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NONCOMMUTATIVE AUSLANDER THEOREM

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ABSTRACT. In the 1960s Maurice Auslander proved the following important result. Let R be the commutative polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, and let G be a finite small subgroup of $\mathrm{GL}_n(\mathbb{C})$ acting on R naturally. Let A be the fixed subring $R^G := \{a \in R | g(a) = a \text{ for all } g \in G\}$. Then the endomorphism ring of the right A-module R_A is naturally isomorphic to the skew group algebra $R * G$. In this paper, a version of the Auslander theorem is proven for the following classes of noncommutative algebras: (a) noetherian PI local (or connected graded) algebras of finite injective dimension, (b) universal enveloping algebras of finite-dimensional Lie algebras, and (c) noetherian graded down-up algebras.

0. INTRODUCTION

The Auslander theorem as stated in the abstract (see also [Au1, Au2] or [IT, Theorem 4.2]) is a fundamental result in the study of the McKay correspondence, isolated singularities, and other homological aspects of commutative algebras. Recently, several researchers or research groups have been interested in the Auslander theorem in the noncommutative setting; see [CKWZ1, HVZ, Ki, Mor, MU, Ue]. Here is a partial list of results concerning noncommutative versions of the Auslander theorem during the last few years:

(1) Mori-Ueyama proved a version of the Auslander theorem when the fixed subring has graded isolated singularities [MU].
(2) Van Oystaeyen-Zhang and the second-named author proved a version of the Auslander theorem for H^*-dense Galois extensions [HVZ].
(3) Chan-Kirkman-Walton and the third-named author proved a version of the Auslander theorem for Hopf actions on Artin-Schelter regular algebras of global dimension two with trivial homological determinant [CKWZ1].

More generally the authors proved some results concerning the Auslander theorem that extends the results in (1)–(3) above; see [BHZ].

This paper is a sequel to [BHZ]. We freely use terminologies introduced in [BHZ]. Our motivation is to understand noncommutative McKay correspondence where one of the important ingredients is the Auslander theorem. Here we apply a main result of [BHZ] to establish the Auslander theorem for several different classes of noncommutative algebras. This research is related to noncommutative algebraic geometry,

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noncommutative invariant theory, and representation theory of noncommutative algebras.

The first family of algebras that we are interested in are noetherian local or connected graded algebras that satisfy a *polynomial identity* (abbreviated as PI). We need to recall some definitions before stating the results.

Let A be an algebra over a base field $\Bbbk$ and H a nontrivial finite-dimensional semisimple Hopf algebra acting on A inner faithfully [BHZ, Definition 3.9]. Let $A\#H$ denote the smash product [Mon, Definition 4.1.3]. Note that both A and H are subalgebras of $A\#H$. Since H is semisimple, the left and the right integrals of H coincide, and is denoted by $\int$.

Let ∂ be a dimension function defined on the right A-modules (or on finitely generated right A-modules) in the sense of [MR, 6.8.4]. In this paper ∂ is either the Gelfand-Kirillov dimension, denoted by GKdim (see [KL] and [BHZ, Definition 1.1]), or the Krull dimension, denoted by Kdim (see [MR, Chapter 6]). Ideas in this paper should apply to other dimension functions.

Definition 0.1. Let ∂ be a dimension function on right A-modules.

(1) [BHZ, Definition 0.1] Suppose that $\partial(A) < \infty$. The *pertinency* of the H-action on A with respect to ∂ is defined to be

$$\mathsf{p}_\partial(A, H) := \partial(A) - \partial((A\#H)/I),$$

where I is the 2-sided ideal of $A\#H$ generated by $1\#\int$. If $\partial = \mathrm{GKdim}$, then $\mathsf{p}_\partial(A, H)$ is denoted by $\mathsf{p}(A, H)$.

(2) [BHZ, Definition 0.2(1)] The *grade* of a right A-module M is defined to be

$$j(M) := \min\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\}.$$

If $\mathrm{Ext}^i_A(M, A) = 0$ for all i, then we say $j(M) = \infty$.

(3) [BHZ, Definition 0.2(2)] If $j((A\#H)/I) \geq 2$, where $(A\#H)/I$ is viewed as a right A-module, we say the H-action on A is *homologically small* or *h.small*.

Pertinency is an essential invariant introduced in [BHZ]. It is an invariant of the H-action on A, not just the pair (A, H). Recall that, for a finite-dimensional $\Bbbk$-vector space V, a finite subgroup $G \subseteq \mathrm{GL}(V)$ is called *small* if G does not contain a pseudo-reflection. When A is the polynomial ring and H is a group algebra, then homological smallness is equivalent to smallness by Lemma 7.2.

Now we are ready to state our first result concerning the Auslander theorem.

Theorem 0.2. *Let R be a noetherian, PI local* (*or connected graded*) *algebra of finite injective dimension ≥ 2, and let H be a semisimple Hopf algebra acting on R. Then the following are equivalent.*

(1) *There is a natural isomorphism of algebras $R\#H \cong \mathrm{End}_{R^H}(R)$.*
(2) $\mathsf{p}_{\mathrm{Kdim}}(R, H) \geq 2$.
(3) *The H-action on R is homologically small.*

In the above theorem, even if A is graded, we do not assume that the H-action is homogeneous. When R is the commutative polynomial ring $\Bbbk[V]$ and G acts on a $\Bbbk$-vector space V inner faithfully, the above theorem agrees with the original Auslander theorem stated at the beginning of the paper. One immediate question is the following.

Question 0.3. Does a version of Theorem 0.2 hold without the PI hypothesis?

Considering an H-action on an algebra A, we say that *the Auslander theorem holds* if there is a natural isomorphism of algebras $A\#H \cong \operatorname{End}_{A^H}(A)$. By Theorem 0.2, the pertinency is ultimately connected with the Auslander theorem. Under some reasonable hypotheses (such as A being Cohen-Macaulay, Artin-Schelter regular, and so on), $\mathsf{p}(A, H) = 0$ if and only if the H-action on A is not inner-faithful. Equivalently, $\mathsf{p}(A, H) \geq 1$ if and only if the H-action on A is inner-faithful. It follows from the Auslander theorem that $\mathsf{p}(A, H) \geq 2$ if and only if $A\#H$ is naturally isomorphic to $\operatorname{End}_{A^H}(A)$. When $\mathsf{p}(A, H)$ is maximal possible, namely, $\mathsf{p}(A, H) = \operatorname{GKdim} A$, the fixed ring A^H has isolated singularities [BHZ]. In general, $\mathsf{p}(A, H)$ controls the dimension of the singularities of A^H; see [BHZ, (E0.5.1)]. Therefore the pertinency is an important invariant related to several properties of the H-action on A.

For different classes of algebras, proofs of the Auslander theorem are different and sometimes require different technologies. This is a main reason why we have different hypotheses in different theorems.

By using the idea of mod-p reduction, we are able to prove a version of Theorem 0.2 for a class of nonlocal noncommutative algebras.

Theorem 0.4. *Suppose* char $\Bbbk = 0$. *Let R be the universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$. Suppose G is a finite small subgroup of* $\operatorname{Aut}_{Lie}(\mathfrak{g})$. *Then there is a natural isomorphism of algebras $R * G \cong \operatorname{End}_{R^G}(R)$.*

Note that $\operatorname{Aut}_{Lie}(\mathfrak{g})$ is the group of Lie algebra automorphisms of $\mathfrak{g}$. The following is an immediate consequence of Theorem 0.4.

Corollary 0.5. *Suppose* char $\Bbbk = 0$. *Let R be the universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g} \neq \mathfrak{g}' \ltimes \Bbbk x$ for a 1-dimensional Lie ideal $\Bbbk x \subseteq \mathfrak{g}$. Then $R * G \cong \operatorname{End}_{R^G}(R)$ for every finite group $G \subseteq \operatorname{Aut}_{Lie}(\mathfrak{g})$.*

Theorem 0.4 and Corollary 0.5 suggest that there should be a version of the McKay correspondence for the universal enveloping algebra of a finite-dimensional Lie algebra, which would be a very interesting future project. Ideas of the proof of Theorem 0.4 apply to other algebras with good filtration; see Theorem 4.10.

Down-up algebras were introduced by Benkart and Roby in [BR] as a tool to study the structure of certain posets. Noetherian graded down-up algebras are Artin-Schelter regular of global dimension three with two generators [KMP]. Their graded automorphism groups were computed in [KK], and are rich enough to provide many nontrivial examples. Some invariant theoretic aspects concerning down-up algebras have been studied in [KK, KKZ1]. We have a version of the Auslander theorem for down-up algebras. For a graded algebra A, let $\operatorname{Aut}_{gr}(A)$ be the group of all graded algebra automorphisms of A.

Theorem 0.6. *Suppose* char $\Bbbk = 0$. *Let R be a noetherian graded down-up algebra $A(\alpha, \beta)$ generated by $V := \Bbbk x + \Bbbk y$; see Definition* 6.1. *Assume that either $\beta \neq -1$ or $(\alpha, \beta) = (2, -1)$. Let G be any nontrivial finite subgroup of* $\operatorname{Aut}_{gr}(R)$. *Then the following hold.*

1. $\mathsf{p}(R, G) \geq 2$.
2. *The G-action on R is homologically small.*
3. *There is a natural isomorphism of graded algebras $R * G \cong \operatorname{End}_{R^G}(R)$.*

Theorem 0.6 suggests that there should be a version of the McKay correspondence for noetherian graded down-up algebras, which would be a good test project for understanding the noncommutative McKay correspondence in dimension three.

It is unfortunate that we need some extra hypothesis on the parameters (α, β) in Theorem 0.6. Our conjecture is the following.

Conjecture 0.7. *Theorem 0.6 also holds in the cases when $\beta = -1$ and $\alpha \neq 2$.*

See [CKZ2, GKMW] for some results related to Conjecture 0.7. For completeness we work out a version of the Auslander theorem for skew polynomial rings (see Example 5.2). Let $\Bbbk^{\times}$ denote the set of units in $\Bbbk$.

Theorem 0.8. *Let R be the skew polynomial ring $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$ for $n \geq 2$ and with $p_{ij} \in \Bbbk^{\times}$ being generic. Let G be a finite group of algebra automorphisms of R. Then G acts on the finite-dimensional $\Bbbk$-space $V := \bigoplus_{s=1}^{n} \Bbbk x_s$ and the following are equivalent.*

(1) *$G \subseteq \mathrm{GL}(V)$ is small.*
(2) *The G-action on R is homologically small.*
(3) *There is a natural isomorphism of graded algebras $R * G \cong \mathrm{End}_{R^G}(R)$.*

The Auslander theorem is one step in establishing the noncommutative McKay correspondence. With this in place one should expect to extend other parts of the McKay correspondence to the noncommutative world [CKWZ1, CKWZ2].

