

A MULTIPLIER ALGEBRA FUNCTIONAL CALCULUS

KELLY BICKEL, MICHAEL HARTZ, AND JOHN E. McCARTHY

ABSTRACT. This paper generalizes the classical Sz.-Nagy–Foias $H^\infty(\mathbb{D})$ functional calculus for Hilbert space contractions. In particular, we replace the single contraction T with a tuple $T = (T_1, \dots, T_d)$ of commuting bounded operators on a Hilbert space and replace $H^\infty(\mathbb{D})$ with a large class of multiplier algebras of Hilbert function spaces on the unit ball in \mathbb{C}^d .

1. INTRODUCTION

1.1. Overview. One seminal result connecting operator theory and complex function theory is the classical Sz.-Nagy–Foias $H^\infty(\mathbb{D})$ functional calculus [29], which says that for a completely nonunitary contraction T on a Hilbert space \mathcal{K} , the polynomial functional calculus

$$(1) \quad \Phi_T : \mathbb{C}[z] \rightarrow B(\mathcal{K}), \quad T \mapsto p(T)$$

extends to a weak-* continuous, contractive algebra homomorphism on $H^\infty(\mathbb{D})$ (with related statements for general contractions). This allows one to make sense of $f(T)$ for fairly general functions f and operators T and has both illuminated the structure of C_0 contractions and related operators (see [8, 10, 28, 29]) and led to breakthroughs related to the invariant subspace problem (see [11, 12]).

In this paper, we generalize the Sz.-Nagy–Foias $H^\infty(\mathbb{D})$ functional calculus in two nontrivial ways: first, we replace the single contraction $T \in B(\mathcal{K})$ with a tuple $T = (T_1, \dots, T_d)$ of commuting (but not necessarily contractive) bounded operators on \mathcal{K} , and second, we replace $H^\infty(\mathbb{D})$ with a multiplier algebra associated to any of a large class of reproducing kernel Hilbert spaces \mathcal{H} on the open unit ball $\mathbb{B}_d \subseteq \mathbb{C}^d$.

Results of this type were proved by J. Eschmeier, who established an $H^\infty(\mathbb{B}_d)$ -functional calculus for completely nonunitary tuples of commuting operators satisfying von Neumann’s inequality over the unit ball [19] and by R. Clouâtre and K. Davidson, who studied the Drury–Arveson space H_d^2 on \mathbb{B}_d and its multiplier algebra $\text{Mult}(H_d^2)$. They proved that completely nonunitary commuting row contractions admit a $\text{Mult}(H_d^2)$ -functional calculus [13]. This current paper generalizes these earlier investigations and provides new arguments that avoid technical analyses of specific multiplier algebras.

1.2. Main result. Here is our setting: Let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on a Hilbert space \mathcal{K} , and let $U = (U_1, \dots, U_d)$ be a *spherical unitary*

Received by the editors March 28, 2017.

2010 *Mathematics Subject Classification.* Primary 47A60; Secondary 47A13, 46E22.

Key words and phrases. Functional calculus, multiplier algebra, unit ball.

The first author was partially supported by National Science Foundation Grant DMS 1448846.

The second author was partially supported by a Feodor Lynen Fellowship.

The third author was partially supported by National Science Foundation Grant DMS 1565243.

on \mathcal{K} , i.e., each U_i is normal and $\sum U_i U_i^* = I$. We say T is *completely nonunitary* if it has no nonzero reducing subspace \mathcal{M} with $T|_{\mathcal{M}}$ a spherical unitary.

Let \mathcal{H} be a reproducing kernel Hilbert space on \mathbb{B}_d whose multiplier algebra, $\text{Mult}(\mathcal{H})$, contains the polynomials $\mathbb{C}[z_1, \dots, z_d]$. Thus, the polynomial functional calculus for T ,

$$(2) \quad \Phi_T : \mathbb{C}[z_1, \dots, z_d] \rightarrow B(\mathcal{K}), \quad T \mapsto p(T),$$

is defined on a subalgebra of $\text{Mult}(\mathcal{H})$.

We are interested in well-behaved extensions of the polynomial functional calculus Φ_T , namely ways to make sense of $f(T)$ for a larger class of functions. Let $A(\mathcal{H})$ denote the norm closure of the polynomials in $\text{Mult}(\mathcal{H})$. Then, we say T *admits an $A(\mathcal{H})$ -functional calculus* if Φ_T extends to a completely contractive algebra homomorphism $A(\mathcal{H}) \rightarrow B(\mathcal{K})$; this loosely says that T admits an $A(\mathcal{H})$ version of von Neumann’s inequality. Moreover, we say T is *$\text{Mult}(\mathcal{H})$ -absolutely continuous* if in addition Φ_T extends to a weak- $*$ continuous algebra homomorphism $\text{Mult}(\mathcal{H}) \rightarrow B(\mathcal{K})$. Thus, the following two questions naturally arise:

- (Q1) When does T admit an $A(\mathcal{H})$ -functional calculus?
- (Q2) If T admits an $A(\mathcal{H})$ -functional calculus, is T $\text{Mult}(\mathcal{H})$ -absolutely continuous?

If \mathcal{H} is a complete Nevanlinna–Pick space, then (Q1) has a concrete answer in terms of the reproducing kernel of \mathcal{H} (see Section 5). However, even in the case when \mathcal{H} is the Hardy space on the unit ball, so that $A(\mathcal{H})$ is the ball algebra, we are not aware of an explicit answer to (Q1). Nevertheless, Arveson’s dilation theorem shows that one can rephrase (Q1) in terms of dilations, and the resulting dilation has a very explicit form if \mathcal{H} satisfies some additional assumptions, which we now describe. We say that \mathcal{H} is a *unitarily invariant space* on \mathbb{B}_d if the reproducing kernel K of \mathcal{H} is of the form

$$K(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n \quad \text{with } a_0 = 1 \text{ and } a_n > 0 \text{ for all } n \in \mathbb{N}.$$

If in addition $\lim_{n \rightarrow \infty} a_n/a_{n+1} = 1$, we say that \mathcal{H} is a *regular unitarily invariant space*. A discussion of this condition can be found in Subsection 2.1. For now, we merely mention that this class of spaces includes many well-known spaces on \mathbb{B}_d such as the Bergman space, the Hardy space, the Dirichlet space, and the Drury–Arveson space.

Then our main result is the following answer to (Q2) for completely nonunitary tuples T .

Theorem 1.1. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d . Let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on a Hilbert space that admits an $A(\mathcal{H})$ -functional calculus. If T is completely nonunitary, then T is $\text{Mult}(\mathcal{H})$ -absolutely continuous.*

This loosely says that if T satisfies an $A(\mathcal{H})$ -version of von Neumann’s inequality and is completely nonunitary, then we can make sense of $f(T)$ for all $f \in \text{Mult}(\mathcal{H})$. This theorem partially extends Corollary 1.7 in [19], which establishes an $H^\infty(\mathbb{B}_d)$ -functional calculus for completely nonunitary tuples which satisfy von Neumann’s inequality over the unit ball. Moreover, if $\mathcal{H} = H_d^2$, the Drury–Arveson space, then a theorem of Müller–Vasilescu [24] and Arveson [6] shows that a tuple of commuting operators T admits an $A(\mathcal{H})$ -functional calculus if and only if T is a

row contraction, i.e., $\sum_{i=1}^d T_i T_i^* \leq I$. Thus, in this setting, we recover Theorem 4.3 in [13]. We remark that in this last case, our proof does not require the detailed description of the dual of $A(H_d^2)$ from [14], which was used in [13].

Even in the one-variable setting, Theorem 1.1 appears to be new. For example, consider the one-variable Dirichlet space \mathcal{D} . Because the Dirichlet space has an irreducible complete Nevanlinna–Pick kernel, there is an explicit characterization (see Theorem 5.1) for when T admits an $A(\mathcal{D})$ -functional calculus, and this characterization does not require T to be a contraction. Then for such T which are completely nonunitary, Theorem 1.1 says that T is $\text{Mult}(\mathcal{D})$ -absolutely continuous.

