On the Chern numbers of a smooth threefold

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Abstract. We study the behaviour of Chern numbers of three-dimensional terminal varieties under divisorial contractions.

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1. Introduction

The main goal of this paper is to study the Chern numbers of a smooth projective threefold, especially in relation with divisorial contractions. To this aim we will investigate the interplay between topological properties and birational properties of threefolds.

The starting point of our research is the following question of Hirzebruch [Hir54]: Which linear combinations of Chern numbers on a smooth complex projective variety are topologically invariant?

Hirzebruch’s question has been answered by Kotschick [Kot08, Kot12], who showed that a rational linear combination of Chern numbers is a homeomorphism invariant of smooth complex projective varieties if and only if it is a multiple of the Euler characteristic. In particular, Kotschick shows the existence of a sequence of infinitely many pairs of smooth projective threefolds $X_i, Y_i$, with $i \in \mathbb{N}$, such that $X_i$ and $Y_i$ are diffeomorphic and

$$c_1c_2(X_i) \neq c_1c_2(Y_i) \quad \text{and} \quad c_3(X_i) \neq c_3(Y_i)$$

for each $i \in \mathbb{N}$.

In view of this, it is natural to ask if the Chern numbers of an $n$-dimensional smooth projective variety can only assume finitely many values after we fix the underlying manifold. In general, $c_n$ is a topological invariant, as it coincides with the Euler characteristic, and therefore if $n = 1$, then the problem is easily settled.
On the other hand, if $X$ and $Y$ are homeomorphic complex surfaces, then either $c_1^3(X) = c_1^3(Y)$ or $c_1^3(X) = 4c_2(Y) - c_1^3(Y)$, depending on whether the homeomorphism between $X$ and $Y$ is orientation preserving or not (cf. [Kot08]). Nevertheless, if $X$ and $Y$ are diffeomorphic surfaces, then $c_1(X)^2 = c_1(Y)^2$.

In dimension 3, the relevant Chern numbers are $c_1c_2$ and $c_1^3$. If $X$ is Kähler, then by the Hirzebruch–Riemann–Roch theorem we have

$$\left| \frac{1}{24} c_1c_2(X) \right| = |\chi(O_X)| = |1 - h^{1,0} + h^{2,0} - h^{3,0}| \leq 1 + b_1 + b_2 + b_3,$$

where $h^{i,0} = h^i(X, O_X)$ and $b_1, b_2$, and $b_3$ denote the topological Betti numbers of $X$. Thus, $c_1c_2(X)$ is bounded by a linear combination of the Betti numbers of $X$. On the other hand, LeBrun [LeB99] shows that the same result does not hold if we drop the assumption of being Kähler, answering a question raised by Okonek and Van de Ven [OVdV95]. In particular, he shows that if $M$ denotes the 4-manifold underlying a $K3$ surface and $S^2$ is the two-dimensional sphere, then there exist infinitely many complex structures $J_m$ on $M \times S^2$ such that $c_1c_2 = 48m$, with $m \in \mathbb{N}$.

More generally, in dimension $n$, Libgober and Wood [LW90] showed that $c_1c_{n-1}$ can be expressed in terms of Hodge numbers and, in particular, it is bounded by a constant that depends only on the Betti numbers of the underlying topological space. Recently, Schreieder and Tasin [ST16] studied the problem in dimension at least 4, proving that in complex dimension $n \geq 4$, the Chern numbers $c_n$, $c_1c_{n-1}$, and $c_2^3$ ($n = 4$) are the only Chern numbers that take on only finitely many values on the complex projective structures with the same underlying smooth $2n$-manifold.

Thus, the motivating question of this paper is the following.

**Question 1.1** ([Kot08] Problem 1)). Does $c_1^3 = -K_X^3$ take only finitely many values on the projective algebraic structures $X$ with the same underlying 6-manifold?

Our aim is to study this problem from a birational point of view.

Let $X$ be a smooth threefold. We first consider Question 1.1 in three extreme cases which arise as building blocks in birational geometry: Fano manifolds, Calabi–Yau and canonically polarized varieties. In the first case, it is known that $X$ belongs to a bounded family and, in particular, $K_X^3$ is bounded [Kol93a]. If $X$ is Calabi–Yau, then by definition $K_X = 0$ and therefore $K_X^3 = 0$. Finally, if $X$ is canonically polarized (i.e., $K_X$ is ample), then the Bogomolov–Miyaoka–Yau inequality implies that $0 < K_X^3 \leq 8/3c_1c_2(X)$. Thus, the arguments above imply that $K_X^3$ is bounded by the Betti numbers of $X$.

We now consider the general case of a smooth projective threefold $X$. Thanks to Mori’s program [KM98], we can run a Minimal Model Program (MMP, in short) on $X$ and obtain a birational map $\varphi: X \to Y$ into a threefold $Y$ such that either $X$ is not uniruled and $Y$ is minimal (i.e., the canonical divisor $K_Y$ is nef) or $X$ is uniruled and $Y$ admits a Mori fibre space structure (i.e., a morphism $Y \to Z$ with connected fibres with relative Picard number equal to one and whose general fibre is a nontrivial Fano variety). Thus, our strategy consists in two steps: we first want to bound $K_Y^3$ and then bound $K_X^3 - K_Y^3$.

One of the difficulties of the first step is due to the fact that in general $Y$ is not smooth, but it admits some mild singularities, called terminal. On the other hand, by [CZ14], we can bound the singularities of $Y$, and in particular the index
of each singularity, by a bound which depends only on the topology of $X$ (see Proposition 2.3).

Recall that a variety of dimension $n$ is said to be uniruled if there exists a variety $Y$ of dimension $n - 1$ and a dominant rational map $Y \times \mathbb{P}^1 \to X$. In particular if $X$ is uniruled, then it is covered by rational curves, i.e., for each $x \in X$ there exists a nontrivial morphism $f: \mathbb{P}^1 \to X$ such that $x \in f(\mathbb{P}^1)$.

Note that if $X$ is not uniruled then $Y$ is minimal and $K_Y^3$ coincides with the volume of $X$ (cf. Definition 2.1), which is a birational invariant of the variety $X$.

Our first result, based on the Bogomolov–Miyaoka–Yau inequality for terminal threefolds, is the following.

**Theorem 1.2.** Let $X$ be a smooth complex projective threefold which is not uniruled. Then

$$\text{vol}(X, K_X) \leq 6b_2(X) + 36b_3(X).$$

An interesting consequence is that the volume only takes finitely many values on the family of smooth projective varieties of general type with fixed underlying 6-manifold (see Corollary 4.1). A second consequence (which follows immediately applying [HM06]) is that the family of all smooth complex projective threefolds of general type with bounded Betti numbers is birationally bounded (see Corollary 4.2). Such questions remain open in higher dimensions. In a forthcoming paper, we plan to study the Chern numbers of a variety $Y$ which admits a Mori fibre space structure.

We now describe the second part of our program: we want to determine how the Chern number $c_3$ varies under the MMP. Recall that if $X$ is a smooth projective threefold and we run an MMP on $X$, then we obtain a birational map $X \to Y$ as a composition of elementary transformations, given by divisorial contractions and flips:

$$X = X_0 \to X_1 \to \cdots \to X_m = Y.$$  

We plan to bound $K_{X_k}^3 - K_{X_{k-1}}^3$ at each step, in terms of the topology of the manifold underlying $X$.

In this paper we consider the case of divisorial contractions. Recall that a divisorial contraction $X_{k-1} \to X_k$ is a birational morphism which contracts a prime divisor $E$ into either a point or a curve. The first case can be easily handled thanks to Kawakita’s classification [Kaw05]. In particular, we can show that

$$0 < K_{X_{k-1}}^3 - K_{X_k}^3 \leq 2^{10}b_2^2,$$

where $b_2 = b_2(X)$ is the second Betti number of $X$ (see Proposition 4.4).

The case of divisorial contractions to curves is much harder. In general, in this case, the difference between the Chern numbers may not be bounded by a combination of Betti numbers (e.g., consider a blowup of a rational curve of degree $d$ in $\mathbb{P}^3$). To deal with this situation, we study the integral cubic form $F_X$, associated to the cup product on $H^2(X, \mathbb{Z})$. The cubic form $F_X$ is one of the most important topological invariants of a smooth threefold $X$ and much topological information of $X$ is encoded in the cubic form $F_X$ (e.g., see [OVdV95]). In the case of a blowdown to a smooth curve $f: W \to Z$, the cubic form $F_W$ assumes a special form

$$F_W(x_0, \ldots, x_n) = ax_0^3 + 3x_0^2\left(\sum_{i=1}^n b_ix_i\right) + F_Z(x_1, \ldots, x_n),$$

where $a$ and $b_i$ are coefficients related to the geometry of the blowup and the curve.
which we call reduced form. The goal of section 3 is to prove a finiteness result on the number of possible reduced forms in the case of cubic forms with nonzero discriminant (see Theorem 3.1).

In particular, we can associate to any projective threefold $X$ a topological invariant $S_X$ which is an integer number depending only on the cubic form $F_X$ of $X$ (see Definition 2.12).

Our main result is Theorem 1.3. It is obtained by combining methods in birational geometry, topology, and arithmetic geometry.

**Theorem 1.3.** Let $Y$ be a terminal $\mathbb{Q}$-factorial threefold with associate cubic form $F_Y$, and let $f : Y \to X$ be a divisorial contraction to a point or to a smooth curve contained in the smooth locus of $X$ (in this last case assume also that $\Delta_{F_Y} \neq 0$).

1. If $f$ contracts a divisor to a point, then $|K^3_Y - K^3_X| \leq 2^{10}b_2(Y)^2$. If $f$ contracts a divisor to a curve, then

   $|K^3_Y - K^3_X| \leq 2S_W + 6(b_3(Y) + 1)$,

   where $S_Y$ is as in Definition 2.12. Moreover, the same inequality is true after replacing $b_3(Y)$ by $Ib_3(Y) = \dim IH^3(Y, \mathbb{Q})$.

2. The cubic form $F_X$ is determined up to finite ambiguity by the cubic form $F_Y$.

We believe that the methods used to prove Theorem 1.3 will have interesting applications to questions concerning the topology and the geography of threefolds (see, for example, [BCT16] and [ST17]).

Let $X$ be a smooth threefold, and let $f : X \to Y$ be a minimal model of $X$. It is very natural to ask which topological invariants of $Y$ are determined by those of $X$. It is known that the Betti numbers of $Y$ are determined up to finite ambiguity by the Betti numbers of $X$ (the case of $b_3$ has been treated very recently in [Che16]).

The same question for the ring structure of the cohomology is very delicate. The following immediate consequence of Theorem 1.3 goes in the positive direction.

**Corollary 1.4.** Let $X$ be a smooth complex projective threefold. Let $f : X \to Y$ be a minimal model program for $X$.

If $f$ is composed only by divisorial contractions to points, then $F_Y$ is determined up to finite ambiguity by $F_X$.

If $\Delta_{F_X} \neq 0$ and $f$ is a composition of divisorial contractions to points and blowdowns to smooth curves in smooth loci, then $F_Y$ is determined up to finite ambiguity by $F_X$.

Finally, we can combine the above results to obtain the following corollary.

**Corollary 1.5.** Let $X$ be a smooth complex projective threefold which is not uniruled, and let $F_X$ be its associated cubic form. Assume that $\Delta_{F_X} \neq 0$ and that there exists a birational morphism $f : X \to Y$ onto a minimal projective threefold $Y$, which is obtained as a composition of divisorial contractions to points and blowdowns to smooth curves in smooth loci. Then there exists a constant $D$ depending only on the topology of the 6-manifold underlying $X$ such that

$|K^3_X| \leq D$.

It remains to study divisorial contractions to singular curves and flips. On the other hand, the MMP of any smooth projective threefold may be also factored into
a sequence of flops, blowups along smooth curves, and divisorial contractions to points (see [CH11, Che15]). Recall that if $W \rightarrow Z$ is a flop, then $K^3_W = K^3_Z$; thus, it is crucial to study how the cubic form $F$ varies under flops. We will study this problem in a forthcoming paper.

2. Preliminary results

2.1. Notations. We work over the field of complex numbers. We refer to [KM98] for the classical notions in birational geometry. In particular, if $X$ is a normal projective variety, we denote by $K_X$ the canonical divisor of $X$. We also denote by $\rho(X)$ the Picard number of $X$, by $N^1_X$ the group of Cartier divisors modulo numerical equivalence, and by $H^i(X, \mathbb{Z})$ the $i$th singular cohomology group of $X$ modulo its torsion subgroup. In particular, $b_i(X) = \dim H^i(X, \mathbb{Q})$ is the $i$th Betti number of $X$. We say that $X$ is $\mathbb{Q}$-factorial if every Weil divisor $D$ on $X$ is $\mathbb{Q}$-Cartier, i.e., there exists a positive integer $m$ such that $mD$ is Cartier. If $f: Y \rightarrow X$ is a birational morphism between normal projective varieties and $K_X$ is $\mathbb{Q}$-Cartier, then we may write

$$K_Y = f^*K_X + \sum_{i=1}^k a_i E_i,$$

where the sum is over all the exceptional divisors $E_1, \ldots, E_k$ of $f$. The number $a_i$ is the discrepancy of $f$ along $E_i$ and it is denoted by $a(E_i, X)$. In particular, $X$ is said to be terminal if for any birational morphism $f: Y \rightarrow X$ and for any exceptional divisor $E$, we have $a(E, X) > 0$. Recall that terminal singularities are rational; i.e., if $f: Y \rightarrow X$ is a resolution, then $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$. A terminal variety $X$ is said to be minimal if it is $\mathbb{Q}$-factorial and $K_X$ is nef.

A contraction $f: Y \rightarrow X$ is a proper birational morphism between normal projective varieties. The contraction $f: Y \rightarrow X$ is said to be divisorial if the exceptional locus of $f$ is an irreducible divisor. It is said to be elementary, if $\rho(Y) = \rho(X) + 1$. Finally, an elementary contraction $f: Y \rightarrow X$ is said to be $K_Y$-negative, if $-K_Y$ is $f$-ample, i.e., the exceptional locus of $f$ is covered by curves $\xi$ such that $K_Y \cdot \xi < 0$. Note that if $Y$ is $\mathbb{Q}$-factorial and $f: Y \rightarrow X$ is an elementary divisorial contraction, then $X$ is also $\mathbb{Q}$-factorial. Moreover, if $Y$ is terminal and $f$ is $K_Y$-negative, then $X$ is also terminal.

**Definition 2.1.** Let $X$ be a projective variety with terminal singularities. Then the volume of $X$ is given by

$$\text{vol}(X) = \limsup_{m \rightarrow \infty} \frac{n! \ h^0(X, mK_X)}{m^n},$$

where $n$ is the dimension of $X$.

In particular, the volume is a birational invariant, and if $X$ is a minimal variety of dimension $n$, then

$$\text{vol}(X) = K^n_X$$

(see [Laz04] Section 2.2.C for more details).
2.2. Terminal singularities on threefolds. We now recall a few known facts about terminal singularities in dimension 3. Let \((X,p)\) be the germ of a three-dimensional terminal singularity. The index of \(p\) is the smallest positive integer \(r\) such that \(rK_X\) is Cartier. In addition, it follows from the classification of terminal singularities [Mor85], that there exists a deformation of \((X,p)\) into a variety with \(h\geq 1\) terminal singularities \(p_1,\ldots, p_h\) which are isolated cyclic quotient singularities of index \(r(p_i)\). The set \(\{p_1,\ldots, p_h\}\) is called the basket \(\mathcal{B}(X,p)\) of singularities of \(X\) at \(p\) [Rei87]. As in [CH11], we define \(\Xi(X,p) = \sum_{i=1}^{h} r(p_i)\).

Thus, if \(X\) is a projective variety of dimension 3 with terminal singularities and \(\text{Sing} X\) denotes the finite set of singular points of \(X\), we may define \(\Xi(X) = \sum_{p \in \text{Sing} X} \Xi(X,p)\).

Lemma 2.2. Let \((X,p)\) be the germ of a three-dimensional terminal singularity, and let \(\mathcal{B}(X,p)\) be the basket at \(p\). Then, for each \(q \in \mathcal{B}(X,p)\), the index \(r(q)\) of \(q\) divides \(4 \cdot \Xi(X,p)\). 

Proof. It follows from the classification of terminal singularities, that the points of the basket \(\mathcal{B}(X,p)\) either have all the same index \(r\) or their index divides 4 when \(r(p) = 4\) and \(p \in X\) is of type \(cAx/4\) (e.g., see [CH11, Remark 2.1]). Thus the claim follows. 

By [CZ14, Proposition 3.3], we have the following.

Proposition 2.3. Let \(X\) be a smooth projective threefold, and assume that 
\[X = X_0 \rightarrow \cdots \rightarrow X_k = Y\]
is a sequence of steps for the \(K_X\) MMP of \(X\). Then
\[\Xi(Y) \leq 2b_2(X)\]
In particular, the inequality holds if \(Y\) is the minimal model of \(X\).