This paper is organized as follows. We provide background material in Section 1. In Section 2, we recall some results in [BHZ]. In Section 3, we prove Theorem 0.2. In Section 4, we prove Theorem 0.4 and Corollary 0.5. Theorem 0.8 is proven in Section 5 and Theorem 0.6 is proven in Section 6. In Section 7 we give some comments about different definitions of smallness and argue that the homological smallness is probably the best replacement of the classical smallness when having Hopf algebra actions.

1. Preliminaries

Throughout let $\Bbbk$ be a base ring that is a noetherian commutative domain. Unless otherwise stated, algebraic objects are over $\Bbbk$. Let A be a (left and right) noetherian algebra. Usually we are working with *right* A-modules and, mostly, with finitely generated (or f.g. for short) *right* A-modules. We write $\mathrm{Mod}\, A$ for the category of all right A-modules, and $\mathrm{mod}\, A$ for the full subcategory of all f.g. right A-modules.

In this paper we mainly use GKdim or Kdim. However, it is a good idea to introduce some definitions for an abstract dimension function. When $\Bbbk$ is not a field, the GK-dimension is given as in [BHZ, Definition 1.1] or [BZ, Lemma 3.1].

Definition 1.1. Let ∂ denote a function on f.g. right A-modules,

$$\partial : \mathrm{mod}\, A \longrightarrow \mathbb{R} \cup \{\pm\infty\}.$$

(1) We say ∂ is a *dimension function* if for all f.g. A-modules M,

$$\partial(M) \geq \max\{\partial(N), \partial(M/N)\} \tag{E1.1.1}$$

whenever N is a submodule of M.

(2) We say ∂ is *exact* if for all f.g. A-modules M,

$$\partial(M) = \max\{\partial(N), \partial(M/N)\} \tag{E1.1.2}$$

whenever N is a submodule of M.

(3) Suppose B is an overring of A such that B is noetherian and B_A is f.g. Assume that the dimension function ∂ is also defined on right B-modules.
 (3a) We say ∂ is *weakly B/A-hereditary* if $\partial(M_A) \leq \partial(M_B)$ for every f.g. right B-module M.
 (3b) We say ∂ is *B/A-hereditary* if, for every f.g. right B-module M,

$$\partial(M_A) = \partial(M_B). \tag{E1.1.3}$$

(4) Let C be another noetherian algebra and suppose ∂ is also a dimension function on *left* C-modules. We say ∂ is *(C, A)-symmetric* if, for every (C, A)-bimodule M that is f.g. over both sides, one has

$$\partial({}_C M) = \partial(M_A). \tag{E1.1.4}$$

(5) Let D be a noetherian algebra and suppose that the dimension function ∂ is also defined on *right* D-modules. We say ∂ is *$(A, D)_i$-torsitive* if, for every (A, D)-bimodule M f.g. on both sides and every f.g. right A-module N, one has

$$\partial(\operatorname{Tor}_j^A(N, M)_D) \leq \partial(N_A) \tag{E1.1.5}$$

for all $j \leq i$.

The definition of a dimension function given in [MR, 6.8.4] is stronger than the definition in Definition 1.1(1). The word "torsitive" stands for "Tor transitive". We collect some facts about GKdim and Kdim. If M is an f.g. graded right module over a noetherian locally finite graded algebra B, then its GK-dimension can be computed by [Zh1, (E7)]

$$\operatorname{GKdim} M = \overline{\lim_{k\to\infty}} \log_k \sum_{j\leq k} \dim(M_j). \tag{E1.1.6}$$

Unless otherwise stated, a graded algebra in this paper means $\mathbb{N}$-graded. Let B be a filtered algebra with an $\mathbb{N}$-filtration $\mathcal{F}$:

$$0 \subseteq F_0 B \subseteq F_1 B \subseteq \cdots \subseteq F_n B \subseteq \cdots .$$

The *associated graded ring* is defined to be

$$\operatorname{gr}_{\mathcal{F}} B := \bigoplus_{i=0}^{\infty} F_i B / F_{i-1} B,$$

where $F_{-1}B = 0$. We say $\mathcal{F}$ is *exhaustive* if $B = \bigcup_i F_i B$. In this paper all filtrations are exhaustive.

Lemma 1.2. *Let A and B be noetherian algebras.*

(1) *If A is an $\mathbb{N}$-filtered algebra such that the associated graded algebra is locally finite and noetherian, then* GKdim *is exact.*
(2) *If B is an overring of A, then* GKdim *is weakly B/A-hereditary.*
(3) *Let A be a subring of B. Suppose that B is $\mathbb{N}$-filtered such that $\operatorname{gr}_{\mathcal{F}} B$ is noetherian and locally finite graded and that $\operatorname{gr}_{\mathcal{F}} A$ induced by the filtration on B is a noetherian and locally finite graded subalgebra of $\operatorname{gr}_{\mathcal{F}} B$. If modules B_A and $(\operatorname{gr}_{\mathcal{F}} B)_{\operatorname{gr}_{\mathcal{F}} A}$ are f.g., then* GKdim *is B/A-hereditary.*
(4) GKdim *is (A, B)-symmetric.*
(5) GKdim *is $(A, B)_0$-torsitive.*
(6) [BHZ, Lemma 1.6] *Let A and B be noetherian and locally finite graded. Then* GKdim *is $(A, B)_\infty$-torsitive in the graded setting.*

Proof. (1) We can pass to the case when $\Bbbk$ is a field. Then the assertion follows from [KL, Theorem 6.14].

(2) This is a consequence of the definition of GKdim.

(3) By localization we pass to the case when $\Bbbk$ is a field. Let M be an f.g. right B-module. There is a finite-dimensional subspace $V \subseteq M$ such that $M = VA = VB$. Let $\mathcal{F}_B := \{F_iB\}_{i\geq 0}$ be the filtration on B and let $\mathcal{F}_A := \{F_iA := F_iB \cap A\}_{i\geq 0}$ be the induced filtration on A. Define a filtration $\mathcal{F}_M$ on M by $F_iM := VF_iB$ for all i. Then $\mathcal{F}_M$ is an $\mathcal{F}_B$-filtration and $\operatorname{gr}_{\mathcal{F}} M$ is an f.g. right $\operatorname{gr}_{\mathcal{F}} B$-module. By [KL, Proposition 6.6], $\operatorname{GKdim} M_B = \operatorname{GKdim}(\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} B}$. Note that $\mathcal{F}_M$ is also an $\mathcal{F}_A$-filtration and $\operatorname{gr}_{\mathcal{F}} M$ is an f.g. right $\operatorname{gr}_{\mathcal{F}} A$-module. Since $(\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} B} = (\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} A}$ as graded vector spaces, by (E1.1.6),

$$\operatorname{GKdim}(\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} B} = \operatorname{GKdim}(\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} A}.$$

Combining these statements we have

$$\operatorname{GKdim} M_B = \operatorname{GKdim}(\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} B} = \operatorname{GKdim}(\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} A} = \operatorname{GKdim} M_A.$$

(4) By localization we pass to the case when $\Bbbk$ is a field. The assertion follows from [KL, Lemma 5.3]

(5) This follows from [KL, Proposition 5.6] with a slight modification. $\square$

The next lemma concerns the Krull dimension, Kdim.

Lemma 1.3.

(1) [KL, Lemma 6.2.4] Kdim *is exact.*
(2) [MR, Proposition 6.4.13] *Suppose A and C are noetherian and PI. Then* Kdim *is (A, C)-symmetric.*

We recall some more definitions. The first one is from [BHZ], which is similar to torsitivity (Definition 1.1(5)) of a dimension function.

Definition 1.4 ([BHZ, Definition 1.2])**.** Let A and B be noetherian algebras and let ∂ be an exact dimension function that is defined on right A-modules and on right B-modules. Let n and i be nonnegative integers. Let ${}_AM_B$ denote any bimodule which is f.g. both as a left A-module and as a right B-module.

(1) We say ∂ *satisfies* $\gamma_{n,i}(M)$ if, for every f.g. right A-module N with $\partial(N_A) \leq n$, one has $\partial(\operatorname{Tor}_j^A(N, M)_B) \leq n$ for all $0 \leq j \leq i$.
(2) We say ∂ *satisfies* $\gamma_{n,i}$ if it satisfies $\gamma_{n,i}(M)$ for all M given as above.

Definition 1.5 ([ASZ1, Definition 0.4])**.** Let B be an algebra and M a right B-module.

(1) Let ∂ be a dimension function. We say B is *∂-Cohen-Macaulay* (or *∂-CM*) if $\partial(B) = d \in \mathbb{N}$, and
$$j(M) + \partial(M) = \partial(B)$$
for every f.g. right B-module $M \neq 0$.
(2) If B is GKdim-Cohen-Macaulay, we just say it is *Cohen-Macaulay* or *CM*.

The CM property together with the Artin-Schelter regularity and the Auslander property (another homological property) has been studied in the noncommutative setting in [ASZ1, ASZ2, SZ, YZ, Zh1].

2. Hypotheses and results from [BHZ]

We repeat the following hypotheses given in [BHZ, Section 2].

Hypothesis 2.1 ([BHZ, Hypothesis 2.1])**.**
(1) *A and B are noetherian algebras.*
(2) *Let e be an idempotent in B and $A = eBe$.*
(3) *∂ is a dimension function defined on right A-modules and right B-modules and $2 \leq d := \partial(B) < \infty$.*
(4) *B is a ∂-CM algebra.*
(5) *The left A-module eB and the right A-module Be are f.g.*
(6) *∂ is exact on right A-modules and right B-modules.*
(7) *For every f.g. right B-module N, $\partial((Ne)_A) \leq \partial(N_B)$.*

The original Hypothesis 2.1(4) in [BHZ] is weaker. We use the present form to avoid another technical definition. Under Hypothesis 2.1(2), Be is a right A-module and there is a natural algebra morphism

$$\varphi: \ B \to \operatorname{End}_A(Be), \quad \varphi(b)(b'e) = bb'e \tag{E2.1.1}$$

induced by the left multiplication. Here is one of the main results in [BHZ]. Some parts of the hypotheses are satisfied automatically.

Lemma 2.2. *Under Hypothesis* 2.1(1)–(6)*, Hypothesis* 2.1(7) *holds if one of the following is true.*
(i) *∂ is $(B, A)_0$-torsitive.*
(ii) $\partial = \operatorname{GKdim}$.
(iii) *B is PI and* $\partial = \operatorname{Kdim}$.

Proof. (i) Hypothesis 2.1(7) can be written as $\partial((N \otimes_B Be)_A) \leq \partial(N_B)$ which is a special case of $(B, A)_0$-torsitivity.

(ii) This follows by case (i) and Lemma 1.2(5).

