TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 11, November 2018, Pages 7743–7759
http://dx.doi.org/10.1090/tran/7221
Article electronically published on May 3, 2018

THE ROPER-SUFFRIDGE EXTENSION OPERATOR AND ITS APPLICATIONS TO CONVEX MAPPINGS IN $\mathbb{C}^2$

JIANFEI WANG AND TAISHUN LIU

Dedicated to the memory of Professor Sheng Gong

Abstract. The purpose of this paper is twofold. The first is to investigate the Roper-Suffridge extension operator which maps a biholomorhic function f on D to a biholomorphic mapping F on

$$\Omega_{n,p_2,\cdots,p_n}(D)=\left\{(z_1,z_0)\in D\times\mathbb{C}^{n-1}:\sum_{j=2}^{n}|z_j|^{p_j}<\frac{1}{\lambda_D(z_1)}\right\},\ \ p_j\geq 1,$$

where $z_0=(z_2,\dots,z_n)$ and λ_D is the density of the Poincaré metric on a simply connected domain $D\subset\mathbb{C}$. We prove this Roper-Suffridge extension operator preserves ε-starlike mapping: if f is ε-starlike, then so is F. As a consequence, we solve a problem of Graham and Kohr in a new method. By introducing the scaling method, the second part is to construct some new convex mappings of domain $\Omega_{2,\,m}=\{(z_1,z_2)\in\mathbb{C}^2:|z_1|^2+|z_2|^m<1\}$ with $m\geq 2$, which can be applied to discuss the extremal point of convex mappings on the domain. This scaling idea can be viewed as providing an alternative approach to studying convex mappings on $\Omega_{2,\,m}$. Moreover, we propose some problems.

Contents

Received by the editors August 9, 2016, and, in revised form, December 18, 2016, and February 5, 2017.

2010 *Mathematics Subject Classification.* Primary 32H02; Secondary 30C55.

The first author's research was supported by the National Natural Science Foundation of China (Nos. 11671362 and 11001246) and the Natural Science Foundation of Zhejiang Province (No. LY16A010004). The first author is the corresponding author.

The second author's research was supported by the National Natural Science Foundation of China (Nos. 11471111 and 11571105).

1. Introduction

In a pioneering work, Roper and Suffridge [25] introduced an extension operator. This operator is defined for a locally biholomorphic function f on the unit disk U by

$$\Phi_n(f)(z) = (f(z_1), \sqrt{f'(z_1)}z_0),$$

where $z = (z_1, z_0)$ belongs to the unit ball B_n in $\mathbb{C}^n$, $z_0 = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}$, and the branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

It is well known that the Roper-Suffridge extension operator has the following remarkable properties:

(i) If f is a normalized convex function on U, then $\Phi_n(f)$ is a normalized convex mapping on B_n.

(ii) If f is a normalized starlike function on U, then $\Phi_n(f)$ is a normalized starlike mapping on B_n.

(iii) If f is a normalized Bloch function on U, then $\Phi_n(f)$ is a normalized Bloch mapping on B_n.

Roper and Suffridge proved the result (i). In 2000, Graham and Kohr in [8] gave a simplified proof of their theorem and proved the results (ii) and (iii). Further, they proposed the following Problem [8, Open Problem 2.8].

Consider the “egg” domain

$$\Omega_{2,\,p} = \{(z_1, z_2) \in \mathbb{C}^2 :\ |z_1|^2 + |z_2|^p < 1\},$$

where $p \geq 1$. Does the operator

$$\Phi_{2,\,\frac{1}{p}}(f)(z) = \big(f(z_1), (f'(z_1))^{1/p} z_2\big)$$

extend convex functions on U to convex mappings on $\Omega_{2,\,p}$?

In 2002, Gong and the second author [9] introduced the definition of ε-starlike mapping, which is a unification of convex and starlike mapping. By using the non-increasing property of Carathéodory metric under holomorphic mappings, they proved the following.

Theorem 1.1. *If $f(z_1)$ is a normalized biholomorphic ε-starlike function on the unit disk U, then*

$$\Phi_{n,\,\frac{1}{p}}(f)(z) = \big(f(z_1), (f'(z_1))^{1/p} z_0\big)$$

is a normalized biholomorphic ε-starlike mapping on

$$\Omega_{n,\,p} = \{(z_1, z_0) \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n} |z_j|^p < 1\},$$

where $z = (z_1, z_0) \in \Omega_{n,p}$, $z_0 = (z_2, \cdots, z_n) \in \mathbb{C}^{n-1}, p \geq 1$, and the branch is chosen so that $(f'(z_1))^{1/p}\,|_{z_1=0} = 1$.

When $\varepsilon = 1$ and $\varepsilon = 0$, $\Phi_{n,\,\frac{1}{p}}$ maps convex function and starlike function on U to convex mapping and starlike mapping on $\Omega_{n,\,p}$, respectively. Further, Gong and Lin answered the above problem of Graham and Kohr. There has been an increase of research interest in this extension operator; the reader is referred to Gong [10], Graham and Kohr [7] for a general presentation of this subject. According to the Roper-Suffridge extension operator, we may construct lots of concrete examples of convex and starlike mappings. This is one important reason why people are interested in this extension operator.

It is well-known that the Riemann Mapping Theorem is one of the most remarkable results in complex analysis. Namely, it states that a proper domain D of $\mathbb{C}$ is biholomorphically equivalent to the unit disk U if and only if D is a simply connected proper subdomain of $\mathbb{C}$. Hence, it is natural to consider the following question.

What is the behavior of the Roper-Suffridge extension operator for some domain $\Omega(D)$ of $\mathbb{C}^n$ generated by a given simply connected proper subdomain $D \subset \mathbb{C}$?

The purpose of this paper is to consider the above question and its applications to convex mappings of several complex varialbes. When D is the unit disk U, we then give new convex construction for a modification of the Roper-Suffridge extension operator via the scaling technique on domain $\Omega_{2,m}$. This scaling idea seems entirely new to investigate convex mappings in higher dimensions.