In the proof of our main results, we will use the Bogomolov–Miyaoka–Yau inequality and the Riemann–Roch formula for terminal threefolds. Recall that, on any terminal threefold \(X\), we may define \(c_1(X)\) as the anticanonical divisor \(-K_X\) and, for any \(\mathbb{Q}\)-Cartier divisor \(D\) on \(X\), we define the number \(D.c_2(X)\) as \(f^*D.c_2(Y)\) where \(f: Y \rightarrow X\) is any resolution of \(X\). It is easy to check that the definition does not depend on the resolution.

Theorem 2.4. Let \(Y\) be a minimal three-dimensional projective variety with terminal singularities. Then 
\[(3c_2 - c_1^2).c_1 \leq 0\]
Proof. It follows from [Miy87] Theorem 1.1].

Theorem 2.5. Let \(Y\) be a three-dimensional projective variety with terminal singularities. Then the holomorphic Euler characteristic of \(Y\) is given by
\[\chi(Y, \mathcal{O}_Y) = \frac{1}{24}(-K_Y \cdot c_2(Y) + e),\]
easy to see that the content is multiplicative.

\[ e = \sum_{p_{\alpha}} \left( r(p_{\alpha}) - \frac{1}{r(p_{\alpha})} \right), \]

and the sum runs over all the points of all the baskets of \( Y \).

\[ \tag{2.3.} \]

Proof. See [Kaw86, Rei87]. \( \square \)

2.3. Cubic forms. For any polynomial \( P \in \mathbb{C}[x_0, \ldots, x_n] \), we denote by \( \partial_i P(x) \) the partial derivative of \( P \) with respect to \( x_i \) at the point \( x \in \mathbb{C}^{n+1} \). For any ring \( R \subseteq \mathbb{C} \) and for any positive integer \( d \), we denote by \( R[x_0, \ldots, x_n]_d \) the set of homogeneous polynomials of degree \( d \) with coefficients in \( R \).

Given a cubic form \( F \in \mathbb{C}[x_0, \ldots, x_n] \) (i.e., an homogeneous polynomial of degree 3), let

\[ \mathcal{H}_F(x) = (\partial_i \partial_j F(x))_{i,j} \]

be the Hessian of \( F \) at the point \( x \in \mathbb{C}^{n+1} \). Note that, for any \( x \in \mathbb{C}^{n+1} \) and for any \( \lambda \neq 0 \), the rank of \( \mathcal{H}_F \) at the point \( \lambda x \) is constant with respect to \( \lambda \) and therefore we will denote, by abuse of notation, \( \text{rk} \mathcal{H}_F(p) \) to be the rank of \( \mathcal{H}_F \) at any point in the class of \( p \in \mathbb{P}^n \). We say that \( F \) is nondegenerate if \( \text{rk} \mathcal{H}_F \) is maximal at the general point of \( \mathbb{P}^n \), i.e., if \( \det \mathcal{H}_F \) is not identically zero.

Let \( F(x_0, \ldots, x_n) = \sum_i c_i x_i^3 \in \mathbb{C}[x_0, \ldots, x_n]_d \). Then the discriminant \( \Delta_F \) of \( F \) is the unique (up to sign) polynomial with integral coefficients in the variables \( c_i \) such that \( \Delta_F \) is irreducible over \( \mathbb{Z} \) and \( \Delta_F = 0 \) if and only if the hypersurface \( \{ F = 0 \} \subseteq \mathbb{P}^n_\mathbb{C} \) is singular (see [GKZ94, p. 433] for more details). In particular, the discriminant is an invariant under the natural \( \text{SL}(n + 1, \mathbb{C}) \)-action.

If \( F \in \mathbb{C}[x, y, z] \) is a ternary cubic form, then we denote by \( S_F \) and \( T_F \) the two \( \text{SL}(3, \mathbb{C}) \)-invariants of \( F \) as defined in [Stu93, 4.4.7 and 4.5.3]. Then the discriminant of \( F \) satisfies

\[ \Delta_F = T_F^2 - 64 S_F^3. \]

Lemma 2.6. Let \( F \in \mathbb{Z}[x_0, \ldots, x_n]_3 \) be an integral cubic form, and assume that

\[ F(x_0, \ldots, x_n) = a x_0^3 + x_0^2 \left( \sum_{i=1}^n b_i x_i \right) + G(x_1, \ldots, x_n) \]

for some \( G \in \mathbb{Z}[x_1, \ldots, x_n]_3 \). Then \( \Delta_G \) divides \( \Delta_F \).

Proof. If \( P \) is a polynomial with integral coefficients, we denote by \( \text{ct}(P) \) the content of \( P \), that is the gcd of the coefficients of \( P \). As in the case of one variable, it is easy to see that the content is multiplicative.

Let \( A, \{ B_i \}_{i=1, \ldots, n} \), and \( \{ C_J \} \) be variables, and consider the cubic form

\[ f = A x_0^3 + x_0^2 \left( \sum_{i=1}^n B_i x_i \right) + g(x_1, \ldots, x_n), \]

where \( g = \sum_J C_J x_J^3 \). Then \( \Delta_f \) and \( \Delta_g \) are polynomial in \( \mathbb{Z}[A, B_i, C_J] \). We want to show that \( \Delta_g \) divides \( \Delta_f \).

Let \( R = \mathbb{C}[A, B_i, C_J] \), and let \( Z(f), Z(g) \subseteq \mathbb{P}^N_\mathbb{C} = \text{Proj} R \) be the closed subsets defined by \( \Delta_f = 0 \) and \( \Delta_g = 0 \), respectively. Note that \( Z(g) \subseteq Z(f) \) because if \( \{ g = 0 \} \) has a singular point \( z = [z_1, \ldots, z_n] \), then \( [0, z_1, \ldots, z_n] \) is a singular point of \( \{ f = 0 \} \). Since \( \Delta_g \) is irreducible over \( \mathbb{Q} \) by definition, and hence \( Z(g) \) is reduced over \( \mathbb{C} \), we deduce that \( \Delta_f = \Delta_g \cdot H \) where \( H \in R \).
We need to show that $H \in \mathbb{Z}[A, B_i, C_j]$. We proceed as in the proof of the Gauss lemma. We start assuming by contradiction that $H \notin \mathbb{Q}[A, B_i, C_j]$. Fix an order on $R$, and consider the maximal monomial $m$ in $H$ such that its coefficient is not rational. Consider now the product between $m$ and the highest monomial in $\Delta_g$ to get a contradiction. Hence $H \in \mathbb{Q}[A, B_i, C_j]$.

The claim follows from the fact that the content of $\Delta_g$ is 1 and that the content is multiplicative. □

We have the following.

**Lemma 2.7.** Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ be a cubic form such that there exists a point $p \in \mathbb{P}^n$ for which $\text{rk} \, H_F(p) = 0$, i.e., $H_F(p)$ is the trivial matrix. Then after a suitable coordinate change, $F$ depends on at most $n$ variables. In particular, $\det H_F$ vanishes identically on $\mathbb{P}^n$.

**Proof.** Euler’s formula for homogeneous polynomials implies that $F(p) = \partial_i F(p) = 0$ for all $i = 0, \ldots, n$. After a suitable coordinate change, we may assume that $p = (1, 0, \ldots, 0)$. Let $f(y_1, \ldots, y_n) = F(1, y_1, \ldots, y_n)$. By Taylor’s formula, $f$ is a homogeneous polynomial of degree 3. Thus, $F(x_0, \ldots, x_n) = f(x_1, \ldots, x_n)$, and the claim follows. □

As mentioned in the introduction, arithmetic geometry will play an important role for the proof of our main theorem. In particular, we need the following.

**Theorem 2.8** (Siegel theorem). Let $R$ be a ring finitely generated over $\mathbb{Z}$. Let $C$ be an affine smooth curve defined over $R$ and of genus $g \geq 1$. Then there are only finitely many $R$-integral points on $C$.

**Proof.** See [Lan83, Ch. 8, Theorem 2.4]. □

2.4. Reduced triples. Given a ring $A$, we denote by $\mathcal{M}(n, A)$ the set of all matrices with coefficients in $A$, by $\text{GL}(n, A)$ the subgroup of invertible matrices, and by $\text{SL}(n, A)$ the subgroup of matrices with determinant 1.

Given a cubic form $F \in \mathbb{C}[x_0, \ldots, x_n]$ and a matrix $T \in \text{GL}(n + 1, \mathbb{C})$, we will denote by $T \cdot F$ the cubic form given by

$$T \cdot F(x) = F(T \cdot x).$$

We define

$$W_F = \{ p \in \mathbb{P}^n \mid \text{rk} \, H_F(p) \leq 1 \}$$

and

$$V_F = \{ p \in \mathbb{P}^n \mid \text{rk} \, H_F(p) \leq 2 \}.$$

**Definition 2.9.** Let $F \in R[x_0, \ldots, x_n]$ be a nondegenerate cubic form where $R$ is a commutative ring. We say that $(a, B, G)$ is a reduced triple associated to $F$ if there exists an element $T \in \text{SL}(n + 1, R)$ such that

$$T \cdot F = ax_0^3 + x_0^2 \sum_{i=1}^n b_i x_i + G(x_1, \ldots, x_n),$$

where $a \in R$, $B = (b_1, \ldots, b_n) \in R^n$ and $G \in R[x_1, \ldots, x_n]$ is a nondegenerate cubic form. For simplicity, we will denote $\Pi$ as

$$T \cdot F = (a, B, G).$$
In this case we also say that \( T \cdot F \) is in reduced form \((a,B,G)\).

We say that two reduced triples \((a,B,G)\) and \((a',B',G')\), are equivalent over \( R \) if \( a = a' \) and there is an element \( M \in \text{SL}(n,R) \) such that \( B' = M \cdot B \) and \( G' = M \cdot G \).

The motivation for studying the loci \( W_F \) and \( V_F \) and reduced forms comes from Propositions 4.7 and 4.8. More precisely, it is easy to see that if \( F \in \mathbb{C}[x_0,\ldots,x_n] \) is a cubic form in reduced form and \( p = [1,0,\ldots,0] \), then \( p \in V_F \) (see for example [BCT16 Lemma 2.1]).

In what follows we will use the following result.

**Theorem 2.10** (Jordan’s theorem). Let \( F \in \mathbb{Z}[x_0,\ldots,x_n]_3 \) be a cubic form with nonzero discriminant \( \Delta_F \), and consider the set\[ A_F = \{ T \cdot F \mid T \in \text{SL}(n+1,\mathbb{C}) \} \subseteq \mathbb{C}[x_0,\ldots,x_n]_3. \]
Then the quotient \( (A_F \cap \mathbb{Z}[x_0,\ldots,x_n]_3)/\text{SL}(n+1,\mathbb{Z}) \) is finite.

**Proof.** It follows from [OVdV95 Corollary 4 and 5]. \( \square \)

### 2.5. Cubic forms on threefolds.

Let \( X \) be a terminal \( \mathbb{Q} \)-factorial projective threefold. Let \( h = (h_1,\ldots,h_n) \) be a basis of \( H^2(X,\mathbb{Z}) \). The intersection cup product induces a symmetric trilinear form \[ \phi_X : H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \to H^6(X,\mathbb{Z}) \cong \mathbb{Z}. \]

Thus, we may define a cubic homogeneous polynomial \( F_X \in \mathbb{Z}[x_1,\ldots,x_n] \) as
\[
 F_X(x) = \sum_{I=(i_1,\ldots,i_n): i_1+\cdots+i_n=3} \binom{3}{I} \phi_X(h^I) x^I.
\]

We call \( F_X \) the cubic form associated to \( X \).

As in the smooth case, we have the following.

**Lemma 2.11.** The cubic form \( F_X \) is nondegenerate, that is \( \det H_{F_X} \) is not identically zero.

**Proof.** Let \( \Sigma \subseteq X \) be the singular locus of \( X \). Since \( X \) is terminal, \( \Sigma \) is a finite set, and there exists a resolution \( \pi : Y \to X \) with divisorial exceptional locus \( E \) such that \( Y \setminus E \) is isomorphic to \( X \setminus \Sigma \).

Let \( \{ \gamma_0,\ldots,\gamma_b \} \) be a basis of \( H^2(X,\mathbb{Q}) \), and let \( B = \{ \beta_i = f^* \gamma_i \} \). After completing \( B \) to a basis of \( H^2(Y,\mathbb{Q}) \), we may write
\[
 F_Y(x_0,\ldots,x_n) = F_X(x_0,\ldots,x_b) + F(x_{b+1},\ldots,x_n),
\]
where we are considering the cubic forms over \( \mathbb{Q} \).

The paper [OVdV95 Proposition 16] implies that \( \det H_{F_Y} \) is not identically zero. Since \( \det H_{F_Y} = \det H_{F_X} \cdot \det H_F \), the claim follows. \( \square \)

**Definition 2.12.** Let \( X \) be a terminal \( \mathbb{Q} \)-factorial projective threefold, and let \( F_X \in \mathbb{Z}[x_1,\ldots,x_n]_3 \) be the cubic form associated to \( X \). We define
\[ S_X := \sup \{|a| \in \mathbb{Z} \mid \text{there exists } T \in \text{SL}(n+1,\mathbb{Z}) \text{ such that } T \cdot F_X = (a,B,G)\}, \]
where we set \( S_X = 0 \) if there are no reduced triples associated to \( F_X \).
Note that $S_X$ is a topological invariant of $X$ since $F_X$ is a topological invariant (modulo the action of $\text{SL}(n + 1, \mathbb{Z})$).

2.6. Topology of threefolds. We now study how the Betti numbers behave under a birational morphism (see [Cai05] for some related results). Note that the singularities of a $\mathbb{Q}$-factorial terminal threefold $X$ are in general not analytically $\mathbb{Q}$-factorial. In particular, $X$ is in general not a $\mathbb{Q}$-homology manifold (see [Kol89, Lemma 4.2]), and the singular cohomology may differ from the intersection cohomology.

In dimension 3, all the Betti numbers behave well under birational transformations except for $b_3$ (see Lemma 2.16). The behaviour of the third Betti number is more subtle and depends on the singularities of $X$ and $Y$ as the following example shows.

Example 2.13. Let $X \subseteq \mathbb{P}^4$ be a quartic threefold with just one node (rational double point) $p \in X$. It is known that $X$ is $\mathbb{Q}$-factorial (e.g., see [Che06]). Locally, the germ $(X, p)$ may be written as
\[
\{xy - wz = 0\} \subseteq \mathbb{C}^4,
\]
which is not analytically $\mathbb{Q}$-factorial. Let $f: Y \rightarrow X$ be the blowup of the singularity, and let $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the exceptional divisor. It follows that
\[b_3(Y) = b_3(X) - 1.\]

In particular, the third Betti number may increase under some of the steps of the MMP. For this reason, it will often be useful to look at the intersection cohomology instead.

Given a projective variety $X$, we denote by $IH^i(X, \mathbb{Q})$ the middle-perversity intersection cohomology group of dimension $i$, and by $Ib_i$ its dimension. Note that if $X$ is smooth, then $IH^i(X, \mathbb{Q})$ coincides with $H^i(X, \mathbb{Q})$ and in particular $Ib_i(X) = b_i(X)$ for all $i$.

We will use the following consequence of the decomposition theorem for intersection cohomology (see [BBD82]).

Theorem 2.14. Let $f: Y \rightarrow X$ be a proper birational morphism between algebraic varieties. Assume that $Y$ is smooth. Then the cohomology $H^*(Y, \mathbb{Q}) = IH^*(Y, \mathbb{Q})$ of $Y$ contains the intersection cohomology $IH^*(X, \mathbb{Q})$ of $X$ as a direct summand.

We now restrict our study to the case of threefolds.

Lemma 2.15. Let $f: Y \rightarrow X$ be a birational morphism between projective threefolds with terminal singularities. Let $E$ be an exceptional divisor of $f$, and let $W = f(E)$. Assume that $f$ induces an isomorphism $Y \setminus E \rightarrow X \setminus W$. Then
\[0 \rightarrow H^i(X, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q}) \oplus H^i(W, \mathbb{Q}) \rightarrow H^i(E, \mathbb{Q}) \rightarrow 0\]
is exact for any $i \geq 4$, and
\[0 \rightarrow IH^i(X, \mathbb{Q}) \rightarrow IH^i(Y, \mathbb{Q}) \oplus IH^i(W, \mathbb{Q}) \rightarrow IH^i(E, \mathbb{Q}) \rightarrow 0\]
is exact for any $i \geq 1$.

