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MULTI-TRAVELLING WAVES FOR THE NONLINEAR KLEIN-GORDON EQUATION

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ABSTRACT. For the nonlinear Klein-Gordon equation in $\mathbb{R}^{1+d}$, we prove the existence of multi-solitary waves made of any number N of decoupled bound states. This extends the work of Côte and Muñoz (Forum Math. Sigma **2** (2014)) which was restricted to ground states, as were most previous similar results for other nonlinear dispersive and wave models.

1. Introduction

In this paper we extend previous constructions of multi-solitary wave solutions for the nonlinear Klein-Gordon equation (NLKG) in $\mathbb{R}^{1+d}$, $d \geqslant 1$,

$$\text{(NLKG)} \qquad \partial_{tt} u - \Delta u + u - f(u) = 0, \quad u(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d.$$

This equation arises in quantum field physics as a model for a self-interacting, nonlinear scalar field, invariant under Lorentz transformations (see below). We focus on the particular case where

$$f(u) = |u|^{p-1}u \quad \text{for } 1 < p < \frac{d+2}{d-2} \quad (p > 1 \text{ for } d = 1 \text{ or } 2) \tag{1}$$

but the arguments can be extended to more general situations. We set

$$F(u) = \int_0^u f(v)dv = \frac{|u|^{p+1}}{p+1}.$$

As usual, we see the (NLKG) equation as a first order system of equations

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} \partial_t u \\ \Delta u - u + f(u) \end{pmatrix}. \tag{2}$$

In this framework, we work with vector data $U = (u, \partial_t u)^\top$. We use upper-case letters to denote vector-valued functions and lower-case letters for scalar functions.

Recall that the corresponding Cauchy problem for (NLKG) is locally well-posed in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ for any $s \geqslant 1$; we refer to Ginibre-Velo [22] and Nakamura-Ozawa [40] (when $d = 2$) for more details.

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Also under the above conditions, the energy and momentum (every integral is taken over $\mathbb{R}^d$),

$$\mathscr{E}[u,u_t](t) = \frac{1}{2}\int \left[|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + |u(t,x)|^2 - 2F(u(t,x))\right] dx, \tag{3}$$

$$\mathscr{P}[u,u_t](t) = \frac{1}{2}\int \partial_t u(t,x)\nabla u(t,x)\, dx, \tag{4}$$

are conserved along the flow. In this paper, we will work in the energy space $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ endowed with the following scalar product: denote $U = (u_1,u_2)^\top$, $V = (v_1,v_2)^\top$, and define

$$\langle U,V\rangle := (u_1,v_1) + (u_2,v_2), \quad \text{where} \quad (u,v) := \int uv dx. \tag{5}$$

We will refer to the orthogonality with respect to $\langle\cdot,\cdot\rangle$ as L^2-orthogonality (for vector-valued functions). We also define the energy norm

$$\|U\|^2 := \langle U,U\rangle + (\nabla u_1,\nabla u_1) = \|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2. \tag{6}$$

Looking for stationary solutions $u(t,x) = q(x)$ of (NLKG) in $H^1(\mathbb{R}^d)$ we reduce to the elliptic PDE

$$-\Delta q + q - f(q) = 0, \quad q\in H^1(\mathbb{R}^d). \tag{7}$$

Let us recall well-known results for equation (7) from [5] (see also the references therein). We call the solutions of (7) *bound states*; the set of bound states is denoted by $\mathscr{B}$,

$$\mathscr{B} = \{q : q \text{ is a nontrivial solution of (7)}\}.$$

Standard elliptic arguments (see e.g. [21] or Theorem 8.1.1 in [5]) show that if $q\in\mathscr{B}$, then q is of class $C^2(\mathbb{R}^d)$ and has exponential decay as $|x|\to+\infty$, as well as its first and second order derivatives.

Let

$$\mathscr{W}(u) = \frac{1}{2}\int(|\nabla u|^2 + |u|^2 - 2F(u))dx.$$

We call *ground states* the solutions of (7) that minimize the functional W; the set of ground states is denoted by $\mathscr{G}$,

$$\mathscr{G} = \{q_{\mathrm{GS}} : q_{\mathrm{GS}}\in\mathscr{B} \text{ and } \mathscr{W}(q_{\mathrm{GS}}) \leqslant \mathscr{W}(q) \text{ for all } q\in\mathscr{B}\}.$$

Ground states are now well understood. In particular, it is well known (Berestycki-Lions [2], Gidas-Ni-Nirenberg [20], Kwong [28], Serrin-Tang [45]) that there exists a radial positive function q_0 of class C^2, exponentially decreasing, along with its first and second derivatives, such that

$$\mathscr{G} = \{q_0(x-x_0) : x_0\in\mathbb{R}^d\}.$$

In dimension 1, it is well known (by ODE arguments) that $\mathscr{B} = \mathscr{G}$. In contrast, for any $d\geqslant 2$, it is known that $\mathscr{G}\subsetneq\mathscr{B}$; see Remark 8.1.16 in [5], we also refer to Ding [15], where it is proven that $\mathscr{B}$ (up to translation) is infinite. Functions $q\in\mathscr{B}\backslash\mathscr{G}$ are referred to as *excited states*. Few papers in the literature deal with excited states. Here are some references on the construction of such solutions. Berestycki-Lions [3] showed the existence of infinitely many radial *nodal* (i.e., sign changing) solutions (see also [25] and the references therein). More recently, Del Pino, Musso, Pacard and Pistoia constructed in [13] solutions to the massless version of equation (7) with a centered soliton crowned with negative spikes (rescaled solitons) at the vertices of a regular polygon of radius 1; in [14], they constructed sign changing, nonradial solutions to (7) on the sphere $\mathbb{S}^d$ ($d \geqslant 4$) whose energy is concentrated along special submanifolds of $\mathbb{S}^d$.

The main difficulty in dealing with excited states in the evolution equation (NLKG) is the lack of information on the linearized operator $-\Delta z + z - f'(q)z$. Whereas for ground states, it is known that the linearized operator has a unique simple negative eigenvalue, and a (nondegenerate) kernel given by $\mathrm{Span}(\partial_{x_j} q;\, j = 1, \ldots, d)$; the detailed spectral properties of the linearized operator around general bound states are not known. See Section 2 of this paper.

Since (NLKG) is invariant under *Lorentz boosts*, given a bound state q, we can define its *boosted* counterpart, with relative velocity $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$, where $|\beta| < 1$ (we denote by $|\cdot|$ the euclidean norm on $\mathbb{R}^d$), as

$$(8) \qquad q_\beta(x) := q(\Lambda_\beta(x)), \quad \Lambda_\beta(x) := x + (\gamma - 1)\frac{\beta(\beta \cdot x)}{|\beta|^2}, \quad \gamma := \frac{1}{\sqrt{1 - |\beta|^2}}.$$

Note that the function q_β satisfies

$$-(\Delta - (\beta \cdot \nabla)^2) q_\beta + q_\beta - f(q_\beta) = 0.$$

In particular,

$$R(t,x) = \begin{pmatrix} q_\beta(x - \beta t) \\ -\beta \cdot \nabla q_\beta(x - \beta t) \end{pmatrix}$$

is a solution of the (first order system form of the) Klein-Gordon equation (2).

It is well known (see e.g. Grillakis-Shatah-Strauss [23]) that the ground state $(q_0, 0)$ is unstable in the energy space (this result is known in the physics literature as Derrick's Theorem [15]). For recent works on the instability properties of q_0 and on general solutions with energy slightly above $\mathscr{E}[(q_0, 0)]$, we refer to Nakanishi-Schlag [41,42] and subsequent works. We also refer to Duyckaerts-Merle [18], in the context of the energy critical nonlinear wave equation for related works.

In this paper, we continue the study of the dynamics of large, quantized energy solutions. Specifically, we deal with solutions describing multi-bound states for (NLKG), i.e., solutions u to (NLKG) defined on a semi-infinite interval of time, such that

$$u(t,x) \sim \sum_{n=1}^{N} q_{n,\beta_n}(x - \beta_n t) \quad \text{as} \quad t \to +\infty$$

for given speeds β_n (all distinct). Such solutions were constructed in the context of the nonlinear Schrödinger equations, the generalized Korteweg-de Vries equations, the Hartree equation, the energy critical wave equation, and the water wave system, by Merle [32], Martel [30], Martel-Merle [33], Côte-Martel-Merle [9], Combet [6,7], Krieger-Martel-Raphaël [26], Martel-Merle [35], and Ming-Rousset-Tzvetkov [37], both in stable and unstable contexts (see also the references in these works). For (NLKG), the same result was proved by Côte-Muñoz [10]: there exist multi-solitary waves based on the ground state for the whole range of parameters $\beta_1, \ldots, \beta_N \in \mathbb{R}^d$ (two by two distinct), with $|\beta_n| < 1$.

We point out that the above results all concern ground states $q \in \mathscr{G}$, and rely on the complete description of the linearized operator around the ground state in these cases. To our knowledge, the only work related to excited states is by Côte-Le Coz [8] for the nonlinear Schrödinger equation. In this work, the lack of information on the linearized operator is counterbalanced by assuming that solitary waves are well separated (high-speed assumption).

The main goal of this paper is to extend the construction of multi-solitary waves to a bound state $q \in \mathscr{B}$ of the (NLKG) equation, *without assumption on the speeds* (besides their being distinct), thus completing Theorem 1 in [10], and opening the way to treat such questions for other models.

Theorem 1. *Let $N \in \mathbb{N} \setminus \{0\}$ and $\beta_1, \beta_2, \dots, \beta_N \in \mathbb{R}^d$ such that*

$$\forall n, \quad |\beta_n| < 1 \quad \textit{and} \quad \forall n' \neq n, \quad \beta_{n'} \neq \beta_n.$$

Let $q_1, q_2, \dots, q_N \in \mathscr{B}$ be any bound state solution of equation (7).

Then there exist $T_0 > 0$, $\omega > 0$, and a solution U of (2) *in the energy space, defined for $t \geqslant T_0$, satisfying*

$$\forall t \geqslant T_0, \quad \left\| U(t) - \sum_{n=1}^{N} R_n(t) \right\| \leqslant e^{-\omega t},$$

where

$$R_n(t,x) = \begin{pmatrix} q_{n,\beta_n}(x - \beta_n t) \\ -\beta_n \cdot \nabla q_{n,\beta_n}(x - \beta_n t) \end{pmatrix}.$$

Using the techniques of this paper, it is possible to extend the main result to more general H^1 subcritical nonlinearities. See e.g. [10] for standard conditions on the nonlinearity.

Recall that for integrable models, like the (KdV) and (mKdV) equations, or the 1D cubic nonlinear Schrödinger equation, multi-solitons are explicitly derived from the inverse scattering method. Such solutions are quite special since they are global multi-solitons, both for $t \to \pm\infty$ and describe elastic collisions of solitons. See e.g. the classical references [39, 50, 51]. For nonintegrable equations, in general, the asymptotic behavior as $t \to -\infty$ of multi-bound states as constructed in Theorem 1 is not known.

Recall also that the importance of multi-solitons among all solutions is clearly established by the so-called *soliton resolution conjecture*, which says roughly speaking that any solution of a nonlinear dispersive equation should decompose in large time as the sum of a certain number of solitons and a dispersive part. See e.g. [44] for a proof in the case of the KdV equation. We refer to recent works of Duyckaerts, Kenig, and Merle [16, 17], and the references therein for general soliton decomposition results in the nonintegrable situation of the energy critical wave equation.

The scheme of the proof is the same as for previous related results, notably [9, 10, 30]: Theorem 1 can be reduced to the existence of solutions to (NLKG) satisfying uniform estimates, which is the following proposition.

Proposition 2. *There exist $T_0 > 0$ and $\omega_0 > 0$ such that for any $S_0 \geqslant T_0$ there exists U_0 such that the solution $U(t)$ of* (2) *with data $U(S_0) = U_0$ is defined in the energy space on the time interval $[T_0, S_0]$ and satisfies*

$$\forall t \in [T_0, S_0], \quad \left\| U(t) - \sum_{n=1}^{N} R_n(t) \right\| \leqslant e^{-\omega_0 t}. \tag{9}$$

Indeed, let $S_m \to +\infty$, and assuming that Proposition 2 holds, let U_m be one solution to (NLKG) satisfying the uniform estimates (9) on the time interval $[T_0, S_m]$. Using the compactness arguments of Section 4 of [10], one observes that $(U_m(T_0))_{m\in\mathbb{N}}$ has a weak-$H^1 \times L^2$ limit U_0^*. Then consider the solution U^* to (NLKG) with data U_0^* at time T_0: the key feature is that the flow of (NLKG) is continuous for the weak-$H^1 \times L^2$ topology, and this allows us to conclude that U^* is the desired multi-soliton. We refer to [10, Section 4] for further details.