The paper is organized as follows. In Section 2, we introduce some definitions and two Lemmas. In Section 3, we will prove the Roper-Suffridge extension operator preserves ε-starlike mapping. As a consequence, we answer the problem of Graham and Kohr in a quite direct and new method. In Section 4, we will give two applications for the Roper-Suffridge extension operator. The first is to establish the lower bound of distortion theorem of convex mappings associated with the Roper-Suffridge extension operator in Theorem 4.1. The second is to construct some new convex mappings on the Thullen domain $\Omega_{2,m}$ in Theorem 4.4. When $m = 2$, this result reduces to [21, Lemma 2.1] and [19, Theorem 3.1]. However, when $m > 2$, there appears a serious difficulty because we cannot proceed in analogy with Muir and Suffridge's idea of the ball. Our way to overcome this obstacle is to introduce the scaling method, which applies to the bounded convex domain $\Omega_{2,m}$ produces a biholomorphism mapping $\Phi : H_{2,m} \to \Omega_{2,m}$, where $H_{2,m} = \{(z, w) \in \mathbb{C}^2 : \Re z > |w|^m\}$ is unbounded. By characterizing exact convexity mappings of $H_{2,m}$ in Theorem 4.3, we obtain some new convex examples of $\Omega_{2,m}$ which play an important role to study the extremal points. Interestingly, Theorem 4.4 can be used to prove convex mapping generated by the Roper-Suffridge operator is not the extremal point of convex mappings, which is quite different from the case in one complex variable. For the best of our knowledge, there seems to be no other convex examples on $\Omega_{2,m}$ except for the Roper-Suffridge construction when $m > 2$. In Section 5, we consider some problems of convex mappings related to the Roper-Suffridge extension operator.

2. Preliminaries

2.1. **Simply connected domain and Poincaré metric.** Let D be a simply connected domain in the complex plane $\mathbb{C}$ with at least two boundary points, and let f be a conformal mapping of the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ onto D. The Poincaré or hyperbolic metric of D is defined by

$$\lambda_D(f(z))|f'(z)| = \lambda_U(z) = \frac{1}{1 - |z|^2}, \qquad z \in U. \tag{2.1}$$

This metric is independent of the choice of conformal mapping. Hence, convenient choices are available for us in this paper. Namely, let $z \in D$ and choose the conformal mapping f obeying $f(0) = z$ and $f'(0) > 0$. Then

$$\lambda_D(z) = \frac{1}{f'(0)}. \tag{2.2}$$

The function $\lambda_D(z)$ is real analytic on D, and the metric $\lambda_D(z)|d(z)|$ has constant Gaussian curvature -4, i.e., satisfying the equation

$$\Delta \log \lambda_D = 4\lambda_D^2,$$

while it is sometimes preferable to use $2\lambda_D$ with curvature -1; see [6].

At this point we will use the following conformal invariance about the Poincaré metric. Namely, if f is a conformal mapping of a domain D onto G, then

$$\lambda_G(f(z))|f'(z)| = \lambda_D(z), \qquad z \in D. \tag{2.3}$$

This follows easily from (2.1) and (2.2).

Accordingly, we give examples of simply connected domains and their Poincaré metrics. The reader is referred to Beardon and Minda [2] for a good overview of this subject.

Example 1. Note that $f(z) = (1+z)/(1-z)$ is a conformal map of the unit disk U onto $H = \{z \in \mathbb{C} : \Re z > 0\}$, so we have

$$\lambda_H(z)|dz| = \frac{|dz|}{2\Re z}.$$

Example 2. As $f(z) = z/(1-z)$ is a conformal map of the unit disk U onto the open half-plane $K = \{z \in \mathbb{C} : \Re z > -1/2\}$, we have found that

$$\lambda_K(z)|dz| = \frac{|dz|}{1+2\Re z}.$$

Example 3. Because $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ is a conformal map of the unit disk U onto the strip $S = \{z \in \mathbb{C} : |\Im z| < \pi/4\}$, we get

$$\lambda_S(z)|dz| = \frac{|dz|}{\cos 2\Im z}.$$

2.2. **Some definitions.** Let us make the following definitions living on $\mathbb{C}^n$:

- Let $B_n = \Big\{z \in \mathbb{C}^n : |z| < 1\Big\}$ and $S_n = \Big\{z \in \mathbb{C}^n : |z| = 1\Big\}$ represent the unit ball and the unit sphere of $\mathbb{C}^n$ under the inner product

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k\overline{w_k}, \quad z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n, \;\; w = (w_1, w_2, \cdots, w_n) \in \mathbb{C}^n$$

  and norm $|z| = \langle z, z\rangle^{\frac{1}{2}}$. In the case of one variable, B_1 and S_1 are always denoted by U and T, respectively.
- A domain $\Omega \subset \mathbb{C}^n$ is said to be starlike (with respect to the origin) if given any $z \in \Omega$, then $(1-t)z \in \Omega$ holds for all $t \in [0, 1]$.
- A domain $\Omega \subset \mathbb{C}^n$ is said to be convex if given any z_1, $z_2 \in \Omega$, then $(1-t)z_1 + tz_2 \in \Omega$ holds for all $t \in [0, 1]$.
- A domain $\Omega \subset \mathbb{C}^n$ containing the origin is said to be ε-starlike if there exists a positive number ε, $0 \le \varepsilon \le 1$, such that given any z_1, $z_2 \in \Omega$, then $(1-t)z_1 + \varepsilon t z_2 \in \Omega$ holds for all $t \in [0, 1]$. Obviously, ε-starlike domain reduces to convex and starlike when $\varepsilon = 1$ and $\varepsilon = 0$, respectively.

- Let $\Omega \subset \mathbb{C}^n$ be a domain containing the origin. A holomorphic mapping $f : \Omega \to \mathbb{C}^n$ is said to be normalized if $f(0) = 0$ and $Df(0) = I_n$, where I_n is the identity matrix. Let $\|Df(z)\|$ denote the norm of the complex Jacobian matrix of f at the point $z \in \Omega$.
- Let $\Omega \subset \mathbb{C}^n$ be a domain and let f be a biholomorphic mapping from Ω into $\mathbb{C}^n$. If $f(\Omega)$ is an ε-starlike domain, then f is called an ε-starlike mapping.
- Suppose that $G \subset \mathbb{C}$ is a domain including the origin and f and g are two holomorphic functions on G. If there is a holomorphic function $\varphi : G \to G$ such that $\varphi(0) = 0$ and $f = g \circ \varphi$, then f is subordinate to g and is denoted by $f \prec g$ on G.
- The Minkowski functional $\rho(z)$ of the Reinhardt domain

$$\Omega_{n,p_2,\cdots,p_n} = \left\{ z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n} |z_j|^{p_j} < 1 \right\}, \qquad p_j \geq 1,\ j = 2, \cdots, n,$$

is defined as

$$\rho(z) = \inf\{t > 0, \frac{z}{t} \in \Omega_{n,p_2,\cdots,p_n}\}, \qquad z \in \mathbb{C}^n.$$

Also, the Minkowski functional $\rho(z)$ is a Banach norm of $\mathbb{C}^n$, and $\Omega_{n,p_2,\cdots,p_n}$ becomes the unit ball in the Banach space $\mathbb{C}^n$ with respect to this norm. $\rho(z)$ is C^1 on $\bar{\Omega}_{n,p_2,\cdots,p_n}$ except for a lower dimensional manifold. When $z \in \Omega_{n,p_2,\cdots,p_n}$, $\rho(z) \geq \max\{|z_1|, |z_2|, ..., |z_n|\}$ follows from the definition of $\rho(z)$.

