Hilbert C^*-bimodule. Put

$$\begin{aligned} N(X) = \{x \in X \mid {}_{\mathcal{A}_1}\langle x d_2 \mid x\rangle \in \mathcal{D}_1 \text{ for all } d_2 \in \mathcal{D}_2, \\ \langle x \mid d_1 x\rangle_{\mathcal{A}_2} \in \mathcal{D}_2 \text{ for all } d_1 \in \mathcal{D}_1\}. \end{aligned}$$

The $\mathcal{A}_1$–$\mathcal{A}_2$-Hilbert C^*-bimodule X is called an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-*relative imprimitivity bimodule* if it satisfies the following conditions.

(1) X is an $\mathcal{A}_1$–$\mathcal{A}_2$-imprimitivity bimodule.
(2) There exists a sequence $x_n \in N(X), n = 1, 2, \dots,$ such that
  (a) $\sum_{n=1}^{\infty} \langle x_n \mid x_n\rangle_{\mathcal{A}_2} = 1$ in the strict topology of $M(\mathcal{A}_2)$.
  (b) ${}_{\mathcal{A}_1}\langle x_n d_2 \mid x_m\rangle = 0$ for all $d_2 \in \mathcal{D}_2$ and $n, m \in \mathbb{N}$ with $n \neq m$.
(3) There exists a sequence $y_n \in N(X), n = 1, 2, \dots,$ such that
  (a) $\sum_{n=1}^{\infty} {}_{\mathcal{A}_1}\langle y_n \mid y_n\rangle = 1$ in the strict topology of $M(\mathcal{A}_1)$.
  (b) $\langle y_n \mid d_1 y_m\rangle_{\mathcal{A}_2} = 0$ for all $d_1 \in \mathcal{D}_1$ and $n, m \in \mathbb{N}$ with $n \neq m$.

Remark 3.2.

(1) Since X is an $\mathcal{A}_1$–$\mathcal{A}_2$-imprimitivity bimodule, norms on X defined by their inner products coincide each other, that is, $\|{}_{\mathcal{A}_1}\langle x \mid x\rangle\|^{\frac{1}{2}} = \|\langle x \mid x\rangle_{\mathcal{A}_2}\|^{\frac{1}{2}}$ for $x \in X$ (cf. [35, Proposition 3.11]). We denote the norm by $\|x\|$.
(2) The above elements $x_n, y_n \in N(X)$ in Definition 3.1 satisfy the inequalities

$${}_{\mathcal{A}_1}\langle x_n \mid x_n\rangle \leq 1, \qquad \langle y_n \mid y_n\rangle_{\mathcal{A}_2} \leq 1 \tag{3.2}$$

because of the inequalities

$${}_{\mathcal{A}_1}\langle x_n \mid x_n\rangle \leq \|{}_{\mathcal{A}_1}\langle x_n \mid x_n\rangle\| = \|\langle x_n \mid x_n\rangle_{\mathcal{A}_2}\| \leq \|\sum_{n=1}^{\infty} \langle x_n \mid x_n\rangle_{\mathcal{A}_2}\| = 1$$

and of similar inequalities for $\langle y_n \mid y_n\rangle_{\mathcal{A}_2}$.
(3) Both the left action of $\mathcal{A}_1$ and the right action of $\mathcal{A}_2$ on X are non-degenerate, that is, $\overline{\mathcal{A}_1 X} = X = \overline{X \mathcal{A}_2}$. More strongly, we see that

$\overline{\mathcal{D}_1 X} = X = \overline{X\mathcal{D}_2}$. In fact, for $d_1 \in \mathcal{D}_1$ and $x \in X$, the following inequalities hold:

$$\begin{aligned}\|x - d_1 x\|^2 &= \|{}_{\mathcal{A}_1}\langle x - d_1 x \mid x - d_1 x\rangle\| \\ &= \|{}_{\mathcal{A}_1}\langle x \mid x\rangle - d_1 {}_{\mathcal{A}_1}\langle x \mid x\rangle - {}_{\mathcal{A}_1}\langle x \mid x\rangle d_1^* + d_1 {}_{\mathcal{A}_1}\langle x \mid x\rangle d_1^*\| \\ &\le \|{}_{\mathcal{A}_1}\langle x \mid x\rangle - d_1 {}_{\mathcal{A}_1}\langle x \mid x\rangle\| + \|{}_{\mathcal{A}_1}\langle x \mid x\rangle - d_1 {}_{\mathcal{A}_1}\langle x \mid x\rangle\|\|d_1^*\|.\end{aligned}$$

As $\mathcal{D}_1$ has a countable approximate unit for $\mathcal{A}_1$, we have a sequence $d_1(n)$ in $\mathcal{D}_1$ such that $\lim_{n\to\infty} \|x - d_1(n)x\| = 0$ so that $\overline{\mathcal{D}_1 X} = X$.

In the following two lemmas, we assume that X will be an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-relative imprimitivity bimodule and $N(X)$ will be the subset of X defined in Definition 3.1.

Lemma 3.3. *For $x \in N(X)$, we have*

(i) ${}_{\mathcal{A}_1}\langle x \mid x\rangle \in \mathcal{D}_1$.
(ii) $\langle x \mid x\rangle_{\mathcal{A}_2} \in \mathcal{D}_2$.

Proof. (i) Let $x \in N(X)$. For $d_2 \in \mathcal{D}_2$, we have

$$\langle x - xd_2 \mid x - xd_2\rangle_{\mathcal{A}_2} = \langle x \mid x\rangle_{\mathcal{A}_2} - \langle x \mid x\rangle_{\mathcal{A}_2} d_2 - d_2^*\langle x \mid x\rangle_{\mathcal{A}_2} + d_2^*\langle x \mid x\rangle_{\mathcal{A}_2} d_2. \tag{3.3}$$

Now $\mathcal{D}_2$ contains an approximate unit for $\mathcal{A}_2$, and the equality (3.3) shows that for any $\epsilon > 0$ there exists an element $d_2 \in \mathcal{D}_2$ such that $\|\langle x - xd_2 \mid x - xd_2\rangle_{\mathcal{A}_2}\| < \epsilon$. Since X is an $\mathcal{A}_1$–$\mathcal{A}_2$-imprimitivity bimodule, we see that $\|{}_{\mathcal{A}_1}\langle x - xd_2 \mid x - xd_2\rangle\| < \epsilon$ by [35, Proposition 3.11]. By the Cauchy–Schwartz inequality (cf. [35, Lemma 2.5]) we have

$$\begin{aligned}\|{}_{\mathcal{A}_1}\langle x - xd_2 \mid x\rangle\|^2 &= \|{}_{\mathcal{A}_1}\langle x - xd_2 \mid x\rangle^* {}_{\mathcal{A}_1}\langle x - xd_2 \mid x\rangle\| \\ &\le \|{}_{\mathcal{A}_1}\langle x - xd_2 \mid x - xd_2\rangle\|\|{}_{\mathcal{A}_1}\langle x \mid x\rangle\| \\ &< \epsilon\|{}_{\mathcal{A}_1}\langle x \mid x\rangle\|.\end{aligned}$$

Hence we have

$$\|{}_{\mathcal{A}_1}\langle x \mid x\rangle - {}_{\mathcal{A}_1}\langle xd_2 \mid x\rangle\|^2 = \|{}_{\mathcal{A}_1}\langle x - xd_2 \mid x\rangle\|^2 < \epsilon\|{}_{\mathcal{A}_1}\langle x \mid x\rangle\|. \tag{3.4}$$

As ${}_{\mathcal{A}_1}\langle xd_2 \mid x\rangle$ belongs to $\mathcal{D}_1$, we conclude that ${}_{\mathcal{A}_1}\langle x \mid x\rangle$ belongs to $\mathcal{D}_1$.
(ii) is proved similarly to (i). ☐

Lemma 3.4.

(i) *We have $z = \sum_{n=1}^{\infty} {}_{\mathcal{A}_1}\langle z \mid x_n\rangle x_n$ for $z \in X$, which converges in the norm of X, and ${}_{\mathcal{A}_1}\langle x_n \mid x_m\rangle = 0$ for $n, m \in \mathbb{N}$ with $n \ne m$.*
(ii) *We have $z = \sum_{n=1}^{\infty} y_n\langle y_n \mid z\rangle_{\mathcal{A}_2}$ for $z \in X$, which converges in the norm of X, and $\langle y_n \mid y_m\rangle = 0$ for $n, m \in \mathbb{N}$ with $n \ne m$.*

Proof. (i) As $X = \overline{X\mathcal{D}_2}$, for $z \in X$ and $\epsilon > 0$ there exists $d_2 \in \mathcal{D}_2$ such that $\|z - zd_2\| < \epsilon$. Since $\sum_{n=1}^{\infty}\langle x_n \mid x_n\rangle_{\mathcal{A}_2} = 1$ in the strict topology of $M(\mathcal{A}_2)$, we may find $K \in \mathbb{N}$ such that $\| \sum_{n=1}^{K} d_2 \langle x_n \mid x_n \rangle_{\mathcal{A}_2} - d_2 \| < \epsilon$. Therefore we have

$$\begin{aligned}
\|z - \sum_{n=1}^{K} {}_{\mathcal{A}_1}\langle z \mid x_n \rangle x_n\| &= \|z - \sum_{n=1}^{K} z \langle x_n \mid x_n \rangle_{\mathcal{A}_2}\| \\
&\le \|z - zd_2\| + \|zd_2 - \sum_{n=1}^{K} zd_2 \langle x_n \mid x_n \rangle_{\mathcal{A}_2}\| \\
&+ \|\sum_{n=1}^{K} zd_2 \langle x_n \mid x_n \rangle_{\mathcal{A}_2} - \sum_{n=1}^{K} z \langle x_n \mid x_n \rangle_{\mathcal{A}_2}\| \\
&\le \|z - zd_2\| + \|z\| \|d_2 - \sum_{n=1}^{K} d_2 \langle x_n \mid x_n \rangle_{\mathcal{A}_2}\| \\
&+ \|(zd_2 - z) \sum_{n=1}^{K} \langle x_n \mid x_n \rangle_{\mathcal{A}_2}\| \\
&= (2 + \|z\|)\epsilon,
\end{aligned}$$

so that $\sum_{n=1}^{\infty} {}_{\mathcal{A}_1}\langle z \mid x_n \rangle x_n$ converges to z in the norm of X.

As in the proof of Lemma 3.3, for $n, m \in \mathbb{N}$ with $n \neq m$, there exists $d_2(k) \in \mathcal{D}_2$ such that

$$\lim_{k \to \infty} \|{}_{\mathcal{A}_1}\langle x_n \mid x_m \rangle - {}_{\mathcal{A}_1}\langle x_n d_2(k) \mid x_m \rangle\|^2 = 0.$$

Since ${}_{\mathcal{A}_1}\langle x_n d_2(k) \mid x_m \rangle = 0$, we have ${}_{\mathcal{A}_1}\langle x_n \mid x_m \rangle = 0$. □

The sequences $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subset N(X)$ satisfying conditions (2) and (3) in Definition 3.1 are called a *relative left basis* or a *relative right basis*, respectively. The pair $(\{x_n\}, \{y_n\})$ is called a *relative basis* for X. We remark that study of finite basis of Hilbert C^*-modules is seen in [42].

We arrive at our definition of the relative version of strong Morita equivalence.

Definition 3.5. Two relative σ-unital pairs of C^*-algebras $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ are said to be *relatively Morita equivalent* if there exists an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-relative imprimitivity bimodule. In this case we write $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$.

Lemma 3.6. *Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras. If there exists an isomorphism $\theta : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ of C^*-algebras such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$, then we have $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$. In particular, for a relative σ-unital pair $(\mathcal{A}, \mathcal{D})$ of C^*-algebras, we have $(\mathcal{A}, \mathcal{D}) \underset{\text{RME}}{\sim} (\mathcal{A}, \mathcal{D})$.*

Proof. Let $a_n \in \mathcal{A}_1, n \in \mathbb{N}$, be a relative approximate unit for $(\mathcal{A}_1, \mathcal{D}_1)$. Put $X_\theta = \mathcal{A}_1$ as vector space having module structure and inner products given by

$$a_1 \cdot x \cdot a_2 := a_1 x \theta^{-1}(a_2) \quad \text{for } a_1 \in \mathcal{A}_1,\ a_2 \in \mathcal{A}_2,\ x \in X_\theta, \tag{3.5}$$

$${}_{\mathcal{A}_1}\langle x \mid y \rangle = xy^*, \qquad \langle x \mid y \rangle_{\mathcal{A}_2} = \theta(x^* y) \quad \text{for } x, y \in X_\theta. \tag{3.6}$$

Put $x_n = a_n, n \in \mathbb{N}$. We have for $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$,

$${}_{\mathcal{A}_1}\langle x_n d_2 \mid x_n \rangle = a_n \theta^{-1}(d_2) a_n^* \in \mathcal{D}_1, \qquad \langle x_n \mid d_1 x_n \rangle_{\mathcal{A}_2} = \theta(a_n^* d_1 a_n) \in \mathcal{D}_2,$$

so that $x_n \in N(X_\theta)$. We also have

$$\sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{\mathcal{A}_2} = \sum_{n=1}^{\infty} \theta(a_n^* a_n) = 1$$

and

$${}_{\mathcal{A}_1}\langle x_n d_2 \mid x_m \rangle = a_n \theta^{-1}(d_2) a_m^* = 0 \quad \text{for all } d_2 \in \mathcal{D}_2 \text{ and } n, m \in \mathbb{N} \text{ with } n \neq m.$$

Similarly, by putting $y_n = a_n^*$, we have

$${}_{\mathcal{A}_1}\langle y_n d_2 \mid y_n \rangle = a_n^* \theta^{-1}(d_2) a_n \in \mathcal{D}_1, \qquad \langle y_n \mid d_1 y_n \rangle_{\mathcal{A}_2} = \theta(a_n d_1 a_n^*) \in \mathcal{D}_2,$$

so that $y_n \in N(X_\theta)$. We also have

$$\sum_{n=1}^{\infty} {}_{\mathcal{A}_1}\langle y_n \mid y_n \rangle = \sum_{n=1}^{\infty} a_n^* a_n = 1$$

and

$$\langle y_n \mid d_1 y_m \rangle = \theta(a_n d_1 a_m^*) = 0 \quad \text{for all } d_1 \in \mathcal{D}_1 \text{ and } n, m \in \mathbb{N} \text{ with } n \neq m.$$

Hence $(\{x_n\}, \{y_n\})$ is a relative basis for X_θ so that X_θ becomes an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-relative imprimitivity bimodule, thus proving $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$. □

We will next show that the relation $\underset{\text{RME}}{\sim}$ is an equivalence relation in relative σ-unital pairs of C^*-algebras. Relative tensor products of Hilbert C^*-bimodules are seen in [38, Section 1] (see also [35, Section 3], [19, Definition 1.20]).

Lemma 3.7. *Suppose that X_{12} is an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-relative imprimitivity bimodule and X_{23} is an $(\mathcal{A}_2, \mathcal{D}_2)$–$(\mathcal{A}_3, \mathcal{D}_3)$-relative imprimitivity bimodule. Then the relative tensor product $X_{12} \otimes_{\mathcal{A}_2} X_{23}$ of bimodules is an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_3, \mathcal{D}_3)$-relative imprimitivity bimodule.*

Proof. Take relative bases $(\{x_n\}, \{y_n\})$ for X_{12} and $(\{z_n\}, \{w_n\})$ for X_{23}. We will show that the pair $(\{x_n \otimes z_m\}_{n,m}, \{y_n \otimes w_m\}_{n,m})$ becomes a relative basis for $X_{12} \otimes_{\mathcal{A}_2} X_{23}$. For $d_3 \in \mathcal{D}_3, d_1 \in \mathcal{D}_1$, we have

$${}_{\mathcal{A}_1}\langle (x_n \otimes z_m) d_3 \mid x_n \otimes z_m \rangle = {}_{\mathcal{A}_1}\langle x_n \otimes (z_m d_3) \mid x_n \otimes z_m \rangle = {}_{\mathcal{A}_1}\langle x_n {}_{\mathcal{A}_2}\langle z_m d_3 \mid z_m \rangle \mid x_n \rangle,$$
$$\langle x_n \otimes z_m \mid d_1 (x_n \otimes z_m) \rangle_{\mathcal{A}_3} = \langle x_n \otimes z_m \mid (d_1 x_n) \otimes z_m \rangle_{\mathcal{A}_3} = \langle z_m \mid \langle x_n \mid d_1 x_n \rangle_{\mathcal{A}_2} z_m \rangle_{\mathcal{A}_3}.$$

As ${}_{\mathcal{A}_2}\langle z_m d_3 \mid z_m \rangle \in \mathcal{D}_2$, we have ${}_{\mathcal{A}_1}\langle x_n {}_{\mathcal{A}_2}\langle z_m d_3 \mid z_m \rangle \mid x_n \rangle \in \mathcal{D}_1$ so that ${}_{\mathcal{A}_1}\langle (x_n \otimes z_m) d_3 \mid x_n \otimes z_m \rangle \in \mathcal{D}_1$. Similarly, we know that $\langle x_n \otimes z_m \mid d_1 (x_n \otimes z_m) \rangle_{\mathcal{A}_3} \in \mathcal{D}_3$.