Proof. From the exact sequence of the pairs we get a long exact sequence in cohomology
\[
\cdots \rightarrow H^i(X, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q}) \oplus H^i(W, \mathbb{Q}) \rightarrow H^i(E, \mathbb{Q}) \rightarrow H^{i+1}(X, \mathbb{Q}) \rightarrow \cdots,
\]
which by [Del74, Prop. 8.3.9] is an exact sequence of mixed Hodge structure.
Since $X, Y$ have isolated singularities, for $i \geq 4$ the Hodge structure on $H^i(X, \mathbb{Q})$ is pure of weight $i$ (see [Ste83]). On the other hand, since $E$ is projective, $H^k(E, \mathbb{Q})$ has weight at most $k$ for any $k$ ([Del74, Thm. 8.2.4]). Thus, the maps

$$H^i(E, \mathbb{Q}) \to H^{i+1}(X, \mathbb{Q})$$

are zero for $i \geq 3$.

The same argument applies for intersection cohomology with the advantage that the Hodge structure on $IH^i(X, \mathbb{Q})$ is pure of weight $i$ for any $i$ by [Sai88]. □

**Lemma 2.16.** Let $f: Y \to X$ be an elementary divisorial contraction between $\mathbb{Q}$-factorial projective threefolds with terminal singularities. Then

1. $b_0(Y) = b_6(Y) = b_0(X) = b_6(Y) = 1$,
2. $b_1(Y) = b_1(X)$,
3. $b_2(Y) = b_2(X) + 1$,
4. $b_4(Y) = b_4(X) + 1$, and
5. $b_5(Y) = b_5(X)$.

**Proof.** (1) is clear. Lemma 2.15 implies (4) and (5).

We now want to show that $R^1 f_* \mathcal{Z} = 0$; it is enough to show it locally around any point $x \in X$. We consider the exact sequence

$$0 \to f_* \mathcal{Z} \to f_* \mathcal{O}_Y \xrightarrow{\exp} f_* \mathcal{O}_Y \to R^1 f_* \mathcal{Z} \to R^1 f_* \mathcal{O}_Y.$$ 

The exponential map is surjective locally around $x \in X$. Since $X$ and $Y$ have rational singularities, it follows that $R^1 f_* \mathcal{O}_Y = 0$. Thus, $R^1 f_* \mathcal{Z} = 0$, as claimed. The Leray spectral sequence implies that $H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$ is an isomorphism, and, in particular, (2) follows.

Let $H_2(Y/X, \mathbb{C}) \subseteq H_2(Y, \mathbb{C})$ be the subspace generated by all the images of $H_2(F, \mathbb{C})$, where $F$ runs through all the fibres of $f$. The paper [KM92] Theorem 12.1.3] implies that $H_2(Y/X, \mathbb{C})$ is generated by algebraic cycles and that there exists an exact sequence,

$$0 \to H^2(Y/X, \mathbb{C}) \to H^2(Y, \mathbb{C}) \to H^2(X, \mathbb{C}) \to 0.$$ 

Since $f$ is an elementary divisorial contraction, it follows that all the nontrivial algebraic cycles contained in the fibre of $f$ are numerically proportional to each other, and, in particular,

$$\dim H_2(Y/X, \mathbb{C}) = 1.$$ 

Thus, (3) follows. □

### 3. Cubic Forms in Reduced Form

The aim of this section is to prove the following.

**Theorem 3.1.** Let $F \in \mathbb{Z}[x_0, \ldots, x_n]$ be a nondegenerate cubic form cf. §2.3 with nonzero discriminant $\Delta_F$. Then there are finitely many triples

$$(a_i, B_i, G_i) \in \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}[x_1, \ldots, x_n]$$

such that any reduced triple associated to $F$ is equivalent to $(a_i, B_i, G_i)$ over $\mathbb{Z}$ for some $i \in \{1, \ldots, k\}$ cf. Definition 2.9 In addition, we have that $\Delta_{G_i} \neq 0$ for all $i = 1, \ldots, k$. 

Before we proceed with the proof of Theorem 3.1, we first sketch some of its main ideas. Note that if \( F \) is in reduced form \( (a, B, G) \), then the point \( p = (1, 0, \ldots, 0) \) is contained in the set \( V_F \), defined in §2.4. Thus, our first goal is to show that the set of points \( p \in V_F \) such that \( F(p) \neq 0 \) is contained in a finite union of points, lines, and plane cubics (cf. Theorem 3.6). Assuming furthermore that the discriminant \( \Delta_F \) of \( F \) is not zero, we characterise the cubic forms \( F \) which contain a line (cf. Corollary 3.9) or plane curves (cf. Corollary 3.10) inside \( V_F \).

The next step is to restrict the cubic form to one of the lines or plane curves contained in \( V_F \). To deal with this situation, we study binary (cf. Proposition 3.13) and ternary cubic forms (cf. Proposition 3.16) with nonzero discriminant. The main tool used in the proof of these results is Siegel’s theorem on the finiteness of integral points in a curve of positive genus. Finally, we conclude the proof of Theorem 3.1 in §3.3.

3.1. Points of low rank for a cubic form. In this subsection, we study the sets \( W_F \) and \( V_F \) associated to a cubic form \( F \in \mathbb{C}[x_0, \ldots, x_n] \) (cf. §2.4). Many of the results below depend on some simple calculations on cubics forms. To illustrate some of the methods presented below, we begin with a basic result.

**Lemma 3.2.** Let \( F = x_0^3 + x_0 Q + R \in \mathbb{C}[x_0, \ldots, x_n] \) be a cubic form, where \( Q, R \in \mathbb{C}[x_1, \ldots, x_n] \) are homogeneous polynomials of degree 2 and 3, respectively. Let \( A \) be the \( n \times n \) symmetric matrix associated to \( Q \). Let \( p = [1, 0, \ldots, 0] \). Then \( \text{rk} \; H_F(p) = \text{rk} \; A + 1 \).

**Proof.** The claim is a simple computation. \( \square \)

We now proceed by studying the set \( W_F \) (cf. §2.4) associated to a nondegenerate cubic form \( F \).

**Proposition 3.3.** Let \( F \in \mathbb{C}[x_0, \ldots, x_n] \) be a nondegenerate cubic form. Then \( W_F \) is a finite set.

**Proof.** Let \( W'_F = W_F \cap \{ F = 0 \} \). We first show that \( W'_F \) is a finite set. Assume by contradiction that there exists an irreducible curve \( C \) inside \( W'_F \) and let \( p \in C \). We say that a hyperplane \( H \subseteq \mathbb{P}^n \) is associated to \( p \) if:

1. \( \det \; H_F \) vanishes along \( H \);
2. \( p \in H \); and
3. if \( G = F|_H \), then \( H_G(p) \) is trivial.

Lemma 2.7 implies that \( \text{rk} \; H_F(p) = 1 \). After taking a suitable coordinate change, we may assume that \( p = [1, 0, \ldots, 0] \). In particular

\[
F(x_0, \ldots, x_n) = x_0^2 \cdot L_1 + x_0 \cdot Q_1 + R_1
\]

for some homogeneous polynomials \( L_1, Q_1, R_1 \in \mathbb{C}[x_1, \ldots, x_n] \) of degrees 1, 2, and 3, respectively. Since \( p \in W_F \), it follows that \( L_1 = 0 \). By assumption, \( Q_1 \) is not zero. Using again the fact that \( p \in W_F \), similarly to Lemma 3.2 it follows that, after taking a suitable coordinate change in \( x_1, \ldots, x_n \), we may assume that \( Q_1 = x_1^2 \).

We may write

\[
R_1(x_1, \ldots, x_n) = x_1^2 \cdot L + x_1 \cdot Q + R
\]
for some homogeneous polynomials \( L \in \mathbb{C}[x_1, \ldots, x_n] \) and \( Q, R \in \mathbb{C}[x_2, \ldots, x_n] \) of degrees 1, 2, and 3, respectively. After replacing \( x_0 \) by \( x_0 + L \), we may assume that \( L = 0 \). Thus, we have

\[
F(x_0, \ldots, x_n) = x_0 \cdot x_1^2 + x_1 \cdot Q + R.
\]

Let \( H_p = \{ x_1 = 0 \} \). An easy computation shows that \( H_p \) is a hyperplane associated to \( p \). We now show that such a hyperplane is unique. Assume that \( H' \subseteq \mathbb{P}^n \) is also a hyperplane associated to \( p \). Since \( p \in H' \), we have \( H' = \{ \ell = 0 \} \) for some linear function \( \ell \in \mathbb{C}[x_1, \ldots, x_n] \). If \( H' \neq H_p \), after a suitable change of coordinates in \( x_2, \ldots, x_n \), we may assume that

\[
\ell = x_n - \alpha x_1
\]

for some \( \alpha \in \mathbb{C} \). Thus if \( G' = F|_{H'} \), we may write

\[
G'(x_0, \ldots, x_{n-1}) = x_0 x_1^2 + x_1 Q(x_2, \ldots, x_{n-1}, \alpha x_1) + R(x_2, \ldots, x_{n-1}, \alpha x_1),
\]

and it follows that

\[
\partial_1 \partial_1 G'(p) \neq 0,
\]

which contradicts (3). Thus, \( H' = H_p \), and the claim follows.

Now let \( q \in C \) be a point such that \( H_p = H_q \). We want to show that \( q = p \). If \( R = 0 \), then it follows easily that \( W'_F = \{ p \} \). Thus, by Lemma 2.7 after a suitable change in coordinates in \( x_2, \ldots, x_n \), we may assume that \( R = R(x_{n-k}, \ldots, x_n) \) for some \( k \geq 0 \) and that there is no point \( z \in \mathbb{P}^k \) such that \( \mathcal{H}_R(z) \) is trivial. If \( q = [y_0, \ldots, y_n] \), it follows by (3) that

\[
y_{n-k} = \cdots = y_n = 0.
\]

Since \( \text{rk} \mathcal{H}_F(q) = 1 \), it follows that the minor spanned by the \( i \)th and \( (n-i) \)th rows and columns of \( \mathcal{H}_F(p) \) must have trivial determinant for any \( i = 0, \ldots, n-2 \) and in particular, since \( y_1 = 0 \) and \( \mathcal{H}_F(y_2, \ldots, y_n) \) is trivial, it follows that \( \partial_1 Q(y_0, \ldots, y_n) = 0 \). It is easy to show that this implies that if \( q \neq p \), then \( \text{det} \mathcal{H}_R \) vanishes identically, a contradiction.

Since by assumption \( \text{det} \mathcal{H}_F \) is a nontrivial function, there exist only finitely many hyperplanes on which \( \text{det} \mathcal{H}_F \) vanishes and (1) implies that \( H_p = H_q \) for infinitely many pairs of points \( p, q \in C \), a contradiction. Thus, \( W'_F \) is a finite set.

Now let \( p \in W_F \) be a point such that \( F(p) \neq 0 \). After a suitable change of coordinates, we may assume that \( p = [1, 0, \ldots, 0] \) and that

\[
F(x_0, \ldots, x_n) = x_0^3 + x_0^2 \cdot L + x_0 \cdot Q + R
\]

for some homogeneous polynomials \( L, Q, R \in \mathbb{C}[x_1, \ldots, x_n] \) of degrees 1, 2, and 3, respectively. After replacing \( x_0 \) by \( x_0 + \frac{1}{3} L \), we may assume that \( L = 0 \). Since \( p \in W_F \), Lemma 3.2 implies that \( Q = 0 \). Let \( q = [z_0, \ldots, z_n] \in W_F \). Then either \( q = p \) or \( z_0 = 0 \) and \( [z_1, \ldots, z_n] \in W_R \). Thus, the result follows by induction on \( n \).

Remark 3.4. Note that the same result does not hold if we replace the assumption that \( F \) is nondegenerate by the weaker assumption that \( \text{rk} \mathcal{H}_F(p) \geq 1 \) for any \( p \in \mathbb{P}^n \) (see Lemma 2.7). For example, consider

\[
F(x_0, \ldots, x_4) = x_4 x_3^2 + x_3 x_1 x_0 + x_2 x_1^2.
\]

Then it is easy to check that \( W'_F \) is not finite.
Theorem 3.5. Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ be a nondegenerate cubic form. Let $C \subseteq V_F$ be a curve such that $F(p) \neq 0$ at the general point of $C$. Then, there exist nonnegative integers $\ell, k$ such that, after a suitable change of coordinates, we may write

$$F = \sum_{i=0}^{\ell} G_i + \sum_{i=1}^{k} (x_{\ell + 2i + 1}^2 + M_i) \cdot x_{\ell + 2i} + R_{\ell + k + 1},$$

where

1. $G_i \in \mathbb{C}[x_i, x_{i+1}]$ is a cubic form for any $i = 0, \ldots, \ell$ with
   $$G_0 = x_0^3 + x_0 x_1^2;$$
2. $M_i = \delta_i x_{\ell + 1}^2$ for any $i = 1, \ldots, k$ with $\delta_i \in \mathbb{C};$
3. $R_{\ell+k+1} \in \mathbb{C}[x_{I_{\ell,k+1}}]$ is a cubic form;
4. $C \subseteq \bigcap_{i \in I_{\ell,k+1}} \{x_i = 0\}.$

Moreover, if $C \not\subseteq \{x_{\ell+2k+2} = 0\}$, we may write

$$R_{\ell+k+1} = M_{k+1} \cdot x_{\ell+2k+2} + R_{l+k+2},$$

where

5. $R_{\ell+k+2} \in \mathbb{C}[x_{I_{\ell,k+1}}]$ is a cubic form and $M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}]$ is a quadric.

Proof. We divide the proof into four steps:

**Step 1.** By Proposition 3.3 there exists $p \in C$ such that $F(p) \neq 0$ and ${\text{rk}} H_F(p) = 2$. Since $F(p) \neq 0$, after a suitable change of coordinates, we may assume that $p = [1, 0, \ldots, 0]$ and

$$F = x_0^3 + x_0^2 L + x_0 Q + R$$

for some homogeneous polynomials $L, Q, R \in \mathbb{C}[x_1, \ldots, x_n]$ of degrees 1, 2, and 3, respectively. After replacing $x_0$ by $x_0 - \frac{1}{3} L$, we may assume that $L = 0$. Since ${\text{rk}} H_F(p) = 2$, by Lemma 3.2 after a suitable change of coordinates in $x_1, \ldots, x_n$, we may assume that $Q = x_1^2$. Thus, we have

$$F = G_0 + R_1,$$

where $G_0 = x_0^3 + x_0 x_1^2$ and $R_1 = R \in \mathbb{C}[x_1, \ldots, x_n]$. We distinguish two cases. If $C$ is contained in the hyperplane $\{x_1 = 0\}$, then we set $k = \ell = 0$ and continue to Step 3; otherwise, we set $\ell = 1$ and proceed to Step 2.
Step 2. We are assuming that

\[ F = \sum_{i=0}^{\ell-1} G_i + R_\ell, \]

where \( G_i \in \mathbb{C}[x_i, x_{i+1}] \) and \( R_\ell \in \mathbb{C}[x_\ell, \ldots, x_n] \) are cubic forms, and \( C \) is not contained in the hyperplane \( \{x_\ell = 0\} \). We claim that after a suitable change of coordinates in \( x_\ell, \ldots, x_n \), we may write

\[ R_\ell = G_\ell + R_{\ell+1}, \]

where \( G_\ell \in \mathbb{C}[x_\ell, x_{\ell+1}] \) and \( R_{\ell+1} \in \mathbb{C}[x_{\ell+1}, \ldots, x_n] \) are cubic forms. Assuming the claim, if \( C \) is contained in the hyperplane \( \{x_{\ell+1} = 0\} \), we set \( k = 0 \) and proceed to Step 3; otherwise, we replace \( \ell \) by \( \ell + 1 \) and repeat Step 2.

We now prove the claim. By assumption, there exists \( q \in C \) such that \( q \notin \{x_\ell = 0\} \). After a suitable change of coordinates in \( x_\ell, \ldots, x_n \), we may assume that

\[ q = [z_0, \ldots, z_{\ell-1}, 1, 0, \ldots, 0] \]

for some \( z_0, \ldots, z_{\ell-1} \in \mathbb{C} \). We may write

\[ R_\ell = \alpha x_\ell^3 + L_\ell x_\ell^2 + Q_\ell x_\ell + R_{\ell+1}, \]

for some homogeneous polynomials \( L_\ell, Q_\ell, R_\ell \in \mathbb{C}[x_{\ell+1}, \ldots, x_n] \) of degrees 1, 2, and 3, respectively. Since \( \text{rk} \mathcal{H}_F(q) \leq 2 \), after a suitable change of coordinates, we may write \( L_\ell = \beta_\ell x_{\ell+1} \) and \( Q_\ell = \gamma_\ell x_{\ell+1}^2 \) for some \( \beta_\ell, \gamma_\ell \in \mathbb{C} \). We may define

\[ G_\ell = \alpha x_\ell^3 + \beta_\ell x_{\ell+1}^2 \cdot x_{\ell+1} + \gamma_\ell x_\ell \cdot x_{\ell+1}^2, \]

and the claim follows.