2.3. **Two lemmas.** To prove Theorem 3.1, we need the following lemma, which is even interesting in complex analysis and geometric function theory about the Poincaré metric.

Lemma 2.1. *Let $D \subset \mathbb{C}$ contain the origin. If D is an ε-starlike domain and D is not the entire complex plane $\mathbb{C}$, then given z_1, $z_2 \in D$,*

$$\frac{1}{\lambda_D((1-t)z_1 + \varepsilon t z_2)} \geq \frac{1-t}{\lambda_D(z_1)} + \frac{\varepsilon t}{\lambda_D(z_2)}$$

holds for all $t \in [0, 1]$.

Proof. By the Riemann Mapping Theorem, there exist two conformal mappings $f_k : U \to D$ so that $f_k(U) = D$, $f_k(0) = z_k$, and $f_k'(0) > 0$, where $k = 1, 2$. For $t \in [0, 1]$, let $z_t = (1-t)z_1 + \varepsilon t z_2$. The condition D, an ε-starlike domain, yields that $z_t \in D$. Similarly, let g be a conformal mapping of U onto D so that $g(0) = z_t$ and $g'(0) > 0$.

Since $(1-t)f_1 + \varepsilon t f_2$ is holomorphic from U into D, we then see that

$$(1-t)f_1 + \varepsilon t f_2 \prec g.$$

Hence,

$$|(1-t)f_1'(0) + \varepsilon t f_2'(0)| \leq |g'(0)|,$$

that is,

$$(1-t)f_1'(0) + \varepsilon t f_2'(0) \leq g'(0).$$

From (2.2), we have

$$\frac{1-t}{\lambda_D(z_1)} + \frac{\varepsilon t}{\lambda_D(z_2)} \leq \frac{1}{\lambda_D((1-t)z_1 + \varepsilon t z_2)},$$

which thereby is proved. □

Remark 1. When $\varepsilon = 1$, the domain D is convex. In this case, Lemma 2.1 is proved by Gustafsson [11] using a coefficient inequality for convex univalent functions and was later found by Kim and Minda [12] in a simplified proof.

The following lemma, which can be found in [15], plays an important role in studying the Bloch constant of several complex variables; see e.g., [5]. For the completeness of this note, we provide a self-contained proof.

Lemma 2.2. *Suppose that $A = (a_{ij})$ is an $n \times n$ complex matrix. If $\|A\| > 0$, then for any unit vector $\xi \in S_n$, the following inequality holds:*

$$|A\xi| \geq \frac{|\det A|}{||A||^{n-1}}.$$

Proof. We need only consider the case $|\det A| \neq 0$. There are two $n \times n$ unitary matrixes P and Q for which A has the polar decomposition

$$A = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} Q = P\Lambda Q,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and $\|A\| = \lambda_1$. Then

$$|\det A| = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Given any $\xi \in S_n$, since P and Q are unitary matrixes, we have $\eta = Q\xi \in S_n$ and

$$|A\xi| = |P^{-1}AQ^{-1}Q\xi| = |\Lambda\eta| \geq \lambda_n = \frac{|\det A|}{\lambda_1 \cdots \lambda_{n-1}} \geq \frac{|\det A|}{||A||^{n-1}}.$$

□

3. Roper-Suffridge extension operator and ε-starlike mapping

In this section we will generalize [9, Theorem 1] from the unit disk to an arbitrary simply connected domain D in $\mathbb{C}$ containing 0. Moreover, our proof is different. In particular, when $D = U$ and $n = 2$, we answer the problem of Graham and Kohr in an alternative proof.

Theorem 3.1. *Assume $D \subset \mathbb{C}$ is a simply connected domain containing 0 and D is not the entire complex plane. If $f(z_1)$ is a biholomorphic ε-starlike function on D, then*

$$\begin{aligned}\Phi_{n,\frac{1}{p_2},\cdots,\frac{1}{p_n}}(f)(z) &= F_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z) \\ &= \big(f(z_1), (f'(z_1))^{1/p_2} z_2, \cdots, (f'(z_1))^{1/p_n} z_n\big)\ p_j \geq 1\end{aligned}$$

is a biholomorphic ε-starlike mapping on domain

$$\Omega_{n,p_2,\cdots,p_n}(D) = \{(z_1, z_0) \in D \times \mathbb{C}^{n-1} : \sum_{j=2}^{n} |z_j|^{p_j} < \frac{1}{\lambda_D(z_1)}\},$$

where $\lambda_D(z_1)$ is the density of the Poincaré metric of D at z_1, $z_0 = (z_2, \cdots, z_n)$ and the branch is chosen so that $(f'(z_1))^{1/p_j}\ |_{z_1=0} = 1$ for $j = 2, \cdots, n$.

Proof. Let

$$(u_1, u_2, \cdots, u_n) = \big(f(z_1), \big(f'(z_1)\big)^{1/p_2} z_2, \cdots, (f'(z_n))^{1/p_n} z_n\big).$$

Under this, for $j = 2, \cdots, n$, we have

$$\begin{cases} u_1 = f(z_1), \\ u_j = (f'(z_1))^{1/p_j} z_j. \end{cases}$$

This implies the relation

$$\begin{cases} z_1 = f^{-1}(u_1), \\ z_j = \frac{u_j}{(f'[f^{-1}(u_1)])^{1/p_j}}. \end{cases} \tag{3.1}$$

A short computation shows that the image of the mapping $F_{\frac{1}{p_2}, \cdots, \frac{1}{p_n}}$ is the set

$$G_{n,p_2,\cdots,p_n}(f, D) = \left\{ (u_1, u_2, \dots, u_n) \in \mathbb{C}^n : \sum_{j=2}^{n} \frac{|u_j|^{p_j}}{|f'[f^{-1}(u_1)]|} < \frac{1}{\lambda_D(f^{-1}(u_1))} \right\}, \tag{3.2}$$

which follows from the above relation (3.1) and the definition of

$$\Omega_{n,p_2,\cdots,p_n}(D) = \{(z_1, z_0) \in D \times \mathbb{C}^{n-1} : \sum_{j=2}^{n} |z_j|^{p_j} < \frac{1}{\lambda_D(z_1)}\}.$$

Hence $F_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}$ is an ε-starlike mapping on $\Omega_{n,p_2,\cdots,p_n}$ if and only if $G_{n,p_2,\cdots,p_n}(f, D)$ is an ε-starlike domain in $\mathbb{C}^n$. Namely, we need to prove that if given any $t \in [0, 1]$, $u = (u_1, \cdots, u_n) \in G_{n,p_2,\cdots,p_n}(f, D)$, and $v = (v_1, \cdots, v_n) \in G_{n,p_2,\cdots,p_n}(f, D)$, then $(1 - t)u + \varepsilon tv \in G_{n,p_2,\cdots,p_n}(f, D)$.