We are therefore left with solely the proof of Proposition 2, to which the remainder of this paper is devoted. To prove it for any bound state, we use two new points:

(1) a general coercivity argument with no a priori knowledge of the spectral properties of the linearized operator (see Section 2);
(2) a simplification of the existence proof so as to deal with possibly multiple degenerate directions, not related to translation invariance (see Section 3).

2. Spectral theory for bound states

We consider a bound state $q \in \mathscr{B}$, a velocity $\beta \in \mathbb{R}^d$, $|\beta| < 1$, and the corresponding Lorentz state q_β defined by (8). In this section we are interested in the linearized flow around the solution $R(t,x)$ of (2),

$$R(t,x) = \begin{pmatrix} q_\beta(x-\beta t) \\ -\beta\cdot\nabla q_\beta(x-\beta t) \end{pmatrix}.$$

Define the matrix operator

$$\mathbf{H} = \begin{pmatrix} -\Delta + 1 - f'(q_\beta) & -\beta\cdot\nabla \\ \beta\cdot\nabla & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The (NLKG) equation around R, i.e., for solutions of the form

$$U(t,x) = R(t,x) + V(t, x-\beta t),$$

where V is a small perturbation, rewrites as

$$\partial_t V = \mathbf{JH}V + \mathscr{N}(V), \tag{10}$$

where $\mathscr{N}(V)$ denotes nonlinear terms in V.

2.1. Spectral analysis of JH.

First, following [10], we study the spectral properties of the operator $\mathbf{JH}$ appearing in equation (10),

$$\mathbf{JH} = \begin{pmatrix} \beta\cdot\nabla & 1 \\ \Delta - 1 + f'(q_\beta) & \beta\cdot\nabla \end{pmatrix},$$

in terms of the spectral properties of the elliptic operator

$$\mathbf{L} = -\Delta + 1 - f'(q).$$

Lemma 1. (i) Spectral properties of $\mathbf{L}$. *The self-adjoint operator* $\mathbf{L}$ *has essential spectrum* $[1,+\infty)$*, a finite number* $\bar{k} \geqslant 1$ *of negative eigenvalues (counted with multiplicity), and its kernel is of finite dimension* $\bar{\ell} \geqslant d$*. Let* $(\phi_k)_{k=1,\dots,\bar{k}}$ *be an* L^2*-orthogonal family of eigenfunctions of* $\mathbf{L}$ *with negative eigenvalues* $(-\lambda_k^2)_{k=1,\dots,\bar{k}}$*, and let* $(\phi_\ell^0)_{\ell=1,\dots,\bar{\ell}}$ *be an* L^2*-orthogonal basis of* $\ker(\mathbf{L})$*, i.e.,*

$$\mathbf{L}\phi_k = -\lambda_k^2\phi_k, \quad \lambda_k > 0, \qquad k = 1,\dots,\bar{k}, \tag{11}$$

$$\mathbf{L}\phi_\ell^0 = 0, \qquad \ell = 1,\dots,\bar{\ell}. \tag{12}$$

Then, there exists $c > 0$ such that for any $v \in H^1$ satisfying $(v, \phi_k) = (v, \phi_0^\ell) = 0$ for all $k = 1, \ldots, \bar{k}$, $\ell = 1, \ldots, \bar{\ell}$, the following holds:

$$(\mathbf{L}v, v) \geqslant c\|v\|_{H^1}^2. \tag{13}$$

(ii) Spectral properties of $\mathbf{JH}$. *For $k = 1, \ldots, \bar{k}$ and $\ell = 1, \ldots, \bar{\ell}$ and signum $\pm$, let*

$$Y_k^\pm(x) = e^{\mp\gamma\lambda_k\beta\cdot x}\begin{pmatrix}\phi_k\\ \mp\gamma\beta\cdot\nabla\phi_k \pm \gamma\lambda_k\phi_k\end{pmatrix}(\Lambda_\beta(x)), \tag{14}$$

$$\Phi_\ell^0(x) = \begin{pmatrix}\phi_\ell^0\\ -\gamma\beta\cdot\nabla\phi_\ell^0\end{pmatrix}(\Lambda_\beta(x)). \tag{15}$$

Then

$$(\mathbf{JH})Y_k^\pm = \pm\frac{\lambda_k}{\gamma}Y_k^\pm, \tag{16}$$

$$\ker\mathbf{H} = \ker(\mathbf{JH}) = \operatorname{Span}(\Phi_\ell^0, \ell = 1, \ldots, \bar{\ell}). \tag{17}$$

Moreover,

$$\langle\mathbf{H}Y_k^+, Y_{k'}^+\rangle = \langle\mathbf{H}Y_k^-, Y_{k'}^-\rangle = 0 \quad \textit{for all } k, k' = 1, \ldots, \bar{k}. \tag{18}$$

Finally, the family $(Y_k^\pm)_{\pm,k=1,\ldots,\bar{k}}$ is linearly independent. As a consequence, the family $(\mathbf{H}Y_k^\pm)_{\pm,k=1,\ldots,\bar{k}}$ is linearly independent.

(iii) Exponential decay. *There exist $C_0 > 0$ and $\omega_0 > 0$, such that for all $\alpha \in \mathbb{N}^d$, $|\alpha| \leqslant 1$, and for all $x \in \mathbb{R}^d$,*

$$|\partial_\alpha q(x)| + |\partial_\alpha\phi_k(x)| + |\partial_\alpha Y_k^\pm(x)| + |\partial_\alpha\phi_\ell^0(x)| \leqslant C_0 e^{-\omega_0|x|}. \tag{19}$$

Proof. We start by noticing that by rotation (with first vector $\beta/|\beta|$ for $\beta \neq 0$), we can assume that the Lorentz boost is of the form $(\beta, 0, \ldots, 0)$, where (with slight abuse of notation) $\beta \in (-1, 1)$. Observe that in this case $\Lambda_\beta(x) = (\gamma x_1, x')$, where $x = (x_1, x')$, $x' = (x_2, \ldots, x_d)$.

(i) The operator $\mathbf{L}$ is a compact perturbation of $-\Delta + 1$, and so the two operators have the same essential spectrum $[1, +\infty)$. In particular, for any $\delta > 0$, both operators have a finite number of eigenvalues (counting their multiplicities) on $(-\infty, 1-\delta]$. We define $(\phi_k)_{k=1,\ldots,\bar{k}}$, $(\lambda_k)_{k=1,\ldots,\bar{k}}$, and $(\phi_\ell^0)_{\ell=1,\ldots,\bar{\ell}}$ as in the statement of the lemma. From the spectral theorem, the following coercivity holds: there exists $c' > 0$ such that for any $v \in H^1$ satisfying $(v, \phi_k) = (v, \phi_0^\ell) = 0$ for all $k = 1, \ldots, \bar{k}$, $\ell = 1, \ldots, \bar{\ell}$, we have $(\mathbf{L}v, v) \geqslant c'\|v\|_{L^2}^2$. Since $f'(q)$ is bounded, a standard argument proves that the coercivity property (13) holds.

Note that by direct computations $(\mathbf{L}q, q) = (1-p)\int|q|^{p+1} < 0$ and thus $\bar{k} \geqslant 1$. Moreover, it is clear by differentiating (7) with respect to x_j that $\partial_j q \in \ker\mathbf{L}$. Since the family $(\partial_j q)_{j=1,\ldots,d}$ is linearly independent, we obtain $\bar{\ell} \geqslant d$.

(ii) Looking for an eigenfunction $Y = (\rho_1, \rho_2)^\top$ of the operator $\mathbf{JH}$, with eigenvalue λ, we are led to the system

$$\begin{cases}\beta\partial_1\rho_1 + \rho_2 = \lambda\rho_1,\\ (\Delta - 1 + f'(q_\beta))\rho_1 + \beta\partial_1\rho_2 = \lambda\rho_2.\end{cases}$$

The first equation gives $\rho_2 = (\lambda - \beta\partial_1)\rho_1$ which we plug into the second equation:

$$(-\Delta + 1 - f'(q_\beta))\rho_1 + (\lambda - \beta\partial_1)^2\rho_1 = 0,$$

which rewrites as

$$-(1-\beta^2)\partial_{11}\rho_1 - \Delta'\rho_1 - 2\lambda\beta\partial_1\rho_1 + \rho_1 - f'(q_\beta)\rho_1 = -\lambda^2\rho_1,$$

where Δ' is the Laplace operator with respect to the variable $x' = (x_2, \dots, x_d)$. Write

$$\rho_1(x) = e^{-\gamma^2\lambda\beta x_1}\sigma_1(\gamma x_1, x') = e^{-\gamma\lambda\beta y_1}\sigma_1(y), \quad y = (\gamma x_1, x').$$

Then the equation on ρ_1 rewrites as

$$\begin{aligned} e^{-\gamma\lambda\beta y_1}\big[-(1-\beta^2)\left(\gamma^2\partial_{11}^2 - 2\gamma^2\lambda\beta\gamma\partial_1 + (\gamma^2\lambda\beta)^2\right) \\ -\Delta' - 2\lambda\beta(\gamma\partial_1 - \gamma^2\lambda\beta) + (1+\lambda^2) - f'(q)\big]\sigma_1(y) = 0, \end{aligned}$$

which simplifies to

$$-\Delta\sigma_1 + \sigma_1 - f'(q)\sigma_1 = -\gamma^2\lambda^2\sigma_1.$$

Therefore σ_1 has to be an eigenfunction of $\mathbf{L}$ with eigenvalue $-\gamma^2\lambda^2 \leqslant 0$.

Reciprocally, if $\sigma_1 = \phi_k$ and $\lambda = \lambda_k/\gamma$, then

$$e^{-\gamma^2\lambda\beta x_1}\begin{pmatrix} \phi_k \\ -\gamma\beta\partial_1\phi_k + \lambda\gamma^2\phi_k \end{pmatrix}(\gamma x_1, x')$$

is an eigenfunction of $\mathbf{JH}$ with eigenvalue λ.

Let us check (18):

$$\begin{aligned} \frac{\lambda_{k'}}{\gamma}\langle \mathbf{H}Y_k^+, Y_{k'}^+\rangle &= \langle \mathbf{H}Y_k^+, \mathbf{JH}Y_{k'}^+\rangle = -\langle \mathbf{JH}Y_k^+, \mathbf{H}Y_{k'}^+\rangle \\ &= -\frac{\lambda_k}{\gamma}\langle Y_k^+, \mathbf{H}Y_{k'}^+\rangle = -\frac{\lambda_k}{\gamma}\langle \mathbf{H}Y_k^+, Y_{k'}^+\rangle. \end{aligned}$$

Since $\lambda_k, \lambda_{k'} > 0$, this implies $\langle \mathbf{H}Y_k^+, Y_{k'}^+\rangle = 0$. All these computations are similar for $(Y_k^-)_k$.

Also observe that if $\lambda_k \neq \lambda_{k'}$, then $\langle \mathbf{H}Y_k^+, Y_{k'}^-\rangle = 0$ with the same argument.

Let us now prove that the $(Y_k^\pm)_{\pm,k=1,\dots,\bar{k}}$ are linearly independent. Assume that we have the dependence relation:

$$\sum_{k=1}^{\bar{k}}(a_k^+Y_k^+ + a_k^-Y_k^-) = 0.$$

Fix $k_* \in \{1, \dots, \bar{k}\}$, and let I_{k_*} be the set of indices $k \in \{1, \dots, \bar{k}\}$ such that $\lambda_k = \lambda_{k_*}$. As spectral spaces associated to different eigenvalues are in direct sum, we infer that

$$\sum_{k\in I_{k_*}} a_k^+Y_k^+ = 0 \quad \text{and} \quad \sum_{k\in I_{k_*}} a_k^-Y_k^- = 0.$$

In the first equality, the first line writes

$$e^{-\gamma\lambda_k\beta\cdot x}\sum_{k\in I_{k_*}} a_k^+\phi_k(\Lambda_\beta x) = 0.$$

As Λ_β is one-to-one, this means that $\sum_{k\in I_{k_*}} a_k^+\phi_k = 0$ and by linear independence of the $(\phi_k)_{k=1,\dots,\bar{k}}$, this relation is trivial: $a_k^+ = 0$ for all $k \in I_{K_*}$, and in particular $a_{k_*}^+ = 0$. A similar argument on the second equality gives that $a_{k_*}^- = 0$. Therefore, the dependence relation is trivial, and the $(Y_k^\pm)_{\pm,k=1,\dots,\bar{k}}$ are linearly independent.