1.3. Outline of the paper. Recall that the Sz.-Nagy–Foias $H^\infty(\mathbb{D})$ functional calculus requires two structural results about contractions: first, each contraction T decomposes as $T = T_{cnu} \oplus U$, with T_{cnu} completely nonunitary and U unitary; and second, Sz.-Nagy’s dilation theorem, namely each contraction admits a unitary dilation. If $T = U$ is unitary, the spectral theorem implies that U has a weak-* continuous $H^\infty(\mathbb{D})$ functional calculus if and only if its scalar spectral measure is absolutely continuous with respect to Lebesgue measure. If $T = T_{cnu}$ is completely nonunitary, then the spectral measure of its minimal unitary dilation is absolutely continuous with respect to Lebesgue measure, from which the general calculus follows [29].

This paper follows the classical outline. In Section 2, we recall some necessary background material on reproducing kernel Hilbert spaces and multivariable operator theory. In particular, Proposition 2.2 notes that every tuple of commuting operators T splits as $T = T_{cnu} \oplus U$, for T_{cnu} completely nonunitary and U a spherical unitary. Using standard dilation theoretic arguments, we show in Theorem 2.4 that (Q1) can be rephrased in terms of concrete dilations of T for regular unitarily invariant spaces \mathcal{H} .

Section 3 develops some properties of $\text{Mult}(\mathcal{H})$ -Henkin measures, the analogue of measures which are absolutely continuous with respect to Lebesgue measure in our setting. This section closely follows part of the work of Clouâtre and Davidson on \mathcal{A}_d -Henkin measures [14]. In particular, adapting their arguments, we extend in Lemma 3.3 a theorem of Henkin by showing that $\text{Mult}(\mathcal{H})$ -Henkin measures form a band. Moreover, Lemma 3.4 gives the following answer to (Q2) for spherical unitaries: a spherical unitary U is $\text{Mult}(\mathcal{H})$ -absolutely continuous if and only if its scalar spectral measure is $\text{Mult}(\mathcal{H})$ -Henkin. In Proposition 3.5, we also provide an answer to (Q2) for operator tuples T in terms of a minimal dilation of T .

Section 4 proves our main result, Theorem 1.1. The proof requires the dilation theorem (Theorem 2.4) as well as Lemma 4.1, which says analytic polynomials are weak-* dense in $L^\infty(\mu)$, for μ a totally singular measure. Then Theorem 4.3 also answers (Q2) for commuting operator tuples which may have spherical unitary summands.

In Section 5, we consider the case when \mathcal{H} is a complete Nevanlinna–Pick space. In this setting, Corollary 5.2 characterizes those operator tuples for which (Q1) and (Q2) have affirmative answers.

2. PRELIMINARIES

2.1. Unitarily invariant spaces. We begin by discussing unitarily invariant reproducing kernel Hilbert spaces on the unit ball \mathbb{B}_d ; in what follows, we only consider the situation $d < \infty$. Background material on reproducing kernel Hilbert

spaces can be found in [3] and [26]. Recall from the introduction that a *unitarily invariant space* on \mathbb{B}_d is a reproducing kernel Hilbert space on \mathbb{B}_d whose reproducing kernel K is of the form

$$K(z, w) = \sum_{n=0}^{\infty} a_n \langle z, w \rangle^n \quad \text{with } a_0 = 1 \text{ and } a_n > 0 \text{ for all } n \in \mathbb{N}.$$

Moreover, we say that \mathcal{H} is a *regular unitarily invariant space* if in addition, we have $\lim_{n \rightarrow \infty} a_n/a_{n+1} = 1$. Since \mathcal{H} is a space on \mathbb{B}_d , it is natural to assume that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n t^n$ is 1 so that the limit $\lim_{n \rightarrow \infty} a_n/a_{n+1}$, if it exists, is necessarily equal to 1. Thus, we regard this condition as a regularity condition on the sequence (a_n) . If \mathcal{H} is a regular unitarily invariant space, then the polynomials are automatically multipliers of \mathcal{H} (see, for example [20, Corollary 4.4 (1)]). Regularity is very useful to us since it guarantees that the tuple $(M_{z_1}, \dots, M_{z_d})$ is a spherical unitary tuple modulo the compacts (see the proof of Theorem 2.4 below).

Examples of regular unitarily invariant spaces are the spaces with reproducing kernels

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\alpha},$$

where $\alpha \in (0, \infty)$, which includes in particular the Drury–Arveson space ($\alpha = 1$), the Hardy space ($\alpha = d$), and the Bergman space ($\alpha = d + 1$). A related scale of spaces is given by the kernels

$$K(z, w) = \sum_{n=0}^{\infty} (n+1)^s \langle z, w \rangle^n,$$

where $s \in \mathbb{R}$. Here, the Drury–Arveson space corresponds to the choice $s = 0$. If $s = -1$, then we obtain the Dirichlet space on \mathbb{B}_d .

2.2. The weak-* topology on the multiplier algebra. Let \mathcal{H} be a unitarily invariant space on \mathbb{B}_d . Since $1 \in \mathcal{H}$, every multiplier $\varphi \in \text{Mult}(\mathcal{H})$ is uniquely determined by its associated multiplication operator M_φ . Thus, the assignment $\varphi \mapsto M_\varphi$ allows us to regard $\text{Mult}(\mathcal{H})$ as a unital subalgebra of $\mathcal{B}(\mathcal{H})$. It is not hard to see that $\text{Mult}(\mathcal{H})$ is closed in the weak operator topology and a fortiori in the weak-* topology. Therefore, $\text{Mult}(\mathcal{H})$ is a dual space in its own right, namely the dual of $\mathcal{B}(\mathcal{H})_* / \text{Mult}(\mathcal{H})_\perp$. We endow $\text{Mult}(\mathcal{H})$ with the resulting weak-* topology.

Since unitarily invariant spaces are separable, so is $\mathcal{B}(\mathcal{H})_* / \text{Mult}(\mathcal{H})_\perp$, hence the weak-* topology on bounded subsets of $\text{Mult}(\mathcal{H})$ is metrizable. Moreover, one easily verifies that on bounded subsets of $\text{Mult}(\mathcal{H})$, the weak-* topology coincides with the topology of pointwise convergence on \mathbb{B}_d .

2.3. Decomposition of operator tuples. The proof of [13, Theorem 4.1] implicitly contains the fact that every commuting operator tuple $T = (T_1, \dots, T_d)$ is the direct sum of a completely nonunitary tuple and a spherical unitary. The authors of [13] attribute the argument to Jörg Eschmeier. For completeness, we repeat the relevant part of the argument, beginning with the following lemma.

Lemma 2.1. *Let $U = (U_1, \dots, U_d)$ be a tuple of commuting operators on a Hilbert space. Then U is a spherical unitary if and only if*

$$(3) \quad \sum_{k=1}^d U_k U_k^* = \sum_{k=1}^d U_k^* U_k = I.$$

Proof. It is obvious that every spherical unitary satisfies (3). Conversely, suppose that U satisfies (3). Then U and U^* are spherical isometries. By a theorem of Athavale [7], U therefore extends to a spherical unitary $V = (V_1, \dots, V_d)$, say

$$V_k = \begin{bmatrix} U_k & A_k \\ 0 & B_k \end{bmatrix}.$$

From $\sum_{k=1}^d V_k V_k^* = I$ and $\sum_{k=1}^d U_k U_k^* = I$, we deduce

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I + \sum_{k=1}^d A_k A_k^* & * \\ * & * \end{bmatrix},$$

so that $A_k = 0$ for $k = 1, \dots, d$. Hence, U is the restriction of the spherical unitary V to a reducing subspace, so U itself is a spherical unitary. \square

The desired decomposition of operator tuples follows easily. Note that there are no contractivity assumptions on the operator tuple.