Step 3. We are assuming that

\[ F = \sum_{i=0}^\ell G_i + \sum_{i=1}^k (x_{i+2i+1}^2 + M_i) \cdot x_{i+2i} + R_{\ell+k+1}, \]

where \( G_i, M_i, \) and \( R_{\ell+k+1} \) satisfy (1), (2), and (3) and

\[ C \subseteq \{x_{\ell+1} = x_{\ell+3} = \cdots = x_{\ell+2k+1} = 0\}. \]

If we also have that

\[ C \subseteq \{x_{\ell+2k+2} = \cdots = x_n = 0\}, \]

then we are done. In particular, if \( n < \ell + 2k + 2 \), then we are done. Otherwise, after a suitable change of coordinates in \( x_{\ell+2k+2}, \ldots, x_n \), we may assume that there exists

\[ q = [z_0, \ldots, z_n] \in C \]

such that \( z_{\ell+2k+2} \neq 0 \) and \( z_{\ell+2k+3} = \cdots = z_n = 0 \). Since

\[ \det(\partial_x \partial_y F(p))_{i,j=0,1} \neq 0, \]

we may assume that the same inequality holds for \( q \). We may write

\[ R_{\ell+k+1} = \alpha_{\ell+k+1} x_{\ell+2k+2}^3 + x_{\ell+2k+2}^2 \cdot L_{\ell+k+1} + x_{\ell+2k+2} \cdot Q_{\ell+k+1} + R_{\ell+k+2}, \]

where \( \alpha_{\ell+k+1} \in \mathbb{C} \), and \( L_{\ell+k+1}, Q_{\ell+k+1}, R_{\ell+k+2} \in \mathbb{C}[x_{\ell+k+1}] \) are homogeneous polynomials of degrees 1, 2, and 3, respectively.

We first assume that \( \alpha_{\ell+k+1} \neq 0 \). After replacing \( x_{\ell+2k+2} \) by

\[ x_{\ell+2k+2} - \frac{1}{3\alpha_{\ell+k+1}} L_{\ell+k+1}, \]
we may assume that $L_{\ell+k+1} = 0$. Since $q \in V_F$, we get a contradiction by considering the minor

$$(\partial_i \partial_j F(q))_{i,j=0,1,\ell+2k+2}.$$  

We now assume that $\alpha_{\ell+k+1} = 0$. Since $z_{\ell+2k+2} \neq 0$ and $q \in V_F$, it follows that $L_{\ell+k+1} = 0$ and, after a suitable change of coordinates,

$$Q_{\ell+k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+3}].$$

We may write

$$Q_{\ell+k+1} = \beta_k x_{\ell+2k+3}^2 + x_{\ell+2k+3} \cdot \ell_k + M_k,$$

where $\beta_k \in \mathbb{C}$ and $\ell_k, M_k \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}]$ are homogeneous polynomials of degrees 1 and 2, respectively. If $\beta_k \neq 0$, then after a suitable change of coordinates we may assume $\beta_k = 1$ and $\ell_k = 0$. By considering the minor

$$(\partial_i \partial_j F(q))_{i,j=0,\ell+2k+2,\ell+2k+3},$$

it follows that $C \subseteq \{x_{\ell+2k+3} = 0\}$. Thus, we may proceed to Step 4.

If $\beta_k = 0$, then since $q \in V_F$ it follows that $\ell_k = 0$. In case $C$ is contained in $\{x_{\ell+2k+3} = \cdots = x_n = 0\}$ we are done, so we may assume that there exists a point

$$q' = [z'_0, \ldots, z'_n] \in C \cap \bigcap_{i \in J} \{x_i = 0\}$$

such that $z'_0 \neq 0$ and $z'_{\ell+2k+3} \neq 0$, where, $J = I_{\ell+1,k+1} \setminus \{\ell + 2k + 3\}$. Proceeding as above, we may write

$$R_{\ell+k+2} = x_{\ell+2k+3} \cdot Q_{\ell+k+2} + R_{\ell+k+3},$$

where $Q_{\ell+k+2} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+3}, x_{\ell+2k+4}]$ and $R_{\ell+k+3} \in \mathbb{C}[x_J]$ are homogeneous polynomials of degrees 2 and 3, respectively. We may write

$$Q_{\ell+k+2} = \beta_{k+1} x_{\ell+2k+4}^2 + x_{\ell+2k+4} \cdot \ell_{k+1} + M_{k+1},$$

where $\beta_{k+1} \in \mathbb{C}$ and $\ell_{k+1}, M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}]$ are homogeneous polynomials of degrees 1 and 2, respectively.

If $\beta_{k+1} = 0$, then $\ell_{k+1} = 0$ because $q' \in V_F$. Denoting by $H'_F$ the $i$th column of $H_F$, it follows that the vectors $H'_{\ell+2}, H'_{\ell+4}, \ldots, H'_{\ell+2k+2}$ and $H'_{\ell+2k+3}$ are linearly dependent. Thus, $H_F$ does not have maximal rank, which contradicts the assumptions.

Hence, we have $\beta_{k+1} \neq 0$. After a suitable change of coordinates, we may assume that $\beta_{k+1} = 1$ and $\ell_{k+1} = 0$. By considering the minor

$$(\partial_i \partial_j F(q'))_{i,j=0,\ell+2k+3,\ell+2k+4},$$

it follows that $C \subseteq \{x_{\ell+2k+4} = 0\}$. Thus, we first exchange $x_{\ell+2k+3}$ and $x_{\ell+2k+4}$, then we exchange $x_{\ell+2k+2}$ and $x_{\ell+2k+4}$. So we may write

$$R_{\ell+k+1} = x_{\ell+2k+2} \cdot (x_{\ell+2k+3}^2 + M_{k+1}) + R_{\ell+k+2},$$

where $M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}]$ is a quadric, $R_{\ell+k+2} \in \mathbb{C}[x_{I_{\ell,k+1}}]$ is a cubic form and $C \subseteq \{x_{\ell+2k+3}\}$. We also may write

$$R_{\ell+k+2} = x_{\ell+2k+4} \cdot M_{k+2} + R_{\ell+k+3},$$

where $M_{k+2} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}], R_{\ell+k+3} \in \mathbb{C}[x_{I_{\ell,k+2}}]$ are homogeneous polynomials of degrees 2 and 3, respectively. Moreover, we have a point

$$q' = [z'_0, \ldots, z'_n] \in C \cap \bigcap_{i \in J} \{x_i = 0\}$$
such that \( z_0 \neq 0 \) and \( z_{\ell+2k+2} \neq 0 \), where \( J = I_{\ell,k+1} \setminus \{ \ell + 2k + 4 \} \). Replacing \( x_{\ell+2k+4} \) by \( x_{\ell+2k+4} + \frac{z_{\ell+2k+4}}{z_{\ell+2k+2}} x_{\ell+2k+2} \), we get a point
\[
q = [z_0, \ldots, z_n] \in C \cap \bigcap_{i \in I_{\ell,k+1}} \{ x_i = 0 \}
\]
such that \( z_0 \neq 0 \), \( z_{\ell+2k+2} \neq 0 \), and we may proceed to Step 4.

**Step 4.** We are assuming that
\[
F = \sum_{i=0}^{\ell} G_i + \sum_{i=1}^{k} (x_{\ell+2i+1}^2 + M_i) \cdot x_{\ell+2i} + R_{\ell+k+1},
\]
where \( G_i, M_i, \) and \( R_{\ell+k+1} \) satisfy (1), (2), and (3) and
\[
C \subseteq \{ x_{\ell+1} = x_{\ell+3} = \cdots = x_{\ell+2k+1} = 0 \}.
\]
By Step 3 we also have that
\[
R_{\ell+k+1} = x_{\ell+2k+2} \cdot (x_{\ell+2k+3}^2 + M_{k+1}) + R_{\ell+k+2},
\]
where \( M_{k+1} \in \mathbb{C}[x_{\ell+1}, x_{\ell+3}, \ldots, x_{\ell+2k+1}] \) is homogeneous of degree 2 and \( C \subseteq \{ x_{\ell+2k+3} = 0 \} \). Moreover, there is a point \( q = [z_0, \ldots, z_n] \) such that \( z_0 \neq 0 \), \( z_{\ell+2k+2} \neq 0 \) and
\[
q \in C \cap \bigcap_{i \in I_{\ell,k+1}} \{ x_i = 0 \}.
\]
We show that we may assume
\[
M_{k+1} = \delta_{k+1} x_{\ell+1}^2,
\]
where \( \delta_k \in \mathbb{C} \). Since \( q \in C \) and \( z_{\ell+2k+2} \neq 0 \), we have \( \det(\partial_i \partial_j F(q))_{i,j=0,1} = 0 \). Considering the minors
\[
(\partial_i \partial_j F(q))_{i=0,m,\ell+2k+3}^{j=0,\ell+2k+3}
\]
for \( h, m = 1, \ldots, n, (h,m) \neq (\ell + 2k+3, \ell + 2k+3) \), we deduce that \( \partial_h \partial_m F(q) = 0 \), and so, since by induction \( M_i = \delta_i x_{\ell+1} \) for \( i = 1, \ldots, k \), we have
\[
M_{k+1} = \sum_{j=0}^{k} \gamma_j^2 x_{\ell+2j+1},
\]
where \( \gamma_k^j \in \mathbb{C} \). Since \( M_j = \delta_j x_{\ell+1} \) for \( j = 1, \ldots, k \), to conclude it is enough to replace \( x_{\ell+2j} \) with \( x_{\ell+2j} - \gamma_j^2 x_{\ell+2k+2} \) for \( j = 1, \ldots, k \). In this way we get
\[
M_{k+1} = \delta_{k+1} x_{\ell+1}^2,
\]
where \( \delta_{k+1} = \gamma_k^0 - \sum_{j=1}^{k} \gamma_k^j \delta_j \).

After replacing \( k \) by \( k + 1 \), we may repeat Step 3. \( \square \)

**Theorem 3.6.** Let \( F \in \mathbb{C}[x_0, \ldots, x_n] \) be a nondegenerate cubic form. Then the set of points \( p \in V_F \) such that \( F(p) \neq 0 \) is a finite union of points, lines, plane conics, and plane cubics.

**Proof.** We may assume that there is an irreducible component \( C \subseteq V_F \) such that \( \dim C \geq 1 \) and \( F(p) \neq 0 \) at the general point \( p \) of \( C \); otherwise, we are done. By Theorem 3.5 we may write
\[
F = \sum_{i=0}^{\ell} G_i + \sum_{i=1}^{k} (x_{\ell+2i+1}^2 + M_i) \cdot x_{\ell+2i} + R_{\ell+k+1},
\]
where \( G_i, M_i, \) and \( R_{\ell + k + 1} \) are as in Theorem 3.5 and
\[
C \subseteq \{x_{\ell + 1} = x_{\ell + 3} = \cdots = x_{\ell + 2k + 1} = 0\}.
\]

By the proof of Theorem 3.5 we may also assume that for any \( i = 1, \ldots, k \), there is a point \( q_i \in C \) such that \( q_i \notin \{x_0 = 0\}, q_i \notin \{x_{\ell + 2i} = 0\}, \) and \( q_i \in \bigcap_{j=2i+1}^n \{x_{\ell + j} = 0\} \).

We distinguish two cases: \( C \subseteq \{x_1 = 0\} \) and \( C \nsubseteq \{x_1 = 0\} \).

If \( C \subseteq \{x_1 = 0\} \), then \( \ell = 0 \). Let \( z = [z_0, \ldots, z_n] \in C \) be a general point in \( C \).

If \( C \subseteq \{x_{2k+2} = 0\} \), then considering
\[
(\partial_i \partial_j F(z))_{j=0,1,2k+1}^{i=0,1,2k+1},
\]
we immediately get a contradiction because \( \text{det}(\partial_i \partial_j F(z))_{i,j=0,1} \neq 0 \) and \( z_{2k} \neq 0 \).

So let \( C \nsubseteq \{x_{2k+2} = 0\} \). Then we may write
\[
R_{\ell + k + 1} = M_{k+1} \cdot x_{\ell + 2k + 2} + R_{\ell + k + 2}
\]
as in [13] of Theorem 3.5. Assume that \( k > 2 \). Then we have
\[
\text{det}(\partial_i \partial_j F(z))_{j=0,3,2k+1}^{i=0,1,2k+1} = 6x_0 \cdot (2\gamma_13x_2x_3x_{2k+2} + \gamma_13\gamma_2k+1,2k+1x_2^2x_{2k+2} - \gamma_12k+1\gamma_3,2k+1x_2^2x_{2k+2} + Q),
\]
where \( Q \in \mathbb{C}[x_1, \ldots, x_n] \) is a quadratic form such that \( C \subseteq \{Q = 0\} \) (because \( C \subseteq \bigcap_{i \in I_{k+1}} \{x_i = 0\} \)) and where \( \gamma_{i,j} \) is the coefficient of \( x_{2k+2} \) in \( \partial_i \partial_j F \). Note that \( \gamma_{1,3} \neq 0 \) (because \( \partial_3 \partial_3 F(z) \neq 0 \), being this last inequality true for \( q_2 \)). Since \( z_0 \neq 0 \) and \( z_{\ell + 2k} \neq 0 \), we conclude that
\[
C \subseteq \{2\gamma_13x_2 + (\gamma_13\gamma_2k+1,2k+1 - \gamma_12k+1\gamma_3,2k+1)x_{2k+2} = 0\},
\]
which contradicts the fact that \( q_k \in C \). Hence we conclude that \( k \leq 2 \). Now it is easy to see that \( C \) is a line or a plane conic.

Assume now that \( C \nsubseteq \{x_1 = 0\} \). Then \( \ell \geq 1 \). Note that for \( j = 3, \ldots, n \), we have \( \partial_1 \partial_j F = 0 \), hence for a general point \( z = [z_0, \ldots, z_n] \in C \), for \( h = 2, \ldots, n \), and for \( m = 3, \ldots, n \), we may consider
\[
(\partial_i \partial_j F(z))_{j=0,1,m}^{i=0,1,h}
\]
to conclude that \( \partial_h \partial_m F(z) = 0 \) (because \( \text{det}(\partial_i \partial_j F(z))_{i,j=0,1} \neq 0 \)). This implies easily that we may assume \( k = 0 \). By Step 2 of the proof of Theorem 3.5 for any \( i = 1, \ldots, \ell \), there is a point \( p_i \in C \) such that \( p_i \notin \{x_0 = 0\}, p_i \notin \{x_i = 0\}, \) and \( p_i \in \bigcap_{j=i+1}^n \{x_j = 0\} \).

Assume first that \( C \subseteq \{x_{\ell + 2} = 0\} \). So we may write
\[
F = \sum_{i=0}^{\ell} G_i + R_{\ell + 1},
\]
where \( G_i \in \mathbb{C}[x_i, x_{i+1}] \) and \( R_{\ell + 1} \in \mathbb{C}[x_{\ell + 1}, \ldots, x_n] \) are cubic forms and \( C \subseteq \bigcap_{i=\ell + 1}^n \{x_i = 0\} \).

Suppose that \( \ell > 2 \). Since \( \partial_3 \partial_3 F(p_2) = 0, \partial_2 \partial_3 F(p_2) = 0, \) and \( \partial_3 \partial_3 F(p_3) = 0 \), we see that the monomials \( x_3x_2^2, x_3^2x_3, \) and \( x_3^3 \) do not appear in \( F \). The same holds for \( x_3x_2^2 \) and \( x_3^2x_4 \), which gives a contradiction. Hence \( \ell \leq 2 \), and it is easy to conclude.
If \( C \not\subseteq \{ x_{\ell+2} = 0 \} \), then we may write

\[
F = \sum_{i=0}^{\ell} G_i + x_{\ell+1}^2 \cdot x_{\ell+2} + R_{\ell+1},
\]

where \( G_i \in \mathbb{C}[x_i, x_{i+1}] \) and \( R_{\ell+1} \in \mathbb{C}[x_{I_{\ell+1}}] \).

Suppose \( \ell \geq 2 \). Since \( \partial_{\ell+1} \partial_{\ell+1} F(p_\ell) = 0 \), we see that \( x_{\ell+1}^2 x_{\ell} \) does not appear in \( F \), and this implies, considering \( \partial_{\ell+1} \partial_{\ell+1} F(z) \), that also \( x_{\ell+1}^2 x_{\ell+2} \) does not appear in \( F \), which is a contradiction. Thus \( \ell < 2 \), and we are done. \( \square \)

Remark 3.7. Note that in general \( V_F \) might contain surfaces, e.g., if

\[
F(x_0, \ldots, x_n) = x_0^3 + x_0 x_1^2 + x_1 \sum_{i=2}^{n} x_i^2,
\]

then \( \dim V_F = n - 2 \).