As they are eigenfunctions for $\mathbf{JH}$ with nonzero eigenvalue, we infer that the family $(\mathbf{JH}Y_k^\pm)_{\pm,k=1,\dots,\bar{k}}$ is linearly independent. As $\mathbf{J}$ is one-to-one (it is an involution), the $(\mathbf{H}Y_k^\pm)_{\pm,k=1,\dots,\bar{k}}$ are linearly independent as well.

(iii) The exponential decay of any bound state q and its derivates is well known, and follows from Agmon type estimates; we refer the reader to [21].

By standard elliptic arguments, we first note that there exist $C>0$ such that for all $\alpha\in\mathbb{N}^d$, $|\alpha|\leqslant 2$,

$$\forall x\in\mathbb{R}^d,\quad |\partial_\alpha\phi_k(x)|\leqslant Ce^{-(1+\lambda_k^2)^{\frac{1}{2}}|x|},\quad |\partial_\alpha\phi_\ell^0(x)|\leqslant Ce^{-|x|}. \tag{20}$$

This and the definition of $Y_k^\pm$ in (14) is enough to prove (19). □

2.2. **Spectral analysis of H.** The eigenfunctions $Y_k^\pm$ of $\mathbf{JH}$ are related to equation (10), as well as the eigenfunctions $\mathbf{H}Y_k^\pm$ of the adjoint operator $\mathbf{HJ}$. In particular, it is straightforward to compute the main order time evolution of the projection of the perturbation V on such directions (see Lemma 8). However, in order to study stability properties of the flow using the energy method (see the next section), the relevant operator turns out to be $\mathbf{H}$.

The operator $\mathbf{H}$ is self-adjoint for the $\langle\cdot,\cdot\rangle$ scalar product and we already know from Lemma 1 that

$$\ker\mathbf{H}=\operatorname{Span}(\Phi_\ell^0,\ell=1,\dots,\bar{\ell}), \tag{21}$$

where the vector-valued functions Φ_ℓ^0 are defined in (15). However, unlike for $\mathbf{JH}$, the eigenfunctions of $\mathbf{H}$ related to negative eigenvalues do not seem to be explicitly related to that of $\mathbf{L}$.

Nonetheless a key observation of this paper is that for any $\beta\in\mathbb{R}^d$, $|\beta|<1$, the number of negative directions for the quadratic form $\langle\mathbf{H}\cdot,\cdot\rangle$ is equal to the number $\bar{k}$ of negative eigenvalues of the operator $\mathbf{L}$.

Lemma 2. *The self-adjoint operator* $\mathbf{H}$ *has a finite number* $\bar{m}\geqslant 1$ *of negative eigenvalues (counted with multiplicity). Let* $(\Upsilon_m)_{m=1,\dots,\bar{m}}$ *be an* L^2*-orthogonal family of eigenfunctions of* $\mathbf{H}$ *with negative eigenvalues, normalized so that*

$$\mathbf{H}\Upsilon_m=-\mu_m^2\Upsilon_m,\quad \mu_m>0,\quad m=1,\dots,\bar{m}, \tag{22}$$

$$\langle\mathbf{H}\Upsilon_m,\Upsilon_m\rangle=-1,\quad \langle\Upsilon_m,\Upsilon_{m'}\rangle=0\quad \textit{for } m\neq m'. \tag{23}$$

Then the following holds:

$$\bar{m}=\bar{k}.$$

Moreover, there exists $c>0$ *such that for all* $V\in H^1\times L^2$,

$$\langle\mathbf{H}V,V\rangle\geqslant c\|V\|^2-\frac{1}{c}\sum_{m=1}^{\bar{m}}\langle V,\Upsilon_m\rangle^2-\frac{1}{c}\sum_{\ell=1}^{\bar{\ell}}\langle V,\Phi_\ell^0\rangle^2. \tag{24}$$

Proof. As before we assume (without loss of generality) that the Lorentz boost is of the form $(\beta,0,\dots,0)$ for some $\beta\in(-1,1)$. Note that

$$\begin{aligned}\langle\mathbf{H}V,V\rangle&=((-\Delta+1-f'(q_\beta))v_1,v_1)+2\beta(\partial_1v_1,v_2)+\|v_2\|_{L^2}^2\\&=(\tilde{\mathbf{L}}v_1,v_1)+\|\beta\partial_1v_1+v_2\|_{L^2}^2,\\&\text{where}\quad \tilde{\mathbf{L}}:=-(1-\beta^2)\partial_{11}-\Delta'+1-f'(q_\beta).\end{aligned}$$

Observe that $\tilde{\mathbf{L}}$ is self-adjoint and that it is a compact perturbation of the operator $-(1-\beta^2)\partial_{11}^2-\Delta'$ so it has essential spectrum $[1-\beta^2,+\infty)$. From there we infer that $\mathbf{H}$ has only finitely many negative eigenvalues, whose eigenfunctions span a vector space of dimension $\bar{m}$; as $(\Phi^0_\ell)_{\ell=1,\dots,\bar{\ell}}$ span $\ker \mathbf{H}$, and this yields (24).

Also notice that if we denote $\tilde{V}(x) := V(\Lambda_\beta(x))$, then $(\tilde{\mathbf{L}}\tilde{V})(x) = (\mathbf{L}V)(\Lambda_\beta(x))$. This means that a basis of the eigenfunctions of $\tilde{\mathbf{L}}$ with negative eigenvalues is given by the $(\phi_k \circ \Lambda_\beta)_{k=1\dots,\bar{k}}$; in particular they span a subspace of dimension $\bar{k}$.

Now, we prove that $\bar{m} = \bar{k}$. On the one hand, for $k = 1, \dots, \bar{k}$, define

$$\Phi_k(x) = \begin{pmatrix} \phi_k \\ -\gamma\beta\partial_1\phi_k \end{pmatrix} (\Lambda_\beta(x)) \quad \text{so that} \quad \mathbf{H}\Phi_k = \begin{pmatrix} -\lambda_k^2\phi_k \\ 0 \end{pmatrix} (\Lambda_\beta(x)).$$

Then the $(\Phi_k)_k$ are linearly independent as a consequence of the linear independence of the $(\phi_k)_k$. Let $W \in \mathrm{Span}(\Phi_k, k = 1, \dots, \bar{k})$ be nonzero, $W = \sum_k \alpha_k \Phi_k$, and, by L^2-orthogonality of the ϕ_k,

$$\langle \mathbf{H}W, W\rangle = -\sum_k \lambda_k^2 \alpha_k^2 < 0.$$

Hence $\langle \mathbf{H}\cdot, \cdot\rangle|_{\mathrm{Span}(\Phi_k, k=1,\dots,\bar{k})}$ is negative definite on $\mathrm{Span}(\Phi_k, k = 1, \dots, \bar{k})$ which is of dimension $\bar{k}$. By the Sylvester inertia theorem, we deduce that $\bar{k} \leqslant \bar{m}$.

On the other hand, denote by $\Upsilon_m = (\upsilon_m^1, \upsilon_m^2)^\top$ a family of L^2-orthogonal eigenfunctions of $\mathbf{H}$ with negative eigenvalues $-\mu_m^2$ ($\mu_k > 0$), i.e., $\mathbf{H}\Upsilon_m = -\mu_m^2\Upsilon_m$. Then $(\upsilon_m^1, \upsilon_m^2)$ satisfy

$$\begin{cases} (-\Delta + 1 - f'(q_\beta))\upsilon_m^1 - \beta\partial_1\upsilon_m^2 = -\mu_m^2\upsilon_m^1, \\ \qquad\qquad\qquad\qquad \beta\partial_1\upsilon_m^1 + \upsilon_m^2 = -\mu_m^2\upsilon_m^2 \end{cases}$$

so that $\upsilon_m^2 = -\frac{\beta}{1+\mu_m^2}\partial_1\upsilon_m^1$ and

$$\left(-\Delta + 1 - f'(q_\beta) + \frac{\beta^2}{1+\mu_m^2}\partial_{11}\right)\upsilon_m^1 = -\mu_m^2\upsilon_m^1.$$

Then,

$$\tilde{\mathbf{L}}\upsilon_m^1 = -\mu_m^2\upsilon_m^1 + \frac{\beta^2\mu_m^2}{1+\mu_m^2}\partial_{11}\upsilon_m^1$$

and so

$$(\tilde{\mathbf{L}}\upsilon_m^1, \upsilon_m^1) = -\mu_m^2\left(\|\upsilon_m^1\|_{L^2}^2 + \frac{\beta^2}{1+\mu_m^2}\|\partial_1\upsilon_m^1\|_{L^2}^2\right).$$

For $m \neq m'$, note first that the orthogonality $\langle \Upsilon_m, \Upsilon_{m'}\rangle = 0$ gives

$$\begin{aligned} 0 = \langle \Upsilon_m, \Upsilon_{m'}\rangle &= \int \upsilon_m^1\upsilon_{m'}^1 + \int \upsilon_m^2\upsilon_{m'}^2 \\ &= \int \upsilon_m^1\upsilon_{m'}^1 + \frac{\beta^2}{(1+\mu_m^2)(1+\mu_{m'}^2)}\int \partial_1\upsilon_m^1\partial_1\upsilon_{m'}^1. \end{aligned}$$

Thus, for $m \neq m'$,

$$\begin{aligned} (\tilde{\mathbf{L}}\upsilon_m^1, \upsilon_{m'}^1) &= -\mu_m^2\left(\int \upsilon_m^1\upsilon_{m'}^1 + \frac{\beta^2}{1+\mu_m^2}\int \partial_1\upsilon_m^1\partial_1\upsilon_{m'}^1\right) \\ &= -\frac{\beta^2\mu_m^2\mu_{m'}^2}{(1+\mu_m^2)(1+\mu_{m'}^2)}\int \partial_1\upsilon_m^1\partial_1\upsilon_{m'}^1. \end{aligned}$$

Let $w = -\beta^2 \sum_{m=1}^{\bar m} \alpha_m v_m^1$ be nonzero. Then, we obtain for w

$$(\tilde{\mathbf{L}}w, w) = -\beta^2 \sum_{m,m'=1}^{\bar m} \frac{\alpha_m \alpha_{m'} \mu_m^2 \mu_{m'}^2}{(1+\mu_m^2)(1+\mu_{m'}^2)} \int \partial_1 v_m^1 \partial_1 v_{m'}^1 \\ - \sum_{m=1}^{\bar m} \alpha_m^2 \mu_m^2 \left(\|v_m^1\|_{L^2}^2 + \frac{\beta^2}{(1+\mu_m^2)^2} \|\partial_1 v_m^1\|_{L^2}^2 \right).$$

For the first term of the right-hand side, we have the identity

$$\sum_{m,m'=1}^{\bar m} \frac{\alpha_m \alpha_{m'} \mu_m^2 \mu_{m'}^2}{(1+\mu_m^2)(1+\mu_{m'}^2)} \int \partial_1 v_m^1 \partial_1 v_{m'}^1 = \left\| \sum_{m=1}^{\bar m} \frac{\alpha_m \mu_m^2}{1+\mu_m^2} \partial_1 v_m^1 \right\|_{L^2}^2,$$

and we obtain

$$(\tilde{\mathbf{L}}w, w) \leqslant - \sum_{m=1}^{\bar m} \alpha_m^2 \mu_m^2 \left(\|v_m^1\|_{L^2}^2 + \frac{\beta^2}{(1+\mu_m^2)^2} \|\partial_1 v_m^1\|_{L^2}^2 \right),$$

which means that $(\tilde{\mathbf{L}}\cdot, \cdot)$ is negative definite on $\mathrm{Span}(v_m^1, m = 1, \ldots, \bar m)$, a subspace of dimension $\bar m$. By the Sylvester inertia theorem, $\bar m \leqslant \bar k$. In conclusion, we have proved $\bar m = \bar k$. $\square$

Even if $\bar m = \bar k$, we will still use $k \in \{1, \ldots, \bar k\}$ and $m \in \{1, \ldots, \bar m\}$ to denote with clarity the ranges of indices of the negative eigenvalues and corresponding eigenfunctions of the operators $\mathbf{L}$ and $\mathbf{H}$.

2.3. **A coercivity property.** The main result of this section is the following proposition, which states a coercivity property for $\mathbf{H}$ related to the eigenfunctions $Y_k^\pm$ and Ψ_ℓ^0 of $\mathbf{JH}$.

Proposition 3. *There exists $c > 0$ such that, for all $V \in H^1 \times L^2$,*

$$\langle \mathbf{H}V, V \rangle \geqslant c\|V\|^2 - \frac{1}{c} \sum_{\pm, k=1}^{\bar k} \langle \mathbf{H}V, Y_k^\pm \rangle^2 - \frac{1}{c} \sum_{\ell=1}^{\bar \ell} \langle V, \Phi_\ell^0 \rangle^2.$$

This result is a generalization of Proposition 2 in [10] to the case of bound states (we also refer to Lemma 5.2 in [18] for a previous similar result for the energy critical NLS equation). In contrast with previous works in the case of ground states, this result is obtained with no a priori information on the spectrum of $\mathbf{L}$.