We also have

$$\begin{aligned}
\sum_{n,m=1}^{\infty} \langle x_n \otimes z_m \mid x_n \otimes z_m \rangle_{\mathcal{A}_3} &= \sum_{n,m=1}^{\infty} \langle z_m \mid \langle x_n \mid x_n \rangle_{\mathcal{A}_2} z_m \rangle_{\mathcal{A}_3} \\
&= \sum_{m=1}^{\infty} \langle z_m \mid (\sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{\mathcal{A}_2}) z_m \rangle_{\mathcal{A}_3} \\
&= \sum_{m=1}^{\infty} \langle z_m \mid z_m \rangle_{\mathcal{A}_3} = 1.
\end{aligned}$$

For $d_3 \in \mathcal{D}_3$, we have

$${}_{\mathcal{A}_1}\langle (x_n \otimes z_m)d_3 \mid x_l \otimes z_k\rangle = {}_{\mathcal{A}_1}\langle x_n \otimes (z_m d_3) \mid x_l \otimes z_k\rangle = {}_{\mathcal{A}_1}\langle x_n {}_{\mathcal{A}_2}\langle z_m d_3 \mid z_k\rangle \mid x_l\rangle.$$

If $m \neq k$, then ${}_{\mathcal{A}_2}\langle z_m d_3 \mid z_k\rangle = 0$. If $n \neq l$, then ${}_{\mathcal{A}_1}\langle x_n {}_{\mathcal{A}_2}\langle z_m d_3 \mid z_k\rangle \mid x_l\rangle = 0$ because ${}_{\mathcal{A}_2}\langle z_m d_3 \mid z_k\rangle \in \mathcal{D}_2$. Hence if $(n,m) \neq (l,k)$, we have ${}_{\mathcal{A}_1}\langle (x_n \otimes z_m)d_3 \mid x_l \otimes z_k\rangle = 0$, thus proving that the sequence $\{x_n \otimes z_m\}_{n,m}$ is a relative left basis for $X_{12} \otimes_{\mathcal{A}_2} X_{23}$. By a similar argument, one can show that $\{y_n \otimes w_m\}_{n,m}$ is a relative right basis for $X_{12} \otimes_{\mathcal{A}_2} X_{23}$, so that $(\{x_n \otimes z_m\}_{n,m}, \{y_n \otimes w_m\}_{n,m})$ is a relative basis for $X_{12} \otimes_{\mathcal{A}_2} X_{23}$. ☐

Therefore, we have

Proposition 3.8. *A relative Morita equivalence $\underset{\mathrm{RME}}{\sim}$ is an equivalence relation in relative σ-unital pairs of C^*-algebras.*

Proof. The reflexive law follows from Lemma 3.6. We will show the symmetric law. Suppose that $(\mathcal{A}_1, \mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$ via relative imprimitivity bimodule X_{12}. Then its conjugate module $\overline{X}_{12}$ denoted by X_{21} becomes an $(\mathcal{A}_2, \mathcal{D}_2)$–$(\mathcal{A}_1, \mathcal{D}_1)$-relative imprimitivity bimodule (see [38, Definition 6.17], cf. [19, p. 3443]), so that $(\mathcal{A}_2, \mathcal{D}_2) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_1, \mathcal{D}_1)$. The transitive law follows from Lemma 3.7. ☐

Lemma 3.9. *Let $(\mathcal{A}, \mathcal{D})$ be a relative σ-unital pair of C^*-algebras. Then we have*

$$(\mathcal{A}, \mathcal{D}) \underset{\mathrm{RME}}{\sim} (\mathcal{A} \otimes \mathcal{K}, \mathcal{D} \otimes \mathcal{C}).$$

Proof. Let $a_n \in \mathcal{A}, n \in \mathbb{N}$ be a relative approximate unit for $(\mathcal{A}, \mathcal{D})$. Recall that $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ denotes the matrix units of $\mathcal{K}$. Define $X = \mathcal{A} \otimes e_{1,1}\mathcal{K}$. By identifying $\mathcal{A}$ with $\mathcal{A} \otimes \mathbb{C}e_{1,1}$, X has the natural structure of an $\mathcal{A}$–$\mathcal{A}\otimes\mathcal{K}$-imprimitivity bimodule. Put $x_{n,m} = a_n \otimes e_{1,m} \in X, n, m \in \mathbb{N}$. For $d_1 \in \mathcal{D}$ and $d_2 = d \otimes f \in \mathcal{D} \otimes \mathcal{C}$, we have

$$\begin{aligned} {}_{\mathcal{A}}\langle x_{n,m} d_2 \mid x_{n,m}\rangle &= a_n d a_n^* \otimes e_{1,m} f e_{1,m}^* \in \mathcal{D} \otimes \mathbb{C}e_{1,1}, \\ \langle x_{n,m} \mid d_1 x_{n,m}\rangle_{\mathcal{A}\otimes\mathcal{K}} &= a_n^* d a_n \otimes e_{m,1} e_{1,1} e_{1,m} \in \mathcal{D} \otimes \mathcal{C}, \end{aligned}$$

so that $x_{n,m}$ belongs to $N(X)$ under the identification between $\mathcal{D}$ with $\mathcal{D} \otimes \mathbb{C}e_{1,1}$. We also have

$$\sum_{n,m=1}^{\infty} \langle x_{n,m} \mid x_{n,m}\rangle_{\mathcal{A}\otimes\mathcal{K}} = \sum_{n,m=1}^{\infty} a_n^* a_n \otimes e_{1,m}^* e_{1,m} = 1 \otimes 1$$

in $M(\mathcal{A} \otimes \mathcal{K})$. For $d_2 = d \otimes f \in \mathcal{D} \otimes \mathcal{C}$, we have

$${}_{\mathcal{A}}\langle x_{n,m} d_2 \mid x_{k,l}\rangle = a_n d a_k^* \otimes e_{1,m} f e_{1,l}^*.$$

If $n \neq k$, we have $a_n d a_k^* = 0$. If $m \neq l$, we have $e_{1,m} f e_{1,l}^* = 0$. Hence if $(n,m) \neq (k,l)$, we have ${}_{\mathcal{A}}\langle x_{n,m} d_2 \mid x_{k,l}\rangle = 0$.

Put $y_n = a_n^* \otimes e_{1,1}$. Then for $d_1 \in \mathcal{D}$ and $d_2 = d \otimes f \in \mathcal{D} \otimes \mathcal{C}$, we have

$$\begin{aligned} {}_{\mathcal{A}}\langle y_n d_2 \mid y_n\rangle &= a_n^* d a_n \otimes e_{1,1} f e_{1,1}^* \in \mathcal{D} \otimes \mathbb{C}e_{1,1}, \\ \langle y_n \mid d_1 y_n\rangle_{\mathcal{A}\otimes\mathcal{K}} &= a_n d a_n^* \otimes e_{1,1} \in \mathcal{D} \otimes \mathcal{C}, \end{aligned}$$

so that y_n belongs to $N(X)$. We also have

$$\sum_{n=1}^{\infty} {}_{\mathcal{A}}\langle y_n \mid y_n\rangle = \sum_{n=1}^{\infty} a_n^* a_n \otimes e_{1,1} = 1 \otimes e_{1,1},$$

and $\langle y_n \mid d_1 y_m\rangle_{\mathcal{A}\otimes\mathcal{K}} = a_n d a_m^* \otimes e_{1,1} = 0$ for $n \neq m$. Therefore, X becomes an $(\mathcal{A},\mathcal{D})$–$(\mathcal{A}\otimes\mathcal{K},\mathcal{D}\otimes\mathcal{C})$-relative imprimitivity bimodule, so that $(\mathcal{A},\mathcal{D}) \underset{\mathrm{RME}}{\sim} (\mathcal{A}\otimes\mathcal{K},\mathcal{D}\otimes\mathcal{C})$. □

Example 3.10. For $m,k \in \mathbb{N}$, let $\mathcal{A}_1 = M_m(\mathbb{C}), \mathcal{D}_1 = \operatorname{diag}(M_m(\mathbb{C})) = \mathbb{C}^m$, and $\mathcal{A}_2 = M_k(\mathbb{C}), \mathcal{D}_2 = \operatorname{diag}(M_k(\mathbb{C})) = \mathbb{C}^k$. Then we have $(\mathcal{A}_1,\mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2,\mathcal{D}_2)$.

We will present an $(\mathcal{A}_1,\mathcal{D}_1)$–$(\mathcal{A}_2,\mathcal{D}_2)$-relative imprimitivity bimodule in the follówung way. Let $\mathcal{A}_0,\mathcal{D}_0$ be $M_{m+k}(\mathbb{C}), \operatorname{diag}(M_{m+k}(\mathbb{C}))$, respectively. Let p_1,p_2 be the projections in $\mathcal{D}_0$ defined by

$$p_1 = (\overbrace{1,\cdots,1}^{m},\overbrace{0,\cdots,0}^{k}), \quad p_2 = (\overbrace{0,\cdots,0}^{m},\overbrace{1,\cdots,1}^{k}).$$

We then have

$$\mathcal{A}_1 = p_1\mathcal{A}_0 p_1, \qquad \mathcal{D}_1 = \mathcal{D}_0 p_1 \quad \text{and} \quad \mathcal{A}_2 = p_2\mathcal{A}_0 p_2, \qquad \mathcal{D}_2 = \mathcal{D}_0 p_2.$$

Put $X = p_1\mathcal{A}_0 p_2$ with a natural $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule structure and inner products such that

$$ {}_{\mathcal{A}_1}\langle x \mid y\rangle = xy^*, \qquad \langle x \mid y\rangle_{\mathcal{A}_2} = x^* y \quad \text{for } x,y \in X. \tag{3.7}$$

It is not difficult to see that X becomes an $(\mathcal{A}_1,\mathcal{D}_1)$–$(\mathcal{A}_2,\mathcal{D}_2)$-relative imprimitivity bimodule so that $(\mathcal{A}_1,\mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2,\mathcal{D}_2)$.

4. Isomorphism of relative stabilizations

This section is devoted to proving the following theorem, which is a relative version of a part of Brown–Green–Rieffel theorem [5, Theorem 1.2].

Theorem 4.1. *Suppose $(\mathcal{A}_1,\mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2,\mathcal{D}_2)$. Then there exists an isomorphism $\Phi : \mathcal{A}_1\otimes\mathcal{K} \longrightarrow \mathcal{A}_2\otimes\mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_1\otimes\mathcal{C}) = \mathcal{D}_2\otimes\mathcal{C}$.*

Suppose that X is an $(\mathcal{A}_1,\mathcal{D}_1)$–$(\mathcal{A}_2,\mathcal{D}_2)$-relative imprimitivity bimodule. Let $\overline{X}$ be the conjugate bimodule of X ([38, Definition 6.17], cf. [19, p. 3443]). The corresponding element in $\overline{X}$ to $y \in X$ is denoted by $\overline{y}$. It is straightforward to see that $\overline{X}$ is an $(\mathcal{A}_2,\mathcal{D}_2)$–$(\mathcal{A}_1,\mathcal{D}_1)$-relative imprimitivity bimodule. We define the relative linking pair $(\mathcal{A}_0,\mathcal{D}_0)$ by setting

$$\mathcal{A}_0 = \left\{ \begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix} \mid a_1 \in \mathcal{A}_1,\, a_2 \in \mathcal{A}_2,\, x,y \in X \right\}, \tag{4.1}$$

$$\mathcal{D}_0 = \left\{ \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \mid d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2 \right\}. \tag{4.2}$$

As in [5, p. 350] the products between two elements of $\mathcal{A}_0$ are defined by

$$\begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix} \begin{bmatrix} b_1 & z \\ \overline{w} & b_2 \end{bmatrix} := \begin{bmatrix} a_1 b_1 + {}_{\mathcal{A}_1}\langle x \mid w\rangle & a_1 z + x b_2 \\ \overline{y} b_1 + a_2 \overline{w} & \langle y \mid z\rangle_{\mathcal{A}_2} + a_2 b_2 \end{bmatrix},$$

and the adjoint of $\begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix} \in \mathcal{A}_0$ is defined by

$$\begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix}^* := \begin{bmatrix} a_1^* & y \\ \overline{x} & a_2^* \end{bmatrix}.$$

Let $X \oplus \mathcal{A}_2$ be the Hilbert C^*-right module over $\mathcal{A}_2$ with the natural right action of $\mathcal{A}_2$ and $\mathcal{A}_2$-valued right inner product defined by

$$\langle \begin{bmatrix} x \\ a_2 \end{bmatrix} \mid \begin{bmatrix} y \\ b_2 \end{bmatrix} \rangle_{\mathcal{A}_2} := \langle x \mid y \rangle_{\mathcal{A}_2} + a_2 b_2.$$

The algebra $\mathcal{A}_0$ acts on $X \oplus \mathcal{A}_2$ by

$$\begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix} \begin{bmatrix} z \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 z + x b_2 \\ \langle y \mid z \rangle_{\mathcal{A}_2} + a_2 b_2 \end{bmatrix}.$$

As seen in [35, Lemma 3.20], $\mathcal{A}_0$ itself is a C^*-subalgebra of all bounded adjointable operators on the Hilbert C^*-right module $X \oplus \mathcal{A}_2$. We set

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.3}$$

They satisfy $P_1 + P_2 = 1$ and

$$P_1 \mathcal{A}_0 P_1 = \mathcal{A}_1, \qquad \mathcal{D}_0 P_1 = \mathcal{D}_1 \quad \text{and} \quad P_2 \mathcal{A}_0 P_2 = \mathcal{A}_2, \qquad \mathcal{D}_0 P_2 = \mathcal{D}_2. \tag{4.4}$$

To prove Theorem 4.1, we provide several lemmas.

Lemma 4.2. *Let $(\{x_n\}, \{y_n\})$ be a relative basis for X.*

(i) *Put $U_n = \begin{bmatrix} 0 & x_n \\ 0 & 0 \end{bmatrix} \in \mathcal{A}_0, n \in \mathbb{N}$. The sequence U_n satisfies the following conditions.*
  (a) *$P_2 = \sum_{n=1}^{\infty} U_n^* U_n$ which converges in the strict topology of $M(\mathcal{A}_0)$.*
  (b) *$U_n U_n^* \le P_1$ and $U_n U_m^* = 0$ for $n \ne m$.*
  (c) *$U_n \mathcal{D}_0 U_n^* \subset \mathcal{D}_0 P_1 = \mathcal{D}_1$.*
  (d) *$U_n^* \mathcal{D}_0 U_n \subset \mathcal{D}_0 P_2 = \mathcal{D}_2$.*

(ii) *Put $T_n = \begin{bmatrix} 0 & 0 \\ \overline{y}_n & 0 \end{bmatrix} \in \mathcal{A}_0, n \in \mathbb{N}$. The sequence T_n satisfies the following conditions.*
  (a) *$P_1 = \sum_{n=1}^{\infty} T_n^* T_n$ which converges in the strict topology of $M(\mathcal{A}_0)$.*
  (b) *$T_n T_n^* \le P_2$ and $T_n T_m^* = 0$ for $n \ne m$.*
  (c) *$T_n \mathcal{D}_0 T_n^* \subset \mathcal{D}_0 P_2 = \mathcal{D}_2$.*
  (d) *$T_n^* \mathcal{D}_0 T_n \subset \mathcal{D}_0 P_1 = \mathcal{D}_1$.*

Proof. (i) For $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$, we have

$$U_n^* \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} U_n = \begin{bmatrix} 0 & 0 \\ 0 & \langle x_n \mid d_1 x_n \rangle_{\mathcal{A}_2} \end{bmatrix}. \tag{4.5}$$

Since $x_n \in N(X)$ and $d_1 \in \mathcal{D}_1$, we have $\langle x_n \mid d_1 x_n \rangle_{\mathcal{A}_2} \in \mathcal{D}_2$, so that $U_n^* \mathcal{D}_0 U_n \subset \mathcal{D}_0 P_2$, which shows (d). Since we have

$$U_n^* U_n = \begin{bmatrix} 0 & 0 \\ 0 & \langle x_n \mid x_n \rangle_{\mathcal{A}_2} \end{bmatrix}, \tag{4.6}$$

the equality $\sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{\mathcal{A}_2} = 1$ implies $\sum_{n=1}^{\infty} U_n^* U_n = P_2$ which shows (a). And also for $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$, we have

$$U_n \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} U_n^* = \begin{bmatrix} {}_{\mathcal{A}_1}\langle x_n d_2 \mid x_n \rangle & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.7}$$

Since $x_n \in N(X)$ and $d_2 \in \mathcal{D}_2$, we have ${}_{\mathcal{A}_1}\langle x_n d_2 \mid x_n \rangle \in \mathcal{D}_1$ so that $U_n \mathcal{D}_0 U_n^* \subset \mathcal{D}_0 P_1$, which shows (c). Since we have

$$U_n U_m^* = \begin{bmatrix} {}_{\mathcal{A}_1}\langle x_n \mid x_m \rangle & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.8}$$

the inequality ${}_{\mathcal{A}_1}\langle x_n \mid x_n \rangle \leq 1$ implies $U_n U_n^* \leq P_1$ and ${}_{\mathcal{A}_1}\langle x_n \mid x_m \rangle = 0$ for $n, m \in \mathbb{N}$ with $n \neq m$, which shows (b).

(ii) is proved similarly to (i). □

Lemma 4.3. *The pair $(\mathcal{A}_0, \mathcal{D}_0)$ is relative σ-unital.*

Proof. Refer to Lemma 4.2 and the notation given there. Put $a_n = \begin{bmatrix} 0 & x_n \\ \overline{y}_n & 0 \end{bmatrix} = U_n + T_n$. It then follows that

$$\sum_{n=1}^{\infty} a_n^* a_n = \sum_{n=1}^{\infty} U_n^* U_n + \sum_{n=1}^{\infty} T_n^* T_n = P_2 + P_1 = 1.$$

For $d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2$, we have

$$\begin{aligned} a_n^* \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} a_n &= U_n^* \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} U_n + T_n^* \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} T_n \\ &= \begin{bmatrix} \langle \overline{y}_n \mid d_2 \overline{y}_n \rangle_{\mathcal{A}_1} & 0 \\ 0 & \langle x_n \mid d_1 x_n \rangle_{\mathcal{A}_2} \end{bmatrix} \in \mathcal{D}_1 \oplus \mathcal{D}_2 = \mathcal{D}_0. \end{aligned}$$

Similarly, we have $a_n \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} a_n^* \in \mathcal{D}_1 \oplus \mathcal{D}_2$. We also have

$$a_n d a_m^* = (U_n + T_n) d (U_m + T_m)^* = U_n d U_m^* + T_n d T_m^* = 0$$

for $d = d_1 + d_2 \in \mathcal{D}_1 \oplus \mathcal{D}_2$ and $n \neq m$. Hence, $\{a_n\}$ is a relative approximate unit for $(\mathcal{A}_0, \mathcal{D}_0)$, thus showing that $(\mathcal{A}_0, \mathcal{D}_0)$ is relative σ-unital. □

Let us decompose the set $\mathbb{N}$ of natural numbers into disjoint infinite subsets $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathbb{N}_j$ and decompose $\mathbb{N}_j$ for each j once again into disjoint infinite sets $\mathbb{N}_j = \bigcup_{k=0}^{\infty} \mathbb{N}_{j,k}$. Recall that $\{e_{i,j}\}_{i,j \in \mathbb{N}}$ denotes the matrix units which generate the algebra $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$. Put the projections $f_j = \sum_{i \in \mathbb{N}_j} e_{i,i}$ and $f_{(j,k)} = \sum_{i \in \mathbb{N}_{j,k}} e_{i,i}$. Take a partial isometry $s_{(j,k),j}$ such that $s_{(j,k),j}^* s_{(j,k),j} = f_j$, $s_{(j,k),j} s_{(j,k),j}^* = f_{(j,k)}$, and put $s_{j,(j,k)} = s_{(j,k),j}^*$. Let P_1, P_2 be the projections of $M(\mathcal{A}_0)$ defined in (4.3). Take sequences $U_n, T_n, n \in \mathbb{N}$, as in Lemma 4.2. We set for $n = 1, 2, \dots,$

$$u_n = \sum_{k=1}^{\infty} U_k \otimes s_{(n,k),n}, \qquad w_n = P_1 \otimes s_{(n,0),n} + u_n, \tag{4.9}$$

$$t_n = \sum_{l=1}^{\infty} T_l \otimes s_{(n,l),n}, \qquad z_n = P_2 \otimes s_{(n,0),n} + t_n. \tag{4.10}$$

Then we have

Lemma 4.4 (cf. [29, Lemma 3.3])**.** *For each $n \in \mathbb{N}$, we have*

(i) *w_n is a partial isometry in $M(\mathcal{A}_0 \otimes \mathcal{K})$ such that*
   (a) *$w_n^* w_n = 1 \otimes f_n$.*
   (b) *$w_n w_n^* \leq P_1 \otimes f_n$.*
   (c) *$w_n (\mathcal{D}_0 \otimes \mathcal{C}) w_n^* \subset \mathcal{D}_1 \otimes \mathcal{C}$.*
   (d) *$w_n^* (\mathcal{D}_0 \otimes \mathcal{C}) w_n \subset \mathcal{D}_2 \otimes \mathcal{C}$.*

(ii) *z_n is a partial isometry in $M(\mathcal{A}_0 \otimes \mathcal{K})$ such that*
  (a) $z_n^* z_n = 1 \otimes f_n$.
  (b) $z_n z_n^* \leq P_2 \otimes f_n$.
  (c) $z_n(\mathcal{D}_0 \otimes \mathcal{C})z_n^* \subset \mathcal{D}_2 \otimes \mathcal{C}$.
  (d) $z_n^*(\mathcal{D}_0 \otimes \mathcal{C})z_n \subset \mathcal{D}_1 \otimes \mathcal{C}$.