Proposition 2.2. *Let T be a tuple of commuting operators on a Hilbert space \mathcal{K} . Then there exist unique complementary reducing subspaces \mathcal{K}_{cnu} and \mathcal{K}_u for T such that $T_{cnu} := T|_{\mathcal{K}_{cnu}}$ is completely nonunitary and $U := T|_{\mathcal{K}_u}$ is a spherical unitary.*

Proof. Let \mathcal{U} be the set of all reducing subspaces M of T with the property that $T|_M$ is a spherical unitary, and let \mathcal{K}_u be the closed linear span of all elements of \mathcal{U} . Then \mathcal{K}_u is reducing for T , and if we define $\mathcal{K}_{cnu} = \mathcal{K} \ominus \mathcal{K}_u$, then $T|_{\mathcal{K}_{cnu}}$ is completely nonunitary by definition.

Let $U = T|_{\mathcal{K}_u}$. Observe that each $M \in \mathcal{U}$ is contained in the space

$$\ker \left(I - \sum_{k=1}^d U_k U_k^* \right) \cap \ker \left(I - \sum_{k=1}^d U_k^* U_k \right),$$

and hence, so is \mathcal{K}_u . This shows that $\sum_{k=1}^d U_k U_k^* = \sum_{k=1}^d U_k^* U_k = I_{\mathcal{K}_u}$ so that U is a spherical unitary by Lemma 2.1.

Finally, if $\mathcal{K} = \mathcal{K}'_u \oplus \mathcal{K}'_{cnu}$ is another decomposition as in the statement, then by definition, $\mathcal{K}'_u \subset \mathcal{K}_u$. On the reducing subspace $\mathcal{K}_u \ominus \mathcal{K}'_u = \mathcal{K}'_{cnu} \cap \mathcal{K}_u$, the tuple T is both spherical unitary and completely nonunitary. Hence, this subspace is trivial, so $\mathcal{K}'_u = \mathcal{K}_u$ and thus $\mathcal{K}'_{cnu} = \mathcal{K}_{cnu}$. \square

Remark 2.3. According to our definition, an operator tuple T is completely nonunitary if and only if T has no nonzero *reducing* subspace M with $T|_M$ a spherical unitary. If T is a row contraction, then it is not hard to check that if T is completely nonunitary in our sense, then it does not even have a nonzero *invariant* subspace M with $T|_M$ a spherical unitary. This is the definition used in [13].

2.4. Dilations. If $T = (T_1, \dots, T_d)$ is a tuple of commuting operators on a Hilbert space \mathcal{K} and if $S = (S_1, \dots, S_d)$ is a tuple of commuting operators on a larger Hilbert space $\mathcal{L} \supset \mathcal{K}$, then we say that T dilates to S if \mathcal{L} decomposes as $\mathcal{L} = \mathcal{L}_- \oplus \mathcal{K} \oplus \mathcal{L}_+$ such that with respect to this decomposition,

$$S_k = \begin{bmatrix} * & 0 & 0 \\ * & T_k & 0 \\ * & * & * \end{bmatrix}, \quad \text{for } k = 1, \dots, d.$$

Recall that the polynomials are multipliers of every regular unitarily invariant space \mathcal{H} . We let M_z denote the d -shift on \mathcal{H} , namely the tuple of multiplication operators $(M_{z_1}, \dots, M_{z_d})$. Abstract dilation theory in the form of Arveson’s dilation theorem yields the following result (cf. [6, Section 8]).

Theorem 2.4. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d , and let T be a tuple of commuting operators on a Hilbert space \mathcal{K} . Then the following are equivalent:*

- (i) T admits an $A(\mathcal{H})$ -functional calculus
- (ii) T dilates to $M_z^\kappa \oplus U$ for some cardinal κ and spherical unitary U .

Moreover, if \mathcal{K} is separable, then the dilation $M_z^\kappa \oplus U$ can be realized on a separable space.

Proof. The implication (ii) \Rightarrow (i) is clear, since M_z^κ tautologically admits an $A(\mathcal{H})$ -functional calculus, and U admits an $A(\mathcal{H})$ -functional calculus since the multiplier norm dominates the supremum norm.

Conversely, suppose that (i) holds. By Arveson’s dilation theorem (see, for example, [25, Corollary 7.7]), the $A(\mathcal{H})$ -functional calculus for T dilates to a $*$ -representation $\pi : C^*(A(\mathcal{H})) \rightarrow \mathcal{B}(\mathcal{L})$, for some Hilbert space \mathcal{L} . A result of Sarason (see, for example, [3, Lemma 10.2]) implies that \mathcal{L} decomposes as $\mathcal{L} = \mathcal{L}_- \oplus \mathcal{K} \oplus \mathcal{L}_+$ such that

$$\pi(M_{z_k}) = \begin{bmatrix} * & 0 & 0 \\ * & T_k & 0 \\ * & * & * \end{bmatrix}$$

for $k = 1, \dots, d$. It remains to show that $\pi(M_z)$ is unitarily equivalent to $M_z^\kappa \oplus U$.

Since $\lim_{n \rightarrow \infty} a_n/a_{n+1} = 1$, an application of [20, Theorem 4.6] shows that there exists a short exact sequence of C^* -algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow C^*(A(\mathcal{H})) \longrightarrow C(\partial\mathbb{B}_d) \longrightarrow 0,$$

where $\mathcal{K}(\mathcal{H})$ is the ideal of all compact operators on \mathcal{H} , the first map is the inclusion map, and the second map sends M_{z_k} to z_k for $k = 1, \dots, d$. In this setting, general results about representations of C^* -algebras (see, for example, the discussion preceding Theorem 1.3.4 and Corollary 2 in Section 1.4 in [5]) show that π splits as an orthogonal direct sum $\pi = \pi_1 \oplus \pi_2$, where π_1 is unitarily equivalent to a multiple of the identity representation and π_2 annihilates the compact operators and hence can be regarded as a representation of $C(\partial\mathbb{B}_d)$. Consequently, $\pi_1(M_z)$ is unitarily equivalent to M_z^κ for some cardinal κ , and $\pi_2(M_z)$ is a spherical unitary tuple.

The additional claim follows because if \mathcal{H} is separable, then the dilation π , which is obtained from Stinespring’s dilation theorem, can be realized on a separable space (see the discussion following [25, Theorem 4.1]). □

In the setting of Theorem 2.4, we say that $M_z^\kappa \oplus U \in \mathcal{B}(\mathcal{L})^d$ is a *minimal dilation* of T if \mathcal{L} is the smallest reducing subspace for $M_z^\kappa \oplus U$ that contains \mathcal{K} . If T acts on a separable Hilbert space, then so does every minimal dilation of T . Note that minimal dilations of this type are generally not unique up to unitary equivalence. Indeed, if $\mathcal{H} = H^2$ (the Hardy space on the unit disc) and if $T = M_z \in \mathcal{B}(H^2)$, then M_z itself and the bilateral shift are both minimal dilations of M_z of the form $M_z \oplus U$, but they are not unitarily equivalent.

2.5. Normal tuples. We briefly recall a few notions from the spectral theory for commuting normal operators. Let $N = (N_1, \dots, N_d)$ be a commuting tuple of normal operators on a Hilbert space \mathcal{K} . By the Putnam–Fuglede theorem, the unital C^* -algebra $C^*(N) = C^*(N_1, \dots, N_d)$ is commutative. Moreover, the maximal ideal space X of $C^*(N)$ can be identified with a compact subset of \mathbb{C}^d via

$$X \rightarrow \mathbb{C}^d, \quad \rho \mapsto (\rho(N_1), \dots, \rho(N_d)).$$

The range of this map is called the *joint spectrum* of N , denoted by $\sigma(N)$. By the Gelfand–Naimark theorem, $C^*(N)$ is isomorphic to $C(\sigma(N))$ via a $*$ -homomorphism that sends N_i to z_i for each $i = 1, \dots, d$.