Proof. As before, we assume that the Lorentz boost is of the form $(\beta, 0, \ldots, 0)$, with $\beta \in (-1, 1)$. By a standard argument, it suffices to prove that there exists $c > 0$ such that for any $V \in H^1 \times L^2$ with $\langle \mathbf{H}V, Y_k^\pm \rangle = \langle V, \Phi_\ell^0 \rangle = 0$ for all $k = 1, \ldots, \bar k$, $\ell = 1, \ldots, \bar \ell$, and signum $\pm$, there holds

$$\langle \mathbf{H}V, V \rangle \geqslant c\|V\|^2. \tag{25}$$

Let V be such a function. We introduce the L^2-orthogonal decompositions of the functions $Y_k^{\pm}$ and of V:

$$Y_k^{\pm} = \tilde{Y}_k^{\pm} + \sum_{m=1}^{\bar{m}} y_{k,m}^{\pm} \Upsilon_m + \sum_{\ell=1}^{\bar{\ell}} y_{k,\ell}^{\pm,0} \Phi_\ell^0, \qquad \langle \tilde{Y}_k^{\pm}, \Upsilon_m \rangle = \langle \tilde{Y}_k^{\pm}, \Phi_\ell^0 \rangle = 0, \tag{26}$$

$$V = \tilde{V} + \sum_{m=1}^{\bar{m}} v_m \Upsilon_m, \qquad \langle \tilde{V}, \Upsilon_m \rangle = \langle \tilde{V}, \Phi_\ell^0 \rangle = 0. \tag{27}$$

Then, expanding by linearity and using (22), (21), and (23), we get, for any k, k',

$$0 = \langle \mathbf{H} Y_k^{\pm}, Y_{k'}^{\pm} \rangle = \langle \mathbf{H} \tilde{Y}_k^{\pm}, \tilde{Y}_{k'}^{\pm} \rangle - \sum_{m=1}^{\bar{m}} y_{k,m}^{\pm} y_{k',m}^{\pm}, \tag{28}$$

$$0 = \langle \mathbf{H} V, Y_k^{\pm} \rangle = \langle \mathbf{H} \tilde{V}, \tilde{Y}_k^{\pm} \rangle - \sum_{m=1}^{\bar{m}} y_{k,m}^{\pm} v_m, \tag{29}$$

$$\langle \mathbf{H} V, V \rangle = \langle \mathbf{H} \tilde{V}, \tilde{V} \rangle - \sum_{m=1}^{\bar{m}} v_m^2. \tag{30}$$

Claim. The family $(\tilde{Y}_k^{\pm})_{\pm, k=1,\dots,\bar{k}}$ is linearly independent.

Indeed, if there is a dependence relation $\sum_{k=1}^{\bar{k}} (a_k^+ \tilde{Y}_k^+ + a_k^- \tilde{Y}_k^-) = 0$, then for some α_m, α_ℓ^0, $m = 1, \dots, \bar{m}$, $\ell = 1, \dots, \bar{\ell}$,

$$\sum_{k=1}^{\bar{k}} (a_k^+ Y_k^+ + \alpha_k^- Y_k^-) = \sum_{m=1}^{\bar{m}} \alpha_m \Upsilon_m + \sum_{\ell=1}^{\bar{\ell}} \alpha_\ell^0 \Phi_\ell^0.$$

Then we have by the orthogonalities in (26)

$$\begin{aligned} &\left\langle \mathbf{H} \sum_{k=1}^{\bar{k}} (a_k^+ \tilde{Y}_k^+ + a_k^- \tilde{Y}_k^-), \sum_{k=1}^{\bar{k}} (a_k^+ \tilde{Y}_k^+ + a_k^- \tilde{Y}_k^-) \right\rangle \\ &\quad = \left\langle \mathbf{H} \left(\sum_{m=1}^{\bar{m}} \alpha_m \Upsilon_m + \sum_{\ell=1}^{\bar{\ell}} \alpha_\ell^0 \Phi_\ell^0 \right), \sum_{k=1}^{\bar{k}} (a_k^+ \tilde{Y}_k^+ + a_k^- \tilde{Y}_k^-) \right\rangle = 0. \end{aligned}$$

But we also have, by the normalization (23),

$$\begin{aligned} &\left\langle \mathbf{H} \sum_{k=1}^{\bar{k}} (a_k^+ \tilde{Y}_k^+ + a_k^- \tilde{Y}_k^-), \sum_{k=1}^{\bar{k}} (a_k^+ \tilde{Y}_k^+ + a_k^- \tilde{Y}_k^-) \right\rangle \\ &\quad = \left\langle \mathbf{H} \left(\sum_{m=1}^{\bar{m}} \alpha_m \Upsilon_m + \sum_{\ell=1}^{\bar{\ell}} \alpha_\ell^0 \Phi_\ell^0 \right), \sum_{m=1}^{\bar{m}} \alpha_m \Upsilon_m + \sum_{\ell=1}^{\bar{\ell}} \alpha_\ell^0 \Phi_\ell^0 \right\rangle = - \sum_{m=1}^{\bar{m}} \alpha_m^2. \end{aligned}$$

Hence $\alpha_m = 0$ for $m = 1, \dots, \bar{m}$, and

$$\sum_{k=1}^{\bar{k}} (a_k^+ Y_k^+ + \alpha_k^- Y_k^-) = \sum_{\ell=1}^{\bar{\ell}} \alpha_\ell^0 \Phi_\ell^0 \in \ker \mathbf{H}.$$

It follows that

$$0 = \mathbf{J}\mathbf{H}\sum_{k=1}^{\bar k}(a_k^+ Y_k^+ + \alpha_k^- Y_k^-) = \sum_{k=1}^{\bar k}\frac{1}{\gamma}(\lambda_k a_k^+ Y_k^+ - \lambda_k \alpha_k^- Y_k^-).$$

As the $(Y_k^\pm)_{\pm,k=1,\dots,\bar k}$ are linearly independent, we infer that $a_k^+ = a_k^- = 0$ for $k=1,\dots,\bar k$. The dependence relation was in fact trivial, and this proves the claim.

Define the $\bar k \times \bar k$ Gram matrix

$$G^+ = \begin{pmatrix} \langle \mathbf{H}\tilde Y_1^+, \tilde Y_1^+\rangle & \cdots & \langle \mathbf{H}\tilde Y_1^+, \tilde Y_{\bar k}^+\rangle \\ \vdots & & \vdots \\ \langle \mathbf{H}\tilde Y_{\bar k}^+, \tilde Y_1^+\rangle & \cdots & \langle \mathbf{H}\tilde Y_{\bar k}^+, \tilde Y_{\bar k}^+\rangle \end{pmatrix}$$

and the matrix $\mathscr{Y}^+$ (with $\bar k$ lines and $\bar m$ rows)

$$\mathscr{Y}^+ = (y_{k,m}^+)_{\substack{k=1,\dots,\bar k \\ m=1,\dots,\bar m}}.$$

Then the relations in (28) rewrite as the matrix property

$$G^+ = \mathscr{Y}^+(\mathscr{Y}^+)^\top.$$

As the restriction of $\langle \mathbf{H}\cdot,\cdot\rangle$ to $\operatorname{Span}(\Upsilon_m, \Phi_\ell^0, m=1,\dots,\bar m, \ell=1,\dots,\bar\ell)^\perp$ (L^2-orthogonality) is coercive due to (24), the matrix G^+ is positive definite, hence its rank is $\bar k$. This implies that the rank of $\mathscr{Y}^+$ is at least $\bar k$, and as $\bar m = \bar k$, it is an invertible matrix.

One can define similarly $G^- = (\langle \mathbf{H}\tilde Y_k^-, \tilde Y_{k'}^-\rangle)_{\substack{k=1,\dots,\bar k \\ k'=1,\dots,\bar k}}$, which is positive definite and $\mathscr{Y}^- = (y_{k,m}^-)_{\substack{k=1,\dots,\bar k \\ m=1,\dots,\bar m}}$ and get that $\mathscr{Y}^-(\mathscr{Y}^-)^\top = G^-$.

We also see that (29) rewrites as, denoting $\mathscr{V} = (v_m)_{m=1,\dots,\bar m}$ ($\bar m$ lines),

$$\forall k=1,\dots,\bar k, \quad \langle \mathbf{H}V, \tilde Y_k^+\rangle = (\mathscr{Y}^+\mathscr{V})_k, \quad \langle \mathbf{H}V, \tilde Y_k^-\rangle = (\mathscr{Y}^-\mathscr{V})_k,$$

and from (30) we get

$$\langle \mathbf{H}V, V\rangle = \langle \mathbf{H}\tilde V, \tilde V\rangle - \sum_{m=1}^{\bar m} v_m^2 = \langle \mathbf{H}\tilde V, \tilde V\rangle - \mathscr{V}^\top\mathscr{V}. \tag{31}$$

Define the $\mathbf{H}$-orthogonal projection Π^+ on $\operatorname{Span}(\tilde Y_k^+, k=1,\dots,\bar k)$ as

$$\Pi^+ V := \sum_{k=1}^{\bar k} w_k^+ \tilde Y_k^+,$$

where $\mathscr{W}^+ := (w_k^+)_k$ is defined as

$$\mathscr{W}^+ := (G^+)^{-1}\mathscr{Y}^+\mathscr{V}.$$

Indeed, we check that this definition implies that, for all $k'=1,\dots,\bar k$,

$$\langle \mathbf{H}\Pi^+ V, \tilde Y_{k'}^+\rangle = \sum_{k=1}^{\bar k}\langle \mathbf{H}\tilde Y_k^+, \tilde Y_{k'}^+\rangle w_k^+ = (G^+\mathscr{W}^+)_{k'} = (\mathscr{Y}^+\mathscr{V})_{k'} = \langle \mathbf{H}V, \tilde Y_{k'}^+\rangle.$$

As a consequence,

$$\langle \mathbf{H}\Pi^+ V, \Pi^+ V\rangle = (\mathscr{W}^+)^\top G^+ \mathscr{W}^+.$$

We also see from the definition of $\mathscr{W}^+$ that $\mathscr{Y}^+\mathscr{V} = G^+\mathscr{W}^+ = \mathscr{Y}^+(\mathscr{Y}^+)^\top\mathscr{W}^+$ and so, since $\mathscr{Y}^+$ is invertible, we obtain $\mathscr{V} = (\mathscr{Y}^+)^\top\mathscr{W}^+$. Thus,

$$\mathscr{V}^\top\mathscr{V} = (\mathscr{W}^+)^\top G^+\mathscr{W}^+ = \langle \mathbf{H}\Pi^+V, \Pi^+V\rangle.$$

Similarly, defining Π^- as the $\mathbf{H}$-orthogonal projection on $\mathrm{Span}(\tilde{Y}_k^-, k = 1, \dots, \bar{k})$, we have $\mathscr{V}^\top\mathscr{V} = \langle \mathbf{H}\Pi^-V, \Pi^-V\rangle$.

Therefore, we rewrite (31) as

$$\langle \mathbf{H}V, V\rangle = \langle \mathbf{H}\tilde{V}, \tilde{V}\rangle - \sqrt{\langle \mathbf{H}\Pi^+V, \Pi^+V\rangle}\sqrt{\langle \mathbf{H}\Pi^-V, \Pi^-V\rangle}. \tag{32}$$

Let F be the $2\bar{k}$-dimensional vector space $F = \mathrm{Span}(\tilde{Y}_k^\pm, \pm, k = 1, \dots, \bar{k})$ and consider

$$a := \sup_{W\in F\setminus\{0\}} \frac{\sqrt{\langle \mathbf{H}\Pi^+W, \Pi^+W\rangle}\sqrt{\langle \mathbf{H}\Pi^-W, \Pi^-W\rangle}}{\langle \mathbf{H}W, W\rangle}.$$

Recall from (24) that the quadratic form $\langle \mathbf{H}\cdot, \cdot\rangle$ is positive definite on the subspace $\mathrm{Span}(\Upsilon_m, \Phi_\ell^0, m = 1, \dots, \bar{m}, \ell = 1, \dots, \bar{\ell},)^\perp$ (L^2-orthogonality), and so $\langle \mathbf{H}\cdot, \cdot\rangle$ is positive definite on F (in particular, the quantity involved in the definition of a is well defined and positive for all $W \in F\setminus\{0\}$). Then on F, Π^+ and Π^- are orthogonal projections with respect to the *positive definite* scalar product $\langle \mathbf{H}\cdot, \cdot\rangle|_{F\times F}$, and therefore we have

$$\forall W \in F, \quad \langle \mathbf{H}\Pi^+W, \Pi^+W\rangle \leqslant \langle \mathbf{H}W, W\rangle, \quad \langle \mathbf{H}\Pi^-W, \Pi^-W\rangle \leqslant \langle \mathbf{H}W, W\rangle.$$

Moreover, there is equality in the left [resp., right] inequality if and only if $W \in \mathrm{Span}(\tilde{Y}_k^+, k = 1, \dots, \bar{k})$ [resp., $W \in \mathrm{Span}(\tilde{Y}_k^-, k = 1, \dots, \bar{k})$]. It follows that $a \leqslant 1$.