Our goal is now to improve Theorems 3.5 and 3.6 and characterise those cubic forms \( F \) such that \( V_F \) contains a curve \( C \) such that \( C \not\subseteq \{ F = 0 \} \). To this end we restrict to the case of cubic forms with nonzero discriminant.

Corollary 3.8. Let \( F \in \mathbb{C}[x_0, \ldots, x_n] \) be a nondegenerate cubic form such that

\[
F = ax_0^3 + bx_0^2 x_1 + G(x_1, \ldots, x_n).
\]

Let \( C \subseteq V_F \) be positive-dimensional irreducible variety such that \( p = [1, 0, \ldots, 0] \in C \), and assume that at least one of the following properties holds:

1. \( C \subseteq \{ x_1 = 0 \} \);
2. \( C \subseteq \{ F = 0 \} \).

Then \( \Delta_F = 0 \).

Proof. We first assume that \( C \subseteq \{ x_1 = 0 \} \). By the proof of Theorem 3.6, we may write

\[
F = x_0^3 + x_0 x_1^2 + (x_2^2 + \delta_1 x_2^2) x_2 + R(x_1, x_3, x_4, \ldots, x_n)
\]

for some \( \delta_1 \in \mathbb{C} \) and \( R \in \mathbb{C}[x_1, x_3, x_4, \ldots, x_n] \). It follows that the hypersurface \( \{ F = 0 \} \subseteq \mathbb{P}^n \) is singular at the point \( [0, 0, 1, 0, \ldots, 0] \) and in particular \( \Delta_F = 0 \), as claimed.

We now suppose that \( C \subseteq \{ F = 0 \} \) and \( C \not\subseteq \{ x_1 = 0 \} \). Since \( [1, 0, \ldots, 0] \in C \), we may write

\[
F = bx_0^2 x_1 + c_1 x_1^3 + L x_1^2 + Q x_1 + R,
\]

where \( b, c_1 \in \mathbb{C} \) and \( L, Q, R \in \mathbb{C}[x_2, \ldots, x_n] \) are homogeneous polynomials of degrees 1, 2, and 3, respectively. Since \( F \) is nondegenerate, we have that \( b \neq 0 \).

After a change of coordinates in \( (x_1, x_2, \ldots, x_n) \), we may assume that there exists a point \( q = [q_0, q_1, 0, \ldots, 0] \in C \) such that \( q_0, q_1 \neq 0 \) and that \( L = c_2 x_2 \) for some \( c_2 \in \mathbb{C} \). Note that since \( C \subseteq \{ F = 0 \} \), it follows that \( C \) is not a line. Furthermore, since \( q \in V_F \), we may assume that \( Q = c_3 x_2^2 \) for some \( c_3 \in \mathbb{C} \), and we may write

\[
F = bx_0^2 x_1 + c_1 x_1^3 + c_2 x_1^2 x_2 + c_3 x_1 x_2^2 + c_4 x_2^3 + R_1,
\]

where \( c_4 \in \mathbb{C} \) and \( R_1 \in \mathbb{C}[x_2, \ldots, x_n] \) is such that the monomial \( x_2^3 \) does not appear in \( R_1 \). It is easy to see that \( \partial_i \partial_j F(z) = 0 \) for \( i = 2, \ldots, n, j = 2, \ldots, n \), with \( (i, j) \neq (2, 2) \) and \( z \in C \). If \( C \subseteq \{ x_2 = 0 \} \), then, after a change of coordinates
in \((x_3, \ldots, x_n)\), we may assume that there is a point \(r = [r_0, r_1, 0, r_3, 0, \ldots, 0] \in C\) such that \(r_3 \neq 0\). It follows that
\[
R_1 = \alpha x_3^2 + R_2(x_2, x_4, \ldots, x_n)
\]
for some \(\alpha \in \mathbb{C}\) and \(R_2 \in \mathbb{C}[x_4, \ldots, x_n]\). In particular, \([0, 0, 0, 1, 0, \ldots, 0]\) is a singular point of \(\{F = 0\} \subseteq \mathbb{P}^n\). Thus, \(\Delta_F = 0\), as claimed.

Thus, we may assume that \(C \not\subseteq \{x_2 = 0\}\) and that there is a point \(s = [s_0, s_1, s_2, 0, \ldots, 0]\) such that \(s_2 \neq 0\). Since \(\partial \partial_j F(s) = 0\) for \(i = 2, \ldots, n, j = 2, \ldots, n\), with \((i, j) \neq (2, 2)\), it follows that \(R_1\) does not depend on \(x_2\). Thus, \(\partial \partial_j F(z) = 0\) for any \(i, j \geq 3\) and \(z \in C\). Lemma 2.7 implies that \(C\) is contained in the plane \(\Pi = \{x_3 = \cdots = x_n = 0\}\). Let \(F_1\) be the restriction of \(F\) to \(\Pi\). Since \(C \subseteq \{F = 0\}\), it follows that if \([x_0, x_1, x_2, 0, \ldots, 0] \in C\), then \(F_1(x_0, x_1, x_2) = 0\) and \(H_{F_1}(x_0, x_1, x_2) = 0\). Thus \(C\) is a line, which gives a contradiction. \(\square\)

**Corollary 3.9.** Let
\[
F(x_0, \ldots, x_n) = \alpha x_3^2 + x_0^2(bx_1 + cx_2) + G(x_1, \ldots, x_n)
\]
be a nondegenerate cubic form with integral coefficients such that \(b \neq 0\). Assume that the line \(C = \{x_2 = x_3 = \cdots = x_n = 0\}\) is contained inside \(V_F\). Then there exists \(T = (t_{ij}), i,j = 0, \ldots, n \in \text{SL}(n+1, \mathbb{Q})\) such that
\[
T \cdot F = \alpha x_3^2 + bx_0^2 x_1 + c_1 x_3^3 + R(x_2, \ldots, x_n),
\]
where \(c_1 \in \mathbb{Z}\) and \(R \in \mathbb{Q}[x_2, \ldots, x_n]\) is a cubic form. Moreover, we may choose \(T\) such that \(t_{00} = t_{11} = 1, t_{0i} = t_{i0} = 0\) for \(i = 1, \ldots, n, t_{ij} = 0\) for \(i = 2, \ldots, n,\) and \(j = 1\).

**Proof.** After replacing \(x_1\) by \(x_1 - cx_2/b\), we may write
\[
F = \alpha x_3^2 + bx_0^2 x_1 + c_1 x_3^3 + L x_1^2 + Q x_1 + R,
\]
where \(c_1 \in \mathbb{Z}\) and \(L, Q, R \in \mathbb{Q}[x_2, \ldots, x_n]\) are homogeneous polynomials of degrees 1, 2, and 3, respectively. After a change of coordinates in \((x_2, \ldots, x_n)\), we may also assume that \(L = c_2 x_2\) for some \(c_2 \in \mathbb{Q}\). Let \(q = [0, 1, 0, \ldots, 0] \in C\). We distinguish two cases: \(c_1 \neq 0\) and \(c_1 = 0\).

If \(c_1 \neq 0\) then, since \(b \neq 0\) and \(\text{rk} \ H_F(q) \leq 2\), we see that \(Q = c_3 x_2^2\) for some \(c_3 \in \mathbb{Q}\) and
\[
|\partial_i \partial_j F(q)_{i=1,2}| = 0.
\]
It follows that \(|\partial_i \partial_j F(z)_{i=1,2}| = 0\) for any \(z \in C\). Since
\[
|\partial_i \partial_j F(z)_{i,j=0,1,2}| = 0,
\]
we have that \(c_2 = c_3 = 0\). Thus, \(L = Q = 0\), and the claim follows.

If \(c_1 = 0\) then since \(b \neq 0\) and \(\text{rk} \ H_F(q) \leq 2\), it follows that \(c_2 = 0\). Since \(\text{rk} \ H_F(z) \leq 2\) for any \(z \in C\), we have \(Q = 0\) and, again, the claim follows. Note that in this case, we have \(\Delta_F = 0\). \(\square\)

**Corollary 3.10.** Let
\[
F(x_0, \ldots, x_n) = \alpha x_3^2 + x_0^2(bx_1 + cx_3) + G(x_1, \ldots, x_n)
\]
be a nondegenerate cubic form with integral coefficients with \(b, c \in \mathbb{Z}\) and \(G \in \mathbb{Z}[x_1, \ldots, x_n]\) such that \(b \neq 0\) and \(\Delta_F \neq 0\). Let \(C \subseteq V_F\) be a positive-dimensional irreducible variety such that \(C \not\subseteq \{F = 0\}\) and \(p = [1, 0, \ldots, 0] \in C\). Assume that \(C\) contains infinitely many rational points. Assume moreover that
$C \subseteq \Pi = \{x_3 = \cdots = x_n = 0\}$ and $C$ is not a line. Then there exists $T = (t_{ij})_{i,j=0,\ldots,n} \in \text{SL}(n+1, \mathbb{Q})$, $R \in \mathbb{Z}[x_1, x_2]_3$, and $S \in \mathbb{Q}[x_3, \ldots, x_n]_3$ such that:

1. $t_{00} = 1, t_{i0} = t_{0i} = 0$ for $i = 1, \ldots, n$, $t_{ij} = 0$ for $i = 3, \ldots, n$, and $j = 1, 2$,

$$\left( t_{ij} \right)_{i,j=0,1,2} \in \text{SL}(3, \mathbb{Z});$$

and

2. $T \cdot F = ax_0^3 + bx_0^2x_1 + R(x_1, x_2) + S(x_3, \ldots, x_n)$.

Proof. We may assume that there is a point $q = [z_0, 1, 0, \ldots, 0] \in C$ such that $z_0 \neq 0$. Indeed, since $C$ is not a line, there exists $m \in \mathbb{Z}$ such that $\{mx_1 + x_2 = 0\} \cap \Pi$ intersect $C$ in a point $[z_0, 1, -m, 0, \ldots, 0]$ with $z_0 \neq 0$. After replacing $x_2$ with $x_2 + mx_1$, we may assume that $m = 0$.

In addition, after replacing $x_1$ with $x_1 - c/bx_3$, we may assume that $c = 0$. Thus, we may write

$$F = ax_0^3 + bx_0^2x_1 + c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3 + x_1^2L + x_1Q + S,$$

where $c_i \in \mathbb{Z}$ and $L \in \mathbb{Q}[x_3, \ldots, x_n]$, and $Q, S \in \mathbb{Q}[x_2, \ldots, x_n]$ are homogeneous polynomials of degrees 1, 2, and 3, respectively, such that the coefficient of $x_2^2$ in $Q$ and the coefficient of $x_3^3$ in $S$ are zero.

If $c_2 \neq 0$, then, after replacing $x_2$ with $x_2 - L/c_2$, we may assume $L = 0$. Since $b \neq 0$ and $q \in V_F$, it follows that $Q = 0$. Now considering a general point $z \in C \subseteq \{x_3 = \cdots = x_n = 0\}$, we see that $\partial_2 \partial_3 S(1, 0, \ldots, 0) = 0$ for all $i, j \geq 2$.

As in the proof of Lemma 2.7 it follows that $S$ does not depend on $x_2$. Thus, (2) holds.

Assume now that $c_2 = 0$ and $L = 0$. Then the Hessian of the quadric $c_3x_2^2 + Q$ has rank not greater than 1, which means that

$$c_3x_2^2 + Q = c_3(x_2 + L_1)^2$$

for some $L_1 \in \mathbb{Q}[x_3, \ldots, x_n]$ of degree 1. Hence, replacing $x_2$ with $x_2 - L_1$, we may assume that $Q = 0$. As in the previous case, it follows that $S$ does depend on $x_2$. Thus, (2) holds.

Finally, assume that $c_2 = 0$ and $L \neq 0$. Acting on $(x_3, \ldots, x_n)$ with $\text{SL}(n-2, \mathbb{Q})$, we may write $L = \alpha x_3$, where $\alpha \neq 0$. In particular, $\partial_3 \partial_1 F(q) \neq 0$. It follows that the first two columns $H_F^0(q)$ and $H_F^1(q)$ of $H_F(q)$ are linearly independent, which implies that $c_3 = 0$. Considering now a general point in $C \subseteq \{x_3 = \cdots = x_n = 0\}$, we see that $c_4 = 0$, and that the only monomial which appears in $x_1Q + S$ with nonzero coefficient and which contains $x_2$ is $x_2x_3^2$. Since $[0, 0, 1, 0, \ldots, 0]$ is a singular point of the hypersurface $\{F = 0\} \subseteq \mathbb{P}^n$, it follows that $\Delta_F = 0$, a contradiction. $\square$

3.2. Binary and ternary cubic forms. We now study the possible reduced forms of a nondegenerate binary or ternary cubic form. We show that if $F$ is a binary cubic form, it admits only finitely many nonequivalent reduced forms (cf. Proposition 3.13). On the other hand, if $F$ is a ternary cubic form, then the same result holds with the extra assumption that the discriminant $\Delta_F$ is nonzero (cf. Proposition 3.16). Example 3.17 shows that this assumption is necessary. We first recall the following known result.

**Proposition 3.11.** Let $\Delta \neq 0$ be an integer. Then there exist

$$F_1, \ldots, F_k \in \mathbb{Z}[x_0, x_1, x_2]_3 \quad (\text{resp. } \mathbb{Z}[x_0, x_1]_3)$$

such that if $F \in \mathbb{Z}[x_0, x_1, x_2]_3$ (resp. $\mathbb{Z}[x_0, x_1]_3$) is such that $\Delta_F = \Delta$, then there exists $i = 1, \ldots, k$ and $T \in \text{SL}(3, \mathbb{Z})$ (resp. $\text{SL}(2, \mathbb{Z})$) such that $F = T \cdot F_i$. 


Lemma 3.12. Let 
\[ F(x, y) = ax^3 + bx^2y + cy^3 \in \mathbb{Z}[x, y] \]
be a binary cubic form with integral coefficients and such that \( c \neq 0 \). Then there are finitely many pairs 
\[ (a_i, b_i) \in \mathbb{Z}^2, \quad i = 1, \ldots, k, \]
such that if \((a', b', cy^3)\) is a reduced triple associated to \( F \) (cf. Definition 2.9), then \( a' = a_i \) and \( b' = b_i \) for some \( i \in \{1, \ldots, k\} \).

Proof. Assume that \( T = (t_{ij})_{i,j=0,1} \in SL(2, \mathbb{Z}) \) is such that \( T \cdot F \) is in reduced form \((a', b', cy^3)\) for some \( a', b' \in \mathbb{Z} \).

Note that \( F(t_{01}, t_{11}) = c \), and, since \( c \neq 0 \), the equation \( F(x, y) = c \) defines a smooth affine plane curve of genus 1. Thus, by Siegel’s Theorem 2.8, it only admits finitely many solutions. Thus, we may assume that \( t_{01} \) and \( t_{11} \) are fixed. Since \( \det T = 1 \) and since the coefficient of \( xy^2 \) is zero, we get the linear system in \( t_{00} \) and \( t_{10} \):
\[
\begin{align*}
1 &= t_{11}t_{00} - t_{01}t_{10}, \\
0 &= (3at_{01}^2 + 2bt_{01}t_{11})t_{00} + (bt_{01}^2 + 3ct_{11}^2)t_{10}.
\end{align*}
\]

Note that the determinant of the system is equal to \( 3F(t_{01}, t_{11}) = 3c \neq 0 \). Thus, the system admits exactly one solution, and the claim follows.

Proposition 3.13. Let 
\[ F(x, y) = ax^3 + bx^2y + cy^3 \in \mathbb{Z}[x, y] \]
be a binary integral cubic form with \( c \neq 0 \). Then there are finitely many triples 
\[ (a_i, b_i, c_i) \in \mathbb{Z}^3, \quad i = 1, \ldots, k, \]
such that \( c_i \neq 0 \), and if \((a', b', cy^3)\) is a reduced triple associated to \( F \) (cf. Definition 2.9), then \( a' = a_i, b' = b_i, \) and \( c' = c_i \) for some \( i \in \{1, \ldots, k\} \).

Proof. By Lemma 3.12 it is enough to show that there are only finitely many \( c_1, \ldots, c_k \in \mathbb{Z} \) such that if \( T \in SL(3, \mathbb{Z}) \) is such that \( T \cdot F \) is in reduced form \((a', b', cy^3)\) with \( c' \neq 0 \), then \( c' = c_i \) for some \( i \in \{1, \ldots, k\} \).

If the discriminant \( \Delta_F = 4b^3c + 27a^2c^2 \) of \( F \) is not zero, then \( c'|\Delta_F \), and the claim follows.