(iii) By Lemma 3.1(2) in the next section, ∂ is $(B, A)_\infty$-torsitive. The assertion follows by case (i). $\square$

Theorem 2.3 ([BHZ, Theorem 2.4])**.** *Let (A, B) satisfy Hypothesis* 2.1(1)–(7)*. Suppose*

$$\partial \text{ satisfies } \gamma_{d-2,1}(eB). \tag{E2.3.1}$$

Then the following statements are equivalent.
(i) *The functor $- \otimes_{\mathcal{B}} \mathcal{B}e: \ \operatorname{qmod}_{d-2} B \longrightarrow \operatorname{qmod}_{d-2} A$ is an equivalence.*
(ii) *The natural map φ of* (E2.1.1) *is an isomorphism of algebras.*
(iii) $\partial(B/(BeB)) \leq d - 2$.

We are mainly concerned with parts (ii) and (iii), so we will not explain part (i) in the above theorem (definitions can be found in [BHZ]). There is also a graded version of Theorem 2.3. For applications to the Auslander theorem, we need to have a Hopf action.

Let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra that is free of finite rank over the base commutative domain $\Bbbk$. Let R be a noetherian algebra and assume that R is a left H-module algebra, i.e., H acts on R. Then the smash product $R\#H$ is a noetherian algebra. Suppose that H has a (left and right) integral $\int$ such that $\varepsilon(\int) = 1$. If $\Bbbk$ is a field, this is equivalent to the fact that H is semisimple. Let $e = 1\# \int \in R\#H$.

One sees that e is an idempotent of $R\#H$. The fixed subring of the H-action on R is

$$R^H := \{r \in R | h \cdot r = \varepsilon(h)r \ \forall \ h \in H\}.$$

Then R^H is a subalgebra of R. We recall the hypothesis in [BHZ, Section 3] in the ungraded setting.

Hypothesis 2.4 ([BHZ, Hypothesis 3.2]).

(1) *R is a noetherian algebra.*
(2) *H is a Hopf algebra given as above with an action on R.*
(3) *Let B be the algebra $R\#H$ with $e := 1\#\int \in B$. Identify R^H with eBe by* [BHZ, Lemma 3.1(3)] *and R with Be by* [BHZ, Lemma 3.1(5)].
(4) *Let ∂ be an exact dimension function on right f.g. B-modules, R-modules, and R^H-modules, and let $\partial(R) =: d \geq 2$.*
(5) *∂ is B/R-hereditary.*
(6) *R is ∂-CM.*

Again Hypothesis 2.4(6) is stronger than the original version in [BHZ, Hypothesis 3.2(6)] to avoid a new concept. On the other hand, the original version requires R to be locally finite graded, which is removed here. We need an ungraded version of [BHZ, Proposition 3.3].

Proposition 2.5 ([BHZ, Proposition 3.3]). *Retain Hypothesis* 2.4(1)–(6). *Assume that $\Bbbk$ is a field. Then $B := R\#H$ is ∂-CM.*

Proof. Repeat the proof of [BHZ, Proposition 3.3] in the ungraded setting. □

Lemma 2.6. *Suppose ∂ is either* GKdim *when algebras are in Lemma* 1.2(1) *or* Kdim *when algebras are PI. Assume that $\Bbbk$ is a field. Under Hypothesis* 2.4*, Hypothesis* 2.1 *holds.*

Proof. We need to check (1)–(7) in Hypothesis 2.1.

(1) Since R is noetherian, so is $B := R\#H$. By [BHZ, Lemma 3.1(1)], $A := R^H$ is noetherian.

(2) This follows from Hypothesis 2.4(3).

(3) Since ∂ is either GKdim or Kdim, this is not hard to check.

(4) This is Proposition 2.5.

(5) This follows from [BHZ, Lemma 3.1(2)].

(6) When $\partial = \text{GKdim}$, this is Lemma 1.2(1), and when $\partial = \text{Kdim}$, this is Lemma 1.3(1).

(7) This is Lemma 2.2. □

We conclude this section by stating a result of [BHZ].

Theorem 2.7 ([BHZ, Theorem 3.5]). *Retain Hypothesis* 2.4(1)–(3) *and assume that R is locally finite and graded over a base field $\Bbbk$. Further assume that R is CM with* $\operatorname{GKdim} R \geq 2$. *Set $(A, B) = (R^H, R\#H)$. Then the following are equivalent.*

(i) *The natural map $\varphi : B \longrightarrow \operatorname{End}_A(R)$ is an isomorphism of algebras.*
(ii) $\mathsf{p}(R, H) \geq 2$.
(iii) *The H-action on R is h.small.*

3. The PI case

In this section the base ring $\Bbbk$ is a field. We start with a few lemmas.

Lemma 3.1. *All algebras mentioned in this lemma are noetherian and PI. Let B be an algebra and let A be a subalgebra of B such that B_A is f.g. Let ∂ be a dimension function in the sense of* [MR, §6.8.4, p. 224] *defined on left and right A- and B-modules such that*

(i) *∂ is exact and symmetric over any two algebras,*
(ii) *∂ is an integer-valued function,*
(iii) *$\partial(M) \geq 0$ for all $M \neq 0$ and $\partial(0) = -1$.*

Then the following hold.

(1) *Assume that $\partial(B) < \infty$. Then ∂ is B/A-hereditary.*
(2) *If C is another algebra and ∂ is defined on right C-modules, then ∂ is $(C, B)_\infty$-torsitive.*

Proof. (1) We show that, if M is an f.g. right B-module, then

$$\partial(M_A) = \partial(M_B),$$

by induction on $\partial(M_B)$. Nothing needs to be proven if $\partial(M_B) < 0$. Suppose that $\partial(N_A) = \partial(N_B)$ for all f.g. right B-modules N with $\partial(N_B) < n$. Let M be an f.g. right B-module with $\partial(M_B) = n$. Since ∂ is exact and B is PI, we can assume that M_B is critical and M_B is a submodule of $B/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq B$ (see [SZ, Lemma 2.1]). Replacing M_B by $M_B^{\oplus s}$ for some s, we can assume that M_B is essential in $T := B/\mathfrak{p}$. Then $\partial((T/M)_B) < n$. So $\partial(M_B) = \partial(T_B)$ by the exactness. By the induction hypothesis, $\partial((T/M)_A) = \partial((T/M)_B) < n$.

Note that T is a B-bimodule and a (B, A)-bimodule, which is f.g. over A and B on both sides. By the symmetry of ∂,

$$\partial(T_A) = \partial({}_BT) = \partial(T_B) = \partial(M_B) = n.$$

Since $\partial(T_A) = n$ and $\partial((T/M)_A) < n$, by the exactness,

$$\partial(M_A) = \partial(T_A) = n = \partial(M_B)$$

which finishes the induction.

(2) Fix a (C, B)-bimodule S that is f.g. on both sides. We show that, if M is an f.g. right C-module, then

$$\partial(\operatorname{Tor}_i^C(M, S)_B) \leq \partial(M_C),$$

by induction on $\partial(M_C)$. Nothing needs to be proven if $\partial(M) < 0$. Suppose that the assertion holds for any M that is an f.g. right C-module with $\partial(M) < n$. Let M be an f.g. right C-module with $\partial(M) = n$. Note that M has a Kdim-critical composition series

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = 0,$$

such that M_j/M_{j+1} is Kdim-critical and $\partial(M_j/M_{j+1}) \leq n$ for all $j = 0, \ldots, n-1$. If we can show $\partial(\operatorname{Tor}_i^C(M_j/M_{j+1}, S)_B) \leq \partial(M_j/M_{j+1})$ for all j, then it follows that $\partial(\operatorname{Tor}_i^C(M, S)_B) \leq n$ by the exactness of ∂ and the long exact sequence of Tor_i^C. Hence we may assume that M itself is Kdim-critical. Now by [SZ, Lemma 2.1(ii)], M_C is a submodule of $C/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq C$. Replacing M by $M^{\oplus s}$ for some s, we can assume that M_C is essential in $T := C/\mathfrak{p}$. Then $\partial(T/M) < n$. So $\partial(M) = \partial(T) = n$ by the exactness. By the induction hypothesis, for all i,

$$\partial(\mathrm{Tor}_i^C((T/M), S)) \le \partial(T/M) < n.$$

By using the long exact sequence of Tor_i^C again,

$$\partial(\mathrm{Tor}_i^C(M, S)) \le \max\{n-1, \partial(\mathrm{Tor}_i^C(T, S))\}_{i\ge 0}.$$

It remains to show that $\partial(\mathrm{Tor}_i^C(T, S)) \le n$ for all i. Since C is noetherian and T and S are f.g. on both sides, the (C, B)-bimodule $W := \mathrm{Tor}_i^C(T, S)$ is f.g. on both sides. By the symmetry of ∂,

$$\partial(W_B) = \partial({}_C W) \le \partial({}_C T) = \partial(T_C) = n$$

as required. □

For the rest of this section we take $\partial = \mathrm{Kdim}$. Following [SZ], an ($\mathbb{N}$-graded) algebra A is called (*graded*) *injectively smooth* if $\mathrm{injdim}\, A = n < \infty$ and $\mathrm{Ext}_A^n(S, A) \neq 0$ for all simple right (graded) A-modules S. We will use the following result.

Lemma 3.2. *Let A be noetherian and PI. The following hold.*

(1) [SZ, Theorem 1.3] *Let A be a (graded) injectively smooth algebra. Then A is* Kdim-*CM.*
(2) [SZ, Corollary 3.13] *If A is local of finite injective dimension, then A is* Kdim-*CM. Consequently, A is injectively smooth.*
(3) [SZ, Theorem 1.1] *If A is connected graded of finite injective dimension, then A is CM. Consequently, A is injectively smooth.*

We have the following version of the Auslander theorem.

Theorem 3.3. *Let R be a noetherian PI and* Kdim-*CM algebra of* $\mathrm{Kdim} \ge 2$ *and let H be a semisimple Hopf algebra acting on R. Then the following are equivalent.*

(1) *There is a natural isomorphism of algebras $R\#H \cong \mathrm{End}_{R^H}(R)$.*
(2) $\mathsf{p}_{\mathrm{Kdim}}(R, H) \ge 2$.
(3) *The H-action on R is h.small.*

Proof. First we check Hypothesis 2.4(1)–(6). (1)–(4) are clear by taking $\partial = \mathrm{Kdim}$. Item (5) follows from Lemmas 1.3(1) and 3.1(1). (6) is a hypothesis. By Lemma 2.6, Hypothesis 2.1 holds. By Lemma 3.1(2), Kdim is $(A, B)_\infty$-torsitive. Therefore (E2.3.1) holds. By Theorem 2.3, (1) is equivalent to (2). Since R is Kdim-CM, (2) is equivalent to (3). □

Now we are ready to prove Theorem 0.2. Recall that $\mathrm{Kdim}\, M = \mathrm{GKdim}\, M$ if M is an f.g. right module over an affine noetherian PI algebra [SZ, Lemma 4.3(i)].