Proof. (i) Since $u_n^* u_n = P_2 \otimes f_n$, we have

$$w_n^* w_n = P_1 \otimes f_n + u_n^* u_n = P_1 \otimes f_n + P_2 \otimes f_n = 1 \otimes f_n.$$

As $u_n(P_1 \otimes s_{n,(n,0)}) = (P_1 \otimes s_{n,(n,0)})u_n^* = 0$, we have

$$w_n w_n^* = P_1 \otimes f_{(n,0)} + u_n u_n^* = P_1 \otimes f_{(n,0)} + \sum_{k=1}^{\infty} U_k U_k^* \otimes f_{(n,k)}.$$

Since $f_{(n,0)}, f_{(n,k)} \leq f_n$, we have

$$w_n w_n^* \leq P_1 \otimes f_n.$$

The assertions (c) and (d) directly follow from (i)(c) and (i)(d) in Lemma 4.2, respectively.

(ii) is proved similarly to (i). □

We will construct and study a unitary V_1 in $M(\mathcal{A}_0 \otimes \mathcal{K})$ such that $\mathrm{Ad}(V_1) : \mathcal{A}_0 \otimes \mathcal{K} \longrightarrow \mathcal{A}_1 \otimes \mathcal{K}$ and $\mathrm{Ad}(V_1)(\mathcal{D}_0 \otimes \mathcal{C}) = \mathcal{D}_1 \otimes \mathcal{C}$.

Let $f_{n,m}$ be a partial isometry satisfying $f_{n,m}^* f_{n,m} = f_m$, $f_{n,m} f_{n,m}^* = f_n$. The following lemma is straightforward.

Lemma 4.5 (cf. [29, Lemma 3.4])**.** *We put*

$$\begin{aligned} v_1 &= w_1 = P_1 \otimes s_{(1,0),1} + u_1, \\ v_{2n} &= (P_1 \otimes f_n - v_{2n-1} v_{2n-1}^*)(P_1 \otimes f_{n,n+1}) \quad \text{for } 1 \leq n \in \mathbb{N}, \\ v_{2n-1} &= w_n(1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \quad \text{for } 2 \leq n \in \mathbb{N}. \end{aligned}$$

Then we have for $n \in \mathbb{N}$
  (a) $v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} = 1 \otimes f_n$.
  (b) $v_{2n-1} v_{2n-1}^* + v_{2n} v_{2n}^* = P_1 \otimes f_n$.
  (c) $v_n(\mathcal{D}_0 \otimes \mathcal{C})v_n^* \subset \mathcal{D}_1 \otimes \mathcal{C}$.
  (d) $v_n^*(\mathcal{D}_1 \otimes \mathcal{C})v_n \subset \mathcal{D}_0 \otimes \mathcal{C}$.

By Lemma 4.5 we have the following proposition.

Proposition 4.6. *Assume that $(\mathcal{A}_1, \mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$. Let $(\mathcal{A}_0, \mathcal{D}_0)$ be the relative linking pair defined in (4.1) and (4.2).*

(i) *There exists an isometry V_1 in $M(\mathcal{A}_0 \otimes \mathcal{K})$ such that*
  (a) $V_1^* V_1 = 1 \otimes 1$.
  (b) $V_1 V_1^* = P_1 \otimes 1$.
  (c) $V_1(\mathcal{D}_0 \otimes \mathcal{C})V_1^* = \mathcal{D}_1 \otimes \mathcal{C}$.
  (d) $V_1^*(\mathcal{D}_1 \otimes \mathcal{C})V_1 = \mathcal{D}_0 \otimes \mathcal{C}$.

(ii) *There exists an isometry V_2 in $M(\mathcal{A}_0 \otimes \mathcal{K})$ such that*
  (a) $V_2^* V_2 = 1 \otimes 1$.
  (b) $V_2 V_2^* = P_2 \otimes 1$.
  (c) $V_2(\mathcal{D}_0 \otimes \mathcal{C})V_2^* = \mathcal{D}_2 \otimes \mathcal{C}$.
  (d) $V_2^*(\mathcal{D}_2 \otimes \mathcal{C})V_2 = \mathcal{D}_0 \otimes \mathcal{C}$.

Proof. (i) Let v_n be the sequence of partial isometries in $M(\mathcal{A}_0 \otimes \mathcal{K})$ defined in Lemma 4.5. Recall that $\{e_{i,j}\}_{i,j\in\mathbb{N}}$ denotes the set of matrix units of the C^*-algebra $\mathcal{K}$ of compact operators on $\ell^2(\mathbb{N})$. For $a \otimes e_{i,j} \in \mathcal{A}_0 \otimes \mathcal{K}$ and $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
\| \sum_{k=1}^{2m-2} v_k(a\otimes e_{i,j}) - \sum_{k=1}^{2n-2} v_k(a\otimes e_{i,j})\|^2 &= \|(\sum_{k=2n-1}^{2m-2} v_k)(a\otimes e_{i,j})\|^2 \\
&= \|(a^*\otimes e_{i,j}^*)(\sum_{k=2n-1}^{2m-2} v_k^* v_k)(a\otimes e_{i,j})\| \\
&= \|(a^*\otimes e_{j,i})(\sum_{k=n}^{m} 1\otimes f_k)(a\otimes e_{i,j})\| \\
&\le \sum_{k=n}^{m} \|a^*a\otimes e_{j,i} f_k e_{i,j})\|
\end{aligned}$$

and

$$\begin{aligned}
\|(a\otimes e_{i,j})\sum_{k=1}^{2m-2} v_k - (a\otimes e_{i,j})\sum_{k=1}^{2n-2} v_k\|^2 &= \|(a\otimes e_{i,j})(\sum_{k=2n-1}^{2m-2} v_k v_k^*)(a^*\otimes e_{i,j}^*)\| \\
&= \|(a\otimes e_{i,j})(\sum_{k=n}^{m} P_1\otimes f_k)(a^*\otimes e_{i,j}^*)\| \\
&\le \sum_{k=n}^{m} \|aa^*\otimes e_{i,j} f_k e_{j,i})\|.
\end{aligned}$$

As $f_k = \sum_{i\in\mathbb{N}_k} e_{i,i}$, we have $e_{j,i} f_k e_{i,j} = e_{i,j} f_k e_{j,i} = 0$ for sufficiently large numbers k. Since the linear span of the form $a \otimes e_{i,j}$ for $a \in \mathcal{A}_0$, $i, j \in \mathbb{N}$ is dense in $\mathcal{A}_0 \otimes \mathcal{K}$, a routine argument shows that the summation $\sum_{n=1}^{\infty} v_n$ converges in $M(\mathcal{A}_0 \otimes \mathcal{K})$ to an element V_1 in the strict topology of $M(\mathcal{A}_0 \otimes \mathcal{K})$. The conditions (a) and (b) in Lemma 4.5 let V_1 satisfy the conditions (a) and (b) in (i) of Proposition 4.6, so that V_1 becomes a partial isometry in $M(\mathcal{A}_0 \otimes \mathcal{K})$. It satisfies the conditions (c) and (d) because of the conditions (c) and (d) in Lemma 4.5.

(ii) We similarly obtain a desired partial isometry V_2 in $M(\mathcal{A}_0 \otimes \mathcal{K})$ from the preceding partial isometries t_n, z_n defined in (4.10) instead of u_n, w_n. □

Therefore, we reach the following theorem.

Theorem 4.7. *Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras. Then $(\mathcal{A}_1, \mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$ if and only if there exists an isomorphism $\Phi : \mathcal{A}_1 \otimes \mathcal{K} \longrightarrow \mathcal{A}_2 \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_1 \otimes \mathcal{C}) = \mathcal{D}_2 \otimes \mathcal{C}$.*

Proof. Suppose $(\mathcal{A}_1, \mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$. Take isometries V_1, V_2 in $M(\mathcal{A}_0 \otimes \mathcal{K})$ as in Proposition 4.6. Put $\Phi = \mathrm{Ad}(V_2 V_1^*)$ which gives rise to an isomorphism $\Phi : \mathcal{A}_1 \otimes \mathcal{K} \longrightarrow \mathcal{A}_2 \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_1 \otimes \mathcal{C}) = \mathcal{D}_2 \otimes \mathcal{C}$.

Converse implication comes from Lemma 3.6, Proposition 3.8, and Lemma 3.9. □

5. Relative full corners

In [4], L. G. Brown introduced the notion of full corner of a C^*-algebra and proved that two σ-unital C^*-algebras are stably isomorphic if and only if they are full corners of some common σ-unital C^*-algebra ([4, Corollary 2.9]). Brown, Green, and Rieffel have further shown that two C^*-algebras are strongly Morita equivalent if and only if they are complementary full corners of some C^*-algebra ([5, Theorem 1.1]). In this section, we will study a relative version of their result.

Definition 5.1. For a relative σ-unital pair $(\mathcal{A}, \mathcal{D})$ of C^*-algebras, a projection $P \in M(\mathcal{D})$ is said to be *relatively full* in $(\mathcal{A}, \mathcal{D})$ if it satisfies the following conditions.

(1) $Pd = dP$ for all $d \in \mathcal{D}$.
(2) There exists a sequence $a_n \in \mathcal{A}, n = 1, 2, \ldots,$ such that
  (a) $a_n^* d a_n \in \mathcal{D}$, $a_n d a_n^* \in \mathcal{D}P$ for all $d \in \mathcal{D}$ and $n = 1, 2, \ldots$.
  (b) $\sum_{n=1}^{\infty} a_n^* P a_n = 1 - P$ in the strict topology of $M(\mathcal{A})$.
  (c) $a_n d a_m^* = 0$ for all $d \in \mathcal{D}$ and $n, m \in \mathbb{N}$ with $n \neq m$.

We call the sequence $\{a_n\}_{n\in\mathbb{N}}$ satisfying the three conditions (a), (b), and (c) a *relative full sequence* for P.

Remark 5.2. By condition (b) above, we know that

(b') $$a_n^* d P a_n \in \mathcal{D}(1 - P) \quad \text{for all } d \in \mathcal{D},$$

because we have

$$(a_n^* d P a_n)^* a_n^* d P a_n = a_n^* P d^* a_n a_n^* d P a_n \leq \|d^* a_n a_n^* d\| a_n^* P a_n \leq 1 - P.$$

Definition 5.3. Relative σ-unital pairs $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ of C^*-algebras are said to be *complementary relative full corners* if there exists a relative σ-unital pair $(\mathcal{A}_0, \mathcal{D}_0)$ of C^*-algebras such that there exist relative full projections $P_1, P_2 \in M(\mathcal{D}_0)$ such that

$$P_1 + P_2 = 1 \quad \text{and} \quad P_i \mathcal{A}_0 P_i = \mathcal{A}_i, \quad \mathcal{D}_0 P_i = \mathcal{D}_i, \quad i = 1, 2. \tag{5.1}$$

Proposition 5.4. *Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras. If they are complementary relative full corners, then we have $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$.*

Proof. Let $(\mathcal{A}_0, \mathcal{D}_0)$ and $P_i \in M(\mathcal{D}_0), i = 1, 2$, be a relative σ-unital pair of C^*-algebras and projections, respectively, satisfying Definition 5.3. Let $\{a_n\}$ and $\{b_n\}$ be relative full sequences for the projections P_1, P_2, respectively. We set $X = P_1 \mathcal{A}_0 P_2$ with natural $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule structure and inner products as in (3.7). Define two sequences by $x_n = P_1 a_n P_2$ and $y_n = P_1 b_n^* P_2$. For $d \in \mathcal{D}_0$, put $d_i = dP_i, i = 1, 2$. It then follows that

$$_{\mathcal{A}_1}\langle x_n d_2 \mid x_n \rangle = P_1 a_n P_2 d_2 P_2 a_n^* P_1 \in \mathcal{D}_0 P_1 = \mathcal{D}_1,$$
$$\langle x_n \mid d_1 x_n \rangle_{\mathcal{A}_2} = P_2 a_n^* P_1 d_1 P_1 a_n P_2 \in \mathcal{D}_0 P_2 = \mathcal{D}_2.$$

Hence, x_n belongs to $N(X)$. We also have

$$\sum_{n=1}^{\infty} \langle x_n \mid x_n \rangle_{\mathcal{A}_2} = \sum_{n=1}^{\infty} P_2 a_n^* P_1 a_n P_2 = P_2$$

and

$$ {}_{\mathcal{A}_1}\langle x_n d_2 \mid x_m \rangle = P_1 a_n P_2 d P_2 a_m^* P_1 = 0 \quad \text{for } n \neq m, $$

because $P_2 d P_2 \in \mathcal{D}_0$ and $a_n P_2 d P_2 a_m^* = 0$ for $n \neq m$. Hence, $\{x_n\}$ is a relative left basis for X. Similarly, we have

$$ {}_{\mathcal{A}_1}\langle y_n d_2 \mid y_n \rangle = P_1 b_n^* P_2 d_2 P_2 b_n P_1 \in \mathcal{D}_0 P_1 = \mathcal{D}_1, $$
$$ \langle y_n \mid d_1 y_n \rangle_{\mathcal{A}_2} = P_2 b_n P_1 d_1 P_1 b_n^* P_2 \in \mathcal{D}_0 P_2 = \mathcal{D}_2. $$

Hence, y_n belongs to $N(X)$. We also have

$$ \sum_{n=1}^{\infty} {}_{\mathcal{A}_1}\langle y_n \mid y_n \rangle = \sum_{n=1}^{\infty} P_1 b_n^* P_2 b_n P_1 = P_1 $$

and

$$ \langle y_n \mid d_1 y_m \rangle_{\mathcal{A}_2} = P_2 b_n P_1 d P_1 b_m^* P_2 = 0 \quad \text{for } n \neq m. $$

Hence, $\{y_n\}$ is a relative right basis for X. Therefore, X is an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-relative imprimitivity bimodule, so that we have $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$. □

We obtain the following theorem.

Theorem 5.5. *Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras. Then $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$ if and only if $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ are complementary relative full corners.*

Proof. The "if" part has been proved in Proposition 5.4. To show the "only if" part, suppose $(\mathcal{A}_1, \mathcal{D}_1) \underset{\text{RME}}{\sim} (\mathcal{A}_2, \mathcal{D}_2)$. Take $(\mathcal{A}_0, \mathcal{D}_0)$ the linking pair defined in (4.1) and (4.2). Let P_1, P_2 be the projections in $M(\mathcal{D}_0)$ defined by (4.3). Take the sequences U_n, T_n as in Lemma 4.2. The proof of Lemma 4.2 shows us that the sequences $a_n := U_n$ and $b_n := T_n$ are relative full sequences for P_1 and P_2, respectively, so that P_1 and P_2 are relative full projections in $(\mathcal{A}_0, \mathcal{D}_0)$. Since $P_1 + P_2 = 1$, the equalities (4.4) show that $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ are complementary relative full corners. □

6. Relative Morita equivalence in Cuntz–Krieger pairs

In this section, we will study relative Morita equivalence, particularly in Cuntz–Krieger algebras, from a viewpoint of symbolic dynamical systems. For a nonnegative matrix $A = [A(i,j)]_{i,j=1}^{N}$, the associated directed graph $G_A = (V_A, E_A)$ consists of the vertex set $V_A = \{v_1^A, \dots, v_N^A\}$ of N-vertices and the edge set $E_A = \{a_1, \dots, a_{N_A}\}$ where there are $A(i,j)$ edges from v_i^A to v_j^A. For $a_i \in E_A$, denote by $t(a_i), s(a_i)$ the terminal vertex of a_i and the source vertex of a_i, respectively. The graph G_A has the $N_A \times N_A$ transition matrix $A^G = [A^G(a_i, a_j)]_{i,j=1}^{N_A}$ of edges defined by

$$ A^G(a_i, a_j) = \begin{cases} 1 & \text{if } t(a_i) = s(a_j), \\ 0 & \text{otherwise} \end{cases} \tag{6.1} $$

for $a_i, a_j \in E_A$. The Cuntz–Krieger algebra $\mathcal{O}_A$ for the matrix A is defined as the Cuntz–Krieger algebra $\mathcal{O}_{A^G}$ for the matrix A^G which is the universal C^*-algebra generated by partial isometries S_{a_i} indexed by edges $a_i, i = 1, \dots, N_A$ subject to the relations

$$\sum_{j=1}^{N_A} S_{a_j} S_{a_j}^* = 1, \qquad S_{a_i}^* S_{a_i} = \sum_{j=1}^{N_A} A^G(a_i, a_j) S_{a_j} S_{a_j}^* \quad \text{for } i = 1, \dots, N_A. \tag{6.2}$$

The subalgebra $\mathcal{D}_A$ is defined as the algebra $\mathcal{D}_{A^G}$. The pair $(\mathcal{O}_A, \mathcal{D}_A)$ is called the Cuntz–Krieger pair for the matrix A. In what follows, we assume that the matrix A is irreducible and non-permutation. Since $1 \in \mathcal{D}_A \subset \mathcal{O}_A$, the pair $(\mathcal{O}_A, \mathcal{D}_A)$ is relative σ-unital. As in [26], the isomorphism class of the pair $(\mathcal{O}_A, \mathcal{D}_A)$ is exactly corresponding to the continuous orbit equivalence class of the underlying one-sided topological Markov shift (X_A, σ_A). Its complete classification result has been obtained in [30, Theorem 3.6] (cf. [8] for more general result).

Let A, B, Z be square irreducible and non-permutation matrices with entries in nonnegative integers.

Definition 6.1. Two Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_Z, \mathcal{D}_Z)$ are said to be *elementary corner isomorphic* if there exists a projection $P \in \mathcal{D}_Z$ and an isomorphism $\Phi : P\mathcal{O}_Z P \longrightarrow \mathcal{O}_A$ such that $\Phi(\mathcal{D}_Z P) = \mathcal{D}_A$. We identify $P\mathcal{O}_Z P$, $\mathcal{D}_Z P$ with $\mathcal{O}_A$, $\mathcal{D}_A$ through Φ, respectively, so that we write

$$P\mathcal{O}_Z P = \mathcal{O}_A, \qquad \mathcal{D}_Z P = \mathcal{D}_A. \tag{6.3}$$

Two Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are said to be *corner isomorphic* if there exists a finite chain of Cuntz–Krieger pairs $(\mathcal{O}_{Z_i}, \mathcal{D}_{Z_i}), i = 0, 1, \dots, n$, such that $Z_0 = A, Z_n = B$, and either $(\mathcal{O}_{Z_i}, \mathcal{D}_{Z_i})$ and $(\mathcal{O}_{Z_{i+1}}, \mathcal{D}_{Z_{i+1}})$ or $(\mathcal{O}_{Z_{i+1}}, \mathcal{D}_{Z_{i+1}})$ and $(\mathcal{O}_{Z_i}, \mathcal{D}_{Z_i})$ are elementary corner isomorphic for all $i = 0, 1, \dots, n$. That is, the equivalence relation generated by elementary corner isomorphisms in Cuntz–Krieger pairs is the corner isomorphism.

This equivalence relation appears in Carlsen, Ruiz, and Sims's paper [9] related to Kakutani equivalence of groupoids introduced by Matui [31]. Carlsen, Ruiz, and Sims are discussing a more general setting, the so-called graph algebra setting. By their result [9, Corollary 4,5], we see that two Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are corner isomorphic if and only if there is a diagonal preserving isomorphism of their stabilized Cuntz–Krieger algebras. Hence the following proposition and Theorem 6.3 are obtained directly from their result. Their methods are due to groupoid technique. We will prove the following proposition and Theorem 6.3 by a functional analytic method.