In fact, let $\tilde{D} = f(D)$. Then $\tilde{D}$ is an ε-starlike domain of $\mathbb{C}$ as f is ε-starlike. In view of (2.3), we have

$$\lambda_{\tilde{D}}(f(z_1))|f'(z_1)| = \lambda_D(z_1), \ z_1 \in D.$$

This yields that

$$|f'(z_1)| = |f'[f^{-1}(u_1)]| = \frac{\lambda_D(f^{-1}(u_1))}{\lambda_{\tilde{D}}(u_1)}. \tag{3.3}$$

A combination of (3.2) and (3.3) gives that we must prove that

$$G_{n,p_2,\cdots,p_n}(f, D) = \left\{ (u_1, u_2, \cdots, u_n) \in \mathbb{C}^n : \sum_{j=2}^{n} |u_j|^{p_j} - \frac{1}{\lambda_{\tilde{D}}(u_1)} < 0 \right\}$$

is an ε-starlike domain of $\mathbb{C}^n$.

By using Lemma 2.1, we have

$$-\frac{1}{\lambda_{\tilde{D}}((1-t)u_1 + \varepsilon tv_1)} \le -\frac{1-t}{\lambda_{\tilde{D}}(u_1)} - \frac{\varepsilon t}{\lambda_{\tilde{D}}(v_1)}. \tag{3.4}$$

Because $p_j \geq 1$ we have x^{p_j} is a real convex function on $x \in [0, \infty)$. Hence, for $j = 2, \cdots, n$, we have

$$\begin{aligned} &|(1-t)u_j + \varepsilon t v_j|^{p_j} \\ &\leq ((1-t)|u_j| + \varepsilon t|v_j|)^{p_j} \\ &\leq (1-t)|u_j|^{p_j} + t|\varepsilon v_j|^{p_j} \\ &\leq (1-t)|u_j|^{p_j} + \varepsilon^{p_j} t|v_j|^{p_j} \\ &\leq (1-t)|u_j|^{p_j} + \varepsilon t|v_j|^{p_j}. \end{aligned}$$

This means that

$$\sum_{j=2}^{n} |(1-t)u_j + \varepsilon t v_j|^{p_j} \leq \sum_{j=2}^{n} (1-t)|u_j|^{p_j} + \varepsilon t \sum_{j=2}^{n} |v_j|^{p_j}. \tag{3.5}$$

According to (3.4) and (3.5), we obtain $(1-t)u + \varepsilon t v \in G_{n,p_2,\cdots,p_n}(f, D)$. Hence, this completes the proof of Theorem 3.1. □

Remark 2.

(i) When $D = U$ and $p_j = p\,(j = 2, \cdots, n)$, Theorem 3.1 reduces to Theorem 1.1 of Gong and the second author [9]. Although results of the Roper-Suffridge operator are stated for the normalized convex (or starlike) univalent function in the unit disk U, they are also valid for any convex (or starlike) univalent function without the normalized condition, respectively.

(ii) To explain Theorem 3.1 clearly, we check that the mapping

$$F(z) = \Big(\frac{z}{1-z}, \frac{w}{1-z}\Big), \qquad z \in U,$$

is a convex mapping on B_2 in some words. Indeed, let

$$\begin{cases} u = \frac{z}{1-z}, \\ v = \frac{w}{1-z} \end{cases}$$

and let G be the image domain of F. Because the image domain of convex function $z/(1-z)$ is the set $\{u \in \mathbb{C} : \Re u > -1/2\}$, by using Example 2, we then obtain that the image domain G is equal to

$$G = \{(u, v) \in \mathbb{C}^2 :\ 1 + 2\Re u > |v|^2\},$$

which is obviously a convex domain in $\mathbb{C}^2$. Similarly,

$$F(z) = \Big(\frac{1}{2}\log\frac{1+z_1}{1-z_1}, \frac{z_2}{\sqrt{1-z_1^2}}, \cdots, \frac{z_n}{\sqrt{1-z_1^2}}\Big)$$

is also convex in the unit ball B_n via Example 3, which is a simple explanation of Roper-Suffridge convex mapping examples; see [25, pp. 334-335].

Also, applying the idea of Theorem 3.1 to the right-plane $D = H = \{z \in \mathbb{C} : \Re z > 0\}$ and using the invariance of affine transformation of convex mappings, we obtain the following corollary. To the best of our knowledge, there is no related result about biholomorphic convex mappings on a generalized half-space of $\mathbb{C}^n$, which is obviously unbounded and is not circular. In this case, we do not need the normalization condition.

Corollary 3.2. *Let $F_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)$ be defined as above. If $f(z_1)$ is a biholomorphic convex function on $H=\{z_1\in\mathbb{C}:\Re z_1>0\}$, then $F_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)$ is a biholomorphic convex mapping on domain*

$$H_{n,p_2,\cdots,p_n}=\{(z_1,z_0)\in H\times\mathbb{C}^{n-1}:\Re z_1>\sum_{j=2}^{n}|z_j|^{p_j}\}.$$

4. Two applications

In this section, we give two applications of the Roper-Suffridge extension operator in the geometric function theory of several complex variables. Let us begin with the distortion theorem of convex mappings.

4.1. **Distortion theorem on convex mappings.** It is well known that, in the case of several complex variables, there are many counterexamples to show that the distortion theorem of normalized biholomorphic mappings does not hold unless we restrict to certain subclasses of biholomorphic mappings; for instance see [10]. In 1933, H. Cartan [4] first suggested the study of convex mappings, starlike mappings, and some other subclasses of biholomorphic mappings in several complex variables. In 1994, the first affirmative result on the estimate of distortion theorem for biholomorphic convex mappings was due to Barnard, FitzGerald, and Gong [1]. Later, many authors made progress on the distortion theorem of convex mappings; see e.g. [18, 24].

In this section, we will apply Theorem 3.1 and Lemma 2.2 to give the lower bound distortion theorem of convex mappings associated with the Roper-Suffridge operator. For simplicity, let $\mathcal{K}^n$ and $\mathcal{S}^n$ respectively represent the class of normalized convex mappings and starlike mappings (with respect to the origin) defined on the unit ball B_n.

For $f\in\mathcal{K}^1$, there holds the well-known distortion theorem

$$\frac{1}{(1+|z|)^2}\le|f'(z)|\le\frac{1}{(1-|z|)^2},\qquad z\in U,\tag{4.1}$$

with equality for the univalent convex function $K(z)=z/(1-e^{i\theta}z)$, $0\le\theta<2\pi$.