A well-known result about representations of algebras of continuous functions (see [16, Theorem IX.1.14]) shows that there exists a *projection-valued spectral measure* E on $\sigma(N)$ such that

$$p(N, N^*) = \int_{\sigma(N)} p(z, \bar{z}) \, dE$$

for every $p \in \mathbb{C}[z_1, \dots, z_d, \bar{z}_1, \dots, \bar{z}_d]$. If \mathcal{K} is separable, then $W^*(N)$, the von Neumann algebra generated by N , admits a separating vector $x \in \mathcal{K}$ (see [16, Corollary IX.7.9]), so the scalar-valued measure $\mu = \langle E(\cdot)x, x \rangle$ has the same null sets as E . Any scalar-valued measure with this property is called a (*scalar-valued spectral measure for N*). It is unique up to mutual absolute continuity. Moreover, if μ is a scalar-valued spectral measure for N , then the isomorphism between $C^*(N)$ and $C(\sigma(N))$ extends to a $*$ -isomorphism between $W^*(N)$ and $L^\infty(\mu)$ which is a weak- $*$ -weak- $*$ homeomorphism (see [18, Theorem II.2.5]).

3. HENKIN MEASURES

We begin by recalling a few definitions of measures on the unit sphere which are related to the ball algebra. Background material on this topic can be found in [27, Chapter 9]. Let $M(\partial\mathbb{B}_d)$ denote the space of complex regular Borel measures on $\partial\mathbb{B}_d$, which can be identified with the dual space of $C(\partial\mathbb{B}_d)$, the set of continuous functions on $\partial\mathbb{B}_d$. A probability measure $\mu \in M(\partial\mathbb{B}_d)$ is called a *representing measure for the origin* if

$$p(0) = \int_{\partial\mathbb{B}_d} p \, d\mu$$

holds for every polynomial p (or, equivalently, for every function in the ball algebra). A Borel set E is said to be *totally null* if it is null for every representing measure of the origin. A measure $\mu \in M(\partial\mathbb{B}_d)$ is said to be *totally singular* if it is singular with respect to every representing measure of the origin.

Let \mathcal{H} be a unitarily invariant space on \mathbb{B}_d such that the polynomials are multipliers of \mathcal{H} . Let $A(\mathcal{H})$ denote the norm closure of the polynomials. Since the supremum norm is dominated by the multiplier norm, $A(\mathcal{H})$ is contained in the

ball algebra, and so every function in $A(\mathcal{H})$ extends uniquely to a continuous function on $\overline{\mathbb{B}_d}$.

Then, we say that a functional $\rho \in A(\mathcal{H})^*$ is *Mult(\mathcal{H})-Henkin* if it extends to a weak-* continuous functional on $\text{Mult}(\mathcal{H})$. We say that a measure $\mu \in M(\partial\mathbb{B}_d)$ is a *Mult(\mathcal{H})-Henkin measure* if the functional $\rho_\mu \in A(\mathcal{H})^*$ defined by

$$\rho_\mu(f) = \int_{\partial\mathbb{B}_d} f d\mu, \quad f \in A(\mathcal{H}),$$

is *Mult(\mathcal{H})-Henkin*.

The classical Henkin measures are precisely the $H^\infty(\mathbb{B}_d)$ -Henkin measures in our terminology. Moreover, $\text{Mult}(H^2_d)$ -Henkin measures are called \mathcal{A}_d -Henkin in [13].

The following lemma is a straightforward generalization of [19, Lemma 1.1]. In particular, specializing to the case $\mathcal{K} = \mathbb{C}$, we see that the notion of *Mult(\mathcal{H})-Henkin* measures (or functionals) only depends on $\text{Mult}(\mathcal{H})$ and not on the particular choice of Hilbert space \mathcal{H} .

Lemma 3.1. *Let \mathcal{H} be a unitarily invariant space on \mathbb{B}_d such that the polynomials are multipliers of \mathcal{H} , let \mathcal{K} be a Hilbert space, and let $\Phi : A(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ be a bounded linear map. Then the following are equivalent:*

- (i) Φ extends to a weak-* continuous linear map from $\text{Mult}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$.
- (ii) Whenever (p_n) is a sequence of polynomials such that $\|p_n\|_{\text{Mult}(\mathcal{H})} \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} p_n(z) = 0$ for all $z \in \mathbb{B}_d$, the sequence $(\Phi(p_n))$ converges to 0 in the weak-* topology of $\mathcal{B}(\mathcal{K})$.

If Φ is multiplicative, then so is its weak-* continuous extension to $\text{Mult}(\mathcal{H})$. Moreover, the set of *Mult(\mathcal{H})-Henkin* functionals forms a norm closed subspace of $A(\mathcal{H})^*$.

Proof. The implication (i) \Rightarrow (ii) is trivial. For the proof of the reverse implication, we first observe that unitary invariance of \mathcal{H} implies that there exists a strongly continuous one-parameter unitary group $(U_t)_{t \in \mathbb{R}}$ on \mathcal{H} defined by $(U_t f)(z) = f(e^{it}z)$ for $f \in \mathcal{H}$ and $z \in \mathbb{B}_d$. If $\varphi \in \text{Mult}(\mathcal{H})$, then $U_t M_\varphi U_t^* = M_{\varphi^{(t)}}$, where $\varphi^{(t)}(z) = \varphi(e^{it}z)$. Consequently, if $\varphi \in \text{Mult}(\mathcal{H})$, then $\varphi^{(t)}$ belongs to $\text{Mult}(\mathcal{H})$ with $\|\varphi^{(t)}\|_{\text{Mult}(\mathcal{H})} = \|\varphi\|_{\text{Mult}(\mathcal{H})}$, and the map $t \mapsto \varphi^{(t)}$ is continuous in the strong operator topology. Using these two facts, a routine argument involving the Fejér kernel (cf. [23, Lemma I 2.5]) shows that if $\varphi \in \text{Mult}(\mathcal{H})$ and if $\varphi = \sum_{n=0}^\infty \varphi_n$ is the homogeneous expansion of φ as an analytic function on \mathbb{B}_d , then the sequence of polynomials (ψ_n) defined by

$$\psi_n = \frac{1}{n+1} \sum_{k=0}^n \sum_{j=0}^k \varphi_j$$

satisfies $\|\psi_n\|_{\text{Mult}(\mathcal{H})} \leq \|\varphi\|_{\text{Mult}(\mathcal{H})}$ and converges to φ in the strong operator topology. With this observation in hand, the proof of [19, Lemma 1.1] carries over to the present setting. Moreover, the first additional claim follows from the fact that multiplication is separately continuous in the weak-* topology.

To show that the set of *Mult(\mathcal{H})-Henkin* functionals is norm closed inside of $A(\mathcal{H})^*$, observe that the argument in the first part of the proof shows that the unit ball of $A(\mathcal{H})$ is weak-* dense in the unit ball of $\text{Mult}(\mathcal{H})$. Thus, the contraction

$$\text{Mult}(\mathcal{H})^* \rightarrow A(\mathcal{H})^*, \quad \rho \mapsto \rho|_{A(\mathcal{H})}$$

restricts to an isometry between the predual of $\text{Mult}(\mathcal{H})$ and the set of $\text{Mult}(\mathcal{H})$ -Henkin functionals. Since the predual of $\text{Mult}(\mathcal{H})$ is complete, it follows that the set of $\text{Mult}(\mathcal{H})$ -Henkin functionals is closed in $A(\mathcal{H})^*$. (Alternatively, this can also be seen by using the characterization (ii) of $\text{Mult}(\mathcal{H})$ -Henkin functionals.) \square

Remark 3.2. Since the multiplier norm dominates the supremum norm over $\overline{\mathbb{B}_d}$, it also follows from Lemma 3.1 that every classical Henkin measure is a $\text{Mult}(\mathcal{H})$ -Henkin measure. The converse is false in general. Indeed, there are $\text{Mult}(H^2_2)$ -Henkin measures that are not classical Henkin measures [21].