Proof of Theorem 0.2. When R is PI local or connected graded of finite injective dimension, R is Kdim-CM by Lemma 3.2. The assertion follows by Theorem 3.3. □

We consider some explicit examples. The next is an immediate consequence of Theorem 3.3.

The skew polynomial ring $\Bbbk_{p_{ij}}[x_1, \dots, x_n]$ is defined in Example 5.2.

Corollary 3.4. *Let p_{ij} be roots of unity for all $1 \leq i < j \leq n$ and let R be the skew polynomial ring $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$. Suppose H is a semisimple Hopf algebra acting on R. Then the following are equivalent.*

(1) *There is a natural isomorphism of algebras $R\#H \cong \operatorname{End}_{R^H}(R)$.*
(2) *$\mathsf{p}(R, H) \geq 2$.*
(3) *The H-action on R is h.small.*

Proof. It is well known that R is CM. The assertion follows from Theorem 3.3 and the fact Kdim = GKdim. □

To study other examples, we need to consider filtered algebras. Let B be a filtered algebra with an $\mathbb{N}$-filtration $\mathcal{F}$:

$$0 \subseteq F_0B \subseteq F_1B \subseteq \cdots \subseteq F_nB \subseteq \cdots .$$

For an element $e \in F_0B$, let $\overline{B}$ be the factor ring $B/(e)$, where (e) is the ideal of B generated by e. Let $\pi : B \to \overline{B}$ be the canonical projection map. Then $\overline{B}$ is also a filtered algebra with the filtration $\overline{\mathcal{F}}$ induced from $\mathcal{F}$. Let $\operatorname{gr} e$ be the element in $\operatorname{gr}_{\mathcal{F}} B$ corresponding to e, which has degree 0 as $e \in F_0B$. The following observation is obvious.

Lemma 3.5. *The projection map induces an epimorphism of graded algebras*

$$(\operatorname{gr}_{\mathcal{F}} B)/(e') \to \operatorname{gr}_{\overline{\mathcal{F}}} \overline{B},$$

where $e' := \operatorname{gr} e$ is viewed as an element of degree 0 in $\operatorname{gr}_{\mathcal{F}} B$.

Proof. First of all, we have a surjective homomorphism $\phi : \operatorname{gr}_{\mathcal{F}} B \to \operatorname{gr}_{\overline{\mathcal{F}}} \overline{B}$ induced by the surjection

$$F_iB/F_{i-1}B \to (F_iB + (e))/(F_{i-1}B + (e))$$

for all i. It is clear that ϕ maps e' to 0. The assertion follows. □

Proposition 3.6. *Let R be a filtered algebra with an $\mathbb{N}$-filtration $\mathcal{F}$ such that $\operatorname{gr}_{\mathcal{F}} R$ is locally finite and noetherian. Assume that both R and $\operatorname{gr}_{\mathcal{F}} R$ are CM. Let H be a semisimple Hopf algebra acting on R. Suppose that the H-action on R preserves the filtration $\mathcal{F}$. Then*

$$\mathsf{p}(R, H) \geq \mathsf{p}(\operatorname{gr}_{\mathcal{F}} R, H).$$

As a consequence, if the Auslander theorem holds for the H-action on $\operatorname{gr}_{\mathcal{F}} R$ and if $\gamma_{d-2,1}({}_{R^H}R_{R\#H})$ holds, where $d = \operatorname{GKdim} R$, then the Auslander theorem holds for the H-action on R.

Proof. First of all, H acts on $\operatorname{gr}_{\mathcal{F}} R$ naturally. Since $\operatorname{gr}_{\mathcal{F}} R$ is locally finite and noetherian, $\operatorname{GKdim} \operatorname{gr}_{\mathcal{F}} R = \operatorname{GKdim} R$ by [KL, Proposition 6.6].

Let $e := 1\#\textstyle\int \in R\#H$, where $\textstyle\int$ is the integral of H, and let $e' := 1\#\textstyle\int \in (\operatorname{gr}_{\mathcal{F}} R)\#H$. We define a filtration $\mathcal{F}'$ on $B := R\#H$ by

$$F'_iB = (F_iR)\#H$$

for all $i \geq 0$. Then the associated graded ring of B is

$$\operatorname{gr}_{\mathcal{F}'} B = \operatorname{gr}_{\mathcal{F}'}(R\#H) \cong (\operatorname{gr}_{\mathcal{F}} R)\#H.$$

Let $\overline{B} = B/(e)$ and let $\overline{\mathcal{F}}$ be the filtration on $\overline{B}$ induced by the filtration $\mathcal{F}'$. Now we have

$$\begin{aligned} \operatorname{GKdim}(((\operatorname{gr}_{\mathcal{F}} R)\#H)/(e')) &= \operatorname{GKdim}((\operatorname{gr}_{\mathcal{F}'} B)/(e')) \\ &\geq \operatorname{GKdim} \operatorname{gr}_{\overline{\mathcal{F}}}(B/(e)) \qquad \text{by Lemma 3.5} \\ &= \operatorname{GKdim} \operatorname{gr}_{\overline{\mathcal{F}}}(\overline{B}) \\ &= \operatorname{GKdim} \overline{B}, \end{aligned}$$

where the last equation follows by [KL, Proposition 6.6]. Therefore

$$\begin{aligned} \mathsf{p}(R,H) &= \operatorname{GKdim} R - \operatorname{GKdim}(R\#H)/(e) = \operatorname{GKdim} R - \operatorname{GKdim} \overline{B} \\ &= \operatorname{GKdim} \operatorname{gr}_{\mathcal{F}} R - \operatorname{GKdim} \overline{B} \\ &\geq \operatorname{GKdim} \operatorname{gr}_{\mathcal{F}} B - \operatorname{GKdim}(((\operatorname{gr}_{\mathcal{F}} R)\#H)/(e')) \\ &= \mathsf{p}(\operatorname{gr}_{\mathcal{F}} R, H). \end{aligned}$$

The consequence follows from Theorems 2.3 and 2.7 and Lemma 2.6. □

In the next example, R is neither local nor connected graded.

Corollary 3.7. *Suppose* char $\Bbbk \nmid 2n$. *Let R be the quantum Weyl algebra generated by $x_1, \ldots, x_n$ and subject to the relations $x_i x_j + x_j x_i = 1$ for all $i \neq j$. Let G be the group generated by $\sigma : x_i \to x_{i+1}$ for all $i < n$ and $x_n \to x_1$. Then $\mathsf{p}(R,G) \geq 2$ and $R * G \cong \operatorname{End}_{R^G}(R)$.*

Proof. Using the standard filtration defined by $F_i R = (\Bbbk + \sum_{s=1}^{n} \Bbbk x_s)^i$, we have $\operatorname{gr}_{\mathcal{F}} R \cong \Bbbk_{-1}[x_1, \ldots, x_n]$. By [BHZ, Theorem 0.5], $\mathsf{p}(\operatorname{gr}_{\mathcal{F}} R, G) \geq 2$. By Proposition 3.6, $\mathsf{p}(R,G) \geq \mathsf{p}(\operatorname{gr}_{\mathcal{F}} R, G) \geq 2$. Since R is affine PI and noetherian, $\operatorname{GKdim} M = \operatorname{Kdim} M$ for every f.g. right R-module. It is well known that both R and $\operatorname{gr}_{\mathcal{F}} R$ are CM. By Theorem 3.3, $R * G \cong \operatorname{End}_{R^G}(R)$. □

4. Reduction mod-p and universal enveloping algebras

The goal of this section is to prove Theorem 0.4 and Corollary 0.5, which requires quite a bit of preparation. In this section we assume that $\Bbbk$ is a field with prime subring $\Bbbk_0 = \mathbb{Z}$ or $\mathbb{F}_p := \mathbb{Z}/(p)$, for some prime p, inside $\Bbbk$. Let D denote any $\Bbbk_0$-affine subalgebra of $\Bbbk$. Such a D is an *admissible domain* in the sense of [ArSZ, p. 580]. Note that if D is a $\Bbbk_0$-affine subalgebra of $\Bbbk$, so is $D[s^{-1}]$ for any nonzero element $s \in D$.

Definition 4.1. Let A be an algebra over $\Bbbk$ and M a right A-module. Let D be a $\Bbbk_0$-affine subalgebra of $\Bbbk$.

(1) A D-subalgebra A_D of A is called an *order* of A if the following hold.
  (1a) A_D is free over D,
  (1b) $A_D \otimes_D \Bbbk = A$.

(2) An A_D-submodule M_D of M is called an *order* of M if the following hold.
  (2a) M_D is free over D,
  (2b) $M_D \otimes_D \Bbbk (= M_D \otimes_{A_D} A) = M$.

The following lemma is easy. If $D \to F$ is an algebra homomorphism of commutative rings, and if A_D is an algebra over D, define A_F to be $A_D \otimes_D F$. The module M_F is defined similarly.

Lemma 4.2. *Let A_D and B_D be two D-algebras. Let N be a right A_D-module and let M be an (A_D, B_D)-bimodule which is D-central. Let $j \geq 0$ be an integer.*

(1) *Assume that F is flat over D (e.g., F is a localization of D). Then*
$$\operatorname{Tor}_i^{A_D}(N, M) \otimes_D F = \operatorname{Tor}_i^{A_F}(N_F, M_F)$$
for all i.

(2) *Suppose N, M, and $\operatorname{Tor}_i^{A_D}(N, M)$ are free over D for all $i \leq j$. Then, for every algebra homomorphism $D \to F$ of commutative rings, we have*
$$\operatorname{Tor}_i^{A_D}(N, M) \otimes_D F = \operatorname{Tor}_i^{A_F}(N_F, M_F)$$
for all $i \leq j$.