For $F\in\mathcal{K}^n$, Pfaltzgraff and Suffridge [24] proved the distortion theorem

$$\frac{1}{(1+|z|)^2}\le||DF(z)||\le\frac{1}{(1-|z|)^2},\qquad z\in B_n,\tag{4.2}$$

where $||DF(z)||=\sup\{|DF(z)\xi|:\xi\in B_n\}$.

Also, they showed that the upper bound of the inequality (4.2) is sharp and equality is obtained by $k\in\mathcal{K}^n$ defined as follows:

$$k(z)=z(1-z_1)^{-1},\qquad z=(z_1,\cdots,z_n)'\in B_n.$$

However, Liczberski and Starkov [13] in 2002 observed the lower bound $(1+|z|)^{-2}$ of the inequality (4.2) is not sharp when $n\ge 2$. They conjectured that the sharp lower bound may be $(1+|z|)^{-1}$, which is still an open problem. However, for the convex mapping generated by the Roper-Suffridge operator, Liczberski and Starkov proved that the sharp lower bound is $(1+|z|)^{-1}$ for z close to zero. In [14], Liczberski and Starkov proved that the sharp lower bound $(1+|z|)^{-1}$ holds in whole ball B_n. In the following, we give the lower bound of the distortion theorem of convex mapping associated with the Roper-Suffridge operator on domain $\Omega_{n,p_2,\cdots,p_n}$. Moreover, our proof is different from the unit ball before.

Theorem 4.1. *If $f(z_1)$ is a normalized biholomorphic convex function in the unit disk $U\subset\mathbb{C}$ and $F(z)$ is defined by*

$$F_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)=\big(f(z_1),(f'(z_1))^{1/p_2}z_2,\cdots,(f'(z_1))^{1/p_n}z_n\big),\qquad p_j\geq 1,$$

then

$$||DF(z)||\geq\frac{1}{(1+\rho(z))^{\frac{2}{n-1}\sum\limits_{j=2}^{n}\frac{1}{p_j}}}$$

for all $z\in\Omega_{n,p_2,\cdots,p_n}$, where $\rho(z)$ is the Minkowski functional on $\Omega_{n,p_2,\cdots,p_n}$.

Proof. Theorem 3.1 implies that

$$F_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)=\big(f(z_1),(f'(z_1))^{1/p_2}z_2,\cdots,(f'(z_1))^{1/p_n}z_n\big)$$

is a normalized biholomorphic convex mapping on $\Omega_{n,p_2,\cdots,p_n}$. It is easy to see that

$$DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)=\begin{pmatrix} f'(z_1) & 0 & \cdots & 0\\ \frac{f''(z_1)}{p_2(f'(z_1))^{1-\frac{1}{p_2}}}z_2 & (f'(z_1))^{1/p_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{f''(z_1)}{p_n(f'(z_1))^{1-\frac{1}{p_n}}}z_n & 0 & \cdots & (f'(z_1))^{1/p_n}\end{pmatrix}.$$

Obviously, $f'(z_1)$ is an eigenvalue of $DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)$ and

$$|\det DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)|=|f'(z_1)|^{1+\sum\limits_{j=2}^{n}\frac{1}{p_j}}.$$

Upon taking a unit eigenvector $e_1=(1,0,\cdots,0)'$, we then get

$$DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)e_1=f'(z_1).$$

In view of Lemma 2.2, we have
(4.3)

$$|f'(z_1)|=|DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)e_1|\geq\frac{|\det DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)|}{||DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)||^{n-1}}=\frac{|f'(z_1)|^{1+\sum\limits_{j=2}^{n}\frac{1}{p_j}}}{||DF_{\frac{2}{p_2},\cdots,\frac{1}{p_n}}(z)||^{n-1}}.$$

Note that $\rho(z)\geq|z_1|$, and putting (4.1) and (4.3) together, we then get

$$||DF_{\frac{1}{p_2},\cdots,\frac{1}{p_n}}(z)||\geq|f'(z_1)|^{\frac{1}{n-1}\sum\limits_{j=2}^{n}\frac{1}{p_j}}\geq\frac{1}{(1+|z_1|)^{\frac{2}{n-1}\sum\limits_{j=2}^{n}\frac{1}{p_j}}}\geq\frac{1}{(1+\rho(z))^{\frac{2}{n-1}\sum\limits_{j=2}^{n}\frac{1}{p_j}}}.$$

$\square$

In the case of the unit ball B_n, we have the following corollary.

Corollary 4.2. *Let $f : B_n \to \mathbb{C}^n$ be a normalized biholomorphic convex mapping and*

$$F(z) = (f(z_1), \sqrt{f'(z_1)}z_0).$$

Then

$$||DF(z)|| \geq \frac{1}{1+|z|}$$

for all $z \in B_n$.

4.2. **New convex mappings on the Thullen domain.** In 1999, Roper and Suffridge [26, Example 7] proved that $F(z, w) = (z + aw^2, w)$ is a convex mapping on B_2 if and only if $|a| \leq 1/2$. Using this result and the compact of the convex mappings, Muir and Suffridge [21, Corollary 2.2] proved that the Roper-Suffridge convex mapping

$$F(z, w) = (\frac{z}{1-z}, \frac{w}{1-z})$$

is not the extremal point of convex mappings class on the unit ball B_2. Hence, for the general domain $\Omega_{2,\,m}$, it is very natural to consider the following problem:

What is the relation between the Roper-Suffridge convex mapping $F(z, w) = (\frac{z}{1-z}, \frac{w}{(1-z)^{2/m}})$ and the extremal point of convex mappings on $\Omega_{2,\,m}$ for $m \geq 2$?

However, when $m > 2$, Suffridge's idea of the unit ball seems not to work on $\Omega_{2,\,m}$. Fortunately, we can overcome this difficulty via the scaling method and answer the above problem. Let us first give the following result, and its proof is also interesting.

Theorem 4.3. *Let $m \in \mathbb{N}$ and let $m \geq 2$. If $F : H_{2,\,m} = \{(z, w) \in \mathbb{C}^2 : \Re z > |w|^m\} \to \mathbb{C}^2$ is defined by*

$$F(z, w) = (z + aw^m,\, w)$$

for constant $a \in \mathbb{C}$, then F is convex on $H_{2,\,m}$ if and only if $|a| \leq \frac{1}{m-1}$.