Thus, we may assume that \( \Delta_F = 0 \). We may also assume that \( a, b, \) and \( c \) do not have a common factor; otherwise, we just consider the cubic form obtained by dividing by the common factor. Suppose that \( T = (t_{ij})_{i,j=0,1} \). Then,
\[
\begin{align*}
a &= a't_{00}^3 + b't_{00}^2t_{10} + c't_{10}^2, \\
b &= 3a't_{00}^2t_{01} + b't_{00}t_{11} + 2b't_{00}t_{01}t_{10} + 3c't_{10}^2t_{11}, \\
o &= 3a't_{00}t_{01}^2 + b't_{01}^2t_{10} + 2b't_{00}t_{01}t_{11} + 3c't_{10}t_{11}^2, \\
c &= a't_{01}^3 + b't_{01}^2t_{11} + c't_{11}^3.
\end{align*}
\]
and \( GCD(a', b', c') = 1 \).

Let \( p \) be a prime factor of \( c' \) such that \( p \neq 2, 3 \), and let \( \alpha \) be a positive integer such that \( p^\alpha|c' \). Then, since \( \Delta_F = 0 \), it follows that \( p^\alpha|\Delta_F \) divides \( b' \). By (4) and since \( \gcd(t_{00}, t_{01}) = 1 \), we have that either \( p^{\alpha/3} \) divides \( t_{00} \) or \( p^{\alpha/6} \) divides \( t_{01} \).
In the first case, (2) implies that $p^\alpha$ divides $a$, and in the second case (5) implies that $p^{\alpha/2^3}$ divides $c$. Since $a, c \neq 0$ are fixed, it follows that $p^\alpha$ is bounded. A similar argument holds for the powers of 2 and 3. Hence $c'$ is bounded, as claimed. \hfill \Box

We now consider ternary cubic forms.

**Proposition 3.14.** Let $R$ be a ring which is finitely generated over $\mathbb{Z}$, and let $F \in R[x, y, z]$ be a cubic form with nonzero discriminant $\Delta_F$. Let $G(y, z) = dy^3 + z^3$ for some nonzero $d \in R$, and assume that $F$ is in reduced form $(a, (b, c), G)$ for some pair $(a, (b, c)) \in R \times R$. Then there are finitely many pairs

$$(a_i, (b_i, c_i)) \in R \times R^2, \quad i = 1, \ldots, k,$$

such that if $(a', (b', c'), G)$ is a reduced triple associated to $F$ (cf. Definition 2.9), then $a' = a_i, b' = b_i$, and $c' = c_i$ for some $i \in \{1, \ldots, k\}$.

**Proof.** Assume that $T \in SL(3, R)$ is such that $T \cdot F$ is in reduced form $(a', (b', c'), G)$. The invariants $S_F$ and $T_F$ (cf. subsection 2.3 and [Stu93, 4.4.7 and 4.5.3]) have the form

$$S_F = dbc \quad \text{and} \quad T_F = 27a^2d^2 + 4b^3d + 4c^3d^2.$$  

We first assume that $S_F \neq 0$, and we consider the curve $C \subseteq \mathbb{P}^3$ given by the ideal

$$I = (S_Fx_3^2 - dx_1x_2, T_Fx_3^3 - 27d^2x_0^3x_3 - 4dx_1^3 - 4d^2x_2^3).$$

We claim that the points $[a', b', c', 1] \in C$, with $a', b', c' \in R$ are in finite number, and hence the claim follows.

Note that the first equation defines a cone over a conic with vertex the point $q = [1, 0, 0, 0] \in C$. If we blow up the point $q$, then it is easy to check the strict transform $\overline{C}$ of the curve $C$ is a connected smooth curve of genus 3. Thus, the claim follows by Siegel’s Theorem 2.8.

We now assume that $S_F = 0$. Then, $b' = 0$ or $c' = 0$. Assume that $c' = 0$. Then the pair $(a', b')$ corresponds to an $R$-integral point in the affine plane curve, defined by the equation

$$27x_0^2d^2 + 4x_1^3d - T_F = 0.$$  

Since, by assumption $\Delta_F \neq 0$, we have that $T_F \neq 0$. Thus, Siegel’s Theorem 2.8 implies the claim. The case $b' = 0$ is similar. \hfill \Box

**Remark 3.15.** Note that if $F \in R[x, y, z]$ is a cubic form such that $\Delta_F = 0$ and $S_F = 0$, and $C$ is the curve defined in the proof of Proposition 3.14, then $C$ is a rational curve.

As a consequence of the previous result we obtain the following.

**Proposition 3.16.** Let $F \in \mathbb{Z}[x, y, z]$ be a cubic form with nonzero discriminant $\Delta_F$. Then there are finitely many triples

$$(a_i, B_i, G_i) \in \mathbb{Z} \times \mathbb{Z}^2 \times \mathbb{Z}[y, z], \quad i = 1, \ldots, k,$$  

such that any reduced triple associated to $F$ is equivalent to $(a_i, B_i, G_i)$ over $\mathbb{Z}$, for some $i \in \{1, \ldots, k\}$ (cf. Definition 2.9).

**Proof.** Let $T \in SL(3, \mathbb{Z})$ such that $T \cdot F$ is in reduced form $(a, B, G)$ for some $a \in \mathbb{Z}$, $B \in \mathbb{Z}^2$, and $G \in \mathbb{Z}[y, z]$ a cubic form. Lemma 2.6 implies that $\Delta_G$ divides $\Delta_F$. Thus, $\Delta_G \neq 0$, we may assume that its value is fixed, and, by Proposition 3.11 we may assume that $G$ is also fixed, up to the action of $SL(2, \mathbb{Z})$.
Let \( d = \sqrt{\Delta_F} \). After possibly replacing the ring of integers \( \mathbb{Z} \) by a finitely generated ring \( R \) over \( \mathbb{Z} \), we may assume, up to a \( \text{SL}(2,R) \)-action, that
\[
G(y,z) = dy^3 + z^3.
\]
Thus, the claim follows from Proposition 3.14. \( \Box \)

Note that Proposition 3.14 does not hold if the discriminant of \( F \) is zero, as the following example shows.

**Example 3.17.** Let
\[
F = ax^3 + bx^2y + x^2z - 3y^2z,
\]
where \( a, b \in \mathbb{Z} \). Note that \( \Delta_F = 0 \), since \([0,0,1]\) is a singular point for \( \{ F = 0 \} \).
Consider Pell’s equation
\[
s^2 - 3t^2 = 1. \tag{6}
\]
For any solution \((\alpha, \beta) \in \mathbb{Z}^2\) of (6), we define the matrix
\[
M = \begin{pmatrix}
\alpha & 3\beta & 0 \\
\beta & \alpha & 0 \\
m_{31} & m_{32} & 1
\end{pmatrix},
\]
where \( m_{31} = \beta(3b\beta^2 + 9a\alpha\beta + 2b\alpha^2) \) and \( m_{32} = 3\beta^2(3a\beta + b\alpha) \). Then \( M \in \text{SL}(3, \mathbb{Z}) \) and
\[
M \cdot F(X,Y,Z) = AX^3 + BX^2Y + X^2Z - 3Y^2Z,
\]
where
\[
A = 3b\alpha^2\beta + 3b\beta^3 + a\alpha^3 + 9a\alpha\beta^2 \quad \text{and} \quad B = 9a\beta^3 + 9b\alpha^2\beta + 9a\alpha^2\beta + b\alpha^3.
\]
Since (6) has infinitely many integral solutions, it follows that there are infinitely many ways to write \( F \) in reduced form.

In the example above, \( \{ F = 0 \} \) defines an irreducible cubic with a node. Note that such cubics can be realised as the cubic form associated to a smooth threefold (the existence of such a threefold was queried in [OVdV95, Proposition 21]):

**Example 3.18.** Let \( W = \mathbb{P}^3 \), let \( h \) be the hyperplane class, and let \( C \) be a line. Note that \( \text{deg} \ N_C/W = 2 \). Let \( \pi : X \to W \) be the blowup of \( W \) along \( C \), and define \( H = \pi^*h \). Let \( \{ L_1, L_2 \} \) be the basis of \( H^2(X, \mathbb{Z}) \) given by
\[
L_1 = H \quad \text{and} \quad L_2 = H - E,
\]
where \( E \) is the exceptional divisor of \( \pi \). The intersection cubic form on \( H^2(X, \mathbb{Z}) \) is
\[
G(y,z) = (yL_1 + zL_2)^3 = y^3 + 3y^2z.
\]
Let \( C' \subseteq \mathbb{P}^3 \) be a line which meets \( C \) transversally in one point, and let \( D \) be the strict transform of \( C' \) in \( X \). Then \( D \cong H^2 - H \cdot E \) and blowing up \( X \) along \( D \), we get a threefold \( Y \) with associated cubic form
\[
F(x,y,z) = x^3 - 3(y + z)x^2 + y^3 + 3y^2z.
\]
Note that \( \{ F = 0 \} \subseteq \mathbb{P}^2 \) defines an irreducible cubic with a node and in particular \( \Delta_F = 0 \).
3.3. **General cubic forms.** We now combine the previous results to give a proof of Theorem 3.1. We begin with the following.

**Lemma 3.19.** Let $F \in \mathbb{Z}[x_0, \ldots, x_n]$ be a nondegenerate cubic form, and let $p \in V_F$ such that $F(p) \neq 0$. Then there are finitely many triples

$$(a_i, B_i, G_i) \in \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}[x_1, \ldots, x_n]_3, \quad i = 1, \ldots, k,$$

such that, for all $T \in \text{SL}(n+1, \mathbb{Z})$ such that $T \cdot p = [1, 0, \ldots, 0]$ and $T \cdot F$ is in reduced form, we have that $T \cdot F$ is equivalent to $(a_i, B_i, G_i)$ over $\mathbb{Z}$ for some $i \in \{1, \ldots, k\}$ (cf. Definition 2.9).

**Proof.** We may assume that $p = [1, 0, \ldots, 0]$ and that $F = (a, b, G)$ is in reduced form, for some $a \in \mathbb{Z}$, $B \in \mathbb{Z}^n$ and $G \in \mathbb{Z}[x_1, \ldots, x_n]_3$. We consider all the matrices $T \in \text{SL}(n+1, \mathbb{Z})$, such that $T \cdot p = p$, and $T \cdot F = (a_T, b_T, G_T)$ is in reduced form, for some $a_T \in \mathbb{Z}$, $B_T \in \mathbb{Z}^n$ and $G_T \in \mathbb{Z}[x_1, \ldots, x_n]$.

If we write $T = (t_{ij})_{i,j=0,\ldots,n}$ with $t_{ij} \in \mathbb{Z}$, then, since $T \cdot p = p$, we have $t_{i0} = 0$ for $1 \leq i \leq n$. Thus, $t_{00} = \pm 1$ and in particular $a_T = \pm a$.

By considering the action of $\text{SL}(n, \mathbb{Z})$ over $(x_1, \ldots, x_n)$, we may assume that $B = (b_1, 0, \ldots, 0)$ and that, for each $T$, $B_T = (b^T_1, 0, \ldots, 0)$, with $b_1, b^T_1 \in \mathbb{Z}$. Note that, by the assumption on $F$, we have that $b$ and $b_1$ cannot be both zero.

By looking at the coefficients of $x_0^2x_i$ and $x_0x_i^2$, we obtain the equations

$$3at_{0i} + b_1t_{1i} = 0 \quad \text{for } i = 2, \ldots, n$$

and

$$3at^2_{0i} + 2b_1t_{0i}t_{1i} = 0 \quad \text{for } i = 1, \ldots, n.$$ 

We now consider three cases.

If $b_1 = 0$, then $a \neq 0$ and (7) implies that $t_{0i} = 0$ for $i = 1, \ldots, n$. In particular, $T \cdot F$ is equivalent to $F$.

If $a = 0$, then $b_1 \neq 0$ and (7) implies that $t_{1i} = 0$ for $i = 2, \ldots, n$. In particular, $t_{11} = \pm 1$. By looking at the coefficients of $x_0x_i^2$ for $i = 1, \ldots, n$, we get the equations

$$b_1t_{0i}t_{11} = 0.$$ 

Thus $t_{0i} = 0$ for $i = 1, \ldots, n$ and, as in the previous case, we obtain that $T \cdot F$ is equivalent to $F$.

Finally, if $a, b \neq 0$, then (7) implies that $t_{0i} = t_{1i} = 0$ for $i = 2, \ldots, n$. In particular, $t_{11} = \pm 1$. By (7), it follows that $t_{01}$ can only acquire finitely many values. Thus, under these assumptions on $T$, it follows that there are only finitely many nonequivalent reduced forms $T \cdot F$ over $\mathbb{Z}$, as claimed. 

In the next lemma we show that under the action of the transformations given by Corollaries 3.9 and 3.10, we may control the last part of a reduced form.

**Lemma 3.20.** Let $s \in \{1, 2\}$, and let $F, F_1 \in \mathbb{Q}[x_0, \ldots, x_n]$ be nondegenerate cubic forms such that

$$F = ax_0^3 + bx_0^2x_1 + R(x_1, x_s) + H(x_{s+1}, \ldots, x_n)$$

and

$$F_1 = a_1x_0^3 + b_1x_0^2x_1 + R_1(x_1, x_s) + H_1(x_{s+1}, \ldots, x_n),$$

where $b, b_1 \neq 0$ and $R, R_1, H, H_1$ are cubic forms.
Assume that there exists \( T = (t_{hk})_{h,k=0,\ldots,n} \in \text{SL}(n+1,\mathbb{Q}) \) such that \( T \cdot F = F_1 \), 
\( t_{hk} = 0 \) for \( h = s+1,\ldots,n \) and \( k = 0,\ldots,s \) and \( \det(t_{hk})_{h,k=0,\ldots,s} = 1 \), i.e.,
\[
T = \begin{pmatrix} S & \ast \\ 0 & \ast \end{pmatrix}
\]
with \( \det S = 1 \). Then there exists \( P \in \text{SL}(n-s,\mathbb{Q}) \) such that \( P \cdot H = H_1 \).

**Proof.** We prove the case \( s = 2 \); the case \( s = 1 \) is similar and easier.

We will show that \( t_{hk} = 0 \) for \( h = 0,1,2 \) and \( k = 3,\ldots,n \), which implies the claim.

Let \( S = (t_{hk})_{h,k=0,1,2} \), and define \( \overline{T} = (\overline{t_{hk}})_{h,k=0,\ldots,n} \in \text{SL}(n+1,\mathbb{Q}) \) as
\[
\overline{T} = \begin{pmatrix} S^{-1} & 0 \\ 0 & I_{n-2} \end{pmatrix},
\]
where \( I_{n-2} \in \text{SL}(n-2,\mathbb{Q}) \) is the identity matrix.

If \( M = (m_{ij})_{i,j=0,\ldots,n} = T \cdot T \) and \( \overline{F_1} = M \cdot F \), then \( F_1 \) is in reduced form with associated triple \( (a,b,0), R + H_1 \). In addition
\[
(m_{hk})_{h,k=0,1,2} = I_3 \quad \text{and} \quad (m_{hk})_{h=3,\ldots,n} = 0.
\]

We want to show that \( m_{hk} = 0 \) for \( h = 0,1,2 \) and \( k = 3,\ldots,n \). Since \( S \) is invertible, it follows that \( t_{hk} = 0 \) for \( h = 0,1,2 \) and \( k = 3,\ldots,n \), as claimed.

We assume first that \( a \neq 0 \). Recall that, by assumption, we have \( b \neq 0 \). For any \( k = 3,\ldots,n \), looking at the coefficients of the monomials \( x_0x_k^2 \) and \( x_0^2x_k \) in \( F_1 \), we obtain the equations
\[
3am_{0k} + bm_{1k} = 0 \quad \text{and} \quad 3am_{0k}^2 + 2bm_{0k}m_{1k} = 0,
\]
which imply that \( m_{0k} = m_{1k} = 0 \) for any \( k = 3,\ldots,n \).

We may write
\[
R(x_1, x_2) = c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3
\]
for some \( c_1,\ldots,c_4 \in \mathbb{Q} \). Looking at the coefficients of the monomials \( x_1^2x_k, x_1x_k^2 \), and \( x_0^2x_k \) in \( F_1 \) we see that
\[
c_2m_{2k} = 0, \quad c_3m_{2k} = 0, \quad \text{and} \quad c_4m_{2k} = 0.
\]
Since \( F \) is a nondegenerate cubic form, it follows that \( m_{2k} = 0 \) for \( k = 3,\ldots,n \). Thus, the claim follows.