According to Henkin’s theorem [22] (see also [27, Theorem 9.3.1]), classical Henkin measures form a band. Clouâtre and Davidson showed that $\text{Mult}(H^2_d)$ -Henkin measures form a band as well [14, Theorem 5.4]. Their proof generalizes to $\text{Mult}(\mathcal{H})$ -Henkin measures. If $\nu, \mu \in M(\partial\mathbb{B}_d)$ are complex measures, then we write $\nu \ll \mu$ if $\nu \ll |\mu|$.

Lemma 3.3. *Let \mathcal{H} be a unitarily invariant space on \mathbb{B}_d such that the polynomials are multipliers of \mathcal{H} . Then the $\text{Mult}(\mathcal{H})$ -Henkin measures form a band. That is, if $\mu, \nu \in M(\partial\mathbb{B}_d)$ such that μ is $\text{Mult}(\mathcal{H})$ -Henkin and $\nu \ll \mu$, then ν is $\text{Mult}(\mathcal{H})$ -Henkin.*

Proof. While the relevant part of the proof of [14, Theorem 5.4] does generalize to this setting, other parts do not easily generalize. Thus, for clarity, we include the proof here. Given a subset E of $\partial\mathbb{B}_d$, let $\|f\|_E$ denote the value $\sup_{z \in E} |f(z)|$.

By the Glicksberg–König–Seever decomposition theorem and Henkin’s theorem (see Theorem 9.4.4, Theorem 9.3.1 and Section 9.8 in [27]), we can decompose $\mu = \mu_a + \mu_s$, where μ_a is $H^\infty(\mathbb{B}_d)$ -Henkin and μ_s is totally singular. Since every $H^\infty(\mathbb{B}_d)$ -Henkin measure is $\text{Mult}(\mathcal{H})$ -Henkin, the assumption implies that $\mu_s = \mu - \mu_a$ is $\text{Mult}(\mathcal{H})$ -Henkin. Since $\nu \ll \mu$, by the Radon–Nikodym theorem, there is a function $h \in L^1(|\mu|)$ such that $\nu = h\mu = h\mu_a + h\mu_s$. By Henkin’s theorem [27, Theorem 9.3.1], $h\mu_a$ is $H^\infty(\mathbb{B}_d)$ -Henkin and hence, $\text{Mult}(\mathcal{H})$ -Henkin. It therefore suffices to show that $h\mu_s$ is $\text{Mult}(\mathcal{H})$ -Henkin.

Let $\varepsilon > 0$, and let g be a continuous function on $\partial\mathbb{B}_d$ such that $\|g - h\|_{L^1(|\mu_s|)} < \varepsilon$. Since μ_s is totally singular, Rainwater’s lemma (see [27, Lemma 9.4.3]) shows that μ_s is concentrated on an F_σ set which is totally null. Thus, there exists a totally null compact set $K \subset \partial\mathbb{B}_d$ such that $\|\mu_s - \mu_s|_K\|_{M(\partial\mathbb{B}_d)} < \varepsilon / (\|g\|_{\partial\mathbb{B}_d} + 1)$. By Bishop’s theorem (see [27, Theorem 10.1.2]), there exists a function in the ball algebra which agrees with g on K and is bounded by $\|g\|_{\partial\mathbb{B}_d}$. Thus, there is a polynomial p such that $\|p - g\|_K < \varepsilon$ and $\|p\|_{\partial\mathbb{B}_d} \leq \|g\|_{\partial\mathbb{B}_d}$. Clearly, $p\mu_s$ is $\text{Mult}(\mathcal{H})$ -Henkin since μ_s is $\text{Mult}(\mathcal{H})$ -Henkin. Moreover,

$$\begin{aligned} \|p\mu_s - h\mu_s\|_{A(\mathcal{H})^*} &\leq \|p\mu_s - h\mu_s\|_{M(\partial\mathbb{B}_d)} \\ &\leq \|(p - g)\mu_s|_K\|_{M(\partial\mathbb{B}_d)} + \|(p - g)(\mu_s - \mu_s|_K)\|_{M(\partial\mathbb{B}_d)} \\ &\quad + \|g\mu_s - h\mu_s\|_{M(\partial\mathbb{B}_d)} \\ &\leq |\mu_s|(K) \|p - g\|_K + \|p - g\|_{\partial\mathbb{B}_d} \|\mu_s - \mu_s|_K\|_{M(\partial\mathbb{B}_d)} \\ &\quad + \|g - h\|_{L^1(|\mu_s|)} \\ &\leq \varepsilon(3 + \|\mu_s\|_{M(\partial\mathbb{B}_d)}). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and since the set of $\text{Mult}(\mathcal{H})$ -Henkin functionals is norm closed in $A(\mathcal{H})^*$ by Lemma 3.1, it follows that integration against $h\mu_s$ is $\text{Mult}(\mathcal{H})$ -Henkin, as desired. \square

The spectral measure of a commuting tuple of normal operators (see Subsection 2.5) is only unique up to mutual absolute continuity. Nevertheless, Lemma 3.3 shows that it is meaningful to say that the spectral measure of a commuting spherical unitary is $\text{Mult}(\mathcal{H})$ -Henkin. Using Lemmas 3.1 and 3.3, the following result can now be proved just like [13, Lemma 3.1].

Lemma 3.4. *Let \mathcal{H} be a unitarily invariant space on \mathbb{B}_d such that the polynomials are multipliers of \mathcal{H} , and let U be a spherical unitary on a separable Hilbert space. Then U is $\text{Mult}(\mathcal{H})$ -absolutely continuous if and only if the spectral measure of U is $\text{Mult}(\mathcal{H})$ -Henkin. \square*

We also obtain an answer to (Q2) in terms of minimal dilations of an operator tuple. Recall from the discussion at the end of Subsection 2.4 that minimal dilations are not necessarily unique. The following result generalizes [13, Theorem 3.2], at least in the separable case. Since we are dealing with dilations as opposed to co-extensions, the proof is somewhat more complicated.

Proposition 3.5. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d . Let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on a separable Hilbert space \mathcal{K} , and let $M_z^\kappa \oplus U \in \mathcal{B}(\mathcal{H}^\kappa \oplus \mathcal{L})$ be a minimal dilation of T , where U is spherical unitary. Then T is $\text{Mult}(\mathcal{H})$ -absolutely continuous if and only if the spectral measure of U is $\text{Mult}(\mathcal{H})$ -Henkin.*

Proof. If the spectral measure of U is $\text{Mult}(\mathcal{H})$ -Henkin, then U is $\text{Mult}(\mathcal{H})$ -absolutely continuous by Lemma 3.4. Since M_z^κ is $\text{Mult}(\mathcal{H})$ -absolutely continuous and since $p(T) = P_{\mathcal{K}}p(M_z^\kappa \oplus U)|_{\mathcal{K}}$ for every polynomial p , it follows that T is $\text{Mult}(\mathcal{H})$ -absolutely continuous as well.

Conversely, suppose that T is $\text{Mult}(\mathcal{H})$ -absolutely continuous. Let E denote the projection-valued spectral measure of U , and let

$$M = \{(x, y) \in \mathcal{L} \oplus \mathcal{L} : \langle E(\cdot)x, y \rangle \text{ is } \text{Mult}(\mathcal{H})\text{-Henkin}\}.$$

We have to show that $M = \mathcal{L} \oplus \mathcal{L}$. For $y \in \mathcal{L}$, let

$$M_y = \{x \in \mathcal{L} : (x, y) \in M\}.$$

Since the map

$$\mathcal{L} \rightarrow A(\mathcal{H})^*, \quad x \mapsto \langle E(\cdot)x, y \rangle$$

is linear and continuous, and since the set of all $\text{Mult}(\mathcal{H})$ -Henkin functionals forms a norm closed subspace of $A(\mathcal{H})^*$ by Lemma 3.1, we see that each M_y is a closed subspace of \mathcal{L} .