Proposition 6.2. *Let A, B be nonnegative irreducible non-permutation matrices. If two Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are corner isomorphic, then they are relatively Morita equivalent, and hence there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$.*

Proof. We assume that $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_Z, \mathcal{D}_Z)$ are elementary corner isomorphic by a projection $P \in \mathcal{D}_Z$ satisfying (6.3), so that we identify $\mathcal{O}_A, \mathcal{D}_A$ with $P\mathcal{O}_Z P, \mathcal{D}_Z P$, respectively. We set $X = P\mathcal{O}_Z$ which has a natural structure of an $\mathcal{O}_A$–$\mathcal{O}_Z$-imprimitivity bimodule in the following way:

$$a \cdot x \cdot b := axb \quad \text{for } a \in \mathcal{O}_A,\ b \in \mathcal{O}_Z,\ x \in X,$$
$${}_{\mathcal{O}_A}\langle x \mid y \rangle = xy^*, \qquad \langle x \mid y \rangle_{\mathcal{O}_Z} = x^* y \quad \text{for } x, y \in X.$$

We will show that X becomes an $(\mathcal{O}_A, \mathcal{D}_A)$–$(\mathcal{O}_Z, \mathcal{D}_Z)$-relative imprimitivity bimodule. We may assume that the projection $Q = 1 - P$ is not zero. Let $S_1, \dots, S_{N_Z}$ be the canonical generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_Z$ satisfying the relations (6.1) for the matrix Z. As $Q \in \mathcal{D}_Z$, one may find a finite family of admissible words $\mu(k), k = 1, \dots, N_1$ of X_Z such that $|\mu(1)| = \cdots = |\mu(N_1)|$ and $Q = \sum_{k=1}^{N_1} S_{\mu(k)} S_{\mu(k)}^*$, where $|\mu(i)|$ denotes the length of $\mu(i)$. Since Z is irreducible, we may find admissible words $\nu(k)$ of X_Z for each $\mu(k)$ such that $|\nu(1)| = \cdots = |\nu(N_1)|$ and

$$P \geq S_{\nu(k)} S_{\nu(k)}^*, \qquad S_{\nu(k)} S_{\mu(k)} \neq 0, \qquad k = 1, \dots, N_1.$$

As $\nu(k)\mu(k)$ is an admissible word in X_Z, we know $S_{\nu(k)}^* S_{\nu(k)} \geq S_{\mu(k)} S_{\mu(k)}^*$. For $k = 0, 1, \dots, N_1$, put

$$x_k = \begin{cases} P & \text{if } k = 0, \\ S_{\nu(k)} S_{\mu(k)} S_{\mu(k)}^* & \text{if } k = 1, \dots, N_1. \end{cases}$$

As $x_k = P x_k, k = 0, 1, \dots, N_1$, the sequence $x_k, k = 0, 1, \dots, N_1$ belongs to X. We then see that for $k = 0, 1, \dots, N_1$,

$$\begin{aligned} {}_{\mathcal{O}_A}\langle x_k d_2 \mid x_k \rangle &= x_k d_2 x_k^* \in \mathcal{D}_Z P \quad \text{ for } d_2 \in \mathcal{D}_Z, \\ \langle x_k \mid d_1 x_k \rangle_{\mathcal{O}_Z} &= x_k^* d_1 x_k \in \mathcal{D}_Z \quad \text{ for } d_1 \in \mathcal{D}_Z P \end{aligned}$$

so that $x_k, k = 0, 1, \dots, N_1$ belong to $N(X)$. We also see that

$$\begin{aligned} \sum_{k=0}^{N_1} \langle x_k \mid x_k \rangle_{\mathcal{O}_Z} &= P + \sum_{k=1}^{N_1} S_{\mu(k)} S_{\mu(k)}^* S_{\nu(k)}^* S_{\nu(k)} S_{\mu(k)} S_{\mu(k)}^* \\ &= P + \sum_{k=1}^{N_1} S_{\mu(k)} S_{\mu(k)}^* = P + Q = 1 \end{aligned}$$

and

$${}_{\mathcal{O}_A}\langle x_k d_2 \mid x_l \rangle = 0 \quad \text{ for } d_2 \in \mathcal{D}_Z,\ k \neq l.$$

Hence the sequence $x_k, k = 0, 1, \dots, N_1$, is a relative left basis for X in the sense of right before Definition 3.5. It is easy to see that the sequence $y_k = P S_k, k = 1, \dots, N_1$ belongs to $N(X)$ and satisfies the equalities

$$\sum_{k=1}^{N_1} {}_{\mathcal{O}_A}\langle y_k \mid y_k \rangle = \sum_{k=1}^{N_1} P S_k S_k^* P = P$$

and

$$\langle y_k \mid d_1 y_l \rangle_{\mathcal{O}_Z} = S_k^* P d_1 P S_l = 0 \quad \text{ for } d_1 \in \mathcal{D}_Z P,\ k \neq l.$$

Hence, the sequence $y_k, k = 1, \dots, N_1$, is a relative right basis for X, so that the pair $(\{x_k\}_{k=0}^{N_1}, \{y_k\}_{k=1}^{N_1})$ is a relative basis for X, proving that X is an $(\mathcal{O}_A, \mathcal{D}_A)$–$(\mathcal{O}_Z, \mathcal{D}_Z)$-relative imprimitivity bimodule, which gives rise to a relative Morita equivalence between $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_Z, \mathcal{D}_Z)$. The assertion that there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_Z \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_Z \otimes \mathcal{C}$ follows from Theorem 4.7. ∎

Therefore we have the following theorem, which has already appeared in Carlsen, Ruiz, and Sims [9].

Theorem 6.3 (Carlsen–Ruiz–Sims [9])**.** *Let A, B be nonnegative irreducible and non-permutation matrices. The Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are corner isomorphic if and only if there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$.*

Proposition 6.2 shows the only if part of the above theorem. The if part directly follows from Carlsen–Ruiz–Sims [9, Corollary 4.5] as well as the only if part. The if part also follows from the discussions in Appendix B with [30, Corollary 3.8]. As a consequence we have a functional analytic proof of the above theorem using [30, Corollary 3.8].

We may summarize our discussions for Cuntz–Krieger algebras in the following way.

Theorem 6.4. *Let A, B be irreducible non-permutation matrices with entries in $\{0, 1\}$. Let $\mathcal{O}_A, \mathcal{O}_B$ be the associated Cuntz–Krieger algebras. Then the following assertions are mutually equivalent.*

1. $(\mathcal{O}_A, \mathcal{D}_A) \underset{\text{RME}}{\sim} (\mathcal{O}_B, \mathcal{D}_B)$.
2. $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \underset{\text{RME}}{\sim} (\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C})$.
3. *There exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$.*
4. *$(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$ are corner isomorphic.*
5. *The two-sided topological Markov shifts $(\overline{X}_A, \bar{\sigma}_A)$ and $(\overline{X}_B, \bar{\sigma}_B)$ are flow equivalent.*

Proof. (1) $\Longleftrightarrow$ (2) comes from Lemma 3.9.
(1) $\Longleftrightarrow$ (3) comes from Theorem 4.7.
(3) $\Longleftrightarrow$ (4) comes from Theorem 6.3 ([9, Corollary 4.5]).
(5) $\Longrightarrow$ (3) comes from [14, 4.1 Theorem].
(3) $\Longrightarrow$ (5) comes from [30, Corollary 3.8]. □

7. Relative Picard groups

Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras. Let X, Y be an $(\mathcal{A}_1, \mathcal{D}_1)$–$(\mathcal{A}_2, \mathcal{D}_2)$-relative imprimitivity bimodule. Then X and Y are said to be *equivalent* if there exists an isomorphism $\varphi : X \longrightarrow Y$ of an $\mathcal{A}_1$–$\mathcal{A}_2$-imprimitivity bimodule such that

$$\langle \varphi(x_1) \mid \varphi(x_2) \rangle = \langle x_1 \mid x_2 \rangle \quad \text{for} \quad x_1, x_2 \in X$$

for both left and right inner products. As $\varphi : X \longrightarrow Y$ preserves the bimodule structures and inner products of X and Y, we know $\varphi(N(X)) = N(Y)$. We denote by $[X]$ the equivalence class of a relative imprimitivity bimodule X. For a relative σ-unital pair $(\mathcal{A}, \mathcal{D})$ of C^*-algebras, we introduce a notion of relative version of Picard group as follows.

Definition 7.1. The *relative Picard group* $\mathrm{Pic}(\mathcal{A}, \mathcal{D})$ for $(\mathcal{A}, \mathcal{D})$ is defined by the group of equivalence classes $[X]$ of an $(\mathcal{A}, \mathcal{D})$–$(\mathcal{A}, \mathcal{D})$-relative imprimitivity bimodule by the product

$$[X] \cdot [Y] := [X \otimes_{\mathcal{A}} Y].$$

We remark that the identity element of the group $\mathrm{Pic}(\mathcal{A},\mathcal{D})$ is the class of the identity $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodule $X=\mathcal{A}$ defined by the module structure and the inner products

$$(7.1)\quad a\cdot x\cdot b = axb, \qquad {}_{\mathcal{A}}\langle x \mid y\rangle := xy^*, \qquad \langle x \mid y\rangle_{\mathcal{A}} := x^*y \quad \text{for} \quad a,b,x,y\in\mathcal{A}.$$

Since $(\mathcal{A},\mathcal{D})$ is relative σ-unital, the above X becomes an $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodule as seen in Lemma 3.6.

Lemma 7.2. *If* $(\mathcal{A}_1,\mathcal{D}_1) \underset{\mathrm{RME}}{\sim} (\mathcal{A}_2,\mathcal{D}_2)$, *we have isomorphic Picard groups* $\mathrm{Pic}(\mathcal{A}_1,\mathcal{D}_1)\cong\mathrm{Pic}(\mathcal{A}_2,\mathcal{D}_2)$. *Hence we have* $\mathrm{Pic}(\mathcal{A},\mathcal{D})\cong\mathrm{Pic}(\mathcal{A}\otimes\mathcal{K},\mathcal{D}\otimes\mathcal{C})$ *for every relative σ-unital pair $(\mathcal{A},\mathcal{D})$ of C^*-algebras.*

Proof. Let X be an $(\mathcal{A}_1,\mathcal{D}_1)$–$(\mathcal{A}_2,\mathcal{D}_2)$-relative imprimitivity bimodule, and let $\overline{X}$ be its conjugate module, which is an $(\mathcal{A}_2,\mathcal{D}_2)$–$(\mathcal{A}_1,\mathcal{D}_1)$-relative imprimitivity bimodule. It is easy to see that the correspondence

$$[Y]\in\mathrm{Pic}(\mathcal{A}_1,\mathcal{D}_1)\longrightarrow[\overline{X}\otimes_{\mathcal{A}_1}Y\otimes_{\mathcal{A}_1}X]\in\mathrm{Pic}(\mathcal{A}_2,\mathcal{D}_2)$$

yields an isomorphism as groups, because $[\overline{X}\otimes_{\mathcal{A}_1}X]$ is the unit of the group $\mathrm{Pic}(\mathcal{A}_2,\mathcal{D}_2)$ and $[X\otimes_{\mathcal{A}_2}\overline{X}]$ is the unit of the group $\mathrm{Pic}(\mathcal{A}_1,\mathcal{D}_1)$. □

If $\theta:\mathcal{A}_1\longrightarrow\mathcal{A}_2$ is an isomorphism of C^*-algebras such that $\theta(\mathcal{D}_1)=\mathcal{D}_2$, then we write $\theta:(\mathcal{A}_1,\mathcal{D}_1)\longrightarrow(\mathcal{A}_2,\mathcal{D}_2)$ and call it an isomorphism of relative σ-unital pairs of C^*-algebras. As in Lemma 3.6, any isomorphism $\theta:(\mathcal{A}_1,\mathcal{D}_1)\longrightarrow(\mathcal{A}_2,\mathcal{D}_2)$ gives rise to an $(\mathcal{A}_1,\mathcal{D}_1)$–$(\mathcal{A}_2,\mathcal{D}_2)$-relative imprimitivity bimodule X_θ. The following lemma holds.

Lemma 7.3. *Let* $\theta_{12}:(\mathcal{A}_1,\mathcal{D}_1)\longrightarrow(\mathcal{A}_2,\mathcal{D}_2)$ *and* $\theta_{23}:(\mathcal{A}_2,\mathcal{D}_2)\longrightarrow(\mathcal{A}_3,\mathcal{D}_3)$ *be isomorphisms of relative σ-unital pairs of C^*-algebras. Then we have*

$$[X_{\theta_{12}}\otimes_{\mathcal{A}_2}X_{\theta_{23}}]=[X_{\theta_{23}\circ\theta_{12}}].$$

Therefore, we have a contravariant functor from the category of relative σ-unital C^*-algebras with isomorphisms $\theta:(\mathcal{A}_1,\mathcal{D}_1)\longrightarrow(\mathcal{A}_2,\mathcal{D}_2)$ *as morphisms into the category of relative σ-unital C^*-algebras with equivalence classes of relative imprimitivity bimodules.*

Proof. As in the proof of Lemma 3.6, for the isomorphism $\theta_{ii+1}:(\mathcal{A}_i,\mathcal{D}_i)\longrightarrow(\mathcal{A}_{i+1},\mathcal{D}_{i+1})$, $i=1,2$, the $(\mathcal{A}_i,\mathcal{D}_i)$–$(\mathcal{A}_{i+1},\mathcal{D}_{i+1})$-relative imprimitivity bimodule $X_{\theta_{ii+1}}$ is defined by $X_{\theta_{ii+1}}=\mathcal{A}_i$ having module structure and inner products given by

$$a_i\cdot x_{ii+1}\cdot a_{i+1}=a_ix_{ii+1}\theta_{ii+1}^{-1}(a_{i+1})\quad \text{for } a_i\in\mathcal{A}_i,\ a_{i+1}\in\mathcal{A}_{i+1},\ x_{ii+1}\in X_{\theta_{ii+1}},$$
$${}_{\mathcal{A}_i}\langle x_{ii+1}\mid y_{ii+1}\rangle = x_{ii+1}y_{ii+1}^*,\qquad \langle x_{ii+1}\mid y_{ii+1}\rangle_{\mathcal{A}_{i+1}}=\theta_{ii+1}(x_{ii+1}^*y_{ii+1})$$

for $x_{ii+1},y_{ii+1}\in X_{\theta_{ii+1}}$, $i=1,2$. We will see that the correspondence

$$\varphi: x_{12}\otimes x_{23}\in X_{\theta_{12}}\otimes_{\mathcal{A}_2}X_{\theta_{23}}\longrightarrow x_{12}\theta_{12}^{-1}(x_{23})\in X_{\theta_{23}\circ\theta_{12}}$$

yields an isomorphism from $X_{\theta_{12}}\otimes_{\mathcal{A}_2}X_{\theta_{23}}$ to $X_{\theta_{23}\circ\theta_{12}}$. For $a_i\in\mathcal{A}_i$, $i=1,3$, we have the equalities

$$\begin{aligned}\varphi(a_1(x_{12}\otimes x_{23})a_3) &= \varphi(a_1x_{12}\otimes x_{23}\theta_{23}^{-1}(a_3))\\ &= a_1x_{12}\theta_{12}^{-1}(x_{23}\theta_{23}^{-1}(a_3))\\ &= a_1x_{12}\theta_{12}^{-1}(x_{23})(\theta_{23}\circ\theta_{12})^{-1}(a_3)\\ &= a_1\varphi(x_{12}\otimes x_{23})a_3.\end{aligned}$$

We also have

$$\begin{aligned}
{}_{\mathcal{A}_1}\langle \varphi(x_{12}\otimes x_{23}) \mid \varphi(y_{12}\otimes y_{23})\rangle &= {}_{\mathcal{A}_1}\langle x_{12}\theta_{12}^{-1}(x_{23}) \mid y_{12}\theta_{12}^{-1}(y_{23})\rangle \\
&= x_{12}\theta_{12}^{-1}(x_{23}y_{23}^*)y_{12}^* \\
&= (x_{12\,\mathcal{A}_2}\langle x_{23} \mid y_{23}\rangle)y_{12}^* \\
&= {}_{\mathcal{A}_1}\langle x_{12\,\mathcal{A}_2}\langle x_{23} \mid y_{23}\rangle \mid y_{12}\rangle \\
&= {}_{\mathcal{A}_1}\langle x_{12}\otimes x_{23} \mid y_{12}\otimes y_{23}\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle \varphi(x_{12}\otimes x_{23}) \mid \varphi(y_{12}\otimes y_{23})\rangle_{\mathcal{A}_3} &= \langle x_{12}\theta_{12}^{-1}(x_{23}) \mid y_{12}\theta_{12}^{-1}(y_{23})\rangle_{\mathcal{A}_3} \\
&= (\theta_{23}\circ\theta_{12})((x_{12}\theta_{12}^{-1}(x_{23})^*)y_{12}\theta_{12}^{-1}(y_{23})) \\
&= \theta_{23}(x_{23}^*(\langle x_{12} \mid y_{12}\rangle_{\mathcal{A}_2}y_{23})) \\
&= \langle x_{23} \mid \langle x_{12} \mid y_{12}\rangle_{\mathcal{A}_2}y_{23}\rangle_{\mathcal{A}_3} \\
&= \langle x_{12}\otimes x_{23} \mid y_{12}\otimes y_{23}\rangle_{\mathcal{A}_3}.
\end{aligned}$$

Hence, we know that $\varphi : X_{\theta_{12}}\otimes_{\mathcal{A}_2} X_{\theta_{23}} \longrightarrow X_{\theta_{23}\circ\theta_{12}}$ yields an isomorphism so that $[X_{\theta_{12}}\otimes_{\mathcal{A}_2} X_{\theta_{23}}] = [X_{\theta_{23}\circ\theta_{12}}]$. □

Let $\mathrm{Aut}(\mathcal{A},\mathcal{D})$ be the group of automorphisms θ on $\mathcal{A}$ such that $\theta(\mathcal{D}) = \mathcal{D}$, that is,

$$\mathrm{Aut}(\mathcal{A},\mathcal{D}) := \{\theta \in \mathrm{Aut}(\mathcal{A}) \mid \theta(\mathcal{D}) = \mathcal{D}\}.$$

We denote by $U(\mathcal{A},\mathcal{D})$ the group of unitaries $u \in M(\mathcal{A})$ satisfying $u\mathcal{D}u^* = \mathcal{D}$. We denote by $\mathrm{Ad}(u)$ the automorphism of $(\mathcal{A},\mathcal{D})$ defined by $\mathrm{Ad}(u)(a) = uau^*$ for $a \in \mathcal{A}$. Let us denote by $\mathrm{Int}(\mathcal{A},\mathcal{D})$ the subgroup of $\mathrm{Aut}(\mathcal{A},\mathcal{D})$ consisting of such automorphisms of $(\mathcal{A},\mathcal{D})$. Each $\theta \in \mathrm{Aut}(\mathcal{A},\mathcal{D})$ can be extended to an automorphism of $M(\mathcal{A})$ in a unique way by [7] and denoted by θ. Hence $\mathrm{Int}(\mathcal{A},\mathcal{D})$ is a normal subgroup of $\mathrm{Aut}(\mathcal{A},\mathcal{D})$. By the preceding lemma, we have an antihomomorphism

$$\theta \in \mathrm{Aut}(\mathcal{A},\mathcal{D}) \longrightarrow [X_\theta] \in \mathrm{Pic}(\mathcal{A},\mathcal{D}).$$

Proposition 7.5 and Corollary 7.6 below are achieved in a similar manner to Brown, Green, and Rieffel's argument [5, Proposition 3.1] and [5, Corollary 3.2], respectively. We will give the proofs for the sake of completeness. We provide a lemma below.

Lemma 7.4. *Let $X_0 = \mathcal{A}$ be the identity element of the group* $\mathrm{Pic}(\mathcal{A},\mathcal{D})$ *which has a natural $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodule structure. Then for $u \in U(\mathcal{A},\mathcal{D})$, the correspondence*

$$\varphi_u : x \in X_0 \longrightarrow xu \in X_{\mathrm{Ad}(u)}$$

yields an isomorphism of $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodules.