Proof. It is easy to see that F is a biholomorphic mapping on $H_{2,\,m}$. For convenience, we may assume $a \geq 0$. Otherwise, if $a = re^{i\theta}$, then we can replace F with $P^{-1} \circ F \circ P$, where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta/m} \end{pmatrix}.$$

Let

$$\begin{cases} u = z + aw^m, \\ v = w. \end{cases}$$

Then

$$\begin{cases} z = u - av^m, \\ w = v. \end{cases}$$

Since $(z, w) \in H_{2,\,m}$, we have

$$\Re(u - av^m) > |v|^m$$

or

$$\Re u > |v|^m + a\Re v^m.$$

Define

$$G_{a,\,m} = \{(u, v) \in \mathbb{C}^2 : \Re u > M_{a,\,m}(v)\},$$

where

$$M_{a,\,m}(v) = |v|^m + a\Re v^m.$$

Since the image domain under the holomorphic mapping F is the set

$$G_{a,\,m} = \{(u, v) \in \mathbb{C}^2 : \Re u > |v|^m + a\Re v^m\},$$

we then need to prove that $G_{a,\,m}$ is a convex set if and only if $a \leq 1/(m-1)$ because of $a \geq 0$ mentioned before. Note that convexity is equivalent to meaning that both eigenvalues of the real Hessian of $M_{a,\,m}$ are all non-negative definite. Namely, it is a question of determining the extent of constant a so that this real Hessian of $M_{a,\,m}(v)$ is non-negative definite. In this case, this condition can be written by complex partial derivatives. Namely, at any point $v \in \mathbb{C}$ and for any $\lambda \in \mathbb{C}$,

$$\frac{\partial^2 h}{\partial v^2}\lambda^2 + 2\frac{\partial^2 h}{\partial v \partial \bar{v}}\lambda\bar{\lambda} + \frac{\partial^2 h}{\partial \bar{v}^2}\bar{\lambda}^2 \geq 0,$$

where $h = M_{a,\,m}$. One needs to consider λ of modulus one, and varying λ then implies that $G_{a,\,m}$ is convex if and only if

$$\left|\frac{\partial^2 h}{\partial v^2}\right| \leq \frac{\partial^2 h}{\partial v \partial \bar{v}}. \tag{4.4}$$

Upon taking the condition of $h(v) = |v|^m + a\Re v^m$, we have the following relation:

$$\begin{cases} \frac{\partial^2 h}{\partial v^2} = \frac{m(m-2)}{4}|v|^{m-4}\bar{v}^2 + \frac{m(m-1)a}{2}v^{m-2}, \\ \frac{\partial^2 h}{\partial v \partial \bar{v}} = \frac{m^2}{4}|v|^{m-2}. \end{cases} \tag{4.5}$$

Substituting (4.5) into (4.4), we get

$$\left|\frac{m(m-1)a}{2}v^{m-2} + \frac{m(m-2)}{4}|v|^{m-4}\bar{v}^2\right| \leq \frac{m^2}{4}|v|^{m-2}.$$

Let $\lambda = v/|v|$. The condition $m \geq 2$ and the above inequality yield that, for any $\lambda \in T$,

$$\left|(m-1)a\lambda^{m-2} + (\frac{m}{2} - 1)\bar{\lambda}^2\right| \leq \frac{m}{2}. \tag{4.6}$$

A simple calculation shows that the inequality (4.6) holds if and only if $a \leq \frac{1}{m-1}$. In fact, if we take $\lambda = 1$, then (4.6) implies that $a \leq \frac{1}{m-1}$. On the other hand, if $a \leq \frac{1}{m-1}$, then

$$\left|(m-1)a\lambda^{m-2} + (\frac{m}{2} - 1)\bar{\lambda}^2\right| \leq (m-1)a + (\frac{m}{2} - 1) = \frac{m}{2},$$

and we prove Theorem 4.3. □

Remark 3. By some computation, we get that the Levi form on the boundary of $H_{2,\,m}$ is equal to

$$\frac{1}{4}\Delta(M_{a,\,m})(v) = m^2|v|^{m-2}.$$

It follows that $G_{a,\,m}$ is pseudoconvex for all $a \in \mathbb{C}$. Thus, geometric convexity is much stronger than pseudoconvex on this domain.

Now, we present our new construction of convex mappings on domain $\Omega_{2,\,m}$.

Theorem 4.4. *Let $m \in \mathbb{N}$, $m \geq 2$, and $\Omega_{2,\,m} = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^m < 1\}$. If $F : \Omega_{2,\,m} \to \mathbb{C}^2$ is defined by*

$$F(z,w) = (\frac{z}{1-z} + a\frac{w^m}{(1-z)^2}, \frac{w}{(1-z)^{2/m}})$$

for some $a \in \mathbb{C}$, then F is convex on $\Omega_{2,\,m}$ if and only if $|a| \leq \frac{1}{2(m-1)}$.

When $m = 2$, then $\Omega_{2,\,2}$ becomes the unit ball B_2. In this case, Theorem 4.4 reduces to Muir and Suffridge in [21, Lemma 2.1] and [19, Theorem 3.1], and our proof is new. When $m \to \infty$, then $\Omega_{2,\,m} \to U^2$ and $a \to 0$. This is a special result of Suffridge [27].

Proof. We begin by testifying that the Thullen domain

$$\Omega_{2,\,m} = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^m < 1\}$$

is biholomorphic to

$$H_{2,\,m} = \{(z, w) \in \mathbb{C}^2 : \Re z > |w|^m\}$$

under the biholomorphic mapping

$$\Phi(z,w) = (\frac{1+z}{1-z}, \frac{w}{(1-z)^{2/m}}), \qquad (z, w) \in \Omega_{2,\,m}.$$

In fact, let

$$\left\{ \begin{array}{l} \tilde{z} = \frac{1+z}{1-z}, \\ \tilde{w} = \frac{w}{(1-z)^{2/m}}. \end{array} \right. \tag{4.7}$$

Then

$$\Re\tilde{z} - |\tilde{w}|^m = \frac{1 - |z|^2 - |w|^m}{|1-z|^2},$$

and

$$\left\{ \begin{array}{l} z = \frac{\tilde{z}-1}{\tilde{z}+1}, \\ w = (\frac{2}{(\tilde{z}+1)})^{2/m}\tilde{w}. \end{array} \right.$$

Hence Φ is a biholomorphic mapping from $\Omega_{2,\,m}$ onto $H_{2,\,m}$.

Theorem 4.3 is employed to derive that

$$F(\tilde{z}, \tilde{w}) = (\tilde{z} + a\tilde{w}^m, \tilde{w}) \tag{4.8}$$

is convex on $H_{2,\,m}$ if and only if $|a| \leq \frac{1}{m-1}$.