Now, assume for the sake of contradiction that $a = 1$. As F is finite dimensional and hence locally compact, there exists $W \in F \setminus \{0\}$ such that

$$\langle \mathbf{H}\Pi^+W, \Pi^+W\rangle\langle \mathbf{H}\Pi^-W, \Pi^-W\rangle = \langle \mathbf{H}W, W\rangle^2,$$

and so $\langle \mathbf{H}\Pi^+W, \Pi^+W\rangle = \langle \mathbf{H}W, W\rangle = \langle \mathbf{H}\Pi^-W, \Pi^-W\rangle$ and

$$W \in \mathrm{Span}(\tilde{Y}_k^+, k = 1, \dots, \bar{k}) \cap \mathrm{Span}(\tilde{Y}_k^-, k = 1, \dots, \bar{k}).$$

By linear independence of the family $(\tilde{Y}_k^\pm)_{\pm, k=1,\dots,\bar{k}}$, this implies that $W = 0$, a contradiction. This proves that $a < 1$, and therefore (32) gives

$$\langle \mathbf{H}V, V\rangle \geqslant (1 - a)\langle \mathbf{H}\tilde{V}, \tilde{V}\rangle. \tag{33}$$

We can now proceed to prove the coercivity announced in (25). Recall (24):

$$\langle \mathbf{H}\tilde{V}, \tilde{V}\rangle \geqslant c(1 - a)\|\tilde{V}\|^2. \tag{34}$$

We infer that $\langle \mathbf{H}V, V\rangle \geqslant 0$ and then (30) implies that

$$\langle \mathbf{H}\tilde{V}, \tilde{V}\rangle \geqslant \sum_{m=1}^{\bar{m}} v_m^2. \tag{35}$$

Now the decomposition (27) and the triangle inequality also give that

$$\begin{aligned}\|V\| &\leqslant \|\tilde{V}\| + \sum_{m=1}^{\bar{m}} |v_m|\|\Upsilon_m\| \leqslant C_1\left(\|\tilde{V}\| + \sum_{m=1}^{\bar{m}} |v_m|\right)\\ &\leqslant C_1(\bar{m} + 1)\left(\|\tilde{V}\|^2 + \sum_{m=1}^{\bar{m}} |v_m|^2\right)^{\frac{1}{2}},\end{aligned} \tag{36}$$

where $C_1 = 1 + \max(\|\Upsilon_m\|, m = 1, \dots, \bar{m})$. Hence, combining (33), (35), (34), and (36), and denoting $C = \frac{4C_1^2(\bar{m}+1)^2}{(1-a)} \max\big((c(1-a))^{-1}, 1\big)$, we get

$$\begin{aligned} C\langle \mathbf{H}V, V\rangle &\geqslant C(1-a)\langle \mathbf{H}\tilde{V}, \tilde{V}\rangle \geqslant \frac{C(1-a)}{2}\left(\langle \mathbf{H}\tilde{V}, \tilde{V}\rangle + \sum_{m=1}^{\bar{m}} v_m^2\right) \\ &\geqslant \frac{C(1-a)}{2}\left(c(1-a)\|\tilde{V}\|^2 + \sum_{m=1}^{\bar{m}} v_m^2\right) \\ &\geqslant C_1^2(\bar{m}+1)^2\left(\|\tilde{V}\|^2 + \sum_{m=1}^{\bar{m}} v_m^2\right) \geqslant \|V\|^2. \qquad \square \end{aligned}$$

3. Proof of Proposition 2

3.1. **Notation.** Let $N \in \mathbb{N} \setminus \{0\}$ and $\beta_1, \dots, \beta_N \in \mathbb{R}^d$ such that

$$\forall n, \quad |\beta_n| < 1 \quad \text{and} \quad \forall n' \neq n, \quad \beta_{n'} \neq \beta_n; \quad \gamma_n := \frac{1}{\sqrt{1-\beta_n^2}},$$

and let $q_1, q_2, \dots, q_N$ be any bound states of equation (7). Denote by I and I^0 the following two sets of indices:

$$I = \{(n,k) : n = 1, \dots, N,\ k = 1, \dots, \bar{k}_n\}, \quad |I| = \operatorname{Card} I = \sum_{n=1}^{N} \bar{k}_n,$$

$$I^0 = \{(n,\ell) : n = 1, \dots, N,\ \ell = 1, \dots, \bar{\ell}_n\}, \quad |I^0| = \operatorname{Card} I^0 = \sum_{n=1}^{N} \bar{\ell}_n.$$

Denote by $\mathbb{B}$ the closed unit ball of $\mathbb{R}^{|I|}$ for the euclidean norm. For any $n \in \{1, \dots, N\}$, we consider the operators $\mathbf{L}_n$ and $\mathbf{H}_n$ for the bound state q_n, along with the eigenvalues and eigenfunctions defined in Lemma 1: $(\lambda_{n,k})_{(n,k)\in I}$, $(\phi_{n,k})_{(n,k)\in I}$, $(\phi^0_{n,\ell})_{(n,\ell)\in I^0}$, $(\Phi^0_{n,\ell})_{(n,\ell)\in I^0}$, and $(Y^\pm_{n,k})_{(n,k)\in I,\pm}$ (with obvious notation). Let

$$r_n(t,x) = q_n(\Lambda_{\beta_n}(x - \beta_n t)), \quad R_n = \begin{pmatrix} r_n \\ -\beta_n \cdot \nabla r_n \end{pmatrix}, \tag{37}$$

$$\psi^0_{n,\ell}(t,x) = \phi^0_{n,\ell}(\Lambda_{\beta_n}(x - \beta_n t)), \quad \Psi^0_{n,\ell} = \begin{pmatrix} \psi^0_{n,\ell} \\ -\beta_n \cdot \nabla \psi^0_{n,\ell} \end{pmatrix} = \Phi_{n,\ell}(x - \beta_n t), \tag{38}$$

$$\text{and} \quad Z^\pm_{n,k}(t,x) = (\mathbf{H}_n Y^\pm_{n,k})(x - \beta_n t) \tag{39}$$

be their travelling-in-time counterparts. We recall the equation for $\psi^0_{n,\ell}$:

$$(\Delta - (\beta_n \cdot \nabla)^2)\psi^0_{n,\ell} - \psi^0_{n,\ell} + f'(r_n)\psi^0_{n,\ell} = 0. \tag{40}$$

Let $T_0 \gg 1$ be fixed later large enough and let $\omega > 0$ be fixed later small enough, independently of T_0. For brevity, we will omit mentioning the fact that ω is taken small so that estimates hold. It will be convenient in the estimates to introduce the following enveloping functions: for any $n = 1, \dots, N$, we set

$$\rho_n(t,x) = e^{-\omega|x - \beta_n t|} \quad \text{and} \quad \rho = \sum_{n=1}^{N} \rho_n.$$

In particular, ω will be so small and T_0 so large that for any $n \neq n'$,

$$\forall t \geqslant T_0, \forall x \in \mathbb{R}^d, \quad e^{-(p_0-1)\omega_0|x-\beta_n t|} e^{-(p_0-1)\omega_0|x-\beta_{n'} t|} \leqslant e^{-10\omega t}\rho(t,x) \tag{41}$$

(the exponential decay rate $\omega_0 > 0$ was defined in (19)).

Fix any $S_0 \geqslant T_0$. To prove Proposition 2, we show that there exists a choice of coefficients $(\theta^{\pm}_{n,k})_{(n,k)\in I,\pm}$, $|\theta| \ll e^{-\omega S_0}$, such that the backward solution $U(t)$ of (2) with data

$$U(S_0) = \sum_{n=1}^{N} R_n(S_0) + \sum_{\pm,(n,k)\in I^0} \theta^{\pm}_{n,k} Z^{\pm}_{n,k}(S_0) \tag{42}$$

exists on $[T_0, S_0]$ and satisfies the properties of Proposition 2.

We consider such a solution U defined on its maximal backwards interval of existence $[S_{\max}, S_0]$, and we first set

$$U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \sum_{n=1}^{N} R_n + V, \quad V = \begin{pmatrix} v \\ \partial_t v \end{pmatrix}. \tag{43}$$

We further decompose V according to the kernel of the linearized operator around each bound state r_n.

Lemma 3. *For $T_0 > 1$ large enough and $t \geqslant T_0$, there exists $b = (b_{n,\ell})_{(n,\ell)\in I^0}$ such that*

$$W = \begin{pmatrix} w \\ z \end{pmatrix} := V - \sum_{(n,\ell)\in I^0} b_{n,\ell} \Psi^0_{n,\ell} \tag{44}$$

satisfies, for all $(n,\ell) \in I^0$ and $C > 0$ independent of t,

$$\langle W, \Psi^0_{n,\ell}\rangle = 0, \quad |b| \leqslant C\|V\|. \tag{45}$$

Proof. The orthogonality condition in (45) is equivalent to a matrix identity

$$(\langle V, \Psi^0_{n,\ell}\rangle)_{(n,\ell)\in I^0} = \mathscr{H} b,$$

where $b = (b_{n,\ell})_{(n,\ell)\in I^0}$ (written in one row) and

$$\mathscr{H} = (\langle \Psi^0_{n,\ell}, \Psi^0_{n',\ell'}\rangle)_{(n,\ell),(n',\ell')\in I^0} = \mathscr{D}^0 + O(e^{-10\omega t}),$$

$$\mathscr{D}^0 = \operatorname{diag}(\mathscr{H}^0_1, \dots, \mathscr{H}^0_n), \quad \mathscr{H}^0_n = (\langle \Psi^0_{n,\ell}, \Psi^0_{n,\ell'}\rangle)_{\ell,\ell'\in(1,\dots,\bar{\ell}_n)}.$$

Note that for fixed n, the family $(\Psi^0_{n,\ell})_{\ell\in(1,\dots,\bar{\ell}_n)}$ being linearly independent (see Lemma 1), the Gram matrix $\mathscr{H}^0_n$ is invertible. Thus, $\mathscr{D}^0$ is invertible: for T_0 large enough, so is $\mathscr{H}$ and $b = \mathscr{H}^{-1}(\langle V, \Psi^0_{n,\ell}\rangle)_{(n,\ell)\in I^0}$ (and $\mathscr{H}^{-1}$ has uniform norm in $t \geqslant T_0$). ☐

Note that (44) is equivalent to

$$w = v - \sum_{n=1}^{N} \sum_{\ell=1}^{\bar{\ell}_n} b_{n,\ell} \psi^0_{n,\ell}, \tag{46}$$

$$z = v_t + \sum_{n=1}^{N} \sum_{\ell=1}^{\bar{\ell}_n} b_{n,\ell} \beta_n \cdot \nabla \psi^0_{n,\ell} = w_t + \sum_{n=1}^{N} \sum_{\ell=1}^{\bar{\ell}_n} \dot{b}_{n,\ell} \psi^0_{n,\ell}. \tag{47}$$

For the sake of brevity, we denote

$$r = \sum_{n=1}^{N} r_n, \quad \psi^0 = \sum_{n=1}^{N} \sum_{\ell=1}^{\bar{\ell}_n} b_{n,\ell} \psi_{n,\ell}^0 \quad \text{so that} \quad u = r + v = r + \psi^0 + w.$$

Finally, we set

$$a_{n,k}^{\pm} = \langle V, Z_{n,k}^{\pm} \rangle. \tag{48}$$

Observe that $\langle \Psi_{n,\ell}^0, Z_{n',k}^{\pm} \rangle = O(e^{-10\omega t})$ for all $n, n' = 1, \ldots, N$, $\ell = 1, \ldots, \bar{\ell}_n$, $k = 1, \ldots, \bar{k}_{n'}$, and signum $\pm$ (it is obvious by separation and decay (19) when $n \neq n'$, and when $n = n'$ it is equal to $\langle \Phi_{n,\ell}^0, \mathbf{H}_n \mathbf{J} Y_{n',k}^{\pm} \rangle = \langle \mathbf{J} \mathbf{H}_n \Phi_{n,\ell}^0, Y_{n',k}^{\pm} \rangle = 0$). Therefore

$$\langle W, Z_{n,k}^{\pm} \rangle = a_{n,k}^{\pm} + O(|b| e^{-10\omega t}). \tag{49}$$

Let

$$p_0 = \min(2, p), \quad 1 < p_0 \leqslant 2.$$

3.2. **Bootstrap setting.** We consider the following bootstrap estimates:

$$\begin{cases} \|W(t)\| \leqslant e^{-\omega t}, & |b(t)| \leqslant e^{-\omega t}, \\ |a^-(t)| \leqslant e^{-\frac{1}{3}(p_0+2)\omega t}, & |a^+(t)| \leqslant e^{-\frac{1}{3}(p_0+2)\omega t}. \end{cases} \tag{50}$$

We claim that any given initial value of $a^+(S_0)$ can be matched by a suitable choice of initial θ in the definition of $U(S_0)$ in (42).