Proof. For $a, b \in \mathcal{A}$ and $x, y \in X_0$, we have

$$\varphi_u(axb) = axbu = axu\cdot u^*bu = axu\,\mathrm{Ad}(u)^{-1}(b) = a\varphi_u(x)b$$

and

$$\begin{gathered}
{}_{\mathcal{A}}\langle \varphi_u(x) \mid \varphi_u(y)\rangle = \varphi_u(x)\varphi_u(y)^* = xu(yu)^* = xy^* = {}_{\mathcal{A}}\langle x \mid y\rangle, \\
\langle \varphi_u(x) \mid \varphi_u(y)\rangle_{\mathcal{A}} = \mathrm{Ad}(u)(\varphi_u(x)^*\varphi_u(y)) = u((xu)^*(yu))u^* = x^*y = \langle x \mid y\rangle_{\mathcal{A}}.
\end{gathered}$$

Hence, $\varphi_u : x \in X_0 \longrightarrow xu \in X_{\mathrm{Ad}(u)}$ gives rise to an isomorphism of $\mathcal{A}$–$\mathcal{A}$-bimodules, which yields an isomorphism of $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodules. ☐

Proposition 7.5 (cf. [5, Proposition 3.1])**.** *The kernel of the antihomomorphism from* $\mathrm{Aut}(\mathcal{A},\mathcal{D})$ *into* $\mathrm{Pic}(\mathcal{A},\mathcal{D})$ *is exactly* $\mathrm{Int}(\mathcal{A},\mathcal{D})$*. That is, we have an exact sequence*

$$1 \longrightarrow \mathrm{Int}(\mathcal{A},\mathcal{D}) \longrightarrow \mathrm{Aut}(\mathcal{A},\mathcal{D}) \longrightarrow \mathrm{Pic}(\mathcal{A},\mathcal{D}).$$

Proof. The identity element of the group $\mathrm{Pic}(\mathcal{A},\mathcal{D})$ is $X_0 = \mathcal{A}$ with a natural $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodule. For any $u \in U(\mathcal{A},\mathcal{D})$, Lemma 7.4 says that the map $\varphi_u : x \in X_0 \longrightarrow xu \in X_{\mathrm{Ad}(u)}$ gives rise to an isomorphism of an $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodule so that $[X_0] = [X_{\mathrm{Ad}(u)}]$. Hence $\mathrm{Ad}(u)$ belongs to the kernel of the antihomomorphism $\mathrm{Aut}(\mathcal{A},\mathcal{D}) \longrightarrow \mathrm{Pic}(\mathcal{A},\mathcal{D})$.

Conversely, for $\theta \in \mathrm{Aut}(\mathcal{A},\mathcal{D})$, suppose that X_θ represents the identity element of $\mathrm{Pic}(\mathcal{A},\mathcal{D})$, which means that X_θ is equivalent to $X_0 = \mathcal{A}$. Hence one may take an isomorphism $\xi : X_0 \longrightarrow X_\theta$ as $(\mathcal{A},\mathcal{D})$–$(\mathcal{A},\mathcal{D})$-relative imprimitivity bimodules. It satisfies for $a, b \in \mathcal{A}$, $x, y \in X_0$

$$\xi(ax) = a\xi(x), \qquad \xi(xb) = \xi(x)\theta^{-1}(b), \tag{7.2}$$

$$\xi(x)\xi(y)^* = xy^*, \qquad \theta(\xi(x)^*\xi(y)) = x^*y. \tag{7.3}$$

We then have by (7.2), $\xi(x)y = \xi(x\theta(y)) = x\xi(\theta(y))$ for $x, y \in \mathcal{A}$, so that the pair $(\xi \circ \theta, \xi)$ gives rise to a double centralizer of $\mathcal{A}$ which is regarded as an element of $M(\mathcal{A})$ denoted by u (cf. [41, Proposition 2.2.11]). This means that $\xi(x) = xu$, $(\xi \circ \theta)(y) = uy$ for $x, y \in \mathcal{A}$. Equation (7.3) implies that $xuu^*y^* = xy^*$ for $x, y \in \mathcal{A}$ so that $uu^* = 1$. Now ξ preserves the right $\mathcal{A}$-valued inner product so that $\theta(u^*x^*yu) = x^*y$ and hence $u^*x^*yu = \theta^{-1}(x^*y)$ for all $x, y \in \mathcal{A}$. This implies that $u^*au = \theta^{-1}(a)$ for all $a \in \mathcal{A}$. By taking an approximate unit $\{a_n\}$ in $\mathcal{A}$, we see that $u^*u = 1$. Therefore, we obtain a unitary $u \in M(\mathcal{A})$. Since $\theta(\mathcal{D}) = \mathcal{D}$, we have $u \in U(\mathcal{A},\mathcal{D})$ and $\mathrm{Ad}(u) \in \mathrm{Int}(\mathcal{A},\mathcal{D})$, thus proving that the sequence

$$1 \longrightarrow \mathrm{Int}(\mathcal{A},\mathcal{D}) \longrightarrow \mathrm{Aut}(\mathcal{A},\mathcal{D}) \longrightarrow \mathrm{Pic}(\mathcal{A},\mathcal{D})$$

is exact. ☐

Corollary 7.6 (cf. [5, Corollary 3.2])**.** *Let* $(\mathcal{A}_1,\mathcal{D}_1)$ *and* $(\mathcal{A}_2,\mathcal{D}_2)$ *be relative σ-unital pairs of* C^**-algebras. Let* $\alpha, \beta : (\mathcal{A}_1,\mathcal{D}_1) \longrightarrow (\mathcal{A}_2,\mathcal{D}_2)$ *be isomorphisms. If* X_α *and* X_β *are equivalent, then there exists a unitary* $u \in U(\mathcal{A},\mathcal{D})$ *such that* $\beta = \mathrm{Ad}(u) \circ \alpha$*.*

Proof. Since $[X_\alpha] = [X_\beta]$, we have

$$\mathrm{id} = [X_\alpha]^{-1}[X_\beta] = [X_{\alpha^{-1}} \otimes_{\mathcal{A}_1} X_\beta] = [X_{\beta\circ\alpha^{-1}}].$$

Hence $\beta^{-1} \circ \alpha \in \mathrm{Int}(\mathcal{A}_2,\mathcal{D}_2)$ so that there exists a unitary $u \in U(\mathcal{A}_2,\mathcal{D}_2)$ such that $\beta^{-1} \circ \alpha = \mathrm{Ad}(u)$. ☐

The following lemma is also a relative version of [5, Lemma 3.3].

Lemma 7.7 (cf. [5, Lemma 3.3])**.** *Let* X *be an* $(\mathcal{A}_1,\mathcal{D}_1)$–$(\mathcal{A}_2,\mathcal{D}_2)$*-relative imprimitivity bimodule. Let* $(\mathcal{A}_0,\mathcal{D}_0)$ *be the linking pair of* X *defined by* (4.1) *and* (4.2)*.*

Then X is equivalent to X_θ for some isomorphism $\theta : (\mathcal{A}_1, \mathcal{D}_1) \longrightarrow (\mathcal{A}_2, \mathcal{D}_2)$ if and only if there exists a partial isometry $v \in M(\mathcal{A}_0)$ such that

$$v^*v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad vv^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.4}$$

and

$$v\mathcal{D}_0 v^* = \mathcal{D}_0 vv^*, \qquad v^*\mathcal{D}_0 v = \mathcal{D}_0 v^* v. \tag{7.5}$$

In this case, θ is defined by $\theta(a) = vav^*$, $a \in \mathcal{A}_1$.

Remark 7.8. Under equality (7.4), the second equality of (7.5) follows from the first equality of (7.5), because the first equality of (7.5) ensures the equality

$$v^*v\mathcal{D}_0 v^* v = v^*\mathcal{D}_0 vv^* v. \tag{7.6}$$

By (7.4), v^*v commutes with any elements of $\mathcal{D}_0$ so that (7.6) goes to the second equality of (7.5).

Proof of Lemma 7.7. Although the proof basically follows the proof of [5, Lemma 3.3], we give it here for the sake of completeness. Suppose that X is equivalent to X_θ for some isomorphism $\theta : (\mathcal{A}_1, \mathcal{D}_1) \longrightarrow (\mathcal{A}_2, \mathcal{D}_2)$. By this isomorphism, the linking algebra $\mathcal{A}_0$ of X is identified with that of X_θ. Hence, $X_\theta = \mathcal{A}_1$ and

$$\mathcal{A}_0 = \left\{ \begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix} \mid a_1 \in \mathcal{A}_1,\, a_2 \in \mathcal{A}_2,\, x, y \in X_\theta \right\}.$$

We define operators v, v^* on $X_\theta \oplus \mathcal{A}_2$ by

$$v \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta(z) \end{bmatrix}, \qquad v^* \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} \theta^{-1}(c_2) \\ 0 \end{bmatrix} \qquad \text{for } z \in X_\theta, c_2 \in \mathcal{A}_2,$$

where X_θ is identified with $\mathcal{A}_1$ so that $\theta(z) \in \mathcal{A}_2$ and $\theta^{-1}(c_2) \in X_\theta$. Put

$$R_v\left(\begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix} v, \qquad L_v\left(\begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix}\right) = v \begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix}.$$

For $z \in X_\theta, c \in \mathcal{A}_2$, we have

$$R_v\left(\begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix}\right) \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix} \begin{bmatrix} 0 \\ \theta(z) \end{bmatrix} = \begin{bmatrix} x \cdot \theta(z) \\ a_2 \cdot \theta(z) \end{bmatrix}.$$

Since $z \in X_\theta$ is regarded as an element of $\mathcal{A}_1$, the first component of the right-hand side above is exactly $x\theta^{-1}(\theta(z)) = xz$ because of the definition of the right $\mathcal{A}_1$-module structure of X_θ as in (3.5). The second component $a_2 \cdot \theta(z)$ equals $a_2\theta(z)$. On the other hand, we have

$$\begin{bmatrix} x & 0 \\ \overline{\theta^{-1}(a_2)} & 0 \end{bmatrix} \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} xz \\ \langle \theta^{-1}(a_2) \mid z \rangle_{\mathcal{A}_2} \end{bmatrix} = \begin{bmatrix} xz \\ a_2\theta(z) \end{bmatrix}$$

so that

$$R_v\left(\begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix}\right) = \begin{bmatrix} x & 0 \\ \overline{\theta^{-1}(a_2)} & 0 \end{bmatrix} \in \mathcal{A}_0.$$

Similarly, we see that

$$\begin{aligned} L_v\left(\begin{bmatrix} a_1 & x \\ \bar{y} & a_2 \end{bmatrix}\right) \begin{bmatrix} z \\ c_2 \end{bmatrix} &= v \begin{bmatrix} a_1 z + x \cdot c_2 \\ \langle y \mid z \rangle_{\mathcal{A}_2} + a_2 c_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \theta(a_1 z + x\theta^{-1}(c_2)) \end{bmatrix} = \begin{bmatrix} 0 \\ \theta(a_1 z) + \theta(x)c_2 \end{bmatrix} \end{aligned}$$

and

$$\begin{bmatrix} 0 & 0 \\ \overline{a}_1 & \theta(x) \end{bmatrix} \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \langle a_1 \mid z \rangle_{\mathcal{A}_2} + \theta(x) c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta(a_1 z) + \theta(x) c_2 \end{bmatrix}$$

so that

$$L_v \left(\begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ \overline{a}_1 & \theta(x) \end{bmatrix} \in \mathcal{A}_0.$$

Hence, both R_v and L_v give rise to operators on $\mathcal{A}_0$. Since the identity

$$R_v \left(\begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix} \right) \begin{bmatrix} a_1' & x' \\ \overline{y}' & a_2' \end{bmatrix} = \begin{bmatrix} a_1 & x \\ \overline{y} & a_2 \end{bmatrix} L_v \left(\begin{bmatrix} a_1' & x' \\ \overline{y}' & a_2' \end{bmatrix} \right)$$

hold, the pair (L_v, R_v) becomes a double centralizer of $\mathcal{A}_0$, which defines an element of $M(\mathcal{A}_0)$ (cf. [41, Proposition 2.2.11]). Similarly (L_{v^*}, R_{v^*}) defines an element of $M(\mathcal{A}_0)$ such that $(L_v, R_v)^* = (L_{v^*}, R_{v^*})$, so that we may write $(L_v, R_v) = v$. It then follows that

$$v^* v \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} z \\ 0 \end{bmatrix} \quad \text{and hence} \quad v^* v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$v v^* \begin{bmatrix} z \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \quad \text{and hence} \quad v v^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For $a_1 \in \mathcal{A}_1, z \in X_\theta, c_2 \in \mathcal{A}_2$, we have

$$\left(v \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} v^* \right) \begin{bmatrix} z \\ c_2 \end{bmatrix} = v \begin{bmatrix} a_1 \theta^{-1}(c_2) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta(a_1) c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \theta(a_1) \end{bmatrix} \begin{bmatrix} z \\ c_2 \end{bmatrix}$$

so that

$$v \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} v^* = \begin{bmatrix} 0 & 0 \\ 0 & \theta(a_1) \end{bmatrix}.$$

This means that $\theta(a_1) = v a_1 v^*$ for $a_1 \in \mathcal{A}_1$ under the identification between $\mathcal{A}_1$ and $\begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $\theta : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ satisfies $\theta(\mathcal{D}_1) = \mathcal{D}_2$ and $\mathcal{D}_1 = \mathcal{D}_0 v^* v$, $\mathcal{D}_2 = \mathcal{D}_0 v v^*$, we have

$$v \mathcal{D}_0 v^* = v \mathcal{D}_0 v^* v v^* = v \mathcal{D}_1 v^* = \theta(\mathcal{D}_1) = \mathcal{D}_2 = \mathcal{D}_0 v v^*$$

and hence

$$v^* \mathcal{D}_0 v = v^* (\mathcal{D}_0 v v^*) v = v^* (v \mathcal{D}_0 v^*) v = \mathcal{D}_0 v^* v.$$

Conversely, suppose that a partial isometry $v \in M(\mathcal{A}_0)$ satisfies the equalities (7.4) and (7.5). For $a \in \mathcal{A}_1$, the equalities

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} v \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} v^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = v v^* v \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} v^* v v^* = v \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} v^*$$

hold, so that there exists an element $\theta(a)$ in $\mathcal{A}_2$ for each $a \in \mathcal{A}_1$ such that

$$v \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} v^* = \begin{bmatrix} 0 & 0 \\ 0 & \theta(a) \end{bmatrix},$$

and the correspondence $a \in \mathcal{A}_1 \longrightarrow \theta(a) \in \mathcal{A}_2$ gives rise to an isomorphism of C^*-algebras. The conditions (7.4) and (7.5) imply that $v \mathcal{D}_0 v^* = \mathcal{D}_0 v v^* = \mathcal{D}_2$ and $v^* \mathcal{D}_0 v = \mathcal{D}_0 v^* v = \mathcal{D}_1$ so that we have $v \mathcal{D}_1 v^* = v v^* \mathcal{D}_0 v v^* = \mathcal{D}_2$. This implies that $\theta(\mathcal{D}_1) = \mathcal{D}_2$.

We will next see that X is equivalent to X_θ. We identify $\mathcal{A}_1$ with its image in $\mathcal{A}_0$, and then we will define a map $\eta : X \longrightarrow \mathcal{A}_1 (= X_\theta)$ by

$$\eta(x) := \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} v \qquad \text{for } x \in X.$$

Since

$$v^* v \eta(x) v^* v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} v v^* v = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} v = \eta(x),$$

we see that $\eta(x) \in \mathcal{A}_1$. By a routine calculation, we know that η is a bimodule homomorphism from X to X_θ which preserves both inner products, and hence η gives rise to an isomorphism between X and X_θ. □

The following theorem is also a relative version of a Brown–Green–Rieffel theorem [5, Theorem 3.4]. We will give its proof for the sake of completeness.

Theorem 7.9 (cf. [5, Theorem 3.4])**.** *Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be relative σ-unital pairs of C^*-algebras. Let X be an $(\mathcal{A}_1 \otimes \mathcal{K}, \mathcal{D}_1 \otimes \mathcal{C})$–$(\mathcal{A}_2 \otimes \mathcal{K}, \mathcal{D}_2 \otimes \mathcal{C})$-relative imprimitivity bimodule. Then there exists an isomorphism $\theta : \mathcal{A}_1 \otimes \mathcal{K} \longrightarrow \mathcal{A}_2 \otimes \mathcal{K}$ satisfying $\theta(\mathcal{D}_1 \otimes \mathcal{C}) = \mathcal{D}_2 \otimes \mathcal{C}$ such that X is equivalent to X_θ. Furthermore, θ is unique up to left multiplication by an element of $\mathrm{Int}(\mathcal{A}_2 \otimes \mathcal{K}, \mathcal{D}_2 \otimes \mathcal{C})$, that is if X is equivalent to X_φ for some isomorphism $\varphi : (\mathcal{A}_1 \otimes \mathcal{K}, \mathcal{D}_1 \otimes \mathcal{C}) \longrightarrow (\mathcal{A}_2 \otimes \mathcal{K}, \mathcal{D}_2 \otimes \mathcal{C})$, then there exists a unitary $u \in U(\mathcal{A}_2 \otimes \mathcal{K}, \mathcal{D}_2 \otimes \mathcal{C})$ such that $\varphi = \mathrm{Ad}(u) \circ \theta$.*

Proof. The uniqueness follows immediately from Corollary 7.6.