Proof. (1) Since M is D-central, $F \otimes_D M = M \otimes_D F =: M_F$. Since F is flat over D, the functor $- \otimes_D F$ is exact. Then, for each $i \geq 0$,
$$\begin{aligned}\operatorname{Tor}_i^{A_D}(N, M) \otimes_D F &= \operatorname{Tor}_i^{A_D}(N, M \otimes_D F) = \operatorname{Tor}_i^{A_D}(N, M_F) \\ &= \operatorname{Tor}_i^{A_D}(N, (A_F \otimes_{A_F} M_F)) = \operatorname{Tor}_i^{A_F}((N \otimes_{A_D} A_F), M_F) \\ &= \operatorname{Tor}_i^{A_F}(N_F, M_F).\end{aligned}$$

(2) In part (2) we do not assume that F is flat over D. By the standard Tor spectral sequence [Ro, Theorem 11.51],
$$\operatorname{Tor}_i^{A_D}(N, M) \otimes_D F = \operatorname{Tor}_i^{A_D}(N, M \otimes_D F) = \operatorname{Tor}_i^{A_D}(N, M_F)$$
since M and $\operatorname{Tor}_i^{A_D}(N, M)$ are free over D for all $i \leq j$. Since N is free over D, $\operatorname{Tor}_s^{A_D}(N, A_F) = 0$ for all $s > 0$. Then, by [Ro, Theorem 11.51], we have
$$\begin{aligned}\operatorname{Tor}_i^{A_D}(N, M_F) &= \operatorname{Tor}_i^{A_D}(N, (A_F \otimes_{A_F} M_F)) = \operatorname{Tor}_i^{A_F}((N \otimes_{A_D} A_F), M_F) \\ &= \operatorname{Tor}_i^{A_F}(N_F, M_F).\end{aligned}$$
Therefore the assertion follows. □

We recall some definitions for D-algebras. As a convention in this paper, a *filtration* of an algebra A is an exhaustive $\mathbb{N}$-filtration. A *filtered* algebra is an algebra with a filtration. Following [ArSZ, p. 580], a D-algebra A is called *strongly noetherian* if for every noetherian commutative D-algebra F, $A \otimes_D F$ is noetherian. A *graded* algebra means an $\mathbb{N}$-graded algebra and *locally finite* means that each homogeneous piece is of finite rank over D. We say a filtered algebra A with filtration $\mathcal{F}$ is *locally finite* if its associated graded ring $\operatorname{gr}_{\mathcal{F}} A$ is locally finite.

Definition 4.3. A $\Bbbk$-algebra A is called *congenial* if the following hold.

(1) A is a noetherian locally finite filtered algebra with an $\mathbb{N}$-filtration $\mathcal{F}$.
(2) A has an order A_D, where D is a $\Bbbk_0$-affine subalgebra of $\Bbbk$ such that A_D is a noetherian locally finite filtered algebra over D with the induced filtration, still denoted by $\mathcal{F}$.
(3) The associated graded ring $\operatorname{gr}_{\mathcal{F}} A_D$ is an order of $\operatorname{gr}_{\mathcal{F}} A$.
(4) $\operatorname{gr}_{\mathcal{F}} A_D$ is a strongly noetherian locally finite graded algebra over D.
(5) If F is a factor ring of D and is a finite field, then $A_D \otimes_D F$ is an affine noetherian PI algebra over F.

Let M be an f.g. right A_D-module. We say M is *generically free* [ArSZ, p. 580] if there is a simple localization $D[s^{-1}]$ for some $0 \neq s \in D$, such that $M[s^{-1}]$ is free over $D[s^{-1}]$. We will use the following generic freeness result in [ArSZ].

Lemma 4.4. *Let A be congenial. The following hold.*

1. *A_D is a strongly noetherian locally finite filtered algebra over D.*
2. *Every f.g. right A_D-module M is generically free. This implies that if D is replaced by $D[s^{-1}]$ for some $0 \neq s \in D$, M becomes free over D.*
3. *$\operatorname{gr}_{\mathcal{F}} A$ is congenial.*

Proof. (1) This follows from [ArSZ, Proposition 4.10] and the fact that $\operatorname{gr}_{\mathcal{F}} A_D$ is a strongly noetherian locally finite graded algebra over D.

(2) This follows by part (1) and [ArSZ, Theorem 0.3].

(3) This is clear. □

We will use generic freeness to prove some properties concerning the GK-dimension of modules over a congenial algebra.

Definition 4.5. Let A be an algebra over $\Bbbk$ and let M be a right A-module. Let D be a $\Bbbk_0$-affine subalgebra of $\Bbbk$ and let A_D be an order of A. We say that M is GKdim-*stable over D* if M has a D-order M_D such that, for every algebra map $D \to F$ to a noetherian commutative domain F,

$$\operatorname{GKdim}(M_D \otimes_D F)_{A_D \otimes_D F} = \operatorname{GKdim} M. \tag{E4.5.1}$$

Lemma 4.6. *If M is* GKdim*-stable over D, then $M_{D'}$ is* GKdim*-stable over D' for every $\Bbbk_0$-affine subalgebra D' of $\Bbbk$ containing D.*

Proof. Let $f : D' \to F$ be an algebra map. Then $D \to D' \to F$ is an algebra map. Applying (E4.5.1), we have

$$\operatorname{GKdim}(M_{D'} \otimes_{D'} F)_{A_{D'} \otimes_{D'} F} = \operatorname{GKdim}(M_D \otimes_D F)_{A_D \otimes_D F} = \operatorname{GKdim} M.$$

The assertion follows. □

Proposition 4.7. *Suppose that A is congenial. Let M be an f.g. right A-module. Then there is an affine $\Bbbk_0$-subalgebra $D \subseteq \Bbbk$ such that M is* GKdim*-stable over D.*

Proof. By definition, there is a $D \subseteq \Bbbk$ such that A_D is an order of A. Since A is noetherian and M is f.g., there is an exact sequence of right A-modules

$$A^{\oplus m} \xrightarrow{\phi} A^{\oplus n} \to M \to 0$$

for some integers m and n. Using a D-basis of A_D as a $\Bbbk$-basis of A, all coefficients of ϕ are in the subring $D' = D[c_i]$ for finitely many $c_i \in \Bbbk$. Replacing D by D' we might assume that all coefficients of ϕ are in D. Let M_D be defined by the exact sequence

$$A_D^{\oplus m} \xrightarrow{\phi_D} A_D^{\oplus n} \to M_D \to 0,$$

where the matrix corresponding to ϕ_D is the same as the matrix corresponding to ϕ. By Lemma 4.4(2), after replacing D by $D[s^{-1}]$, one can assume that M_D is free over D. Since $\phi = \phi_D \otimes_D \Bbbk$, $M_D \otimes_D \Bbbk = M$. Since $D \to \Bbbk$ is injective (and flat), we can view M_D as an A_D-submodule of M. Hence, M_D is an order of M.

Since A has a filtration $\mathcal{F}$, we have the induced filtration, still denoted by $\mathcal{F}$, of A_D. Let M_D be an f.g. right A_D-module with a finite set of generators G. Define a filtration on M_D by $F_i M_D = G(F_i A_D)$ for all $i \geq 0$. The associated graded module is

$$\operatorname{gr}_{\mathcal{F}} M_D := \bigoplus_{i=0}^{\infty} F_i M_D / F_{i-1} M_D.$$

Then the graded right module $\operatorname{gr}_{\mathcal{F}} M_D$ is f.g. over $\operatorname{gr}_{\mathcal{F}} A_D$. In Definition 4.3, one can freely replace D by a finitely generated extension of D. Since $\operatorname{gr}_{\mathcal{F}} A_D$ is strongly noetherian and locally finite by definition, $\operatorname{gr}_{\mathcal{F}} M_D$ is generic free by Lemma 4.4(2). By replacing D by $D[s^{-1}]$ for some s, we assume that $\operatorname{gr}_{\mathcal{F}} M_D$ is free over D. As a consequence, each $F_i M_D$ is free of finite rank over D. In this case, the GKdim of M_D is only dependent on the Hilbert series of $\operatorname{gr}_{\mathcal{F}} M_D$. A similar comment holds true for M. One can check that $\operatorname{gr}_{\mathcal{F}} M_D$ is an order of $\operatorname{gr}_{\mathcal{F}} M$, whence, the Hilbert series of $\operatorname{gr}_{\mathcal{F}} M_D$ over D is equal to the Hilbert series of $\operatorname{gr}_{\mathcal{F}} M$ over $\Bbbk$. Therefore

$$\operatorname{GKdim}(M_D)_{A_D} = \operatorname{GKdim}(\operatorname{gr}_{\mathcal{F}} M_D)_{\operatorname{gr}_{\mathcal{F}} A_D} = \operatorname{GKdim}(\operatorname{gr}_{\mathcal{F}} M)_{\operatorname{gr}_{\mathcal{F}} A} = \operatorname{GKdim} M_A.$$

Now consider an algebra map $D \to F$, where F is a noetherian commutative domain. Since $\operatorname{gr}_{\mathcal{F}} M_D$ is free over D, $(\operatorname{gr}_{\mathcal{F}} M_D) \otimes_D F$ is free over F. Consequently, $M_F := M_D \otimes_D F$ is free over F. The GKdim of M_F can be computed by the Hilbert series of $\operatorname{gr}_{\mathcal{F}} M_F$ since $\operatorname{gr}_{\mathcal{F}} M_F$ is an f.g. right module over the noetherian graded ring $\operatorname{gr}_{\mathcal{F}} A_F$. Therefore

$$\operatorname{GKdim}(M_F)_{A_F} = \operatorname{GKdim}(M_D)_{A_D} = \operatorname{GKdim} M_A,$$

and thus M is GKdim-stable over D. □

Proposition 4.8. *Let R be a congenial algebra with a filtration $\mathcal{F}$. Let H be a semisimple Hopf algebra acting on R with H-action preserving the filtration. Let $B := R\#H$ and $A := eBe$ as in Hypothesis* 2.4. *Then the following hold.*

(1) *B is congenial.*
(2) *A is congenial.*
(3) *Let M be $R(= eB)$. There is a $\Bbbk_0$-affine subalgebra $D \subseteq \Bbbk$ and an order M_D of M such that M_D is an (A_D, B_D)-bimodule that is D-central and f.g. on both sides.*

Proof. (1) Since R is congenial, there is an order R_D satisfying the conditions listed in Definition 4.3. Remember that we can freely replace D by a D-affine subalgebra $D' \subseteq \Bbbk$.

Since H is finite dimensional with normalized integral $\textstyle\int$, there is an order H_D of H such that H_D is a Hopf D-algebra and $\textstyle\int \in H_D$. Then $R_D\#H_D$ is an order of $R\#H$. The filtration on R extends to a filtration on $R\#H$ by setting elements of H in F_0. Then Definition 4.3(1)(2) holds for the algebra $R\#H$.

Since $\operatorname{gr}_{\mathcal{F}} R$ is congenial, we can assume that $F_i(R_D)$ and $F_i(R_D)/F_{i-1}(R_D)$ are free over D for all i because $\operatorname{gr}_{\mathcal{F}} R_D$ is free over D. Thus every $F_i(R_D\#H_D)$ and $F_i(R_D\#H_D)/F_{i-1}(R_D\#H_D)$ are free over D for all i. Therefore we have $\operatorname{gr}_{\mathcal{F}}(R_D\#H_D) = (\operatorname{gr}_{\mathcal{F}} R_D)\#H_D$. Since $\operatorname{gr}_{\mathcal{F}} R_D$ satisfies Definition 4.3(3)(4), so does $(\operatorname{gr}_{\mathcal{F}} R_D)\#H_D$. Therefore $\operatorname{gr}_{\mathcal{F}}(R_D\#H_D)$ satisfies Definition 4.3(3)(4). Note that Definition 4.3(5) is clear for B. The assertion follows by combining the last two paragraphs.