Substituting (4.7) into (4.8), we have that

$$G(z,w) = (\frac{1+z}{1-z} + a\frac{w^m}{(1-z)^2}, \frac{w}{(1-z)^{2/m}})$$

is convex on $\Omega_{2,\,m}$ if and only if $|a| \leq \frac{1}{m-1}$.

Since

$$G(0,0) = (1,0), \qquad DG(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

we then normalize the mapping G given by

$$\tilde{G}(z,w) = (DF(0,0))^{-1}(G(z,w) - G(0,0))'.$$

Hence,

$$\tilde{G}(z,w) = (\frac{z}{1-z} + \frac{a}{2}\frac{w^m}{(1-z)^2}, \frac{w}{(1-z)^{2/m}})$$

is a convex mapping on $\Omega_{2,m}$ if and only if $|\frac{a}{2}| \leq \frac{1}{2(m-1)}$. Therefore, we achieve the proof of Theorem 4.4 only when we replace $a/2$ by a. $\square$

Remark 4. Theorem 4.4 is somewhat surprising, because it seems very difficult to show that F is convex if we use the characterization of convex mappings defined on the domain $\Omega_{2,m}$ for $m > 2$. However, Theorem 4.4 can be obtained by using the scaling technique. As a consequence, we are able to construct many unbounded convex mappings which cannot be obtained by the Roper-Suffridge extension operator on $\Omega_{2,m}$.

Denote $\mathcal{K}_\Omega$ by the family of normalized biholomorphic convex mappings on $\Omega_{2,m}$. Recall that a function of $\mathcal{K}_\Omega$ is called an extreme point of $\mathcal{K}_\Omega$ if it cannot be written as a proper convex combination of two other members of $\mathcal{K}_\Omega$. As an application of Theorem 4.4, we have the following interesting result about the extremal points of convex mappings of $\Omega_{2,m}$.

Corollary 4.5. *Let $F : \Omega_{2,m} \to \mathbb{C}^2$ be defined as*

$$F(z,w) = (\frac{z}{1-z} + a\frac{w^m}{(1-z)^2}, \frac{w}{(1-z)^{2/m}}), \qquad (z,w) \in \mathbb{C}^2.$$

If $|a| < 1/(2m-2)$, then the mapping F is not an extreme point of $\mathcal{K}_\Omega$. That is, if F is the extreme point, then $|a| = 1/(2m-2)$.

Proof. For simplicity, we need only to consider the case $a = 0$, because the general case can be proved by the argument of [19, Theorem 3.2].

Let

$$R(z,w) = (\frac{z}{1-z} + \frac{1}{(2m-2)}\frac{w^m}{(1-z)^2}, \frac{w}{(1-z)^{2/m}})$$

and

$$S(z,w) = (\frac{z}{1-z} - \frac{1}{(2m-2)}\frac{w^m}{(1-z)^2}, \frac{w}{(1-z)^{2/m}}).$$

Then R and S are both convex mapping from Theorem 4.4. Obviously, $F = \frac{R+S}{2}$ when $a = 0$. Hence, F is not an extreme point of $\mathcal{K}_\Omega$. $\square$

In 1971, Brickman and MacGregor [3] proved that the extreme points of convex mappings on the unit disk comprise the Koebe function $K(z_1) = z_1/(1 - e^{i\theta} z_1)$, $0 \leq \theta < 2\pi$. Corollary 4.5 tells us that this does not hold in dimension $n \geq 2$. It shows that the higher order terms w^m will also play an important role.

Remark. Owing to the work in [27], [22], and [16], we have found the theory of biholomorphic convex mappings on bounded symmetric domains is well known for cases of rank at least two, and so it is meaningful only on the unit ball of $\mathbb{C}^n$ under bounded symmetric domains. In view of Theorems 3.1 and 4.4, we think it is also meaningful to investigate convex mappings on $\Omega_{2,m}$. This domain has non-compact automorphism group and is not holomorphically equivalent to the unit ball B_2 of $\mathbb{C}^2$ when $m > 2$.

5. Some problems

Finally, we would like to mention some problems that are naturally suggested from the results in this note.

The first obvious problem is to ask whether the mapping F defined in Theorem 4.4 comprises the extremal points of $\mathcal{K}_\Omega$ if and only if $|a| = \frac{1}{2(m-1)}$. Although we do not currently determine the extremal points of convex mappings $\mathcal{K}_\Omega$, we believe this will be true.

The proof of Theorem 4.4 is a little surprising because the convex mapping F of the unbounded domain $H_{2,m}$ plays an important role. However, our results in Section 4 are only concerned with the complex dimension $n = 2$, so the second natural problem is how to generalize Theorem 4.4 to dimension $n > 2$.

In [23, Theorem 3.1], Muir proved the following interesting result for a modification of the Roper-Suffridge operator on the unit ball B_n. Here, we state the result in B_2.

Theorem 5.1. *Let*

$$F(z,w) = \big(f(z) + af'(z)w^2, \sqrt{f'(z)}w\big).$$

If $f(z)$ is a normalized biholomorphic convex mapping on the unit disk U, then $F(z,w)$ is a convex mapping on B_2 if and only if $|a| \le 1/2$.

Inspired by Theorems 4.4 and 5.1, we will propose the following conjecture.

Conjecture 1. *Let $m \in \mathbb{N}$, $m \ge 2$, and $\Omega_{2,m} = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^m < 1\}$. If $f(z)$ is a normalized biholomorphic convex mapping, then*

$$F(z,w) = \big(f(z) + af'(z)w^m, (f'(z))^{1/m}w\big)$$

is convex on $\Omega_{2,m}$ if and only if $|a| \le 1/2(m-1)$.

As for the p-ball $B_p = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : \sum\limits_{j=1}^{n} |z_j|^p < 1\}$, Liu and Zhang [17] proved the following interesting result in 1997.

Theorem 5.2. *If $f : B_p \to \mathbb{C}^n$ is a normalized biholomorphic convex mapping and k is the natural number such that $2 \le k < p < k+1$, then*

$$f(z) = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} a_{12}z_1^2 \\ a_{22}z_2^2 \\ \vdots \\ a_{n2}z_n^2 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1k}z_1^k \\ a_{2k}z_2^k \\ \vdots \\ a_{nk}z_n^k \end{pmatrix} + O(|z|^{k+1}),$$

where $|a_{ij}| \le 1$ for $1 \le i \le n$, $2 \le j \le k$.

By Theorem 5.2, we obtain that $\Phi_{n,\frac{1}{p}}$ defined by Theorem 1.1 is not convex mapping on B_p when $2 < p < \infty$. In [20], Muir and Suffridge gave some bounded convex mappings construction on B_p. Until now, we have not found any unbounded convex mapping on B_p, so it is natural to ask the following question, which was first considered in [20].