Now let X be an $(\mathcal{A}_1 \otimes \mathcal{K}, \mathcal{D}_1 \otimes \mathcal{C})$–$(\mathcal{A}_2 \otimes \mathcal{K}, \mathcal{D}_2 \otimes \mathcal{C})$-relative imprimitivity bimodule. We put $\bar{\mathcal{A}}_i = \mathcal{A}_i \otimes \mathcal{K}$, $\bar{\mathcal{D}}_i = \mathcal{D}_i \otimes \mathcal{C}$ for $i = 1, 2$. Let $(\bar{\mathcal{A}}_0, \bar{\mathcal{D}}_0)$ be the linking pair for X defined from $\bar{\mathcal{A}}_i, \bar{\mathcal{D}}_i, i = 1, 2$ and X by (4.1) and (4.2). By the assumption that $(\bar{\mathcal{A}}_1, \bar{\mathcal{D}}_1) \underset{\mathrm{RME}}{\sim} (\bar{\mathcal{A}}_2, \bar{\mathcal{D}}_2)$ with Proposition 4.6 and Theorem 4.7, we know that there exists $v_i \in M(\bar{\mathcal{A}} \otimes \mathcal{K}), i = 1, 2$, such that

$$v_i^* v_i = 1 \otimes 1 \quad \text{in} \quad M(\bar{\mathcal{A}}_0 \otimes \mathcal{K}), \quad i = 1, 2,$$

$$v_1 v_1^* = P_1 \otimes 1 \quad \text{where } P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{in } M(\bar{\mathcal{A}}_0),$$

$$v_2 v_2^* = P_2 \otimes 1 \quad \text{where } P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{in } M(\bar{\mathcal{A}}_0)$$

and

$$v_i(\bar{\mathcal{D}}_0 \otimes \mathcal{C}) v_i^* = \bar{\mathcal{D}}_i \otimes \mathcal{C}, \qquad v_i^*(\bar{\mathcal{D}}_i \otimes \mathcal{C}) v_i = \bar{\mathcal{D}}_0 \otimes \mathcal{C}, \qquad i = 1, 2.$$

Put a partial isometry $w = v_2 v_1^* \in M(\bar{\mathcal{A}} \otimes \mathcal{K})$ so that we have

$$w^* w = P_1 \otimes 1 = \begin{bmatrix} 1 \otimes 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad w w^* = P_2 \otimes 1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{bmatrix} \quad \text{in } M(\bar{\mathcal{A}}_0 \otimes \mathcal{K})$$

and

$$w(\bar{\mathcal{D}}_1 \otimes \mathcal{C}) w^* = \bar{\mathcal{D}}_2 \otimes \mathcal{C}, \qquad w^*(\bar{\mathcal{D}}_2 \otimes \mathcal{C}) w = \bar{\mathcal{D}}_1 \otimes \mathcal{C}.$$

Let $p \in \mathcal{C}$ be the rank one projection $p = e_{1,1}$ on $\ell^2(\mathbb{N})$. Regard $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ as the C^*-algebra of compact operators on $\ell^2(\mathbb{N})$. As $p\mathcal{K}p \cong \mathcal{C}p \cong \mathbb{C}$ and hence

$p\mathcal{K}p\otimes\mathcal{K}\cong\mathcal{K}$, $\mathcal{C}p\otimes\mathcal{C}\cong\mathcal{C}$, one may find an isometry t on $\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N})\cong\ell^2(\mathbb{N}\times\mathbb{N})$ such that $tt^*=p\otimes 1$ and $t^*t=1\otimes 1$ and

$$t(\mathcal{K}\otimes\mathcal{K})t^*=p\mathcal{K}p\otimes\mathcal{K},\qquad t^*(p\mathcal{K}p\otimes\mathcal{K})t=\mathcal{K}\otimes\mathcal{K},$$
$$t(\mathcal{C}\otimes\mathcal{C})t^*=\mathcal{C}p\otimes\mathcal{C},\qquad t^*(\mathcal{C}p\otimes\mathcal{C})t=\mathcal{C}\otimes\mathcal{C}.$$

Put $\bar{v}_1=1\otimes t\in M(\bar{\mathcal{A}}_1\otimes\mathcal{K}\otimes\mathcal{K})$ so that

$$\bar{v}_1^*\bar{v}_1=1\otimes 1\otimes 1,\qquad \bar{v}_1\bar{v}_1^*=1\otimes p\otimes 1.$$

By the construction of $\bar{v}_1$, we see that

$$\bar{v}_1(\bar{\mathcal{D}}_1\otimes\mathcal{C}\otimes\mathcal{C})\bar{v}_1^*=\bar{\mathcal{D}}_1\otimes\mathcal{C}p\otimes\mathcal{C},\qquad \bar{v}_1^*(\bar{\mathcal{D}}_1\otimes\mathcal{C}p\otimes\mathcal{C})\bar{v}_1=\bar{\mathcal{D}}_1\otimes\mathcal{C}\otimes\mathcal{C}.$$

We identify $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{D}}_1$ with $\bar{\mathcal{A}}_1\otimes\mathbb{C}\otimes\mathcal{K}$ and $\bar{\mathcal{D}}_1\otimes\mathbb{C}\otimes\mathcal{C}$, respectively, so that we have $\bar{v}_1\in M(\bar{\mathcal{A}}_1\otimes\mathcal{K})$ and

$$\bar{v}_1^*\bar{v}_1=1\otimes 1,\qquad \bar{v}_1\bar{v}_1^*=1\otimes p,$$
$$\bar{v}_1(\bar{\mathcal{D}}_1\otimes\mathcal{C})\bar{v}_1^*=\bar{\mathcal{D}}_1\otimes\mathcal{C}p,\qquad \bar{v}_1^*(\bar{\mathcal{D}}_1\otimes\mathcal{C}p)\bar{v}_1=\bar{\mathcal{D}}_1\otimes\mathcal{C}.$$

Similarly we have $\bar{v}_2\in M(\bar{\mathcal{A}}_2\otimes\mathcal{K})$ and

$$\bar{v}_2^*\bar{v}_2=1\otimes 1,\qquad \bar{v}_2\bar{v}_2^*=1\otimes p,$$
$$\bar{v}_2(\bar{\mathcal{D}}_2\otimes\mathcal{C})\bar{v}_2^*=\bar{\mathcal{D}}_2\otimes\mathcal{C}p,\qquad \bar{v}_2^*(\bar{\mathcal{D}}_2\otimes\mathcal{C}p)\bar{v}_2=\bar{\mathcal{D}}_2\otimes\mathcal{C}.$$

Define $\bar{v}\in M(\bar{\mathcal{A}}_0\otimes\mathcal{K})$ by

$$\bar{v}=\begin{bmatrix}0&0\\0&\bar{v}_2\end{bmatrix}w\begin{bmatrix}\bar{v}_1^*&0\\0&0\end{bmatrix}\quad\text{in } M(\bar{\mathcal{A}}_0\otimes\mathcal{K}).$$

We then have

$$\begin{aligned}\bar{v}^*\bar{v}&=\begin{bmatrix}\bar{v}_1&0\\0&0\end{bmatrix}w^*\begin{bmatrix}0&0\\0&1\otimes 1\end{bmatrix}w\begin{bmatrix}\bar{v}_1^*&0\\0&0\end{bmatrix}\\&=\begin{bmatrix}\bar{v}_1&0\\0&0\end{bmatrix}w^*w\begin{bmatrix}\bar{v}_1^*&0\\0&0\end{bmatrix}\\&=\begin{bmatrix}1\otimes p&0\\0&0\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\bar{v}\bar{v}^*&=\begin{bmatrix}0&0\\0&\bar{v}_2\end{bmatrix}w\begin{bmatrix}1\otimes 1&0\\0&0\end{bmatrix}w^*\begin{bmatrix}0&0\\0&\bar{v}_2^*\end{bmatrix}\\&=\begin{bmatrix}0&0\\0&\bar{v}_2\end{bmatrix}ww^*\begin{bmatrix}0&0\\0&\bar{v}_2^*\end{bmatrix}\\&=\begin{bmatrix}0&0\\0&1\otimes p\end{bmatrix}.\end{aligned}$$

We will next show that $\bar{v}(\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}^*=(\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}\bar{v}^*$. For $\begin{bmatrix}d_1&0\\0&d_2\end{bmatrix}\in\bar{\mathcal{D}}_0$ with $d_i\in\bar{\mathcal{D}}_i, i=1,2$, we have

$$\bar{v}\begin{bmatrix}d_1\otimes p&0\\0&d_2\otimes p\end{bmatrix}\bar{v}^*=\begin{bmatrix}0&0\\0&\bar{v}_2\end{bmatrix}w\begin{bmatrix}\bar{v}_1^*(d_1\otimes p)\bar{v}_1&0\\0&0\end{bmatrix}w^*\begin{bmatrix}0&0\\0&\bar{v}_2^*\end{bmatrix}.$$

Since $\bar{v}_1^*(d_1\otimes p)\bar{v}_1 \in \bar{\mathcal{D}}_1\otimes\mathcal{C}$, we have $w\begin{bmatrix}\bar{v}_1^*(d_1\otimes p)\bar{v}_1 & 0\\ 0 & 0\end{bmatrix}w^* \in w(\bar{\mathcal{D}}_1\otimes\mathcal{C})w^* = \bar{\mathcal{D}}_2\otimes\mathcal{C}$ so that

$$\bar{v}\begin{bmatrix}d_1\otimes p & 0\\ 0 & d_2\otimes p\end{bmatrix}\bar{v}^* \in \bar{v}_2(\bar{\mathcal{D}}_2\otimes\mathcal{C})\bar{v}_2^* = \bar{\mathcal{D}}_2\otimes\mathcal{C}p = (\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}\bar{v}^*.$$

Therefore, we have $\bar{v}(\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}^* \subset (\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}\bar{v}^*$ and, similarly, $\bar{v}^*(\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v} \subset (\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}^*\bar{v}$ so that we have

$$\bar{v}(\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}^* = (\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}\bar{v}^* \quad\text{and}\quad \bar{v}^*(\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v} = (\bar{\mathcal{D}}_0\otimes\mathcal{C}p)\bar{v}^*\bar{v}.$$

By the equalities

$$\bar{v}^*\bar{v} = \begin{bmatrix}1\otimes p & 0\\ 0 & 0\end{bmatrix}, \qquad \bar{v}\bar{v}^* = \begin{bmatrix}0 & 0\\ 0 & 1\otimes p\end{bmatrix},$$

we know that $\bar{v}$ commutes with $1\otimes p$ so that we can regard $\bar{v}$ as an element of $M(\bar{\mathcal{A}}_0\otimes p\mathcal{K}p) = M(\bar{\mathcal{A}}_0)$. Thus, we obtain a partial isometry $\bar{v}$ in $M(\bar{\mathcal{A}}_0)$ such that

$$\bar{v}^*\bar{v} = \begin{bmatrix}1_{\bar{\mathcal{A}}_1} & 0\\ 0 & 0\end{bmatrix}, \qquad \bar{v}\bar{v}^* = \begin{bmatrix}0 & 0\\ 0 & 1_{\bar{\mathcal{A}}_2}\end{bmatrix},$$

and

$$\bar{v}\bar{\mathcal{D}}_0\bar{v}^* = \bar{\mathcal{D}}_0\bar{v}\bar{v}^* \quad\text{and}\quad \bar{v}^*\bar{\mathcal{D}}_0\bar{v} = \bar{\mathcal{D}}_0\bar{v}^*\bar{v}.$$

Therefore, by Lemma 7.7, we conclude that X is equivalent to X_θ for some isomorphism $\theta : (\bar{\mathcal{A}}_1, \bar{\mathcal{D}}_1) \longrightarrow (\bar{\mathcal{A}}_2, \bar{\mathcal{D}}_2)$. ☐

Recall that subgroups $\mathrm{Aut}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})$, $\mathrm{Int}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})$ of automorphism group $\mathrm{Aut}(\mathcal{A}\otimes\mathcal{K})$ are defined by

$$\mathrm{Aut}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C}) = \{\beta\in\mathrm{Aut}(\mathcal{A}\otimes\mathcal{K}) \mid \beta(\mathcal{D}\otimes\mathcal{C}) = \mathcal{D}\otimes\mathcal{C}\},$$
$$\mathrm{Int}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C}) = \{\beta\in\mathrm{Int}(\mathcal{A}\otimes\mathcal{K}) \mid \beta(\mathcal{D}\otimes\mathcal{C}) = \mathcal{D}\otimes\mathcal{C}\}.$$

Corollary 7.10. *Let $(\mathcal{A}, \mathcal{D})$ be a relative σ-unital pair of C^*-algebras. For any relative imprimitivity bimodule $[X] \in \mathrm{Pic}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})$, there exists an automorphism $\theta\in\mathrm{Aut}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})$ such that $[X] = [X_\theta]$. Thus, we have an exact sequence*

$$1 \longrightarrow \mathrm{Int}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C}) \longrightarrow \mathrm{Aut}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C}) \longrightarrow \mathrm{Pic}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C}) \longrightarrow 1.$$

Let us denote by $\mathrm{Out}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})$ the quotient group $\mathrm{Aut}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})/\mathrm{Int}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})$. We then have

Corollary 7.11. *Let $(\mathcal{A}, \mathcal{D})$ be a relative σ-unital pair of C^*-algebras. We have*

$$\mathrm{Pic}(\mathcal{A}, \mathcal{D}) = \mathrm{Out}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C}).$$

Proof. By Lemma 7.2, we see that $\mathrm{Pic}(\mathcal{A}, \mathcal{D}) = \mathrm{Pic}(\mathcal{A}\otimes\mathcal{K}, \mathcal{D}\otimes\mathcal{C})$ so that we have the desired equality by the preceding corollary. ☐

8. Relative Picard groups of Cuntz–Krieger pairs

In this section, we will study the relative Picard group $\mathrm{Pic}(\mathcal{A}, \mathcal{D})$ for the Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$. We are assuming that the matrix A is irreducible and non-permutation. By [22, Lemma 1.1], for a unitary $u\in M(\mathcal{O}_A\otimes\mathcal{K})$, the automorphism $\mathrm{Ad}(u)$ acts trivially on $K_0(\mathcal{O}_A\otimes\mathcal{K})$. We will first show the following proposition which is a relative version of [22, Lemma 3.13] (Lemma 9.1 in Appendix A).

Proposition 8.1. *Let* $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ *satisfy* $\beta(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}$*, and let* $\beta_* = \mathrm{id}$ *on* $K_0(\mathcal{O}_A)$*. Then there exists a unitary* $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ *and an automorphism* $\alpha \in \mathrm{Aut}(\mathcal{O}_A)$ *such that*

$$\beta = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}) \quad \text{and} \quad \alpha_* = \mathrm{id} \ \text{on } K_0(\mathcal{O}_A),$$
$$u(\mathcal{D}_A \otimes \mathcal{C})u^* = \mathcal{D}_A \otimes \mathcal{C}, \qquad \alpha(\mathcal{D}_A) = \mathcal{D}_A.$$

To show the above proposition, we provide several lemmas.

Lemma 8.2. *Let* $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ *satisfy* $\beta(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}$*, and let* $\beta_* = \mathrm{id}$ *on* $K_0(\mathcal{O}_A)$*. Then for each* $k \in \mathbb{N}$*, there exists a partial isometry* $w_k \in \mathcal{O}_A \otimes \mathcal{K}$ *such that*

$$w_k^* w_k = 1 \otimes e_{k,k}, \qquad w_k w_k^* = \beta(1 \otimes e_{k,k}), \tag{8.1}$$
$$w_k(\mathcal{D}_A \otimes \mathcal{C})w_k^* \subset \mathcal{D}_A \otimes \mathcal{C}, \qquad w_k^*(\mathcal{D}_A \otimes \mathcal{C})w_k \subset \mathcal{D}_A \otimes \mathcal{C}. \tag{8.2}$$

Proof. Let us denote by $N_s(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ the normalizer semigroup

$$\{v \in \mathcal{O}_A \otimes \mathcal{K} \mid v \text{ is a partial isometry}; v(\mathcal{D}_A \otimes \mathcal{C})v^* \subset \mathcal{D}_A \otimes \mathcal{C},\ v^*(\mathcal{D}_A \otimes \mathcal{C})v \subset \mathcal{D}_A \otimes \mathcal{C}\}$$

of partial isometries in $\mathcal{O}_A \otimes \mathcal{K}$. Denote by $K_0(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ the Murray–von Neumann equivalence classes of projections in $\mathcal{D}_A \otimes \mathcal{C}$ by partial isometries in $N_s(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$. It has been proved in [27, Proposition 3.6] that there exists a natural isomorphism between $K_0(\mathcal{O}_A)$ and $K_0(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$. Since $[\beta(1 \otimes e_{k,k})] = \beta_*([1 \otimes e_{k,k}]) = [1 \otimes e_{k,k}]$, we have $\beta(1 \otimes e_{k,k}) \sim 1 \otimes e_{k,k}$ in $K_0(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$. We may find a partial isometry $w_k \in \mathcal{O}_A \otimes \mathcal{K}$ satisfying the desired conditions. □

Lemma 8.3. *Let* $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ *satisfy* $\beta(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}$*, and let* $\beta_* = \mathrm{id}$ *on* $K_0(\mathcal{O}_A)$*. Then there exists a unitary* $w \in M(\mathcal{O}_A \otimes \mathcal{K})$ *such that*

$$(\mathrm{Ad}(w^*) \circ \beta)(1 \otimes e_{k,k}) = 1 \otimes e_{k,k},$$
$$(\mathrm{Ad}(w^*) \circ \beta)(\mathcal{O}_A \otimes e_{k,k}) = \mathcal{O}_A \otimes e_{k,k},$$
$$(\mathrm{Ad}(w^*) \circ \beta)(\mathcal{D}_A \otimes e_{k,k}) = \mathcal{D}_A \otimes e_{k,k}$$

for all $k \in \mathbb{N}$*.*

Proof. Take a partial isometry $w_k \in \mathcal{O}_A \otimes \mathcal{K}$ for each $k \in \mathbb{N}$ satisfying (8.1) and (8.2). It is easy to see that the summation $\sum_{k=1}^\infty w_k$ converges to an element w in $M(\mathcal{O}_A \otimes \mathcal{K})$ in the strict topology of $M(\mathcal{O}_A \otimes \mathcal{K})$. By (8.1) and (8.2), we have $w^*w = ww^* = 1$ and $w(\mathcal{D}_A \otimes \mathcal{C})w^* = w^*(\mathcal{D}_A \otimes \mathcal{C})w = \mathcal{D}_A \otimes \mathcal{C}$. We then see that

$$w(1 \otimes e_{k,k})w^* = ww_k^*w_kw^* = w_kw_k^* = \beta(1 \otimes e_{k,k})$$

so that $(\mathrm{Ad}(w^*) \circ \beta)(1 \otimes e_{k,k}) = 1 \otimes e_{k,k}$. For $x \in \mathcal{O}_A$, we have

$$\begin{aligned}(\mathrm{Ad}(w^*) \circ \beta)(x \otimes e_{k,k}) =&(\mathrm{Ad}(w^*) \circ \beta)((1 \otimes e_{k,k})(x \otimes e_{k,k})(1 \otimes e_{k,k}))\\ =&(1 \otimes e_{k,k}) \cdot (\mathrm{Ad}(w^*) \circ \beta)(x \otimes e_{k,k}) \cdot (1 \otimes e_{k,k})\end{aligned}$$

so that $(\mathrm{Ad}(w^*) \circ \beta)(\mathcal{O}_A \otimes e_{k,k}) = \mathcal{O}_A \otimes e_{k,k}$. As $\beta(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}$ and $w^*(\mathcal{D}_A \otimes \mathcal{C})w = \mathcal{D}_A \otimes \mathcal{C}$, we have $(\mathrm{Ad}(w^*) \circ \beta)(\mathcal{D}_A \otimes e_{k,k}) = \mathcal{D}_A \otimes e_{k,k}$. □

Proof of Proposition 8.1. Suppose that $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ satisfies $\beta(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}$ and $\beta_* = \mathrm{id}$ on $K_0(\mathcal{O}_A)$. Take a unitary $w \in M(\mathcal{O}_A \otimes \mathcal{K})$ satisfying the conditions of Lemma 8.3. Put $\beta_w = \mathrm{Ad}(w^*) \circ \beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$. Since

$(\mathrm{Ad}(w^*)\circ\beta)(\mathcal{O}_A\otimes e_{k,k}) = \mathcal{O}_A\otimes e_{k,k}$, we may find an automorphism $\alpha_k \in \mathrm{Aut}(\mathcal{O}_A)$ for $k\in\mathbb{N}$ such that

$$\alpha_k(x)\otimes e_{k,k} = \beta_w(x\otimes e_{k,k}) \quad \text{for } x\in\mathcal{O}_A.$$

By replacing β_w with β, we may assume that $\beta(x\otimes e_{k,k}) = \alpha_k(x)\otimes e_{k,k}$. For $j,k\in\mathbb{N}$, we have

$$\beta(x\otimes e_{j,k}) = \beta((1\otimes e_{j,k})(x\otimes e_{k,k})) = \beta(1\otimes e_{j,k})\cdot(\alpha_k(x)\otimes e_{k,k}).$$