(2) We identify A with R^H, together with the filtration induced by $\mathcal{F}$. Then

$$\operatorname{gr}_{\mathcal{F}} A = \operatorname{gr}_{\mathcal{F}}(R^H) = (\operatorname{gr}_{\mathcal{F}} R)^H = \textstyle\int \cdot(\operatorname{gr}_{\mathcal{F}} R) \tag{E4.8.1}$$

by [CWWZ, Lemma 3.1]. Define $A_D := (R_D)^{H_D} = \textstyle\int \cdot R_D$ with the filtration induced by the filtration on R. Recall that R_D is strongly noetherian, so is A_D by [BHZ, Lemma 3.1]. Since R is congenial, the hypothesis of [ArSZ, Theorem 0.2] holds and, hence, we can assume that A_D is free over D after changing D if necessary. The equation $A_D = \int \cdot R_D$ also shows that A_D is an order of A. Then Definition 4.3(1)(2) holds for A.

Similarly, using (E4.8.1), one can check that Definition 4.3(3)(4) holds for A. Definition 4.3(5) is clear for $A = \int \cdot R$. Therefore the assertion follows.

(3) Let D be such that B_D (respectively, H_D and R_D) is an order of B (respectively, H and R sitting inside B). Identify eB (respectively, eBe) with R (respectively, A) by [BHZ, Lemma 3.1(4,2)]. Since A is congenial, by Lemma 4.4(2), one can assume that $eB_D = R_D$ (respectively, eB_De) is an order of $eB = R$ (respectively, eBe). By [BHZ, Lemma 3.1(2)], eB_D is f.g. in both sides. Finally it is clear that eB_D is D-central. The assertion holds. □

Theorem 4.9. *Let R be a congenial algebra with a filtration $\mathcal{F}$. Let H be a semisimple Hopf algebra acting on R with H-action preserving the filtration. Let $B = R\#H$ and $A = eBe$ as in Hypothesis* 2.4. *Then $\gamma_{n,j}(eB)$ holds for all n, j.*

Proof. It suffices to show that, for every right A-module N,

$$\operatorname{GKdim}(\operatorname{Tor}_j^A(N, eB))_B \leq \operatorname{GKdim} N_A.$$

By Lemma 4.4(2) and Proposition 4.7, we can pick a D such that all of N_A, ${}_A(eB)$, $(eB)_B$, and $(\operatorname{Tor}_i^A(N, eB))_B$, for $i \leq j$, are free and GKdim-stable over D. Then, for every algebra homomorphism $D \longrightarrow F$ of commutative rings,

$$\begin{aligned} \operatorname{GKdim}(\operatorname{Tor}_j^A(N, eB))_B &= \operatorname{GKdim}(\operatorname{Tor}_j^A(N, eB) \otimes_D F) && \text{Definition 4.5}\\ &= \operatorname{GKdim}(\operatorname{Tor}_j^{A_F}(N_F, (eB)_F)) && \text{Lemma 4.2(2).} \end{aligned}$$

Note that Lemma 4.2(2) can be applied in the last equation since all modules involved are free over D. By Definition 4.3(5), for every finite field quotient F of D, R_F is noetherian affine PI. In this case, $\operatorname{Kdim} M = \operatorname{GKdim} M$ for all f.g modules M over R_F (or over A_F, B_F). Then

$$\begin{aligned} \operatorname{GKdim}(\operatorname{Tor}_j^{A_F}(N_F, (eB)_F)) &= \operatorname{Kdim}(\operatorname{Tor}_j^{A_F}(N_F, (eB)_F)) && \\ &\leq \operatorname{Kdim} N_F && \text{Lemma 3.1(2)}\\ &= \operatorname{GKdim} N_F = \operatorname{GKdim} N_B && \text{Definition 4.5.} \end{aligned}$$

Therefore the assertion follows. □

Now we have an Auslander theorem for congenial algebras as a consequence of Theorems 2.3 and 4.9.

Theorem 4.10. *Let R be a congenial algebra with a filtration $\mathcal{F}$. Let H be a semisimple Hopf algebra acting on R with H-action preserving the filtration. Let $B = R\#H$ and $A = eBe$ as in Hypothesis* 2.4 *and suppose that R is CM. Then the following are equivalent.*

(1) *There is a natural isomorphism of algebras $R\#H \cong \operatorname{End}_{R^H}(R)$.*
(2) $\mathsf{p}(R, H) \geq 2$.
(3) *The H-action on A is h.small.*

There are ample examples of congenial algebras, one of which is the universal enveloping algebra of any finite-dimensional Lie algebra.

Lemma 4.11. *The universal enveloping algebra $R := U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is congenial.*

Proof. We need to show (1)–(5) in Definition 4.3.

(1) R has a standard filtration $\mathcal{F}$ defined by $F_i R = (\Bbbk + \mathfrak{g})^i$ for all $i \geq 0$. It is well known that R is a noetherian locally finite filtered algebra.

(2) Pick any $\Bbbk$-linear basis $\{x_1, \ldots, x_n\}$ of the Lie algebra $\mathfrak{g}$ and consider the Lie product $[x_i, x_j] = \sum_{s=1}^n c_{i,j}^s x_s$ with coefficients $c_{i,j}^s \in \Bbbk$. Let D be the $\Bbbk_0$-subalgebra of $\Bbbk$ generated by these $c_{i,j}^s$.

Let $\mathfrak{g}_D = \bigoplus_{s=1}^n Dx_i$. Then $\mathfrak{g}_D$ is a D-Lie algebra with $\mathfrak{g}_D \otimes_D \Bbbk = \mathfrak{g}$. One can define the universal enveloping algebra $R_D := U(\mathfrak{g}_D)$ to be

$$D\langle x_1, \ldots, x_n\rangle/([x_i, x_j] = \sum_{s=1}^n c_{i,j}^s x_s \ \forall i, j).$$

Then $U(\mathfrak{g}_D)$ has a filtration $\mathcal{F}'$ which is compatible with the filtration $\mathcal{F}$ such that $\operatorname{gr}_{\mathcal{F}'} R_D$ is isomorphic to the commutative polynomial ring $D[x_1, \ldots, x_n]$. Hence, R_D is free over D and R_D is an order of R. Therefore (2) holds.

The proof of (2) also shows that $\operatorname{gr}_{\mathcal{F}'} R_D$ is an order of $\operatorname{gr}_{\mathcal{F}} R$, which is (3).

(4) This follows from the fact that $\operatorname{gr}_{\mathcal{F}'} R_D$ is the commutative polynomial ring over D.

(5) One can check that $R_D \otimes_D F$ is the universal enveloping algebra $U(\mathfrak{g}_D \otimes_D F)$. Since F is a finite field, $\operatorname{char} F > 0$. By [Ja], $U(\mathfrak{g}_D \otimes_D F)$ is a finite module over its affine center. Therefore it is an affine noetherian PI domain over F. □

In order to prove Theorem 0.4, we need a further lemma.

Lemma 4.12. *Let V be a finite-dimensional $\Bbbk$-vector space of dimension at least 2, and let G be a finite subgroup of $\mathrm{GL}(V)$ acting on the polynomial ring $R := \Bbbk[V]$ naturally. Then the following are equivalent.*

1. *$G \subseteq \mathrm{GL}(V)$ is small.*
2. *The natural algebra map $R * G \to \operatorname{End}_{R^G}(R)$ is an isomorphism of algebras.*
3. *$\mathsf{p}(R, G) \geq 2$.*
4. *The G-action on R is h.small.*

Proof. (1) $\Longrightarrow$ (2) This is the commutative Auslander theorem [Au1, Au2]. See [IT, Theorem 4.2] for a more recent proof.

(2) $\Longrightarrow$ (1) Suppose G is not small. Let W be the subgroup of G generated by a pseudo-reflection in G. Then W is nontrivial and R^W is a commutative polynomial ring by the Shephard-Todd-Chevalley theorem. In this case R is a free module over R^W of rank at least 2 and write $R \cong A^{\oplus d}$, where $A = R^W$ and $d \geq 2$. Since W is a subgroup of G, $R^G \subseteq A$. Thus $\operatorname{End}_{R^G}(R) \cong \operatorname{End}_{R^G}(A^{\oplus d}) = M_d(\operatorname{End}_{R^G}(A))$. This implies that every factor ring of $\operatorname{End}_{R^G}(R)$ has dimension at least $d^2 > 1$. However, $R * G$ has a factor ring $\Bbbk$ by sending $g \mapsto 1$ and $R_{\geq 1} \mapsto 0$. Therefore $R * G \ncong \operatorname{End}_{R^G}(R)$. The assertion follows.

(2) ⇔ (3) This follows from Theorem 2.7.

(3) ⇔ (4) This follows from the fact that $\Bbbk[V]$ is CM. □

Proof of Theorem 0.4. As in the proof of Lemma 4.11, R is congenial and has a standard filtration $\mathcal{F}$. The G-action on R induces naturally a G-action on $\operatorname{gr}_{\mathcal{F}} R \cong \Bbbk[x_1, \ldots, x_n]$. Then, by Proposition 3.6 and Lemma 4.12,

$$\mathsf{p}(R, G) \geq \mathsf{p}(\operatorname{gr}_{\mathcal{F}} R, G) \geq 2.$$

It is well known that $U(\mathfrak{g})$ is CM. The assertion follows from Theorem 4.10. □

To prove Corollary 0.5 we need the following easy observation.

Lemma 4.13. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra such that it is not a semidirect product $\mathfrak{g}' \ltimes \Bbbk x$ for a 1-dimensional ideal $\Bbbk x$. Then every finite subgroup $G \subseteq \mathrm{Aut}_{Lie}(\mathfrak{g})$ is small.*

Proof. If G is not small, there is a $1 \neq g \in G$ which is a pseudo-reflection. Then there is a decomposition $\mathfrak{g} = V \oplus \Bbbk x$ such that $g\mid_V$ is the identity and $g(x) = ax$ for some $a \neq 1$. Then it is easy to show that V is a Lie subalgebra of $\mathfrak{g}$ and $\Bbbk x$ is a 1-dimensional Lie ideal of $\mathfrak{g}$. Therefore $\mathfrak{g} = V \ltimes \Bbbk x$, a contradiction. $\square$

Proof of Corollary 0.5. Under the hypothesis, every finite subgroup $G \subseteq \mathrm{Aut}_{Lie}(\mathfrak{g})$ is small by Lemma 4.13. The assertion follows from Theorem 0.4. $\square$

5. Twisting and skew polynomial rings

In the rest of the paper we assume that $\Bbbk$ is a field.

Let Γ be a group and let A be an $\mathbb{N} \times \Gamma$-graded algebra. Assume that A is locally finite when considered as an $\mathbb{N}$-graded algebra. We first recall Γ-twisting systems and twisted algebras from [Zh2].