Problem 1. Do there exist some unbounded convex mappings on B_p when $2 < p < \infty$? If they do not exist, then all convex mappings on B_p must be bounded.

Acknowledgment

The authors wish to express their gratitude to the referees for many valuable suggestions and corrections.

References

[1] Roger W. Barnard, Carl H. FitzGerald, and Sheng Gong, *A distortion theorem for biholomorphic mappings in* $\mathbf{C}^2$, Trans. Amer. Math. Soc. **344** (1994), no. 2, 907–924, DOI 10.2307/2154515. MR1250815

[2] A. F. Beardon and D. Minda, *The hyperbolic metric and geometric function theory*, Quasiconformal mappings and their applications, Narosa, New Delhi, 2007, pp. 9–56. MR2492498

[3] L. Brickman, T. H. MacGregor, and D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. **156** (1971), 91–107, DOI 10.2307/1995600. MR0274734

[4] H. Cartan, Sur la possibilité d'éntendre aux fonctions de plusieurs variables complexes la thérie des fonctions univalentes. In: Montel, P.(ed.), *Leçons sur les Fonctions Univalentes ou Multivalentes*, Gauthier-Villar, Cambridge, MA, 1933, pp. 129-155.

[5] Carl H. FitzGerald and Sheng Gong, *The Bloch theorem in several complex variables*, J. Geom. Anal. **4** (1994), no. 1, 35–58, DOI 10.1007/BF02921592. MR1274137

[6] F. W. Gehring, *Spirals and the universal Teichmüller space*, Acta Math. **141** (1978), no. 1-2, 99–113, DOI 10.1007/BF02545744. MR0499134

[7] Ian Graham and Gabriela Kohr, *Geometric function theory in one and higher dimensions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 255, Marcel Dekker, Inc., New York, 2003. MR2017933

[8] Ian Graham and Gabriela Kohr, *Univalent mappings associated with the Roper-Suffridge extension operator*, J. Anal. Math. **81** (2000), 331–342, DOI 10.1007/BF02788995. MR1785287

[9] Sheng Gong and Taishun Liu, *On the Roper-Suffridge extension operator*, Dedicated to the memory of Tom Wolff, J. Anal. Math. **88** (2002), 397–404, DOI 10.1007/BF02786583. MR1979778

[10] Sheng Gong, *Convex and starlike mappings in several complex variables*, with a preface by David Minda, Mathematics and its Applications (China Series), vol. 435, Kluwer Academic Publishers, Dordrecht; Science Press Beijing, Beijing, 1998. MR1689825

[11] Björn Gustafsson, *On the convexity of a solution of Liouville's equation*, Duke Math. J. **60** (1990), no. 2, 303–311, DOI 10.1215/S0012-7094-90-06012-0. MR1047755

[12] Seong-A. Kim and David Minda, *The hyperbolic and quasihyperbolic metrics in convex regions*, J. Anal. **1** (1993), 109–118. MR1230512

[13] Piotr Liczberski and Victor V. Starkov, *Distortion theorems for biholomorphic convex mappings in* $\mathbb{C}^n$, J. Math. Anal. Appl. **274** (2002), no. 2, 495–504, DOI 10.1016/S0022-247X(02)00266-4. MR1936712

[14] Piotr Liczberski and Victor V. Starkov, *On two conjectures for convex biholomorphic mappings in* $\mathbb{C}^n$, J. Anal. Math. **94** (2004), 377–383, DOI 10.1007/BF02789055. MR2124468

[15] Xiangyang Liu, *Bloch functions of several complex variables*, Pacific J. Math. **152** (1992), no. 2, 347–363. MR1141801

[16] Taishun Liu and Guangbin Ren, *Decomposition theorem of normalized biholomorphic convex mappings*, J. Reine Angew. Math. **496** (1998), 1–13, DOI 10.1515/crll.1998.030. MR1605807

[17] Taishun Liu and Wenjun Zhang, *Homogeneous expansions of normalized biholomorphic convex mappings over* B^p, Sci. China Ser. A **40** (1997), no. 8, 799–806, DOI 10.1007/BF02878918. MR1481679

[18] Taishun Liu, Jianfei Wang, and Xiaomin Tang, *Schwarz lemma at the boundary of the unit ball in* $\mathbb{C}^n$ *and its applications*, J. Geom. Anal. **25** (2015), no. 3, 1890–1914, DOI 10.1007/s12220-014-9497-y. MR3358077

[19] Jerry R. Muir Jr. and Ted J. Suffridge, *Extreme points for convex mappings of* B_n, J. Anal. Math. **98** (2006), 169–182, DOI 10.1007/BF02790274. MR2254484

[20] Jerry R. Muir Jr. and Ted J. Suffridge, *Construction of convex mappings of p-balls in* $\mathbb{C}^2$, Comput. Methods Funct. Theory **4** (2004), no. 1, 21–34, DOI 10.1007/BF03321052. MR2081662

[21] Jerry R. Muir Jr. and Ted J. Suffridge, *A generalization of half-plane mappings to the ball in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **359** (2007), no. 4, 1485–1498, DOI 10.1090/S0002-9947-06-04205-X. MR2272135

[22] Ngaiming Mok and I Hsun Tsai, *Rigidity of convex realizations of irreducible bounded symmetric domains of rank* ≥ 2, J. Reine Angew. Math. **431** (1992), 91–122. MR1179334

[23] Jerry R. Muir Jr., *A modification of the Roper-Suffridge extension operator*, Comput. Methods Funct. Theory **5** (2005), no. 1, 237–251, DOI 10.1007/BF03321096. MR2174355

[24] J. A. Pfaltzgraff and T. J. Suffridge, *Norm order and geometric properties of holomorphic mappings in* $\mathbf{C}^n$, J. Anal. Math. **82** (2000), 285–313, DOI 10.1007/BF02791231. MR1799667

[25] Kevin A. Roper and Ted J. Suffridge, *Convex mappings on the unit ball of* $\mathbf{C}^n$, J. Anal. Math. **65** (1995), 333–347, DOI 10.1007/BF02788776. MR1335379

[26] Kevin A. Roper and Ted J. Suffridge, *Convexity properties of holomorphic mappings in* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **351** (1999), no. 5, 1803–1833, DOI 10.1090/S0002-9947-99-02219-9. MR1475692

[27] T. J. Suffridge, *The principle of subordination applied to functions of several variables*, Pacific J. Math. **33** (1970), 241–248. MR0261040

School of Mathematical Sciences, Huaqiao University, Quanzhou, Fujian, 362021, People's Republic of China–and–Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, People's Republic of China

Email address: `wangjf@mail.ustc.edu.cn`

Department of Mathematics, Huzhou University, Huzhou, Zhejiang, 313000, People's Republic of China

Email address: `lts@ustc.edu.cn`