By putting $x=1$, we see that

$$\beta(1\otimes e_{j,k}) = (1\otimes e_{j,j})\beta(1\otimes e_{j,k})(1\otimes e_{k,k})$$

so that there exists $w_{j,k}\in\mathcal{O}_A$ such that $w_{j,k}^* = w_{k,j}$ and $\beta(1\otimes e_{j,k}) = w_{j,k}\otimes e_{j,k}$. Since

$$w_{j,k}^* w_{j,k}\otimes e_{k,k} = \beta(1\otimes e_{j,k})^*\beta(1\otimes e_{j,k}) = \beta(1\otimes e_{k,k}) = 1\otimes e_{k,k},$$

we have $w_{j,k}^*w_{j,k}=1$ and similarly $w_{j,k}w_{j,k}^*=1$. We also have for $a\in\mathcal{D}_A$

$$\begin{aligned} w_{j,k}aw_{j,k}^*\otimes e_{j,j} =& (w_{j,k}\otimes e_{j,k})(a\otimes e_{k,k})(w_{j,k}\otimes e_{j,k})^* \\ =&\beta((1\otimes e_{j,k})(\alpha_k^{-1}(a)\otimes e_{k,k})(1\otimes e_{k,j})) \\ =&\beta(\alpha_k^{-1}(a)\otimes e_{j,j}) \\ =&\alpha_j(\alpha_k^{-1}(a))\otimes e_{j,j} \end{aligned}$$

so that $w_{j,k}\mathcal{D}_A w_{j,k}^* = \mathcal{D}_A$. Since

$$\beta(x\otimes e_{j,k}) = \beta(1\otimes e_{j,k})\cdot(\alpha_k(x)\otimes e_{k,k}) = w_{j,k}\alpha_k(x)\otimes e_{j,k}$$

and similarly $\beta(x\otimes e_{j,k}) = \alpha_j(x)w_{j,k}\otimes e_{j,k}$, we see $w_{j,k}\alpha_k(x)\otimes e_{j,k} = \alpha_j(x)w_{j,k}\otimes e_{j,k}$ and hence $\alpha_k(x) = w_{j,k}^*\alpha_j(x)w_{j,k}$ for $x\in\mathcal{O}_A$. Put $u=\sum_{k=1}^\infty w_{1,k}\otimes e_{k,k}$, which is easily proved to be a unitary in $M(\mathcal{O}_A\otimes\mathcal{K})$. It then follows that

$$\begin{aligned} \beta(x\otimes e_{j,k}) =&\beta((1\otimes e_{j,1})(x\otimes e_{1,1})(1\otimes e_{1,k})) \\ =&(w_{j,1}\otimes e_{j,1})(\alpha_1(x)\otimes e_{1,1})(w_{1,k}\otimes e_{1,k}) \\ =&(w_{j,1}\alpha_1(x)w_{1,k})\otimes e_{j,k} \\ =&u^*(\alpha_1(x)\otimes e_{j,k})u \end{aligned}$$

for $x\in\mathcal{O}_A$ so that $\beta = \mathrm{Ad}(u^*)\circ(\alpha_1\otimes\mathrm{id})$. Since $w_{1,k}\mathcal{D}_A w_{1,k}^* = \mathcal{D}_A$, we have $u(\mathcal{D}_A\otimes\mathcal{C})u^* = \mathcal{D}_A\otimes\mathcal{C}$. By [22, Lemma 1.1], we know that $\mathrm{Ad}(u)_* = \mathrm{id}$ on $K_0(\mathcal{O}_A)$ so that $\alpha_{1*} = (\beta^{-1})_* = \mathrm{id}$ on $K_0(\mathcal{O}_A)$. □

We thus have the following theorem.

Theorem 8.4. *Let $\beta\in\mathrm{Aut}(\mathcal{O}_A\otimes\mathcal{K})$. Then β satsifies the condition*

$$\beta(\mathcal{D}_A\otimes\mathcal{C}) = \mathcal{D}_A\otimes\mathcal{C} \quad \textit{and } \beta_* = \mathrm{id} \textit{ on } K_0(\mathcal{O}_A) \tag{8.3}$$

if and only if there exists an automorphism $\alpha\in\mathrm{Aut}(\mathcal{O}_A)$ and a unitary $u\in M(\mathcal{O}_A\otimes\mathcal{K})$ such that

$$\beta = \mathrm{Ad}(u)\circ(\alpha\otimes\mathrm{id}) \quad \textit{and} \quad \alpha_* = \mathrm{id} \textit{ on } K_0(\mathcal{O}_A), \tag{8.4}$$

$$u(\mathcal{D}_A\otimes\mathcal{C})u^* = \mathcal{D}_A\otimes\mathcal{C}, \qquad \alpha(\mathcal{D}_A) = \mathcal{D}_A. \tag{8.5}$$

The following proposition shows that the expression β in the form (8.4) and (8.5) is unique up to inner automorphisms on $\mathcal{O}_A$ keeping $\mathcal{D}_A$ globally.

Proposition 8.5. *Suppose that* $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ *is of the form* $\beta = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}) = \mathrm{Ad}(u') \circ (\alpha' \otimes \mathrm{id})$ *for some automorphisms* $\alpha, \alpha' \in \mathrm{Aut}(\mathcal{O}_A, \mathcal{D}_A)$ *and unitaries* $u, u' \in M(\mathcal{O}_A \otimes \mathcal{K})$ *satisfying both the conditions* (8.4) *and* (8.5). *Then there exists a unitary* $V \in \mathcal{O}_A$ *such that*

$$u = u'(V \otimes 1), \qquad \alpha = \mathrm{Ad}(V^*) \circ \alpha' \quad \textit{and} \quad V\mathcal{D}_A V^* = \mathcal{D}_A. \tag{8.6}$$

Proof. For $x \otimes K \in \mathcal{O}_A \otimes \mathcal{K}$, we have $u(\alpha(x) \otimes K)u^* = u'(\alpha'(x) \otimes K)u'^*$. Put $v = u'^* u \in M(\mathcal{O}_A \otimes \mathcal{K})$ which is unitary satisfying $v(\alpha(x) \otimes K) = (\alpha'(x) \otimes K)v$. We particularly see that $v(1 \otimes e_{j,k}) = (1 \otimes e_{j,k})v$ for all $j, k \in \mathbb{N}$. Define $V \in \mathcal{O}_A$ by setting $V \otimes e_{1,1} = (1 \otimes e_{1,1})v(1 \otimes e_{1,1})$. As v commutes with $1 \otimes e_{1,1}$, we know that V is a unitary in $\mathcal{O}_A$. We then have

$$\begin{aligned} v(1 \otimes e_{k,k}) &= v(1 \otimes e_{k,1})(1 \otimes e_{1,k}) \\ &= (1 \otimes e_{k,1})v(1 \otimes e_{1,k}) \\ &= (1 \otimes e_{k,1})(V \otimes e_{1,1})(1 \otimes e_{1,k}) \\ &= V \otimes e_{k,k} \end{aligned}$$

for all $k \in \mathbb{N}$. Hence we have $v = V \otimes 1$. As we have for $x \in \mathcal{O}_A$

$$\begin{aligned} \alpha(x) \otimes e_{1,1} &= v^*(\alpha'(x) \otimes e_{1,1})v \\ &= v^*(1 \otimes e_{1,1})(\alpha'(x) \otimes e_{1,1})(1 \otimes e_{1,1})v \\ &= (V^* \otimes e_{1,1})(\alpha'(x) \otimes e_{1,1})(V \otimes e_{1,1}) \\ &= V^* \alpha'(x) V \otimes e_{1,1} \end{aligned}$$

so that $\alpha(x) = V^*\alpha'(x)V$ for $x \in \mathcal{O}_A$. As $\alpha(\mathcal{D}_A) = \alpha'(\mathcal{D}_A) = \mathcal{D}_A$, we have $V\mathcal{D}_A V^* = \mathcal{D}_A$. ∎

Corollary 8.6. *Let* $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$. *Then* β *satsifies the conditions*

$$\beta(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C} \quad \textit{and} \quad \beta_*([1 \otimes e_{1,1}]) = [1 \otimes e_{1,1}] \textit{ on } K_0(\mathcal{O}_A \otimes \mathcal{K})$$

if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_A)$ *and a unitary* $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ *such that*

$$\begin{gathered} \beta = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}) \quad \textit{and} \quad \alpha_* = \beta_* \textit{ on } K_0(\mathcal{O}_A), \\ u(\mathcal{D}_A \otimes \mathcal{C})u^* = \mathcal{D}_A \otimes \mathcal{C}, \qquad \alpha(\mathcal{D}_A) = \mathcal{D}_A. \end{gathered}$$

Proof. The if part is clear. It suffices to show the only if part. Suppose that $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ satsifies the following conditions

$$\beta(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C} \quad \text{and} \quad \beta_*([1 \otimes e_{1,1}]) = [1 \otimes e_{1,1}] \text{ in } K_0(\mathcal{O}_A \otimes \mathcal{K}).$$

Since $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ satisfies $\beta_*([1 \otimes e_{1,1}]) = [1 \otimes e_{1,1}]$, by [39, Theorem 6.5] and its proof, there exists an automorphism $\alpha_\circ$ of $\mathcal{O}_A$ such that $\alpha_{\circ *} = \beta_*$ on $K_0(\mathcal{O}_A)$. Hence by using [27, Proposition 5.1], we may find an automorphism α_1 of $\mathcal{O}_A$ such that $\alpha_1(\mathcal{D}_A) = \mathcal{D}_A$ and $\alpha_{1*} = \alpha_{\circ *}$ on $K_0(\mathcal{O}_A)$. Put $\beta_1 := \beta \circ (\alpha_1^{-1} \otimes \mathrm{id}) \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$. We have $\beta_1(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}$ and $\beta_{1*} = \beta_* \circ \alpha_{1*}^{-1} = \mathrm{id}$ on $K_0(\mathcal{O}_A)$. By Theorem 8.4, one may take an automorphism $\alpha_2 \in \mathrm{Aut}(\mathcal{O}_A)$ and a unitary $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ such that

$$\begin{gathered} \beta_1 = \mathrm{Ad}(u) \circ (\alpha_2 \otimes \mathrm{id}) \quad \text{and} \quad \alpha_{2*} = \mathrm{id} \text{ on } K_0(\mathcal{O}_A), \\ u(\mathcal{D}_A \otimes \mathcal{C})u^* = \mathcal{D}_A \otimes \mathcal{C}, \qquad \alpha_2(\mathcal{D}_A) = \mathcal{D}_A. \end{gathered}$$

Put $\alpha := \alpha_2 \circ \alpha_1 \in \mathrm{Aut}(\mathcal{O}_A)$. We then have

$$\beta = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}), \qquad \alpha_* = \beta_* \text{ on } K_0(\mathcal{O}_A), \qquad \alpha(\mathcal{D}_A) = \mathcal{D}_A. \qquad \square$$

Let

$$\mathrm{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A) = \{\alpha \in \mathrm{Aut}(\mathcal{O}_A) \mid \alpha(\mathcal{D}_A) = \mathcal{D}_A, \alpha_* = \mathrm{id} \text{ on } K_0(\mathcal{O}_A)\}.$$

For a unitary $u \in \mathcal{O}_A$ satisfying $u\mathcal{D}_A u^* = \mathcal{D}_A$, and $\alpha \in \mathrm{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A)$, we have $\alpha \circ \mathrm{Ad}(u) \circ \alpha^{-1} = \mathrm{Ad}(\alpha(u))$. Since $(\alpha \circ \mathrm{Ad}(u) \circ \alpha^{-1})(\mathcal{D}_A) = \mathcal{D}_A$ and $\mathrm{Ad}(u)_* = \mathrm{id}$ on $K_0(\mathcal{O}_A)$, the automorphism $\alpha \circ \mathrm{Ad}(u) \circ \alpha^{-1}$ belongs to $\mathrm{Int}(\mathcal{O}_A, \mathcal{D}_A)$ so that $\mathrm{Int}(\mathcal{O}_A, \mathcal{D}_A)$ is a normal subgroup of $\mathrm{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A)$, and we may consider the quotient group $\mathrm{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A)/\mathrm{Int}(\mathcal{O}_A, \mathcal{D}_A)$ which we denote by $\mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A)$.

Let $\Psi : \mathrm{Aut}(\mathcal{O}_A, \mathcal{D}_A) \longrightarrow \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ be the homomorphism defined by $\Psi(\alpha) = \alpha \otimes \mathrm{id}$. Since $\Psi(\mathrm{Int}(\mathcal{O}_A, \mathcal{D}_A)) \subset \mathrm{Int}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$, it induces a homomorphism from $\mathrm{Out}(\mathcal{O}_A, \mathcal{D}_A)$ to $\mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ written $\bar{\Psi}$. The following is a corollary of Proposition 8.5.

Corollary 8.7. *The homomorphism* $\bar{\Psi} : \mathrm{Out}(\mathcal{O}_A, \mathcal{D}_A) \longrightarrow \mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ *is injective.*

Proof. Suppose that $\alpha \in \mathrm{Aut}(\mathcal{O}_A, \mathcal{D}_A)$ satisfies $\alpha \otimes \mathrm{id} = \mathrm{Ad}(u')$ for some $u' \in U(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$. Put $\alpha' = \mathrm{id}$ and $u = 1$ in the statement of Proposition 8.5 to have a unitary $V \in U(\mathcal{O}_A, \mathcal{D}_A)$ such that $u' = V \otimes 1$ and $\alpha = \mathrm{Ad}(V)$. $\square$

By [22, Lemma 1.1], we may define a homomorphism

$$K_* : \mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \longrightarrow \mathrm{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K}))$$

by setting $K_*([\alpha]) = \alpha_*$ for $[\alpha] \in \mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$. Thanks to Theorem 8.4 and Corollary 8.6, we know the following theorem on the relative Picard group $\mathrm{Pic}(\mathcal{O}_A, \mathcal{D}_A)$, which is a relative version of the results shown in Appendix A.

Theorem 8.8. *Let A be an irreducible and non-permutation matrix. Then the following short exact sequence holds:*

$$(8.7)\quad 1 \longrightarrow \mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\bar{\Psi}} \mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}) \xrightarrow{K_*} \mathrm{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K})) \longrightarrow 1.$$

Hence, there exists a short exact sequence,

$$(8.8)\qquad 1 \longrightarrow \mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{\bar{\Psi}} \mathrm{Pic}(\mathcal{O}_A, \mathcal{D}_A) \xrightarrow{K_*} \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N) \longrightarrow 1.$$

Proof. We will show the exactness of (8.7). The injectivity of the homomorphism $\bar{\Psi} : \mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A) \longrightarrow \mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ follows from Corollary 8.7. The inclusion relation $\bar{\Psi}(\mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A)) \subset \mathrm{Ker}(K_*)$ is clear. Conversely, for any $[\beta] \in \mathrm{Ker}(K_*)$, we know $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ satisfies $\beta_* = \mathrm{id}$ on $K_0(\mathcal{O}_A)$. By Theorem 8.4 there exist a unitary $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ and an automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_A, \mathcal{D}_A)$ such that $\beta = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id})$ and $\alpha_* = \mathrm{id}$ on $K_0(\mathcal{O}_A)$. Hence we have $[\beta] = [\alpha \otimes \mathrm{id}] = \bar{\Psi}([\alpha])$ and $[\alpha] \in \mathrm{Aut}_\circ(\mathcal{O}_A, \mathcal{D}_A)/\mathrm{Int}(\mathcal{O}_A, \mathcal{D}_A)$, so that we have $\bar{\Psi}(\mathrm{Out}_\circ(\mathcal{O}_A, \mathcal{D}_A)) = \mathrm{Ker}(K_*)$

For any $\xi \in \mathrm{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K}))$, ξ gives rise to an automorphism of the abelian group $\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$. The group $\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$ is isomorphic to the Bowen–Franks group $BF(A) = \mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N$ of the matrix A. By Huang's theorem [16, Theorem 2.15] and its proof, any automorphism of the Bowen–Franks group $BF(A)$ comes from a flow equivalence of the underlying topological Markov shift $(\overline{X}_A, \bar{\sigma}_A)$. It implies that there exists an automorphism $\psi \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$

such that $\psi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}$ and $\psi_* = \xi$ on $K_0(\mathcal{O}_A)$. Hence, ψ belongs to $\mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C})$ such that $K_*(\psi) = \xi$. Consequently, the sequence (8.7) is exact. $\square$

Let $\mathrm{Aut}_1(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N)$ be a subgroup of $\mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N)$ defined by

$$\mathrm{Aut}_1(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N) = \{\xi \in \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N) \mid \xi([1]) = [1]\},$$

where $[1] \in \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$ denotes the class of the vector $(1, \dots, 1)$ in $\mathbb{Z}^N$.

Theorem 8.9. *Let A be an irreducible and non-permutation matrix. Then there exists a short exact sequence,*

$$\begin{aligned} 1 \longrightarrow \mathrm{Out}(\mathcal{O}_A, \mathcal{D}_A) &\xrightarrow{\bar{\Psi}} \mathrm{Pic}(\mathcal{O}_A, \mathcal{D}_A) \\ &\xrightarrow{K_*} \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N)/\,\mathrm{Aut}_1(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N) \longrightarrow 1. \end{aligned}$$

Proof. It suffices to show the exactness at the middle. The inclusion relation $\bar{\Psi}(\mathrm{Out}(\mathcal{O}_A)) \subset \mathrm{Ker}(K_*)$ is clear. Conversely, by Rørdam's result [39, Theorem 6.5] and its proof again, for any $\xi \in \mathrm{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K}))$ with $\xi([1]) = [1]$, there exists an automorphism $\alpha_\circ$ of $\mathcal{O}_A$ such that $\alpha_{\circ *} = \xi$ on $K_0(\mathcal{O}_A)$. By [27, Proposition 5.1], we may find an automorphism α_1 of $\mathcal{O}_A$ such that $\alpha_1(\mathcal{D}_A) = \mathcal{D}_A$ and $\alpha_{1*} = \alpha_{\circ *}$ on $K_0(\mathcal{O}_A)$. Hence $\alpha_1 \in \mathrm{Aut}(\mathcal{O}_A, \mathcal{D}_A)$ such that $\bar{\Psi}([\alpha_1]) = \xi$ so that $\bar{\Psi}(\mathrm{Out}(\mathcal{O}_A)) = \mathrm{Ker}(K_*)$, and the sequence is exact. $\square$

Related studies of automorphism groups of Cuntz algebras have been done by R. Conti, J. H. Hong, and W. Szymański (cf. [10], [11], etc).

9. Appendix A: Picard groups of Cuntz–Krieger algebras

In this appendix, we will refer to the Picard groups of Cuntz–Krieger algebras and especially Cuntz algebras. As examples of the Picard groups for some interesting class of C^*-algebras, K. Kodaka has studied the Picard groups $\mathrm{Pic}(A_\theta)$ for irrational rotation C^*-algebras A_θ to show that $\mathrm{Pic}(A_\theta)$ is isomorphic to $\mathrm{Out}(A_\theta)$ if θ is not quadratic, and a semidirect product $\mathrm{Out}(A_\theta) \rtimes \mathbb{Z}$ if θ is quadratic ([21], [22]). He also studied the Picard groups of certain Cuntz algebras in [22]. He proved that $\mathrm{Pic}(\mathcal{O}_N) = \mathrm{Out}(\mathcal{O}_N)$ for $N = 2, 3$. He also showed that there exists a short exact sequence

$$1 \longrightarrow \mathrm{Out}(\mathcal{O}_N) \xrightarrow{\bar{\Psi}} \mathrm{Pic}(\mathcal{O}_N) \xrightarrow{K_*} \mathrm{Aut}(\mathbb{Z}/(1-N)\mathbb{Z}) \longrightarrow 1 \tag{9.1}$$

for $N = 4, 6$. Since $\mathrm{Aut}(\mathbb{Z}/(1-N)\mathbb{Z})$ is trivial for $N = 2, 3$, Kodaka's results say that the exact sequence (9.1) holds for $N = 2, 3, 4, 6$.