Assume now that \( a = 0 \). Then, looking at the coefficients of \( x_0x_k^2 \) and \( x_0x_1x_k \), we obtain \( m_{0k} = m_{1k} = 0 \) for \( k = 3,\ldots,n \). Thus, as in the previous case, the claim follows. \( \square \)

**Proposition 3.21.** Let \( F \in \mathbb{Z}[x_0,\ldots,x_n] \) be a nondegenerate cubic form in reduced form,
\[
F(x_0,\ldots,x_n) = ax_0^3 + bx_0^2x_1 + G(x_1,\ldots,x_n),
\]
where \( G \in \mathbb{Z}[x_1,\ldots,x_n]_3 \). Assume that \( \Delta_F \neq 0 \). Let \( C \subseteq V_F \) be an irreducible component of positive dimension such that
\[
p = [1,0,\ldots,0] \in C, \quad C \not\subseteq \{F = 0\}, \quad \text{and} \quad C \not\subseteq \{x_1 = 0\}.
\]

Then there are finitely many triples
\[
(a_i, b_i, G_i) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}[x_1,\ldots,x_n]_3, \quad i = 1,\ldots,k,
\]
such that for all $T \in \text{SL}(n+1, \mathbb{Z})$ such that $[1,0,\ldots,0] \in T(C)$ and $T \cdot F$ is in reduced form, we have that $T \cdot F$ is equivalent to $(a_i, (b_i, 0), G_i)$ over $\mathbb{Z}$ for some $i \in \{1, \ldots, k\}$ (cf. Definition 3.9).

**Proof.** Suppose not. Then there exist an infinite sequence $T_i \in \text{SL}(n+1, \mathbb{Z})$ with $i = 1, 2, \ldots$ such that $[1,0,\ldots,0] \in T_i(C)$, $T_i \cdot F$ is in reduced form, and $T_i \cdot F$ and $T_j \cdot F$ are not equivalent over $\mathbb{Z}$ for any $i \neq j$.

Lemma 3.19 implies that the set $\{T_i^{-1}([1,0,\ldots,0])\}$ is infinite. In particular, $C$ admits infinitely many rational points. By Proposition 3.3 we have that $b \neq 0$, as otherwise $p \in W_F$.

We first assume that $C$ is a line. After acting on $(x_1,\ldots,x_n)$ with $\text{SL}(n, \mathbb{Z})$, we may assume that $C = \{x_2 = x_3 = x_4 = \cdots = x_n = 0\}$, and we may write

$$F = ax_0^3 + (bx_1 + cx_2)x_0^2 + G(x_1, \ldots, x_n),$$

where $b, c \in \mathbb{Z}$, $b \neq 0$, and $G \in \mathbb{Z}[x_1, \ldots, x_n]$ is a cubic form. Since reduced forms are considered modulo the action of $\text{SL}(n, \mathbb{Z})$ on $(x_1,\ldots,x_n)$, we may assume that for any $i = 1, 2, \ldots$, the cubic form $F_i = T_i \cdot F$ satisfies the same property, that is $F_i = a_i x_0^3 + (b_i x_1 + c_i x_2)x_0^2 + G_i(x_1, \ldots, x_n)$, where $b_i, c_i \in \mathbb{Z}$ are such that $b_i \neq 0$, $G_i \in \mathbb{Z}[x_1, \ldots, x_n]$, and $T_i(C) = \{x_2 = x_3 = x_4 = \cdots = x_n = 0\}$.

Fix $i$, and let $T_i = (t_{hh})_{h,k=0,\ldots,n}$. Since $\{x_2 = x_3 = x_4 = \cdots = x_n = 0\}$ is fixed by $T_i$, we have $t_{hh} = 0$ for $h = 2, \ldots, n$ and $k = 0, 1$. Since $\det T_i = 1$, we may assume $\det(t_{hh})_{h,k=0,1} = 1$.

We may find $M_i, M_i' \in \text{SL}(n, \mathbb{Q})$ as in Corollary 3.9 such that

$$\hat{F} = M \cdot F = ax_0^3 + bx_0^2x_1 + dx_1^3 + H(x_2, \ldots, x_n)$$

and

$$\hat{F}_i = M_i \cdot F_i = a_i x_0^3 + b_i x_0^2x_1 + d_i x_1^3 + H_i(x_2, \ldots, x_n),$$

where $d, d_i \in \mathbb{Z}$ and $H, H_i \in \mathbb{Q}[x_2, \ldots, x_n]$ are cubic forms.

In addition, if $\hat{T}_i = (\hat{t}_{hh})_{h,k=0,\ldots,n} = M_i \cdot T_i \cdot M_i^{-1}$, we have that $\hat{T}_i \cdot \hat{F} = \hat{F}_i$. Let

$$U_i := (\hat{t}_{hh})_{h,k=0,1}.$$

Note that, by Corollary 3.9, it follows that $\hat{t}_{hh} = 0$ for $h = 2, \ldots, n$ and $k = 0, 1$ and $U_i \in \text{SL}(2, \mathbb{Z})$.

Let

$$F' = \hat{F}_{i|C} = ax_0^3 + bx_0^2x_1 + dx_1^3$$

and

$$F'_i = \hat{F}_{i|C} = a_i x_0^3 + b_i x_0^2x_1 + d_i x_1^3.$$

Then $F', F'_i \in \mathbb{Z}[x_0, x_1]$ are binary cubic forms such that $U_i \cdot F' = F'_i$. In particular $\Delta_{F'} = \Delta_{F'_i} \neq 0$, as otherwise the hypersurface $\{\hat{F} = 0\} \subseteq \mathbb{P}^n$ would be singular and $\Delta_F = \Delta_\hat{F} = 0$, which contradicts the assumption on $F$. Thus, by Proposition 3.13 we may assume that

$$a_i = a, \quad b_i = b, \quad d_i = d \quad \text{for} \quad i = 1, 2, \ldots.$$

On the other hand, by Lemma 3.20 for each $i = 1, 2, \ldots$ there exists $P_i \in \text{SL}(n-1, \mathbb{Q})$ such that $H_i = P_i \cdot H$. Since the hyperplane $\{x_0 = 0\}$ is invariant with respect to $M_i$, there exist $M, M'_i \in \text{SL}(m, \mathbb{Q})$ such that if

$$H'(x_1, \ldots, x_n) = dx_1^3 + H(x_2, \ldots, x_n)$$

and

$$H'_i(x_1, \ldots, x_n) = dx_1^3 + H_i(x_2, \ldots, x_n),$$
then \( M' \cdot G = H' \) and \( M'_i \cdot G = H'_i \) for \( i = 1, 2, \ldots \). Thus, there exist \( P'_i \in \text{SL}(n, \mathbb{Q}) \) such that \( G_i = P'_i \cdot G \) for all \( i = 1, 2, \ldots \). By Jordan’s Theorem \(2.10\) it follows that, after possibly taking a subsequence, the reduced forms \( F_1, F_2, \ldots \) are equivalent over \( \mathbb{Z} \). Thus, we obtain a contradiction.

Assume now that \( C \) is not a line. Theorem \(3.6\) implies that \( C \) spans a plane \( P \).

After acting on \((x_1, \ldots, x_n)\) with \( \text{SL}(n, \mathbb{Z}) \), we may assume \( \Pi = \{x_3 = x_4 = \cdots = x_n = 0\} \), and we may write

\[
F = ax_0^3 + bx_0^2 x_1 + cx_0 x_1^2 + G(x_1, \ldots, x_n),
\]

where \( b, c \in \mathbb{Z}, b \neq 0, \) and \( G \in \mathbb{Z}[x_1, \ldots, x_n] \) is a cubic form.

Since reduced forms are considered modulo the action of \( \text{SL}(n, \mathbb{Z}) \) on \((x_1, \ldots, x_n)\), we may assume that this holds for any \( i = 1, 2, \ldots \), the cubic form \( F_i = T_i \cdot F \) satisfies the same property, that is

\[
F_i = a_i x_0^3 + x_0^2(b_i x_1 + c_i x_3) + G_i(x_1, \ldots, x_n),
\]

where \( b_i, c_i \in \mathbb{Z} \) are such that \( b_i \neq 0, G_i \in \mathbb{Z}[x_1, \ldots, x_n], \) and \( T_i(C) \subseteq \Pi = \{x_3 = x_4 = \cdots = x_n = 0\} \).

Fix \( i = 1, 2, \ldots \), and let \( T_i = (t_{hk})_{h,k=0,\ldots,n} \). Since \( \Pi = \{x_3 = \cdots = x_n = 0\} \) is fixed by \( T_i \), we have \( t_{hk} = 0 \) for \( h = 3, \ldots, n \) and \( k = 0, 1, 2 \). Since \( \det T_i = 1 \), we may assume \( \det(t_{hk})_{h,k=0,1,2} = 1 \).

By Corollary \(3.10\) we may find \( M, M_i \in \text{SL}(n, \mathbb{Q}) \) such that

\[
\tilde{F} = M \cdot F = ax_0^3 + bx_0^2 x_1 + R(x_1, x_2) + H(x_3, \ldots, x_n)
\]

and

\[
\tilde{F}_i = M_i \cdot \tilde{F} = a_i x_0^3 + b_i x_0^2 x_1 + R_i(x_1, x_2) + H_i(x_3, \ldots, x_n),
\]

where \( R, R_i \in \mathbb{Z}[x_1, x_2] \) and \( H, H_i \in \mathbb{Q}[x_3, \ldots, x_n] \) are cubic forms. In addition, if \( T_i = (\hat{t}_{hk})_{h,k=0,\ldots,n} = M_i \cdot T_i \cdot M_i^{-1} \), we have that \( \tilde{T}_i \cdot \tilde{F} = \tilde{F}_i \).

Let

\[
U_i := (\hat{t}_{hh})_{h,k=0,1,2}.
\]

Note that, by Corollary \(3.10\) it follows that \( \hat{t}_{hk} = 0 \) for \( h = 3, \ldots, n \), and \( k = 0, 1, 2 \) and \( U_i \in \text{SL}(3, \mathbb{Z}) \).

Let

\[
F' = \tilde{F}_{\Pi} = ax_0^3 + bx_0^2 x_1 + R(x_1, x_2) \quad \text{and} \quad F'_i = \tilde{F}_{i\Pi} = a_i x_0^3 + b_i x_0^2 x_1 + R_i(x_1, x_2).
\]

Then \( F', F_i \in \mathbb{Z}[x_0, x_1, x_2] \) are ternary cubic forms such that \( U_i \cdot F' = F'_i \). In particular \( \Delta_{F'} = \Delta_{F'_i} \neq 0 \), as otherwise the hypersurface \( \{\tilde{F} = 0\} \subseteq \mathbb{P}^n \) would be singular and \( \Delta_F = \Delta_{\tilde{F}} = 0 \), which contradicts the assumption on \( F \). Thus, by Proposition \(3.16\) we may assume that \( a_i, b_i, \) and \( R_i \) do not depend on \( i = 1, 2, \ldots \).

As in the previous case we obtain that, after possibly taking a subsequence, \( F_1, F_2, \ldots \) are equivalent over \( \mathbb{Z} \), a contradiction. \( \square \)

We are now ready to prove Theorem \(3.1\)

**Proof of Theorem 3.1** We may assume that \( F \) is in reduced form,

\[
F = ax_0^3 + bx_0^2 x_1 + G,
\]

where \( a, b \in \mathbb{Z} \) and \( G \in \mathbb{Z}[x_1, \ldots, x_n] \).

We assume that there exist \( T_i \in \text{SL}(n + 1, \mathbb{Z}) \), with \( i = 1, 2, \ldots \) such that \( F_i = T_i \cdot F \) is in reduced form \((a_i, B_i, G_i)\) for some \( a_i \in \mathbb{Z}, B_i \in \mathbb{Z}^n, \) and \( G_i \in \mathbb{Z}[x_1, \ldots, x_n] \) and \( F_i \) and \( F_j \) are not equivalent over \( \mathbb{Z} \) for any \( i \neq j \). Acting on \((x_1, \ldots, x_n)\) with \( \text{SL}(n, \mathbb{Z}) \), we may assume that \( B_i = (b_i, 0, \ldots, 0) \), for some \( b_i \in \mathbb{Z} \).
Let $p = [1, 0, \ldots, 0]$, and let $C_1, \ldots, C_k \subseteq V_F$ be all the irreducible components. Then, after possibly replacing $p$ by $T_j(p)$ for some $j$, we may assume that $p, T_j(p) \in C = C_1$ for all $i$ (possibly passing to an infinite subsequence). Lemma 3.19 implies that $C$ is of positive dimension.

Since by assumption $\Delta_F \neq 0$, Corollary 3.8 implies that

$$C \not\subseteq \{x_1 = 0\} \quad \text{and} \quad C \not\subseteq \{F = 0\}.$$  

Thus, Proposition 3.21 implies a contradiction. \hfill $\square$

We conclude the section proving a finiteness result on a special class of reduced forms. The result will be used in 4.2.

**Proposition 3.22.** Let $F \in \mathbb{Z}[x_0, \ldots, x_n]$ be a nondegenerate cubic form such that $\Delta_F \neq 0$. Fix an integer $r \neq 0$. Then there are finitely many pairs

$$(a_i, G_i) \in \mathbb{Z} \times \mathbb{Z}[x_1, \ldots, x_n], \quad i = 1, \ldots, k,$$

such that for all $T \in \text{GL}(n+1, \mathbb{Z})$ such that $\det T = r$ and $T \cdot F$ is in reduced form, we have that $T \cdot F$ is equivalent to $(a_i, 0, G_i)$ over $\mathbb{Z}$ for some $i \in \{1, \ldots, k\}$ (cf. Definition 2.9). Moreover, $\Delta_{G_i} \neq 0$ for all $i = 1, \ldots, k$.

**Proof.** Suppose not. Then there exist infinitely many $T_1, T_2, \ldots \in \text{GL}(n+1, \mathbb{Z})$ such that $\det T_i = r$, $T_i \cdot F = (a_i, 0, G_i)$ is in reduced form for each $i$, and $T_i$ and $T_j$ are not equivalent over $\mathbb{Z}$ for each $i \neq j$. We denote $S_{i,j} = T_i^{-1}T_j$. Note that $T_i([1, 0, \ldots, 0]) \in W_F$ for all $i$. Thus, by Proposition 3.3 we may assume that $[1, 0, \ldots, 0]$ is fixed by $S_{i,j}$ for each $i, j$. It follows easily that if $S_{i,j} = (s_{hk})$, then $s_{h0} = s_{0k} = 0$ for any $h, k = 1, \ldots, n$.

Since $\det T_i = r$, it follows that the denominators of the coefficients of $S_{i,j}$ are bounded, and since $\det S_{i,j} = 1$, it follows that $s_{00}$ is bounded and in particular there exist $i \neq j$ such that $T_i \cdot F$ is equivalent to $T_j \cdot F$ over $\mathbb{Z}$.

Finally, Lemma 2.6 implies that, for each $i$ we have $\Delta_{G_i} \neq 0$. \hfill $\square$

4. **Proof of the main results**

4.1. **Proof of Theorem 1.2** Let $X$ be a smooth projective threefold of general type. In this section we prove Theorem 1.2 i.e., we show that the volume of $X$ (cf. Definition 2.1) is bounded by a constant which depends only on the topological Betti numbers of $X$.

**Proof of Theorem 1.2** We may assume that $X$ is of general type, as otherwise $\text{vol}(X) = 0$. Let $X \dasharrow Y$ be a minimal model of $X$. Then $Y$ has only terminal singularities, and in particular it is smooth outside a finite number of points. In addition,

$$\text{vol}(X, K_X) = \text{vol}(Y, K_Y) = K_Y^3.$$  

Theorem 2.5 implies that

$$\chi(Y, \mathcal{O}_Y) = \frac{1}{24}(-K_Y \cdot c_2(Y) + e),$$

where

$$e = \sum_{p_{\alpha}} \left(r(p_{\alpha}) - \frac{1}{r(p_{\alpha})}\right),$$

where
and the sum runs over all the baskets $B(Y,p)$ of singularities of $Y$. Note that $e \leq \Xi(Y)$. Thus,
\[
\text{vol}(X, K_X) = K_Y^3 \leq 3K_Y \cdot c_2(Y) \\
= 3(-24\chi(Y, \mathcal{O}_Y) + e) \\
= 3(24(-h^{0,0}(X) + h^{1,0}(X) - h^{2,0}(X) + h^{3,0}(X)) + e) \\
\leq 3(12b_3(X) + \Xi(Y)),
\]
where the first inequality follows from Theorem 2.4 and the second inequality follows from the fact that $h^{1,0}(X) \leq h^{2,1}(X)$ by Hard Lefschetz and $h^{2,1}(X) + h^{3,0}(X) \leq b_3(X)/2$ by Hodge decomposition.

Thus, Proposition 2.3 implies the claim. \hfill \Box

Two immediate applications of Theorem 1.2 are the following corollaries.

**Corollary 4.1.** The volume only takes finitely many values on the set of three-dimensional projective varieties with a fixed underlying 6-manifold.

**Proof.** Let $X$ be a smooth projective threefold. The volume $\text{vol}(X, K_X)$ is a rational number whose denominator is bounded by the cube of the index of a minimal model of $X$. By Lemma 2.2, the index of any minimal model of $X$ is less than or equal to $4 \cdot \Xi(X)$. The claim follows now from Proposition 2.3 and Theorem 1.2. \hfill \Box

**Corollary 4.2.** The family of all smooth projective threefolds of general type with bounded Betti numbers is birationally bounded.