Definition 5.1. Let A be an $\mathbb{N} \times \Gamma$-graded algebra. A set of $\mathbb{N} \times \Gamma$-graded algebra automorphisms of A, denoted by $\tau := \{\tau_\gamma \mid \gamma \in \Gamma\}$, is called a *twisting system* of A if

$$\tau_{\gamma_1}\tau_{\gamma_2} = \tau_{\gamma_1\gamma_2}$$

for all $\gamma_1, \gamma_2 \in \Gamma$.

The original definition of a twisting system [Zh2, Definition 2.1] is slightly more general. Given a twisting system, we define a new multiplication of A by

$$x * y = x\tau_\gamma(y)$$

if x is homogeneous of degree (n, γ). By [Zh2, Proposition and Definition 2.3], the *twisted algebra* $A^\tau := (A, *)$ is another $\mathbb{N} \times \Gamma$-graded algebra and $A^\tau = A$ as a graded $\Bbbk$-space.

Example 5.2. Let $\{p_{ij} \mid 1 \leq i < j \leq n\}$ be a subset of $\Bbbk^\times$ and let A be the skew polynomial ring $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$ and subject to the relations

$$x_j x_i = p_{ij} x_i x_j \tag{E5.2.1}$$

for all $i < j$. Define $\deg x_i = (1, e_i)$, where e_i is the ith unit vector in $\mathbb{Z}^n$. Then A is $\mathbb{N} \times \Gamma$-graded, where $\Gamma = \mathbb{Z}^n$. By [Zh2, p. 310], A is a twisted algebra of the commutative polynomial ring $\Bbbk[x_1, \ldots, x_n]$.

Now we put a Hopf action into the picture.

Hypothesis 5.3.

(1) *Let A be an $\mathbb{N} \times \Gamma$-graded algebra with a twisting system $\tau = \{\tau_\gamma \mid \gamma \in \Gamma\}$.*
(2) *Let H be a semisimple Hopf algebra acting on A and preserving the $\mathbb{N} \times \Gamma$-grading.*
(3) *The H-action commutes with τ in the sense that*

$$h \cdot (\tau_\gamma(x)) = \tau_\gamma(h \cdot x)$$

for all $h \in H$, $\gamma \in \Gamma$, and $x \in A$.

The following lemma is easy and we skip most of the proof.

Lemma 5.4. *Retain Hypothesis* 5.3.

(1) *The H-action on A induces naturally an H-action on A^τ which preserves the grading.*
(2) *The smash product $A\#H$ is an $\mathbb{N}\times\Gamma$-graded algebra where* $\deg(1\#h) = (0, 1_\Gamma)$ *for all $h \in H$.*
(3) *The τ extends to a twisting system τ' of $A\#H$ by $\tau'_\gamma(x\#h) = \tau_\gamma(x)\#h$ for all $x \in A$ and $h \in H$, and $(A\#H)^{\tau'} = (A^\tau)\#H$.*
(4) *Let $e = 1\#\int$. Then $\tau'_\gamma(e) = e$ for all $\gamma \in \Gamma$. As a consequence, τ' induces a twisting system τ'' of $(A\#H)/(e)$ such that*
$$((A\#H)/(e))^{\tau''} \cong (A\#H)^{\tau'}/(e) \cong (A^\tau\#H)/(e)$$
as graded algebras.
(5) *Assume that both A and A^τ are noetherian and locally finite as $\mathbb{N}$-graded algebras. Then $\mathsf{p}(A, H) = \mathsf{p}(A^\tau, H)$.*

Proof. (5) Since A and A^τ are noetherian and locally finite as $\mathbb{N}$-graded algebras, we can use (E1.1.6) to compute the GK-dimension, this means that GKdim is independent of their algebra structure. In particular, $\operatorname{GKdim} A = \operatorname{GKdim} A^\tau$ and $\operatorname{GKdim}(A\#H)/(e) = \operatorname{GKdim}(A^\tau\#H)/(e)$ by part (4). The assertion follows. $\square$

Now we are ready to prove Theorem 0.8. An automorphism ϕ of $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$ is called *diagonal* if $\phi(x_i) = a_i x_i$ for some $a_i \in \Bbbk^\times$.

Theorem 5.5. *Suppose* char $\Bbbk = 0$. *Let R be the ring $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$ for $n \geq 2$ and $\{p_{ij}\} \subseteq \Bbbk^\times$. Let G be a finite group of diagonal algebra automorphisms of R. Then the following are equivalent.*

(1) *$G \subseteq \mathrm{GL}(V)$ is small, where $V = \bigoplus_{s=1}^n \Bbbk x_s$.*
(2) *The G-action on R is h.small.*
(3) $\mathsf{p}(R, G) \geq 2$.
(4) *There is a natural isomorphism of algebras $R * G \cong \operatorname{End}_{R^G}(R)$.*

Proof. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) This follows from [BHZ, Theorem 3.5].

(1) $\Leftrightarrow$ (2) By [Zh2, p. 310], $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$ is a twisted algebra of the commutative polynomial ring by τ, where τ consists of diagonal algebra automorphisms.

Let $H = \Bbbk G$. Then one can easily check Hypothesis 5.3. By Lemma 5.4(5), $\mathsf{p}(R, H) = \mathsf{p}(R^{\tau^{-1}}, H)$, where $R^{\tau^{-1}}$ is the commutative polynomial ring. The smallness of G is only dependent on the G-action on the vector space $V = \bigoplus_{s=1}^n \Bbbk x_s$. So we can assume that R is the commutative polynomial ring $R^{\tau^{-1}}$. Therefore the assertion follows by Lemma 4.12. $\square$

Proof of Theorem 0.8. When p_{ij} are generic, every algebra automorphism is diagonal by [AlC]. The assertion follows by Theorem 5.5. $\square$

6. Down-up algebras

First we recall a definition, and refer to [BR, KK, KMP] for basic properties of down-up algebras.

Definition 6.1 ([BR])**.** A *graded down-up algebra*, denoted by $A(\alpha, \beta)$, with parameters $\alpha, \beta \in \Bbbk$, is generated by x and y and subject to the relations

$$\begin{aligned} x^2y &= \alpha xyx + \beta yx^2, \\ xy^2 &= \alpha yxy + \beta y^2x. \end{aligned}$$

The universal enveloping algebra of the 3-dimensional Heisenberg Lie algebra is $A(2, -1)$, and there are other interesting special cases; see [BR, KK]. It is well known that $A(\alpha, \beta)$ is noetherian if and only if it is AS regular if and only if $\beta \neq 0$ [KMP].

Let R be a noetherian graded down-up algebra $A(\alpha, \beta)$, where $\beta \neq 0$. We let a and b be the roots of the "character polynomial"

$$t^2 - \alpha t - \beta = 0.$$

Then

$$\Omega_1 := xy - ayx \quad \text{and} \quad \Omega_2 := xy - byx$$

are normal regular elements in R. We recall a result of Kirkman-Kuzmanovich.

Lemma 6.2 ([KK, Proposition 1.1])**.** *Let R be $A(\alpha, \beta)$. Then the group of graded algebra automorphisms of R is given by*

$$\operatorname{Aut}_{gr}(R) = \begin{cases} \mathrm{GL}(\Bbbk^{\oplus 2}) & \text{if } (\alpha, \beta) = (0, 1), \\ \mathrm{GL}(\Bbbk^{\oplus 2}) & \text{if } (\alpha, \beta) = (2, -1), \\ U := \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} : a_{ij} \in \Bbbk^{\times} \right\} & \text{if } \beta = -1, \alpha \neq 2, \\ O := \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} : a_{ij} \in \Bbbk^{\times} \right\} & \text{otherwise.} \end{cases}$$

As a consequence, $\operatorname{Aut}_{gr}(R)$ *can be realized as a subgroup of* $\mathrm{GL}(\Bbbk^{\oplus 2})$.

Lemma 6.3. *Let $R = A(\alpha, \beta)$ and suppose that $\beta \neq -1$ or $(\alpha, \beta) = (2, -1)$. Let G be a subgroup of $Aut_{gr}(R)$, which can be realized as a subgroup of* $\mathrm{GL}(\Bbbk^{\oplus 2})$ *(see Lemma 6.2)." Then there is a normal regular element $\Omega = xy - ayx$ such that $g(\Omega) = \det(g)\Omega$ for all $g \in G$.*

Proof. There are four cases according to Lemma 6.2. The hypotheses rule out the third case. In the first two cases ($(\alpha, \beta) = (0, 1)$ or $(2, -1)$), we take $a = 1$. It is easy to check that $g(\Omega) = \det(g)\Omega$ for all $g \in \operatorname{Aut}_{gr}(R)$. In the fourth case, we take a to be any of the roots of the character polynomial. Since g is represented by a diagonal matrix, $g(\Omega) = \det(g)\Omega$ when g is in $\operatorname{Aut}_{gr}(R)$. □

When $\beta = -1$ and $\alpha \neq 2$, and if g is represented by a matrix of the form $\left(\begin{smallmatrix} 0 & a_{12} \\ a_{21} & 0 \end{smallmatrix}\right)$, then g switches $xy - ayx$ and $xy - byx$, where a and b are two different roots of the character polynomial. This is the case that we cannot handle.

We are now ready to show Theorem 0.6.

Proof of Theorem 0.6. It is well known that R is CM. By Theorem 2.7, it suffices to show that $\mathsf{p}(R, G) \geq 2$. The idea of the proof is that we construct some filtration of the down-up algebra R and apply Proposition 3.6.

Let R be a down-up algebra $A(\alpha, \beta)$ in the situation of Lemma 6.3, where $\beta \neq -1$ if $\alpha \neq 2$. By Lemma 6.3, this algebra is generated by x and y and has a normal element $\Omega := xy - ayx$ such that $g(\Omega) = \det(g)\Omega$ for all $g \in G$. Consider R as an ungraded algebra and define a filtration $\mathcal{F}$ by setting

$$F_n R = (\Bbbk \oplus \Bbbk x \oplus \Bbbk y \oplus \Bbbk \Omega)^n \subseteq R$$

for all $n \geq 0$ [KKZ1, Lemma 7.2(2)]. Then $F_n R$ is G-stable, and consequently $A := \operatorname{gr}_{\mathcal{F}} R$ is a connected graded algebra with G-action. As a G-module, A is isomorphic to R; then the G-action on A is inner faithful and homogeneous. We will see soon that A is noetherian. By Proposition 3.6, it suffices to show that $\mathsf{p}(A, G) \geq 2$.