We will show that the above exact sequence holds for all $1 < N \in \mathbb{N}$ (Theorem 9.4). As a corollary we know that the Picard group $\mathrm{Pic}(\mathcal{O}_N)$ of the Cuntz algebra $\mathcal{O}_N$ is a semidirect product $\mathrm{Out}(\mathcal{O}_N) \rtimes \mathbb{Z}/(N-2)\mathbb{Z}$ if $N-1$ is a prime number.

We first refer to the Picard groups of Cuntz–Krieger algebras. Let $u \in M(\mathcal{A})$ be unitary in the multiplier C^*-algebra $M(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$. The automorphism $\mathrm{Ad}(u)$ on $\mathcal{A}$ acts trivially on its K-group $K_0(\mathcal{A})$ by [22, Lemma 1.1]. Throughout Appendix A the matrix A of the Cuntz–Krieger algebra $\mathcal{O}_A$ is assumed to be an irreducible and non-permutation matrix.

Lemma 9.1 (Kodaka [22, Lemma 1.3]). *Let* $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ *satisfy* $\beta_* = \mathrm{id}$ *on* $K_0(\mathcal{O}_A)$. *Then there exists a unitary* $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ *and an automorphism* $\alpha \in \mathrm{Aut}(\mathcal{O}_A)$ *such that*

$$\beta = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}) \quad \textit{and} \quad \alpha_* = \mathrm{id} \ \textit{on}\ K_0(\mathcal{O}_A).$$

For a C^*-algebra $\mathcal{A}$, we put

$$\mathrm{Aut}_\circ(\mathcal{A}) = \{\alpha \in \mathrm{Aut}(\mathcal{A}) \mid \alpha_* = \mathrm{id} \text{ on } K_0(\mathcal{A})\},$$

which is a subgroup of $\mathrm{Aut}(\mathcal{A})$. Since $\mathrm{Ad}(u)_* = \mathrm{id}$ on $K_0(\mathcal{A})$ for a unitary $u \in M(\mathcal{A})$, we see that $\mathrm{Int}(\mathcal{A})$ is a subgroup of $\mathrm{Aut}_\circ(\mathcal{A})$. The quotient group $\mathrm{Aut}_\circ(\mathcal{A})/\mathrm{Int}(\mathcal{A})$ is denoted by $\mathrm{Out}_\circ(\mathcal{A})$.

Let $\mathcal{A}$ be a unital C^*-algebra. Let $\Psi : \mathrm{Aut}(\mathcal{A}) \longrightarrow \mathrm{Aut}(\mathcal{A} \otimes \mathcal{K})$ be the homomorphism defined by $\Psi(\alpha) = \alpha \otimes \mathrm{id}$ for $\alpha \in \mathrm{Aut}(\mathcal{A})$. It induces a homomorphism $\bar{\Psi} : \mathrm{Out}(\mathcal{A}) \longrightarrow \mathrm{Out}(\mathcal{A} \otimes \mathcal{K})$. If $\bar{\Psi}([\alpha]) = \mathrm{id}$ for some $\alpha \in \mathrm{Aut}(\mathcal{A})$, we have $\Psi(\alpha) = \mathrm{Ad}(W)$ for some unitary $W \in M(\mathcal{A} \otimes \mathcal{K})$. Hence we see that

$$\alpha(x) \otimes e_{i,j} = W(x \otimes e_{i,j})W^* \quad \text{for all } x \in \mathcal{A},\ i,j \in \mathbb{N}. \tag{9.2}$$

Since

$$1 \otimes e_{1,1} = \alpha(1) \otimes e_{1,1} = W(1 \otimes e_{1,1})W^*,$$

the unitary W commutes $1 \otimes e_{1,1}$ so that there exists a unitary $w \in \mathcal{A}$ such that $w \otimes e_{1,1} = (1 \otimes e_{1,1})W(1 \otimes e_{1,1})$. We then have

$$\alpha(x) \otimes e_{1,1} = (1 \otimes e_{1,1})W(x \otimes e_{1,1})(1 \otimes e_{1,1}) = wxw^* \otimes e_{1,1} \text{ for all } x \in \mathcal{A}.$$

Hence, $\alpha = \mathrm{Ad}(w) \in \mathrm{Int}(\mathcal{A})$. This means that the map $\bar{\Psi} : \mathrm{Out}(\mathcal{A}) \longrightarrow \mathrm{Out}(\mathcal{A}\otimes\mathcal{K})$ is injective. Any automorphism $\beta \in \mathrm{Aut}(\mathcal{A} \otimes \mathcal{K})$ induces an automorphism β_* of $K_0(\mathcal{A} \otimes \mathcal{K})$, which we denote by $K_*(\beta) \in \mathrm{Aut}(\mathcal{A} \otimes \mathcal{K})$. By [5, Theorem 1.2] with [5, Corollary 3.5], we know $\mathrm{Pic}(\mathcal{A}) = \mathrm{Pic}(\mathcal{A} \otimes \mathcal{K}) = \mathrm{Out}(\mathcal{A} \otimes \mathcal{K})$.

Proposition 9.2. *Let A be an irreducible and non-permutation matrix. Then the following short exact sequence holds:*

$$1 \longrightarrow \mathrm{Out}_\circ(\mathcal{A}) \xrightarrow{\bar{\Psi}} \mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K}) \xrightarrow{K_*} \mathrm{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K})) \longrightarrow 1. \tag{9.3}$$

Hence, there exists a short exact sequence

$$1 \longrightarrow \mathrm{Out}_\circ(\mathcal{O}_A) \xrightarrow{\bar{\Psi}} \mathrm{Pic}(\mathcal{O}_A) \xrightarrow{K_*} \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N) \longrightarrow 1. \tag{9.4}$$

Proof. We will show the exactness of (9.3). We have already known that the injectivity of $\bar{\Psi} : \mathrm{Out}_\circ(\mathcal{O}_A) \longrightarrow \mathrm{Out}(\mathcal{O}_A \otimes \mathcal{K})$. By definition of the group $\mathrm{Aut}_\circ(\mathcal{O}_A)$, the inclusion relation $\bar{\Psi}(\mathrm{Out}_\circ(\mathcal{O}_A)) \subset \mathrm{Ker}(K_*)$ is clear. Conversely, for any $[\beta] \in \mathrm{Ker}(K_*)$, we know that $\beta \in \mathrm{Aut}(\mathcal{O}_A \otimes \mathcal{K})$ satisfies $\beta_* = \mathrm{id}$ on $K_0(\mathcal{O}_A)$. By Lemma 9.1 there exists a unitary $u \in M(\mathcal{O}_A \otimes \mathcal{K})$ and an automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_A)$ such that

$$\beta = \mathrm{Ad}(u) \circ (\alpha \otimes \mathrm{id}) \quad \text{and} \quad \alpha_* = \mathrm{id} \text{ on } K_0(\mathcal{O}_A).$$

Hence, we have $[\beta] = [\alpha \otimes \mathrm{id}] = \bar{\Psi}([\alpha])$ and $[\alpha] \in \mathrm{Aut}_\circ(\mathcal{O}_A)/\mathrm{Int}(\mathcal{O}_A)$. Therefore, we have $\bar{\Psi}(\mathrm{Out}_\circ(\mathcal{O}_A)) = \mathrm{Ker}(K_*)$

By Rørdam's result [39], for any $\xi \in \mathrm{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K}))$, there exists an automorphism β of $\mathcal{O}_A \otimes \mathcal{K}$ such that $\beta_* = \xi$. Therefore the map K_* is surjective, thus proving the exactness of the sequence (9.3). ☐

Recall that $\mathrm{Aut}_1(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N)$ denotes a subgroup of $\mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N)$ consisting of $\xi \in \mathrm{Aut}(\mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N)$ satisfying $\xi([1]) = [1]$.

Proposition 9.3. *Let A be an irreducible and non-permutation matrix. Then there exists a short exact sequence*

$$\begin{aligned} 1 \longrightarrow \operatorname{Out}(\mathcal{O}_A) &\xrightarrow{\bar{\Psi}} \operatorname{Pic}(\mathcal{O}_A) \\ &\xrightarrow{K_*} \operatorname{Aut}(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N)/\operatorname{Aut}_1(\mathbb{Z}^N/(\mathrm{id}-A^t)\mathbb{Z}^N) \longrightarrow 1. \end{aligned}$$

Proof. It suffices to show the exactness at the middle. The inclusion relation $\bar{\Psi}(\operatorname{Out}_\circ(\mathcal{O}_A)) \subset \operatorname{Ker}(K_*)$ is clear. Conversely, by Rørdam's result [39] again, for any $\xi \in \operatorname{Aut}(K_0(\mathcal{O}_A \otimes \mathcal{K}))$ with $\xi([1]) = [1]$, there exists an automorphism β of $\mathcal{O}_A$ such that $\beta_* = \xi$. The sequence is exact. □

Before closing Appendix A, we will mention the Picard groups of Cuntz algebras. By using Proposition 9.2, we know the following theorem. Kodaka has already shown it for $N = 2, 3, 4, 6$ in [21, Corollary 15, Remark 17].

Theorem 9.4. *For each $1 < N \in \mathbb{N}$, there exists a short exact sequence:*

$$1 \longrightarrow \operatorname{Out}(\mathcal{O}_N) \xrightarrow{\bar{\Psi}} \operatorname{Pic}(\mathcal{O}_N) \xrightarrow{K_*} \operatorname{Aut}(\mathbb{Z}/(1-N)\mathbb{Z}) \longrightarrow 1. \tag{9.5}$$

Proof. Since $K_0(\mathcal{O}_N) = \mathbb{Z}/(1-N)\mathbb{Z}$ by [12] and the unit 1 of the C^*-algebra $\mathcal{O}_N$ corresponds to the generator [1] of the cyclic group $\mathbb{Z}/(1-N)\mathbb{Z}$, the fact $\alpha(1) = 1$ for any automorphism $\alpha \in \operatorname{Aut}(\mathcal{O}_N)$ ensures us that $\alpha_* = \mathrm{id}$ on $K_0(\mathcal{O}_N)$. Hence we see that $\operatorname{Aut}_\circ(\mathcal{O}_N) = \operatorname{Aut}(\mathcal{O}_N)$ and hence $\operatorname{Out}_\circ(\mathcal{O}_N) = \operatorname{Out}(\mathcal{O}_N)$. Therefore, the exact sequence (9.4) goes to (9.5). □

As a corollary, we have

Corollary 9.5. *Suppose that $N-1$ is a prime number. Then the Picard group $\operatorname{Pic}(\mathcal{O}_N)$ of the Cuntz algebra $\mathcal{O}_N$ is a semidirect product $\operatorname{Out}(\mathcal{O}_N) \rtimes \mathbb{Z}/(N-2)\mathbb{Z}$ of the outer automorphism group by the cyclic group $\mathbb{Z}/(N-2)\mathbb{Z}$:*

$$\operatorname{Pic}(\mathcal{O}_N) = \operatorname{Out}(\mathcal{O}_N) \rtimes \mathbb{Z}/(N-2)\mathbb{Z},$$

where for $N = 2$, the group $\mathbb{Z}/(N-2)\mathbb{Z}$ means $\{0\}$.

Proof. Since Kodaka has shown that $\operatorname{Pic}(\mathcal{O}_2) = \operatorname{Out}(\mathcal{O}_2)$ ([22]), we may assume $N \geq 3$. As $N-1$ is a prime number, an automorphism η of the cyclic group $\mathbb{Z}/(1-N)\mathbb{Z}$ is determined by $\eta(1)$ which can take its value in $\{1, 2, \dots, N-2\}$, so that we have $\operatorname{Aut}(\mathbb{Z}/(1-N)\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(N-2)\mathbb{Z}$. Since N is not prime, by [21, Theorem 16], for any $k \in \mathbb{N}$ with $1 \leq k \leq N-1$, there exists $\beta_k \in \operatorname{Aut}(\mathcal{O}_N \otimes \mathcal{K})$ such that $(\beta_k)_* = k \cdot \mathrm{id}$ on $K_0(\mathcal{O}_N)$. Hence, the correspondence $k \in \{1, 2, \dots, N-1\} \longrightarrow [\beta_k] \in \operatorname{Pic}(\mathcal{O}_N)$ gives rise to a cross section for the exact sequence (9.5). Therefore the exact sequence (9.5) splits and yields a decomposition of $\operatorname{Pic}(\mathcal{O}_N)$ into a semidirect product $\operatorname{Out}(\mathcal{O}_N) \rtimes \mathbb{Z}/(N-2)\mathbb{Z}$. □

10. Appendix B: Relative imprimitivity bimodules from flow equivalent topological Markov shifts

In [9] it has been shown that flow equivalent topological Markov shifts give rise to corner isomorphic Cuntz–Krieger pairs. In this appendix we will concretely construct a finite chain of relative imprimitivity bimodules, which give rise to corner isomorphic Cuntz–Krieger pairs from flow equivalent topological Markov shifts by using the underlying matrix relations. Concrete construction of such imprimitivity bimodules has not been discussed in [9]. We will see how flow equivalent topological

Markov shifts are related to relatively imprimitivity bimodules from the viewpoint of matrix relations. We remark that closely related results have been seen in T. Bates [1] and Bates and Pask [2].

Let A, B be irreducible square matrices with entries in nonnegative integers. In [43] R. F. Williams proved that the two-sided topological Markov shifts $(\overline{X}_A, \bar{\sigma}_A)$ and $(\overline{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if the matrices A, B are strongly shift equivalent. Two nonnegative matrices A, B are said to be elementary equivalent if there exist nonnegative rectangular matrices C, D such that $A = CD, B = DC$. If there exists a finite sequence of nonnegative matrices $A_0, A_1, \dots, A_n$ such that $A = A_0, B = A_n$ and A_i is elementary equivalent to A_{i+1} for $i = 1, 2, \dots, n-1$, then A and B are said to be strongly shift equivalent ([43]). Hence, topological conjugacy of two-sided topological Markov shifts is generated by a finite sequence of elementary equivalence of underlying matrices (see [20], [24] for general theory of symbolic dynamics).

In the first part of Appendix B, we will directly construct an $(\mathcal{O}_A, \mathcal{D}_A)$–$(\mathcal{O}_B, \mathcal{D}_B)$-relative imprimitivity bimodule X from the assumption $A = CD$ and $B = DC$, although the condition $(\mathcal{O}_A, \mathcal{D}_A) \underset{\text{RME}}{\sim} (\mathcal{O}_B, \mathcal{D}_B)$ is deduced from the assumption $A = CD$ and $B = DC$ through Theorem 4.7 and [28, Proposition 4.3].

Suppose that $A = CD$ and $B = DC$ with the size of A is N and that of B is M so that C is an $N \times M$ matrix and D is an $M \times N$ matrix, respectively. We assume that A and B are irreducible matrices that are not any permutations. We set the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ as block matrix. Let us consider the Cuntz–Krieger algebra $\mathcal{O}_Z$ for the matrix Z. Recall the notations given in section 6. We have denoted by E_Z the edge set of the associated directed graph $G_Z = (V_Z, E_Z)$ to the matrix Z. Since E_Z is the disjoint union $E_C \cup E_D$ of the edge sets E_C and E_D for the matrices C and D, respectively, we may write the canonical generating partial isometries of $\mathcal{O}_Z$ as $S_c, S_d, c \in E_C, d \in E_D$ so that $\sum_{c \in E_C} S_c S_c^* + \sum_{d \in E_D} S_d S_d^* = 1$ and

$$S_c^* S_c = \sum_{d \in E_D} Z(c,d) S_d S_d^*, \qquad S_d^* S_d = \sum_{c \in E_C} Z(d,c) S_c S_c^*$$

for $c \in E_C, d \in E_D$. Define the projections in $\mathcal{O}_Z$ by $P_A = \sum_{c \in E_C} S_c S_c^*$ and $P_B = \sum_{d \in E_D} S_d S_d^*$. Both of them belong to $\mathcal{D}_Z$ and satisfy $P_A + P_B = 1$. It has been shown in [25] (cf. [29]) that

$$(10.1) \quad P_A \mathcal{O}_Z P_A = \mathcal{O}_A, \qquad P_B \mathcal{O}_Z P_B = \mathcal{O}_B, \qquad \mathcal{D}_Z P_A = \mathcal{D}_A, \qquad \mathcal{D}_Z P_B = \mathcal{D}_B.$$

It is not difficult to see that $X = P_A \mathcal{O}_Z P_B$ has a natural structure of an $(\mathcal{O}_A, \mathcal{D}_A)$–$(\mathcal{O}_B, \mathcal{D}_B)$-relative imprimitivity bimodule which gives rise to a relative Morita equivalence between the Cuntz–Krieger pairs $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_B, \mathcal{D}_B)$.

In [33] Parry and Sullivan proved that the flow equivalence relation of topological Markov shifts is generated by strong shift equivalences and expansions $A \to \tilde{A}$ defined bellow. In the second part of Appendix B, we will directly construct an $(\mathcal{O}_A, \mathcal{D}_A)$–$(\mathcal{O}_{\tilde{A}}, \mathcal{D}_{\tilde{A}})$-relative imprimitivity bimodule X, although the condition $(\mathcal{O}_A, \mathcal{D}_A) \underset{\text{RME}}{\sim} (\mathcal{O}_{\tilde{A}}, \mathcal{D}_{\tilde{A}})$ is deduced from Theorem 4.7 and [14, 4.1 Theorem].

For an $N \times N$ matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$, put

$$\tilde{A} = \begin{bmatrix} 0 & A(1,1) & \cdots & A(1,N) \\ 1 & 0 & \cdots & 0 \\ 0 & A(2,1) & \cdots & A(2,N) \\ \vdots & \vdots & & \vdots \\ 0 & A(N,1) & \cdots & A(N,N) \end{bmatrix},$$

which is called the expansion of A at the vertex 1. The expansion of A at other vertices are similarly defined. The arguments below follow the proof of [14, 4.1 Theorem].

Let $\{0,1,\dots,N\}$ be the set of symbols for the topological Markov shifts $(\overline{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ defined by the matrix $\tilde{A}$. Let us denote by $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_N$ the canonical generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_{\tilde{A}}$ satisfying $\sum_{j=0}^N \tilde{S}_j \tilde{S}_j^* = 1, \tilde{S}_i^* \tilde{S}_i = \sum_{j=0}^N \tilde{A}(i,j)\tilde{S}_j \tilde{S}_j^*$ for $i = 0,1,\dots,N$. Put $P = \sum_{i=1}^N \tilde{S}_i \tilde{S}_i^*$. Cuntz and Krieger have shown in the proof of [14, 4.1 Theorem] the equalities:

$$P\mathcal{O}_{\tilde{A}}P = \mathcal{O}_A, \qquad \mathcal{D}_{\tilde{A}}P = \mathcal{D}_A.$$

Hence, $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_{\tilde{A}}, \mathcal{D}_{\tilde{A}})$ are corner isomorphic, and $X = P\mathcal{O}_{\tilde{A}}$ becomes an $(\mathcal{O}_A, \mathcal{D}_A)$–$(\mathcal{O}_{\tilde{A}}, \mathcal{D}_{\tilde{A}})$-relative imprimitivity bimodule which gives rise to a relative Morita equivalence between $(\mathcal{O}_A, \mathcal{D}_A)$ and $(\mathcal{O}_{\tilde{A}}, \mathcal{D}_{\tilde{A}})$.

Remark. After the first draft of this paper was completed, a paper appeared by Kazunori Kodaka and Tamotsu Teruya, *The strong Morita equivalence for inclusions of C^*-algebras and conditional expectations for equivalence bimodules* (arXiv:1609.08263).

They defined Morita equivalence for pairs of C^*-algebras in their paper; however, their definition of Morita equivalence for pairs of C^*-algebras is different from ours.

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