We claim that each M_y is reducing for U . To this end, note that

$$\int_{\partial\mathbb{B}_d} p d\langle E(\cdot)U_i x, y \rangle = \langle p(U)U_i x, y \rangle = \int_{\partial\mathbb{B}_d} p z_i d\langle E(\cdot)x, y \rangle$$

and

$$\int_{\partial\mathbb{B}_d} p d\langle E(\cdot)U_i^* x, y \rangle = \langle p(U)U_i^* x, y \rangle = \int_{\partial\mathbb{B}_d} p \bar{z}_i d\langle E(\cdot)x, y \rangle$$

holds for every polynomial p . Clearly, the measures $z_i \langle E(\cdot)x, y \rangle$ and $\bar{z}_i \langle E(\cdot)x, y \rangle$ are absolutely continuous with respect to $\langle E(\cdot)x, y \rangle$, hence Lemma 3.3 shows that if $x \in M_y$, then also $U_i x \in M_y$ and $U_i^* x \in M_y$. Thus, each M_y is reducing for U .

Next, we show that if $x, y \in \mathcal{K}$, then $(P_{\mathcal{L}}x, P_{\mathcal{L}}y) \in M$. Let p be a polynomial. Then

$$\begin{aligned} \int_{\mathbb{B}_d} p d\langle E(\cdot)P_{\mathcal{L}}x, P_{\mathcal{L}}y \rangle &= \langle p(U)P_{\mathcal{L}}x, P_{\mathcal{L}}y \rangle = \langle p(M_z^{\kappa} \oplus U)x, y \rangle - \langle p(M_z^{\kappa})P_{\mathcal{H}^{\kappa}}x, P_{\mathcal{H}^{\kappa}}y \rangle \\ &= \langle p(T)x, y \rangle - \langle p(M_z^{\kappa})P_{\mathcal{H}^{\kappa}}x, P_{\mathcal{H}^{\kappa}}y \rangle. \end{aligned}$$

Since T and M_z^{κ} are both $\text{Mult}(\mathcal{H})$ -absolutely continuous, we see that $\langle E(\cdot)P_{\mathcal{L}}x, P_{\mathcal{L}}y \rangle$ is $\text{Mult}(\mathcal{H})$ -Henkin so that $(P_{\mathcal{L}}x, P_{\mathcal{L}}y) \in M$.

Finally, since $M_z^{\kappa} \oplus U$ is a minimal dilation of T , it follows in particular that \mathcal{L} is the smallest reducing subspace for U that contains $P_{\mathcal{L}}\mathcal{K}$. From the preceding two paragraphs, we therefore deduce that if $y \in P_{\mathcal{L}}\mathcal{K}$, then $M_y = \mathcal{L}$. Another application of Lemma 3.3 shows that if $(x, y) \in M$, then also $(y, x) \in M$, since $\langle E(\cdot)x, y \rangle = \overline{\langle E(\cdot)y, x \rangle}$. Thus, for every $y \in \mathcal{L}$, the space M_y contains $P_{\mathcal{L}}\mathcal{K}$. Using minimality of the dilation again, we conclude that $M_y = \mathcal{L}$ for every $y \in \mathcal{L}$ so that $M = \mathcal{L} \oplus \mathcal{L}$, as desired. \square

4. $\text{Mult}(\mathcal{H})$ -ABSOLUTE CONTINUITY

The goal of this section is to prove our main result, Theorem 1.1. Before proceeding with the proof, we require a lemma about totally singular measures on $\partial\mathbb{B}_d$. It is a well-known consequence of the classical F. and M. Riesz theorem that if μ is a measure on the unit circle which is singular with respect to Lebesgue measure, then the analytic polynomials are weak-* dense in $L^{\infty}(\mu)$ (see, for example, [17, Exercise VI.7.10]). The following lemma is a generalization of this fact to higher dimensions.

Lemma 4.1. *Let μ be a positive totally singular measure on $\partial\mathbb{B}_d$. Then the analytic polynomials are weak-* dense in $L^{\infty}(\mu)$.*

Proof. Let $f \in L^1(\mu)$ such that

$$\int_{\partial\mathbb{B}_d} pf \, d\mu = 0$$

for all analytic polynomials p . By the Hahn–Banach theorem, we need to show that $f = 0 \in L^1(\mu)$. To this end, observe that the measure $f\mu$ annihilates the ball algebra, hence the Cole–Range theorem (see [27, Theorem 9.6.1]) shows that there exists a representing measure ρ for the origin such that $f\mu \ll \rho$. On the other hand, since μ is totally singular, μ , and hence $f\mu$, is singular with respect to ρ . Thus, $f\mu = 0$, as desired. \square

Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d . To prove Theorem 1.1, we will use the dilation of Theorem 2.4 and so require information about operator tuples of the form $M_z^{\kappa} \oplus U$, where $M_z = (M_{z_1}, \dots, M_{z_d}) \in \mathcal{B}(\mathcal{H})^d$ and $U \in \mathcal{B}(\mathcal{L})^d$ is a spherical unitary. Let us assume that \mathcal{L} is separable, and let $\mu \in M(\partial\mathbb{B}_d)$ be a scalar-valued spectral measure of U . By the Glicksberg–König–Seever decomposition theorem [27, Theorem 9.4.4] and Henkin’s theorem [27, Theorem 9.3.1] (see also the discussion in [27, Section 9.8]), we can decompose $\mu = \mu_a + \mu_s$, where μ_a is $H^{\infty}(\mathbb{B}_d)$ -Henkin and μ_s is totally singular. Correspondingly, $U = U_a \oplus U_s$, where μ_a is a spectral measure of U_a and μ_s is a spectral measure of U_s . Let $\mathcal{L}_a, \mathcal{L}_s$ denote the spaces on which U_a, U_s act, respectively.

The following lemma plays a key role in the proof of Theorem 1.1.

Lemma 4.2. *In the setting of the preceding paragraph, let \mathcal{M} be the unital weak- $*$ closed subalgebra of $\mathcal{B}(\mathcal{H}^\kappa \oplus \mathcal{L}_a \oplus \mathcal{L}_s)$ generated by $M_z^\kappa \oplus U_a \oplus U_s$, and let*

$$\mathcal{M}_s = \{N \in W^*(U_s) : 0 \oplus 0 \oplus N \in \mathcal{M}\}.$$

Then \mathcal{M}_s is a weak- $$ closed ideal of the von Neumann algebra $W^*(U_s)$.*

Proof. It is clear that \mathcal{M}_s is a weak- $*$ closed subspace of $W^*(U_s)$. To show that it is an ideal, let $N \in \mathcal{M}_s$. For $p \in \mathbb{C}[z_1, \dots, z_d]$,

$$0 \oplus 0 \oplus p(U_s)N = p(M_z^\kappa \oplus U_a \oplus U_s)(0 \oplus 0 \oplus N)$$

belongs to \mathcal{M} , and hence $p(U_s)N \in \mathcal{M}_s$. The spectral measure μ_s of U_s is totally singular, and so Lemma 4.1 implies that the analytic polynomials are weak- $*$ dense in $L^\infty(\mu_s)$. Recall that $W^*(U_s)$ and $L^\infty(\mu_s)$ are isomorphic as von Neumann algebras via an isomorphism that sends the tuple U_s to (z_1, \dots, z_d) . Thus, operators of the form $p(U_s)$, where p is an analytic polynomial, are weak- $*$ dense in $W^*(U_s)$. Since \mathcal{M}_s is weak- $*$ closed, we conclude that $RN \in \mathcal{M}_s$ for all $R \in W^*(U_s)$, and by the commutativity of $W^*(U_s)$, NR is also in \mathcal{M}_s . Thus, \mathcal{M}_s is an ideal. \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $T = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{K})^d$ be a commuting operator tuple on a Hilbert space \mathcal{K} that admits an $A(\mathcal{H})$ -functional calculus. Suppose that T is not $\text{Mult}(\mathcal{H})$ -absolutely continuous. We will show that T has a spherical unitary summand.