Lemma 4. *There exists a $\mathscr{C}^1$ map $\Theta : \mathbb{B} \to (e^{-\frac{1}{4}(p_0+3)\omega S_0} \mathbb{B})^2$ such that for any $\mathfrak{a}^+ = (\mathfrak{a}_{n,k}^+)_{(n,k) \in I} \in \mathbb{B}$, if we take $\theta = (\theta_{n,k}^{\pm})_{\pm,(n,k) \in I} = \Theta(\mathfrak{a}^+)$ in the definition of $V(S_0)$ from* (42)*–*(43)*, there holds, for $a^{\pm}(S_0) = (a_{n,k}^{\pm}(S_0))_{(n,k) \in I}$ (defined in* (48)*), that*

$$a^+(S_0) = e^{-\frac{1}{3}(p_0+2)\omega S_0} \mathfrak{a}^+ \quad \text{and} \quad a^-(S_0) = 0. \tag{51}$$

Moreover,

$$\|W(S_0)\| \leqslant e^{-\frac{1}{4}(p_0+3)\omega S_0}, \quad |b(S_0)| \leqslant C e^{-10\omega S_0}. \tag{52}$$

Proof. The proof of this result is similar to that of Lemma 6 in [10] and Lemma 3 in [9]. In view of (42) and (43), it holds that

$$V(S_0) = \sum_{(n',k') \in I, \pm} \theta_{n',k'}^{\pm} Z_{n',k'}^{\pm}(S_0),$$

and so we are looking for a solution $(\theta_{n,k}^{\pm})_{\pm,(n,k) \in I}$ of the equalities:

$$e^{-\frac{1}{3}(p_0+2)\omega S_0} \mathfrak{a}_{n,k}^+ = a_{n,k}^+(S_0) = \sum_{(n',k') \in I, \pm} \theta_{n',k'}^{\pm} \langle Z_{n',k'}^{\pm}(S_0), Z_{n,k}^+(S_0) \rangle,$$

$$0 = a_{n,k}^-(S_0) = \sum_{(n',k') \in I, \pm} \theta_{n',k'}^{\pm} \langle Z_{n',k'}^{\pm}(S_0), Z_{n,k}^-(S_0) \rangle,$$

which rewrites as $a = \mathscr{Z} \theta$, where

$$a = (a_{1,1}^+, a_{1,1}^-, a_{1,2}^+, a_{1,2}^-, \ldots)^\top = (a^{\pm})_{\pm,(n,k) \in I},$$
$$\theta = (\theta_{1,1}^+, \theta_{1,1}^-, \theta_{1,2}^+, \theta_{1,2}^-, \ldots)^\top = (\theta^{\pm})_{\pm,(n,k) \in I},$$

and where $\mathscr{Z}$ is the $2|I| \times 2|I|$ matrix

$$\mathscr{Z} = (\langle Z^{\pm}_{n,k}(S_0), Z^{\pm'}_{n',k'}(S_0)\rangle)_{\pm,\pm',(n,k),(n',k')\in I}.$$

In particular, by (19), for $\omega > 0$ small enough, we note that

$$\mathscr{Z} = \mathscr{D} + O(e^{-10\omega S_0}), \quad \text{where} \quad \mathscr{D} = \operatorname{diag}\left(\mathscr{Z}_1, \dots, \mathscr{Z}_N\right), \quad \text{with}$$

$$\mathscr{Z}_n = \begin{pmatrix} (Z^+_{n,1}(S_0), Z^+_{n,1}(S_0)) & (Z^-_{n,1}(S_0), Z^+_{n,1}(S_0)) & \dots & (Z^-_{n,1}(S_0), Z^+_{n,\bar{k}}(S_0)) \\ (Z^+_{n,1}(S_0), Z^-_{n,1}(S_0)) & (Z^-_{n,1}(S_0), Z^-_{n,1}(S_0)) & \dots & (Z^-_{n,1}(S_0), Z^-_{n,\bar{k}}(S_0)) \\ \vdots & & \ddots & \\ (Z^+_{n,\bar{k}}(S_0), Z^-_{n,1}(S_0)) & (Z^-_{n,1}(S_0), Z^-_{n,1}(S_0)) & \dots & (Z^-_{n,\bar{k}_n}(S_0), Z^-_{n,\bar{k}}(S_0)) \end{pmatrix}.$$

For any $n = 1, \dots, N$, $\mathscr{Z}_n$ is the Gram matrix of the linearly independent family of size $2\bar{k}_n$ $(Z^{\pm}_{n,k}(S_0))_{\pm,k=1,\dots\bar{k}_n}$. Thus, $\mathscr{Z}_n$ is invertible and $\mathscr{D}$ is invertible. It follows that $\mathscr{Z}$ is invertible for T_0 large enough.

Moreover, from (51), for T_0 large enough,

$$|\theta| \leqslant C|a(S_0)| \leqslant Ce^{-\frac{1}{3}(p_0+2)\omega S_0}, \tag{53}$$

and so from the definition of $V(S_0)$ above and the fact that $\langle Z^{\pm}_{n,k}, \Psi^0_{n,\ell}\rangle = 0$ for any n, $k = 1, \dots \bar{k}_n$, $\ell = 1, \dots, \bar{\ell}_n$, and signum $\pm$, we infer that $|\langle V(S_0), \Psi^0_{n,\ell}\rangle| \leqslant Ce^{-10\omega S_0}$.

Recalling the definition of b (at the end of the proof of Lemma 3), we deduce that

$$|b(S_0)| = |\mathscr{H}^{-1}(\langle V, \Psi^0_{n,k}\rangle)_{(n,\ell)\in I^0}| \leqslant C|(\langle V, \Psi^0_{n,k}\rangle)_{(n,\ell)\in I^0}| \leqslant Ce^{-10\omega S_0}.$$

From (53) and (45), we get

$$\|V(S_0)\|, \|W(S_0)\| \leqslant Ce^{-\frac{1}{3}(p_0+2)\omega S_0}.$$

To conclude, simply observe that for large T_0, $Ce^{-\frac{1}{3}(p_0+2)\omega S_0} \leqslant e^{-\frac{1}{4}(p_0+3)\omega S_0}$. □

We define the following backward exit time $S_\star = S_\star(\mathfrak{a}^+)$ related to the bootstrap estimates (50):

$$S_\star = \inf\{T \in [T_0, S_0] \text{ such that } U \text{ is defined and satisfies (50) on } [T, S_0]\}.$$

Note that in view of Lemma 4, $U(S_0)$ satisfies (50) so that $T_0 \leqslant S_\star \leqslant S_0$ is well defined. Our goal is to find a specific choice of $\mathfrak{a}^+ \in \mathbb{B}$ so that $S_\star = T_0$.

The argument goes by contradiction of this condition.

In the next subsections, we fix a choice of $\mathfrak{a}^+ \in \mathbb{B}$, such that

$$T_0 < S_\star(\mathfrak{a}^+) \leqslant S_0. \tag{54}$$

We now derive estimates on $\|W\|$, $|b|$, and $|a^{\pm}|$ on $[S^\star, S_0]$, so as to prove—in Lemma 10—that the flow issued from $\mathfrak{a}^+$ is transverse at the exit time $S_\star = S_\star(\mathfrak{a}^+)$.

3.3. **Equation of W and preliminary estimates.**

Lemma 5. *The function W satisfies*

$$\partial_t W = \begin{pmatrix} z \\ \Delta w - w + f\left(r + \psi^0 + w\right) - f\left(r + \psi^0\right) \end{pmatrix} + \sum_{(n,\ell)\in I^0} \dot{b}_{n,\ell}\Psi^0_{n,\ell} + G, \tag{55}$$

where

$$G = \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad g := f\left(r + \psi^0\right) - \sum_{n=1}^{N} f(r_n) - \sum_{(n,\ell)\in I^0} b_{n,\ell} f'(r_n)\psi^0_{n,\ell}. \tag{56}$$

Proof. First, since U and R_n solve (2), it is direct to check the following equation for v:

$$\partial_t^2 v = \Delta v - v + f\left(r + v\right) - \sum_{n=1}^{N} f(r_n). \tag{57}$$

Next, the first line of (55) follows from the definition of z. For the second line, we observe from the equation of V,

$$\begin{aligned}
\partial_t z &= \partial_t^2 v + \sum_{(n,\ell)\in I^0} \dot{b}_{n,\ell}\beta_n \cdot \nabla\psi^0_{n,\ell} - \sum_{(n,\ell)\in I^0} b_{n,\ell}(\beta_n\cdot\nabla)^2\psi^0_{n,\ell} \\
&= \Delta v - v + f(r+v) - \sum_{n=1}^{N} f(r_n) + \sum_{(n,\ell)\in I^0} \dot{b}_{n,\ell}\beta_n\cdot\nabla\psi^0_{n,\ell} \\
&\quad - \sum_{(n,\ell)\in I^0} b_{n,\ell}(\beta_n\cdot\nabla)^2\psi^0_{n,\ell}.
\end{aligned}$$

Inserting $v = w + \sum_{(n,\ell)\in I^0} b_{n,\ell}\psi^0_{n,\ell} = w + \psi^0$ and using (40), we find the second line of (55). □

Now, we derive some preliminary estimates related to the equation of W. Recall that $p_0 = \min(2,p)$, $1 < p_0 \leqslant 2$.

First, note that for $K > 0$ and any real numbers $(s_j)_{j=1,\dots,\bar{j}}$ such that $|s_j| \leqslant K$, the following hold:

$$\Big| f\Big(\sum_{j=1}^{\bar{j}} s_j\Big) - \sum_{j=1}^{\bar{j}} f(s_j)\Big| \leqslant C(K)\sum_{j\neq j'} |s_j|^{p_0-1}|s_{j'}|^{p_0-1}, \tag{58}$$

$$|f(s_1+s_2) - f(s_1) - s_2 f'(s_1)| \leqslant C(K)|s_2|^{p_0}, \tag{59}$$

$$|f'(s_1+s_2) - f'(s_1)| \leqslant C(K)|s_2|^{p_0-1}. \tag{60}$$

Second, applying these estimates to various situations, using (19) and (41), we obtain

$$\Big| f(r) - \sum_n f(r_n)\Big| \leqslant C\sum_{n\neq n'} |r_n|^{p_0-1}|r_{n'}|^{p_0-1} \leqslant Ce^{-10\omega t}\rho, \tag{61}$$

$$\Big| f(r+\psi^0) - f(r) - f'(r)\psi^0\Big| \leqslant C|\psi^0|^{p_0} \leqslant C|b|^{p_0}\rho, \tag{62}$$

(63)

$$\Big| f'(r)\psi^0 - \sum_{(n,\ell)\in I^0} b_{n,\ell} f'(r_n)\psi^0_{n,\ell}\Big| \leqslant \sum_{(n,\ell)\in I^0} \Big|\psi^0_{n,\ell} b_{n,\ell}(f'(r) - f'(r_n))\Big| \leqslant C|b|e^{-10\omega t}\rho$$

(the implicit constant does essentially depend on $\max(\|q_n\|_{L^\infty}, n = 1,\dots,N)$). In particular, we obtain

$$|g(t)| \leqslant C\left(e^{-10\omega t} + |b(t)|^{p_0}\right)\rho(t). \tag{64}$$

Moreover, we also have

$$|f(r+\psi^0+w)-f(r+\psi^0)| \leqslant C\left(\rho^{p_0-1}|w|^{p_0-1}+|w|^p\right), \tag{65}$$

$$|f(r+\psi^0+w)-f(r+\psi^0)-wf'(r+\psi^0)| \leqslant C\left(|w|^{p_0}+|w|^p\right). \tag{66}$$

Since $p>1$, a similar estimate for F holds:

$$\left|F\left(r+\psi^0+w\right)-F\left(r+\psi^0\right)-wf\left(r+\psi^0\right)\right| \leqslant C\left(|w|^2+|w|^{p+1}\right). \tag{67}$$

3.4. **Degenerate directions. Estimates for** $(b_{n,\ell})$.