Let x and y denote the corresponding elements of x and y in A, and let z be the image of Ω in A. Since Ω is normal in R, we obtain an isomorphism of graded algebras:

$$A := \operatorname{gr}_{\mathcal{F}}(R) \cong (\Bbbk_{a^{-1}}[x, y])[z; \sigma], \tag{E6.3.1}$$

where σ is a graded algebra isomorphism of $\Bbbk_{a^{-1}}[x, y] = R/(\Omega)$ determined by the equations $\Omega x = \sigma(x)\Omega$ and $\Omega y = \sigma(y)\Omega$ in R. Let V' be the $\Bbbk$-subspace $\Bbbk x + \Bbbk y + \Bbbk z$ of A. For $g \in G$, we have seen $g(z) = \det(g)z$. Since G is a finite group, there is a basis $\{x', y'\}$ of $\Bbbk x + \Bbbk y$ such that $g(x') = \xi_1 x'$ and $g(y') = \xi_2 y'$, where ξ_1 and ξ_2 are roots of unity. Then $g(z) = \xi_1 \xi_2 z$. Hence, for any $1 \neq g \in G$, at least two of the three numbers $\{\xi_1, \xi_2, \xi_1\xi_2\}$ are not 1, that is, g is not a pseudo-reflection. Therefore G is small. Next we prove that $\mathsf{p}(A, G) \geq 2$ in the following three cases.

Case 1: $(\alpha, \beta) = (0, 1)$, which is the first case in Lemma 6.2. Then $a = 1$ and $\sigma : x \mapsto -x, y \mapsto -y$. In this case, A is a graded twist of $\Bbbk[x, y][z]$ with a twisting system commuting with the G-action. By Lemma 5.4(5), it suffices to show that $\mathsf{p}(A^\tau, G) \geq 2$, where A^τ is the commutative polynomial ring. When A^τ is the commutative polynomial ring, $\mathsf{p}(A^\tau, G) \geq 2$ is equivalent to the fact that G is small by Lemma 4.12.

Case 2: $(\alpha, \beta) = (2, -1)$, which is the second case in Lemma 6.2. Then A is the commutative polynomial ring $\Bbbk[x, y, z]$. The assertion follows by Lemma 4.12.

Case 3: The "otherwise" case in Lemma 6.2. In this case, every $g \in G$ is diagonal and A is a skew polynomial ring $\Bbbk_{p_{ij}}[x, y, z]$. As a consequence, g acts on A diagonally. Then the assertion follows from Theorem 5.5. □

7. Some comments on smallness

In this section we provide some easy examples and comments on different definitions of smallness. For simplicity, assume that $\operatorname{char} \Bbbk = 0$.

Recall that a finite subgroup G of the general linear group $\mathrm{GL}(V)$ is called *small* if G does not contain a pseudo-reflection of V (except for the identity). We now recall the definition of conventionally small.

Let g be a graded algebra automorphism of a connected graded algebra R. Recall from [JZ] that the *trace function* of g is defined to be

$$Tr_R(g, t) := \sum_{i=0}^{\infty} \operatorname{tr}(g\mid_{R_i}) t^i \in \Bbbk[[t]].$$

Definition 7.1 ([BHZ, Definition 0.8])**.** Let g be a graded algebra automorphism of a noetherian Koszul AS regular algebra R of finite order and let G be a finite subgroup of $\mathrm{Aut}_{gr}(R)$.

(1) The *reflection number* of g is defined to be

$$\mathsf{r}(g) := \mathrm{GKdim}\, R - \text{the order of the pole of } Tr_R(g,t) \text{ at } t = 1.$$

(2) [KKZ3, Definition 2.2] g is called a *quasi-reflection* if $\mathsf{r}(g) = 1$.
(3) [KKZ4, Definition 3.6(b)] g is called a *quasi-bireflection* if $\mathsf{r}(g) = 2$.
(4) The *reflection number* of the G-action on R is defined to be

$$\mathsf{r}(R, G) := \min\{\mathsf{r}(g) \mid 1 \neq g \in G\}.$$

(5) The G-action on R is called *conventionally small* or *c.small* if $\mathsf{r}(R, G) \geq 2$ or, equivalently, G does not contain any quasi-reflection.

In view of Lemma 4.12, we have the following equivalences of different smallness.

Lemma 7.2. *Let V be a finite-dimensional $\Bbbk$-vector space of dimension at least 2, and let G be a finite subgroup of $\mathrm{GL}(V)$ acting on the polynomial ring $R := \Bbbk[V]$ naturally. Then the following are equivalent.*

(1) *$G \subseteq \mathrm{GL}(V)$ is small.*
(2) *$\mathsf{p}(R, G) \geq 2$.*
(3) *The G-action on R is h.small.*
(4) *The G-action on R is c.small.*

Proof. (1) $\Leftrightarrow$ (4) For polynomial algebra $\Bbbk[V]$, a quasi-reflection in Definition 7.1(2) agrees with the classical definition of a pseudo-reflection. So the assertion follows. See Lemma 4.12 for other parts. □

Example 7.3. Let R be the skew polynomial ring $\Bbbk_{-1}[x, y]$ generated by x and y subject to the relation $xy = -yx$. Let G be the group of algebra automorphisms of R generated by $\sigma : x \mapsto y, y \mapsto x$. We claim the following.

(1) $G \subseteq \mathrm{GL}(V)$ is NOT small, where $V = \Bbbk x + \Bbbk y$.
(2) The G-action on R is h.small.
(3) The G-action on R is c.small.

(1) is obvious. (2) is a special case of [BHZ, Theorem 0.5]. (3) follows from the formula in [KKZ2, Lemma 1.7(1)].

If B is the commutative polynomial ring $\Bbbk[x, y]$ and G is the group of algebra automorphisms of B generated by $\sigma : x \mapsto y, y \mapsto x$, then we have the following by Lemma 7.2.

(4) $G \subseteq \mathrm{GL}(V)$ is not small, where $V = \Bbbk x + \Bbbk y$.
(5) The G-action on B is not h.small.
(6) The G-action on B is not c.small.

Example 7.4. Let $R := \Bbbk_{-1}[x, y]$ be as in Example 7.3, and let G' be the group of algebra automorphisms of R generated by $\sigma : x \mapsto iy, y \mapsto ix$, where $i^2 = -1$. By a $\Bbbk$-linear base change, we are in the situation of [KKZ3, Example 2.3(b)]. We claim the following.

(1) $G' \subseteq \mathrm{GL}(V)$ is small, where $V = \Bbbk x + \Bbbk y$.
(2) The G'-action on R is NOT h.small.
(3) The G'-action on R is NOT c.small.

Indeed, statement (1) is obvious.

(2) By [KKZ3, Example 2.3(b)], $A := R^{G'}$ is AS regular. Then R is a free module over A of rank $d \geq 2$. So $\mathrm{End}_A(R)$ is the $d \times d$-matrix algebra over A. Hence, $\mathrm{End}_A(R)$ cannot be isomorphic to $R * G'$. By Theorem 2.7, the G'-action on R is NOT h.small.

(3) It follows from the formula in [KKZ2, Lemma 1.7(1)] that $Tr_R(\sigma, t) = \frac{1}{(1-t^2)}$. Or one can use the trace formula given in [KKZ3, Example 2.3(b)]. Then the G'-action on R is NOT c.small.

If B is the commutative polynomial ring $\Bbbk[x, y]$ and G' is the group of algebra automorphisms of A generated by $\sigma : x \mapsto iy, y \mapsto ix$, then we have the following by Lemma 7.2.

(4) $G' \subseteq \mathrm{GL}(V)$ is small, where $V = \Bbbk x + \Bbbk y$.
(5) The G'-action on B is h.small.
(6) The G'-action on B is c.small.

Remark 7.5. Let R be a noetherian AS regular, CM, and Koszul algebra.

(1) The above two examples show that the smallness is different from the h.smallness in the general noncommutative setting.
(2) We conjecture that the c.smallness is equivalent to the h.smallness.
(3) By "(2) $\Longrightarrow$ (1)" in the proof of Lemma 4.12, h.smallness is stronger than c.smallness. Therefore the conjecture in part (2) follows from [BHZ, Conjecture 0.9].
(4) Let R be the commutative polynomial ring as in Lemma 7.2. By using the definition of smallness, one sees that the Auslander theorem holds if and only if

(E7.5.1) the fixed subring $R^{G'}$ is not AS regular for all $1 \neq G' \subseteq G$.

In this case, the smallness of G can be characterized as the property (E7.5.1).

The next example is given in [CKZ1, Example 2.1].

Example 7.6. Let R be the down-up algebra $A(0, 1)$ generated by x and y and subject to relations

$$x^2y = yx^2 \quad \text{and} \quad xy^2 = y^2x.$$

This is a noetherian, connected graded AS regular, CM, and PI domain. Let H be the Hopf algebra $(\Bbbk D_8)^\circ$, where D_8 is the dihedral group of order 8. There is an H-action on R defined as in [CKZ1, Example 2.1]. Every Hopf subalgebra H' of H is of the form $(\Bbbk(D_8/N))^\circ$, where N is a normal subgroup of D_8. By [CKZ1, Theorem 0.1], $R^{H'}$ is not AS regular for all nontrivial Hopf subalgebras $H' \subseteq H$. Suggested by Remark 7.5(4), the Auslander theorem should hold for the H-action on R. However, this is not true by the next paragraph, which indicates that the "smallness" (of a Hopf action) should not be defined by using the *failure of the AS regularity of the fixed subrings $R^{H'}$ for all nontrivial $H' \subseteq H$*. So this is different from the commutative case (E7.5.1).

To see that the Auslander theorem does not hold, we use [CKZ1, Lemma 2.2]. Let $A := R^H$. By [CKZ1, Example 2.1], $A \cong \Bbbk[x, y, z, t]/(xy - zt^2)$, where we view x, y, z of degree 4 and t of degree 2. Hence it is AS-Gorenstein (see [KKZ4, Theorem 0.4], for instance). Note that each component in [CKZ1, Lemma 2.2(1,2,3)] is a right A-module, and R, as a right A-module, is the direct sum of all the components in

[CKZ1, Lemma 2.2]. Hence $R = A \oplus A(-1) \oplus A(-2) \oplus M$ for some A-module M since the second and the third components in [CKZ1, Lemma 2.2(2,3)] are isomorphic to $A(-1)$ and $A(-2)$, respectively. Then $\mathrm{End}_A(R)$ contains nonzero elements of degree -1. Thus $\mathrm{End}_A(R)$ cannot be isomorphic to $R\#H$. This means that the Auslander theorem fails for this H-action on R.

As a consequence of Theorem 0.2, $\mathsf{p}(R, H) = 1$.

Finally we include another example which could be a test example for the conjecture in Remark 7.5(2).

Example 7.7. Let R be the non-PI Sklyanin algebra of dimension at least 3. By [KKZ3, Corollary 6.3], R has no quasi-reflection of finite order. By definition, every finite subgroup G in $\mathrm{Aut}_{gr}(R)$ is c.small. Following Remark 7.5(2) we conjecture that every such G is h.small.

What we have learned from Theorems 0.2 and 2.7 and the examples and comments in this section is: *in the general Hopf action setting, the homological smallness is the most reasonable replacement of the classical smallness.*

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