**Proof.** By [HM06, Cor. 1.2] we know that the family all smooth projective threefolds of general type with bounded volume is birationally bounded. The result then follows from Theorem 1.2. \hfill \Box

**4.2. Divisorial contractions.** Let $Y$ be a $\mathbb{Q}$-factorial projective threefold, and let $f: Y \to X$ be an elementary $K_Y$-negative birational contraction. By Lemma 2.16 we have that $b_2(Y) - b_2(X) = 1$. Let $\{\gamma_1, \ldots, \gamma_b\}$ be a basis of $\bar{H}^2(X, \mathbb{Z})$, and let $\beta_i = f^* \gamma_i$.

If $f$ is a divisorial contraction, then we have a natural choice for a class $\alpha \in \bar{H}^2(Y, \mathbb{Z})$ such that $\{\alpha, \beta_1, \ldots, \beta_b\}$ is a basis of $\bar{H}^2(Y, \mathbb{Q})$. Indeed, we can choose $\alpha = c_1(rE)$, where $E$ is the exceptional divisor and $r$ is the smallest positive integer such that $rE$ is Cartier.

If $f$ is a contraction to a point, by the projection formula we get
\[
\alpha \cdot \beta_i \cdot \beta_j = 0
\]
and
\[
\alpha^2 \cdot \beta_i = 0
\]
for any $i, j = 1, \ldots, b$. On the other hand, in general, we do not have an isomorphism $\bar{H}^2(X, \mathbb{Z}) = \mathbb{Z}\langle \alpha, \beta_1, \ldots, \beta_b \rangle$, as the following example shows.

**Example 4.3.** Let $Z = \mathbb{P}^2$, and consider the $\mathbb{P}^1$-bundle
\[
Y = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(2))
\]
over $Z$ with induced morphism $\pi: Y \to Z$. Then there exists a birational morphism $f: Y \to X$ which contracts a section $E$ of $\pi$ into a point. In particular, $X$ is the
cone over \( \mathbb{P}^2 \) associated to \( O_{\mathbb{Z}}(2) \). Note that \( X \) is terminal and \( \mathbb{Q} \)-factorial, and \( K_Y = f^* K_X + 1/2 E \). Let \( \ell \) be a line in \( Z \), and let \( F = \pi^* \ell \). Then \( \{ E, F \} \) is a basis of \( H^2(Y, \mathbb{Z}) \). On the other hand, \( F' = f_* F \) is not Cartier, and therefore it is not an element of \( H^2(X, \mathbb{Z}) \), while \( 2F' \) is a generator of \( H^2(X, \mathbb{Z}) \).

Given a divisorial contraction to a point \( f: Y \to X \) between terminal threefolds, our goal is to first bound the difference \( K_3^Y - K_3^X \) and then compute the cubic form \( F_X \) associated to \( X \) from the cubic form \( F_Y \) associated to \( Y \). We begin with the following.

**Proposition 4.4.** Let \( X_0 \) be a smooth projective threefold, and let
\[
X_0 \to X_1 \to \cdots \to X_{k-1} \to X_k
\]
be a sequence of steps of the minimal model program for \( X_0 \). Assume that \( f: Y = X_{k-1} \to X = X_k \) is a divisorial contraction to a point \( p \in X \). Then
\[
0 < K_3^Y - K_3^X \leq 2^{10} b_2^2,
\]
where \( b_2 = b_2(X_0) \) is the second Betti number of \( X_0 \).

**Proof.** Let \( E \) be the exceptional divisor of \( f \), and let \( a = a(E, X) \) be the discrepancy of \( f \) along \( E \). Since \( X \) is terminal, we have that \( a > 0 \). Since \( K_3^Y - K_3^X = a^3 E^3 \), it is enough to bound \( a^3 E^3 \). The possible values of \( a E^3 \) are listed in [Kaw05] Tables 1 and 2. In particular, we have
\[
0 < aE^3 \leq 4.
\]
Let \( B(X, p) = \{ p_1, \ldots, p_k \} \) be the basket of \( X \) at \( p \) with indices \( r_1 = r(p_1), \ldots, r_k = r(p_k) \) (cf. [2.2]), and let \( R \) be the least common multiple of \( r_1, \ldots, r_k \). Then, [Kaw05] Lemma 2.3, implies that \( E^3 \geq 1/R \). Thus,
\[
0 < (aE)^3 \leq 64 \left( \frac{E^3}{(E^3)^2} \right) \leq 64R^2.
\]
Let \( \Xi = \Xi(X, p) \leq \Xi(X) \). Then Lemma 2.2 implies that
\[
R \leq 4 \cdot \Xi,
\]
and Proposition 2.3 implies
\[
(aE)^3 \leq 2^{10} b_2^2.
\]
Thus, the claim follows. \( \square \)

We now study the behaviour of the cubic form associated to a terminal threefold under a divisorial contraction to a point. We begin with the following elementary fact.

**Lemma 4.5.** Let \( A \) be a maximal rank submodule of \( \mathbb{Z}^m \), and let \( r \) be a positive integer. Assume that for any \( b \in \mathbb{Z}^m \), we have that \( r \cdot b \in A \). Let \( T \in \mathcal{M}(m, \mathbb{Z}) \) be a matrix whose columns form a basis of \( A \). Then \( 0 < |\det T| \leq r^m \).

**Proof.** By assumption, there exists \( X \in \mathcal{M}(m, \mathbb{Z}) \) such that \( T \cdot X = r I_m \), where \( I_m \in \text{SL}(m, \mathbb{Z}) \) is the identity matrix. Thus, \( \det T \) divides \( r^m \), and the claim follows. \( \square \)
Lemma 4.6. Let \( X \) and \( Y \) be \( \mathbb{Q} \)-factorial projective threefolds with terminal singularities, and let \( f : Y \to X \) be a divisorial contraction onto a point \( x \in X \) with exceptional divisor \( E \). Then \( \pi_1(E) = 1 \) and, in particular, \( H^2(E, \mathbb{Z}) \) is torsion free.

Proof. Let \( U \) be an analytic neighborhood of \( x \) such that \( U \) retracts to \( x \), and consider the morphism \( f_U : V = f^{-1}(U) \to U \). Then [Kol93b] Theorem 7.8 implies that \( \pi_1(V) = \pi_1(U) = 1 \). Since \( V \) retracts to \( E \), it follows that \( \pi_1(E) = 1 \).

The universal coefficient theorem implies that \( H^2(E, \mathbb{Z}) \) is torsion free. \( \square \)

Thus, we have the following.

Proposition 4.7. Let \( X \) and \( Y \) be \( \mathbb{Q} \)-factorial projective threefolds with terminal singularities, and let \( f : Y \to X \) be a divisorial contraction onto a point with exceptional divisor \( E \). Let \( \alpha \in \check{H}^2(Y, \mathbb{Z}) \) be a generator of the ray \( \mathbb{R}_{>0}[E] \) in \( N^1(Y) \otimes \mathbb{R} \). Let \( n = b_2(Y) \), and let \( \alpha, \alpha_2, \ldots, \alpha_n \) be a basis of \( \check{H}^2(Y, \mathbb{Z}) \). Let \( r = |\alpha|^3 \). Then there exists \( T \in \mathcal{M}(n, \mathbb{Z}) \) such that \( 0 < |\det T| \leq r^n \), and \( \alpha, T(\alpha_2), \ldots, T(\alpha_n) \) is a basis of the submodule of \( \check{H}^2(Y, \mathbb{Z}) \) spanned by \( f^* \check{H}^2(X, \mathbb{Z}) \) and \( \alpha \). In particular, it follows that

\[
T \cdot F_Y = a x_0^3 + F_X(x_1, \ldots, x_n),
\]

where \( a = \alpha^3 \).

Proof. Fix an isomorphism \( \check{H}^2(Y, \mathbb{Z}) \cong \mathbb{Z}^n \), and consider the submodule \( A \) of \( \mathbb{Z}^n \) spanned by \( f^* \check{H}^2(X, \mathbb{Z}) \) and \( \alpha \). Let \( \beta \in \check{H}^2(Y, \mathbb{Z}) \). Then there exist integers \( c, b \) with \( |b| \leq r \) such that

\[
(c \alpha + b \beta) \cdot \alpha^2 = 0.
\]

Set \( \gamma = c \alpha + b \beta \). As in the proof of Lemma 2.10, it follows that \( R^1 f_* \mathcal{O} = 0 \) and, in particular, \( H^1(E, \mathcal{O}) = 0 \). Thus, as in Lemma 2.15 we get the exact sequence

\[
0 \to f^* \check{H}^2(X, \mathbb{Z}) \to \check{H}^2(Y, \mathbb{Z}) \xrightarrow{\rho} H^2(E, \mathbb{Z}).
\]

Possibly passing to a desingularization, we can apply [KM92] Proposition 12.1.6 to obtain that \( p(E) \) is a multiple of \( p(\gamma) \) in \( H^2(E, \mathbb{Q}) \). Since \( \gamma \cdot \alpha^2 = 0 \), it follows that \( p(\gamma) \) is a torsion element of \( H^2(E, \mathbb{Z}) \), which implies that \( p(\gamma) = 0 \) by Lemma 4.6 and so \( \gamma \in f^* \check{H}^2(X, \mathbb{Z}) \). Thus, \( b \beta \in A \) and Lemma 4.5 implies the claim. \( \square \)

We now consider divisorial contraction to a smooth curve. We begin with the following well-known result (e.g., see [OVdV95] Prop. 14).

Proposition 4.8. Let \( X \) be a \( \mathbb{Q} \)-factorial projective threefold, and let \( C \) be a smooth curve of genus \( g \) contained in the smooth locus of \( X \). Let \( f : Y \to X \) be the blowup of \( X \) along \( C \), and let \( \alpha = c_1(E) \). Then \( H^2(Y, \mathbb{Z}) \cong \mathbb{Z}[\alpha] \oplus H^2(X, \mathbb{Z}) \) and

\[
K_Y^3 - K_X^3 = -2K_X \cdot C + 2 - 2g = -2E^3 + 6 - 6g.
\]

In particular, if \( \beta_1, \ldots, \beta_n \) is a basis of \( H^2(X, \mathbb{Z}) \), then \( \alpha, f^* \beta_1, \ldots, f^* \beta_n \) is a basis of \( H^2(Y, \mathbb{Z}) \), and with respect to these bases we have

\[
F_Y(x_0, \ldots, x_n) = ax_0^3 + 3x_0^2 \left( \sum_{i=1}^n b_i x_i \right) + F_X(x_1, \ldots, x_n),
\]

where \( a = \alpha^3 \) and \( b_i = -\beta_i \cdot C \).
4.3. Proof of Theorem 1.3 We can finally prove our main result.

Proof of Theorem 1.3 If \( f \) is a divisorial contraction to a point, then (1) is the content of Proposition 4.4. Assume hence that \( f \) contracts a divisor \( E \) to a smooth curve \( C \). Then \( E \) is a \( \mathbb{P}^1 \)-bundle over \( C \), and, in particular, if \( g \) is the genus of \( C \), then \( b_3(E) = 2g \). Thus, by Lemma 2.15 and Lemma 2.16 and since \( E \) and \( C \) are smooth, we have that

\[
b_3(Y) - b_3(X) = I_3(Y) - I_3(X) = 2g.
\]

Moreover, considering the cubic form \( F_Y \) associated to \( Y \) and applying Proposition 4.8 we have that \( |E^3| \leq S_Y \). Hence

\[
|K_Y^3 - K_X^3| = |-2E^3 + 6 - 6g| \\
\leq 2S_Y + 6(b_3(Y) + 1).
\]

This finishes the proof of (1).

We now prove (2).

Let us first assume that \( f \) is a divisorial contraction to a point with exceptional divisor \( E \). Let \( \alpha \in H^2(Y, \mathbb{Z}) \) be a generator of the ray \( \mathbb{R}_{\geq 0}[E] \). By Proposition 4.7 \( \alpha \) is a point of rank 1 for the Hessian of the cubic form \( F_Y \). Then, by Proposition 3.3 \( \alpha \) is determined up to finite ambiguity by \( F_Y \) and so it is \( r = \alpha^3 \). By Proposition 3.22 there are finitely many pairs

\[
(a_i, G_i) \in \mathbb{Z} \times \mathbb{Z}[x_1, \ldots, x_n]_3, \quad i = 1, \ldots, k,
\]

such that for all \( T \in \mathcal{M}(n+1, \mathbb{Z}) \) such that \( 0 < \det T \leq r^n \), we have that \( T \cdot F \) is equivalent to \( (a_i, 0, G_i) \) over \( \mathbb{Z} \) for some \( i \in \{1, \ldots, k\} \).

By Proposition 4.7 there exists \( T \in \mathcal{M}(n+1, \mathbb{Z}) \) such that \( 0 < \det T \leq r^n \) and \( T \cdot F \) is in reduced form \( (a, 0, F_Y) \), where \( a = \alpha^3 \). Thus, there exists \( M \in \text{SL}(n, \mathbb{Z}) \) such that \( F_Y = M \cdot G_i \) for some \( i \in \{1, \ldots, k\} \).

Let us assume now that \( f \) is a divisorial contraction to a smooth curve. By Theorem 3.1 there exist finitely many triples

\[
(a_i, B_i, G_i) \in \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}[x_0, \ldots, x_n]_3, \quad i = 1, \ldots, k,
\]

such that any reduced triple associated to \( F \) is equivalent to \( (a_i, B_i, G_i) \) over \( \mathbb{Z} \) for some \( i \in \{1, \ldots, k\} \). By Proposition 4.8 there exist \( a \in \mathbb{Z} \) and \( B \in \mathbb{Z}^n \) such that \( (a, B, F_Y) \) is a reduced triple associated to \( F \). Thus, there exists \( M \in \text{SL}(n, \mathbb{Z}) \) such that \( F_Y = M \cdot G_i \) for some \( i \in \{1, \ldots, k\} \).

Proof of Corollary 1.4 This is a simple iteration of point (2) of Theorem 1.3 keeping in mind that if \( g : W \to Z \) is a step of an MMP as in the statement and \( \Delta_{F_W} \neq 0 \), then also \( \Delta_{F_Z} \neq 0 \) (this follows by combining Proposition 4.8 and Proposition 4.7 with Lemma 2.6).

Proof of Corollary 1.5 Let

\[
X = X_0 \to X_1 \to \cdots \to X_k
\]

be an MMP for \( X \) such that each \( f_i : X_i \to X_{i+1} \) is a divisorial contraction to a point or to a smooth curve contained in the smooth locus of \( X_{i+1} \).

Denote by \( F_i \) the cubic form associated to \( X_i \), and let \( S_i = S_{X_i} \) (cf. Definition 2.12). Theorem 3.1 implies that \( S_{X_0} < +\infty \).
We proceed by induction on $i = 0, \ldots, k$. Proceeding as in the proof of Theorem 1.3 by combining Proposition 4.8, Proposition 4.7, Proposition 3.22, and Theorem 3.1, it follows that, for any $i = 0, \ldots, k$,

$$\Delta F_i \neq 0 \quad \text{and} \quad S_i < +\infty.$$

Moreover, each $S_i$ depends only on $F_X$ and, therefore, only on the topology of the manifold underlying $X$.

We define

$$D_k = 6b_2(X) + 36b_3(X),$$

and for any $i < k$, let

$$D_i = D_{i+1} + \max\{210b_2(X)^2, 2S_i + 6(Ib_3(X_i) + 1)\}.$$

We claim that

$$|K^3_X| \leq D_i$$

for any $i = 0, \ldots, k$.

The proof is by descending induction on $i = k, \ldots, 0$. If $i = k$, the result is exactly Theorem 1.2. Assume now that $i < k$ and $|K^3_{X,i+1}| \leq D_{i+1}$. Then the claim follows by combining Proposition 4.4 and Theorem 1.3. In particular, we have that $|K^3_X| \leq D_0$.

Finally, we need to show that for any $i = 1, \ldots, k$, we have that $Ib_3(X_i) \leq Ib_3(X_{i-1})$. If $f_{i-1}: X_{i-1} \rightarrow X_i$ is a divisorial contraction to a point, then the claim follows immediately from Lemma 2.15. On the other hand, if $f_{i-1}: X_{i-1} \rightarrow X_i$ is a divisorial contraction to a smooth curve $C_i \subseteq X_i$ with exceptional divisor $E_i$, then $E_i$ is a $\mathbb{P}^1$-bundle over $C_i$ and if $g(C_i)$ is the genus of $C_i$, then Lemma 2.15 implies

$$Ib_3(X_{i-1}) - Ib_3(X_i) = Ib_3(E_i) = b_3(E_i) = 2g(C_i) \geq 0,$$

as claimed. Thus, $Ib_3(X_i) \leq Ib_3(X) = b_3(X)$ for any $i = 1, \ldots, k$, and the theorem follows.

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