Lemma 6. *For all* $t\in[S_\star,S_0]$,

$$|\dot b(t)| \leqslant C\left(\|W(t)\|+e^{-10\omega t}+|b(t)|^{p_0}\right) \leqslant Ce^{-\omega t}. \tag{68}$$

Proof. We differentiate the orthogonality $\langle W,\Psi^0_{n,\ell}\rangle=0$ from (45), using (55),

$$\begin{aligned}
0=\frac{d}{dt}\langle W,\Psi^0_{n,\ell}\rangle &= \langle \partial_t W,\Psi^0_{n,k}\rangle-\beta_n\langle W,\nabla\Psi^0_{n,k}\rangle\\
&=\left\langle\begin{pmatrix}\beta_n\cdot\nabla w+z\\ \Delta w-w+f(r+\psi^0+w)-f(r+\psi^0)+\beta_n\cdot\nabla z\end{pmatrix},\Psi^0_{n,k}\right\rangle\\
&\quad+\sum_{(n',\ell')\in I^0}\dot b_{n',\ell'}\langle\Psi^0_{n',\ell'},\Psi^0_{n,\ell}\rangle+\langle G,\Psi^0_{n,k}\rangle.
\end{aligned}$$

We see that the first term of the equality is bounded by $C\|W\|$, by performing integration by parts so that derivatives fall on the components of $\Psi^0_{n,k}$ and using (58). Using (64) and the notation of the proof of Lemma 3, we obtain

$$|\mathscr{H}b| \leqslant C\left(\|W\|+e^{-10\omega t}+|b|^{p_0}\right),$$

and the first estimate in (68) follows from the the matrix $\mathscr{H}^{-1}$ being uniformly bounded, and the second, from the bootstrap estimate (50). $\square$

3.5. **Energy properties.** We let

$$\mathscr{E}_W=\frac12\int |z|^2+|\nabla w|^2+|w|^2-2\left[F\left(r+\psi^0+w\right)-F\left(r+\psi^0\right)-wf\left(r+\psi^0\right)\right].$$

We consider a C^∞ radial function $\chi:\mathbb{R}^d\to\mathbb{R}$ such that

$$\chi(x)=0 \text{ for } |x|\geqslant 2,\quad \chi(x)=1 \text{ for } |x|<1,\quad 0\leqslant\chi\leqslant 1 \text{ on } \mathbb{R}^d. \tag{69}$$

We set

$$\mathscr{P}_n=\frac12\int\chi_n z\nabla w,\quad \chi_n(t,x)=\chi\left(\frac{x-\beta_n t}{\delta t}\right), \tag{70}$$

where

$$\delta=\frac{1}{10}\min\{|\beta_n-\beta_{n'}|:\ 1\leqslant n,n'\leqslant N,\ n\neq n'\}$$

and

$$\mathscr{F}(t)=\mathscr{E}_W(t)+2\sum_{n=1}^N\beta_n\cdot\mathscr{P}_n(t). \tag{71}$$

Lemma 7. *For all* $t\in[S_\star,S_0]$,

$$\left|\frac{d}{dt}\mathscr{F}(t)\right| \leqslant \frac{C}{t}e^{-2\omega t}, \tag{72}$$

$$\|W(t)\|^2 \leqslant C\left(\mathscr{F}(t)+|a(t)|^2+e^{-10\omega t}\right). \tag{73}$$

Proof of (72). First, we see that

$$\frac{d\mathscr{E}_W}{dt} = \int z\partial_t z + \int \partial_t w \left[-\Delta w + w - f\left(r+\psi^0+w\right) + f\left(r+\psi^0\right)\right] \\ - \int \partial_t(r+\psi^0)\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right].$$

Using (55),

$$\int z\partial_t z = \int (\partial_t w + \sum_{n,\ell} \dot b_{n,\ell}\psi^0_{n,\ell})[\Delta w - w + f(r+\psi^0+w) - f(r+\psi^0)] \\ + \int z(\sum_{n,\ell} \dot b_{n,\ell}\beta_n\cdot\nabla\psi^0_{n,\ell} + g).$$

Now, we claim

$$\begin{aligned} &\int \psi^0_{n,\ell}[\Delta w - w + f(r+\psi^0+w) - f(r+\psi^0)] \\ &\quad = \int w(\beta_n\cdot\nabla)^2\psi^0_{n,\ell} + O(\|w\|_{L^2}^{p_0} + \|w\|_{L^2}\left(e^{-10\omega t} + |b|\right)). \end{aligned} \tag{74}$$

Indeed, using (40),

$$\begin{aligned} &\int \psi^0_{n,\ell}[\Delta w - w + f(r+\psi^0+w) - f(r+\psi^0)] \\ &\quad = \int \psi^0_{n,\ell}[\Delta w - w + wf'(r_n)] \\ &\qquad + \int \psi^0_{n,\ell}\left[f(r+\psi^0+w) - f(r+\psi^0) - wf'(r_n)\right] \\ &\quad = \int w(\beta_n\cdot\nabla)^2\psi^0_{n,\ell} + \int \psi^0_{n,\ell}\left[f(r+\psi^0+w) - f(r+\psi^0) - wf'(r_n)\right]. \end{aligned}$$

Moreover, by (66) and (58),

$$\begin{aligned} &\left|\int \psi^0_{n,\ell}\left[f(r+\psi^0+w) - f(r+\psi^0) - wf'(r_n)\right]\right| \\ &\quad \leqslant C\int \rho_n\left|f(r+\psi^0+w) - f(r+\psi^0) - wf'(r+\psi^0)\right| \\ &\qquad + C\int \rho_n|w|\left|f'(r+\psi^0) - f'(r)\right| + C\int \rho_n|w|\,|f'(r) - f'(r_n)| \\ &\quad \leqslant C\|w\|_{L^2}^{p_0} + C\|w\|_{L^2}\left(e^{-10\omega t} + |b|\right), \end{aligned}$$

which proves (74).

Next, using (64), we have

$$\left|\int zg\right| \leqslant C(e^{-10\omega t} + |b|^{p_0})\|z\|_{L^2}. \tag{75}$$

Thus,

$$\int z\partial_t z + \int \partial_t w \left[-\Delta w + w - f\left(r+\psi^0+w\right) + f\left(r+\psi^0\right)\right]$$
$$= \int w \left(\sum_{(n,\ell)\in I^0} \dot{b}_{n,\ell}(\beta_n\cdot\nabla)^2\psi^0_{n,\ell}\right) + \int z\left(\sum_{(n,\ell)\in I^0}\dot{b}_{n,\ell}(\beta_n\cdot\nabla\psi^0_{n',\ell})\right)$$
$$+ O\left(|\dot{b}|\left[\|w\|^{p_0}_{L^2} + \|w\|_{L^2}\left(e^{-10\omega t}+|b|\right)\right]\right) + O\left(\|z\|_{L^2}(e^{-10\omega t}+|b|^{p_0})\right).$$

Finally, we have

$$-\int \partial_t r\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right]$$
$$= \sum_{n=1}^N \int (\beta_n\cdot\nabla r_n)\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right],$$

and, since $\partial_t(b_{n,\ell}\psi^0_{n,\ell}) = \dot{b}_{n,\ell}\psi^0_{n,\ell} - b_{n,\ell}(\beta_n\cdot\nabla\psi^0_{n,\ell})$, by (66),

$$-\int \partial_t\psi^0\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right] = O\left((|b|+|\dot{b}|)\|W\|^2\right).$$

Thus, in conclusion, using also (68) to control $|\dot{b}| \leqslant C\left(\|W\| + e^{-10\omega t} + |b|^{p_0}\right)$,

$$\begin{aligned}\frac{d\mathscr{E}_W}{dt} &= \int w\left(\sum_{(n,\ell)\in I^0}\dot{b}_{n,\ell}(\beta_n\cdot\nabla)^2\psi^0_{n,\ell}\right) + \int z\left(\sum_{(n,\ell)\in I^0}\dot{b}_{n,\ell}(\beta_n\cdot\nabla\psi^0_{n,\ell})\right)\\ &\quad+\sum_n\int(\beta_n\cdot\nabla r_n)\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right]\\ &\quad+ O\left(\|W\|^{p_0+1} + |b|\|W\|^{p_0} + (e^{-10\omega t}+|b|^{p_0})\|W\|\right).\end{aligned} \tag{76}$$

Now, we compute

$$\frac{d\mathscr{P}_n}{dt} = \frac{1}{2}\int(\partial_t\chi_n)z\nabla w + \frac{1}{2}\int\chi_n z\nabla\partial_t w + \frac{1}{2}\int \chi_n\nabla w\,\partial_t z$$
$$= -\frac{1}{2}\int\left(\frac{x}{t}\cdot\nabla\chi_n\right)z\nabla w + \frac{1}{2}\int\chi_n z\nabla z - \frac{1}{2}\int\chi_n z\left(\sum_{n',\ell}\dot{b}_{n',\ell}\nabla\psi^0_{n',\ell}\right)$$
$$+\frac{1}{2}\int\chi_n\Delta w\nabla w - \frac{1}{2}\int\chi_n w\nabla w$$
$$+\frac{1}{2}\int\chi_n\nabla w\left(f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right)\right)$$
$$+\frac{1}{2}\int\chi_n\nabla w\left(\sum_{n',\ell}\dot{b}_{n',\ell}(\beta_{n'}\cdot\nabla\psi^0_{n',\ell})\right) + \frac{1}{2}\int\chi_n g\nabla w.$$

Integrating by parts, this writes

$$\begin{aligned}
\frac{d\mathscr{P}_n}{dt} &= -\frac{1}{2}\int \left(\frac{x}{t}\cdot\nabla\chi_n\right) z\nabla w - \frac{1}{4}\int \nabla\chi_n z^2 - \frac{1}{4}\int \nabla\chi_n |\nabla w|^2 \\
&\quad + \frac{1}{2}\int (\nabla\chi_n\cdot\nabla w)\nabla w + \frac{1}{4}\int w^2\nabla\chi_n \\
&\quad - \frac{1}{2}\int \nabla\chi_n \left[F\left(r+\psi^0+w\right) - F\left(r+\psi^0\right) - wf\left(r+\psi^0\right)\right] \\
&\quad - \frac{1}{2}\int \chi_n\nabla(r+\psi^0)\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right] \\
&\quad - \frac{1}{2}\int \chi_n z\left(\sum_{n',\ell}\dot b_{n',\ell}\nabla\psi^0_{n',\ell}\right) + \frac{1}{2}\int \chi_n\nabla w\left(\sum_{n',\ell}\dot b_{n',\ell}(\beta_{n'}\cdot\nabla\psi^0_{n',\ell})\right) \\
&\quad + \frac{1}{2}\int \chi_n g\nabla w.
\end{aligned}$$

For the terms on the first three lines, we use (67) to bound

$$\begin{aligned}
&\left|\int \left(\frac{x}{t}\cdot\nabla\chi_n\right) z\nabla w\right| + \left|\int \nabla\chi_n z^2\right| + \left|\int \nabla\chi_n|\nabla w|^2\right| + \left|\int(\nabla\chi_n\cdot\nabla w)\nabla w\right| \\
&+ \left|\int \nabla\chi_n w^2\right| + \left|\int \nabla\chi_n\left[F\left(r+\psi^0+w\right) - F\left(r+\psi^0\right) - wf\left(r+\psi^0\right)\right]\right| \\
&\qquad \leqslant C\|\nabla\chi_n\|_{L^\infty}\int\left(z^2+|\nabla w|^2+w^2+|w|^{p+1}\right) \leqslant \frac{C}{t}\|W\|^2.
\end{aligned}$$

For the fourth line, using (66), we have

$$\begin{aligned}
&\left|\int(1-\chi_n)\nabla r_n\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right]\right| \\
&\qquad \leqslant C\|\nabla q_{n,\beta_n}\|_{L^\infty(|x|>\delta t)}\|W\|^{p_0} \leqslant Ce^{-10\omega t}\|W\|^{p_0},
\end{aligned}$$

and for $n'\neq n$, using (19),

$$\left|\int \chi_n z\nabla\psi^0_{n',\ell}\right| + \left|\int \chi_n\nabla w(\beta_{n'}\cdot\nabla\psi^0_{n',\ell})\right| \leqslant Ce^{-10\omega t}\|W\|.$$

Moreover, by (64)

$$\left|\int \chi_n g\nabla w\right| \leqslant C(e^{-10\omega t}+|b|^{p_0})\|W\|.$$