In a first step, we observe that it suffices to consider the case where \mathcal{K} is separable. Indeed, since T is not $\text{Mult}(\mathcal{H})$ -absolutely continuous and since the weak- $*$ and weak operator topologies coincide on bounded subsets of $\mathcal{B}(\mathcal{K})$, there exists by Lemma 3.1 a sequence of polynomials (p_n) in the unit ball of $\text{Mult}(\mathcal{H})$ that converges to 0 in the weak- $*$ topology of $\text{Mult}(\mathcal{H})$ and $x, y \in \mathcal{K}$ such that the sequence $((p_n(T)x, y))$ does not converge to 0. Let \mathcal{K}_0 be the smallest reducing subspace for T that contains x and y . Then \mathcal{K}_0 is separable, and $T|_{\mathcal{K}_0}$ is not $\text{Mult}(\mathcal{H})$ -absolutely continuous. Moreover, if $T|_{\mathcal{K}_0}$ has a spherical unitary summand, then so does T . Thus, we may assume that \mathcal{K} is separable.

By Theorem 2.4, the tuple T dilates to $M_z^\kappa \oplus U$, where κ is a countable cardinal and U is a spherical unitary on a separable Hilbert space. As in the discussion preceding Lemma 4.2, we can decompose $U = U_a \oplus U_s$. Since T is not $\text{Mult}(\mathcal{H})$ -absolutely continuous, Lemma 3.1 guarantees the existence of a sequence of polynomials (p_n) in the unit ball of $\text{Mult}(\mathcal{H})$ such that (p_n) converges to 0 in the weak- $*$ topology of $\text{Mult}(\mathcal{H})$ but the bounded sequence $(p_n(T))$ does not converge to 0 in the weak- $*$ topology of $\mathcal{B}(\mathcal{K})$. Since \mathcal{K} is separable, the weak- $*$ topology on the unit ball of $\mathcal{B}(\mathcal{K})$ is metrizable. Then, by passing to a subsequence, we can assume that $(p_n(T))$ actually converges to a nonzero operator $A \in \mathcal{B}(\mathcal{K})$.

Observe that $(p_n(M_z^\kappa))$ converges to 0 weak- $*$. Moreover, (p_n) is bounded in the supremum norm on \mathbb{B}_d and converges to 0 pointwise in \mathbb{B}_d . Hence, as the spectral measure of U_a is $H^\infty(\mathbb{B}_d)$ -Henkin, the sequence $(p_n(U_a))$ converges to 0 weak- $*$ as well. From

$$p_n(T) = P_{\mathcal{K}} p_n(M_z^\kappa \oplus U_a \oplus U_s)|_{\mathcal{K}}$$

and the fact that $(p_n(T))$ converges to the nonzero operator A , we deduce that the bounded sequence $(p_n(U_s))$ does not converge to 0 weak-*. By passing to another subsequence, we can assume that $(p_n(U_s))$ converges to a nonzero contraction $R \in W^*(U_s)$ so that $A = P_{\mathcal{K}}(0 \oplus 0 \oplus R)|_{\mathcal{K}}$.

Let \mathcal{M} and \mathcal{M}_s be defined as in Lemma 4.2. By our previous arguments, R belongs to \mathcal{M}_s . By Lemma 4.2, \mathcal{M}_s is a weak-* closed ideal of the von Neumann algebra $W^*(U_s)$. Thus, there exists an orthogonal projection $P \in \mathcal{M}_s$ such that $PS = S$ for all $S \in \mathcal{M}_s$ (see, for example, [9, Section III.1.1.13]). Since $M_z^{\kappa} \oplus U_a \oplus U_s$ dilates T , the space \mathcal{K} is semi-invariant for $M_z^{\kappa} \oplus U_a \oplus U_s$ and hence for every operator in \mathcal{M} . Therefore, the map

$$\mathcal{M} \rightarrow \mathcal{B}(\mathcal{K}), \quad S \mapsto P_{\mathcal{K}}S|_{\mathcal{K}}$$

is multiplicative, which we will use repeatedly. Since \mathcal{K} is semi-invariant and thus reducing for the self-adjoint operator $0 \oplus 0 \oplus P \in \mathcal{M}$, we see that $Q = P_{\mathcal{K}}0 \oplus 0 \oplus P|_{\mathcal{K}}$ is an orthogonal projection. Recall that \mathcal{M} is commutative, and so Q and T_k commute for $1 \leq k \leq d$. Moreover, from $PR = R$, we deduce that $QA = A$, which implies $Q \neq 0$.

We finish the proof by showing that the restriction of T to the range of Q is a spherical unitary. Applying Lemma 4.2 again, we see that $P(U_s)_k$ and $P(U_s)_k^*$ belong to \mathcal{M}_s for $1 \leq k \leq d$, hence \mathcal{K} is reducing for each operator $0 \oplus 0 \oplus P(U_s)_k$. Moreover,

$$QT_k = P_{\mathcal{K}}(0 \oplus 0 \oplus P(U_s)_k)|_{\mathcal{K}}$$

for $1 \leq k \leq d$. In particular, each QT_k is the compression of a normal operator to a reducing subspace and thus, is normal. Moreover,

$$\sum_{k=1}^d QT_k(QT_k)^* = P_{\mathcal{K}}(0 \oplus 0 \oplus P)|_{\mathcal{K}} = Q,$$

which completes the proof. □

Moreover, in the presence of a spherical unitary summand, we obtain the following characterization of $\text{Mult}(\mathcal{H})$ -absolute continuity.

Theorem 4.3. *Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d , and let $T = (T_1, \dots, T_d)$ be a commuting tuple of operators on a separable Hilbert space that admits an $A(\mathcal{H})$ -functional calculus. Let $T = T_{cnu} \oplus U$ be the decomposition of T from Proposition 2.2. Then T is $\text{Mult}(\mathcal{H})$ -absolutely continuous if and only if the spectral measure of U is $\text{Mult}(\mathcal{H})$ -Henkin.*

Proof. This follows immediately from Theorem 1.1 and Lemma 3.4. □

5. COMPLETE NEVANLINNA–PICK KERNELS

In this section, we obtain a refined version of Theorem 1.1 when \mathcal{H} is a complete Nevanlinna–Pick space. Specifically, we use the fact that in this case, there exists a more checkable condition for T to admit an $A(\mathcal{H})$ -functional calculus. Background material on Nevanlinna–Pick spaces can be found in the book [3].

Let \mathcal{H} be a regular unitarily invariant space on \mathbb{B}_d with reproducing kernel $K(z, w) = \sum_{n=0}^\infty a_n \langle z, w \rangle^n$. A straightforward generalization of [3, Theorem 7.33] shows that \mathcal{H} is an irreducible complete Nevanlinna–Pick space if and only if the sequence (b_n) given by

$$(4) \quad \sum_{n=1}^\infty b_n t^n = 1 - \frac{1}{\sum_{n=0}^\infty a_n t^n}$$

satisfies $b_n \geq 0$ for all $n \geq 1$.

For example, the class of regular unitarily invariant complete Nevanlinna–Pick spaces on \mathbb{B}_d includes the spaces with reproducing kernels

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\alpha}$$

for $\alpha \in (0, 1]$ and the spaces with kernels

$$K(z, w) = \sum_{n=0}^\infty (n + 1)^s \langle z, w \rangle^n$$

for $s \leq 0$, and hence in particular the Drury–Arveson space and the Dirichlet space (see Subsection 2.1).

Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna–Pick space on \mathbb{B}_d with kernel K , and let (b_n) be the sequence of equation (4). We write $1/K(T, T^*) \geq 0$ if

$$\sum_{n=1}^N b_n \sum_{|\alpha|=n} \binom{n}{\alpha} T^\alpha (T^*)^\alpha \leq I$$

for all $N \in \mathbb{N}$. This definition goes back to work of Agler [1]. For more discussion, see [15, Section 5]. The following result is [15, Theorem 5.4].

Theorem 5.1. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna–Pick space on \mathbb{B}_d with kernel K . Let $T = (T_1, \dots, T_d)$ be a tuple of commuting operators on a Hilbert space. Then the following are equivalent:*

- (i) *The tuple T satisfies $1/K(T, T^*) \geq 0$.*
- (ii) *The tuple T admits an $A(\mathcal{H})$ -functional calculus.*

We remark that if \mathcal{H} is not a complete Nevanlinna–Pick space, then one can often still make sense of the condition $1/K(T, T^*) \geq 0$; see [2, 4]. In general, however, not every tuple which admits an $A(\mathcal{H})$ -functional calculus satisfies $1/K(T, T^*) \geq 0$, as the example of the Bergman space and the unilateral shift shows. Rather, the condition $1/K(T, T^*) \geq 0$ is related to the existence of co-extensions, whereas admitting an $A(\mathcal{H})$ -functional calculus is equivalent to the existence of dilations (see Theorem 2.4). In the complete Nevanlinna–Pick setting, there is no difference; see [15] for more discussion.

The following refinement of Theorem 4.3 in the complete Nevanlinna–Pick setting is almost immediate.

Corollary 5.2. *Let \mathcal{H} be a regular unitarily invariant complete Nevanlinna–Pick space on \mathbb{B}_d with kernel K . Let $T = (T_1, \dots, T_d)$ be a commuting tuple of opera-*

tors on a separable Hilbert space, and let $T = T_{cnu} \oplus U$ be the decomposition of Proposition 2.2. Then the following are equivalent:

- (i) The tuple T admits an $A(\mathcal{H})$ -functional calculus and is $\text{Mult}(\mathcal{H})$ -absolutely continuous.
- (ii) The tuple T satisfies $1/K(T, T^*) \geq 0$ and the spectral measure of U is $\text{Mult}(\mathcal{H})$ -Henkin.

Proof. This follows from Theorems 5.1 and 4.3. □

REFERENCES

- [1] Jim Agler, *The Arveson extension theorem and coanalytic models*, Integral Equations Operator Theory **5** (1982), no. 5, 608–631, DOI 10.1007/BF01694057. MR697007
- [2] Jim Agler and John E. McCarthy, *Nevanlinna–Pick kernels and localization*, Operator theoretical methods (Timișoara, 1998), Theta Found., Bucharest, 2000, pp. 1–20. MR1770310
- [3] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002. MR1882259
- [4] C.-G. Ambrozie, M. Engliš, and V. Müller, *Operator tuples and analytic models over general domains in \mathbb{C}^n* , J. Operator Theory **47** (2002), no. 2, 287–302. MR1911848
- [5] William Arveson, *An invitation to C^* -algebras*, Graduate Texts in Mathematics, No. 39, Springer-Verlag, New York-Heidelberg, 1976. MR0512360
- [6] William Arveson, *Subalgebras of C^* -algebras. III. Multivariable operator theory*, Acta Math. **181** (1998), no. 2, 159–228, DOI 10.1007/BF02392585. MR1668582
- [7] Ameer Athavale, *On the intertwining of joint isometries*, J. Operator Theory **23** (1990), no. 2, 339–350. MR1066811
- [8] Hari Bercovici, *Operator theory and arithmetic in H^∞* , Mathematical Surveys and Monographs, vol. 26, American Mathematical Society, Providence, RI, 1988. MR954383
- [9] B. Blackadar, *Operator algebras*, Theory of C^* -algebras and von Neumann algebras; Operator Algebras and Non-commutative Geometry, III, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006. MR2188261
- [10] Scott Brown and Bernard Chevreau, *Toute contraction à calcul fonctionnel isométrique est réflexive* (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 5, 185–188. MR955549
- [11] S. Brown, B. Chevreau, and C. Pearcy, *Contractions with rich spectrum have invariant subspaces*, J. Operator Theory **1** (1979), no. 1, 123–136. MR526294
- [12] Scott W. Brown, Bernard Chevreau, and Carl Pearcy, *On the structure of contraction operators. II*, J. Funct. Anal. **76** (1988), no. 1, 30–55, DOI 10.1016/0022-1236(88)90047-X. MR923043
- [13] Raphaël Clouâtre and Kenneth R. Davidson, *Absolute continuity for commuting row contractions*, J. Funct. Anal. **271** (2016), no. 3, 620–641, DOI 10.1016/j.jfa.2016.04.012. MR3506960
- [14] Raphaël Clouâtre and Kenneth R. Davidson, *Duality, convexity and peak interpolation in the Drury–Arveson space*, Adv. Math. **295** (2016), 90–149, DOI 10.1016/j.aim.2016.02.035. MR3488033
- [15] Raphaël Clouâtre and Michael Hartz, *Multiplier algebras of complete Nevanlinna–Pick spaces: Dilations, boundary representations and hyperrigidity*, J. Funct. Anal. **274** (2018), no. 6, 1690–1738, DOI 10.1016/j.jfa.2017.10.008. MR3758546
- [16] John B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR1070713
- [17] John B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs, vol. 36, American Mathematical Society, Providence, RI, 1991. MR1112128
- [18] Kenneth R. Davidson, *C^* -algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. MR1402012
- [19] Jörg Eschmeier, *Invariant subspaces for spherical contractions*, Proc. London Math. Soc. (3) **75** (1997), no. 1, 157–176, DOI 10.1112/S0024611597000300. MR1444317

- [20] Kunyu Guo, Junyun Hu, and Xianmin Xu, *Toeplitz algebras, subnormal tuples and rigidity on reproducing $\mathbf{C}[z_1, \dots, z_d]$ -modules*, J. Funct. Anal. **210** (2004), no. 1, 214–247, DOI 10.1016/j.jfa.2003.06.003. MR2052120
- [21] Michael Hartz, *Henkin measures for the Drury-Arveson space*, Michigan Math. J. (to appear), arXiv:1701.07777.
- [22] G. M. Henkin, *The Banach spaces of analytic functions in a ball and in a bicylinder are nonisomorphic* (Russian), Funkcional. Anal. i Priložen. **2** (1968), no. 4, 82–91. MR0415288
- [23] Yitzhak Katznelson, *An introduction to harmonic analysis*, 3rd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004. MR2039503
- [24] V. Müller and F.-H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 979–989, DOI 10.2307/2159525. MR1112498
- [25] Vern Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002. MR1976867
- [26] Vern I. Paulsen and Mrinal Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, vol. 152, Cambridge University Press, Cambridge, 2016. MR3526117
- [27] Walter Rudin, *Function theory in the unit ball of \mathbf{C}^n* , Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2008. MR2446682
- [28] Béla Sz.-Nagy and Ciprian Foiaş, *Sur les contractions de l'espace de Hilbert. VII. Triangulations canoniques. Fonction minimum* (French), Acta Sci. Math. (Szeged) **25** (1964), 12–37. MR0170216
- [29] Béla Sz.-Nagy, Ciprian Foias, Hari Bercovici, and László Kérchy, *Harmonic analysis of operators on Hilbert space*, Revised and enlarged edition, Universitext, Springer, New York, 2010. MR2760647

DEPARTMENT OF MATHEMATICS, BUCKNELL UNIVERSITY, 701 MOORE AVE, LEWISBURG, PENNSYLVANIA 17837

Email address: `kelly.bickel@bucknell.edu`

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY IN ST. LOUIS, ONE BROOKINGS DRIVE, ST. LOUIS, MISSOURI 63130

Email address: `mphartz@wustl.edu`

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY IN ST. LOUIS, ONE BROOKINGS DRIVE, ST. LOUIS, MISSOURI 63130

Email address: `mccarthy@wustl.edu`