Thus, in conclusion for this term

$$\begin{aligned}
\frac{d\mathscr{P}_n}{dt} &= -\frac{1}{2}\int z\left(\sum_{\ell=1}^{\bar\ell_n}\dot b_{n,\ell}\nabla\psi^0_{n,\ell}\right) + \frac{1}{2}\int\nabla w\left(\sum_{\ell=1}^{\bar\ell_n}\dot b_{n,\ell}(\beta_n\cdot\nabla\psi^0_{n,\ell})\right) \\
&\quad + \frac{1}{2}\int \nabla r_n\left[f\left(r+\psi^0+w\right) - f\left(r+\psi^0\right) - wf'\left(r+\psi^0\right)\right] \\
&\quad + O\left(\frac{1}{t}\|W\|^2 + (e^{-10\omega t}+|b|^{p_0})\|W\|\right).
\end{aligned} \tag{77}$$

Combining (76) and (77), we find

$$\left|\frac{d\mathscr{F}}{dt}\right| \leqslant C\left(\|W\|^{p_0+1} + \left(|b| + t^{-1}\right)\|W\|^2 + (e^{-10\omega t} + |b|^{p_0})\|W\|\right). \tag{78}$$

Using (50), we find (72). □

Proof of (73). Expanding $\mathscr{F}(t)$ we get that

$$\mathscr{F}(t) = \sum_{n=1}^{N}\langle \tilde{\mathbf{H}}_n(t)W, W\rangle + O(\|W\|^{p_0+1} + e^{-10\omega t}),$$

where $\tilde{\mathbf{H}}_n(t)$ is the analog of $\mathbf{H}_n$, localized on the ball $B(0,\delta t)$ and translated by $\beta_n t$. Using standard localization arguments and Proposition 3, we infer the following property:
(79)

$$\|W(t)\|^2 \leqslant C\left(\mathscr{F}(t) + \sum_{(n,k)\in I,\pm}|\langle W, Z^{\pm}_{n,k}\rangle|^2 + \sum_{(n,\ell)\in I^0}|\langle W, \Psi^0_{n,\ell}\rangle|^2 + e^{-10\omega t}\right).$$

We refer e.g. to [36] for further details.

Recall that by construction (45), $\langle W, \Psi^0_{n,\ell}\rangle = 0$. Finally by (49) (and the bootstrap (50)), we have

$$|\langle W, Z^{\pm}_{n,k}\rangle| \leqslant |a^{\pm}_{n,k}(t)| + C|b(t)|e^{-10\omega t} \leqslant C(|a| + e^{-10\omega t}).$$ □

3.6. **Negative directions. Transversality at $S_\star$.** Recall that we have set $a^{\pm}_{n,k} = \langle V, Z^{\pm}_{n,k}\rangle$.

Lemma 8 (Negative directions)**.** *For all $t \in [S_\star, S_0]$,*

$$\left|\dot{a}^{\pm}_{n,k}(t) \pm \frac{\lambda_{n,k}}{\gamma_n}a^{\pm}_{n,k}(t)\right| \leqslant Ce^{-\frac{1}{2}(p_0+1)\omega t}. \tag{80}$$

Proof. From (57), we rewrite the equation of V as

$$\partial_t V = \begin{pmatrix} \partial_t v \\ \Delta v - v + f(r+v) - f(r)\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - 1 + f'(r_n) & 0\end{pmatrix} V + G_n,$$

where

$$G_n = \begin{pmatrix} 0 \\ g_n \end{pmatrix}, \quad g_n = f(r+v) - f(r) - f'(r_n)v.$$

Then, by (16)

$$\begin{aligned}
\frac{d}{dt}a^{\pm}_{n,k} &= \langle \partial_t V, Z^{\pm}_{n,k}\rangle - \beta_n\langle V, \nabla Z^{\pm}_{n,k}\rangle \\
&= \left\langle \begin{pmatrix} \beta_n\cdot\nabla & 1 \\ \Delta - 1 + f'(r_n) & \beta_n\cdot\nabla\end{pmatrix} V, Z^{\pm}_{n,k}\right\rangle + \langle G_n, Z^{\pm}_{n,k}\rangle \\
&= \left\langle \mathbf{J}\mathbf{H}_n V(\cdot + \beta_n t), \mathbf{H}_n Y^{\pm}_{n,k}\right\rangle + \langle G_n, Z^{\pm}_{n,k}\rangle \\
&= -\left\langle V(\cdot + \beta_n t), \mathbf{H}_n(\mathbf{J}\mathbf{H}_n Y^{\pm}_{n,k})\right\rangle + \langle G_n, Z^{\pm}_{n,k}\rangle = \mp\frac{\lambda_{n,k}}{\gamma_n}a^{\pm}_{n,k} + \langle G_n, Z^{\pm}_{n,k}\rangle.
\end{aligned}$$

Now, we estimate $\langle G_n, Z_{n,k}^{\pm}\rangle$. By (58) and (59), one has

$$\begin{aligned}|g_n| &= |f(r+v) - f(r_n) - f'(r_n)v| \\ &\leqslant |f(r+v) - f(r_n+v)| + |f(r_n+v) - f(r_n) - f'(r_n)v| \\ &\leqslant \sum_{n',n'\neq n} |f(r'_n)| + Ce^{-10\omega t}\rho + \sum_{n',n'\neq n} |v|^{p_0-1}|r_{n'}|^{p_0-1} + C|v|^{p_0}.\end{aligned}$$

Thus,

$$|\langle G_n, Z_{n,k}^{\pm}\rangle| \leqslant C\|V\|^{p_0} + Ce^{-10\omega t} \leqslant C\|W\|^{p_0} + C|b|^{p_0} + Ce^{-10\omega t}.$$

Therefore, we have obtained, using the bootstrap estimates (50) for the final bound,

$$\left|\dot a_{n,k}^{\pm} \pm \frac{\lambda_{n,k}}{\gamma_n} a_{n,k}^{\pm}\right| \leqslant C\left(\|W\|^{p_0} + |b|^{p_0} + Ce^{-10\omega t}\right) \leqslant Ce^{-\frac{1}{2}(p_0+1)\omega t}. \qquad \square$$

We close the estimates for $\|W\|$, $|b|$, and $|a^-|$ in the following result.

Lemma 9. *For all $t \in [S_\star, S_0]$,*

$$\|W(t)\| \leqslant \frac{1}{2}e^{-\omega t}, \quad |b(t)| \leqslant \frac{1}{2}e^{-\omega t}, \quad |a^-(t)| \leqslant \frac{1}{2}e^{-\frac{1}{3}(p_0+2)\omega t}. \tag{81}$$

Moreover,

$$|a^+(S_\star)| = e^{-\frac{1}{3}(p_0+2)\omega S_\star}. \tag{82}$$

Proof. Let $t \in [S_\star, S_0]$. First, $\mathscr{F}(0) \leqslant C\|W(S_0)\|^2$ so that with (52), integrating (72) on $[t, S_0]$ yields

$$\mathscr{F}(t) \leqslant \frac{C}{t}e^{-\omega t}.$$

It follows from (73) and the bootstrap assumption (50) that

$$\begin{aligned}\|W(t)\|^2 &\leqslant C\mathscr{F}(t) + C|a(t)|^2 + Ce^{-10\omega t} \\ &\leqslant \frac{C}{t}e^{-2\omega t} + Ce^{-\frac{2}{3}(p_0+2)\omega S_0} + Ce^{-10\omega t} \leqslant \frac{C}{t}e^{-2\omega t}.\end{aligned}$$

For $T_0 \geqslant 4C$ large enough, we get $\|W(t)\| \leqslant \frac{1}{2}e^{-\omega t}$.

Then we integrate (68) on $[t, S_0]$, using (52) and our improved bound $\|W(t)\| \leqslant \frac{C}{\sqrt{t}}e^{-\omega t}$, and we find

$$|b(t)| \leqslant C\left(\frac{1}{\sqrt{t}}e^{-\omega t} + e^{-10\omega S_0}\right) \leqslant \frac{1}{2}e^{-\omega t}.$$

In view of (51) and (80), we have

$$\left|\frac{d}{dt}\left[e^{-\frac{\lambda_{n,k}}{\gamma_n}t}a_{n,k}^-(t)\right]\right| \leqslant Ce^{-\frac{\lambda_{n,k}}{\gamma_n}t - \frac{1}{2}(p_0+1)\omega t}, \quad a_{n,k}^-(S_0) = 0.$$

Thus, by integration on $[t, S_0]$, $|a_{n,k}^-(t)| \leqslant Ce^{-\frac{1}{2}(p_0+1)\omega t}$. Taking the $\ell^2(\mathbb{R}^{|I|})$ norm, we get

$$|a^-(t)| \leqslant \frac{1}{2}e^{-\frac{1}{3}(p_0+1)\omega t}.$$

From the contradiction assumption (54) and a continuity argument, it follows that there must be at least one equality in the bootstrap assumptions (50) at time $S_\star$: in view of the above, the only possibility is $|a^+(S_\star)| = e^{-\frac{1}{3}(p_0+2)\omega S_\star}$. $\square$

Now, we are in a position to state the transversality condition on a^+ at $S_\star$.

Lemma 10. *Let* $\mathscr{A}(\mathfrak{a}^+, t) = e^{\frac{2}{3}(p_0+2)\omega t}|a^+(t)|^2$. *Then*

$$\frac{\partial}{\partial t}\mathscr{A}(\mathfrak{a}^+, t)|_{t=S_\star(\mathfrak{a}^+)} < 0. \tag{83}$$

Proof. Let $c_0 = \min\left(\frac{\lambda_{n,k}}{\gamma_n}, (n,k) \in I\right)$ and consider $\omega > 0$ so small that $\frac{8}{3}\omega < c_0$. We compute from (80) and (50), for any $t \in [S_\star, S_0]$, that

$$\begin{aligned}
\frac{d}{dt}\mathscr{A}(\mathfrak{a}^+, t) &= \frac{2}{3}(p_0+2)\omega\mathscr{A}(\mathfrak{a}^+, t) + 2e^{\frac{2}{3}(p_0+2)\omega t}\sum_{(n,k)\in I}\dot{a}^+_{n,k}(t)a^+_{n,k}(t) \\
&= \frac{2}{3}(p_0+2)\omega\mathscr{A}(\mathfrak{a}^+, t) - 2e^{\frac{2}{3}(p_0+2)\omega t}\sum_{(n,k)\in I}\frac{\lambda_{n,k}}{\gamma_n}|a^\pm_{n,k}(t)|^2 \\
&\quad + O\left(e^{\frac{2}{3}(p_0+2)\omega t}e^{-\frac{1}{2}(p_0+1)\omega t}e^{-\frac{1}{3}(p_0+2)t}\right) \\
&\leqslant -c_0\mathscr{A}(\mathfrak{a}^+, t) + O\left(e^{-\frac{1}{6}(p_0-1)\omega t}\right)
\end{aligned}$$

(because for all $(n,k) \in I$, $2\frac{\lambda_{n,k}}{\gamma_n} \geqslant 2c_0$ and $\frac{2}{3}(p_0+2)\omega \leqslant \frac{8}{3}\omega \leqslant c_0$). Now, at time $t = S_\star$, by (82), we have $\mathscr{A}(\mathfrak{a}^+, S_\star) = 1$ and so

$$\frac{d}{dt}\mathscr{A}(\mathfrak{a}^+, t)|_{t=S_\star(\mathfrak{a})} \leqslant -c_0 + O\left(e^{-\frac{1}{6}(p_0-1)\omega t}\right) \leqslant -\frac{c_0}{2} < 0$$

for T_0 large enough. ☐

3.7. **Conclusion: Topological argument.** To conclude the proof of Proposition 2, we argue by contradiction, and assume that for any $\mathfrak{a}^+ \in \mathbb{B}$, (54) holds, that is, $S_\star(\mathfrak{a}^+) > T_0$.

Then Lemma 10 applies to any $\mathfrak{a}^+ \in \mathbb{B}$: as a standard consequence of this transversality result, the map

$$\mathscr{M} : \mathbb{B} \to \mathbb{S}, \quad \mathfrak{a}^+ \mapsto \mathscr{M}(\mathfrak{a}^+) := e^{\frac{1}{3}(p_0+2)\omega S_\star}a^+(S_\star)$$

is well defined on the unit ball $\mathbb{B}$ with values in the unit sphere $\mathbb{S}$ of $\mathbb{R}^{|I|}$, continuous, and its restriction to $\mathbb{S}$ is the identity. A contradiction is reached from Brouwer's theorem. Hence there exists at least one $\mathfrak{a}^+ \in \mathbb{B}$ such that $S_\star(\mathfrak{a}^+) = T_0$, and it provides the sought for solution U of (2).

We refer to [9] and [10] for more details.

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